Delft University of Technology
Faculty Electrical Engineering, Mathematics and Computer Science
Delft Institute of Applied Mathematics

Continued fractions: Properties and invariant measure

(Kettingbreuken: Eigenschappen en invariante maat)

Bachelor of science thesis

by

Angelina C. Lemmens

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Inhoudsopgave

1 Introduction

An irrational number can be represented in many ways. For example the representation could be a rational approximation and could be more convenient to work with than the original number. Representations can be for example the decimal expansion or the binary expansion. These representations are used in everyday life.

Less known is the representation by continued fractions. Continued fractions give the best approximation of irrational numbers by rational numbers. This is something that Christiaan Huygens already knew (and used to construct his planetarium). Number representations like the decimal expansion and continued fractions have relations with probability theory and dynamical systems. In this thesis this relation is the so-called ergodic theory, which can roughly be characterized as the probability theory of dynamical systems.

The history of continued fractions dates back to the Greeks. The (regular) continued fraction of a rational number is exactly equivalent to determining the greatest common divisor of the numberator and denominator of that rational via Euclid's divisor algorithm. The modern history of continued fractions started with Gauss who found the invariant measure of the so-called regular continued fraction. Other famous probabilists like Paul Lévy and Wolfgang Doeblin also contributed to what is nowadays called the "metric theory of continued fractions."

Through the centuries many variants on the continued fraction algorithm were invented. In this thesis we will look at the metric properties of some of these continued fractions, and we will introduce and study a new continued fraction algorithm.

First of all, in the following chapter, Chapter 2, we will look in particular at the convergence of several continued fraction algorithms. In Chapter 3 we will give the definition of an invariant measure followed by an explanation on the ergodic theorem in Chapter 4. In Chapter 5 we will introduce a new continued fraction and study some properties, such as convergence and ergodicity.

2 Continued fractions

A number $x \in \mathbb{R}$ can be represented in many different ways: One could write x as a fraction (when x is rational) or as a decimal number in which case one constructs a sequence of numbers with elements from $\{0, 1, ..., 9\}$. In this thesis we will represent a number as a finitly or infinitely repeated fraction of non-negative numbers in the following way:

$$x = a_0 + \cfrac{N}{a_1 + \cfrac{N}{a_2 + \cfrac{N}{\ddots}}}$$

This representation is called a continued fraction, with $a_0 \in \mathbb{Z}$ such that $x - a_0 \in [0, 1)$ and $a_i \in \mathbb{N}$ for $i \geq 1$. So for $x \in [0, 1)$ finding a_0 is easy, because then $a_0 = 0$. For $N = 1$ the representation is called a *regular continued fraction.* Regular continued fractions are the most common form of a continued fraction. Note that this continued fraction is finite if and only if $x \in \mathbb{Q}$. This follows from Euclid's algorithm (for details, see Chapter 1 in [13]). The regular continued fraction is infinite when x is irrational and is also unique.

2.1 Regular continued fractions

So how does one find an approximation for a number $x \in \mathbb{R}$ by rational numbers? How do we find for a number $x \in \mathbb{R}$ its (regular) continued fraction? For regular continued fractions we can define a map $T : [0, 1) \to [0, 1)$ by

$$T(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor \qquad \text{with } x \neq 0 \text{ and } T(0) = 0.$$

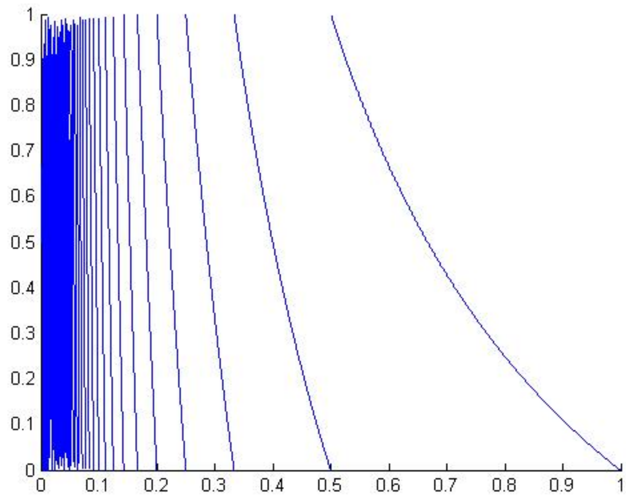
We can take $a_0 = \lfloor x \rfloor$, but for the rest of the elements a_i we use the map T.

First we define:

$$a_1(x) = \left\lfloor \frac{1}{x} \right\rfloor \qquad \text{if } x \neq 0, \text{ and } a_1(0) = \infty$$

It follows that now we can write:

$$x = \frac{1}{a_1 + T(x)}$$



Figuur 1: The regular continued fraction

Now for $T^{n-1}(x) \neq 0$, we define the next partial quotient of x by:

$$a_n(x) = \left\lfloor \frac{1}{T^{n-1}(x)} \right\rfloor .$$

Now that we have defined everything we need, we can find the digits (also known as partial quotients) $a_1, a_2, ...$ by repeatedly applying our function $T(x)$. However, the continued fraction is finite for rational numbers. This follows from Euclid's algorithm:

$$x \text{ has finitely many partial quotiënts} \Leftrightarrow x \in \mathbb{Q}$$

This means that there is an $n \geq 0$ so $T^n(x) = 0$. Now write $x = \frac{p_0}{q_0}$ where x is relatively prime and $0 < p_0 < q_0$. Now we define $T^n(x) = \frac{p_n}{q_n}$, so $T(\mathbb{Q} \cap [0,1]) = \mathbb{Q} \cap [0,1]$. Now:

$$T\left(\frac{p_0}{q_0}\right) = \frac{1}{\frac{p_0}{q_0}} - \left\lfloor \frac{1}{\frac{p_0}{q_0}} \right\rfloor = \frac{q_0}{p_0} - a_1 = \frac{q_0 - a_1 p_0}{p_0} = \frac{p_1}{q_1}.$$

Here p_1 and q_1 are relatively prime.

We can now see that the following inequalities hold:

$$p_1 \leq q_0 - a_1 p_0 \leq p_0.$$

Equality will occur when $p_0 = q_1$. So we find that the continued fraction in this case is finite and in other cases we find that $p_1 < p_0$. In the same way we find $p_2 < p_1$. Thus:

$$0 \leq \cdots < p_n < p_{n-1} < \cdots < p_1 < p_0.$$

for $p_i \in \mathbb{N}_{\geq 0}$ with $i \geq 0$. But then there exists a n for which $p_n = 0$. Here we stop the procedure of determining a_n and thus we find a finite continued fraction.

2.1.1 Convergence of the regular continued fraction

In the previous section we have seen how one can determine the partial quotients a_n given an x. The question now is: How good is this approximation? I.e. does the finite regular continued fraction really converge to x?
Let $x \in [0,1) \setminus \mathbb{Q}$. We now write:

$$c_n = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}.$$

Our goal now is to show that $\lim_{n\to\infty} c_n = x$. In order to do this we use the so called Möbius transformations. Define a map from $\mathbb{R}^* \to \mathbb{R}^*$, associated to a 2×2 matrix A as follows. Let $A \in SL(2,\mathbb{Z})$ be given by:

$$A(x) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Then we define the map $A : \mathbb{R}^* \to \mathbb{R}^*$ by

$$A(x) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}(x) = \frac{ax+b}{cx+d}.$$

where $a, b, c, d \in \mathbb{Z}$. Note that since $A \in SL(2,\mathbb{Z})$ we have that $\det(A) = ad - bc \in \{-1, +1\}$.
Claim: For two matrices $A, B \in SL(2,\mathbb{Z})$ it holds that $(AB)(x) = A(B(x))$.

It is not hard to see that this statement holds. Using freshman linear algebra we find that:

$$\begin{aligned}
(AB)(x) &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} (x) \\
&= \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix} (x) \\
&= \frac{(a_{11}b_{11} + a_{12}b_{21})x + a_{11}b_{12} + a_{12}b_{22}}{(a_{21}b_{11} + a_{22}b_{21}) + a_{21}b_{12} + a_{22}b_{22}}.
\end{aligned}$$

and

$$\begin{aligned}
A(B(x)) &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \left(\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} (x) \right) \\
&= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \left(\frac{b_{11}x + b_{12}}{b_{21}x + b_{22}} \right) \\
&= \frac{a_{11}\frac{b_{11}x+b_{12}}{b_{21}x+b_{22}} + a_{12}}{a_{21}\frac{b_{11}x+b_{12}}{b_{21}x+b_{22}} + a_{22}} \\
&= \frac{a_{11}(b_{11}x + b_{12}) + a_{12}(b_{21}x + b_{22})}{a_{21}(b_{11}x + b_{12}) + a_{22}(b_{21}x + b_{22})} \\
&= \frac{(a_{11}b_{11} + a_{12}b_{21})x + a_{11}b_{12} + a_{12}b_{22}}{(a_{21}b_{11} + a_{22}b_{21}) + a_{21}b_{12} + a_{22}b_{22}}.
\end{aligned}$$

So we see that for the corresponding linear transformation we have that

$$(AB)(x) = A(B(x)).$$

Now let $M_n = A_1 \cdot A_2 \cdot, \cdots, \cdot A_n$ where

$$A_n = \begin{bmatrix} 0 & 1 \\ 1 & a_n \end{bmatrix}, \qquad n \geq 1.$$

Now

$$\begin{aligned}
M_n(0) &= M_{n-1}A_n(0)\\
&= M_{n-1}\left(\frac{1}{a_n}\right)\\
&= M_{n-2}A_{n-1}\left(\frac{1}{a_n}\right)\\
&= M_{n-2}\left(\frac{1}{a_{n-1}+\frac{1}{a_n}}\right)\\
&\vdots\\
&= \cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{1}{\ddots+\cfrac{1}{a_n}}}}\\
&= c_n.
\end{aligned}$$

So we see that we can construct c_n in this way. Now we want to set up a reccurence relation. We do this by using M_n. We write:

$$M_n = \begin{bmatrix} r_n & p_n \\ s_n & q_n \end{bmatrix} \qquad \text{with } r_n, p_n, s_n, q_n \in \mathbb{Z} \text{ and } n \geq 1.$$

Obviously,

$$\begin{aligned}
M_n = M_{n-1}A_n &= \begin{bmatrix} r_{n-1} & p_{n-1} \\ s_{n-1} & q_{n-1} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & a_n \end{bmatrix}\\
&= \begin{bmatrix} p_{n-1} & a_np_{n-1}+r_{n-1} \\ q_{n-1} & a_nq_{n-1}+s_{n-1} \end{bmatrix}.
\end{aligned}$$

So now we find the following reccurence relation

$$\begin{aligned}
r_n &= p_{n-1} & \qquad p_n &= a_np_{n-1}+r_{n-1}\\
s_n &= q_{n-1} & \qquad q_n &= a_nq_{n-1}+s_{n-1}.
\end{aligned}$$

Since $r_n = p_{n-1}$ we can also write $r_{n-1} = p_{n-2}$. In the same way we can write $s_{n-1} = q_{n-2}$. Thus we find:

$$p_n = a_n p_{n-1} + p_{n-2} \quad (1)$$

$$q_n = a_n q_{n-1} + q_{n-2} \quad (2)$$

We start these recurrence relations by setting $M_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ i.e.

$$\begin{array}{ll} p_{-1} = 1, & p_0 = 0 \\ q_{-1} = 0, & q_0 = 1. \end{array}$$

We find that $c_n = \frac{p_n}{q_n}$, and since $\det(M_n) = (-1)^n = p_{n-1}q_n - p_n q_{n-1}$, it follows that p_n and q_n are relatively prime i.e. $gcd(p_n, q_n) = 1$. The next step is to write our x in a different way and follow the steps analogue to that of c_n. Define:

$$A_n^* = \begin{bmatrix} 0 & 1 \\ 1 & a_n + T_n \end{bmatrix} \qquad \text{and } M_n^* = M_{n-1}A_n^*.$$

here $T_n = T^n(x)$. This yields:

$$M_n^*(0) = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots \cfrac{1}{a_n + T_n}}} = x.$$

Also we see:

$$\begin{aligned} M_n^*(0) &= M_{n-1} \begin{bmatrix} 0 & 1 \\ 1 & a_n + T_n \end{bmatrix} (0) \\ &= \begin{bmatrix} p_{n-2} & p_{n-1} \\ q_{n-2} & q_{n-1} \end{bmatrix} \left(\frac{1}{a_n + T_n} \right) \\ &= \frac{p_{n-2}\frac{1}{a_n+T_n} + p_{n-1}}{q_{n-2}\frac{1}{a_n+T_n} + q_{n-1}} \\ &= \frac{p_{n-2} + (a_n + T_n)p_{n-1}}{q_{n-2} + (q_n + T_n)q_{n-1}}. \end{aligned}$$

From the recurrence relations(1) and (2) we find that

$$x = A_n^*(0) = \frac{p_n + p_{n-1}T_n}{q_n + q_{n-1}T_n}.$$

Now that we have found an expression for x, we want to know how good our estimation is. Note that:

$$\begin{aligned} x - c_n &= \frac{p_n + p_{n-1}T_n}{q_n + q_{n-1}T_n} - \frac{p_n}{q_n} \\ &= \frac{(p_n + p_{n-1}T_n)q_n - p_n(q_n + q_{n-1}T_n)}{(q_n + q_{n-1}T_n)q_n} \\ &= \frac{(p_{n-1}q_n - p_n q_{n-1})T_n}{(q_n + q_{n-1}T_n)q_n}. \end{aligned}$$

Sine we just saw that $p_{n-1}q_n - p_n q_{n-1} = (-1)^n$ is the determinant of the matrix M_n we find

$$x - c_n = \frac{(-1)^n T_n}{(q_n + q_{n-1}T_n)q_n}.$$

Thus we get for our estimation:

$$|x - c_n| = \frac{T_n}{(q_n + q_{n-1}T_n)q_n} = \frac{T_n}{q_n^2(1 + \frac{q_{n-1}}{q_n}T_n)} < \frac{1}{q_n^2}.$$

Because $T_n \in [0, 1)$ and the sequence (q_n) grows exponentially it follows that:

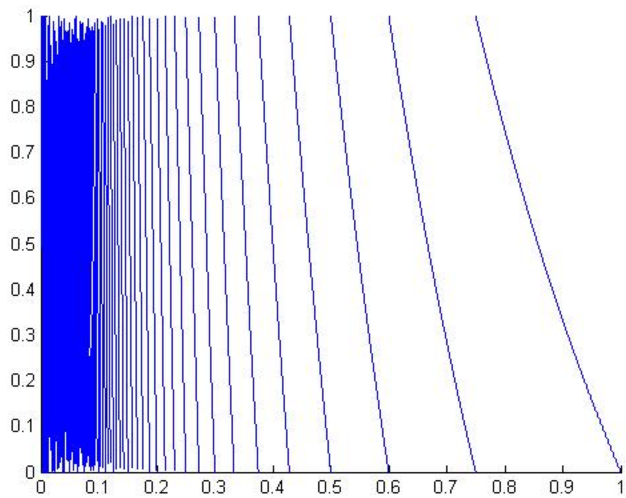
$$\lim_{n\to\infty} |x - c_n| = 0.$$

So we have found a good estimation and we see that (c_n) converges to x.

2.2 N-expansion

In the previous section we have seen how we can construct the regular continued fraction of a number $x \in [0, 1)$ using maps. Instead of taking $N = 1$ for the continued fraction, one could also take $N \in \mathbb{N}$ fixed. In this case we use the map:

$$T_N(x) = \frac{N}{x} - \left\lfloor \frac{N}{x} \right\rfloor .$$



Figuur 2: The new continued fraction with $N = 3$.

We determine the partial quotiënts by:

$$a_n(x) = \left\lfloor \frac{N}{T_n^{n-1}(x)} \right\rfloor \geq N.$$

Essentially the set-up from the previous section can be used. What makes things more difficult is the fact that in case $N \geq 2$ the matrices A_n defined by

$$A_n = \begin{bmatrix} 0 & N \\ 1 & a_n \end{bmatrix}$$

have determinant $\det(A_n) = -N$. However,the N- expansion converges, is ergodic and has an invariant measure. The density of this invariant measure can be given explicitly. More details can be found in the thesis by Niels van der Wekken [6]; see also [12]. The definition of *ergodicity* and *invariant measure* will be given in the following chapters.

2.3 a/b-continued fractions

This continued fraction is also called the generalised continued fraction and is another variation of the regular continued fraction map T. This variation uses two sequences of natural numbers to correspond to a $x \in \mathbb{R}$. The continued fraction map $T : [0,1) \to [0,1)$ is now defined by

$$T(x) = \frac{\lfloor \frac{1}{x} \rfloor}{x} - \left\lfloor \frac{\lfloor \frac{1}{x} \rfloor}{x} \right\rfloor \qquad \text{with } x \in (0,1], T(0) = 0.$$

Defining

$$a_1(x) = \left\lfloor \frac{1}{x} \right\rfloor \text{ and } b_1(x) = \left\lfloor \frac{a_1(x)}{x} \right\rfloor,$$

and setting, in case $T^{n-1}(x) \neq 0$:

$$a_n(x) = a_1(T^{n-1}(x)),\ b_n(x) = b_1(T^{n-1}(x)).$$

it follows that:

$$T(x) = \frac{a_1}{x} - b_1,$$

which yields that

$$x = \frac{a_1}{b_1 + T(x)} \text{ and } T(x) = \frac{a_2}{b_2 + T^2(x)}.$$

A trivial substitution gives that:

$$x = \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + T^2(x)}}.$$

We see that the generalised continued fraction can be constructed using two sequences of natural numbers. From Niels Langeveld's thesis [5] we know that this continued fraction convergences. Also, T is ergodic and a T-invariant measure μ exists, but the density of the invariant measure is unknown.

3 Invariant measures

In this section the definition of invariant measure will be given. In particular we will investigate the invariant measure of the regular continued fraction.

3.1 Definition of invariant measure

Before the definition of an invariant measure will be given, we will first look at some definitions related to the invariant measure starting with the definition of a *σ-algebra*.

Definition 1. *Define a set X. A collection Λ of subsets of X is called a* ***σ-algebra*** *when the following conditions hold:*

1. $\emptyset, X \in \Lambda$
2. $A \in \Lambda \Rightarrow X \setminus A \in \Lambda$
3. $A_n \in \Lambda \ (n = 1, 2, ...) \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \Lambda$

The pair (X, Λ) is called a *measurable space*. As the name suggest, we can define a *measure* on this space.

Definition 2. *A* ***measure*** *on Λ is a function $\mu : \Lambda \to [0, \infty)$ so that:*

1. $\mu(\emptyset) = 0$
2. *If $A \in \Lambda$ and $A_n \in \Lambda(n = 1, 2, ...)$ disjoint so that $A = \bigcup_{n=1}^{\infty} A_n$, then $\mu(A) = \sum_{n=1}^{\infty} \mu(A_n)$*

We call the triple (X, Λ, μ) a *measure space*. Thus, a measure space consists of a measurable space and a measure. If $\mu(X) = 1$, then μ is called a *probability measure*. A measurable set X is called a null set if $\mu(X) = 0$.

Now we know the definition of a measure we can look at invariant measures.

Definition 3. *Let (X, Λ, μ) be a probability space and define the map $T : X \to X$. We call μ* ***T-invariant*** *if $\mu(T^{-1}(A)) = \mu(A)$, where $A \subset \Lambda$.*

The T-invariant measure was discovered by Gauss in 1800. We will use this measure on the interval $[0, 1)$ on the Borel σ-algebra.

Since the Borel σ-algebra is brought forth by intervals it is enough to show that T is μ-measure preserving on intervals.

Let us start with looking at an example. Let

$$T(x) = \begin{cases} 2x & \text{if } x \in [0, \frac{1}{2}) \\ 2x - 1 & \text{if } x \in [\frac{1}{2}, 1). \end{cases}$$

To show that the Lebesgue measure μ of this map is T-invariant, we want that $\mu(T^{-1}(A)) = \mu(A)$ holds for any interval A. So take a measurable set A and lets look at $T^{-1}(A)$. We see that $T^{-1}(A)$ divides into two equal parts which we will call A_1 and A_2. So we can say that $\mu(A_1) = \mu(A_2) = L$ and $T(A_1 \cup A_2) = A$. Note that A_1 is a subset of $[0, \frac{1}{2})$ and A_2 is a subset of $[\frac{1}{2}, 1)$. This gives us:

$$\mu(T^{-1}(A)) = \mu(A_1 \cup A_2) = \mu(A_1) + \mu(A_2) = 2L = \mu(A)$$

Thus we have shown that the measure of this map is T-invariant, because $\mu(T^{-1}(A)) = \mu(A)$ holds.

3.2 Invariant measure of the regular continued fraction

As mentioned earlier, Gauss discovered the invariant measure in 1800. This was before most of probability theory had been created. In modern notation, Gauss claimed that for the regular continued fraction the invariant measure is given by:

$$\mu(A) = \frac{1}{\ln(2)} \int_A \frac{1}{1+x} dx$$

where A is a measurable set. Here μ is a probability measure with $\mu(A) \geq 0$, because the measure is given by a density (which is always positive). Also on the unit interval $I = [0, 1]$,

$$\mu(I) = \frac{1}{\ln(2)} \int_0^1 \frac{1}{1+x} dx = \frac{\ln(2) - \ln(1)}{\ln(2)} = 1.$$

Untill today nobody knows how Gauss came up with the invariant measure of the regular continued fraction. However, we can check on intervals that this measure satisfies the property of the invariant measure, so $\mu(T^{-1}(A)) = \mu(A)$ must hold.

Take $A = (a, b)$. Here we use that it is enough to show that T is μ-measure preserving on intervals since the Borel σ-algebra is brought forth by intervals. Then:

$$\mu((a, b)) = \frac{1}{\ln(2)} \int_a^b \frac{1}{1+x} dx = \frac{1}{\ln(2)} \left[\ln(1+x)\right]_a^b = \frac{1}{\ln(2)} \ln\left(\frac{1+b}{1+a}\right),$$

and

$$\begin{aligned}
\mu(T^{-1}(a,b)) &= \mu\left(\bigcup_{n=1}^{\infty}\left(\frac{1}{n+b},\frac{1}{n+a}\right)\right) \\
&= \sum_{n=1}^{\infty}\mu\left(\left(\frac{1}{n+b},\frac{1}{n+a}\right)\right) \\
&= \sum_{n=1}^{\infty}\frac{1}{\ln(2)}\int_{\frac{1}{n+b}}^{\frac{1}{n+a}}\frac{1}{1+x}dx \\
&= \sum_{n=1}^{\infty}\frac{1}{\ln(2)}\ln\left(\frac{\frac{1}{n+a}+1}{\frac{1}{n+b}+1}\right) \\
&= \frac{1}{\ln(2)}\sum_{n=1}^{\infty}\ln\left(\frac{n+a+1}{n+b+1}\cdot\frac{n+b}{n+a}\right) \\
&= \frac{1}{\ln(2)}\lim_{n\to\infty}\sum_{k=1}^{n}\ln\left(\frac{k+a+1}{k+b+1}\cdot\frac{k+b}{k+a}\right) \\
&= \frac{1}{\ln(2)}\lim_{n\to\infty}\ln(n+a+1)-\ln(a+1)+\ln(b+1)-\ln(n+b+1) \\
&= \frac{1}{\ln(2)}\ln(b+1)-\ln(a+1) \\
&= \frac{1}{\ln(2)}\ln\left(\frac{b+1}{a+1}\right).
\end{aligned}$$

Note that in the second equation the second property of a measure is used. We have now showed for intervals $A=[a,b]$ that $\mu(T^{-1}(A))=\mu(A)$, so the property of the invariant measure holds on intervals. It now follows that μ is T-invariant on all Borel measurable sets $A\in\Lambda$ and we have that $\mu(T^{-1}(A))=\mu(A)$ (see [13]). Thus the Gauss measure is invariant for the regular continued fraction.

4 The Ergodic Theorem

Earlier we looked at a T-invariant measure. The existence of the T-invariant measure is important, because it allows us to use ergodic theory to study the behavior of the map T. In this section we will take a look at the definition of ergodicity and the Ergodic Theorem relevant to this thesis. Also we will look at a way to show ergodicity.

Definition 4. *Let $T : X \to X$. We call T* ***ergodic*** *if for every μ-measurable set A satisfying $T^{-1}(A) = A$ one has that $\mu(A) = 0$ or $\mu(A) = 1$.*

This basically means that the set X can be seen as one coherent whole. This property will turn out to be usefull by the Ergodic Theorem.

The Ergodic theorem, also known as Birkhoff's Ergodic Theorem or the Individual Ergodic Theorem (1930), is in fact a generalization of the strong law of large numbers. The strong law of large numbers states that the sample average converges almost surely to the expected value, so $P(\lim_{n\to\infty} \bar{X}_n = \mu) = 1$.

Theorem 1. *(Birkhoff's Ergodic Theorem)*
Let (X, Λ, μ) be a probability space and $T : X \to X$ a measure that is μ-invariant (i.e. T is a measure preserving transformation). Then for any f in $L^1(\mu)$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f \circ T^i(x) = f^*(x)$$

exists a.e., is T-invariant and $\int_X f d\mu = \int_X f * d\mu$. If moreover T is ergodic, then f^* is a constant c a.e. and $f^* = \int_X f d\mu$.

Note that a property holds *almost everywhere* (in short, a.e.) if it holds except for a set of measure zero. The Ergodic Theorem is especially interesting when T is ergodic. It turns out that for example the regular continued fraction is ergodic. This was proven by W. Doeblin in 1940 and independently by C. Ryll-Nardzewski in 1951.

4.1 Adler's Folklore Theorem

Nowadays there are many proofs of ergodicity, especially that of the regular continued fraction. A general way to proof ergodicity is by showing that the conditions of *Adler's Folklore Theorem* (1973) are satisfied as described in *Schweiger* [1]. Prior to looking at this theorem we will give some basic definitions used in Schweiger.

Let B be a set and $T : B \to B$ be a map. Here the pair (B, T) is called a *fibred system* and the following conditions are satisfied:

- There is a finits or countable set I called the digit set.
- There is a map $k : B \to I$. Then the sets

  $$B(i) = k^{-1}\{i\} = \{x \in B : k(x) = i\}$$

  form a partition of B.
- The restriction of T to any $B(i)$ is an injective map.

Note that $B = [0, 1]$ can be divided into fundamental intervals.

Theorem 2. *(Adler's Folklore Theorem)*
Let $B = [0, 1]$ and $\lambda : \mathcal{F} \to \mathbb{R}$ Lebesgue measure. Suppose that

1. *There exist intervals $]a_k, b_k[$ such that $]a_k, b_k[\subseteq B(k) \subseteq [a_k, b_k] = I_k$ $\forall k \in I$.*
2. *The map $T : B(k) \to B = [0, 1]$ can be extended to a function class C^2 on $I(k)$.*
3. *The map T restricted to $B(k)$ is full, i.e. $TI(k) = B$.*
4. *There exists a constant $\theta > 1$ such that*

   $$\left|T'(x)\right| \geq \theta, \qquad x \in I(k), \qquad k \in I.$$

5. *There exists a constant $M > 0$ such that*

   $$\left|\frac{T''(x)}{T'(x)^2}\right| \leq M, \qquad x \in I(k), \qquad k \in I.$$

Then, along with the convergence of the continued fraction, we find that the continued fraction T is ergodic. Conditions (1) and (2) together say that T is piecewise of class C^2 ([1]). Once the conditions of Adler's Folklore Theorem are satisfied, the so-called *Rényi Condition* holds. This condition is condition (c) of Rényi's Theorem 9.5.3 in Schweiger's famous monograph [1]. Furthermore, if the conditions of Rényi's Theorem are satisfied, Theorem 15.1.2 in [1] now yields that there exists a unique invariant probability measure μ such that for some positive constant C and for all measurable sets E

$$C^{-1}\lambda(E) \leq \mu(E) \leq C\lambda(E)$$

So both theorems together imply that, if satisfied, T is ergodic and has an invariant measure. Note that the conditions on Adler's Folklore Theorem are conditions on cylinder sets.

4.2 Ergodicity of the regular continued fraction

Using *Adler's Folklore Theorem* we will show ergodicity of the regular continued fraction. Conditions (1), (2) and (3) are immediatly satisfied. Take

$$I(k) = \left[\frac{1}{k+1}, \frac{1}{k}\right] \qquad k \in I$$

Note that in this case $I = \mathbb{N}$. For the first condition we see that $\left[\frac{1}{k+1}, \frac{1}{k}\right) \subseteq B(k) \subset \left[\frac{1}{k+1}, \frac{1}{k}\right] = I(k)$. So the first condition is satisfied. The map T is injective, so also the second condition is satisfied. Also the map of the regular continued fraction is obviously *full*, so the third condition is also satisfied.

The other two remaining conditions are a bit tricky to satisfy. Let's look at the fourth condition:

(4) There exists a constant $\theta > 1$ such that

$$\left|T'(x)\right| \geq \theta, \qquad x \in I(k), \qquad k \in I.$$

The map of our regular continued fraction is defined as:

$$T(x) = \frac{1}{x} - k \quad \text{with} \quad x \in I(k).$$

But then

$$T'(x) = -\frac{1}{x^2}$$

and thus there exists no $\theta > 1$ so that $|T'(x)| \geq \theta$, $\forall x \in I(k)$. Therefore condition (4) is not satisfied. We can solve this by refining the fundamental intervals $I(k)$ and use the weakened version of condition (4) as stated in section 15.2.3 of Schweiger [1]:

Theorem. *(4*) There is an $N \geq 1$ such that $|(T^N)'(x)| \geq \theta > 1$ $\forall x \in I(k_1, ..., k_N$, the closure of $B(k_1, ..., k_N)$.*

Using (4*) we now want to check whether this condition holds for T^2 instead. Define

$$I(k_1, k_2) = \{x \in [0, 1) : k_1(x) = k_1, k_2(x) = k_2\} \quad \text{with} \quad k_1, k_2 \in \mathbb{N}.$$

Then for $x \in I(k_1, k_2)$

$$T^2(x) = \frac{x}{1 - k_1 x} - k_2$$

So we find using the inequality $k_1 x \geq \frac{k_1 k_2}{k_1 k_2 + 1}$ that

$$(T^2(x))' = \frac{1}{(1 - k_1 x)^2} \geq 4$$

This satisfies condition (4*) and we see that this condition is satisfied with $N = 2$.

What remains to check now is whether condition (5) of *Adler's Folklore Theorem* holds. We see

$$\left| \frac{T''(x)}{(T'(x))^2} \right| = \left| \frac{\frac{2}{x^3}}{\frac{1}{x^4}} \right| = |2x| \leq 2$$

So condition (5) holds.

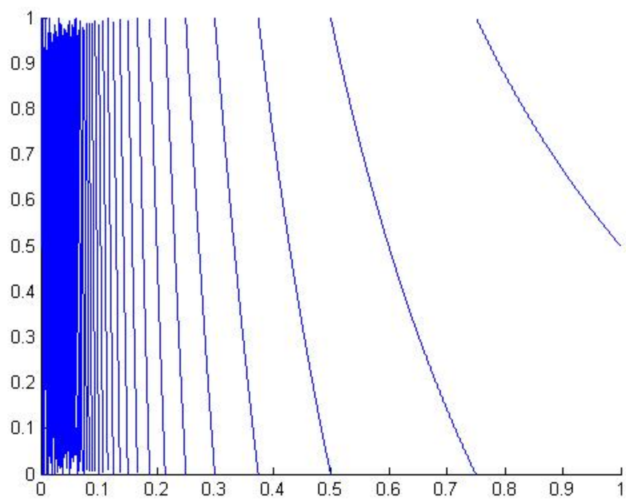
Since all the conditions of Adler's Folklore Theorem have been satisfied, we have shown that the regular continued fraction is ergodic and, with Renyi's theorem, that it has an invariant measure. Our earlier calculation (on p. 13) show that this invariant measure is indeed the one found by Gauss.

5 A new continued fraction

After the regular continued fraction, the N-expansion and the generalized continued fraction, a new continued fraction is introduced. This new continued fraction looks as follows:

$$x = a_0 + \cfrac{\beta}{a_1 + \cfrac{\beta}{a_2 + \cfrac{\beta}{\ddots}}}$$

with $a_0 \in \mathbb{Z}$ and $a_i \in \mathbb{N}$ for $i \geq 1$. What is new about this continued fraction is that $\beta > 1$ and $\beta \notin \mathbb{N}$. If $\beta \in \mathbb{N}$ then we would have the N-expansion.



Figuur 3: The new continued fraction with $\beta = \frac{3}{2}$.

The map $T : [0, 1) \to [0, 1)$ of this new continued fraction is defined by:

$$T(x) = \frac{\beta}{x} - \left\lfloor \frac{\beta}{x} \right\rfloor \qquad \text{with } x \neq 0 \text{ and } T(0) = 0$$

This new continued fraction has a limitation on its digits. Take for example $\beta = \frac{3}{2}$. The map of this continued fraction is obviously not full as can be seen in figure 3.

Now we want to define our digits. Take $a_0 = \lfloor x \rfloor$ and define for $x \in [0,1)$

$$a_1(x) = \left\lfloor \frac{\beta}{x} \right\rfloor \qquad \text{if } x \neq 0, \text{ and } a_1(0) = \infty.$$

Since

$$T(x) = \frac{\beta}{x} - a_1(x),$$

it follows[1] that we can write

$$x = \frac{\beta}{a_1 + T(x)}$$

In general, if $T^{n-1}(x) \neq 0$ we define the digit $a_n = a_n(x)$ by:

$$a_n(x) = \left\lfloor \frac{\beta}{T^{n-1}(x)} \right\rfloor.$$

Lets look once again at figure 3. After a digit $a_n = 1$ we will always get digit $a_{n+1} = 1$ or $a_{n+1} = 2$. In this section we will look at some properties of this new continued fraction. Does the new continued fraction converge? Does it has an invariant measure? Is the new continued fraction ergodic?

5.1 Convergence of the new continued fraction

Since the regular continued fraction, the N-expansion and the generalized continued fraction all converge, the question arises whether this new continued fraction also converges. In order to check whether the new continued fraction converges or not, we will start working in the same way as in Section 2.1.1. where we have shown that the regular continued fraction converges.
We start by defining

$$c_n = \cfrac{\beta}{a_1 + \cfrac{\beta}{a_2 + \cfrac{\beta}{\ddots + \cfrac{\beta}{a_n}}}}.$$

Now let $M_n = A_1 \cdot A_2 \cdot, \cdots, \cdot A_n$, where

$$A_n = \begin{bmatrix} 0 & \beta \\ 1 & a_n \end{bmatrix}, \qquad n \geq 1.$$

[1] As before we suppress whenever possible the dependence of the digits $a_n(x)$ on x.

Again we use the same Möbius transformation. Then in the same way as with the regular continued fraction we find that $c_n = M_n(0) = \frac{p_n}{q_n}$ To set up a reccurence relation we write:

$$M_n = \begin{bmatrix} r_n & p_n \\ s_n & q_n \end{bmatrix} \quad \text{with} \quad r_n, p_n, s_n, q_n \in \mathbb{Z} \quad \text{and} \quad n \geq 1.$$

Now we see that

$$\begin{aligned} M_n = M_{n-1}A_n &= \begin{bmatrix} r_{n-1} & p_{n-1} \\ s_{n-1} & q_{n-1} \end{bmatrix} \begin{bmatrix} 0 & \beta \\ 1 & a_n \end{bmatrix} \\ &= \begin{bmatrix} p_{n-1} & a_n p_{n-1} + \beta r_{n-1} \\ q_{n-1} & a_n q_{n-1} + \beta s_{n-1} \end{bmatrix} \\ &= \begin{bmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{bmatrix}. \end{aligned}$$

So now we find the following reccurence relation

$$\begin{aligned} r_n &= p_{n-1} & p_n &= a_n p_{n-1} + \beta r_{n-1} \\ s_n &= q_{n-1} & q_n &= a_n q_{n-1} + \beta s_{n-1}. \end{aligned}$$

Since $r_n = p_{n-1}$ we can also write $r_{n-1} = p_{n-2}$. In the same way we can write $s_{n-1} = q_{n-2}$. Thus:

$$\begin{aligned} p_n &= a_n p_{n-1} + \beta p_{n-2} \\ q_n &= a_n q_{n-1} + \beta q_{n-2} \end{aligned}$$

We start these recurrence relations in the same way as with the regular continued fraction, thus by setting $M_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ i.e.

$$p_{-1} = 1,\ p_0 = 0$$
$$q_{-1} = 0,\ q_0 = 1.$$

We have found

$$M_n = \begin{bmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{bmatrix}.$$

Now let us define A_n^* and M_n^* as follows:

$$A_n^* = \begin{bmatrix} 0 & \beta \\ 1 & a_n + T_n \end{bmatrix} \qquad \text{and } M_n^* = M_{n-1}A_n^*.$$

where $T_n = T^n(x)$ for convenience. This yields:

$$x = M_n^*(0) = \cfrac{\beta}{a_1 + \cfrac{\beta}{a_2 + \ddots + \cfrac{\beta}{a_n + T_n}}},$$

and we see that

$$\begin{aligned} M_n^*(0) &= M_{n-1}\begin{bmatrix} 0 & \beta \\ 1 & a_n + T_n \end{bmatrix}(0) \\ &= \begin{bmatrix} p_{n-2} & p_{n-1} \\ q_{n-2} & q_{n-1} \end{bmatrix}\left(\frac{\beta}{a_n + T_n}\right) \\ &= \frac{\beta p_{n-2} + (a_n + T_n)p_{n-1}}{\beta q_{n-2} + (q_n + T_n)q_{n-1}} \\ &= \frac{(\beta p_{n-2} + a_n p_{n-1}) + T_n p_{n-1}}{(\beta q_{n-2} + a_n q_{n-1}) + T_n q_{n-1}} \\ &= \frac{p_n + p_{n-1}T_n}{q_n + q_{n-1}T_n}. \end{aligned}$$

Then for convergence:

$$\begin{aligned} x - \frac{p_n}{q_n} &= \frac{p_n + p_{n-1}T_n}{q_n + q_{n-1}T_n} - \frac{p_n}{q_n} \\ &= \frac{(p_{n-1}q_n - p_n q_{n-1})T_n}{q_n^2(1 + \frac{q_{n-1}}{q_n}T_n)}. \end{aligned}$$

Now

$$\begin{aligned} \det(M_n) &= \begin{vmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{vmatrix} \\ &= \det(A_1) \cdot ... \cdot \det(A_n) \\ &= (-\beta)^n. \end{aligned}$$

However in this case, unlike with the regular continued fraction, we cannot tell whether our c_n is relatively prime , i.e. $gcd(p_n, q_n) = 1$), or not. In fact, since $\beta > 1$ we do not even know whether our p_n and q_n are integers or not. What we do know is that since $T_n \in [0, 1]$ and (q_n) grows exponentially we get that

$$\left| x - \frac{p_n}{q_n} \right| = \frac{\beta^n T_n}{q_n^2(1 + \frac{q_{n-1}}{q_n}T_n)} \le \frac{\beta^n}{q_n^2} \tag{3}$$

However, this inequality does not immediately show us that the new continued fraction converges. So, lets start with looking at the sequence of denominators (q_n). We know that $a_n \geq 1$ and $q_n = a_n q_{n-1} + \beta q_{n-2}$. So in the slowest case of convergence:

$$\begin{aligned} q_n &= a_n q_{n-1} + \beta q_{n-2} \geq q_{n-1} + \beta q_{n-2} \\ &= (\beta + a_{n-1}) q_{n-2} + \beta q_{n-3} \geq (\beta + 1) q_{n-2} + \beta q_{n-3} \\ &= (\beta + 1)(a_{n-2} q_{n-3} + \beta q_{n-4}) + \beta q_{n-3} \geq (2\beta + 1) q_{n-3} + \beta(\beta + 1) q_{n-4} \\ &= \ldots \end{aligned}$$

We could carry on with these inequalities for a very long time, but we could also use induction. For convenience, let us show convergence by choosing a fixed number $\beta > 1$, for example the fraction $\beta = \frac{3}{2}$. Note that if we had chosen a whole number we would have an N-expansion and convergence has already been proven.

Step 1: For $n \geq 2$, $q_n \geq \left(\frac{3}{2}\right)^n$ holds. This is also our induction hypothesis. Using the slowest case $q_n = q_{n-1} + \frac{3}{2} q_{n-2}$ and the starting conditions $q_{-1} = 0$ and $q_0 = 1$ this can be seen. In table 1 the few first values are shown. So we see that step 1 is satisfied.

	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$\cdots$
$q_{-1} = 0$, $q_0 = 1$	$q_1 = 1$	$q_2 = \frac{5}{2}$	$q_3 = \frac{8}{2}$	$\frac{31}{4}$	$\cdots$
$\left(\frac{3}{2}\right)^n$	$\frac{3}{2}$	$\frac{9}{4}$	$\frac{27}{16}$	$\frac{81}{16}$	$\cdots$

Tabel 1: Step 1 of induction for $\beta = \frac{3}{2}$.

Step 2: In this step we want to show that $q_{N+1} \geq \left(\frac{3}{2}\right)^{N+1}$ for $n = 2, \cdots, N$. Using $q_n = q_{n-1} + \frac{3}{2} q_{n-2}$ and our induction hypothesis $q_n \geq \left(\frac{3}{2}\right)^n$ we see that

$q_{N+1} \geq \left(\frac{3}{2}\right)^{N+1}$ equals:

$$\begin{aligned} q_N + \frac{3}{2} q_{N-1} &\geq \left(\frac{3}{2}\right)^N + \frac{3}{2}\left(\frac{3}{2}\right)^{N-1} \\ &\geq \left(\frac{3}{2}\right)^N + \left(\frac{3}{2}\right)^N \\ &\geq 2 \cdot \left(\frac{3}{2}\right)^N \\ &\geq \left(\frac{3}{2}\right)^N \end{aligned}$$

So we see that for $\beta = \frac{3}{2}$ our continued fraction converges.

It is nice to know that the new continued fraction converges for $\beta = \frac{3}{2}$. However, it would be nicer if we could proof this for $\beta > 1$. So lets look at the convergence of the new continued fraction in the general case.

Step 1: Again our starting conditions are $q_{-1} = 0$ and $q_0 = 1$. For $n = 0$ we get that $q_0 = 1 \geq \beta^0 = 1$. So, $q_n \geq \beta^n$ holds. This is also our induction hypothesis.

Step 2: In this step we want to show that our induction hypothesis holds for $N + 1$, so $q_{N+1} \geq \beta^{N+1}$ for $n = 0, \cdots, N$. Using the recurrence relation of q_n we have that

$$q_{N+1} = a_{N+1} q_N + \beta q_{N-1} \geq a_{N+1} \beta^N + \beta \beta^{N-1} = (a_{N+1} + 1)\beta^N$$

and because $a_{N+1} + 1 \geq \beta$ we get that $q_{N+1} \geq \beta^{N+1}$.

Now that we have completed our two induction steps we can look back at our previous equation, equation (3). Using (3) and using induction we see that

$$\left| x - \frac{p_n}{q_n} \right| \leq \frac{\beta^n}{q_n^2} \leq \frac{\beta^n}{(\beta^n)^2} = \frac{1}{\beta^n}$$

It now follow that:

$$\lim n \to \infty \left| x - \frac{p_n}{q_n} \right| = 0.$$

So we have found that $c_n = \frac{p_n}{q_n}$ converges to x for $\beta > 1$ in our new continued fraction.

5.2 Invariant measure and ergodicity of the new continued fraction

In a previous section we saw that the regular continued fraction has an invariant measure. The following questions then arise: Does the new continued fraction also have an invariant measure? If so, can we find the invariant measure of this new continued fraction? In general, finding an invariant measure for continued fractions other than the regular continued fraction is hard. In this bachelor thesis we will restrict ourselves to the case $\beta = \frac{3}{2}$ and show that this new continued fraction *has* an invariant measure. Also we will show that the new continued fraction is ergodic.

We will show ergodicity by checking if this new continued fraction satisfies the conditions of *Adler's Folklore Theorem.*

For the first condition of *Adler's Folklore Theorem* let $B = [0, 1]$. Define $\Delta_n(a_k) = \{x \in [0, 1] \mid a_1(x), \cdots, a_n(x) = a_k\}$. Here $a_k \in \mathbb{N}$ is a digit. It is not hard to see that $\Delta_1(a_k) = \left[\frac{\beta}{k+1}, \frac{\beta}{k}\right]$ for $k \in \mathbb{N}$. However note that $\Delta_1(1) = \left[\frac{\beta}{2}, \frac{1}{n}\right]$.

So we see that $[\frac{\beta}{k+1}, \frac{\beta}{k}) \subseteq B(k) = \Delta_n(a_k) \subset \left[\frac{\beta}{k+1}, \frac{\beta}{k}\right] = I(k)$ and the first condition is satisfied. Since our map $T(x) = \frac{\beta}{x} - \left\lfloor\frac{\beta}{x}\right\rfloor$ is injective it follows that T can be extended to a funtion of class C^2 (so the derivatives exist and are continuous) on $I(k)$. So the second condition of *Adler's Folklore Theorem* is also satisfied.

For the third condition we need to show that $TI(k) = B$, i.e. that our map is full. Obviously the map is not full if $\beta > 1$, $\beta \neq \mathbb{N}$. However, if $\beta = \frac{3}{2}$. the partition of T is a so-called Markov-property (so the future depends only on information about the current time and not on information from the past).
To show that our map is "full"we will be using an article written by Shunji Ito and Michiko Yuri [7]. For the convenience of the reader we will once again take $\beta = \frac{3}{2}$.

> Let Y be a bounded measurable subset with piecewise smooth boundary of $\mathbb{R}^n$, in this case $Y = [0, 1]$. Also, let λ be the normalized Lebesgue measure on Y and let I be a countable set. Now let us consider the map $S : Y \to Y$ which satisfies the following condition:
> (0) There exists a countable partition $\xi = \{Y_a : a \in I\}$ with an index set I elements Y_a of which are measurable and connected subsets of Y with piecewise smooth boundary such that $S|_{Y_a}$ is injective, of class C^1 and $\det(DS|_{Y_a}) \neq 0$. The last condition, $\det(DS|_{Y_a}) \neq 0$, means that the derivative exists. The following step is to introduce some notations and definitions: *A cylinder of*

rank n with respect to T is defined by

$$Y_{a_1\cdots a_n} = Y_{a_1} \cap S^{-1}Y_{a_2} \cap \cdots \cap S^{-(n-1)}Y_{a_n}$$

$$\text{if } (Y_{a_1})^\circ \cap (S^{-1}Y_{a_2})^\circ \cap \cdots \cap (S^{-(n-1)}Y_{a_n})^\circ \neq \emptyset$$

Now let us denote $\mathscr{L}^{(n)}$ as the family of all cylinders $Y_{a_1\cdots a_n}$ of rank n and $\mathscr{L} = \cup_{n=1}^{\infty}\mathscr{L}^{()}$. The sequence $(a_1, \cdots, a_n)$ is called *S-admissable* if $Y_{a_1\cdots a_n} \in \mathscr{L}^{(n)}$ and $A(n)$ denotes the set of all S-admissable sequences of length n. We call S a *number theoretical transformation with finite range structure* if the map S satisfies (0) and if there exists a finite number of subsets $V_0, \cdots, V_N$ of Y with positive measure such that for each n and for all $(a_1 \cdots a_n) \in A(n)$

$$S^n Y_{a_1\cdots a_n} \in V_0, \cdots, V_N.$$

This system is denoted by $(Y, S, \{V_0, \cdots, V_N\}, \{Y_a : a \in I\})$. We call such a system a *number theoretical Markov system* and we call S a *number theoretical transformation with Markov structure* if there exists a $k \geq 0$ such that all range sets are $\mathscr{L}^k$-measurable, so

$$V_i = \cup_{Y_{a_1\cdots a_k} \subset V_i} Y_{a_1\cdots a_k} \qquad \text{for all } i \text{ with } 0 \leq i \leq N.$$

Given a constant $C \geq 1$, a cylinder $Y_{a_1\cdots a_n}$ is called an *R.C-cylinder* if it satisfies Renyi's condition. The set of all R.C-cylinders is denoted by R(C.S).

All of this by Shunji Ito and Michiko Yuri fits in our story when $\beta = \frac{3}{2}$. First, our intervals $Y_a \neq 0$ with $a = 1, 2, \cdots$ are on cylinder sets. Note that Y_a are intervals where our digits are $1, 2, \cdots$. So for example in this case $Y_1 = \left[\frac{3}{4}, 1\right)$ is the interval where our digit is 1. Secondly, the intervals $T|_{Y_a}$ are all injective and the derivative exists since $\det DT|_{Y_a}$. In case $\beta = \frac{3}{2}$ we have $V_0 = [0, 1)$ and $V_1 = [\frac{1}{2}, 1)$ to fit in the story.

To satisfy the fourth condition we want to show that there exists a constant $\theta > 1$ such that $|T'(x)| \geq \theta, x \in I(k), k \in I$. We have $T(x) = \frac{\beta}{x} - \left\lfloor \frac{\beta}{x} \right\rfloor$, so

$$T'(x) = \frac{-\beta}{x^2}$$

and thus

$$|T'(x)| = \frac{\beta}{x^2} \text{ for } x \in (0, 1).$$

Since $\beta \geq 1$ and $x \in (0,1)$ we get that

$$|T'(x)| = \frac{\beta}{x^2} \geq \frac{1}{x^2} > 1.$$

Thus there exists a constant $\theta > 1$ such that $|T'(x)| \geq \theta,\ x \in I(k), k \in I$.

For the last and fifth condition we want to show that there exists a constant $M > 0$ such that $\left|\frac{T''(x)}{T'(x)^2}\right| \leq M,\ x \in I(k), k \in I$. We know that $T'(x) = \frac{-\beta}{x^2}$ and thus $T''(x) = \frac{-2\beta}{x^3}$. Knowing that $x \in (0,1)$ and $\beta \geq 1$ we get

$$\left|\frac{T''(x)}{(T'(x))^2}\right| = \frac{\frac{2\beta}{x^3}}{\frac{\beta^2}{x^4}} = \frac{2x}{\beta} < 2.$$

So there exists a constant $M > 0$ such that $\left|\frac{T''(x)}{T'(x)^2}\right| \leq M,$ for $x \in I(k),$ where $k \in I$.

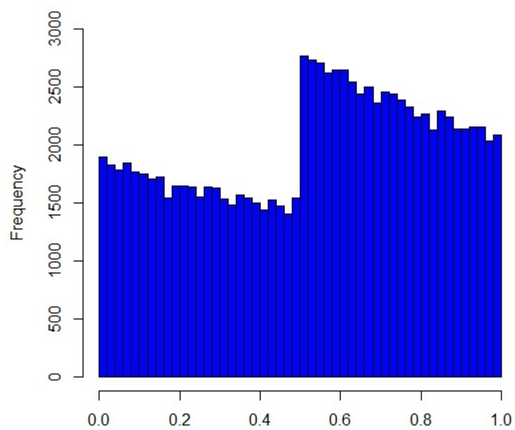
Now that we have satisfied the five conditions of *Adler's Folklore Theorem* and we know that this new continued fraction converges we can finally say that our new continued fraction is ergodic. Knowing that this new continued fraction is ergodic we can use *Rényi's Theorem*, see Theorem 15.1.2 in [1]. This theorem tells us that there exists a unique invariant probability measure μ such that

$$C^{-1}\lambda(E) \leq \mu(E) \leq C\lambda(E).$$

Here $\lambda(E)$ is the Lebesque measure of E and $C > 0$. So our new continued fraction is ergodic and has an invariant measure.

5.3 Simulation of the density of the new continued fraction

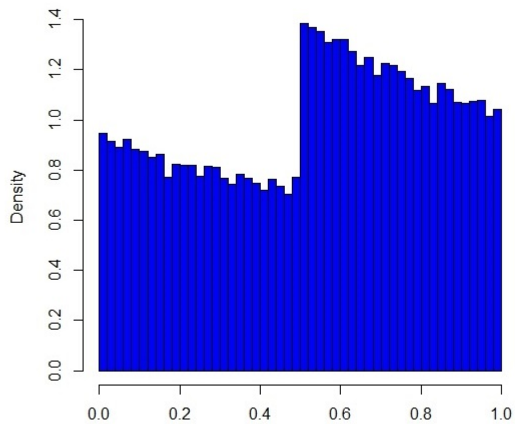
We now know that for $\beta = \frac{3}{2}$ our new continued fraction converges, is ergodic and that there exists an invariant measure. However, finding the invariant measure is a daunting task so we use simulation to get an idea of the shape of the invariant measure. The invariant measure can be seen as a probability distribution. So we could take any $x \in [0, 1)$ and look at its orbit under T [9]. To solve this we will take a different approach in this simulation: First of all, instead of one starting point x we will take more starting points. Secondly, instead of waiting untill a starting point reaches $T^n(x) = 0$ we will decide our own n. Then the data of the path that is followed during the iterations will be plotted in a histogram.



Figuur 4: A histogram for $\beta = \frac{3}{2}$.

In the histogram above 10 starting points are randomly chosen with $x \in [0, 1]$. Every starting point was then iterated 10^4 times resulting in the histogram. The shape of the histogram is remarkable, because the shape looks like that of the regular continued fraction but shifted at $x = \frac{1}{2}$. The result is actually what was expected, because when you have digit 1 the image is sent to $\left[\frac{1}{2}, 1\right)$. After a digit 1 you can get either a digit 2, lying in interval $\left[\frac{1}{2}, \frac{3}{4}\right)$, or a digit 1 again. This situation can repeat itself several times causing the bar to raise in the interval $[\frac{1}{2}, 1)$.

A graph like in figure 4 gives us a good idea of the shape of the invariant measure. After normalization we obtain the probability density which can be seen in figure 5.



Figuur 5: A histogram for $\beta = \frac{3}{2}$.

Knowing the intervals of the digits for $\beta = \frac{3}{2}$ we can estimate the corresponding percentages. In the table below the percentages of how often a digit can be found is given. This is stil done for 10 random starting points iterated 10^4 times.

a_n	1	2	3	4	5	6	7	8	9	the rest
percentage (%)	27.86	32.03	9.11	5.77	4.22	2.89	2.18	1.80	1.37	12.97

Tabel 2: Percentages of the digits for $\beta = \frac{3}{2}$

6 Appendix

This Matlab-code used to generate the new continued fraction.

```
hold on;
x = 0:0.0001:1;
Tx = T(1.5,x);

hulpsize = 1;
hulpvec = 0;
for k = 1:(length(x)-1)
    diff = Tx(k+1) - Tx(k);
    hulpvec(k+1-hulpsize) = Tx(k);
    if(diff>0.9)
        plot(x(hulpsize:k),hulpvec);
        hulpvec = 0;
        hulpsize = k+1;
    end
end
plot(x(hulpsize:k),hulpvec);
```

Referenties

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