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Non-intrusive Reliability Analysis of Multi-layered Slopes in Spatially Variable Soils

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Abstract. Stochastic finite element method and random finite element method can provide rigorous tools for slope reliability analysis incorporating spatial variability of soil properties. However, both of them are difficult to be applied into practice due to the modification of finite-element codes and the low efficiency, respectively. To address these problems, this paper develops a more practical approach called non-intrusive stochastic finite element method (NISFEM) for slope reliability analysis in spatially variable soils. In the NISFEM, the random fields of spatially variable soils are generated using Karhunen-Loeve expansion, and the safety factor of slope stability is calculated using commercial finite-element package. After that, the Hermite polynomial chaos expansion is used to express the safety factor explicitly for slope reliability analysis. In addition, this paper suggests an easy dimension reduction technique to further improve the efficiency of NISFEM, namely, to adopt a relatively small truncated ratio in Karhunen-Loeve expansion. The proposed method is illustrated and verified using a multi-layered soil slope example. Through the sensitivity study, it is found that the vertical spatial variability affects the slope failure probability and the sensitivity of uncertain soil properties significantly.

Keywords. Slope Reliability, Spatial Variability, Non-intrusive Reliability Analysis, Stochastic Finite Element Method, Hermite Polynomial Chaos Expansion

1. Introduction

Spatial variability of geotechnical properties is one of the most significant uncertainties that affect the reliability of slope stability (Li et al., 2014). Stochastic finite element method (SFEM) (Sudret and Der Kiureghian, 2000) and random finite element method (RFEM) (Griffiths and Fenton, 2004, 2009) have provided rigorous tools for slope reliability analysis incorporating spatial variability of soil properties. Both of them are based on FEM which requires no a prior assumption of the location or shape of the critical slip surface (Griffiths and Lane, 1999). However, they are still difficult to be applied into practice. The SFEM fails since it requires modification of deterministic finite-element codes. This is known as "intrusive analysis". Although the RFEM is non-intrusive, the Monte Carlo Simulation used for uncertainty propagation in RFEM needs extensive computational efforts.

A more practical approach to perform finite-element-based reliability analysis in a non-intrusive manner is based on surrogate models or meta-models, which is also referred as non-intrusive SFEM (NISFEM). For example, Jiang et al. (2014) adopted the Hermite polynomial chaos expansion (HPCE) for slope reliability analysis. Nevertheless, surrogate models often suffer from so-called curse of dimensionality in geotechnical reliability analysis considering spatially variable soil properties using random field theory (Vanmarcke, 2010). Some efforts have been made to reduce the uncertainty dimension, such as using sparse polynomial chaos expansion (SPCE) to truncate useless PCE terms (Al-Bittar and Soubra, 2013), and using series expansion methods (e.g., Karhunen-Loeve expansion (KLE), expansion optimal linear estimation (EOLE)) for random field discretization to reduce random variables in uncertainty propagation. These efforts are appreciated but still do not meet the practical demand, particularly when involving reliability analysis of large scale fields in high spatially variable soils.

This study aims to develop a NISFEM for slope reliability analysis considering spatial variability of soil properties and suggest an easy dimension reduction technique to further improve the efficiency. The paper starts with a description of NISFEM, including KLE for modelling the spatial variability of soil properties, finite-element analysis of slope stability, and HPCE used as the surrogate model to express the performance function explicitly. Thereafter, the reliability analysis, including moment analysis, failure probability calculation and global sensitivity analysis, is performed based on the HPCE. Lastly, the proposed method is illustrated through a two-layered soil slope example.

2. Non-intrusive Stochastic Finite Element Method

2.1. Random Field Modeling of Spatially Variable Soils

Random field theory (Vanmarcke, 2010) is adopted in the NISFEM to model the spatial variability of soil properties. Among the commonly-used random field generation technique, the KLE is often used, with the square exponential autocorrelation function to reduce the number of random variables for random field discretization, namely, the uncertainty dimension.

Based on KLE, a stationary lognormal random field $H_k(\boldsymbol{x})$ of k-th uncertain soil property X_k, $k = 1, 2, \ldots, np$, can be expressed as follows (Sudret and Der Kiureghian, 2000):

$$H_k\left(\boldsymbol{x}\right)=\exp\left[\mu_{\ln,k}+\sum_{i=1}^{nt_k}\sigma_{\ln,k}\sqrt{\lambda_{k,i}}\xi_{k,i}\varphi_{k,i}\left(\boldsymbol{x}\right)\right] \tag{1}$$

where $\mu_{\ln,k}$ and $\sigma_{\ln,k}$ are the mean and standard deviation of $\ln(X_k)$, respectively; $\{\xi_{k,i}, i = 1, 2, \ldots, nt_k\}$ is a set of independent standard normal variables; $\{\lambda_{k,i}, i = 1, 2, \ldots, nt_k\}$ and $\{\varphi_{k,i}(\boldsymbol{x}), i = 1, 2, \ldots, nt_k\}$ are the eigenvalues and eigenfunctions of autocorrelation function of X_k, respectively, and they can be calculated using the wavelet-Galerkin technique (Phoon et al., 2002); $\boldsymbol{x}$ is the centroid coordinates of random field mesh that coincides with the finite-element mesh in this study; nt_k is the number of truncated terms, and it is determined to guarantee the truncated ratio $\varepsilon=\sum_{i=1}^{nt_k}\lambda_{k,i}\Big/\sum_{i=1}^{\infty}\lambda_{k,i}$ is larger than a particular ratio ε_0, e.g., 0.95 (Jiang et al., 2014). The total number of random variables of all uncertain soil properties is $nt=\sum_{k=1}^{np}nt_k$.

As pointed out in Sudret and Der Kiureghian (2000), the KLE always under-represents the true variance of the random field. It will behave more significantly as ε decreases. Thus, ε represents the accuracy of random field discretization.

2.2. Finite-Element Analysis of Slope Stability

After a realization of all random fields $\{H_k(\boldsymbol{x}), k = 1, 2, \ldots, np\}$ is generated, it is mapped to the finite-element mesh adopted in the finite-element analysis of slope stability. The safety factor (*FS*) of slope stability can be calculated using shear strength reduction technique, in which the slope failure is defined by the occurrence of non-convergence of solution in finite-element analysis (Griffiths and Lane, 1999).

Note that, in the framework of NISFEM, the finite-element analysis can be easily performed in commercial finite-element package, which is repeatedly invoked to calculate the responses (i.e., *FSs*) and returns them to the reliability analysis. This allows the practitioners to use the NISFEM without being compromised by reliability theory, and also enhances the application of NISFEM in complex slope problems.

2.3. Hermite Polynomial Chaos Expansion as Surrogate Models

To obtain higher computational efficiency on *FS*, the surrogate model, instead of the deterministic finite-element analyses, is often used to express the implicit relationship between *FS* and random variables explicitly. The 2nd order HPCE (Li et al., 2011) is employed herein to express the *FS* as follows:

$$FS\left(\boldsymbol{\xi}\right)=a_0\Gamma_0+\sum_{i=1}^{nt}a_i\Gamma_1\left(\xi_i\right)+\sum_{i=1}^{nt}\sum_{j=1}^{i}a_{ij}\Gamma_2\left(\xi_i,\xi_j\right) \tag{2}$$

where $\boldsymbol{\xi} = \{\xi_{k,i}, k = 1, 2, \ldots, np, i = 1, 2, \ldots, nt_k\}$; Γ_p (p = 0, 1, 2) is Hermite polynomial with p degrees of freedom; a_0, a_i and a_{ij} are unknown coefficients, which can be determined through the regression-based method. Note that the number of unknown coefficients in Eq. (2) is $(nt+1)(nt+2)/2$.

Obviously, the computational efforts depend on the number of unknown coefficients. Some means can be used to reduce the number of unknown coefficients to improve the efficiency of NISFEM, such as truncating some useless PCE terms (e.g., SPCE). A more direct way is to decrease the number of random variables, say *nt*, and this can be easily achieved using a relatively small ε_0 value in KLE. However, the small ε_0 value will, undoubtedly, affect the accuracy of reliability analysis. Their relationship will be explored in the next subsection.

3. Reliability Analysis Based on Hermite Polynomial Chaos Expansion

NISFEM performs reliability analysis based on the HPCE obtained from the previous subsection. Generally, it contains failure probability (P_f) calculation (for slope stability problems, P_f is defined as $P(FS<1)$) and global sensitivity analysis. In this study, the moment analysis on *FS* is also performed.

3.1. Moment analysis on safety factor

Based on the HPCE, it is easy to calculate the first and second moment (i.e., mean and variance) of *FS* using First Order Second Moment (FOSM) method. Note that Eq. (2) can be rewritten as

$$FS(\boldsymbol{\xi}) = b_0 + \sum_{i=1}^{nt} b_i \xi_i + \sum_{i=1}^{nt}\sum_{j=1}^{i} b_{ij}\xi_i\xi_j \tag{3}$$

where b_0, b_i and b_{ij} are unknown coefficients corresponding to a_0, a_i and a_{ij}. Assume Eq. (3) is the accurate expansion of *FS* when *nt* trends to infinity. Its mean and variance can be estimated as

$$\begin{aligned} &E\left[FS(\boldsymbol{\xi})\right] = b_0 \\ &D\left[FS(\boldsymbol{\xi})\right] \approx \sum_{i=1}^{nt}\left(\frac{\partial FS}{\partial \xi_i}\right)^2 \sigma_{\xi_i}^2 = \sum_{i=1}^{nt} b_i^2 \end{aligned} \tag{4}$$

where $E[FS(\xi)]$ and $D[FS(\xi)]$ are the mean and variance of *FS*, respectively, which are derived from the fact that $\{\xi_i, i = 1, 2, \dots, nt\}$ is a set of independent standard normal variables. In addition, using Eqs. (1) and (3), and the empirical linear assumption on *FS*, i.e., $FS \propto H_k(\boldsymbol{x})$, it can be inferred that

$$|b_i| \propto \sqrt{\lambda_i} \tag{5}$$

Eq. (5) implies that the random variable with a small eigenvalue may have a small coefficient, and consequently has minimal effect on the variance of *FS*. Similar conclusion can be drawn by other series expansion methods (e.g., EOLE) or higher order PCE. Furthermore, the accuracy of variance of *FS* (donated by ζ), which indicates the accuracy of reliability analysis, can be defined as

$$\zeta = \frac{D\left[FS(\boldsymbol{\xi})_{nt}\right]}{D\left[FS(\boldsymbol{\xi})_{\infty}\right]} = \frac{\sum_{i=1}^{nt} b_i^2}{\sum_{i=1}^{\infty} b_i^2} \propto \frac{\sum_{i=1}^{nt}\lambda_i}{\sum_{i=1}^{\infty}\lambda_i} = \varepsilon \tag{6}$$

Eq. (6) indicates that ζ and ε have an approximate linear relationship. Although the FOSM lacks of accuracy on the estimation of P_f, it provides an interesting viewpoint to link the accuracy of random field discretization (i.e., ε) with the accuracy of reliability analysis (i.e., ζ). From this aspect, the accuracy of reliability analysis can be reflected by the selected ε_0 a prior. Therefore, a reasonable ε_0 value should prefer to guarantee sufficient accuracy of reliability analysis rather than sufficient accuracy of random field discretization. The rationality will be verified by the illustrative example in section 4.

Table 1. Summary of statistics of uncertain soil properties (Modified from Kim et al. (2002))

Soil Parameters		Mean	COV	Distribution
Upper layer	c'_1(kPa)	28.7	0.3	Lognormal
	ϕ'_1(°)	20	0.2	Lognormal
Weak layer	c'_2(kPa)	0	/	/
	ϕ'_2(°)	10	0.2	Lognormal

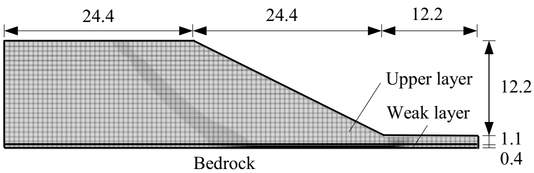

Figure 1. Finite-element mesh of the slope example and its deterministic critical slip surface (Unit: m)

3.2. Subset Simulation for Failure Probability Calculation

The reliability analysis based on the surrogate model, i.e., HPCE, is much easier than that based on deterministic finite-element analyses. Hence, some rigorous methods (e.g., MCS) are feasible. To improve the accuracy at small probability levels, an advanced MCS called Subset Simulation (Au and Wang, 2014) is adopted with a relatively large N (e.g., 10^6) and $p_0 = 0.1$.

3.3. Global Sensitivity Analysis on Sobol's Indices

Global sensitivity of each input random variable, i.e., ξ_i in Eq. (2), can be explored using Sobol's indices based on HPCE (Sudret, 2008). Besides, the sensitivity index of each uncertain soil property X_k, which is of interest in fact, is defined as (Al-Bittar and Soubra, 2013)

$$S_{X_k} = \sum_{\xi_i \leftarrow X_k} S_{T,i} \Big/ \sum_{all} S_{T,i} \tag{7}$$

where $S_{T,i}$ = the total Sobol's index of ξ_i, and its calculation detail can be referred to Sudret (2008). According to the rank of S_{Xk}, the most sensitive soil property can be identified.

4. Illustrative Example

The proposed NISFEM is illustrated using a multi-layered soil slope in this section, as shown in Fig. 1. The example was also studied by Kim et al. (2002) on the deterministic slope stability analysis with limit analysis method. Table 1 summarizes the statistics of uncertain soil properties. In addition, the unit weight, Young's modulus and Poisson's ratio of both layer soils are assumed to be 18.8kN/m^3, 100MPa and 0.3, respectively.

The finite-element model of the slope is created using the commercial finite-element package ABAQUS in this study. Based on the mean values of uncertain soil properties, the deterministic finite-element analysis of slope stability is performed using an elastic-perfectly plastic constitute model with a Mohr-Coulomb failure criterion. The corresponding FS is 1.336, which is between the lower-bound (i.e., 1.25) and upper-bound (i.e., 1.37) using limit analysis method as reported in Kim et al. (2002). As shown in Fig. 1, the deterministic critical slip surface (represented by its plastic strain) is obviously non-circular and passes along the weak layer.

4.1. Nominal Case Study

To incorporate the spatial variability of soil properties into slope reliability analysis, the square exponential autocorrelation function is adopted, with vertical autocorrelation distance l_v and horizontal autocorrelation distance l_h being 4m and 40m, respectively. Fig. 2 shows a typical realization of cohesion random fields and its slope stability analysis results.

As a thumb of rule, ε_0 is take as 0.95. The corresponding nt = 20 (nt_k = 9, 9 and 2 for c'_1, ϕ'_1 and ϕ'_2, respectively), leading to 231 unknown coefficients in the HPCE. A total of 500 runs of deterministic finite-element analyses are then performed based on random samples generated by Latin Hypercube Sampling method to calculate these unknown coefficients. Finally, the P_f is estimated as 0.496% using Subset Simulation. The sensitivity indices $S_{T,i}$ of each random variables ξ_i and S_{Xk} of c'_1, ϕ'_1 and ϕ'_2 are shown in Fig. 3. Obviously, the sensitivity index decays rapidly with nt_k, and c'_1 is most sensitive parameter, followed by ϕ'_2 and ϕ'_1.

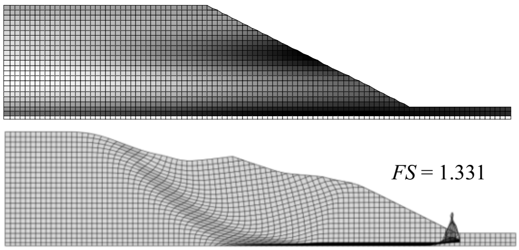

Figure 2. A typical realization of cohesion random fields and its slope stability analysis results

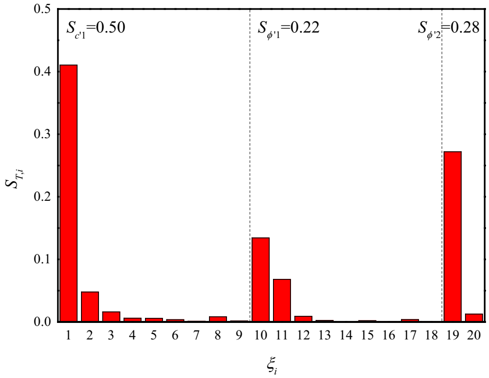

Figure 3. Sensitivity indices of random variables and uncertain soil properties

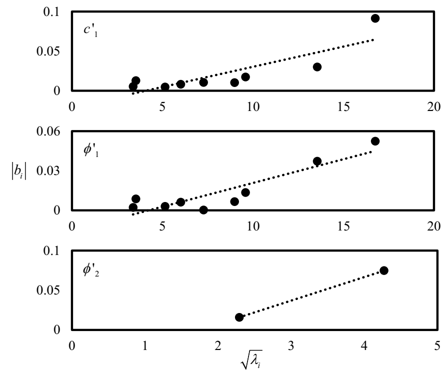

Figure 4. The relationship between $|b_i|$ and $\sqrt{\lambda_i}$

4.2. Effect of Truncated Ratio ε_0

According to the obtained HPCE, the approximate linear relationship between $|b_i|$ and $\sqrt{\lambda_i}$ (see Eq. (5)) can be verified from Fig. 4. Additionally, different ε values are considered in the nominal case, as shown in Fig. 5, to verify the assumption that the accuracy of random field discretization (i.e., ε) is linked with that of reliability analysis (i.e., ζ) (see Eq. (6)). The result obtained from ε = 0.99 is treated as the exact solution. Definitely, there exists an approximate linear relationship between ε and ζ, and ζ increases more gently in comparison with ε. As a result, when ε = (0.75~0.85), it achieves sufficient accuracy to guarantee ζ > 0.95, and larger ε will not improve the accuracy significantly. Although this depends on the problems studied, it provides a reference to choose a reasonable ε_0.

To further verify the results, six representative ε_0 values (i.e., 0.99, 0.95, 0.9, 0.85, 0.75 and 0.5) are considered in Fig. 6. It can be observed from Fig. 6 that ε_0 has slight effect on failure probability when ε_0 is larger than 0.85. In contrast, ε_0 has minimal effect on Sobol's indices as ε_0 varies from 0.5 to 0.99.

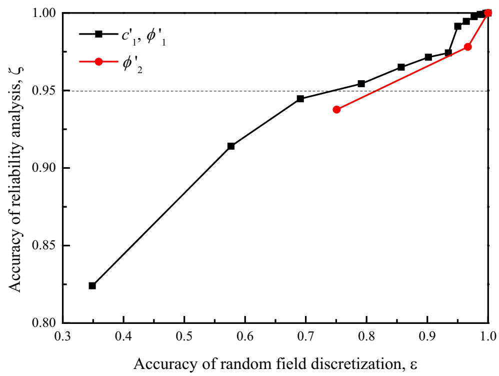

Figure 5. Effect of accuracy of random field discretization on accuracy of reliability analysis

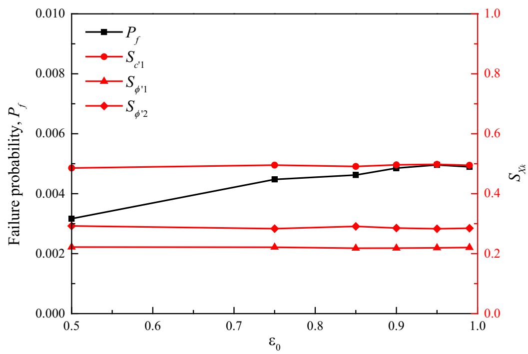

Figure 6. Effect of ε_0 on failure probability and Sobol's indices

In conclusion, ε_0 can be properly loosened (e.g., 0.85) to balance the efficiency and accuracy of the NISFEM. In this example, as ε_0 decreases from 0.95 to 0.85, the *nt* and number of unknown coefficients correspondingly reduce from 20 to 12 and from 231 to 91, respectively, which significantly improves the efficiency but slightly affects the accuracy.

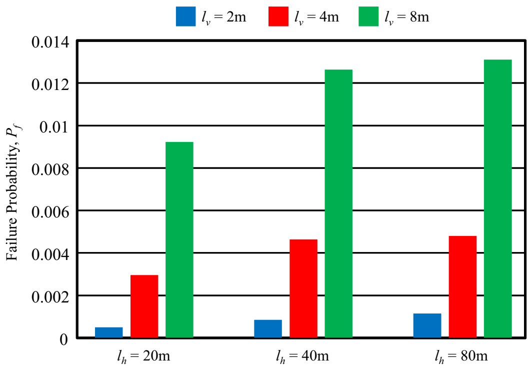

Figure 7. Effects of spatial variability on failure probability

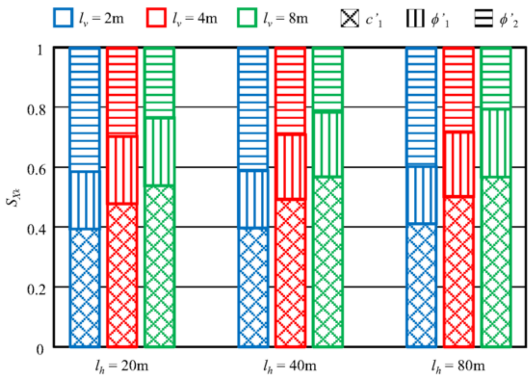

Figure 8. Effects of spatial variability on Sobol's indices

4.3. Effects of Spatial Variability

With the aid of NISFEM, a sensitivity study is performed to explore the effects of spatial variability on the slope reliability analysis. Three typical l_v values (i.e., 2m, 4m and 8m) and three typical l_h values (i.e., 20m, 40m and 80m) are considered, respectively. Note that the ε_0 value is taken as 0.85, as discussed in previous subsection. Consequently, the number of unknown coefficients is reduced by 2~3 times compared with that for $\varepsilon_0 = 0.95$.

As shown in Fig. 7, the vertical spatial variability has more significant effects on slope failure probability than the horizontal spatial variability. As the autocorrelation distance increases (i.e., the spatial variability becomes weaker), the estimated P_f increases. Fig. 8 shows similar results that vertical spatial variability has significant effects on Sobol's indices of uncertain soil properties. For the same l_h, $S_{c'1}$ increases and $S_{\phi'2}$ decreases as l_v increases, while $S_{\phi'1}$ almost remains unchanged. In addition, c'_1 is the most sensitivity parameters, followed by ϕ'_2, in most cases, and ϕ_1' is the least important parameter that affects the slope failure probability since its sensitivity index always ranks the last. The global sensitivity analysis results can provide an important reference for working out effective remedial measures to mitigate slope failure risk.

5. Summary and Conclusion

This paper proposed a non-intrusive stochastic finite element method (NISFEM) for slope reliability analysis in spatially variable soils, which was illustrated through a multi-layered soil slope example. In the NISFEM, the random fields of spatially variable soil properties are discretized using Karhunen-Loeve expansion (KLE), and the safety factor of slope stability is calculated using the commercial finite-element package. This allows the practitioners to use NISFEM without being compromised by reliability theory, and also enhances the application of NISFEM in complex slope problems. The Hermite polynomial chaos expansion is then used to express the *FS* explicitly. Thereafter, the slope reliability analysis, including moment analysis, failure probability calculation and global sensitivity analysis, is performed.

Meanwhile, this paper suggests an easy dimension reduction technique to further improve the efficiency of the NISFEM. Specifically, it suggests a relatively small truncated ratio (ε_0) in KLE, which could lead to a less number of random variables with slight influence on the accuracy. Generally, a proper ε_0 value should be taken to balance the efficiency and the accuracy, and it is recommended to be 0.85 in this study. The rationality is verified through the numerical results in the illustrative example. More rigorous and systematic exploration on the selection of the ε_0 value should be further investigated.

A sensitivity study was also performed using NISFEM to explore the effects of spatial variability on slope reliability analysis. It was found that the vertical spatial variability affects the slope failure probability and the sensitivity of uncertain soil properties significantly.

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