ON GENERALIZATIONS OF FOURIER AND LAPLACE TRANSFORMS

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PROEFSCHRIFT

TER VERKRIJGING VAN DE GRAAD VAN DOCTOR IN DE TECHNISCHE WETENSCHAPPEN AAN DE TECHNISCHE HOGESCHOOL DELFT, OP GEZAG VAN DE RECTOR MAGNIFICUS IR. H.R. VAN NAUTA LEMKE, HOOGLERAAR IN DE AFDELING DER ELEKTROTECHNIEK, VOOR EEN COMMISSIE UIT DE SENAAT TE VERDEDIGEN OP WOENSDAG 24 MAART 1971 TE 16 UUR

DOOR

HENDRIK SIMON VICTOR DE SNOO

wiskundig ingenieur

geboren te Dordrecht

DIT PROEFSCHRIFT IS GOEDGEKEURD DOOR DE PROMOTOR

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This dissertation consists of two separate parts, which may be read independently.

In part I we consider Watson's generalization of the Fourier transform. In particular, we give the spectral theory of this transform in $L^2(-\infty,\infty)$.

In part II we consider a generalization of the Laplace transform. Specifically we prove an inversion formula for integral transforms of which the kernel is a solution of a certain differential equation. This section has already appeared in Proc. Kon. Ned. Ak. Wet. 73, 222-244 (1970).

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PART I. SPECTRAL THEORY OF WATSON TRANSFORMS

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PART I

SPECTRAL THEORY OF WATSON TRANSFORMS

0. Introduction

The purpose of this paper is to study the spectral theory of Watson transforms. If the function $k_1(y)$ satisfies certain requirements, especially $\frac{k_1(y)}{y} \in L^2(0,\infty)$, then the Watson transform $g = Tf$ in $L^2(0,\infty)$ is defined by

$$g(x) = \frac{d}{dx}\int_0^\infty \frac{k_1(xy)}{y}\, f(y)dy.$$

The mean-convergence theory for this type of integral transform can be found in TITCHMARSH [18, 8.5]. Actually we will consider a generalization of the Watson transform to integral transforms in the space $L^2(-\infty,\infty)$, given by BRAAKSMA [1], see section 2.

The spectral theory consists of three problems: determination of the spectrum, the resolvent and the spectral resolution.
The determination of the spectrum of T can be found in section 3, where it turns out that the spectrum can be described completely in terms of the Mellin transform of the kernel $k_1(y)/y$. For the Mellin transform, see section 1. The methods used to obtain the spectrum go back to CARLEMAN [3]. The eigenvalues of T were inverstigated by DOETSCH [6].

The problem of the determination of the resolvent is to be found in section 4. In particular, we are able to determine under what circumstances the Watson transform provides a one to one correspondence between the space $L^2(0,\infty)$ and itself.

Untill now we did not require T to be normal; we were able to determine the spectrum because Watson transforms have the helpful property that their residual spectrum is void. Finally, the problem of finding the spectral resolution of T, forces us to require T to be normal. POLLARD [14] has given a method for studying the spectral resolution of self-adjoint convolution transforms in $L^2(-\infty,\infty)$, which also could be used here. However, we prefer to use the more general method of DUNFORD [7], who based the spectral resolution of convolution transforms in $L^2(-\infty,\infty)$ on an integral representation of bounded, normal operators in a Hilbert space. Therefore we first give necessary and sufficient conditions under which T is normal (section 5) and then we proceed to calculate the spectral resolution of T, following Dunford, in section 6.

In section 8 we define a special class of transforms, which includes the one-sided Laplace transform. These transforms are neither unitary nor involutory. It turns out that all results obtained in the previous sections apply to this class.

Self-reciprocal functions are discussed in section 9. We include an obvious generalization of a theorem, due to BUSBRIDGE [2].
Examples of the transforms considered here, are given in section 10.
The spectral resolution of the two-sided Fourier transform, due to RIESZ and SZ-NAGY [15] is included in our examples, now derived from the general formulas for the spectral resolution of the transforms, defined by BRAAKSMA. Another example, involving an integral transform of MEIJER [12] generalizes results due to POLLARD.

1. Notations and theorems about Fourier and Mellin transforms

Functions which are equal to each other almost everywhere will be identified.

If $a < b$, a and b finite or infinite, then $L^2(a,b)$ or $L^2(a < x < b)$ denotes the class of functions $f(x)$ for which $\int_b^a |f(x)|^2 dx$ exists.

$L^2(\frac{1}{2}-i\infty, \frac{1}{2}+i\infty)$ is the class of functions $f(s)$ defined for $\operatorname{Re} s = \frac{1}{2}$ and such that $f(\frac{1}{2}+ix) \in L^2(-\infty < x < \infty)$.

If $f(a,x) \in L^2(-\infty < x < \infty)$ for sufficiently large values of a, and if there exists a function $f(x)$ such that

$$\lim_{a\to\infty} \int_{-\infty}^{\infty} |f(x)-f(a,x)|^2 dx = 0,$$

then we write

$$f(x) = \underset{a\to\infty}{\text{l.i.m.}}\ f(a,x).$$

Then the function $f(x)$ is determined uniquely and belongs to $L^2(-\infty,\infty)$.

Analogous definitions will be used with $L^2(-\infty < x < \infty)$ replaced by $L^2(0 < x < \infty)$ or $L^2(\frac{1}{2}-i\infty, \frac{1}{2}+i\infty)$. In these cases we add to the expression $\underset{a\to\infty}{\text{l.i.m.}}\ f(a,x)$: l.i.m. in $L^2(0,\infty)$ or l.i.m. in $L^2(\frac{1}{2}-i\infty, \frac{1}{2}+i\infty)$. Instead of $a \to \infty$ we may also have $a \to a_0$, $a \downarrow a_0$ or $a \uparrow a_0$.

In the following powers will have the principal value. Further, if in a formula the signs $\pm$ or $\mp$ occur, then we assume that the upper signs and also the lower signs belong together.

For the sake of convenience we quote some theorems on Fourier and Mellin transforms which will be used frequently in the following.

Theorem A (PLANCHEREL, cf. [18, p. 69]). If $f(x) \in L^2(-\infty,\infty)$, then the F^2-transform (Fourier transform) $F(x)$ of $f(x)$ exists:

$$(1.1) \qquad F(x) = \underset{a \to \infty}{\text{l.i.m.}} \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} f(y) e^{ixy} dy.$$

For almost all real values of x we have

$$(1.2) \qquad F(x) = \frac{d}{dx} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y) \frac{e^{ixy}-1}{iy} dy.$$

Further $f(x)$ is the F^{-2}-transform of $F(x)$:

$$(1.3) \qquad f(x) = \underset{a \to \infty}{\text{l.i.m.}} \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} F(y) e^{-ixy} dy,$$

whereas

$$(1.4) \qquad f(x) = \frac{d}{dx} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(y) \frac{e^{-ixy}-1}{-iy} dy$$

for almost all real values of x.

If $f(x) \in L^2(-\infty,\infty)$ and $g(x) \in L^2(-\infty,\infty)$ and if $F(x)$ and $G(x)$ are their F^2-transforms, then

$$(1.5) \qquad \int_{-\infty}^{\infty} f(x) \overline{g(x)} dx = \int_{-\infty}^{\infty} F(x) \overline{G(x)} dx,$$

where both integrals exist (formula of Parseval).

By a change of variables Plancherel's theorem may be used to prove the following theorem on the Mellin transform.

Theorem B (cf. [18, pp. 94,95]). If $f(x) \in L^2(0,\infty)$, then the M^2-transform (Mellin transform) of $f(x)$ defined by

$$(1.6) \qquad F(s) = \underset{a \to \infty}{\text{l.i.m.}} \int_{1/a}^{a} f(x) x^{s-1} dx$$

(l.i.m. in $L^2(\frac{1}{2}-i\infty, \frac{1}{2}+i\infty)$), exists for $\operatorname{Re} s = \frac{1}{2}$. Conversely, $f(x)$ is the M^{-2}-transform of $F(s)$:

$$f(x) = \underset{A \to \infty}{\mathrm{l.i.m.}} \frac{1}{2\pi i} \int_{\frac{1}{2}-iA}^{\frac{1}{2}+iA} F(s)x^{-s}ds \tag{1.7}$$

(l.i.m. in $L^2(0,\infty)$), and

$$\int_0^x f(y)dy = \frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} F(s)x^{1-s} \frac{ds}{1-s} \tag{1.8}$$

for all non-negative values of x and

$$f(x) = \frac{d}{dx} \frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} F(s)x^{1-s} \frac{ds}{1-s} \tag{1.9}$$

for almost all positive values of x.

If $f(x) \in L^2(0,\infty)$ and $g(x) \in L^2(0,\infty)$, if $F(s)$ and $G(s)$ are their M^2-transforms ($\mathrm{Re}\ s = \frac{1}{2}$) and if $a > 0$, then

$$\int_0^\infty f(ax)g(x)dx = \frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} F(s)G(1-s)a^{-s}ds \tag{1.10}$$

where both integrals exist (formula of Parseval).

If there is given a function $F(s) \in L^2(\frac{1}{2}-i\infty, \frac{1}{2}+i\infty)$, then the M^{-2}-transform defined by (1.7) exists, $f(x) \in L^2(0,\infty)$ and (1.8), (1.9) and (1.6) hold.

Theorem C. Let $f(x) \in L^2(0,\infty)$ and let $F(s)$ be the M^2-transform of $f(x)$ ($\mathrm{Re}\ s = \frac{1}{2}$).

Then the M^2-transform of the function $\overline{f(x)}$, the complex conjugate of $f(x)$, is given by:

$$\overline{F(1-s)} = M^2\,[\,\overline{f(x)}\,], \tag{1.11}$$

and the M^2-transform of the function $\frac{1}{x} f(\frac{1}{x})$ 1) is given by

$$F(1-s) = M^2\,[\frac{1}{x} f(\frac{1}{x})]. \tag{1.12}$$

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1) Note that $f(x) \in L^2(0,\infty)$ if and only if $\frac{1}{x} f(\frac{1}{x}) \in L^2(0,\infty)$.

2. Some properties of generalized Watson transforms

The following definition and theorem are due to BRAAKSMA [1].

Definition 1. Suppose $K_+(s)$ and $K_-(s)$ are measurable, essentially bounded functions on Re $s = \frac{1}{2}$. Let $k_1(\pm x)/\pm x$ be the M^{-2}-transform of $\dfrac{K_\pm(s)}{1-s}$:

$$(2.1) \qquad k_1(\pm x) = \pm x \underset{A\to\infty}{\text{l.i.m.}} \frac{1}{2\pi i} \int_{\frac{1}{2}-iA}^{\frac{1}{2}+iA} K_\pm(s)\, \frac{x^{-s}}{1-s}\, ds$$

(l.i.m. in $L^2(0,\infty)$) if $x > 0$. Suppose $f(x) \in L^2(-\infty,\infty)$. Then the transform $g = Tf$ is defined by

$$(2.2) \qquad g(x) = \frac{d}{dx} \int_{-\infty}^{\infty} \frac{k_1(xy)}{y}\, f(y)dy.$$

Theorem 1. The mapping T, defined by (2.2) is a bounded transform on $L^2(-\infty,\infty)$.

Proof. Let $F_\pm(s)$ be the M^2-transform of $f(\pm x)$ $(x > 0)$. Then $F_\pm(s) \in L^2(\frac{1}{2}-i\infty, \frac{1}{2}+i\infty)$. From this and the properties of $K_\pm(s)$ we deduce that the functions $G_\pm(s)$ defined by

$$(2.3) \qquad \begin{cases} G_+(s) = K_+(s)F_+(1-s)+K_-(s)F_-(1-s), \\ G_-(s) = K_-(s)F_+(1-s)+K_+(s)F_-(1-s), \end{cases}$$

also belong to $L^2(\frac{1}{2}-i\infty,\frac{1}{2}+i\infty)$. Let $g(\pm x)$ $(x > 0)$ be the M^{-2}-transform of $G_\pm(s)$. Then $g(x) \in L^2(-\infty,\infty)$, and if $x > 0$

$$\int_0^x g(y)dy = \frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \{K_+(s)F_+(1-s)+K_-(s)F_-(1-s)\}\, \frac{x^{1-s}}{1-s}\, ds,$$

$$\int_0^x g(-y)dy = \frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \{K_-(s)F_+(1-s)+K_+(s)F_-(1-s)\}\, \frac{x^{1-s}}{1-s}\, ds.$$

Since $\dfrac{K_\pm(s)}{1-s}$ is the M^2-transform of $k_1(\pm x)/\pm x$ $(x > 0)$ we may deduce from the preceding formulas and (1.10)

$$\int_0^x g(y)dy = \int_0^\infty \{k_1(xy)f(y)-k_1(-xy)f(-y)\}\, y^{-1}dy = \int_{-\infty}^{\infty} \frac{k_1(xy)}{y}\, f(y)dy,$$

and

$$\int_0^x g(-y)dy = \int_0^\infty \{-k_1(-xy)f(y)+k_1(xy)f(-y)\}\ y^{-1}dy = -\int_{-\infty}^\infty \frac{k_1(-xy)}{y}\ f(y)dy$$

if $x > 0$. These formulas imply (2.2) for almost all real values of x.

In order to show, that there exists a positive constant c independent of f(x), such that

$$\int_{-\infty}^\infty |g(x)|^2dx \leqq c \int_{-\infty}^\infty |f(x)|^2dx,$$

we use the M^2-transform of $\overline{g(x)}$ and $\overline{g(-x)}$ and calculate

$$\int_0^\infty g(x)\ \overline{g(x)}\ dx + \int_0^\infty g(-x)\ \overline{g(-x)}\ dx$$

by means of (1.10) and (1.11). Then we obtain

$$\frac{1}{2\pi i}\int_{\frac12-i\infty}^{\frac12+i\infty} [\{|K_+(s)|^2 + |K_-(s)|^2\}\{|F_+(1-s)|^2 + |F_-(1-s)|^2\} +$$

$$+ \{K_+(s)\ \overline{K_-(s)} + K_-(s)\ \overline{K_+(s)}\}\ \{F_+(1-s)\ \overline{F_-(1-s)} + F_-(1-s)\ \overline{F_+(1-s)}\}]ds.$$

Now let M be the maximum of $|K_+(s)|^2 + |K_-(s)|^2$ on Re $s = \frac12$. Then we easily see that the last integral is bounded by

$$\frac{M}{\pi i}\int_{\frac12-i\infty}^{\frac12+i\infty} \{|F_+(1-s)|^2 + |F_-(1-s)|^2\}ds = 2M\int_{-\infty}^\infty |f(x)|^2dx.$$

This implies that T is bounded. Hence the theorem is proved.

As the transform T is defined everywhere in $L^2(-\infty,\infty)$ and bounded by the above theorem, the adjoint T* exists and is a bounded transform as well. We wish to find the relation between T and T*. Therefore we first define the conjugation operator J by $Jf = \bar f$ for all $f \in L^2(-\infty,\infty)$.

Theorem 2. The adjoint of the operator T is given by

(2.4) $\qquad T^* = JTJ$

or equivalently

$$(T^*f)(x) = \frac{d}{dx}\int_{-\infty}^\infty \frac{\overline{k_1(xy)}}{y}\ f(y)dy.$$

Proof. Define the function e_x by:

$$\begin{cases} e_x(y) = 1 & \text{if} \quad y \in [0,x], \\ \quad\;\; = 0 & \text{if} \quad y \notin [0,x], \end{cases}$$

if $x > 0$ and by

$$\begin{cases} e_x(y) = -1 & \text{if} \quad y \in [x,0], \\ \quad\;\; = \;\;0 & \text{if} \quad y \notin [x,0], \end{cases}$$

if $x < 0$. Now we use (2.2) in the form

$$\int_0^x g(y)dy = \int_{-\infty}^{\infty} \frac{k_1(xy)}{y} f(y)dy, \tag{2.5}$$

which may be written as:

$$((Tf)(y),\ e_x(y)) = (f(y),\ \frac{\overline{k_1(xy)}}{y})$$

or

$$(f(y),\ (T^*\ e_x)(y)) = (f(y),\ \frac{\overline{k_1(xy)}}{y}).$$

Since this last relation is valid for all $f(y) \in L^2(-\infty,\infty)$, we obtain

$$(T^*\ e_x)(y) = \frac{\overline{k_1(xy)}}{y} \quad \text{a.e.}$$

The special choice of $f(y) = e_x(y)$ in (2.5) yields:

$$(T\ e_x)(y) = \frac{k_1(xy)}{y} \quad \text{a.e.}$$

We now note, that the linear manifold spanned by finite, real linear combinations of functions e_x is dense in the <u>real</u> space $L_r^2\ (-\infty,\infty)$. The result is that $f_1 \in L_r^2\ (-\infty,\infty)$ implies $g_1 = Tf_1$ if and only if $\overline{g_1} = T^*f_1$. Now every $f \in L^2\ (-\infty,\infty)$ can be written as $f = f_1 + if_2$, where $f_1, f_2 \in L_r^2\ (-\infty,\infty)$. If $g_2 = Tf_2$, then we have the following relations:

$$T(f_1+if_2) = g_1+ig_2,$$

$$T^*(f_1+if_2) = \overline{g_1}+i\overline{g_2},$$

because T^* is a linear operator. By decomposing g_1 and g_2 into real and imaginary parts, the result to be proved follows immediately.

The last theorem in this section deals with a situation, where the differentiation in (2.2) may be performed under the integral sign. However, the integral itself must be given another meaning.

Theorem 3. Let the function $k_1(x)$ be defined by (2.1). Suppose that $k_1(x)$ is differentiable and let $k(x) = \frac{d}{dx} k_1(x)$ belong to $L^2(-N,N)$ for all $N > 0$. Then an equivalent form of the transform (2.2) is given by:

$$g(x) = \underset{N \to \infty}{\text{l.i.m.}} \int_{-N}^{N} k(xy)\, f(y)dy, \tag{2.6}$$

(l.i.m. in $L^2(-\infty,\infty)$).

Proof. Let $g(x)$ be defined by (2.2). Let $f_N(y)$ be defined by

$$\begin{cases} f_N(y) = f(y), & -N \leqq y \leqq N, \\ \qquad 0 \quad , & y \notin [-N,N]. \end{cases}$$

Then $f_N(y) \in L^1(-\infty,\infty)$ and we obtain for $x > 0$:

$$\begin{aligned}(Tf_N)(x) &= \frac{d}{dx} \int_{-\infty}^{\infty} \frac{k_1(xy)}{y} f_N(y)dy \\ &= \frac{d}{dx} \int_{-\infty}^{\infty} \frac{f_N(y)}{y} \left\{ \int_0^{xy} k(u)du \right\} dy = \frac{d}{dx} \int_{-\infty}^{\infty} f_N(y) \left\{ \int_0^{x} k(yt)dt \right\} dy \\ &= \frac{d}{dx} \int_0^{x} \left\{ \int_{-\infty}^{\infty} f_N(y)k(yt)dy \right\} dt = \int_{-\infty}^{\infty} f_N(y)k(yx)dy \\ &= \int_{-N}^{N} f(y)k(yx)dy.\end{aligned}$$

The change of the order of integration above is justified by Fubini's theorem and the inequality:

$$\left| \int_{-\infty}^{\infty} f_N(y)k(yt)dy \right| \leqq \frac{1}{\sqrt{t}} \left\{ \int_{-\infty}^{\infty} |f(y)|^2 dy. \int_{-Nt}^{Nt} |k(y)|^2 dy \right\}^{\frac{1}{2}} .$$

If $x < 0$ we use a similar argument. As the transform T is bounded by theorem 1, we have

$$\lim_{N\to\infty} ||Tf-Tf_N|| = 0, \tag{2.7}$$

which is equivalent to (2.6).

Conversely, if we assume (2.6) e.g. (2.7), then for any $h(x) \in L^2(-\infty,\infty)$ we have

$$(2.8) \qquad \lim_{N\to\infty} (Tf_N,h) = (Tf,h)$$

Let $h(y) = e_x(y)$, where $e_x(y)$ is the function defined in the proof of theorem 2. Then for $x > 0$ (2.8) can be rewritten as

$$\lim_{N\to\infty} \int_0^x \left\{ \int_{-N}^N k(uy)f(u)du \right\} dy = \int_0^x g(y)dy,$$

or

$$\lim_{N\to\infty} \int_{-N}^N f(y) \left\{ \int_0^x k(uy)du \right\} dy = \int_0^x g(y)dy.$$

Hence we obtain (2.2) for $x > 0$. The case $x < 0$ is dealt with similarly. This completes the proof.

3. Determination of the spectrum

From TAYLOR [7, chapter 5] we first quote the definition of the spectrum of an operator in a Hilbert space X. Let T be a linear operator, whose domain D(T) and range R(T) lie in X.

If the complex number λ is such that the range of $T-\lambda$ is dense in X and if $T-\lambda$ has a continuous inverse, then λ is said to be in the resolvent set of T, denoted by $\rho(T)$. All scalar values of λ not in $\rho(T)$ comprise the set called the spectrum of T, denoted by $\sigma(T)$.

A finer description of the spectrum is necessary for our purposes. Therefore we remark the following. For the range of $T-\lambda$ there are three possibilities:

I. $R(T-\lambda) = X$,

II. $R(T-\lambda) \neq X$, but $R(T-\lambda)$ is dense in X,

III. $R(T-\lambda)$ is not dense in X.

As regards $(T-\lambda)^{-1}$, the inverse of $T-\lambda$, there are also three possibilities:

1. $(T-\lambda)^{-1}$ exists and is continuous,
2. $(T-\lambda)^{-1}$ exists, but is not continuous,
3. $(T-\lambda)^{-1}$ does not exist.

By combining the above possibilities, we obtain nine different situations.

According to the definition of the resolvent set $\lambda \in \rho(T)$ if and only if λ is in class I_1 of II_1.

The remaining classes characterize the various parts of the spectrum:

I_2 and II_2 : the continuous spectrum, denoted by $C\sigma(T)$;

III_1 and III_2 : the residual spectrum, denoted by $R\sigma(T)$;

I_3, II_3 and III_3: the pointspectrum (eigenvalues), denoted by $P\sigma(T)$.

If T is a bounded, linear operator, defined everywhere in X, then general theorems about linear operators state that the classes I_2 and II_1 do not occur. It also can be proved that in this case $\sigma(T)$ is a non-empty, compact set in the complex plane.

It has been shown in theorem 1 that the integral transforms with which we are dealing are bounded linear operators, defined everywhere in the Hilbert space $L^2(-\infty,\infty)$.

It is our purpose to determine the spectrum of the integral transforms defined by (2.2).

In the remainder of this paper we will use the following abbreviations:

$$\begin{cases} K^*(s) = K_+(s) + K_-(s), \\ K_*(s) = K_+(s) - K_-(s). \end{cases}$$

The next theorem gives a complete characterization of the resolvent set $\rho(T)$ of T in terms of the functions $K_+(s)$ and $K_-(s)$.

Theorem 4. The number λ belongs to the resolvent set $\rho(T)$ of T if and only if the functions $\{\lambda^2-K^*(s)K^*(1-s)\}$ and $\{\lambda^2-K_*(s)K_*(1-s)\}$ have essentially bounded inverses on Re $s = \frac{1}{2}$.

Proof. Suppose $\lambda \in \rho(T)$. Then the equation

$$(3.1) \qquad (T-\lambda I)f = g$$

has a solution $f(x) \in L^2(-\infty,\infty)$ for every $g(x) \in L^2(-\infty,\infty)$. It can easily be seen, that the equation (3.1) is equivalent to the system:

$$(3.2) \qquad \begin{cases} -\lambda F_+(s) + K_+(s)F_+(1-s)+K_-(s)F_-(1-s) = G_+(s), \\ -\lambda F_-(s)+K_-(s)F_+(1-s)+K_+(s)F_-(1-s) = G_-(s). \end{cases}$$

By changing s into 1-s in (3.2) we obtain two new equations:

$$(3.3) \qquad \begin{cases} K_+(1-s)F_+(s)+K_-(1-s)F_-(s)-\lambda F_+(1-s) = G_+(1-s), \\ K_-(1-s)F_+(s)+K_+(1-s)F_-(s) -\lambda F_-(1-s) = G_-(1-s). \end{cases}$$

Let $\Delta_\lambda(s)$ be the determinant of the coefficients of the system, formed by (3.2) and (3.3):

$$(3.4)\qquad \Delta_\lambda(s) = \begin{vmatrix} -\lambda & 0 & K_+(s) & K_-(s) \\ 0 & -\lambda & K_-(s) & K_+(s) \\ K_+(1-s) & K_-(1-s) & -\lambda & 0 \\ K_-(1-s) & K_+(1-s) & 0 & -\lambda \end{vmatrix}$$

$$= \{\lambda^2 - K^*(s)K^*(1-s)\}\{\lambda^2 - K_*(s)K_*(1-s)\}.$$

Thus if the equation (3.1) has a unique solution $f \in L^2(-\infty,\infty)$ for every $g \in L^2(-\infty,\infty)$, then it is necessary that $\lambda^2-K^*(s)K^*(1-s)$ and $\lambda^2-K_*(s)K_*(1-s)$ have essentially bounded inverses on $\mathrm{Re}\ s = \frac{1}{2}$. This proves one part of the theorem.

Conversely we assume the functions $\{\lambda^2-K^*(s)K^*(1-s)\}$ and $\{\lambda^2-K_*(s)K_*(1-s)\}$ have essentially bounded inverses on $\mathrm{Re}\ s = \frac{1}{2}$. Then $\{\Delta_\lambda(s)\}^{-1}$ is essentially bounded on $\mathrm{Re}\ s = \frac{1}{2}$, and the system

$$\begin{aligned}
-\lambda A(s) \qquad\qquad\qquad\qquad &+K_+(s)C(s)+K_-(s)D(s) = G_+(s),\\
-\lambda B(s) \qquad &+K_-(s)C(s)+K_+(s)D(s) = G_-(s),\\
K_+(1-s)A(s)+K_-(1-s)B(s)-\lambda C(s) \qquad &= G_+(1-s),\\
K_-(1-s)A(s)+K_+(1-s)B(s) \qquad -\lambda D(s) \qquad &= G_-(1-s),
\end{aligned}$$

has a unique set of solutions $A(s)$, $B(s)$, $C(s)$ and $D(s)$, each of which belongs to $L^2(\frac{1}{2}-i\infty,\frac{1}{2}+i\infty)$, for any choice of $G_\pm(s) \in L^2(\frac{1}{2}-i\infty,\frac{1}{2}+i\infty)$. Now let $\mathrm{Im}\ s > 0$ and define

$$\begin{aligned} F_+(s) &= A(s), & F_+(1-s) &= C(s),\\ F_-(s) &= B(s), & F_-(1-s) &= D(s). \end{aligned}$$

These functions $F_\pm(s)$ certainly satisfy (3.2) on $\mathrm{Re}\ s = \frac{1}{2}$, which is equivalent to (3.1). Thus for every $g(x) \in L^2(-\infty,\infty)$, there exists a solution $f(x) \in L^2(-\infty,\infty)$ of the equation (3.1). This completes the proof.

Corollary 1. The resolvent set $\rho(T)$ is symmetric with respect to the origin in the complex plane.

Theorem 5. The complex number λ belongs to the point spectrum $P\sigma(T)$ if and only if the roots on $\mathrm{Re}\ s = \frac{1}{2}$ of at least one of the following equations

$$(3.5)\qquad \lambda^2-K^*(s)K^*(1-s) = 0,$$

$$(3.6)\qquad \lambda^2-K_*(s)K_*(1-s) = 0,$$

form a set of positive measure. Such a set is symmetric with respect to $s = \frac{1}{2}$. The point spectrum is symmetric with respect to the origin.

Proof. As the equations (3.5) and (3.6) do not change under the substitution $s \to 1-s$, we see that the roots of these equations are symmetric on $\mathrm{Re}\, s = \frac{1}{2}$ with respect to $s = \frac{1}{2}$.

Suppose λ is an eigenvalue of T. Then in the system formed by (3.2) and (3.3), we have $G_{\pm}(s) \equiv 0$. In order that there exist non-trivial functions $F_+(s)$ and $F_-(s)$ satisfying (3,2) and (3.3), we must have $\Delta_\lambda(s) = 0$ on a subset M_λ of $\mathrm{Re}\, s = \frac{1}{2}$ of positive measure.
Then the functions $F_+(s)$ and $F_-(s)$ will have their support in M_λ. This proves the necessity of the conditions in the theorem.

Conversely, suppose that one of the euations (3.5) or (3.6) has roots on $\mathrm{Re}\, s = \frac{1}{2}$, which form a set M_λ of positive measure on $\mathrm{Re}\, s = \frac{1}{2}$. To be definite, assume the roots of the equation (3.5) have this property.

If $\lambda \neq 0$ we choose on $\mathrm{Re}\, s = \frac{1}{2}$:

$$(3.7) \qquad \begin{cases} F_+(s) = F_-(s) = 0, & s \notin M_\lambda, \\ F_+(s) = F_-(s) = H(s), & s \in M_\lambda,\ \mathrm{Im}\, s > 0, \\ F_+(s) = F_-(s) = \frac{1}{\lambda} K^*(s)H(1-s), & s \in M_\lambda,\ \mathrm{Im}\, s < 0, \end{cases}$$

where $H(s)$ is a function from $L^2(\frac{1}{2}-i\infty, \frac{1}{2}+i\infty)$, non-trivial on M_λ. Such a functions $F_+(s)$ $(= F_-(s))$ is non-trivial and belongs to $L^2(\frac{1}{2}-i\infty, \frac{1}{2}+i\infty)$.
In addition, $F_+(s)$ satisfies the equations (3.2) with $G_{\pm}(s) \equiv 0$ a.e.
If $s \notin M_\lambda$ and if $s \in M_\lambda$, $\mathrm{Im}\, s < 0$ this is clear. If $s \in M_\lambda$, $\mathrm{Im}\, s > 0$ then it follows from the last equation in (3.7) that

$$F_+(1-s) = F_-(1-s) = \frac{1}{\lambda} K^*(1-s)H(s),$$

which, with the equality $H(s) = F_+(s) = F_-(s)$ and the equality (3.5) forces (3.2) to be satisfied with $G_{\pm}(s) \equiv 0$ a.e.

If $\lambda = 0$, then (3.5) shows that $K^*(1-s) = 0$ on a subset $N \subset M_0$ of positive measure. Now we choose on $\mathrm{Re}\, s = \frac{1}{2}$:

$$\begin{cases} F_+(s) = F_-(s) = 0, & s \notin N, \\ F_+(s) = F_-(s) = H(s), & s \in N, \end{cases}$$

where $H(s)$ is a function from $L^2(\frac{1}{2}-i\infty, \frac{1}{2}+i\infty)$, non-trivial on N.
This function $F_+(s)$ $(= F_-(s))$ satisfies the equations (3.2) with $\lambda = 0$ and $G_{\pm}(s) = 0$ a.e.
In both cases $\lambda = 0$ and $\lambda \neq 0$ it follows, that λ is an eigenvalue of T.

A similar argument can be used to prove this theorem, in case the equation (3.6) has a set of roots on $\mathrm{Re}\, s = \frac{1}{2}$, of positive measure.

Corollary 2. Every eigenvalue of the operator T has an infinite dimensional eigenspace. Thus if there are any non-zero eigenvalues the operator T is not compact.

Proof. From the arguments, used in the proof of the above theorem it follows that the function $H(s)$ can be chosen arbitrarily from $L^2(\frac{1}{2}-i\infty,\frac{1}{2}+i\infty)$, such that it is non-trivial on M_λ (or on N in case $\lambda = 0$).

Remark 1. If the positive numbers K_+ and K_- are such that

$$|K_+(s)| \leqq K_+, \quad |K_-(s)| \leqq K_- \quad \text{a.e. on } \operatorname{Re} s = \tfrac{1}{2},$$

then we obtain from (3.5) or (3.6) a bound for the eigenvalues:

$$|\lambda| \leqq K_+ + K_- \ .$$

Remark 2. If $K_+(s) = \pm K_-(s)$ on a set of positive measure of $\operatorname{Re} s = \frac{1}{2}$, then $0 \in P\sigma(T)$. Thus T does not have an inverse. In case $K_+(s) = \pm K_-(s)$ this can also seen by a simple argument. Suppose $K_+(s) = K_-(s)$. Then it follows that $\frac{k_1(x)}{x}$ is an even function, and every non-trivial odd function from $L^2(-\infty,\infty)$ is an eigenfunction of T, belonging to the eigenvalue 0.

Remark 3. The transform T has at most countably many different eigenvalues in $L^2(-\infty,\infty)$. This well known fact can be proved in the following way. From (3.5) and (3.6) it follows for eigenvalues λ and μ with $\lambda^2 \neq \mu^2$, that the corresponding sets M_λ and M_μ are disjoint on $\operatorname{Re} s = \frac{1}{2}$. There are at most countably many disjoint sets of positive measure on $\operatorname{Re} s = \frac{1}{2}$ and this proves the remark. This argument is due to Doetsch [6].

Theorem 6. For the eigenvalues of T we have:

(i) λ is an eigenvalue of T with eigenfunction φ if and only if $\bar{\lambda}$ is an eigenvalue of T with eigenfunction $\bar{\varphi}$,

(ii) if λ_1 and λ_2 are different eigenvalues of T with corresponding eigenfunctions φ_1 and φ_2, then $(\varphi_1,\bar{\varphi}_2) = 0$.

Proof. (i) Let φ be an eigenfunction of T, belonging to the eigenvalue λ, then $T\varphi = \lambda\varphi$, which according to theorem 2, is equivalent to $JT^*J\varphi = \lambda\varphi$. Applying the operator J to both sides of the last equation, we obtain $T^*\bar{\varphi} = \bar{\lambda}\bar{\varphi}$. This argument may also be read in reverse order.

(ii) We have $\lambda_1(\varphi_1,\bar{\varphi}_2) = (T\varphi_1,\bar{\varphi}_2) = (\varphi_1,T^*\bar{\varphi}_2) = (\varphi_1,\bar{\lambda}_2\bar{\varphi}_2) = \lambda_2(\varphi_1,\bar{\varphi}_2)$, thus $(\lambda_1-\lambda_2)(\varphi_1,\bar{\varphi}_2) = 0$. Now if $\lambda_1 \neq \lambda_2$, then $(\varphi_1,\bar{\varphi}_2) = 0$.

Theorem 7. The residual spectrum $R\sigma(T)$ is empty.

Proof. A result in TAYLOR [17, p. 333, problem 6] states: if $\lambda \in R\sigma(T)$ then $\bar{\lambda} \in P\sigma(T^*)$. But theorem 6 (i) shows $\bar{\lambda} \in P\sigma(T^*)$ if and only if $\lambda \in P\sigma(T)$. This leads to a contradiction.

Theorem 8. The complex number λ belongs to the continuous spectrum $C\sigma(T)$ if and only if the roots on $\mathrm{Re}\ s = \frac{1}{2}$ of the equations (3.5) and (3.6) form a set of measure 0, while at least one of the functions $\{\lambda^2 - K^*(s)K^*(1-s)\}^{-1}$ or $\{\lambda^2 - K_*(s)K_*(1-s)\}^{-1}$ is not essentially bounded on $\mathrm{Re}\ s = \frac{1}{2}$.

Proof. Combine theorems 4, 5 and 7.

Corollary 3. Suppose the functions $K^*(s)K^*(1-s)$ and $K_*(s)K_*(1-s)$ are analytic in a region including the line $\mathrm{Re}\ s = \frac{1}{2}$ and suppose they are not constant. Then the point spectrum $P\sigma(T)$ is empty and the spectrum $\sigma(T)$ is entirely made up of the continuous spectrum $C\sigma(T)$.

Proof. This is a direct consequence of theorem 5.

Remark 4. According to theorem 4, the complex number λ belongs to the spectrum $\sigma(T)$ if and only if at least one of the functions $\{\lambda^2 - K^*(s)K^*(1-s)\}$ and $\{\lambda^2 - K_*(s)K_*(1-s)\}$ does not have an essentially bounded inverse on $\mathrm{Re}\ s = \frac{1}{2}$. Suppose for instance that the function $\{\lambda^2 - K^*(s)K^*(1-s)\}$ does not have an essentially bounded inverse on $\mathrm{Re}\ s = \frac{1}{2}$. This means that for each $\varepsilon > 0$ there exists a set of positive measure on $\mathrm{Re}\ s = \frac{1}{2}$, such that on this set

$$|\lambda^2 - K^*(s)K^*(1-s)| < \varepsilon.$$

Hence if the functions $K_+(s)$ and $K_-(s)$ are continuous on $\mathrm{Re}\ s = \frac{1}{2}$, then the above argument shows that the spectrum $\sigma(T)$ is equal to the union of the closure of the set:

$$\{\pm\sqrt{K^*(s)K^*(1-s)} : -\infty < \mathrm{Im}\ s < \infty,\ \mathrm{Re}\ s = \tfrac{1}{2}\} \tag{3.8}$$

and the closure of the set:

$$\{\pm\sqrt{K_*(s)K_*(1-s)} : -\infty < \mathrm{Im}\ s < \infty,\ \mathrm{Re}\ s = \tfrac{1}{2}\}. \tag{3.9}$$

In chapter 6 we will need functions $K_+(s)$ and $K_-(s)$ such that the union of the closure of the sets (3.8) and (3.9) is equal to $\sigma(T)$ almost everywhere in the complex plane. Thus, in particular, continuous functions $K_+(s)$ and $K_-(s)$ satisfy this requirement.

4. The resolvent and inverse

If $\lambda \in \rho(T)$ then the equation $(T-\lambda I)f = g$ has a solution $f(x) \in L^2(-\infty,\infty)$ for any choice of $g(x) \in L^2(-\infty,\infty)$. We are able to compute the resolvent $(T-\lambda I)^{-1}$ explicitly by means of (3.4) and the equations (3.2) and (3.3). This yields $f(x) = M^{-2}[F_+(s)]$ and $f(-x) = M^{-2}[F_-(s)]$ with $x > 0$, where $F_\pm(s)$ are given by:

$$
\begin{aligned}
(4.1)\qquad F_+(s) = {} & \{\lambda^2-K^*(s)K^*(1-s)\}^{-1}\,\{\lambda^2-K_*(s)K_*(1-s)\}^{-1}\cdot \\
& \cdot\,[-\lambda G_+(s)\{\lambda^2-K_+(s)K_+(1-s)-K_-(s)K_-(1-s)\} \\
& -\lambda G_-(s)\{\quad K_+(s)K_-(1-s)+K_-(s)K_+(1-s)\} \\
& +G_+(1-s)\{-\lambda^2K_+(s)+K_+(s)^2K_+(1-s)-K_+(1-s)K_-(s)^2\} \\
& +G_-(1-s)\{-\lambda^2K_-(s)-K_+(s)^2K_-(1-s)+K_-(1-s)K_-(s)^2\}]
\end{aligned}
$$

and

$$
\begin{aligned}
(4.2)\qquad F_-(s) = {} & \{\lambda^2-K^*(s)K^*(1-s)\}^{-1}\,\{\lambda^2-K_*(s)K_*(1-s)\}^{-1}. \\
& [-\lambda G_+(s)\{\lambda^2+K_+(s)K_-(1-s)+K_-(s)K_+(1-s)\} \\
& +\lambda G_-(s)\{\quad K_+(s)K_+(1-s)+K_-(s)K_-(1-s)\} \\
& +G_+(1-s)\ \{-\lambda^2K_-(s)-K_+(s)^2K_-(1-s)+K_-(1-s)K_-(s)^2\} \\
& +G_-(1-s)\ \{-\lambda^2K_+(s)+K_+(s)^2K_+(1-s)-K_+(1-s)K_-(s)^2\}].
\end{aligned}
$$

Of course these formulas are still valid if $\lambda \in C\sigma(T)$, but then $(T-\lambda I)^{-1}$ is only defined on a dense, proper subset of $L^2(-\infty,\infty)$.

Substituting $\lambda = 0$ in (4.1) and (4.2) we obtain expressions for the inverse T^{-1} of T. Suppose $f(x) \in L^2(-\infty,\infty)$ and $0 \notin P\sigma(T)$, let $g = Tf$, then $f = T^{-1}g$ is given by

$$
(4.3)\qquad f(x) = \underset{A\to\infty}{\mathrm{l.i.m.}}\ \frac{1}{2\pi i}\int_{\frac12-iA}^{\frac12+iA}\left\{\frac{K_+(1-s)G_+(1-s)-K_-(1-s)G_-(1-s)}{K^*(1-s)K_*(1-s)}\right\}x^{-s}\,ds,
$$

and

$$
(4.4)\qquad f(-x) = \underset{A\to\infty}{\mathrm{l.i.m.}}\ \frac{1}{2\pi i}\int_{\frac12-iA}^{\frac12+iA}\left\{\frac{-K_-(1-s)G_+(1-s)+K_+(1-s)G_-(1-s)}{K^*(1-s)K_*(1-s)}\right\}x^{-s}\,ds
$$

(l.i.m. in $L^2(0,\infty)$) if $x > 0$.

In the next theorem due to BRAAKSMA [1], we will show that these formulas have a more elegant form in case $0 \in \rho(T)$. Then $R(T) = L^2(-\infty,\infty)$ and the inverse T^{-1} is bounded. We define the functions $H_{\pm}(s)$ on $\mathrm{Re}\, s = \frac{1}{2}$ by

$$(4.5) \qquad H_{\pm}(s) = \pm \frac{K_{\pm}(1-s)}{K^*(1-s)K_*(1-s)} .$$

Theorem 4 shows that these functions $H_{\pm}(s)$ are essentially bounded, measurable functions on $\mathrm{Re}\, s = \frac{1}{2}$.

Theorem 9. Suppose $0 \in \rho(T)$. Let $h_1(\pm x)/\pm x$ be the M^{-2}-transform of $\frac{H_{\pm}(s)}{1-s}$. Suppose $f(x) \in L^2(-\infty,\infty)$. Let $g = Tf$ be defined by (2.2) Then

$$(4.6) \qquad f(x) = \frac{d}{dx} \int_{-\infty}^{\infty} \frac{h_1(xy)}{y}\, g(y)dy,$$

for almost all real values of x.

If a function $g(x) \in L^2(-\infty,\infty)$ is given, then $f(x)$ defined by (4.6) exists for almost all real values of x, $f(x) \in L^2(-\infty,\infty)$ and (2.2) holds for almost all real values of x.

By means of (2.2) and (4.6) $L^2(-\infty,\infty)$ is mapped one to one onto itself.

Proof. From (4.3), (4.4), (4.5), (1.7) and (1.8) it follows that

$$\int_0^x f(y)dy = \frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \{H_+(s)G_+(1-s)+H_-(s)G_-(1-s)\} \frac{x^{1-s}}{1-s}\, ds$$

and

$$\int_0^x f(-y)dy= \frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \{H_-(s)G_+(1-s)+H_+(s)G_+(1-s)\} \frac{x^{1-s}}{1-s}\, ds$$

if $x > 0$. As $\frac{H_{\pm}(s)}{1-s}$ is the M^2-transform of $h_1(\pm x)/\pm x$, we may proceed as in theorem 2. This completes the proof.

Remark 5. Let $H^*(s) = H_+(s)+H_-(s)$ and $H_*(s) = H_+(s)-H_-(s)$. Then, in case $0 \in \rho(T)$, the spectrum of T^{-1} is determined by the functions $H^*(s)H^*(1-s)$ and $H_*(s)H_*(1-s)$. By means of (4.5) one can then relate the spectrum of T^{-1} to the spectrum of T.

5. Characterization of normal and involutory transforms

In this section we derive necessary and sufficient conditions for generalized Watson transforms to be normal, unitary, self-adjoint of involutory. In the first two cases we use theorem 2, and for the last two cases we use theorems 2 and 9.

Theorem 10. The transform T is normal if and only if $|K^*(s)| = |K^*(1-s)|$ and $|K_*(s)| = |K_*(1-s)|$ on $\mathrm{Re}\ s = \frac{1}{2}$.

Proof. From theorem 2 it follows that T is normal if and only if $(TJ)^2 = (JT)^2$.

Suppose T is normal, e.g. $(TJ)^2 = (JT)^2$. Let $f \in L^2(-\infty,\infty)$ and put $u = (TJ)^2 f$. Let $F_\pm(s)$ and $U_\pm(s)$ be the M^2-transforms of $f(\pm x)$ and $u(\pm x)$ $(x > 0)$ respectively. Repeated application of (2.3) and (1.11) yields

$$U_+(s) = \{K_+(s)\overline{K_+(s)}+K_-(s)\overline{K_-(s)}\}\ F_+(s)+ \{K_+(s)\overline{K_-(s)}+\overline{K_+(s)}K_-(s)\}\ F_-(s),$$

$$U_-(s) = \{K_+(s)\overline{K_-(s)}+\overline{K_+(s)}K_-(s)\}\ F_+(s)+ \{K_+(s)\overline{K_+(s)}+K_-(s)\overline{K_-(s)}\}\ F_-(s).$$

However, we also have $u = (JT)^2 f$ which implies

$$U_+(s) = \{K_+(1-s)\overline{K_+(1-s)}+K_-(1-s)\overline{K_-(1-s)}\}F_+(s)+\{K_+(1-s)\overline{K_-(1-s)}+\overline{K_+(1-s)}K_-(1-s)\}F_-(s)$$

$$U_-(s) = \{K_+(1-s)\overline{K_-(1-s)}+\overline{K_+(1-s)}K_-(1-s)\}F_+(s)+\{K_+(1-s)\overline{K_+(1-s)}+K_-(1-s)\overline{K_-(1-s)}\}F_-(s).$$

From these expressions for the functions $U_+(s)$ and $U_-(s)$ we find the equalities:

$$(5.1)\qquad \begin{cases} K_+(s)\overline{K_+(s)} + K_-(s)\overline{K_-(s)} = K_+(1-s)\overline{K_+(1-s)} + K_-(1-s)\overline{K_-(1-s)}, \\ K_+(s)\overline{K_-(s)} + \overline{K_+(s)}K_-(s) = K_+(1-s)\overline{K_-(1-s)} + \overline{K_+(1-s)}K_-(1-s). \end{cases}$$

Adding and subtracting these relations yield the desired equalities.

Next we assume these equalities are satisfied:

$$|K^*(s)| = |K^*(1-s)|\ , \qquad |K_*(s)| = |K_*(1-s)|.$$

They imply (5.1). For a function $f \in L^2(-\infty,\infty)$ we define the functions $u^{(1)}$ and $u^{(2)}$ by $u^{(1)} = (TJ)^2 f$ and $u^{(2)} = (JT)^2 f$ respectively. Let $U_\pm^{(1)}(s)$ and $U_\pm^{(2)}(s)$ be the M^2-transforms of $u^{(1)}(\pm x)$ and $u^{(2)}(\pm x)$ $(x > 0)$ respectively. Express the functions $U_\pm^{(1)}(s)$ and $U_\pm^{(2)}(s)$ in terms of the M^2-transforms $F_\pm(s)$ of $f(\pm x)$ $(x > 0)$. Then it is easily seen that $U_+^{(1)}(s) = U_+^{(2)}(s)$ and $U_-^{(1)}(s) = U_-^{(2)}(s)$; hence $u^{(1)}(x) = u^{(2)}(x)$, which implies $(JT)^2 = (TJ)^2$. Thus the transform T is normal.

Theorem 11. The transform T is unitary if and only if $|K^*(s)| = |K_*(s)| = 1$ on Re $s = \frac{1}{2}$.

Proof. Suppose T is unitary, then $0 \in \rho(T)$ and we may apply theorem 9. On account of theorem 2 it is clear that T^* defines a generalized Watson transform with kernel $\overline{k_1}$. The M^2-transform of $\dfrac{\overline{k_1(\pm x)}}{\pm x}$ $(x > 0)$ can be expressed in terms of the functions $K_+(s)$ and $K_-(s)$ by theorem C; as $T^* = T^{-1}$ we find with (4.5) the relations

$$\overline{K_{\pm}(1-s)} = \pm \frac{K_{\pm}(1-s)}{K^*(1-s)K_*(1-s)} \quad \text{on Re } s = \tfrac{1}{2}, \tag{5.2}$$

This implies $|K^*(s)| = |K_*(s)| = 1$ on Re $s = \frac{1}{2}$. Inverting our steps we can prove the other part of the theorem.

Remark 6. In view of theorem 9, (4.5) and (5.2) we have for unitary transforms the relations

$$H_{\pm}(s) = \overline{K_{\pm}(1-s)} \text{ on Re } s = \tfrac{1}{2},$$

and

$$h_1(x) = \overline{k_1(x)},$$

where $H_{\pm}(s)$ and $h_1(x)$ are defined in theorem 9.

Theorem 12. The transform T is self-adjoint if and only if $K^*(s) = \overline{K^*(1-s)}$ and $K_*(s) = \overline{K_*(1-s)}$ on Re $s = \frac{1}{2}$.

Proof. Combine theorem 2 and theorem C.

Remark 7. A statement equivalent to the above is: the transform T is self-adjoint if and only if k_1 is a real function. In this case we have JT = TJ. The transform is then said to be real with respect to the conjugation J, cf. RIESZ and SZ-NAGY [15, p. 239].

Theorem 13. The transform T is involutory if and only if $K^*(s)K^*(1-s) = K_*(s)K_*(1-s) = 1$ on Re $s = \frac{1}{2}$.

Proof. If the transform T is involutory then we have $T = T^{-1}$. The theorem is a consequence of the relation (4.5) and theorem 9.

Remark 8. If the transform T is involutory, then k_1 is sometimes called a symmetrical Fourier kernel, cf. TITCHMARSH [18, p. 212]. The only points in the spectrum are the eigenvalues ± 1, which follows from theorems 5 and 8, cf. HARDY and TITCHMARSH [11, p. 118].

6. Spectral resolution of normal transforms.

As we have stated in the introduction we will follow a method, used by DUNFORD [7] to obtain the spectral resolution of the identity of the transform (2.2). In order to do this we will require two additional conditions of the transform (2.2):

(6.1) $\quad |K^*(s)| = |K^*(1-s)|$ and $|K_*(s)| = |K_*(1-s)|$ on $\operatorname{Re} s = \frac{1}{2}$, (hence the transform is normal) cf. theorem 10,

(6.2) $\quad K_+(s)$ and $K_-(s)$ are continuous on $\operatorname{Re} s = \frac{1}{2}$, cf. remark 4.

For a bounded normal operator T in a Hilbert space X there exists a resolution of the identity. This is stated in theorem D, which may be found for instance in DUNFORD [7]. The symbol B denotes the class of Borel subsets of the compact set $\sigma(T)$ in the complex plane.

Theorem D. For each $e \in B$ there is a uniquely determined bounded linear operator E_e in X with the properties:

(i) For $x \in X$ the function $E_e x$ is countably additive on B.

(ii) For every pair $x,y \in X$ the scalar product $(E_e x,y)$ is a regular, countably additive set function on B whose total variation is at most $||x||\,||y||$.

(iii) For every e, e_1 and $e_2 \in B$ we have:
$E_e T = TE_e$, $E_{e_1}E_{e_2} = E_{e_2}E_{e_1} = E_{e_1 e_2}$, $E_e^2 = E_e$, $E_e^* = E_e$.

(iv) To every complex valued bounded Borel function $\varphi(\lambda)$, defined on $\sigma(T)$, there corresponds a bounded linear operator $\varphi(T)$ such that:

$$(\varphi(T)x,y) = \int_{\sigma(T)} \varphi(\lambda)\; d(E_\lambda x,y), \qquad x,y \in X.$$

and

$$||\varphi(T)x||^2 = \int_{\sigma(T)} |\varphi(\lambda)|^2\; d(E_\lambda x,x), \qquad x \in X.$$

Before actually calculating the resolution of the identity for normal transforms (2.2) we state some simple observations in the form of lemmas.

First we introduce some notations:

$$\underline{F}(s) = \begin{pmatrix} F_+(s) \\ F_-(s) \end{pmatrix} , \qquad \underline{G}(s) = \begin{pmatrix} G_+(s) \\ G_-(s) \end{pmatrix} ,$$

and

$$\underline{K}(s) = \begin{pmatrix} K_+(s) & K_-(s) \\ K_-(s) & K_+(s) \end{pmatrix} .$$

Let $f(x)$ and $g(x)$ belong to $L^2(-\infty,\infty)$. Let $F_\pm(s)$ and $G_\pm(s)$ be the M^2-transforms of $f(\pm x)$ and $g(\pm x)$ $(x > 0)$ respectively. If $g = Tf$, then we have seen before that (2.3) is valid, which may now be written as:

(6.3) $$\underline{G}(s) = \underline{K}(s)\underline{F}(1-s).$$

Defining the matrix $\overline{\underline{K}(s)}$ by:

$$\overline{\underline{K}(s)} = \begin{pmatrix} \overline{K_+(s)} & \overline{K_-(s)} \\ \overline{K_-(s)} & \overline{K_+(s)} \end{pmatrix} ,$$

we can show that $g = T^*f$ yields the equation

(6.4) $$\underline{G}(s) = \overline{\underline{K}(1-s)}\underline{F}(1-s).$$

This follows from theorem 2 and (1.11).

Lemma 1. Let $f(x) \in L^2(-\infty,\infty)$ then the relations

(i) $g = T^{2n}f$,
(ii) $g = T^{2n+1}f$,
(iii) $g = T^{*2m}f$,
(iv) $g = T^{*2m+1}f$,

imply respectively

(i) $\underline{G}(s) = \underline{K}(s)^n\underline{K}(1-s)^n\underline{F}(s)$,

(ii) $\underline{G}(s) = \underline{K}(s)^{n+1}\underline{K}(1-s)^n\underline{F}(1-s)$,

(iii) $\underline{G}(s) = \overline{\underline{K}(s)}^m\overline{\underline{K}(1-s)}^m\underline{F}(s)$,

(iv) $\underline{G}(s) = \overline{\underline{K}(s)}^m\overline{\underline{K}(1-s)}^{m+1}\underline{F}(1-s)$.

The relations

(v) $\quad g = T^{2n}T^{*2m}f$,

(vi) $\quad g = T^{2n}T^{*2m+1}f$,

(vii) $\quad g = T^{2n+1}T^{*2m}f$,

(viii) $\quad g = T^{2n+1}T^{*2m+1}f$,

imply respectively

(v) $\quad \underline{G}(s) = \underline{K}(s)^n\ \underline{K}(1-s)^n\ \overline{\underline{K}(s)}^m\ \overline{\underline{K}(1-s)}^m\ \underline{F}(s)$,

(vi) $\quad \underline{G}(s) = \underline{K}(s)^n\ \underline{K}(1-s)^n\ \overline{\underline{K}(s)}^m\ \overline{\underline{K}(1-s)}^{m+1}\ \underline{F}(1-s)$,

(vii) $\quad \underline{G}(s) = \underline{K}(s)^{n+1}\ \underline{K}(1-s)^n\ \overline{\underline{K}(s)}^m\ \overline{\underline{K}(1-s)}^m\ \underline{F}(1-s)$,

(viii) $\quad \underline{G}(s) = \underline{K}(s)^{n+1}\ \underline{K}(1-s)^n\ \overline{\underline{K}(s)}^{m+1}\ \overline{\underline{K}(1-s)}^m\ \underline{F}(s)$.

Proof. To prove the first four relations we use (6.3), (6.4), induction and the fact that matrices of the form

$$\begin{pmatrix} p & q \\ q & p \end{pmatrix}$$

commute. The last four relations are immediate consequences of the first four relations.

Lemma 2. For arbitrary complex numbers p, q, r and s and for n,m = 0,1,2,.... we have the relations

$$\begin{pmatrix} \frac{1}{2}(p+q) & \frac{1}{2}(p-q) \\ \frac{1}{2}(p-q) & \frac{1}{2}(p+q) \end{pmatrix}^n$$

$$= \begin{pmatrix} \frac{1}{2}(p^n+q^n) & \frac{1}{2}(p^n-q^n) \\ \frac{1}{2}(p^n-q^n) & \frac{1}{2}(p^n+q^n) \end{pmatrix}$$

and

$$\begin{pmatrix} \frac{1}{2}(p+q) & \frac{1}{2}(p-q) \\ \frac{1}{2}(p-q) & \frac{1}{2}(p+q) \end{pmatrix}^n \begin{pmatrix} \frac{1}{2}(r+s) & \frac{1}{2}(r-s) \\ \frac{1}{2}(r-s) & \frac{1}{2}(r+s) \end{pmatrix}^m$$

$$= \begin{pmatrix} \frac{1}{2}(p^nr^m+q^ns^m) & \frac{1}{2}(p^nr^m-q^ns^m) \\ \frac{1}{2}(p^nr^m-q^ns^m) & \frac{1}{2}(p^nr^m+q^ns^m) \end{pmatrix}$$

Proof. The first relation can be proved by induction, while the second relation follows from the first one.

Lemma 3. Let the matrix $\underline{A}(s)$ be defined by

$$(6.5)\quad \underline{A}(s) = \begin{pmatrix} \frac{1}{2}\{(K^*(s)K^*(1-s))^{\frac{1}{2}} + (K_*(s)K_*(1-s))^{\frac{1}{2}}\} & \frac{1}{2}\{(K^*(s)K^*(1-s))^{\frac{1}{2}} - (K_*(s)K_*(1-s))^{\frac{1}{2}}\} \\ \frac{1}{2}\{(K^*(s)K^*(1-s))^{\frac{1}{2}} - (K_*(s)K_*(1-s))^{\frac{1}{2}}\} & \frac{1}{2}\{(K^*(s)K^*(1-s))^{\frac{1}{2}} + (K_*(s)K_*(1-s))^{\frac{1}{2}}\} \end{pmatrix}$$

Then

$$(6.6)\qquad \underline{A}(s)^2 = \underline{K}(s)\underline{K}(1-s),$$

$$(6.7)\qquad \overline{\underline{A}(s)}^2 = \overline{\underline{K}(s)}\,\overline{\underline{K}(1-s)},$$

and

$$(6.8)\qquad \underline{A}(s)\overline{\underline{A}(s)} = \underline{K}(s)\overline{\underline{K}(s)}.$$

For values of s on Re $s = \frac{1}{2}$ for which $K^*(s) \neq 0$ and $K_*(s) \neq 0$ the matrix $\underline{R}(s)$ is defined by

$$(6.9)\quad \underline{R}(s) = \begin{pmatrix} \frac{1}{2}\left\{\frac{K^*(s)}{(K^*(s)K^*(1-s))^{\frac{1}{2}}} + \frac{K_*(s)}{(K_*(s)K_*(1-s))^{\frac{1}{2}}}\right\} & \frac{1}{2}\left\{\frac{K^*(s)}{(K^*(s)K^*(1-s))^{\frac{1}{2}}} - \frac{K_*(s)}{(K_*(s)K_*(1-s))^{\frac{1}{2}}}\right\} \\ \frac{1}{2}\left\{\frac{K^*(s)}{(K^*(s)K^*(1-s))^{\frac{1}{2}}} - \frac{K_*(s)}{(K_*(s)K_*(1-s))^{\frac{1}{2}}}\right\} & \frac{1}{2}\left\{\frac{K^*(s)}{(K^*(s)K^*(1-s))^{\frac{1}{2}}} + \frac{K_*(s)}{(K_*(s)K_*(1-s))^{\frac{1}{2}}}\right\} \end{pmatrix}$$

Then

$$(6.10)\qquad \underline{A}(s)\underline{R}(s) = \underline{K}(s)$$

and

$$(6.11)\qquad \overline{\underline{A}(s)}\underline{R}(s) = \overline{\underline{K}(1-s)}.$$

Furthermore let $f(x) \in L^2(-\infty,\infty)$ and $g = T^n T^{*m} f$ and let $F_{\pm}(s)$ and $G_{\pm}(s)$ be the M^2-transforms of $f(\pm x)$ and $g(\pm x)$ $(x > 0)$ respectively.

If n+m is an even number then

$$(6.12)\qquad \underline{G}(s) = \underline{A}(s)^n\overline{\underline{A}(s)}^m\underline{F}(s).$$

If n+m is an odd number and if s is a point on Re $s = \frac{1}{2}$ for which $K^*(s) \neq 0$ and $K_*(s) \neq 0$ then

$$(6.13)\qquad \underline{G}(s) = \underline{A}(s)^n\overline{\underline{A}(s)}^m\underline{R}(s)\underline{F}(1-s).$$

Proof. The relations (6.6), (6.7) and (6.10) follow by direct verification and lemma 2. To prove (6.8) and (6.11) we use the condition (6.1) and lemma 2.

If n+m is even, then we have either

$$g = T^{2i}T^{*2j}f, \tag{6.14}$$

or

$$g = T^{2i+1}T^{*2j+1}f. \tag{6.15}$$

In the case (6.14) it follows from lemma 1 (v) that

$$\underline{G}(s) = \underline{K}(s)^i\underline{K}(1-s)^i\overline{\underline{K}(s)}^j\overline{\underline{K}(1-s)}^j\underline{F}(s). \tag{6.16}$$

Now using (6.6) and (6.7) in (6.16) we obtain (6.12). Since the matrices under consideration commute the case (6.15) can be dealt with similarly with lemma 1 (viii), (6.6), (6.7) and (6.8).

If n+m is odd, then we have either

$$g = T^{2i}T^{*2j+1}f, \tag{6.17}$$

or

$$g = T^{2i+1}T^{*2j}f. \tag{6.18}$$

According to lemma 1 (vi) the case (6.17) implies

$$\underline{G}(s) = \underline{K}(s)^i\underline{K}(1-s)^i\overline{\underline{K}(s)}^j\overline{\underline{K}(1-s)}^{j+1}\underline{F}(1-s). \tag{6.19}$$

For values of s for which $K^*(s) \neq 0$ and $K_*(s) \neq 0$ the relation (6.13) follows from (6.19) in view of (6.6), (6.7) and (6.11). Using lemma 1 (vii), (6.6), (6.7) and (6.10) we can prove similarly that (6.18) implies (6.13).

Let $\varphi(\lambda)$ be a continuous complex valued function defined on the spectrum $\sigma(T)$ of the transform T. In order to find an expression for the operator $\varphi(T)$ associated with the function $\varphi(\lambda)$ we shall approximate $\varphi(\lambda)$ by polynomials $P(\lambda,\bar{\lambda})$. Then we determine the operator $P(T,T^*)$ associated with a polynomial $P(\lambda,\bar{\lambda})$ and we find an expression for the function $P(T,T^*)f$ ($f \in L^2(-\infty,\infty)$) as an inverse Mellin-transform. By a limiting process the same can be done for the operator $\varphi(T)$.

If $P(\lambda,\bar{\lambda})$ is a polynomial in λ and $\bar{\lambda}$:

$$P(\lambda,\bar{\lambda}) = \Sigma\ \alpha_{nm}\ \lambda^n\ \bar{\lambda}^m$$

we associate the operator

$$P(T,T^*) = \Sigma\ \alpha_{nm}T^nT^{*m}$$

with this polynomial. Lemma 1 shows that it is convenient to consider polynomials with even and odd factors seperately.

As the spectrum $\sigma(T)$ is a set symmetric with respect to the origin (corollary 1), every function $\varphi(\lambda)$ defined on $\sigma(T)$ can obviously be decomposed into the sum of an even function $\varphi_e(\lambda)$ and an odd function $\varphi_o(\lambda)$, both defined on $\sigma(T)$:

$$\varphi(\lambda) = \varphi_e(\lambda) + \varphi_o(\lambda), \quad \lambda \in \sigma(T).$$

In the following we will consider even and odd functions respectively and we will collect our results in theorem 14.

Even functions. Let $\varphi_e(\lambda)$ be an even continuous complex valued function defined on $\sigma(T)$. Such a function can be approximated uniformly on $\sigma(T)$ by polynomials $P(\lambda,\bar{\lambda})$ of the form:

$$P(\lambda,\bar{\lambda}) = \sum_{m+n=\mathrm{even}} \alpha_{n,m}\, \lambda^n\, \bar{\lambda}^m.$$

Now let $f(x) \in L^2(-\infty,\infty)$, then $g = P(T,T^*)f$ implies

$$g = \sum_{m+n=\mathrm{even}} \alpha_{n,m}\, T^n\, T^{*m} f. \tag{6.20}$$

Applying lemma 3 to (6.20) we obtain

$$\underline{G}(s) = P(\underline{A}(s), \overline{\underline{A}(s)})\underline{F}(s). \tag{6.21}$$

From (6.21), the definition of $\underline{A}(s)$ (6.5) and lemma 2 with

$$p = (K^*(s)K^*(1-s))^{\frac{1}{2}}, \qquad q = (K_*(s)K_*(1-s))^{\frac{1}{2}},$$
$$r = \bar{p}, \qquad s = \bar{q},$$

it follows that

$$G_+(s) = \tfrac{1}{2} P\left((K^*(s)K^*(1-s))^{\frac{1}{2}}, \overline{(K^*(s)K^*(1-s))^{\frac{1}{2}}} \right) \{F_+(s)+F_-(s)\}$$
$$+ \tfrac{1}{2} P\left((K_*(s)K_*(1-s))^{\frac{1}{2}}, \overline{(K_*(s)K_*(1-s))^{\frac{1}{2}}} \right) \{F_+(s)-F_-(s)\}, \tag{6.22}$$

and

$$G_-(s) = \tfrac{1}{2} P\left((K^*(s)K^*(1-s))^{\frac{1}{2}}, \overline{(K^*(s)K^*(1-s))^{\frac{1}{2}}} \right) \{F_+(s)+F_-(s)\}$$
$$- \tfrac{1}{2} P\left((K_*(s)K_*(1-s))^{\frac{1}{2}}, \overline{(K_*(s)K_*(1-s))^{\frac{1}{2}}} \right) \{F_+(s)-F_-(s)\}. \tag{6.23}$$

The polynomials $P_n(\lambda,\bar{\lambda})$ approximate the function $\varphi_e(\lambda)$ boundedly on $\sigma(T)$:

$$|P_n(\lambda,\bar{\lambda})| \leqq K, \qquad K > 0,\ \lambda \in \sigma(T).$$

Then it follows from Lebesgue's theorem on dominated convergence and theorem D (iv) that

$$P_n(T,T^*)f \to \varphi_e(T)f, \qquad f \in L^2(-\infty,\infty).$$

On account of condition (6.2) (cf. remark 4) we also have on Re $s = \frac{1}{2}$:

$$P_n\left((K^*(s)K^*(1-s))^{\frac{1}{2}}, \overline{(K^*(s)K^*(1-s))^{\frac{1}{2}}} \right) \to \varphi_e\left((K^*(s)K^*(1-s))^{\frac{1}{2}} \right),$$

$$\left|P_n\left((K^*(s)K^*(1-s))^{\frac{1}{2}}, \overline{(K^*(s)K^*(1-s))^{\frac{1}{2}}} \right)\right| \leqq K,$$

and

$$P_n\left((K_*(s)K_*(1-s))^{\frac{1}{2}}, \overline{(K_*(s)K_*(1-s))^{\frac{1}{2}}} \right) \to \varphi_e\left((K_*(s)K_*(1-s))^{\frac{1}{2}} \right),$$

$$\left|P_n\left((K_*(s)K_*(1-s))^{\frac{1}{2}}, \overline{(K_*(s)K_*(1-s))^{\frac{1}{2}}} \right)\right| \leqq K.$$

Thus the functions

$$P_n\left((K^*(s)K^*(1-s))^{\frac{1}{2}}, \overline{(K^*(s)K^*(1-s))^{\frac{1}{2}}} \right) \{F_+(s)+F_-(s)\}$$

are dominated by a function of $L^2(\frac{1}{2}-i\infty,\frac{1}{2}+i\infty)$ and hence converge in $L^2(\frac{1}{2}-i\infty,\frac{1}{2}+i\infty)$ to

$$\varphi_e\left((K^*(s)K^*(1-s))^{\frac{1}{2}} \right)\{F_+(s)+F_-(s)\}.$$

Also the functions

$$P_n\left((K_*(s)K_*(1-s))^{\frac{1}{2}}, \overline{(K_*(s)K_*(1-s))^{\frac{1}{2}}} \right) \{F_+(s)-F_-(s)\}$$

converge to

$$\varphi_e\left((K_*(s)K_*(1-s))^{\frac{1}{2}} \right)\{F_+(s)-F_-(s)\}.$$

Since the M^2-transform is a continuous mapping from $L^2(0,\infty)$ onto $L^2(\frac{1}{2}-i\infty,\frac{1}{2}+i\infty)$ (cf. (1.10)) we obtain from (6.22) and (6.23) an expression for the operator $\varphi_e(T)$, applied to a function f:

$$(6.24) \qquad (\varphi_e(T)f)(x) = \tfrac{1}{2}M^{-2}\left[\varphi_e\left((K^*(s)K^*(1-s))^{\frac{1}{2}} \right)\{F_+(s)+F_-(s)\} + \varphi_e\left((K_*(s)K_*(1-s))^{\frac{1}{2}} \right)\{F_+(s)-F_-(s)\}\right]$$

and

$$(6.25) \qquad (\varphi_e(T)f)(-x) = \tfrac{1}{2}M^{-2}\left[\varphi_e\left((K^*(s)K^*(1-s))^{\frac{1}{2}} \right)\{F_+(s)+F_-(s)\} - \varphi_e\left((K_*(s)K_*(1-s))^{\frac{1}{2}} \right)\{F_+(s)-F_-(s)\}\right]$$

if $x > 0$.

Odd functions. Let $\varphi_o(\lambda)$ be an odd continuous complex valued function defined on $\sigma(T)$. Such a function can be approximated uniformly by polynomials $P(\lambda,\bar\lambda)$ of the form:

$$P(\lambda,\bar\lambda) = \sum_{m+n=\text{odd}} \alpha_{n,m}\, \lambda^n\, \bar\lambda^m .$$

Let $f(x) \in L^2(-\infty,\infty)$, then $g = P(T,T^*)f$ implies

$$g = \sum_{m+n=\text{odd}} \alpha_{n,m}\, T^n\, T^{*m} f. \tag{6.26}$$

Applying lemma 3 to (6.26) we obtain

$$\underline{G}(s) = P(\underline{A}(s), \overline{\underline{A}(s)})\, \underline{R}(s)\, \underline{F}(1-s) \tag{6.27}$$

for values of s with $K^*(s) \neq 0$ and $K_*(s) \neq 0$. In view of lemma 2 it then follows from (6.27) that

$$\begin{aligned} G_+(s) = {} & \tfrac12 P\left((K^*(s)K^*(1-s))^{\frac12}, \overline{(K^*(s)K^*(1-s))^{\frac12}} \right) \frac{K^*(s)}{(K^*(s)K^*(1-s))^{\frac12}} \{F_+(1-s)+F_-(1-s \\ & + \tfrac12 P\left((K_*(s)K_*(1-s))^{\frac12}, \overline{(K_*(s)K_*(1-s))^{\frac12}} \right) \frac{K_*(s)}{(K_*(s)K_*(1-s))^{\frac12}} \{F_+(1-s)-F_-(1-s) \end{aligned} \tag{6.28}$$

and

$$\begin{aligned} G_-(s) = {} & \tfrac12 P\left((K^*(s)K^*(1-s))^{\frac12}, \overline{(K^*(s)K^*(1-s))^{\frac12}} \right) \frac{K^*(s)}{(K^*(s)K^*(1-s))^{\frac12}} \{F_+(1-s)+F_-(1-s) \\ & - \tfrac12 P\left((K_*(s)K_*(1-s))^{\frac12}, \overline{(K_*(s)K_*(1-s))^{\frac12}} \right) \frac{K_*(s)}{(K_*(s)K_*(1-s))^{\frac12}} \{F_+(1-s)-F_-(1-s)\}. \end{aligned} \tag{6.29}$$

for values of s under consideration.

Remark 9. Now consider values of s for which $K^*(s) \neq 0$ and $K_*(s) = 0$. Modifying the arguments used above to obtain the formulas (6.28) and (6.29), we can then show

$$G_+(s) = G_-(s) = \tfrac12 P\left((K^*(s)K^*(1-s))^{\frac12}, \overline{(K^*(s)K^*(1-s))^{\frac12}} \right) \frac{K^*(s)}{(K^*(s)K^*(1-s))^{\frac12}} \{F_+(1-s)+F_-(1-s$$

This can also be seen from (6.28) and (6.29), since the polynomial P is odd and the functions $K^*(s)$ and $K_*(s)$ are continuous. A limiting procedure establishes the result. A similar remark applies to other excluded values of s. Hence the formulas (6.28) and (6.29) may be used for all s on Re $s = \frac12$.

Now arguing in the same way as in the case of even functions we find an expression for the operator $\varphi_o(T)$, applied to a function f:

$$(6.30)\quad (\varphi_o(T)f)(x) = \tfrac{1}{2}M^{-2}[\varphi_o((K^*(s)K^*(1-s))^{\frac{1}{2}})\frac{K^*(s)}{(K^*(s)K^*(1-s))^{\frac{1}{2}}}\{F_+(1-s)+F_-(1-s)\}$$

$$+\varphi_o((K_*(s)K_*(1-s))^{\frac{1}{2}})\frac{K_*(s)}{(K_*(s)K_*(1-s))^{\frac{1}{2}}}\{F_+(1-s)-F_-(1-s)\}]$$

and

$$(6.31)\quad (\varphi_o(T)f)(-x)= \tfrac{1}{2}M^{-2}[\varphi_o((K^*(s)K^*(1-s))^{\frac{1}{2}})\frac{K^*(s)}{(K^*(s)K^*(1-s))^{\frac{1}{2}}}\{F_+(1-s)+F_-(1-s)\}$$

$$-\varphi_o((K_*(s)K_*(1-s))^{\frac{1}{2}})\frac{K_*(s)}{(K_*(s)K_*(1-s))^{\frac{1}{2}}}\{F_+(1-s)-F_-(1-s)\}]$$

if $x > 0$. At values of s where $K^*(s) = 0$ the function

$$\varphi_o((K^*(s)K^*(1-s))^{\frac{1}{2}})\frac{K^*(s)}{(K^*(s)K^*(1-s))^{\frac{1}{2}}}$$

occurring in (6.30) and (6.31) has the limit value 0. At values of s where $K_*(s) = 0$ the function

$$\varphi_o((K_*(s)K_*(1-s))^{\frac{1}{2}})\frac{K_*(s)}{(K_*(s)K_*(1-s))^{\frac{1}{2}}}$$

occurring in (6.30) and (6.31) has the limit value 0 (cf. remark 9).

We have proved the formulas (6.24), (6.25), (6.30) and (6.31) under the assumption that the function $\varphi(\lambda)$ is continuous. Since a bounded Borel function defined on $\sigma(T)$ is the pointwise limit a.e. of a bounded sequence of continuous functions on $\sigma(T)$ (see for instance RUDIN [16, p. 54]) these formulas can be shown to hold for any complex valued bounded Borel function defined on $\sigma(T)$.

We will state the results found above in the form of a theorem.

Theorem 14. Let $\varphi(\lambda)$ be a complex valued bounded Borel function defined on the spectrum $\sigma(T)$ of the transform T, given by definition 1. Assume the conditions (6.1) and (6.2) hold. Define

$$\varphi(\lambda) = \varphi_e(\lambda) + \varphi_o(\lambda) \qquad (\varphi_e \text{ even, } \varphi_o \text{ odd}).$$

The operator $\varphi_e(T)$ associated with the function $\varphi_e(\lambda)$ is given by (6.24) and (6.25). The operator $\varphi_o(T)$ associated with the function $\varphi_o(\lambda)$ is given by (6.30) and (6.31).

Remark 10. We will show that the operator $\varphi_o(T)$ is a transform of type (2.2). Define the functions $L_+(s)$ and $L_-(s)$ by

$$(6.32)\quad L_+(s) = \tfrac{1}{2}\left\{\varphi_o((K^*(s)K^*(1-s))^{\frac{1}{2}})\frac{K^*(s)}{(K^*(s)K^*(1-s))^{\frac{1}{2}}} + \varphi_o((K_*(s)K_*(1-s))^{\frac{1}{2}})\frac{K_*(s)}{(K_*(s)K_*(1-s))^{\frac{1}{2}}}\right\}$$

and

$$(6.33)\quad L_-(s) = \tfrac{1}{2}\left\{\varphi_o((K^*(s)K^*(1-s))^{\frac{1}{2}})\frac{K^*(s)}{(K^*(s)K^*(1-s))^{\frac{1}{2}}} - \varphi_o((K_*(s)K_*(1-s))^{\frac{1}{2}})\frac{K_*(s)}{(K_*(s)K_*(1-s))^{\frac{1}{2}}}\right\}$$

Let $l_1(\pm x)/\pm x$ $(x > 0)$ be the M^{-2}-transform of $\frac{L_\pm(s)}{1-s}$, then we may write $\varphi_o(T) = U_L$, where the generalized Watson transform U_L is given by

$$(6.34)\qquad (U_L f)(x) = \frac{d}{dx}\int_{-\infty}^{\infty}\frac{l_1(xy)}{y}\, f(y)dy.$$

In the same way it can be shown that the operator $\varphi_e(T)$ given by (6.24) and (6.25) is a generalized Watson transform, but now unlike (6.34) the function to be transformed is not $f(y)$, but $\frac{1}{y}\, f(\frac{1}{y})$ (cf. theorem C).

7. Watson transforms

The Watson transforms can be found from definition 1, by choosing $K_-(s) \equiv 0$ on $\mathrm{Re}\, s = \frac{1}{2}$. It follows that $k_1(x) \equiv 0$ if $x < 0$ and the transform $g = Tf$ is now defined by:

$$g(x) = \frac{d}{dx}\int_0^{\infty}\frac{k_1(xy)}{y}\, f(y)dy.$$

The results of the preceding sections remain valid in this particular case, if we replace $L^2(-\infty,\infty)$ by $L^2(0,\infty)$ and if we replace all lower limits of integration $-\infty$ by 0. The formulation of several of the theorems becomes somewhat simpler, because now we have

$$K^*(s) = K_*(s) = K_+(s).$$

Of this transform the case $0 \in \rho(T)$ has been discussed in the literarure, e.g. TITCHMARSH [18, ch. 8], but we can get these results by requiring $H_-(s) = 0$ in theorem 9, so that $h_1(x) \equiv 0$ if $x < 0$. Thus if $0 \in \rho(T)$, then T is a one to one correspondence between $L^2(0,\infty)$ and itself. Theorem 5 for the Watson transforms is due to DOETSCH [6] (who used a different notation).

The formulas (6.24), (6.25), (6.30) and (6.31) become much simpler:

$$(7.1)\qquad (\varphi_e(T)f)(x) = M^{-2}[\varphi_e((K_+(s)K_+(1-s))^{\frac{1}{2}})F_+(s)],$$

and

$$(7.2)\qquad (\varphi_o(T)f)(x) = M^{-2}[\varphi_o((K_+(s)K_+(1-s))^{\frac{1}{2}})\frac{K_+(s)}{(K_+(s)K_+(1-s))^{\frac{1}{2}}}F_+(1-s)],$$

if $x > 0$.

Let the function $L_+(s)$ be defined by

$$(7.3)\qquad L_+(s) = \varphi_o((K_+(s)K_+(1-s))^{\frac{1}{2}})\,\frac{K_+(s)}{(K_+(s)K_+(1-s))^{\frac{1}{2}}}\,.$$

Let $l_1(x)/x$ be the M^{-2}-transform of $\frac{L_+(s)}{1-s}$. Then (7.2) can be written as $\varphi_o(T) = U_L$, where the Watson transform U_L is given by

$$(7.4)\qquad (U_L f)(x) = \frac{d}{dx}\int_0^\infty \frac{l_1(xy)}{y}\,f(y)dy.$$

8. Generalized Laplace transforms

By imposing one more condition on the functions $K_+(s)$ and $K_-(s)$ in definition 1, we obtain an integral transform with a simpler form than (2.2). We shall show that the Laplace transform

$$(8.1)\qquad g(x) = \int_0^\infty e^{-xy}f(y)dy$$

is a special case of this new transform.

Definition 2. Suppose $K_+(s)$ and $K_-(s)$ are essentially bounded, measurable functions defined on Re $s = \frac{1}{2}$ and $K_\pm(s) \in L^2(\frac{1}{2}-i\infty,\frac{1}{2}+i\infty)$. Let $k(\pm x)$ be the M^{-2}-transform of $K_\pm(s)$:

$$(8.2)\qquad k(\pm x) = \underset{A\to\infty}{\text{l.i.m.}}\ \frac{1}{2\pi i}\int_{\frac{1}{2}-iA}^{\frac{1}{2}+iA} K_\pm(s)x^{-s}ds$$

(l.i.m. in $L^2(0,\infty)$) for $x > 0$. Suppose $f(x) \in L^2(-\infty,\infty)$. Then the transform $g = Tf$ is defined by

$$(8.3)\qquad g(x) = \int_{-\infty}^\infty k(xy)f(y)dy.$$

Theorem 15. The mapping T, defined by (8.3) is a bounded transform on $L^2(-\infty,\infty)$.

Proof. The proof of theorem 1 can be used with some modifications.

Theorem 16. The adjoint of the operator T is given by

$$(8.4) \qquad (T^*f)(x) = \int_{-\infty}^{\infty} \overline{k(xy)}\, f(y)dy,$$

for any $f(x) \in L^2(-\infty,\infty)$.

Proof. This can be shown by means of Fubini's theorem.

The equation $(T-\lambda)\, f = g$ is equivalent to

$$\begin{cases} -\lambda F_+(s) + K_+(s)F_+(1-s) + K_-(s)F_-(1-s) = G_+(s), \\ -\lambda F_-(s) + K_-(s)F_+(1-s) + K_+(s)F_-(1-s) = G_-(s), \end{cases}$$

where $F_{\pm}(s)$ and $G_{\pm}(s)$ are the M^2-transforms of $f(\pm x)$ and $g(\pm x)$ $(x > 0)$ respectively. This shows that all results proved before for the generalized Watson transforms carry over without any change to the transforms defined by (8.3).

In view of the conditions on $K_{\pm}(s)$ it follows that $K^*(s)K^*(1-s) \in L^1(\frac{1}{2}-i\infty,\frac{1}{2}+i\infty)$ and $K_*(s)K_*(1-s) \in L^1(\frac{1}{2}-i\infty,\frac{1}{2}+i\infty)$. Combining this with theorem 11 and theorem 13 we see that unitary and involutory transforms in $L^2(-\infty,\infty)$ can never be represented by the integral transforms of definition 2.

By a change of variables the transform (8.3) with lower limit 0 in place of $-\infty$ can be written in the form

$$(8.5) \qquad g(x) = \int_{-\infty}^{\infty} a(x+y)\, f(y)dy,$$

with $f(y) \in L^2(-\infty,\infty)$. The transform (8.5) has been investigated by CARLEMAN [3]. POLLARD [14] has studied the spectral properties of (8.5) under suitable conditions on the kernel a.
For a study of Watson transforms in a form analogous to (8.5) we refer to DOETSCH [5], [6].

Finally we wish to point out that in view of the properties of the Fourier transform (1.1) the spectral properties of the convolution-transform

$$(8.6) \qquad g(x) = \int_{-\infty}^{\infty} a(x-y)\, f(y)dy$$

are somewhat easier to find then in the case (8.5). For (8.6) we refer to POLLARD [14] and DUNFORD [7].

9. Self-reciprocal functions

A *non-trivial function* $f \in L^2(-\infty,\infty)$ is said to be a self-reciprocal function of the transform T if $Tf = f$; a non-trivial function $f \in L^2(-\infty,\infty)$ is said to be a skew-reciprocal function of the transform T if $Tf = -f$. It is clear that self-reciprocal functions are eigenfunctions of T belonging to the eigenvalue 1, while skew-reciprocal functions are eigenfunctions of T belonging to the eigenvalue -1. The question whether a given transform T has self- or skew-reciprocal functions is answered by the following corollary of theorem 5.

Corollary 4. *The transform T possesses self- and skew-reciprocal functions if and only if*

$$K^*(s)K^*(1-s) = 1,$$

or

$$K_*(s)K_*(1-s) = 1,$$

on a set of positive measure (on Re $s = \frac{1}{2}$).

Having established conditions for the existence of self- and skew-reciprocal functions for a transform T, we now wish to find requirements which guarantee that a function $f(x) \in L^2(-\infty,\infty)$ is a self- or skew-reciprocal function of T. The next result is a consequence of the equations (2.3).

Corollary 5. *Suppose the condition of corollary 4 is satisfied. Then the function* $f(x)$ *is a self-reciprocal function of the transform T if and only if the functions* $F_{\pm}(s) \in L^2(\frac{1}{2}-i\infty,\frac{1}{2}+i\infty)$ *and satisfy the system:*

$$\begin{cases} F_+(s) = K_+(s)F_+(1-s) + K_-(s)F_-(1-s), \\ F_-(s) = K_-(s)F_-(1-s) + K_+(s)F_-(1-s). \end{cases}$$

The function $f(x)$ is a skew-reciprocal function of the transform T if and only if the functions $F_{\pm}(s) \in L^2(\frac{1}{2}-i\infty,\frac{1}{2}+i\infty)$ and satisfy the system:

$$\begin{cases} -F_+(s) = K_+(s)F_+(1-s) + K_-(s)F_-(1-s), \\ -F_-(s) = K_-(s)F_+(1-s) + K_+(s)F_-(1-s). \end{cases}$$

Remark 11. For Watson transforms the above result is due to BUSBRIDGE [2, theorem 5], while a special case of her theorem can be found in TITCHMARSH [18, theorem 136]. It should be pointed out, that these and similar problems were also considered by DOETSCH [4] and [5].

Let us consider the following loosely stated problem. Suppose the non-trivial function $f \in L^2(-\infty,\infty)$ is invariant under a transform T_1, and suppose f is transformed by an operator T_2 : $g=T_2f$. What are the conditions such that g is invariant under a transform T_3?
The following theorem deals with such a situation, where the transform defined by (8.3), plays a rôle. It will be clear that it is possible to give more theorems of this type.

Theorem 17. Let $K_+(s)$ and $N_+(s)$ be essentially bounded, measurable functions on Re $s = \frac{1}{2}$. Let $\frac{k_1(x)}{x}$ and $\frac{n_1(x)}{x}$ $(x > 0)$ be the M^{-2}-transforms of $\frac{K_+(s)}{1-s}$ and $\frac{N_+(s)}{1-s}$ respectively. Let $H_+(s)$ be an essentially bounded, measurable function on Re $s = \frac{1}{2}$ and $H_+(s) \in L^2(\frac{1}{2}-i\infty, \frac{1}{2}+i\infty)$.
Let $h(x)$ be the M^{-2}-transform of $H_+(s)$.

Suppose $f(x) \in L^2(0,\infty)$ is a self-reciprocal function of the k_1-transform:

$$f(x) = \frac{d}{dx} \int_0^\infty \frac{k_1(xy)}{y} f(y)dy, \tag{9.1}$$

and define $g(x)$ by

$$g(x) = \int_0^\infty h(xy)f(y)dy. \tag{9.2}$$

Suppose on Re $s = \frac{1}{2}$:

$$H(s)K(1-s) = H(1-s)N(s), \tag{9.3}$$

then $g(x) \in L^2(0,\infty)$ is a self-reciprocal function of the n_1-transform:

$$g(x) = \frac{d}{dx} \int_0^\infty \frac{n_1(xy)}{y} g(y)dy. \tag{9.4}$$

Proof. The relation (9.1) implies

$$F(s) = K(s)F(1-s),$$

or equivalently

$$F(1-s) = K(1-s)F(s). \tag{9.5}$$

The relation (9.2) implies

$$G(s) = H(s)F(1-s), \tag{9.6}$$

or equivalently

$$G(1-s) = H(1-s)F(s). \tag{9.7}$$

Substitute (9.5) in (9.6), then

$$G(s) = H(s)K(1-s)F(s). \tag{9.8}$$

Multiply $G(1-s)$ by $N(s)$ then (9.7) yields

$$N(s)G(1-s) = N(s)H(1-s)F(s), \tag{9.9}$$

Comparing (9.8) and (9.9) and using (9.3) we conclude

$$G(s) = N(s)G(1-s),$$

which is equivalent to (9.4).

Remark 12. A more special result of TITCHMARSH [18, theorem 144] is related to the above theorem. In example 5 we will give a corollary of theorem 17 connecting Hankel transforms and Meijer transforms.

10. Examples

Before giving several examples to illustrate the foregoing theorems, we state some auxiliary results which will be needed in some of our arguments.

Lemma 4. (BRAAKSMA [1, lemma 1]). If $0 < \operatorname{Re} s < 1$ and $|\arg y| \leqq \frac{\pi}{2}$, then

$$\int_0^\infty x^{s-1} \frac{e^{-xy}-1}{x} dx = y^{1-s} \frac{\Gamma(s)}{1-s} .$$

Lemma 5. The function

$$w = \frac{Az+B}{Cz+D}, \quad AD-BC \neq 0,$$

maps the real line in the z-plane onto a circle in the w-plane. The center of this circle is given by

$$\frac{B\bar{C}-A\bar{D}}{\bar{C}D-C\bar{D}},$$

and the radius by

$$\left| \frac{BC-AD}{C\bar{D}-C\bar{D}} \right|.$$

Remark 13. In the theory of convolution transforms (8.6) in $L^2(-\infty,\infty)$ one frequently encounters the Dirichlet transform D_α, given by:

$$(10.1) \qquad (D_\alpha f)(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin \alpha(x-t)}{x-t} f(t)dt, \qquad \alpha > 0.$$

We introduce a one to one mapping A from $L^2(0,\infty)$ onto $L^2(-\infty,\infty)$ by

$$(Af)(x) = e^{\frac{1}{2}x} f(e^x), \qquad f \in L^2(0,\infty).$$

Using this mapping A we define a transform B_α in $L^2(0,\infty)$, associated with the Dirichlet transform by

$$(10.2) \qquad B_\alpha = A^{-1} D_\alpha A, \qquad \alpha > 0.$$

The relation $g = B_\alpha f$ then stands for

$$(10.3) \qquad g(x) = \frac{1}{\pi} \int_0^\infty \frac{\sin\{\alpha \log(x/y)\}}{(xy)^{\frac{1}{2}}\log(x/y)} f(y)dy$$

$$= \frac{1}{\pi} \int_0^\infty \frac{\sin\{\alpha \log(xy)\}}{(xy)^{\frac{1}{2}}\log(xy)} \frac{1}{y} f\left(\frac{1}{y}\right) dy.$$

It can easily be seen that B_α is an orthogonal projection in $L^2(0,\infty)$. We will encounter this transform B_α in one of our examples.

Example 1 ([18, 8.4]). A simple transform of type (2.2) is obtained by the choice

$$K_+(s) = 1, \quad K_-(s) = 0.$$

Then the transform $g = Tf$ in the space $L^2(0,\infty)$ has the form

$$g(x) = \frac{1}{x} f\left(\frac{1}{x}\right).$$

Of course this transform is onto $L^2(0,\infty)$. Application of theorems 5 and 8 show that the continuous spectrum $C\sigma(T)$ is empty and that the point spectrum $P\sigma(T)$ consists of the eigenvalues ± 1.

An elementary argument shows that the projections P_k onto the eigenspaces belonging to the eigenvalues $(-1)^k$ $(k=0,1)$ respectively are given by

$$(10.4) \qquad \begin{cases} P_o = \frac{1}{2}(I+T), \\ P_1 = \frac{1}{2}(I-T). \end{cases}$$

It can easily be verified that all eigenfunctions of the transform T belonging to the eigenvalues ± 1 are given by

$$f(x) \pm \frac{1}{x} f\left(\frac{1}{x}\right).$$

Example 2 ([18, 8.4]). Another transform is obtained by the choice of the functions

$$(10.5) \qquad \begin{cases} K_+(s) = 2^{s-\frac{1}{2}} \dfrac{\Gamma(\frac{1}{2}s + \frac{1}{2}\nu + \frac{1}{4})}{\Gamma(-\frac{1}{2}s + \frac{1}{2}\nu + \frac{3}{4})}, \qquad \operatorname{Re} \nu > -1, \\ K_-(s) = 0. \end{cases}$$

The asymptotic behaviour of the Γ-function is given by

$$(10.6) \qquad \Gamma(s) = e^{-s}\, e^{(s-\frac{1}{2})\log s}\, O(1),$$

as $|s| \to \infty$ on $|\arg s| < \pi$ (cf. ERDÉLYI [8, 1.18]).

Hence $K_+(s) = O(1)$ on $\operatorname{Re} s = \frac{1}{2}$ and we may write

$$k_1(x) = x \frac{d}{dx} \frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{\Gamma(\frac{1}{2}s + \frac{1}{2}\nu + \frac{1}{4})}{\Gamma(-\frac{1}{2}s + \frac{1}{2}\nu + \frac{3}{4})} \frac{1}{(1-s)^2} x^{1-s}\, ds,$$

on account of (2.1), (1.7) and (1.9).

From the calculus of residues and the asymptotic behaviour of the Γ-function we obtain

$$(10.7) \qquad k_1(x) = \sum_{n=0}^{\infty} \frac{2^{-2n-\nu}}{\Gamma(n+\nu+1)} \frac{x^{2n+\nu+\frac{3}{2}}}{2n+\nu+\frac{3}{2}} \frac{(-1)^n}{n!}.$$

Differentiation of (10.7) yields:

$$k(x) = \frac{d}{dx} k_1(x) = x^{\frac{1}{2}} J_\nu(x).$$

In order to apply theorem 3 we remark that $x^{\frac{1}{2}}J_\nu(x) \in L^2(0,N)$ for all $N > 0$ if $\mathrm{Re}\,\nu > -1$. Hence the transform $g = Tf$ satisfies

$$g(x) = \underset{N\to\infty}{\mathrm{l.i.m.}} \int_0^N (xy)^{\frac{1}{2}} J_\nu(xy)f(y)dy, \qquad \mathrm{Re}\,\nu > -1. \tag{10.8}$$

Therefore the transform T is the Hankel transform.

Clearly

$$K_+(s)K_+(1-s) \equiv 1, \tag{10.9}$$

which implies that the eigenvalues -1 and 1 are the only points of the spectrum (cf. theorem 5). Theorem 13 shows that the transform T is involutory for all ν with $\mathrm{Re}\,\nu > -1$. According to theorem 12 the transform T is self-adjoint for real ν.

In case $\nu = \pm\frac{1}{2}$ (10.8) reduces to the Fourier sine and cosine transform, respectively given by

$$g(x) = \underset{N\to\infty}{\mathrm{l.i.m.}} \sqrt{\frac{2}{\pi}} \int_0^N \sin xy\ f(y)dy, \tag{10.10}$$

and

$$g(x) = \underset{N\to\infty}{\mathrm{l.i.m.}} \sqrt{\frac{2}{\pi}} \int_0^N \cos xy\ f(y)dy. \tag{10.11}$$

The projections P_k associated with the eigenvalues $(-1)^k$ $(k=0,1)$ can be determined from the formulas (10.4).

Example 3. The two-sided Fourier transform (1.2) is an example of the generalized Watson transform. To this end we choose:

$$\begin{cases} K_+(s) = \dfrac{1}{\sqrt{2\pi}}\, e^{i\frac{\pi}{2}s}\,\Gamma(s), \\[2ex] K_-(s) = \dfrac{1}{\sqrt{2\pi}}\, e^{-i\frac{\pi}{2}s}\,\Gamma(s); \end{cases} \tag{10.12}$$

that these functions satisfy all requirements of definition 1 follows from the formula (10.6). Using lemma 4 we see that (10.12) gives rise to the two-sided Fourier transform (1.1).

Computation yields the following results:

$$K^*(s)K^*(1-s) = 1, \tag{10.13}$$

$$K_*(s)K_*(1-s) = -1, \tag{10.14}$$

$$(10.15)\qquad \frac{K^*(s)}{(K^*(s)K^*(1-s))^{\frac{1}{2}}} = 2^{s-\frac{1}{2}}\,\frac{\Gamma(\frac{1}{2}s)}{\Gamma(\frac{1}{2}-\frac{1}{2}s)}\,,$$

$$(10.16)\qquad \frac{K_*(s)}{(K_*(s)K_*(1-s))^{\frac{1}{2}}} = 2^{s-\frac{1}{2}}\,\frac{\Gamma(\frac{1}{2}s+\frac{1}{2})}{\Gamma(1-\frac{1}{2}s)}\,.$$

It follows from (10.13), (10.14) and theorems 4, 5 and 8 that the spectrum of the Fourier transform consists of four eigenvalues:

$$1,\ i,\ -1,\ -i.$$

Next we proceed by calculating the resolution of the identity or more precisely the projections P_k associated with the eigenvalues i^k $(k=0,1,2,3)$. In the formula

$$\varphi(T) = \int_{\sigma(T)} \varphi(\lambda)dE_\lambda$$

we choose the function $\varphi(\lambda)$ by:

$$\begin{cases} \varphi(\lambda) = 1 & \lambda=1 \\ \qquad\ \ = 0, & \lambda=i,-1,-i. \end{cases}$$

Hence

$$P_o \int_{\sigma(T)} \varphi(\lambda)dE_\lambda.$$

With this function $\varphi(\lambda)$ we associate two functions $\varphi_e(\lambda)$ and $\varphi_o(\lambda)$:

$$\begin{cases} \varphi_e(\lambda) = \frac{1}{2}, & \lambda=-1,1, \\ \qquad\ \ = 0, & \lambda=i,-i. \end{cases}$$

$$\begin{cases} \varphi_o(\lambda) = \frac{1}{2} & \lambda=1, \\ \qquad\ \ = -\frac{1}{2}, & \lambda=-1, \\ \qquad\ \ = 0, & \lambda=i,-i. \end{cases}$$

Now (10.13), (10.14), (6.24) and (6.25) show that the operator $\varphi_e(T)$ associated with the function $\varphi_e(\lambda)$ is given by

$$(\varphi_e(T)f)(x) = \tfrac{1}{4}\,\{f(x)+f(-x)\}.$$

In the same way a combination of (10.13), (10.14), (10.15), (10.16), (6.32), (6.33), (6.34) and formulas (10.10) and (10.11) of example 2 show that the operator $\varphi_o(T)$ associated with the function $\varphi_o(\lambda)$ is given by

$$(\varphi_o(T)f)(x) =$$

$$= \tfrac{1}{4}\left[\sqrt{\frac{2}{\pi}}\int_0^\infty \cos xy\ f(y)dy + \sqrt{\frac{2}{\pi}}\int_0^\infty \cos xy\ f(-y)dy\right]$$

$$= \tfrac{1}{4}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ixy}\ f(y)dy + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ixy}\ f(y)dy\right].$$

Hence we find for the projection P_o the formula:

$$(P_o f)(x) = \tfrac{1}{4}\left[f(x)+f(-x) + \{(Tf)(x) + (Tf)(-x)\}\right].$$

Similar considerations lead to formulas for the other projections:

$$(P_1 f)(x) = \tfrac{1}{4}\left[f(x)-f(-x) - i\ \{(Tf)(x)-(Tf)(-x)\}\right],$$

$$(P_2 f)(x) = \tfrac{1}{4}\left[f(x)+f(-x) - \ \{(Tf)(x)+(Tf)(-x)\}\right],$$

$$(P_3 f)(x) = \tfrac{1}{4}\left[f(x)-f(-x) + i\ \{(Tf)(x)-(Tf)(-x)\}\right].$$

All results, given here can be found in RIESZ and SZ.-NAGY [15].

Example 4 ([18, 8.4]). Titchmarsh has found a pair of inversion formulas (2.2) and (4.6) involving Neumann's Y_ν function and Struve's H_ν functions. We are led to these transforms by the choice

$$(10.17)\qquad \begin{cases} K_+(s) = -\dfrac{2^{s-\frac{1}{2}}}{\pi}\Gamma(\tfrac{1}{2}s+\tfrac{1}{2}\nu+\tfrac{1}{4})\Gamma(\tfrac{1}{2}s-\tfrac{1}{2}\nu+\tfrac{1}{4})\cos(\tfrac{1}{2}s-\tfrac{1}{2}\nu+\tfrac{1}{4})\pi, & |\mathrm{Re}\ \nu| < 1, \\ K_-(s) = 0. \end{cases}$$

Application of (10.6) shows that the function $K_+(s)$ is bounded on $\mathrm{Re}\ s = \tfrac{1}{2}$. By putting $s = \tfrac{1}{2} + it$ $(-\infty < t < \infty)$ we find

$$(10.18)\qquad K_+(s)K_+(1-s) = \frac{\cosh \pi t - \cos \pi \nu}{\cosh \pi t + \cos \pi \nu}, \qquad -\infty < t < \infty.$$

Thus if $\nu = \pm\tfrac{1}{2}$, then $K_+(s)K_+(1-s) \equiv 1$; hence $\pm 1 \in P\sigma(T)$ and $C\sigma(T)$ is empty. These cases will not be considered here. For all other values of ν with $|\mathrm{Re}\ \nu| < 1$, (10.18) yields that $P\sigma(T)$ is empty. We will now try to determine $C\sigma(T)$ or more precisely $\{C\sigma(T)\}^2$. We know that $\lambda \in C\sigma(T)$ if and only if λ^2 belongs to the closure of the range of the function

$$\frac{\cosh \pi t - \cos \pi \nu}{\cosh \pi t + \cos \pi \nu}.$$

Put z=cosh t and consider the mapping:

(10.19) $$w = \frac{z-\cos\pi\nu}{z+\cos\pi\nu}.$$

First we exclude the case $\mathrm{Re}\,\nu = 0$ and the case $\mathrm{Im}\,\nu = 0$. Using lemma 5 we find that the image of the real line in the z-plane under the mapping (10.19) is a circle with center M_ν, given by:

$$\frac{\mathrm{Re}\,\cos\pi\nu}{\mathrm{Im}\,\cos\pi\nu}\,i\ ,$$

and radius r_ν:

$$\sqrt{1+\left(\frac{\mathrm{Re}\,\cos\pi\nu}{\mathrm{Im}\,\cos\pi\nu}\right)^2}\ .$$

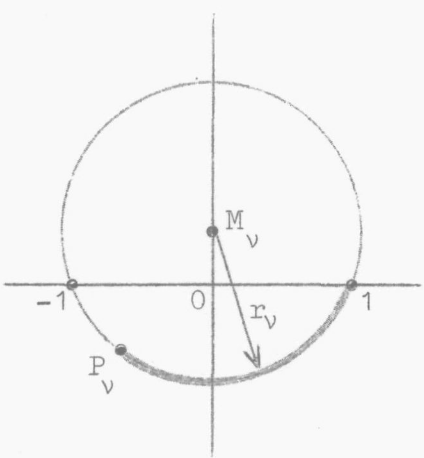
Note that the points w=-1 and w=1 lie on this circle. From the equality

$$\cos\pi\nu = \cos(\pi\mathrm{Re}\,\nu)\cosh(\pi\,\mathrm{Im}\,\nu) - i\sin(\pi\mathrm{Re}\,\nu)\sinh(\pi\,\mathrm{Im}\,\nu),$$

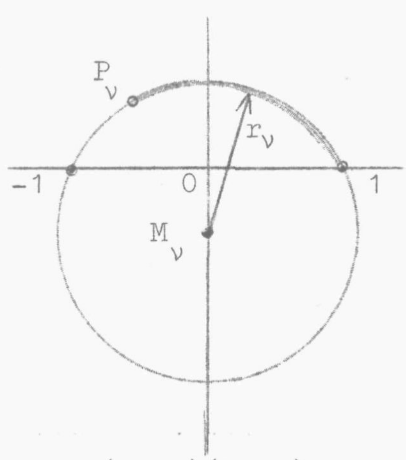
we see that the center M_ν lies in the lower half w-plane if $\mathrm{Re}\,\nu$ and $\mathrm{Im}\,\nu$ have the same sign, and that the center M_ν lies in the upper half w-plane if $\mathrm{Re}\,\nu$ and $\mathrm{Im}\,\nu$ have opposite signs. Now $\{C\sigma(T)\}^2$ is the part of the circle, which is the image of the half line $z \geqq 1$.

Let the point P_ν in the w-plane be given by the complex number $\tan^2\frac{1}{2}\pi\nu$, then $\{C\sigma(T)\}^2$ is indicated in the following diagrams.

w-plane



$(\mathrm{Re}\,\nu)(\mathrm{Im}\,\nu) < 0$



$(\mathrm{Re}\,\nu)(\mathrm{Im}\,\nu) > 0$

If $\mathrm{Im}\,\nu = 0, \nu \neq 0$, then $\{C\sigma(T)\}^2$ lies on the positive part of the real line in the w-plane. Let $-\frac{1}{2} < \nu < \frac{1}{2}$, $\nu \neq 0$, then $\{C\sigma(T)\}^2$ is equal to the interval $[\tan^2 \frac{1}{2}\pi\nu, 1]$; let $-1 < \nu < -\frac{1}{2}$ or $\frac{1}{2} < \nu < 1$ then $\{C\sigma(T)\}^2$ is equal to the interval $[1, \tan^2 \frac{1}{2}\pi\nu]$.

In all cases, considered above, we see $0 \notin C\sigma(T)$ and this implies that the transform defined by (10.17) then provides a one to one correspondence between $L^2(0,\infty)$ and itself.

Now consider $\mathrm{Re}\,\nu = 0$. Then $\{C\sigma(T)\}^2$ is equal to the interval $[-\tanh^2(\frac{\pi}{2}\,\mathrm{Im}\,\nu), 1]$; and $\{C\sigma(T)\}^2$ is equal to the interval $[0,1]$, only if $\nu = 0$. Thus if ν is pure imaginary, then $0 \in C\sigma(T)$ and the transform defined by (10.17) is no longer onto. [1]

We will not go into the details of the spectral resolution of this transform, since the calculations seem to be rather complicated.

Example 5. MEIJER [12] has introduced an integral transform $g = Tf$, having a modified Bessel function as kernel:

$$g(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty (xy)^{\frac{1}{2}} K_\nu(xy)\, f(y)dy. \tag{10.20}$$

If $x > 0$ and $|\mathrm{Re}\,\nu| < 1$, then we will show that this transform is an example of the transform, defined by (8.3).

From the definition of $K_\nu(x)$ (cf. ERDÉLYI [9, 7.2.2. (13)]) it is clear that

$$\sqrt{\frac{2}{\pi}}\, x^{\frac{1}{2}} K_\nu(x) \in L^2(0,\infty),$$

if $|\mathrm{Re}\,\nu| < 1$. Under the same condition we have (cf. ERDÉLYI [10,6.8 (26)]

$$\sqrt{\frac{2}{\pi}} \int_0^\infty x^{\frac{1}{2}} K_\nu(x)\, x^{s-1}\, dx = \frac{2^{s-1}}{\sqrt{\pi}}\, \Gamma(\tfrac{1}{2}s+\tfrac{1}{2}\nu+\tfrac{1}{4})\Gamma(\tfrac{1}{2}s-\tfrac{1}{2}\nu+\tfrac{1}{4}),$$

with $\mathrm{Re}\, s = \frac{1}{2}$. From

$$\Gamma(\tfrac{1}{2}s+\tfrac{1}{2}\nu+\tfrac{1}{4})\Gamma(\tfrac{1}{2}s-\tfrac{1}{2}\nu+\tfrac{1}{4}) = O(e^{-\frac{1}{2}\pi|\mathrm{Im}\, s|})$$

as $|s| \to \infty$ on $\mathrm{Re}\, s = \frac{1}{2}$, it follows that (10.20) is a transform of type (8.3), with

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1) In this case example 2 in TITCHMARSH [18, 8.4] does not provide an illustration of a pair of unsymmetrical Fourierkernels. This can also be seen immediately, because then the function $H_+(s) = K_+(1-s)^{-1}$ is not bounded on $\mathrm{Re}\, s = \frac{1}{2}$.

$$(10.21) \qquad \begin{cases} K_+(s) = \dfrac{2^{s-1}}{\sqrt{\pi}} \; \Gamma(\tfrac{1}{2}s+\tfrac{1}{2}\nu+\tfrac{1}{4})\Gamma(\tfrac{1}{2}s-\tfrac{1}{2}\nu+\tfrac{1}{4}), \\ K_-(s) = 0. \end{cases}$$

Putting $s = \frac{1}{2}+it$ we obtain

$$(10.22) \qquad K_+(s)K_+(1-s) = \frac{\pi}{\cosh \pi t + \cos \pi \nu} \, .$$

From this we see that $P\sigma(T)$ is empty. We shall determine $\{C\sigma(T)\}^2$. Put $z=\cosh \pi t$ in (10.22) and consider the mapping

$$w = \frac{\pi}{z+\cos \pi \nu} \, .$$

First we exclude the case $\operatorname{Re} \nu = 0$ and the case $\operatorname{Im} \nu = 0$.

Using lemma 5 we see that the image of the real line in the z-plane is a circle in the w-plane with center M_ν given by:

$$\frac{\pi}{2 \sin (\pi \operatorname{Re} \nu) \sinh (\pi \operatorname{Im} \nu)} \, i,$$

and radius r_ν:

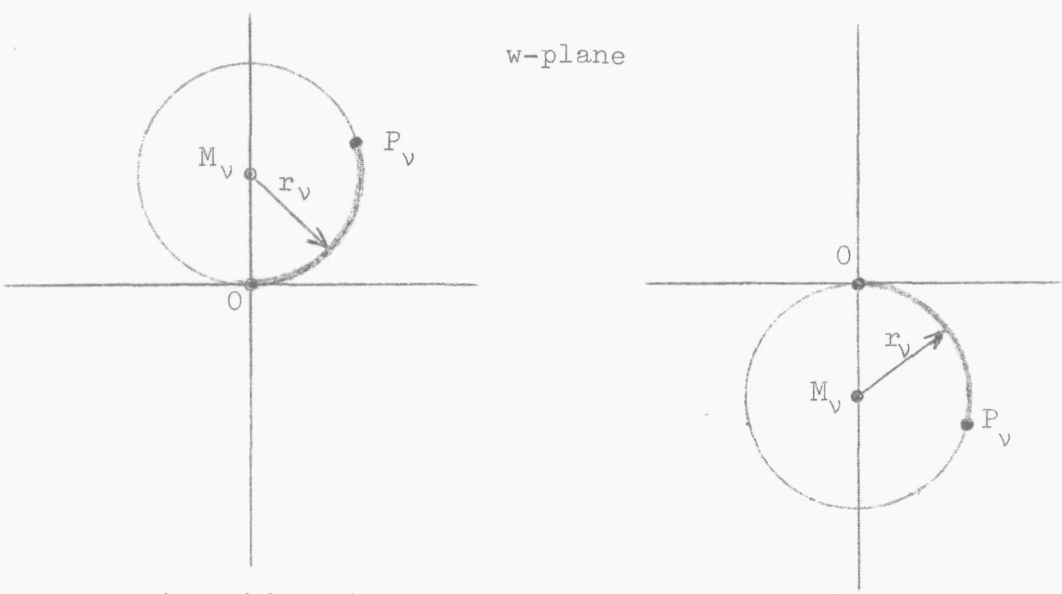
$$\frac{\pi}{2}\left|\frac{1}{\sin (\pi \operatorname{Re} \nu) \sinh (\pi \operatorname{Im} \nu)}\right| \, .$$

$\{C\sigma(T)\}^2$ is the part of the circle, which is the image of the half line $z \geqq 1$.

Let the point P_ν in the w-plane be given by the complex number

$$\frac{\pi}{1+\cos \pi \nu} \, .$$

$\{C\sigma(T)\}^2$ is indicated in the following diagrams.



$(\operatorname{Re} \nu)(\operatorname{Im} \nu) > 0$ $\qquad\qquad$ $(\operatorname{Re} \nu)(\operatorname{Im} \nu) < 0$

If $\mathrm{Re}\,\nu = 0$ or $\mathrm{Im}\,\nu = 0$, then $\{C\sigma(T)\}^2$ is lying on the real axis of the w-plane: it is equal to the interval $[0, \frac{\pi}{1+\cos\pi\nu}]$.

Note that in all cases $0 \in C\sigma(T)$, thus the range of T is a proper, dense subset of $L^2(0,\infty)$ and the Meijer transform does not provide a one to one correspondence between $L^2(0,\infty)$ and itself.

From (10.21) and theorem 12 it follows that the transform T is self-adjoint if $\mathrm{Re}\,\nu = 0$ or $\mathrm{Im}\,\nu = 0$. This can also be seen from the integral representation of the function $K_\nu(x)$ (cf. ERDÉLYI [9, 7.12 (21)]):

$$K_\nu(x) = \int_0^\infty e^{-x\cosh t} \cosh(\nu t)dt$$

and theorem 16.

We now wish to determine the spectral resolution for real ν.
Let μ be a point belonging to the spectrum

$$\sigma(T) = C\sigma(T) = \left[-\sqrt{\frac{\pi}{1+\cos\pi\nu}}, \sqrt{\frac{\pi}{1+\cos\pi\nu}} \right].$$

First we take $\mu > 0$. In the relation

$$\varphi(T) = \int_{\sigma(T)} \varphi(\lambda)dE_\lambda,$$

we take the function $\varphi(\lambda)$ defined by

$$\begin{cases} \varphi(\lambda) = 1, & \lambda \geqq \mu, \\ \qquad\quad 0, & \lambda < \mu. \end{cases}$$

Then $\varphi(T) = I-E_\mu$ or $E_\mu = I-\varphi(T)$. We decompose $\varphi(\lambda)$ into the even function $\varphi_e(\lambda)$ and the odd function $\varphi_o(\lambda)$:

$$\begin{cases} \varphi_e(\lambda) = \frac{1}{2}, & |\lambda| \geqq \mu, \\ \qquad\quad = 0, & |\lambda| < \mu. \end{cases}$$

and

$$\begin{cases} \varphi_o(\lambda) = \frac{1}{2}, & \lambda \geqq \mu, \\ \qquad\quad = 0, & |\lambda| < \mu, \\ \qquad\quad = -\frac{1}{2}, & \lambda \leqq -\mu. \end{cases}$$

In order to use formula (7.1) we see that

$$\sqrt{K_+(s)K_+(1-s)} \geqq \mu$$

if and only if

$$|t| \leqq \frac{1}{\pi} \operatorname{arcosh} \left(\frac{\pi}{\mu^2} - \cos\pi\nu\right).$$

Hence the operator $\varphi_e(T)$, associated with the function $\varphi_e(\lambda)$, is given by:

$$(\varphi_e(T)f)(x) =$$

$$= \tfrac{1}{2}\,\frac{1}{2\pi}\int_{-\frac{1}{\pi}\operatorname{arcosh}\left(\frac{\pi}{\mu^2}-\cos\pi\nu\right)}^{\frac{1}{\pi}\operatorname{arcosh}\left(\frac{\pi}{\mu^2}-\cos\pi\nu\right)} x^{-\frac{1}{2}+it}\left(\underset{a\to\infty}{\mathrm{l.i.m.}}\int_{1/a}^{a} f(u)\,u^{-\frac{1}{2}+it}\,du\right)dt$$

$$= \tfrac{1}{2}\,\frac{1}{2\pi}\lim_{a\to\infty}\int_{1/a}^{a} f(u)(ux)^{-\frac{1}{2}}\left\{\int_{-\frac{1}{\pi}\operatorname{arcosh}\left(\frac{\pi}{\mu^2}-\cos\pi\nu\right)}^{\frac{1}{\pi}\operatorname{arcosh}\left(\frac{\pi}{\mu^2}-\cos\pi\nu\right)}\left(\frac{u}{x}\right)^{it}dt\right\}du$$

$$= \frac{1}{2\pi}\lim_{a\to\infty}\int_{1/a}^{a}(ux)^{-\frac{1}{2}}\frac{1}{\ln(u/x)}\sin\left\{\left(\frac{1}{\pi}\operatorname{arcosh}\left(\frac{\pi}{\mu^2}-\cos\pi\nu\right)\right)\ln(u/x)\right\}f(u)\,du$$

Using the notation of (10.2) we find:

$$(10.23)\qquad \varphi_e(T) = \tfrac{1}{2}\,B_{\frac{1}{\pi}\operatorname{arcosh}\left(\frac{\pi}{\mu^2}-\cos\pi\nu\right)}\,.$$

The operator $\varphi_o(T)$, associated with the function $\varphi_o(\lambda)$, can be found from formula (7.2). From (10.21) and (10.22) we find:

$$(10.24)\qquad \frac{K_+(s)}{(K_+(s)K_+(1-s))^{\frac{1}{2}}} \equiv 2^{s-\frac{1}{2}}\,\frac{\Gamma(\frac{1}{2}s+\frac{1}{2}\nu+\frac{1}{4})\ (\frac{1}{2}s-\frac{1}{2}\nu+\frac{1}{4})}{|\Gamma(\frac{1}{2}s+\frac{1}{2}\nu+\frac{1}{4})\ (\frac{1}{2}s-\frac{1}{2}\nu+\frac{1}{4})|}\,.$$

Combination of (7.2), (7.3) and (7.4) yields

$$(10.25)\qquad \varphi_o(T) = U_L$$

where $L_+(s)$ is given by

$$L_+(s) = 2^{s-\frac{3}{2}}\,\frac{\Gamma(\frac{1}{2}s+\frac{1}{2}\nu+\frac{1}{4})\Gamma(\frac{1}{2}s-\frac{1}{2}\nu+\frac{1}{4})}{|\Gamma(\frac{1}{2}s+\frac{1}{2}\nu+\frac{1}{4})\Gamma(\frac{1}{2}s-\frac{1}{2}\nu+\frac{1}{4})|}\,,$$

$$(10.26)\qquad \left\{\begin{array}{l} |t| \leqq \frac{1}{\pi}\operatorname{arcosh}\left(\frac{\pi}{\mu^2}-\cos\pi\nu\right), \\ \\ = 0, \\ |t| > \frac{1}{\pi}\operatorname{arcosh}\left(\frac{\pi}{\mu^2}-\cos\pi\nu\right). \end{array}\right.$$

Hence for $\mu > 0$ we find:

$$E_\mu = I - \tfrac{1}{2} B_{\frac{1}{\pi}\operatorname{arcosh}\left(\frac{\pi}{\mu^2} - \cos\pi\nu\right)} - U_L. \tag{10.27}$$

Completely similar we find

$$E_o = \tfrac{1}{2} I - U_L, \tag{10.28}$$

where $L_+(s)$ is given by

$$L_+(s) = 2^{s-\frac{3}{2}} \frac{\Gamma(\frac{1}{2}s+\frac{1}{2}\nu+\frac{1}{4})\Gamma(\frac{1}{2}s-\frac{1}{2}\nu+\frac{1}{4})}{|\Gamma(\frac{1}{2}s+\frac{1}{2}\nu+\frac{1}{4})\Gamma(\frac{1}{2}s-\frac{1}{2}\nu+\frac{1}{4})|}.$$

Note that now $2U_L$ is a unitary operator.

If $\mu < 0$ then we define the function $\varphi(\lambda)$ by:

$$\begin{cases} \varphi(\lambda) = 1, & \lambda \leqq \mu, \\ = 0, & \lambda > \mu. \end{cases}$$

Thus the operator $\varphi(T)$ associated with $\varphi(\lambda)$ is given by:

$$\varphi(T) = E_\mu.$$

Again we find

$$\varphi_e(T) = \tfrac{1}{2} B_{\frac{1}{\pi}\operatorname{arcosh}\left(\frac{\pi}{\mu^2} - \cos\pi\nu\right)},$$

but $\varphi_o(T)$ is given by:

$$\varphi_o(T) = U_L,$$

where $L_+(s)$ is given by (10.26) with a - sign.

Hence we find for $\mu < 0$:

$$E_\mu = \tfrac{1}{2} B_{\frac{1}{\pi}\operatorname{arcosh}\left(\frac{\pi}{\mu^2} - \cos\pi\nu\right)} + U_L, \tag{10.29}$$

where $L_+(s)$ is given by (10.26) with a - sign.

In addition to the above results we state a corollary of theorem 17, which seems interesting enough to state separately.

Corollary 6. Let $f(x) \in L^2(0,\infty)$ be invariant under the Hankel transform of order ν with $|\mathrm{Re}\,\nu| < 1$; let $g(x)$ be the Meijer transform of $f(x)$, given by (10.20). Then $g(x)$ is invariant under the Hankel transform of order $-\nu$.

Proof. In theorem 17 we take:

$$K_+(s) = 2^{s-\frac{1}{2}} \frac{\Gamma(\frac{1}{2}s+\frac{1}{2}\nu+\frac{1}{4})}{\Gamma(-\frac{1}{2}s+\frac{1}{2}\nu+\frac{3}{4})}\text{ , cf. (10.5),}$$

$$H_+(s) = \frac{2^{s-1}}{\sqrt{\pi}}\, \Gamma(\tfrac{1}{2}s+\tfrac{1}{2}\nu+\tfrac{1}{4})\Gamma(\tfrac{1}{2}s-\tfrac{1}{2}\nu+\tfrac{1}{4}),\text{ cf. (10.21),}$$

and

$$N_+(s) = 2^{s-\frac{1}{2}} \frac{\Gamma(\frac{1}{2}s-\frac{1}{2}\nu+\frac{1}{4})}{\Gamma(-\frac{1}{2}s-\frac{1}{2}\nu+\frac{3}{4})}\text{ , cf. (10.5).}$$

For $\nu = \pm\frac{1}{2}$ the Meijer transform reduces to the Laplace transform $g = Tf$ defined by (8.1). The spectrum of this transform is given by

$$\sigma(T) = C\sigma(T) = [-\sqrt{\pi}, \sqrt{\pi}\,]. \tag{10.30}$$

This result and the spectral resolution (which can be found from (10.27), (10.28) and (10.29) with $\nu = \pm\frac{1}{2}$) are due to POLLARD [14].

Remark 14. Corollary 6 with $\nu = \pm\frac{1}{2}$ given by TITCHMARSH [18, p. 267] and also DOETSCH [6, p. 121].

The Stieltjes transform $g = Sf$, defined by

$$g(x) = \int_0^\infty \frac{1}{x+t}\, f(t)dt, \tag{10.31}$$

can be obtained from the Laplace transform T. It follows that

$$S = T^2.$$

From this the spectrum of the Stieltjes transform may be found:

$$\sigma(S) = [0,\pi].$$

Also a complex inversion formula for the Stieltjes transform can be given:

$$f(x) = \underset{A\to\infty}{\text{l.i.m.}}\ \frac{1}{2\pi i} \int_{\frac{1}{2}-iA}^{\frac{1}{2}+iA} \frac{\sin\pi s}{\pi}\, G(s)\, x^{-s}\, ds. \tag{10.32}$$

Here $G(s)$ is the M^2-transform of the function $g(x)$, defined by (10.31). Another inversion formula for the Stieltjes transform can be found in PALEY-WIENER [13, § 14].

Example 6. For the sake of completeness we include an example, given by DOETSCH [6] for the Watson transforms. He pointed out that it is possible to obtain Watson transforms which have as eigenvalues the elements of a given, bounded symmetric set of countably many points in the complex plane. Let $\{\lambda_j\}$ be a set with $|\lambda_j| \leqq C$, where C is a positive constant. If $s = \frac{1}{2}+it$, then let

$$\begin{cases} K_+(s) = \lambda_j, & j-1 < |t| \leqq j \quad (j=1,2,\ldots.), \\ K_-(s) \quad 0. & \end{cases}$$

Thus

$$K_+(s)K_+(1-s) = \lambda_j^2,$$

on a set of positive measure on Re $s = \frac{1}{2}$. According to theorem 5 the transform, defined by this choice of the functions $K_{\pm}(s)$ has the points λ_j and $-\lambda_j$ as eigenvalues. As the spectrum $\sigma(T)$ is a compact set, the limit points λ' of the set $\{\pm\lambda_j\}$ also belong to $\sigma(T)$. It follows that

$$\lambda' \in C\sigma(T).$$

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PART II

INVERSION THEOREMS FOR SOME GENERALIZED LAPLACE TRANSFORMS

0. Introduction

Recently BRAAKSMA, MEULENBELD and LEMEI [2] have derived complex inversion formulas under L_1 conditions for integral transforms with kernels, defined as solutions of the differential equation

$$(0.1) \qquad \frac{d^2y}{dx^2} - \{\lambda^2+q(x)\}y = 0, \qquad -\infty < x < \infty,$$

with some integrability conditions on the continuous function $q(x)$, satisfying initial conditions in $\pm\infty$. We will consider similar problems associated with the differential equation

$$(0.2) \qquad \frac{d^2y}{dx^2} - \{\lambda^2 + \frac{\nu^2-\frac{1}{4}}{x^2} + q(x)\}y = 0, \qquad 0 < x < \infty,$$

where $q(x)$ is a complex valued, continuous function for positive values of x and

$$(0.3) \qquad q(x) \in L_1(0,\infty).$$

The values of the parameter ν in (0.2) are to be restricted by

$$(0.4) \qquad -\tfrac{1}{2} \leqq \operatorname{Re} \nu \leqq \tfrac{1}{2}.$$

We will construct two solutions $e(x,\lambda)$ and $\tilde{e}(x,\lambda)$ of (0.2), which satisfy the following initial conditions in ∞ and 0 respectively:

$$(0.5) \qquad \begin{cases} e(x,\lambda) \sim e^{-\lambda x}, \\ \dfrac{d}{dx}\, e(x,\lambda) \sim -\lambda e^{-\lambda x}, \end{cases}$$

as $x \to \infty$ and $\operatorname{Re} \lambda \geqq 0$, $\lambda \neq 0$; and

$$(0.6) \qquad \begin{cases} \tilde{e}(x,\lambda) \sim x^{\nu+\frac{1}{2}}, \\ \dfrac{d}{dx}\, \tilde{e}(x,\lambda) \sim (\nu+\tfrac{1}{2})x^{\nu-\frac{1}{2}}, \end{cases}$$

as $x \downarrow 0$ and $\operatorname{Re} \lambda \geqq 0$, $\lambda \neq 0$. We denote the Wronskian of the functions $e(x,\lambda)$ and $\tilde{e}(x,\lambda)$ by $W(\lambda)$. The functions $e(x,\lambda)$ and $\tilde{e}(x,\lambda)$ will be used as kernels of integral transforms. Our main result is a complex inversion theorem for such transforms.

Theorem 1. Let x and λ_1 be real, positive numbers. Let $f(t)$ be a function, defined for positive values of t and of bounded variation in a neighbourhood of $t=x$.

If

$$(0.7) \qquad f(t)e^{-\lambda_1 t} \in L_1(0,\infty),$$

then

$$(0.8)\qquad \lim_{\mu\to\infty} \frac{1}{\pi i} \int_{\lambda_1-i\mu}^{\lambda_1+i\mu} \frac{\lambda}{W(\lambda)}\, \tilde{e}(x,\lambda)d\lambda \int_0^\infty f(t)e(t,\lambda)dt = \tfrac{1}{2}\{f(x-0)+f(x+0)\}.$$

If

$$(0.9)\qquad f(t)e^{\lambda_1 t} \in L_1(0,\infty),$$

then

$$(0.10)\qquad \lim_{\mu\to\infty} \frac{1}{\pi i} \int_{\lambda_1-i\mu}^{\lambda_1+i\mu} \frac{\lambda}{W(\lambda)}\, e(x,\lambda)d\lambda \int_0^\infty f(t)\tilde{e}(t,\lambda)dt = \tfrac{1}{2}\{f(x-0)+f(x+0)\}.$$

In (0.8) and (0.10) the path of integration in the λ-plane, has been chosen, such that the zeros of $W(\lambda)$ are to the left of it.

Remark 1. It will follow from the approximation for $W(\lambda)$ in section 1, that it is possible to choose the path of integration as required in the theorem.

Remark 2. In case $q(x)$ in (0.2) depends analytically on λ, then (0.3) should be replaced by $|q(x,\lambda)| \leqq q_1(x)$ on Re $\lambda \geqq 0$, where $q_1(x) \in L_1(0,\infty)$.

Remark 3. The differential equation (0.2) reduces to (0.1) for $\nu = \pm\frac{1}{2}$. The conditions (0.6) may then be written as

$$(0.11)\qquad \begin{cases} \tilde{e}(0,\lambda) = 1, \\ \dfrac{d}{dx}\tilde{e}(0,\lambda) = 0, \end{cases}$$

for $\nu = -\frac{1}{2}$, and as

$$(0.12)\qquad \begin{cases} \tilde{e}(0,\lambda) = 0, \\ \dfrac{d}{dx}\tilde{e}(0,\lambda) = 1, \end{cases}$$

for $\nu = \frac{1}{2}$. The differential equation (0.1) on the positive axis with initial conditions such as (0.11) or (0.12) in case $q(x)$ is a real valued function has been considered by TITCHMARSH [14, section 9.6] and SEARS [12].

Integral transforms acting on $L_2(0,\infty)$ functions, related to a class of differential equations similar to (0.2) have been introduced by GASYMOV [5].

Our method of proof is quite analogous to the one given by BRAAKSMA, MEULENBELD and LEMEI [2]. In section 1 two linearly independent solutions of (0.2) will be constructed, where also an asymptotic approximation of

$W(\lambda)$ will be given. Section 2 contains two lemmas about integral transforms related to (0.8) and (0.10), acting on functions $f(t)$, which vanish on $0 < t < x$ and $t > x$ respectively $(x > 0)$. Furthermore we will prove a corollary, which shows a relation between these lemmas and certain formulas proved by TITCHMARSH [14, section 9.6]. In section 3 the preceding lemmas are used to prove theorem 1. Finally several applications of theorem 1 are given in section 4, including complex inversion formulas for the one-sided Laplace transform, and transforms considered by MEYER, KOH-ZEMANIAN, TITCHMARSH and BRAAKSMA-MEULENBELD.

1. Construction of solutions of the differential equation

In order to construct solutions of (0.2) we first consider solutions of the equation

$$\frac{d^2y}{dx^2} - \left\{\lambda^2 + \frac{\nu^2-\frac{1}{4}}{x^2}\right\}y = 0. \tag{1.1}$$

Then we may regard (0.2) as the "inhomogeneous" equation associated with (1.1). Solutions of (1.1) are $(\lambda x)^{\frac{1}{2}}K_\nu(\lambda x)$ and $(\lambda x)^{\frac{1}{2}}I_\nu(\lambda x)$, where $K_\nu(z)$ and $I_\nu(z)$ are modified Bessel functions.

For completeness sake we give some well-known properties of these Bessel functions (see for instance Watson [15]):

$$I_\nu(z) = \sum_{m=0}^{\infty} \frac{1}{m!\Gamma(m+\nu+1)} \left(\frac{z}{2}\right)^{2m+\nu}, \quad |\arg z| < \pi, \tag{1.2}$$

$$K_\nu(z) = \frac{\pi}{2 \sin \nu\pi} [I_{-\nu}(z)+I_\nu(z)], \quad |\arg z| < \pi,\ \nu \neq 0,\pm1,\pm2,\ldots, \tag{1.3}$$

$$K_o(z) = - I_o(z) \log \frac{z}{2} + \sum_{m=0}^{\infty} \frac{\Psi(m+1)}{\{m!\}^2} \left(\frac{z}{2}\right)^{2m}, \quad |\arg z| < \pi, \tag{1.4}$$

where Ψ is the logarithmic derivative of the Γ-function. Asymptotic approximations as $|z| \to \infty$ are [1]):

$$\left\{ \begin{array}{l} z^{\frac{1}{2}}I_\nu(z) = (2\pi)^{-\frac{1}{2}}\, e^z(1+O(\frac{1}{z}))+(2\pi)^{-\frac{1}{2}}e^{-z-(\nu+\frac{1}{2})\pi i}(1+O(\frac{1}{z})), \\ \qquad\qquad\qquad (-\frac{3}{2}\pi+\varepsilon \leqq \arg z \leqq \frac{1}{2}\pi-\varepsilon), \end{array} \right. \tag{1.5a}$$

[1]) ε must be chosen such that $0 < \varepsilon < \frac{1}{2}\pi$.

$$(1.5b)\quad \begin{cases} z^{\frac{1}{2}}I_\nu(z) = (2\pi)^{-\frac{1}{2}}e^z(1+O(\frac{1}{z}))+(2\pi)^{-\frac{1}{2}}e^{-z+(\nu+\frac{1}{2})\pi i}(1+O(\frac{1}{z})), \\ \qquad\qquad (-\frac{1}{2}\pi+\varepsilon \leqq \arg z \leqq \frac{3}{2}\pi-\varepsilon), \end{cases}$$

$$(1.6a)\quad \begin{cases} \frac{d}{dz}\{z^{\frac{1}{2}}I_\nu(z)\} = (2\pi)^{-\frac{1}{2}}e^z(1+O(\frac{1}{z}))-(2\pi)^{-\frac{1}{2}}e^{-z-(\nu+\frac{1}{2})\pi i}(1+O(\frac{1}{z})), \\ \qquad\qquad (-\frac{3}{2}\pi+\varepsilon \leqq \arg z \leqq \frac{1}{2}\pi-\varepsilon), \end{cases}$$

$$(1.6b)\quad \begin{cases} \frac{d}{dz}\{z^{\frac{1}{2}}I_\nu(z)\} = (2\pi)^{-\frac{1}{2}}e^z(1+O(\frac{1}{z}))-(2\pi)^{-\frac{1}{2}}e^{-z+(\nu+\frac{1}{2})\pi i}(1+O(\frac{1}{z})), \\ \qquad\qquad (-\frac{1}{2}\pi+\varepsilon \leqq \arg z \leqq \frac{3}{2}\pi-\varepsilon), \end{cases}$$

$$(1.7)\quad \begin{cases} z^{\frac{1}{2}}K_\nu(z) = (\frac{\pi}{2})^{\frac{1}{2}}\, e^{-z}(1+O(\frac{1}{z})), \\ \qquad\qquad (-\frac{3}{2}\pi+\varepsilon \leqq \arg z \leqq \frac{3}{2}\pi-\varepsilon), \end{cases}$$

$$(1.8)\quad \begin{cases} \frac{d}{dz}\{z^{\frac{1}{2}}K_\nu(z)\} = -\,(\frac{\pi}{2})^{\frac{1}{2}}\, e^{-z}(1+O(\frac{1}{z})), \\ \qquad\qquad (-\frac{3}{2}\pi+\varepsilon \leqq \arg z \leqq \frac{3}{2}\pi-\varepsilon). \end{cases}$$

The formulas (1.6a and b) and (1.8) follow from (1.5a and b) and (1.7) respectively by applying a theorem on the differentiation of asymptotic expansions of analytic functions (cf. ERDÉLYI [4]). The Wronskian of $K_\nu(z)$ and $I_\nu(z)$ is given by

$$(1.9)\qquad W_z[K_\nu(z),I_\nu(z)] = \frac{1}{z}\,.$$

From (0.4), (1.2), (1.3), (1.4), (1.5a en b) and (1.7) it follows that there exists a positive constant A such that for $\mathrm{Re}\, z \geqq 0$, $z \neq 0$ we have

$$(1.10)\qquad |z^{\frac{1}{2}}I_\nu(z)| \leqq Ae^{\mathrm{Re}\, z}$$

and

$$(1.11)\qquad |z^{\frac{1}{2}}K_\nu(z)| \leqq Ae^{-(\mathrm{Re}\, z)}.$$

Rather than using the functions $e(x,\lambda)$ and $\tilde{e}(x,\lambda)$ we will use multiples of these functions. This does not alter theorem 1 because of the Wronskian in the denominator of (0.8) and (0.10).

Put $y_1(x,\lambda) = (\frac{\pi}{2})^{\frac{1}{2}}\, e(x,\lambda)$, then because of (1.7) and (1.8) the initial value problem (0.5) for the differential equation (0.2) is equivalent to the integral equation

$$(1.12)\quad \begin{cases} y_1(x,\lambda) = (\lambda x)^{\frac{1}{2}}K_\nu(\lambda x) + \\ + \frac{1}{\lambda}\int\limits_x^\infty \{(\lambda x)^{\frac{1}{2}}K_\nu(\lambda x)(\lambda t)^{\frac{1}{2}}I_\nu(\lambda t)-(\lambda x)^{\frac{1}{2}}I_\nu(\lambda x)(\lambda t)^{\frac{1}{2}}K_\nu(\lambda t)\}q(t)y_1(t,\lambda)dt, \end{cases}$$

if we observe from (1.9) that $W_x[(\lambda x)^{\frac{1}{2}}K_\nu(\lambda x),\ (\lambda x)^{\frac{1}{2}}I_\nu(\lambda x)] = \lambda$. Using (1.10) and (1.11) we have for $t \geqq x$, $\mathrm{Re}\,\lambda \geqq 0$, $\lambda \neq 0$:

$$|(\lambda x)^{\frac{1}{2}}K_\nu(\lambda x)(\lambda t)^{\frac{1}{2}}I_\nu(\lambda t)-(\lambda x)^{\frac{1}{2}}I_\nu(\lambda x)(\lambda t)^{\frac{1}{2}}K_\nu(\lambda t)| \leqq 2A^2e^{(\mathrm{Re}\,\lambda)(t-x)}.$$

We solve (1.12) by the method of successive approximations:

$$y_{1,0}(x,\lambda) = (\lambda x)^{\frac{1}{2}}K_\nu(\lambda x),$$

$$y_{1,n}(x,\lambda) = (\lambda x)^{\frac{1}{2}}K_\nu(\lambda x) +$$

$$+ \frac{1}{\lambda}\int\limits_x^\infty \{(\lambda x)^{\frac{1}{2}}K_\nu(\lambda x)(\lambda t)^{\frac{1}{2}}I_\nu(\lambda t)-(\lambda x)^{\frac{1}{2}}I_\nu(\lambda x)(\lambda t)^{\frac{1}{2}}K_\nu(\lambda t)\}q(t)y_{1,n-1}(t,\lambda)dt$$

$$(n=1,2,\dots).$$

Setting

$$(1.13)\qquad \theta_1(x) = \int\limits_x^\infty |q(t)|dt$$

we obtain

$$|y_{1,n}(x,\lambda)-y_{1,n-1}(x,\lambda)| \leqq \frac{2^nA^{2n+1}}{n!}\,\frac{\{\theta_1(x)\}^n}{|\lambda|^n}\,e^{-(\mathrm{Re}\,\lambda)x},$$

$$(n=1,2,\dots).$$

Thus $\lim\limits_{\mu\to\infty} y_{1,n}(x,\lambda) = y_1(x,\lambda)$ exists and satisfies

$$(1.14)\qquad |y_1(x,\lambda)-(\lambda x)^{\frac{1}{2}}K_\nu(\lambda x)| \leqq A\{-1+e^{2A^2\theta_1(x)/|\lambda|}\}e^{-(\mathrm{Re}\,\lambda)x}$$

for $x > 0$, $\mathrm{Re}\,\lambda \geqq 0$, $\lambda\neq 0$ and (0.4).

Put $y_2(x,\lambda) = \dfrac{\lambda^{\nu+\frac{1}{2}}}{\Gamma(\nu+1)2^\nu}\,\tilde{e}(x,\lambda)$, then because of (1.2) the initial value problem (0.6) for the equation (0.2) is equivalent to the integral equation

$$(1.15)\quad \begin{aligned} &y_2(x,\lambda) = (\lambda x)^{\frac{1}{2}}I_\nu(\lambda x) + \\ &+ \frac{1}{\lambda}\int\limits_0^\infty \{(\lambda x)^{\frac{1}{2}}I_\nu(\lambda x)(\lambda t)^{\frac{1}{2}}K_\nu(\lambda t)-(\lambda x)^{\frac{1}{2}}K_\nu(\lambda x)(\lambda t)^{\frac{1}{2}}I_\nu(\lambda t)\}q(t)y_2(t,\lambda)dt. \end{aligned}$$

With

$$(1.16)\qquad \theta_2(x) = \int\limits_0^x |q(t)|dt$$

we find in the same way as above

$$(1.17)\qquad |y_2(x,\lambda)-(\lambda x)^{\frac{1}{2}}I_\nu(\lambda x)| \leqq A\{-1+e^{2A^2\theta_2(x)/|\lambda|}\}e^{(\mathrm{Re}\,\lambda)x}$$

for $x > 0$, $\mathrm{Re}\,\lambda \geqq 0$, $\lambda\neq 0$ and (0.4).

Remark 4. The construction of $y_2(x,\lambda)$ is similar to a construction of TITCHMARSH [13].

Now we wish to approximate the Wronskian of $y_1(x,\lambda)$ and $y_2(x,\lambda)$:

$$W_{12}(\lambda) = y_1(x,\lambda)\,\frac{d}{dx}\,y_2(x,\lambda) - y_2(x,\lambda)\,\frac{d}{dx}\,y_1(x,\lambda) \tag{1.18}$$

as $\lambda \to \infty$ on $\mathrm{Re}\,\lambda \geqq 0$. In order to do this we first mention that for positive x it follows from (1.8) and (1.6a and b) respectively that

$$\frac{d}{dx}\,\{(\lambda x)^{\frac{1}{2}}K_\nu(\lambda x)\} = -\,(\tfrac{\pi}{2})^{\frac{1}{2}}\,\lambda e^{-\lambda x}\,(1+O(\tfrac{1}{\lambda x})) \tag{1.19}$$

as $\lambda x \to \infty$ with $\mathrm{Re}\,\lambda \geqq 0$; and

$$\left\{\begin{aligned} \frac{d}{dx}\,\{(\lambda x)^{\frac{1}{2}}I_\nu(\lambda x)\} &= \frac{\lambda}{(2\pi)^{\frac{1}{2}}}\,e^{\lambda x}\,(1+O(\tfrac{1}{\lambda x})) - \\ &\quad - \frac{\lambda}{(2\pi)^{\frac{1}{2}}}\,e^{-\lambda x \pm (\nu+\frac{1}{2})\pi i}\,(1+O(\tfrac{1}{\lambda x}))\end{aligned}\right. \tag{1.20}$$

as $\lambda x \to \infty$ with $\mathrm{Re}\,\lambda \geqq 0$, $\pm\,\mathrm{Im}\,\lambda \geqq 0$; the upper or lower sign in the exponential has to be taken depending on $\mathrm{Im}\,\lambda \lesseqgtr 0$.

Differentiation of (1.12) with respect to x yields the equation

$$\begin{aligned}\frac{d}{dx}\,y_1(x,\lambda) &= \frac{d}{dx}\,\{(\lambda x)^{\frac{1}{2}}K_\nu(\lambda x)\} + \frac{1}{\lambda}\int_0^\infty \Big[\frac{d}{dx}\,\{(\lambda x)^{\frac{1}{2}}K_\nu(\lambda x)\}\,. \\ &\quad .(\lambda t)^{\frac{1}{2}}I_\nu(\lambda t) - (\lambda t)^{\frac{1}{2}}K_\nu(\lambda t)\,\frac{d}{dx}\,\{(\lambda x)^{\frac{1}{2}}I_\nu(\lambda x)\}\Big]\,q(t)y_1(t,\lambda)dt.\end{aligned} \tag{1.21}$$

It follows from (1.20), (1.19), (1.11) and (1.10) that there exists a positive constant B, such that for the [...] term in (1.21) we have

$$|[\ldots]| \leq C|\lambda|e^{(\mathrm{Re}\,\lambda)(t-x)}\,(1+\frac{D}{|\lambda|x}) \tag{1.22}$$

for $|\lambda|x \geqq B$ and $t \geqq x$; here C and D are positive constants. From (1.14), (1.13) and (1.7) we find

$$y_1(t,\lambda) = (\tfrac{\pi}{2})^{\frac{1}{2}}\,e^{-\lambda t}\,(1 + O(\tfrac{1}{\lambda}) + O(\tfrac{1}{\lambda t})) \tag{1.23}$$

as $\lambda \to \infty$ and $\lambda t \to \infty$ on $\mathrm{Re}\,\lambda \geqq 0$, uniformly in t for $t > 0$. Combining (1.23), (1.22) and (0.3) we obtain for the last term in (1.21) the following approximation

$$\frac{1}{\lambda}\int_x^\infty [\ldots]q(t)y_1(t,\lambda)dt = e^{-\lambda x}\,O(1) \tag{1.24}$$

as $\lambda \to \infty, \lambda x \to \infty$ with $\mathrm{Re}\,\lambda \geqq 0$. From this, (1.21) and (1.19) we conclude

$$\frac{d}{dx}\,y_1(x,\lambda) = -\,(\tfrac{\pi}{2})^{\frac{1}{2}}\,\lambda e^{-\lambda x}\,(1+O(\tfrac{1}{\lambda}) + O(\tfrac{1}{\lambda x})) \tag{1.25}$$

as $\lambda \to \infty$ and $\lambda x \to \infty$ on $\mathrm{Re}\,\lambda \geqq 0$, $x > 0$.

Differentiating (1.15) with respect to x we get the equation

$$(1.26)\quad \begin{cases} \dfrac{d}{dx}\, y_2(x,\lambda) = \dfrac{d}{dx}\,\{(\lambda x)^{\frac{1}{2}} I_\nu(\lambda x)\} + \dfrac{1}{\lambda}\int\limits_0^x \Big[\dfrac{d}{dx}\,\{(\lambda x)^{\frac{1}{2}} I_\nu(\lambda x)\}. \\ \quad .(\lambda t)^{\frac{1}{2}} K_\nu(\lambda t) - \dfrac{d}{dx}\,\{(\lambda x)^{\frac{1}{2}} K_\nu(\lambda x)\}(\lambda t)^{\frac{1}{2}} I_\nu(\lambda t)\Big]\, q(t) y_2(t,\lambda)dt. \end{cases}$$

For the [...] term in (1.26) we use (1.20), (1.19), (1.11) and (1.10); this yields:

$$(1.27)\qquad [\ldots] = \lambda e^{\lambda(x-t)}\; 0(1)$$

as $\lambda x \to \infty$ with Re $\lambda \geqq 0$. From (1.17), (1.16) and (1.10) it can be seen that

$$(1.28)\qquad y_2(x,\lambda) = e^{\lambda x}\; 0(1)$$

as $\lambda \to \infty$, uniformly in x on $x > 0$. This, together with (1.27) and (0.3) gives for the last term in (1.26):

$$(1.29)\qquad \frac{1}{\lambda}\int\limits_0^x [\ldots] q(t)\; y_2(t,\lambda)dt = e^{\lambda x}\; 0(1)$$

as $\lambda x \to \infty$ with Re $\lambda \geqq 0$. Combining this with (1.20) in (1.26) we get

$$(1.30)\quad \begin{cases} \dfrac{d}{dx}\, y_2(x,\lambda) = \dfrac{\lambda}{(2\pi)^{\frac{1}{2}}}\, e^{\lambda x}\, (1+0(\tfrac{1}{\lambda})+0(\tfrac{1}{\lambda x})) - \\ \qquad\qquad - \dfrac{\lambda}{(2\pi)^{\frac{1}{2}}}\, e^{-\lambda x \pm (\nu+\frac{1}{2})\pi i}\, (1+0(\tfrac{1}{\lambda x})) \end{cases}$$

as $\lambda \to \infty$ and $\lambda x \to \infty$ on Re $\lambda \geqq 0$, $x > 0$.

From (1.17), (1.16) and (1.5) we find

$$(1.31)\quad \begin{cases} y_2(t,\lambda) = \dfrac{1}{(2\pi)^{\frac{1}{2}}}\, e^{\lambda t}\, (1+0(\tfrac{1}{\lambda})+0(\tfrac{1}{\lambda t})) + \\ \qquad\qquad + \dfrac{1}{(2\pi)^{\frac{1}{2}}}\, e^{-\lambda t \pm (\nu+\frac{1}{2})\pi i}\, (1+0(\tfrac{1}{\lambda t})) \end{cases}$$

as $\lambda \to \infty$ and $\lambda t \to \infty$ on Re $\lambda \geqq 0$, $\pm$ Im $\lambda \geqq 0$, uniformly in t on $t > 0$.

Fixing x, putting t=x in (1.31) and (1.23) and combining this with (1.30) and (1.25) we find an approximation for the Wronskian $W_{12}(\lambda)$:

$$(1.32)\qquad W_{12}(\lambda) = \lambda(1+0(\tfrac{1}{\lambda}))$$

as $\lambda \to \infty$ on Re $\lambda \geqq 0$. From this approximation for $W_{12}(\lambda)$ we see that the function $W(\lambda) = \dfrac{2^{\frac{1}{2}+\nu}\Gamma(\nu+1)}{\pi^{\frac{1}{2}}\, \lambda^{\nu+\frac{1}{2}}}\, W_{12}(\lambda)$ has no zeros in the righthalf λ-plane if $|\lambda|$ is chosen sufficiently large; thus it is possible to choose the path of integration in the λ-plane as required in theorem 1 (cf. remark 1).

Finally, we mention an approximation for $y_2(t,\lambda)$, which we will need in the next section. From (1.17), (1.16) and (1.2) it follows that

$$y_2(t,\lambda) = O((\lambda t)^{\nu+\frac{1}{2}}) + O(\tfrac{1}{\lambda}) \tag{1.33}$$

as $\lambda \to \infty$, $\lambda t \to 0$ on $\operatorname{Re}\lambda \geqq 0$, uniformly in t on $t > 0$.

2. Inversion formulas for functions vanishing in a neighbourhood of ∞ and 0

Our first result in this section is a generalization of theorem 1 in case the function $f(t)$ vanishes for $0 < t < x$ (cf. [2, Theorem 1]).

Lemma 1. Let x and λ_1 be real positive numbers. Let $f(t)$ be a function defined for $t > x$, be of bounded variation in a righthand neighbourhood of $t=x$ and

$$f(t)e^{-\lambda_1 t} \in L_1(x,\infty). \tag{2.1}$$

Let $y_1(t,\lambda)$ be the solution of (0.2), defined in section 1, satisfying (1.14). Let $\varphi(\lambda)$ be a function analytic in $\operatorname{Re}\lambda > \lambda_1$ and continuous on $\operatorname{Re}\lambda \geqq \lambda_1$ with

$$\varphi(\lambda) = e^{\lambda x}\,(1+O(\tfrac{1}{\lambda})) + e^{-\lambda x}\,(\rho e^{\pm\sigma} + O(\tfrac{1}{\lambda})) \tag{2.2}$$

as $\lambda \to \infty$ on $\operatorname{Re}\lambda \geqq \lambda_1$, $\pm \operatorname{Im}\lambda \geqq 0$, where σ and ρ are complex constants.

Then

$$\lim_{\mu\to\infty} \frac{1}{\pi i} \int_{\lambda_1 - i\mu}^{\lambda_1 + i\mu} \varphi(\lambda)d\lambda \int_x^\infty f(t)y_1(t,\lambda)dt = (\tfrac{\pi}{2})^{\frac{1}{2}}\, f(x+0). \tag{2.3}$$

Proof. The function $g(t,\lambda)$ defined by

$$g(t,\lambda) = \varphi(\lambda)y_1(t,\lambda) \tag{2.4}$$

is an analytic function of λ in $\operatorname{Re}\lambda > \lambda_1$ for positive t and x. Let the function $I(t,\mu)$ be defined by

$$I(t,\mu) = \int_{\lambda_1 - i\mu}^{\lambda_1 + i\mu} g(t,\lambda)d\lambda \tag{2.5}$$

with $\mu > 0$. From (1.23) and (2.2) we obtain

$$g(t,\lambda) = (\tfrac{\pi}{2})^{\frac{1}{2}}\; e^{-\lambda(t-x)}(I+\Psi_1(t,\lambda))+e^{-\lambda(t+x)}(\rho(\tfrac{\pi}{2})^{\frac{1}{2}} e^{\pm\sigma}+\Psi_2(t,\lambda)) \tag{2.6}$$

where

$$\Psi_1(t,\lambda) = O(\tfrac{1}{\lambda}),\ \Psi_2(t,\lambda) = O(\tfrac{1}{\lambda}) \tag{2.7}$$

as $\lambda \to \infty$ on $\mathrm{Re}\,\lambda \geqq \lambda_1$, $\pm\,\mathrm{Im}\,\lambda \geqq 0$, uniformly in t on $t \geqq x$. From this and Cauchy's theorem we find for $t > x$:

$$I(t,\mu) = \left\{ \int_{\lambda_1 - i\mu}^{\infty - i\mu} - \int_{\lambda_1 + i\mu}^{\infty + i\mu} \right\} g(t,\lambda)d\lambda. \tag{2.8}$$

We define

$$\begin{aligned} I_1(t,\mu) &= \left(\tfrac{\pi}{2}\right)^{\frac{1}{2}} \left\{ \int_{\lambda_1 - i\mu}^{\infty - i\mu} - \int_{\lambda_1 + i\mu}^{\infty + i\mu} \right\} e^{-\lambda(t-x)} d\lambda \\ &\quad + \rho\left(\tfrac{\pi}{2}\right)^{\frac{1}{2}} \int_{\lambda_1 - i\mu}^{\infty - i\mu} e^{-\lambda(t+x)-\sigma} d\lambda - \rho\left(\tfrac{\pi}{2}\right)^{\frac{1}{2}} \int_{\lambda_1 + i\mu}^{\infty + i\mu} e^{-\lambda(t+x)+\sigma} d\lambda \end{aligned} \tag{2.9}$$

and

$$I_2(t,\mu) = \left\{ \int_{\lambda_1 - i\mu}^{\infty - i\mu} - \int_{\lambda_1 + i\mu}^{\infty + i\mu} \right\} \{ e^{-\lambda(t-x)} \Psi_1(t,\lambda) + e^{-\lambda(t+x)} \Psi_2(t,\lambda) \} d\lambda \tag{2.10}$$

so that we have from (2.9) and (2.10)

$$I(t,\mu) = I_1(t,\mu) + I_2(t,\mu). \tag{2.11}$$

From (2.9) we see

$$I_1(t,\mu) = i(2\pi)^{\frac{1}{2}} \frac{\sin \mu(t-x)}{t-x} e^{-\lambda_1(t-x)} + i\rho(2\pi)^{\frac{1}{2}} \frac{\sin\{\mu(t+x)+i\sigma\}}{t+x} e^{-\lambda_1(t+x)}. \tag{2.12}$$

From this we obtain

$$\begin{aligned} \lim_{\mu\to\infty} \frac{1}{\pi i} \int_x^\infty f(t) I_1(t,\mu) dt &= \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}} \lim_{\mu\to\infty} \Big[\int_x^\infty f(t) \frac{\sin \mu(t-x)}{t-x} \,. \\ &\quad . e^{-\lambda_1(t-x)} dt + \rho \int_x^\infty f(t) \frac{\sin\{\mu(t+x)+i\sigma\}}{t+x} e^{-\lambda_1(t+x)} dt \Big] = \left(\tfrac{\pi}{2}\right)^{\frac{1}{2}} f(x+0), \end{aligned} \tag{2.13}$$

which follows from (2.1) and the lemmas of Dirichlet and Riemann-Lebesgue; f(t) being of bounded variation in a righthand neighbourhood of x.

Next we prove

$$\lim_{\mu\to\infty} \int_x^\infty f(t)\, I_2(t,\mu) dt = 0 \tag{2.14}$$

which implies with (2.13) and (2.11):

$$\lim_{\mu\to\infty} \frac{1}{\pi i} \int_x^\infty f(t)\, I(t,\mu) dt = \left(\tfrac{\pi}{2}\right)^{\frac{1}{2}} f(x+0). \tag{2.15}$$

Let f(t) be of bounded variation on $x \leqq t \leqq a$. Then we write the integral in (2.14) as the sum of $\int_x^a$ and $\int_a^\infty$. From (2.10) and (2.7) it follows that

$$I_2(t,\mu) = e^{-\lambda_1 t}\, O(\frac{1}{\mu})$$

as $\mu \to \infty$ uniformly in t on $t \geqq a$. Hence from (2.1) we obtain

$$(2.16) \qquad \lim_{\mu\to\infty} \int_a^\infty f(t)\, I_2(t,\mu)dt = 0.$$

As f(t) is of bounded variation on $x \leqq t \leqq a$, the real and imaginary parts of f(t) may be written as the difference of two monotonic functions on $x \leqq t \leqq a$. We will prove

$$(2.17) \qquad \lim_{\mu\to\infty} \int_x^s I_2(t,\mu)dt = 0$$

uniformly in s on $x \leqq s \leqq a$. Then applying Bonnet's mean value theorem and (2.17) we may conclude

$$\lim_{\mu\to\infty} \int_x^a f(t)\, I_2(t,\mu)dt = 0$$

which together with (2.16) implies (2.14).

In order to prove (2.17) we deduce from (2.10) and (2.7)

$$\left| \int_x^s I_2(t,\mu)dt \right| \leqq C \left\{ \int_{\lambda_1 - i\mu}^{\infty - i\mu} + \int_{\lambda_1 + i\mu}^{\infty + i\mu} \right\} \frac{1}{|\lambda|\operatorname{Re}\lambda} \{1 - e^{(\operatorname{Re}\lambda)(x-s)}\} |d\lambda|$$

where C is a positive constant. As $0 \leqq \{1-e^{(\operatorname{Re}\lambda)(x-s)}\} \leqq 1$ the last integrals are $o(1)$ as $\mu \to \infty$, uniformly in s on $x \leqq s \leqq a$, which proves (2.17). The equality

$$\int_x^\infty f(t)\, I(t,\mu)dt = \int_{\lambda_1 - i\mu}^{\lambda_1 + i\mu} \varphi(\lambda)d\lambda \int_x^\infty f(t)y_1(t,\lambda)dt$$

is justified by (2.1), (2.2), (2.4) and (2.5). From this, (2.15) and passing to the limit $\mu \to \infty$ we obtain the inversion formula (2.3). This completes the proof.

Next we prove an inversion formula in case f(t) vanishes for $t > x$; now however we use the function $y_2(t,\lambda)$ as the kernel of our transform.

Lemma 2. Let x and λ_1 be real, positive numbers. Let f(t) be a function, defined on $0 < t < x$, be of bounded variation in a left hand neighbourhood of x and

(2.18) $\qquad f(t) \in L_1(0,x).$

Let $y_2(t,\lambda)$ be the solution of (0.2), defined in section 1, satisfying (1.17). Let $\varphi(\lambda)$ be a function, analytic in $\mathrm{Re}\,\lambda > \lambda_1$ and continuous on $\mathrm{Re}\,\lambda \geqq \lambda_1$ with

(2.19) $$\varphi(\lambda) = e^{-\lambda x}\,(1+0(\tfrac{1}{\lambda}))$$

as $\lambda \to \infty$ on $\mathrm{Re}\,\lambda \geqq \lambda_1$. Then

(2.20) $$\lim_{\mu\to\infty} \frac{1}{\pi i} \int_{\lambda_1-i\mu}^{\lambda_1+i\mu} \varphi(\lambda)d\lambda \int_0^x f(t)y_2(t,\lambda)dt = \frac{1}{(2\pi)^{\frac{1}{2}}}\, f(x-0).$$

Proof. We define the function $g(t,\lambda)$ by

(2.21) $$g(t,\lambda) = \varphi(\lambda)\; y_2(t,\lambda)$$

which is an analytic function of λ in $\mathrm{Re}\,\lambda > \lambda_1$ for positive t and x. Let the function $I(t,\mu)$ be defined by

(2.22) $$I(t,\mu) = \int_{\lambda_1-i\mu}^{\lambda_1+i\mu} g(t,\lambda)d\lambda.$$

From (1.31) and (2.19) it follows that

(2.23) $$g(t,\lambda) = \frac{1}{(2\pi)^{\frac{1}{2}}}\, e^{\lambda(t-x)}(1+\Psi_1(t,\lambda)) + \frac{1}{(2\pi)^{\frac{1}{2}}}\, e^{-\lambda(t+x)}(e^{\pm(\nu+\frac{1}{2})\pi i}+\Psi_2(t,\lambda))$$

where

(2.24) $$\Psi_1(t,\lambda) = 0(\tfrac{1}{\lambda}) + 0(\tfrac{1}{\lambda t}),\ \Psi_2(t,\lambda) = 0(\tfrac{1}{\lambda}) + 0(\tfrac{1}{\lambda t})$$

as $\lambda t \to \infty$ and $\lambda \to \infty$ on $\mathrm{Re}\,\lambda \geqq \lambda_1$, $\pm\,\mathrm{Im}\,\lambda \geqq 0$, uniformly in t on $0 < t \leqq x$. From this and Cauchy's theorem we find for $0 < t < x$:

(2.25) $$I(t,\mu) = \left\{ \int_{\lambda_1-i\mu}^{\infty-i\mu} - \int_{\lambda_1+i\mu}^{\infty+i\mu} \right\} g(t,\lambda)d\lambda.$$

We define

$$I_1(t,\mu) = \frac{1}{(2\pi)^{\frac{1}{2}}} \left\{ \int_{\lambda_1-i\mu}^{\infty-i\mu} - \int_{\lambda_1+i\mu}^{\infty+i\mu} \right\} e^{\lambda(t-x)}\,d\lambda$$

(2.26)

$$+\frac{1}{(2\pi)^{\frac{1}{2}}} \int_{\lambda_1-i\mu}^{\infty-i\mu} e^{-\lambda(t+x)-(\nu+\frac{1}{2})\pi i}d\lambda - \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{\lambda_1+i\mu}^{\infty+i\mu} e^{-\lambda(t+x)+(\nu+\frac{1}{2})\pi i}d\lambda$$

and

$$(2.27)\qquad I_2(t,\mu) = \left\{ \int_{\lambda_1-i\mu}^{\infty-i\mu} - \int_{\lambda_1+i\mu}^{\infty+i\mu} \right\}\{e^{\lambda(t-x)}\Psi_1(t,\lambda)+e^{-\lambda(t+x)}\Psi_2(t,\lambda)\}d\lambda$$

so that from (2.26) and (2.27) we have

$$(2.28)\qquad I(t,\mu) = I_1(t,\mu) + I_2(t,\mu).$$

From (2.26) we have

$$(2.29)\qquad \begin{cases} I_1(t,\mu) = i(\frac{2}{\pi})^{\frac{1}{2}} \dfrac{\sin \mu(t-x)}{t-x}\, e^{\lambda_1(t-x)} + \\ \qquad\qquad + i(\frac{2}{\pi})^{\frac{1}{2}} \dfrac{\sin \{\mu(t+x)-(\nu+\frac{1}{2})\pi\}}{t+x}\, e^{-\lambda_1(t+x)}, \end{cases}$$

From this we obtain

$$(2.30)\qquad \begin{cases} \lim\limits_{\mu\to\infty} \int_0^x f(t)I_1(t,\mu)dt = \lim\limits_{\mu\to\infty} i(\frac{2}{\pi})^{\frac{1}{2}} \int_0^x f(t)\, \dfrac{\sin \mu(t-x)}{t-x}\, e^{\lambda_1(t-x)} dt + \\ \quad + \lim\limits_{\mu\to\infty} i(\frac{2}{\pi})^{\frac{1}{2}} \int_0^x f(t)\, \dfrac{\sin \{\mu(t+x)-(\nu+\frac{1}{2})\pi\}}{t+x}\, e^{-\lambda_1(t+x)} dt = i(\frac{\pi}{2})^{\frac{1}{2}} f(x-0) \end{cases}$$

which follows from (2.18) and the lemmas of Dirichlet and Riemann-Lebesgue; $f(t)$ being of bounded variation in a left hand neighbourhood of x. We choose a such that $f(t)$ is of bounded variation on $a \leqq t \leqq x$.

Next we prove

$$(2.31)\qquad \lim_{\mu\to\infty} \int_a^x f(t)\, I_2(t,\mu)dt = 0.$$

The real and imaginary parts of $f(t)$ can be written as the difference of two monotonic functions on $a \leqq t \leqq x$. We will prove

$$(2.32)\qquad \lim_{\mu\to\infty} \int_s^x I_2(t,\mu)dt = 0$$

uniformly in s on $a \leqq s \leqq x$. Then applying Bonnet's mean value theorem and (2.32) we obtain (2.31).

In order to prove (2.32) we deduce from (2.27) and (2.24):

$$\left| \int_s^x I_2(t,\mu)dt \right| \leqq C \left\{ \int_{\lambda_1-i\mu}^{\infty-i\mu} + \int_{\lambda_1+i\mu}^{\infty+i\mu} \right\} \frac{1}{|\lambda|\mathrm{Re}\ \lambda} \{1-e^{(\mathrm{Re}\lambda)(s-x)}\}\, |d\lambda|$$

where C is a positive constant. Since $0 \leqq 1-e^{(\mathrm{Re}\ \lambda)(s-x)}\} \leqq 1$ the last integrals are $o(1)$ as $\mu \to \infty$ uniformly in s on $a \leqq s \leqq x$. This proves (2.31).

From (2.27) and (2.24) we have

$$(2.33) \qquad I_2(t,\mu) = O\left(\frac{1}{\mu t}\right) + O\left(\frac{1}{\mu}\right)$$

as $\mu t \to \infty$, $\mu \to \infty$ uniformly in t on $0 < t \leqq a$. From this we obtain

$$(2.34) \qquad \int_{G/\mu}^{a} f(t)I_2(t,\mu)dt = \frac{1}{G}\int_{G/\mu}^{a} f(t)O(1)dt + \int_{G/\mu}^{a} f(t)O\left(\frac{1}{\mu}\right)dt$$

for a positive G, $\mu \to \infty$ and $G/\mu < a$. Using (2.18) it follows from (2.34) that we may choose a number G_1 so that given $\varepsilon > 0$ we have

$$(2.35) \qquad \left|\int_{G_1/\mu}^{a} f(t)I_2(t,\mu)dt\right| < \frac{1}{3}\varepsilon$$

for $\mu > \mu_1(\varepsilon)$. On account of (2.30), (2.31) and (2.35) we obtain

$$(2.36) \qquad \left|\int_{G_1/\mu}^{x} f(t)I(t,\mu)dt - i\left(\frac{\pi}{2}\right)^{\frac{1}{2}} f(x-0)\right| < \varepsilon$$

if $\mu > \mu_1(\varepsilon)$.

Finally we want to prove

$$(2.37) \qquad \lim_{\mu\to\infty} \int_{0}^{G_1/\mu} f(t)I(t,\mu)dt = 0.$$

Using (1.33) in (2.21) we obtain

$$(2.38) \qquad g(t,\lambda) = e^{-\lambda x}\left(O((\lambda t)^{\nu+\frac{1}{2}}) + O\left(\frac{1}{\lambda}\right)\right)$$

as $\lambda \to \infty$, $\lambda t \to 0$ on $\mathrm{Re}\,\lambda \geqq \lambda_1$, uniformly in t on $0 \leqq t \leqq x$. This yields with (2.25)

$$(2.39) \qquad I(t,\mu) = O((\mu t)^{\nu+\frac{1}{2}}) + O\left(\frac{1}{\mu}\right)$$

as $\mu \to \infty$ and $\mu t \to 0$. The approximation (2.39) with the condition (2.18) proves (2.37). Combining (2.36) and (2.37) we have proved

$$(2.40) \qquad \lim_{\mu\to\infty} \frac{1}{\pi i} \int_{0}^{x} f(t)I(t,\mu)dt = \frac{1}{(2\pi)^{\frac{1}{2}}} f(x-0).$$

The equality

$$\int_0^x f(t)I(t,\mu)dt = \int_{\lambda_1-i\mu}^{\lambda_1+i\mu} \varphi(\lambda)d\lambda \int_0^x f(t)y_2(t,\lambda)dt$$

is now established in view of (2.18), (2.19), (2.21) and (2.22). From this, (2.40) and passing to the limit $\mu \to \infty$ we obtain the inversion formula (2.20). This completes the proof.

In lemma 1 we may substitute $\varphi(\lambda) = (2\pi)^{\frac{1}{2}}y_2(x,\lambda)$ on account of (1.31), while in lemma 2 we may use $\varphi(\lambda) = (\frac{2}{\pi})^{\frac{1}{2}}\, y_1(x,\lambda)$ because of (1.23). Adding (2.3) and (2.20) in these different cases and introducing the function $\Phi(\lambda)$ by

$$\Phi(\lambda) = y_1(x,\lambda) \int_0^x f(t)y_2(t,\lambda)dt + y_2(x,\lambda) \int_x^\infty f(t)y_1(t,\lambda)dt, \tag{2.41}$$

we may state the following corollary.

Corollary. Let x and λ_1 be real, positive numbers. Let f(t) be a function defined for positive values of t, be of bounded variation in a neighbourhood of t=x and $f(t)e^{-\lambda_1 t} \in L_1(0,\infty)$. Then

$$\lim_{\mu\to\infty} \frac{1}{\pi i} \int_{\lambda_1-i\mu}^{\lambda_1+i\mu} \Phi(\lambda)d\lambda = \tfrac{1}{2}\,\{f(x-0) + f(x+0)\}. \tag{2.42}$$

This corollary is closely related to results of TITCHMARSH [14, section 9.6] where formulas are derived resembling (2.42), however under the conditions of remark 3.

3. Proof of theorem 1

In this section the inversion formulas (0.8) and (0.10) will be proved respectively. The formula (0.8) is equivalent to

$$\lim_{\mu\to\infty} \frac{1}{\pi i} \int_{\lambda_1-i\mu}^{\lambda_1+i\mu} \frac{\lambda}{W_{12}(\lambda)}\, y_2(x,\lambda)d\lambda \int_0^\infty f(t)y_1(t,\lambda)dt = \tfrac{1}{2}\,\{f(x-0)+f(x+0)\}. \tag{3.1}$$

To prove (3.1) we consider two cases:

I. $f(t) \equiv 0$ if $0 < t < x$,

II. $f(t) \equiv 0$ if $t > x$.

Case I is contained in lemma 1, because we may specify

$$\varphi(\lambda) = (2\pi)^{\frac{1}{2}} \frac{\lambda}{W_{12}(\lambda)} y_2(x,\lambda)$$

in view of (1.31) and (1.32).

To treat case II we remark that the function $K(x,t;\lambda)$ satisfying (0.2) as a function of x with initial conditions

$$K(t,t;\lambda) = 0, \ \frac{\partial}{\partial x} K(x,t;\lambda)\Big|_{x=t} = 1, \tag{3.2}$$

is equal to

$$K(x,t;\lambda) = \frac{y^*(x,\lambda)y(t,\lambda)-y(x,\lambda)y^*(t,\lambda)}{W(y,y^*;\lambda)}$$

for any pair y and y* of linearly independent solutions of (0.2), where $W(y,y^*;\lambda)$ is the Wronskian of y and y*. This means that case II is equivalent to

$$\begin{aligned} &\lim_{\mu\to\infty} \int_{\lambda_1-i\mu}^{\lambda_1+i\mu} \frac{\lambda}{W_{12}(\lambda)} y_1(x,\lambda)d\lambda \int_0^x f(t)y_2(t,\lambda)dt \\ &+ \lim_{\mu\to\infty} \int_{\lambda_1-i\mu}^{\lambda_1+i\mu} \lambda d\lambda \int_0^x f(t)K(x,t;\lambda)dt = \tfrac{1}{2}\pi i f(x-0). \end{aligned} \tag{3.3}$$

In view of (1.23) and (1.32) we may apply lemma 2 with

$$\varphi(\lambda) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{\lambda}{W_{12}(\lambda)} y_1(x,\lambda)$$

to the first limit in (3.3), because (0.7) implies (2.18).

Hence it remains to show that

$$\lim_{\mu\to\infty} \int_{\lambda_1-i\mu}^{\lambda_1+i\mu} \lambda d\lambda \int_0^x f(t)K(x,t;\lambda)dt = 0. \tag{3.4}$$

First we consider the integral in the λ-plane

$$\int_{\lambda_1-i\mu}^{\lambda_1+i\mu} \lambda K(x,t;\lambda)d\lambda = \int_{-i\mu}^{i\mu} + \int_{\lambda_1-i\mu}^{-i\mu} + \int_{i\mu}^{\lambda_1+i\mu} = I_1(t,\mu)+I_2(t,\mu)+I_3(t,\mu). \tag{3.5}$$

On the line $\mathrm{Re}\,\lambda = 0$ the function $K(x,t;\lambda)$ is an even function of λ, because as a solution of the differential equation (0.2) with the initial conditions (3.2) it is an entire function of λ^2. Hence $I_1(t,\mu)$ vanishes.

For $I_2(t,\mu)$ and $I_3(t,\mu)$ we use

$$(3.6) \qquad K(x,t;\lambda) = \frac{y_1(x,\lambda)y_2(t,\lambda)-y_2(x,\lambda)y_1(t,\lambda)}{W_{12}(\lambda)} .$$

From (1.23),(1.31) and (1.32) we deduce

$$(3.7) \qquad \begin{cases} K(x,t\lambda) = -\tfrac{1}{2}e^{\lambda(x-t)}(1+O(\frac{1}{\mu})+O(\frac{1}{\mu t})) \\ \quad + \tfrac{1}{2}e^{\lambda(t-x)}(1+O(\frac{1}{\mu})+O(\frac{1}{\mu t}))+e^{-\lambda(x+t)\pm(\nu+\frac{1}{2})\pi i}(O(\frac{1}{\mu})+O(\frac{1}{\mu t})) \end{cases}$$

as $\mu t \to \infty$, $\mu \to \infty$ uniformly in t on $0 < t \leqq x$, uniformly in λ on the line segments joining $-i\mu$ with $\lambda_1-i\mu$ and $i\mu$ with $\lambda_1+i\mu$; the upper or lower sign in the exponential has to be taken as $\operatorname{Im}\lambda \lesseqgtr 0$. Hence

$$(3.8) \qquad \begin{cases} I_2(t,\mu)+I_3(t,\mu) = \frac{1}{x-t}[\cos(\mu+i\lambda_1)(t-x)-\cos(\mu-i\lambda_1)(t-x)] \\ \qquad\qquad + e^{\lambda_1(x-t)}(O(\frac{1}{\mu})+O(\frac{1}{\mu t})) \end{cases}$$

as $\mu \to \infty$, $\mu t \to \infty$ uniformly in t on $0 < t \leqq x$. From this, the lemma of Riemann-Lebesgue, (3.5) and (0.7) we obtain

$$(3.9) \qquad \lim_{\mu\to\infty}\int_{G/\mu}^{x} f(t)dt \int_{\lambda_1-i\mu}^{\lambda_1+i\mu} \lambda K(x,t;\lambda)d\lambda = 0.$$

From the formulas (1.14), (1.13) and (1.3) we obtain in case $\nu \neq 0$

$$(3.10) \qquad y_1(t,\lambda) = O((\lambda t)^{\frac{1}{2}+\nu}) + O((\lambda t)^{\frac{1}{2}-\nu}) + O(\tfrac{1}{\lambda})$$

as $\lambda t \to 0$, $\lambda \to \infty$ on $\operatorname{Re}\lambda \geqq 0$, uniformly in t on $t > 0$. If $\nu=0$ the first two O-functions in (3.10) should be replaced by $O((\lambda t)^{\frac{1}{2}}\log(\lambda t))$. On account of (1.23), (1.31), (1.33), (3.10) and (1.32) we have

$$(3.11) \qquad \lambda K(x,t;\lambda) = O((\mu t)^{\frac{1}{2}+\nu})+O((\mu t)^{\frac{1}{2}-\nu})+O(\tfrac{1}{\mu})$$

as $\mu t \to 0$, $\mu \to \infty$, uniformly in t on $0 \leqq t \leqq x$, uniformly in λ on the line segments joining $-i\mu$ with $\lambda_1-i\mu$ and $i\mu$ with $\lambda_1+i\mu$. Hence if $\nu \neq 0$:

$$(3.12) \qquad I_2(t,\mu)+I_3(t,\mu) = O((\mu t)^{\frac{1}{2}+\nu})+O((\mu t)^{\frac{1}{2}-\nu})+O(\tfrac{1}{\mu})$$

as $\mu t \to 0$, $\mu \to \infty$ uniformly in t on $0 \leqq t \leqq x$. In case $\nu=0$ the terms $O((\mu t)^{\frac{1}{2}+\nu})+O((\mu t)^{\frac{1}{2}-\nu})$ in (3.12) have to be replaced by $O((\mu t)^{\frac{1}{2}}\log(\mu t))$.

From this, (3.5) and (0.7) we obtain

$$\lim_{\mu\to\infty} \int_0^{G/\mu} f(t)dt \int_{\lambda_1-i\mu}^{\lambda_1+i\mu} \lambda K(x,t;\lambda)d\lambda = 0 \tag{3.13}$$

for all ν satisfying (0.4). Combining this with (3.9) gives

$$\lim_{\mu\to\infty} \int_0^{x} f(t)dt \int_{\lambda_1-i\mu}^{\lambda_1+i\mu} \lambda K(x,t;\lambda)d\lambda = 0. \tag{3.14}$$

Changing the order of integration in (3.14) is justified by (3.6), (1.14), (1.17) and (0.7); which shows (3.4). This completes the proof of formula (0.8).

The formula (0.10) is equivalent to

$$\left\{ \begin{aligned} &\lim_{\mu\to\infty} \frac{1}{\pi i} \int_{\lambda_1-i\mu}^{\lambda_1+i\mu} \frac{\lambda}{W_{12}(\lambda)} y_1(x,\lambda)d\lambda \int_0^{\infty} f(t)y_2(t,\lambda)dt \\ &\qquad = \tfrac{1}{2}\{f(x-0)+f(x+0)\}. \end{aligned} \right. \tag{3.15}$$

In order to prove (3.15) under the conditions of theorem 1, we consider again the two cases, where f(t) vanishes identically at the left or at the right of t=x respectively. Now case II is contained in lemma 2, if we observe that (0.9) implies (2.18). Hence it remains to show that (3.15) holds in case I or equivalently

$$\begin{aligned} &\lim_{\mu\to\infty} \int_{\lambda_1-i\mu}^{\lambda_1+i\mu} \frac{\lambda}{W_{12}(\lambda)} y_2(x,\lambda)d\lambda \int_x^{\infty} f(t)y_1(t,\lambda)dt \\ &- \lim_{\mu\to\infty} \int_{\lambda_1-i\mu}^{\lambda_1+i\mu} \lambda d\lambda \int_x^{\infty} f(t)K(x,t;\lambda)dt = \tfrac{1}{2}\pi i f(x+0). \end{aligned} \tag{3.16}$$

Functions which satisfy (0.9) will certainly satisfy (2.1) and this means that lemma 1 can be applied to the first term in (3.16). So we have to show

$$\lim_{\mu\to\infty} \int_{\lambda_1-i\mu}^{\lambda_1+i\mu} \lambda d\lambda \int_x^{\infty} f(t)K(x,t;\lambda)dt = 0. \tag{3.17}$$

The rest of the proof is very similar to the treatment of (3.4). We assume (3.5), where $I_1(t,\mu)$ vanishes. To treat $I_2(t,\mu)$ and $I_3(t,\mu)$ we use (3.6) again, but now (3.7) takes the form

$$(3.18)\quad \begin{cases} \lambda K(x,t;\lambda) = -\frac{1}{2}e^{\lambda(x-t)}(1+O(\frac{1}{\mu}))+\frac{1}{2}e^{\lambda(t-x)}(1+O(\frac{1}{\mu})) \\ \qquad\qquad + e^{-\lambda(x+t)\pm(\nu+\frac{1}{2})\pi i}\ O(\frac{1}{\mu}) \end{cases}$$

as $\mu \to \infty$, uniformly in t on $t \geqq x$, uniformly in λ on the line segments joining $-i\mu$ with $\lambda_1-i\mu$ and $i\mu$ with $\lambda_1+i\mu$; the upper or lower sign in the exponential has to be taken as $\operatorname{Im}\lambda \gtrless 0$. This can be seen from (1.14), (1.17) and (1.32). Hence

$$(3.19)\quad \begin{cases} I_2(t,\mu)+I_3(t,\mu) = \frac{1}{x-t}\,[\cos(\mu+i\lambda_1)(t-x)-\cos(\mu-i\lambda_1)(t-x)] \\ \qquad\qquad + e^{\lambda_1(t-x)}\ O(\frac{1}{\mu}) \end{cases}$$

as $\mu \to \infty$, uniformly in t on $t \geqq x$. Using the lemma of Riemann-Lebesgue and (0.9) it follows that

$$(3.20)\qquad \lim_{\mu\to\infty}\int_x^\infty f(t)dt \int_{\lambda_1-i\mu}^{\lambda_1+i\mu} \lambda K(x,t;\lambda)d\lambda = 0.$$

Changing the order of integration in (3.20) is justified by (3.6), (1.14), (1.17) and (0.9); this shows (3.17). This completes the proof of formula (0.10).

4. Some applications

Example A. The simplest application of the preceding theory may be obtained, when we consider the differential equation

$$(4.1)\qquad \frac{d^2y}{dx^2} - \lambda^2 y = 0, \quad x > 0.$$

Here we may take $e(x,\lambda) = e^{-\lambda x}$, while we find $\tilde{e}(x,\lambda) = \cosh \lambda x$ or $\tilde{e}(x,\lambda) = \frac{1}{\lambda}\sinh \lambda x$, depending on the initial conditions (0.11) of (0.12) respectively.

Applying theorem 1 in these two cases and adding the results, we obtain a complex inversion formula for transforms related to (4.1).

Theorem 2. Let the conditions on x,λ_1 and $f(t)$ of theorem 1 be satisfied. If (0.7) holds, then

$$\lim_{\mu\to\infty} \frac{1}{2\pi i} \int_{\lambda_1-i\mu}^{\lambda_1+i\mu} e^{\lambda x}\, d\lambda \int_0^\infty f(t)e^{-\lambda t}\, dt = \tfrac{1}{2}\{f(x-0)+f(x+0)\}. \tag{4.2}$$

If (0.9) holds, then

$$\lim_{\mu\to\infty} \frac{1}{2\pi i} \int_{\lambda_1-i\mu}^{\lambda_1+i\mu} e^{-\lambda x} d\lambda \int_0^\infty f(t)e^{\lambda t} dt = \tfrac{1}{2}\{f(x-0)+f(x+0)\}. \tag{4.3}$$

The inversion formula (4.2) for the one-sided Laplace transform can be found in WIDDER [16, Ch. II, Theorem 7.3] under somewhat more general conditions on x and λ_1.

Example B. Taking $q(x) \equiv 0$ in (0.2) we obtain the differential equation (1.1) for the modified Bessel functions. In one of the foregoing sections their properties have been shown and the following theorem is immediate.

Theorem 3. Let the conditions on x, λ_1 and $f(t)$ of theorem 1 be satisfied and let $-\frac{1}{2} \leqq \operatorname{Re} \nu \leqq \frac{1}{2}$.

If (0.7) holds, then

$$\left\{\begin{aligned} &\lim_{\mu\to\infty} \frac{1}{\pi i} \int_{\lambda_1-i\mu}^{\lambda_1+i\mu} (\lambda x)^{\frac{1}{2}} I_\nu(\lambda x) d\lambda \int_0^\infty f(t)(\lambda t)^{\frac{1}{2}} K_\nu(\lambda t) dt = \\ &\qquad \tfrac{1}{2}\{f(x-0)+f(x+0)\}. \end{aligned}\right. \tag{4.4}$$

If (0.9) holds, then

$$\left\{\begin{aligned} &\lim_{\mu\to\infty} \frac{1}{\pi i} \int_{\lambda_1-i\mu}^{\lambda_1+i\mu} (\lambda x)^{\frac{1}{2}} K_\nu(\lambda x) d\lambda \int_0^\infty f(t)(\lambda t)^{\frac{1}{2}} I_\nu(\lambda t) dt = \\ &\qquad \tfrac{1}{2}\{f(x-0)+f(x+0)\}. \end{aligned}\right. \tag{4.5}$$

The formula (4.4) is essentially due to MEYER [11, Satz 1], while (4.5) seems to be new, although KOH and ZEMANIAN [9] have proved a formula similar to (4.5) for certain generalized functions.

Example C. We now turn to integral transforms with a hypergeometric function as kernel. The theory of this type of transform acting on $L_2(0,\infty)$ functions has been given by TITCHMARSH [14, section 4.18], while we will treat this transform under L_1 conditions. Therefore we consider the hypergeometric differential equation (see [14]):

$$(4.6) \qquad X(X+1)\frac{d^2Y}{dX^2} + \{\gamma + (\alpha+1)X\}\frac{dY}{dX} - (\lambda^2 - \tfrac{1}{4}\alpha^2)Y = 0, \qquad X > 0,$$

which has solutions

$$Y_1 = X^{-a}\, F(a, 1+a-c; 1+a-b; -\tfrac{1}{X}), \quad Y_2 = F(a,b;c;-X)$$

where $a=\lambda+\frac{1}{2}\alpha$, $b=-\lambda+\frac{1}{2}\alpha$ and $c=\gamma$. Putting

$$X = \sinh^2 \tfrac{1}{2}x, \quad Y = r(x)y,$$

then the equation (4.6) becomes

$$(4.7) \qquad \frac{d^2y}{dx^2} - \left\{\lambda^2 + \frac{(\gamma-1)^2 - \frac{1}{4}}{x^2} + q(x)\right\} y = 0$$

where

$$q(x) = \frac{2(\alpha-1)(2\gamma-\alpha-1)\cosh x + 2\alpha^2 - 4\gamma\alpha + (1-2\gamma)^2}{4\sinh^2 x} - \frac{(\gamma-1)^2 - \frac{1}{4}}{x^2}$$

and

$$r(x) = \left(\frac{e^x - 1}{e^x + 1}\right)^{\frac{1}{2}+\frac{1}{2}\alpha-\gamma} \sinh^{-\frac{1}{2}\alpha} x.$$

The function $q(x)$ thus obtained is continuous for positive x and $q(x) \in L_1(0,\infty)$. The solution y_1 corresponding to Y_1 has an asymptotic behaviour

$$y_1(x,\lambda) \sim Ae^{-\lambda x} \text{ as } x \to \infty,$$

while the function y_2, corresponding to Y_2 behaves as

$$y_2(x,\lambda) \sim Bx^{\gamma-\frac{1}{2}} \text{ as } x \downarrow 0,$$

where A and B are parameters, not depending on x. Thus if $\frac{1}{2} \leqq \operatorname{Re}\gamma \leqq \frac{3}{2}$ the solutions $y_1(x,\lambda)$ and $y_2(x,\lambda)$ of (4.7) satisfy the conditions (0.5) and (0.6) respectively. Using some well-known properties of the hypergeometric function (cf. ERDELYI [3,2.10(2)]) we find for the Wronskian of y_1 and y_2:

$W_x[y_1(x,\lambda),y_2(x,\lambda)] =$

$$= \frac{1}{\{r(x)\}^2} W_x[\sinh^{-2\alpha}\tfrac{1}{2}x F(a,1+a-c;1+a-b;-\sinh^{-2}\tfrac{1}{2}x),\ F(a,b;c;-\sinh^2\tfrac{1}{2}x)]$$

$$= \frac{1}{\{r(x)\}^2} W_x\ [X^{-a}F(a,1+a-c;1+a-b;-\tfrac{1}{X}),\ F(a,b;c;-X)]$$

$$= \frac{1}{\{r(x)\}^2} \{X(X+1)\}^{\frac{1}{2}} W_X \Big[X^{-a}F(a,1+a-c;1+a-b;-\tfrac{1}{X}),$$

$$X^{-a}\frac{\Gamma(c)\Gamma(b-a)}{\Gamma(c-a)\Gamma(b)} F(a,1+a-c;1+a-b;-\tfrac{1}{X}) + X^{-b}\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(c-b)\Gamma(a)} \cdot F(b,1+b-c;1+b-a;-\tfrac{1}{X})\Big]$$

$$= \frac{1}{\{r(x)\}^2} \{X(X+1)\}^{\frac{1}{2}} \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(c-b)\Gamma(a)} \cdot$$

$$W_X\Big[X^{-a}F(a,1+a-c;1+a-b;-\tfrac{1}{X}),\ X^{-b}F(b,1+b-c;1+b-a;-\tfrac{1}{X})\Big]$$

$$\sim \frac{1}{\{r(x)\}^2} \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(c-b)\Gamma(a)} (a-b)X^{-a-b} \sim 2^{a+b}\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(a-b)$$

as $X \to \infty$. Since $W_x[y_1(x,\lambda),y_2(x,\lambda)]$ is independent of x, we obtain

$$W_x\ [y_1(x,\lambda),y_2(x,\lambda)] = 2^{a+b}\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(a-b) = 2^{\alpha+1}\frac{\Gamma(\gamma)\Gamma(2\lambda)}{\Gamma(\lambda+\frac{1}{2}\alpha)\Gamma(\lambda-\frac{1}{2}\alpha+\gamma)}\lambda\ .$$

We are now ready to formulate our theorem.

Theorem 4. Let the conditions on x, λ_1 and f(t) of theorem 1 be satisfied and let $\frac{1}{2} \leqq \operatorname{Re}\gamma \leqq \frac{3}{2}$. If (0.7) holds, then

$$\text{(4.8)}\quad \left\{ \begin{aligned} &\lim_{\mu\to\infty}\frac{1}{\pi i}\int_{\lambda_1-i\mu}^{\lambda_1+i\mu}\frac{\Gamma(\lambda+\frac{1}{2}\alpha)\Gamma(\lambda+\gamma-\frac{1}{2}\alpha)}{\Gamma(2\lambda)}\,y_2(x,\lambda)d\lambda\int_0^\infty f(t)y_1(t,\lambda)dt \\ &\qquad = 2^\alpha\ \Gamma(\gamma)\ \{f(x-0)+f(x+0)\}. \end{aligned}\right.$$

If (0.9) holds, then

$$\text{(4.9)}\quad \left\{ \begin{aligned} &\lim_{\mu\to\infty}\frac{1}{\pi i}\int_{\lambda_1-i\mu}^{\lambda_1+i\mu}\frac{\Gamma(\lambda+\frac{1}{2}\alpha)\Gamma(\lambda+\gamma-\frac{1}{2}\alpha)}{\Gamma(2\lambda)}\,y_1(x,\lambda)d\lambda\int_0^\infty f(t)y_2(t,\lambda)dt \\ &\qquad = 2^\alpha\ \Gamma(\gamma)\ \{f(x-0)+f(x+0)\}. \end{aligned}\right.$$

Example D. Our last example deals with the generalized differential equation of Legendre

$$(4.10)\qquad (1-u^2)\,\frac{d^2w}{du^2} -2u\,\frac{dw}{du} +\left\{k(k+1) - \frac{m^2}{2(1-u)} - \frac{n^2}{2(1+u)}\right\} w=0, \qquad u > 1,$$

of which linearly independent solutions $P_k{}^{m,n}(u)$ and $Q_k{}^{m,n}(u)$ have been defined by KUIPERS and MEULENBELD [6]. After the substitution $u=\cosh x$ and some transformations (4.10) becomes

$$(4.11)\qquad \frac{d^2y}{dx^2} - \left\{(k+\tfrac{1}{2})^2 + \frac{m^2-\frac{1}{4}}{x^2} + q(x)\right\} y=0,$$

where

$$q(x) = \frac{2(m^2-n^2)\cosh x +2(m^2+n^2)-1}{4\sinh^2 x} - \frac{m^2-\frac{1}{4}}{x^2}$$

and

$$y(x) = (\sinh x)^{\frac{1}{2}}\, w(\cosh x).$$

Hence the function $q(x)$ is continuous for positive values of x and $q(x) \in L_1(0,\infty)$. The equation (4.11) is satisfied by $(\sinh x)^{\frac{1}{2}} P_k{}^{m,n}(\cosh x)$ and $(\sinh x)^{\frac{1}{2}} Q_k{}^{m,n}(\cosh x)$. From KUIPERS and MEULENBELD [8, (1) and (7)] it can be seen that

$$e^{\pi im}\,(\sinh x)^{\frac{1}{2}}\, Q_k^{-m,-n}(\cosh x) \sim Ae^{-(k+\frac{1}{2})x} \text{ as } x \to \infty,$$

and

$$(\sinh x)^{\frac{1}{2}}\, P_k^{m,n}\,(\cosh x) \sim Bx^{\frac{1}{2}-m} \text{ as } x \downarrow 0,$$

where A and B parameters, not depending on x; cf. (0.5) and (0.6).

In order to find the Wronskian of the solutions we refer to MEULENBELD [10, (8)]:

$$W_z[P_k^{m,n}(z),\ Q_k^{m,n}(z)] =$$

$$= e^{\pi im}\, 2^{-m+n}\ \frac{\Gamma(k + \frac{m+n}{2} + 1)\ \Gamma(k + \frac{m-n}{2} + 1)}{\Gamma(k - \frac{m+n}{2} + 1)\ \Gamma(k - \frac{m-n}{2} + 1)}\ \frac{1}{1-z^2}$$

(z not lying in the cut $(-\infty,1]$)

and to KUIPERS and MEULENBELD [7, (6)]:

$$e^{\pi im}\, Q_k^{-m,-n}(z) = 2^{m-n}\ \frac{\Gamma(k - \frac{m+n}{2} + 1)\ \Gamma(k - \frac{m-n}{2} + 1)}{\Gamma(k + \frac{m+n}{2} + 1)\ \Gamma(k + \frac{m-n}{2} + 1)}\ e^{-m\pi i}\, Q_k^{m,n}(z).$$

Hence

$$W_x[(\sinh x)^{\frac{1}{2}} e^{\pi im}\, Q_k^{-m,-n}(\cosh x), (\sinh x)^{\frac{1}{2}}\, P_k^{m,n}(\cosh x)] = 1.$$

Theorem 5. Let x and k_1 be real numbers with $x > 0$ and $k_1 > -\frac{1}{2}$. Let $f(t)$ be a function, defined for positive values of t and of bounded variation in a neighbourhood of $t=x$. Let $-\frac{1}{2} \leqq \operatorname{Re} m \leqq \frac{1}{2}$.

If

$$e^{-(k_1+\frac{1}{2})t} f(t) \in L_1(0,\infty), \tag{4.12}$$

then

$$\left\{\begin{aligned} &\lim_{\mu\to\infty} \frac{1}{2\pi i} \int_{k_1-i\mu}^{k_1+i\mu} (2k+1)(\sinh x)^{\frac{1}{2}}\, P_k^{m,n}(\cosh x)dk \\ &\int_0^\infty f(t)\, e^{\pi i m}(\sinh t)^{\frac{1}{2}}\, Q_k^{-m,-n}(\cosh t)dt = \tfrac{1}{2}\{f(x-0)+f(x+0)\}. \end{aligned}\right. \tag{4.13}$$

If

$$e^{(k_1+\frac{1}{2})t} f(t) \in L_1(0,\infty), \tag{4.14}$$

then

$$\left\{\begin{aligned} &\lim_{\mu\to\infty} \frac{1}{2\pi i} \int_{k_1-i\mu}^{k_1+i\mu} (2k+1)e^{\pi i m}(\sinh x)^{\frac{1}{2}}\, Q_k^{-m,-n}(\cosh x)dk \\ &\int_0^\infty f(t)\, (\sinh t)^{\frac{1}{2}}\, P_k^{m,n}\, (\cosh t)dt = \tfrac{1}{2}\{f(x-0)+f(x+0)\}. \end{aligned}\right. \tag{4.15}$$

Restating this result in another way, we obtain a theorem which has been found by BRAAKSMA and MEULENBELD [1, Theorem 1] under somewhat different conditions [1]).

Theorem 5a. Let x and k_1 be real numbers with $x > 1$ and $k_1 > -\frac{1}{2}$. Let $f(t)$ be a function, defined for $t > 1$ and of bounded variation in a neighbourhood of $t=x$. Let $-\frac{1}{2} \leqq \operatorname{Re} m \leqq \frac{1}{2}$.

If

$$(t-1)^{-\frac{1}{4}} f(t) \in L_1(1,a),\; t^{-k_1-1} f(t) \in L_1(a,\infty) \text{ with } a > 1, \tag{4.16}$$

then

$$\left\{\begin{aligned} &\lim_{\mu\to\infty} \frac{1}{2\pi i} \int_{k_1-i\mu}^{k_1+i\mu} (2k+1)P_k^{m,n}(x)dk \int_1^\infty f(t)e^{\pi i m}\, Q_k^{-m,-n}(t)dt \\ &\qquad = \tfrac{1}{2}\{f(x-0)+f(x+0)\}. \end{aligned}\right. \tag{4.17}$$

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1) In their statement of the formula (4.19) the second condition in (4.18) should be added to ensure convergence of the integral with respect to t.

If

$$(t-1)^{-\frac{1}{4}}f(t) \in L_1(1,a), t^{k_1}f(t) \in L_1(a,\infty) \text{ with } a > 1, \tag{4.18}$$

then

$$\left\{ \begin{aligned} &\lim_{\mu\to\infty} \frac{1}{2\pi i} \int_{k_1-i\mu}^{k_1+i\mu} (2k+1)e^{\pi i m} Q_k^{-m,-n}(x)dk \int_1^{\infty} f(t)\ P_k^{m,n}(t)dt \\ &\qquad = \tfrac{1}{2}\{f(x-0)+f(x+0)\}. \end{aligned} \right. \tag{4.19}$$

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SAMENVATTING

Dit proefschrift bestaat uit twee gedeelten, I en II, die onderling onafhankelijk gelezen kunnen worden.

In het eerste gedeelte onderzoeken we de spectraal theorie van de door Braaksma uitgebreide Watson transformaties in $L^2(-\infty,\infty)$. Het spectrum van zo'n transformatie blijkt volledig bepaald te worden door de Mellin getransformeerden van de kern (I, § 3). De resolvente van een Watson transformatie is berekend in § 4 van I. Dit levert de symmetrische opbouw van de theorie voor transformaties die $L^2(-\infty,\infty)$ één-éénduidig op zichzelf afbeelden. Door enige additionele eisen aan de kern op te leggen kunnen we de spectraal ontbinding van de Watson transformatie bepalen (I, § 5), op een manier zoals door Dunford voor convolutie transformaties werd gebruikt. Een van die voorwaarden is dat de transformatie normaal moet zijn; het residu spectrum blijkt altijd leeg te zijn. Een klasse van integraal transformaties wordt gedefinieerd in § 8 van I. De eenzijdige Laplace transformatie is een speciaal geval. Voor transformaties uit deze klasse gelden alle eerder gevonden resultaten. Functies die invariant zijn onder een Watson transformatie komen in § 9 van I aan de orde. Tenslotte beschouwen we in I, § 10 een aantal speciale gevallen die het voorgaande illustreren.

Een geheel andere klasse van integraal transformaties wordt behandeld in het tweede gedeelte. Nu worden kernen beschouwd die oplossingen zijn van zekere tweede orde differentiaalvergelijkingen. De kernen worden bepaald door asymptotische voorwaarden op te leggen in een omgeving van de singuliere punten van de vergelijking (II, § 2). Braaksma, Meulenbeld en Lemei hebben inverse stellingen afgeleid voor het geval dat de differentiaalvergelijking op de gehele reële rechte wordt beschouwd. Wij beperken ons tot de positieve half-as. Als speciaal geval (II, § 4) wordt behandeld een transformatie, die door Braaksma en Meulenbeld is gedefinieerd en waarbij de gegeneralizeerde Legendre functies van Kuipers en Meulenbeld een rol spelen.

Levensbeschrijving.

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STELLINGEN

I. De theorie van unitaire Watson transformaties op locaal compacte groepen kan uitgebreid worden tot meer algemene transformaties.

vgl. R.R. Goldberg, Watson transforms on groups, Ann. Math. 71, 355-362 (1960).

II. De stelling van Bochner en Chandrasekharan, dat iedere unitaire Watson transformatie in $L^2(0,\infty)$ kan worden ontbonden in het product van een "elementaire" Watson transformatie en een begrensde unitaire transformatie die met alle translatie operatoren commuteert, geldt ook omgekeerd.

vgl. S. Bochner en K. Chandrasekharan, Fourier transforms, Annals of Mathematical Studies 19, Princeton, 1949.

III. Voor differentiaal vergelijkingen van het type $y''-(\lambda^2+q(x))y=0$ op $0\leqq x<\infty$ kunnen inversie stellingen voor integraal transformaties worden afgeleid. De kernen van deze transformaties worden gevonden door één oplossing met behulp van randvoorwaarden in ∞ en de andere oplossing met behulp van (eventueel van λ afhankelijke) voorwaarden in 0 te bepalen.

vgl. dit proefschrift, deel II.

IV. De Jackson - de la Vallée Poussin operator T op $L^2(-\infty,\infty)$, gedefinieerd door

$$(Tf)(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin^4(x-t)}{(x-t)^4}\, f(t)dt,$$

heeft een rechts-continue spectraal ontbinding, gegeven door:

$$E_\mu = I - D_{\psi(\mu)}, \quad 0 \leqq \mu \leqq \tfrac{2}{3},$$
$$= I \quad , \quad \mu \geqq \tfrac{2}{3} \quad ,$$
$$= 0 \quad , \quad \mu < 0 \quad .$$

Hierin is $D_{\psi(\mu)}$ de Dirichlet transformatie en $\psi(\mu)$ de oplossing op $2 \leqq x \leqq 4$ van de vergelijking $(4-x)^3 - 48\mu = 0$ als $0 \leqq \mu \leqq \frac{1}{6}$, en de oplossing op $0 \leqq x \leqq 2$ van de vergelijking $x^3-4x^2+16(\frac{2}{3}-\mu) = 0$ als $\frac{1}{6} < \mu \leqq \frac{2}{3}$.

vgl. N.I. Achieser, Vorlesungen über Approximationstheorie, Berlin, 1967.

V. De beschouwingen van Soni over zekere integraal transformaties kunnen eenvoudig uitgebreid worden tot een grotere klasse van integraal transformaties. Het is mogelijk om voor deze transformaties de spectraal ontbinding te geven.

vgl. K. Soni, A unitary transform related to some integral equations, SIAM J. Math. Anal. 1, 426-436 (1970).

VI. De transformatie A gedefinieerd door

$$g(z) = (Af)(z) = \frac{1}{\pi} \int_0^\infty f(t) \frac{t}{t^2-z^2}\, dt, \qquad \text{Im } z > 0,$$

levert een één-één correspondentie tussen $L^2(0,\infty)$ en H^2 (de klasse van functies $\psi(z)$, analytisch in $y > 0$, waarvoor $\int_{-\infty}^{\infty} |\psi(x+iy)|^2 dx \leqq C, z=x+iy$).

vgl. D.V. Widder, A transform related to the Poisson integral for a half-plane, Duke Math. J. 33, 355-362 (1966).

VII. Zij $g(z)$ de functie gedefinieerd in stelling VI en zij $g(x)$ de randwaarde van $g(z)$. De transformatie A uit de vorige stelling kan op eenvoudige wijze geinverteerd worden:

$$f(x) = -i(g(x)-g(-x)), \qquad x > 0.$$

VIII. Het resultaat van Boas dat de Meijertransformatie een één-één correspondentie tussen $L^2(0,\infty)$ en H^2 levert, kan ook bewezen worden door gebruik te maken van een stelling van Braaksma.

vgl. R.P. Boas, Generalized Laplace Integrals, Bull. Am. Math. Soc. 48, 286-294 (1942).
B.L.J. Braaksma, Inversion theorems for some generalized Fourier transforms, Proc. Kon. Ned. Ak. Wet. 28, 275-299 (1966).

IX. Als Re $m < 1$, Re $(k+\frac{m+n}{2}) > -1$, Re $(k+\frac{m-n}{2}) > -1$ en als z niet op $(-\infty,1]$ ligt, dan geldt voor de functies van Kuipers-Meulenbeld de relatie:

$$\frac{(z+1)^{k+\frac{1}{2}m}}{(z-1)^{\frac{1}{2}m}}\, e^{-m\pi i}\, Q_k^{m,n}(z) = \frac{1}{2}\int_{-1}^{1} \frac{(1+x)^{k+\frac{1}{2}m}}{(1-x)^{\frac{1}{2}m}}\, \frac{P_k^{m,n}(x)}{z-x}\, dx.$$

vgl. H.S.V. de Snoo, An extension of Neumann's integral relation for generalized Legendre functions, Duke Math. J. 37, 71-75 (1970).

X. Door Jacobi polynomen te karakterizeren als "spherical harmonics" in q dimensies, invariant onder zekere orthoganale transformaties, vinden Dijksma en Koornwinder voor Re $\alpha > -\frac{1}{2}$, Re $\beta > -\frac{1}{2}$:

$$P_n^{(\alpha,\beta)}(1-2t^2) = 2\frac{(-1)^n}{\sqrt{\pi}}\, \frac{\Gamma(\alpha+\beta+1)\Gamma(n+\alpha+1)}{\Gamma(\alpha+\frac{1}{2})\Gamma(n+\alpha+\beta+1)} \int_0^1 C_{2n}^{\alpha+\beta+1}(ut)(1-u^2)^{\alpha-\frac{1}{2}}\, du.$$

Deze relatie kan ook op elementaire wijze worden afgeleid.

vgl. A. Dijksma en T.H. Koornwinder, Spherical harmonics and the product of two Jacobi polynomials.

XI. Bij de overgang van het systeem van grote hoorcolleges naar het gecombineerd college-instructie systeem, zoals op het ogenblik bij het wiskunde onderwijs aan enkele afdelingen van de T.H. Delft plaats vindt, dient het studie-materiaal aangepast te worden.