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Stability analysis of nonlinear vibrations of imperfect thin-walled cylinders

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LIST OF SYMBOLS

A	Nondimensional amplitude function of the assumed driven mode
$\bar{A}$	Average value (over one period) of $A_o(t)$
$A(\tau)$	Nondimensional generalized coordinate associated with A; see equation (2)
B	Nondimensional amplitude function of the assumed companion mode
$\bar{B}$	Average value (over one period) of $B_o(t)$
$B(\tau)$	Nondimensional generalized coordinate associated with B; see equation (2)
C	Nondimensional amplitude function of the assumed vibration mode
c	Damping coefficient
E	Young's modulus
F_D	Generalized excitation function; see equation (1-1)
$\bar{F}_D$	Average value of F_D (over one period)
F_{sD}	Generalized excitation function in equation (2-1), F_D/β_2
h	Wall thickness of the shell
L	Length of the shell
R	Radius of the shell
t	Time
$\bar{\alpha}_1, \dots, \bar{\alpha}_{13}$	Coefficients in equation (7-1), defined in Appendix A of Ref [1]
$\alpha_{s1}, \dots, \alpha_{s7}$	Coefficients in equation (2-1), defined in Appendix A
$\bar{\alpha}_{s1}, \dots, \bar{\alpha}_{s6}$	Coefficients in equation (5-1), defined in Appendix B
$\beta_1, \dots, \beta_{10}$	Coefficients defined in Appendix C of Ref. [2]
$\bar{\beta}_1, \dots, \bar{\beta}_{10}$	Coefficients in equation (7-2), defined in Appendix A of Ref. [1]
$\beta_{s1}, \dots, \beta_{s6}$	Coefficients in equation (2-2), defined in Appendix A
$\bar{\beta}_{s1}, \dots, \bar{\beta}_{s6}$	Coefficients in equation (5-2), defined in Appendix B
γ	Nondimensional damping coefficient
γ_s	Percentage of critical damping
$\delta_{n,\ell}$	Kronecker delta function, $\begin{cases} 0 & n \neq \ell \\ 1 & n \neq \ell \end{cases}$

ε	Small parameter defined in Ref [3], $\varepsilon = (\frac{n^2 h}{R})^2$
ν	Poisson's ratio
δ_2	Nondimensional amplitude of the asymmetric imperfection
$\hat{\delta}_2$	Nondimensional amplitude of the radial displacement for the fundamental solution
$\bar{\rho}$	Specific mass density, defined in Appendix B of Ref [1]
τ	Nondimensional time, $(\tau = \bar{W}_{mn} t)$
ϕ	Phase angle of the driven mode
$\bar{\phi}$	Average value of ϕ
ψ	Phase angle of the companion mode
$\bar{\psi}$	Average value of ψ
$\zeta(\tau)$	Small perturbation in the amplitude of the driven mode
$\zeta_1(\tau)$	Slowly varying component of $\zeta(\tau)$; see equation (6-1)
$\zeta_2(\tau)$	Slowly varying component of $\zeta(\tau)$; see equation (6-1)
$\eta(\tau)$	Small perturbation in the amplitude of the companion mode
$\eta_1(\tau)$	Slowly varying component of $\eta(\tau)$; see equation (6-2)
$\eta_2(\tau)$	Slowly varying component of $\eta(\tau)$; see equation (6-2)
χ_1	Frequency parameter of the driven mode, $(\chi_1 = \Omega_s \tau + \bar{\phi})$
χ_2	Frequency parameter of the companion mode $(\chi_2 = \Omega_s \tau + \bar{\psi})$
ω	Vibration frequency
$\bar{\omega}_{mn}$	Natural frequency of the imperfect shell (Linear theory), $\sqrt{\frac{1}{2} \frac{\beta_2 E}{\bar{\rho} R^2}}$
Ω	Nondimensional frequency, $\omega R \sqrt{\frac{2\rho}{E}}$
Ω_s	Nondimensional frequency, $\omega / \bar{\omega}_{mn}$
Δ	Difference of the phase angles, $\phi - \psi$
$\bar{\Delta}$	Average value of Δ
[M]	Matrix used in the stability analysis; defined in Appendix E
[N]	Matrix used in the stability analysis, defined in Appendix E
$\{\phi\}$	Column matrix in equation (9); see equation (9)

INTRODUCTION

The preceeding results (see ref. [1] and [2]) indicate that the frequency-amplitude relationships admit more than one solution for some values of frequency. One solution that is always possible is $\bar{B} = 0$, namely, single mode response, in which case the nodal lines of radial displacement form a stationary spatial pattern and the motion is symmetric about the planes y = 0. We may also sometimes find real, nonzero values of $\bar{B}$ which satisfy the frequency-amplitude relationships. Such solutions represent the traveling wave moving around the circumference resulting in a moving nodal pattern. In order to ascertain whether the single or coupled mode response will actually occur, we must consider the stability of the responses obtained by theoretical analysis.

The first work dealing with the stability of response was done by Evensen for rings in 1964 [4]. He used the method of slowly varying parameters [3] in his investigation. This method was also used later by Ginsberg in the stability analysis of the nonlinear vibration of shells [6].

The purpose of this phase of the present study is to investigate the stability characteristics of the frequency-amplitude relationship derived for various cases, which are presented in ref. [1] and [2]. The method of slowly varying parameters is used. The stability of both the single mode as well as the coupled mode responses are investigated. In order to ensure the necessary accuracy the Newton-Rapson method was used. The results of the calculations are presented and discussed in the ref. [1] and [2]. Only the process of the investigation and the associated equations are presented here

ANALYSIS

It has been reported in ref. [1] that Galerkin's procedure yielded two coupled nonlinear differential equations for A(t) and B(t), which are the amplitude of the driven mode and of the companion mode, respectively. These equations are:

$$\bar{\alpha}_1 \frac{d^2A}{dt^2} + \bar{\alpha}_2 \frac{dA}{dt} + \bar{\alpha}_3 A + \bar{\alpha}_4 \frac{d^2C}{dt^2} [A+\delta_{n,\ell}(\delta_2+\hat{\delta}_2)] + \bar{\alpha}_5 \frac{dC}{dt} [A+\delta_{n,\ell}(\delta_2+\hat{\delta}_2)] + \\ + \bar{\alpha}_6 A^2 + \bar{\alpha}_7 (A^2+B^2) + \bar{\alpha}_8 (A^2-B^2) + \bar{\alpha}_9 A^3 + \bar{\alpha}_{10}(A^2+B^2)A + \bar{\alpha}_{11}(A^2+B^2)A^2 + \\ + \bar{\alpha}_{12}(A^2+B^2)^2 + \bar{\alpha}_{13}(A^2+B^2)^2 A = F_D \qquad (1-1)$$

$$\bar{\beta}_1 \frac{d^2B}{dt^2} + \bar{\beta}_2 \frac{dB}{dt} + \bar{\beta}_3 B + \bar{\beta}_4 \frac{d^2C}{dt^2} + \bar{\beta}_5 B \frac{dC}{dt} + \bar{\beta}_6 AB + \bar{\beta}_7 A^2B + \bar{\beta}_8 (A^2+B^2)B + $$
$$+ \bar{\beta}_9 (A^2+B^2)AB + \bar{\beta}_{10}(A^2+B^2)^2B = 0 \qquad (1\text{-}2)$$

where $\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_{13}$ and $\bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_{10}$ are coefficients which are defined in Appendix A of ref. [1]. Also F_D is the generalized dynamic force defined in ref. [1].

To study stability of the solutions reported in ref. [1] and [2], equations (1) are rewritten as follows:

$$\frac{d^2A}{d\tau^2} + 2\gamma_s \frac{dA}{d\tau} + A + \frac{3}{8}\varepsilon[(\frac{dA}{d\tau})^2 + A\frac{d^2A}{d\tau^2} + (\frac{dB}{d\tau})^2 + B\frac{d^2B}{d\tau^2} +$$
$$+ \delta_{n,\ell}(\delta_2+\hat{\delta}_2)\frac{d^2A}{d\tau^2}][A+\delta_{n,\ell}(\delta_2+\hat{\delta}_2)] +$$
$$+ \frac{3}{4}\gamma_s\varepsilon[A\frac{dA}{d\tau} + B\frac{dB}{d\tau} + \delta_{n,\ell}(\delta_2+\hat{\delta}_2)\frac{dA}{d\tau}][A+\delta_{n,\ell}(\delta_2+\hat{\delta}_2)] + \alpha_{s1}A^2 + \alpha_{s2}B^2 +$$
$$+ \alpha_{s3}A^3 + \alpha_{s4}AB^2 + \alpha_{s5}(A^2+B^2)A^2 + \alpha_{s6}(A^2+B^2)^2 + \alpha_{s7}(A^2+B^2)^2A =$$
$$F_{sD}\cos\Omega_s\tau \qquad (2\text{-}1)$$

$$\frac{d^2B}{d\tau^2} + 2\gamma_s\frac{dB}{d\tau} + \beta_{s1}B + \frac{3}{8}\varepsilon B[(\frac{dA}{d\tau})^2 + A\frac{d^2A}{d\tau^2} + (\frac{dB}{d\tau})^2 + B\frac{d^2B}{d\tau^2} + \delta_{n,\ell}(\delta_2+\hat{\delta}_2)\frac{d^2A}{d\tau^2}] +$$
$$+ \frac{3}{4}\gamma_s\varepsilon B[A\frac{dA}{d\tau} + B\frac{dB}{d\tau} + \delta_{n,\ell}(\delta_2+\hat{\delta}_2)\frac{dA}{d\tau}] + \beta_{s2}AB + \beta_{s3}A^2B + \beta_{s4}B^3 +$$
$$+ \beta_{s5}(A^2+B^2)AB + \beta_{s6}(A^2+B^2)^2B = 0 \qquad (2\text{-}2)$$

where Ω_s, γ_s, τ and F_{sD} are nondimensional frequency, damping, time and excitation, defined as follows:

$$\Omega_s = \omega/\bar{\omega}_{mn}$$
$$\gamma_s = \frac{c}{2\bar{\rho}\bar{\omega}_{mn}}$$
$$\tau = \bar{\omega}_{mn}t$$
$$F_{sD} = F_D/\beta_2$$

$\bar{\omega}^2_{mn} = \frac{1}{2}\frac{\beta_2 E}{\bar{\rho}R^2}$ is the linear natural frequency of free vibration of the imperfect shell, and

$$\varepsilon = (\frac{n^2 h}{R})^2$$

The coefficients $\alpha_{s1}, \alpha_{s2}, \ldots, \alpha_{s7}$ and $\beta_{s1}, \beta_{s2}, \ldots, \beta_{s6}$ are defined in Appendix A. β_2 is defined in Appendix C of Ref. [2].

As a test of the stability of the response, the following small perturbations $\zeta(\tau)$ and $\eta(\tau)$ are introduced

$$A(\tau) = \bar{A}\cos\chi_1 + \zeta(\tau) \tag{3-1}$$

$$B(\tau) = \bar{B}\cos\chi_2 + \eta(\tau) \tag{3-2}$$

where

$$\chi_1 = \Omega_s\tau + \bar{\phi} \tag{4-1}$$

$$\chi_2 = \Omega_s\tau + \bar{\psi} \tag{4-2}$$

The $\bar{\phi}$ and $\bar{\psi}$ are the average value (over one period) of the phase angle defined in [1].

These expressions are then substituted into equations (1-1) and (1-2). The first order terms in the perturbations are retained. This procedure results in two coupled differential equations for $\zeta(\tau)$ and $\eta(\tau)$, namely:

$$\bar{\alpha}_{s1}\frac{d^2\zeta}{d\tau^2} + \bar{\alpha}_{s2}\frac{d\zeta}{d\tau} + \bar{\alpha}_{s3}\zeta + \bar{\alpha}_{s4}\frac{d^2\eta}{d\tau^2} + \bar{\alpha}_{s5}\frac{d\eta}{d\tau} + \bar{\alpha}_{s6}\eta = 0 \tag{5-1}$$

$$\bar{\beta}_{s1}\frac{d^2\eta}{d\tau^2} + \bar{\beta}_{s2}\frac{d^2\eta}{d\tau} + \bar{\beta}_{s3}\eta + \bar{\beta}_{s4}\frac{d^2\zeta}{d\tau^2} + \bar{\beta}_{s5}\frac{d\zeta}{d\tau} + \bar{\beta}_{s6}\zeta = 0 \tag{5-2}$$

where the coefficients $\bar{\alpha}_{s1}, \bar{\alpha}_{s2}, \ldots, \bar{\alpha}_{s6}$ and $\bar{\beta}_{s1}, \bar{\beta}_{s2}, \ldots, \bar{\beta}_{s6}$ are defined in Appendix B. The details of derivation of the equations (5) can be found in Appendix C.

It is obvious that no closed form solutions of these equations are known. However, approximate solutions can be obtained by using numerical integration procedures directly or indirect numerical procedures. In the present analysis, the Method of Slowly Varying Perturbations [3] is employed. In this method, the perturbations $\zeta(\tau)$ and $\eta(\tau)$ are assumed in the form

$$\zeta(\tau) = \zeta_1(\tau)\cos\chi_1 + \zeta_2(\tau)\sin\chi_1 \tag{6-1}$$

$$\eta(\tau) = \eta_1(\tau)\sin\chi_2 + \eta_2(\tau)\cos\chi_2 \tag{6-2}$$

where ζ_1, ζ_2, η_1 and η_2 are assumed to be slowly varying functions of τ. Then the derivatives $\frac{d\zeta}{d\tau}$, $\frac{d^2\zeta}{d\tau^2}$, $\frac{d\eta}{d\tau}$ and $\frac{d^2\eta}{d\tau^2}$ are replaced by

$$\frac{d\zeta}{d\tau} = -\zeta_1\Omega_s\sin\chi_1 + \zeta_2\Omega_s\cos\chi_1 \tag{7-1}$$

$$\frac{d\eta}{d\tau} = \eta_1\Omega_s\cos\chi_2 - \eta_2\Omega_s\sin\chi_2 \tag{7-2}$$

$$\frac{d^2\zeta}{d\tau^2} = -\zeta_1\Omega_s^2\cos\chi_1 - \zeta_2\Omega_s^2\sin\chi_1 - \frac{d\zeta_1}{d\tau}\Omega_s\sin\chi_1 + \frac{d\zeta_2}{d\tau}\Omega_s\cos\chi_1 \tag{7-3}$$

$$\frac{d^2\eta}{d\tau^2} = \frac{d\eta_1}{d\tau}\Omega_s\cos\chi_2 - \frac{d\eta_2}{d\tau}\Omega_s\sin\chi_2 - \eta_1\Omega_s^2\sin\chi_2 - \eta_2\Omega_s^2\cos\chi_2 \tag{7-4}$$

together with the auxiliary conditions

$$\frac{d\zeta_1}{d\tau}\cos\chi_1 + \frac{d\zeta_2}{d\tau}\sin\chi_1 = 0 \tag{8-1}$$

$$\frac{d\eta_1}{d\tau}\sin\chi_2 + \frac{d\eta_2}{d\tau}\cos\chi_2 = 0 \tag{8-2}$$

These expressions for the derivatives are then substituted into equations (5) and the procedure described in Appendix D is used. This procedure yields four linear differential equations for $\bar{\zeta}_1$, $\bar{\zeta}_2$, $\bar{\eta}_1$ and $\bar{\eta}_2$, which can be put in matrix form as follows:

$$[M]\{\phi\} = [N]\{\frac{d\phi}{d\tau}\} \tag{9}$$

where

$$\{\phi\} = \begin{Bmatrix} \bar{\zeta}_1 \\ \bar{\zeta}_2 \\ \bar{\eta}_1 \\ \bar{\eta}_2 \end{Bmatrix}$$

and [M] and [N] are 4x4 matrices respectively. The elements that are contained in these matrices are functions of $\bar{A}$, $\bar{B}$, Ω and Υ, which are defined in Appendix E. Equations (9) are derived for the case of the coupled mode response. The equations for the case of single mode response can be obtained easily from this matrix equation by letting $\bar{B} = 0$ in the matrix elements and replacing $\sin 2\bar{\Delta}$ and $\cos 2\bar{\Delta}$ by $\sin\bar{\phi}$ and $\cos\bar{\phi}$ respectively as shown in Appendix E.

It is obvious that $\{\phi\} = \{\phi_0\}\, e^{\lambda\tau}$ is a possible solution of the matrix equation (9), where $\{\phi_0\}$ is a constant column matrix whose matrix elements will be determined.

Substituting $\{\phi\} = \{\phi_0\}\, e^{\lambda\tau}$ into equation (9) leads to a standard eigenvalue problem for determining the λ's,

$$|N-\lambda M| = 0 \tag{10}$$

It is demonstrated in Appendix E that the matrices [M] and [N] are nonsymmetric. That means that the solutions of equation (10) are complex. If any of the λ's have a positive real part, then the corresponding perturbations will increase exponentially with time. In this case, the associated response is said to be unstable; conversely, if none of the λ's have a positive real part, the response is said to be stable.

Stability of both the single mode response as well as the coupled mode response for various values of damping, excitation and shell geometry were investigated by using equation (10). In order to ensure the required accuracy in the present analysis, the Newton-Raphson method was used to refine the values of $\bar{A}$ and $\bar{B}$ obtained by cross-plotting. These refined values are then substituted into equation (10) along with the Ω, Υ and associated parameters $\sin 2\bar{\Delta}$ and $\cos 2\bar{\Delta}$. For each case the eigenvalues were examined to determine whether or not they had a positive real part. In this manner, the stability of the responses plotted in ref. [1] and [2] was determined. The results of the calculations are discussed in ref. [1] and [2], respectively.

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APPENDIX A COEFFICIENTS OF EQUATIONS (2)

$$\alpha_{s1} = (\beta_{15} + \beta_{16} + \beta_{17})/\beta_2$$
$$\alpha_{s2} = (\beta_{16} - \beta_{17})/\beta_2$$
$$\alpha_{s3} = \frac{4}{3}\beta_3/\beta_2$$
$$\alpha_{s4} = 4\beta_4/\beta_2$$
$$\alpha_{s5} = \beta_{110}/\beta_2$$
$$\alpha_{s6} = \beta_{111}/\beta_2$$
$$\alpha_{s7} = 8\beta_5/\beta_2$$

$$\beta_{s1} = \beta_7/\beta_2$$
$$\beta_{s2} = \beta_{25}/\beta_2$$
$$\beta_{s3} = 4\beta_8/\beta_2$$
$$\beta_{s4} = \frac{4}{3}\beta_9/\beta_2$$
$$\beta_{s5} = \beta_{28}/\beta_2$$
$$\beta_{s6} = 8\beta_{10}/\beta_2$$

where β_2, β_3, ..., β_{10} are defined in Appendix C of Ref. [2]. β_{15}, β_{16}, β_{17}, β_{25}, β_{28}, β_{110} and β_{111} are defined as follows:

$$\beta_{15} = \frac{2R^2}{Eh^2}\bar{c}_5$$
$$\beta_{16} = \frac{2R^2}{Eh^2}\bar{c}_6$$
$$\beta_{17} = \frac{2R^2}{Eh^2}\bar{c}_7$$
$$\beta_{110} = \frac{2R^2}{Eh^2}\bar{c}_{10}$$
$$\beta_{111} = \frac{2R^2}{Eh^2}\bar{c}_{11}$$
$$\beta_{25} = \frac{2R^2}{Eh^2}\bar{d}_4$$

$$\beta_{28} = \frac{2R^2}{Eh^2} \bar{d}_7$$

where $\bar{c}_5$, $\bar{c}_6$, $\bar{c}_7$, $\bar{c}_{10}$, $\bar{c}_{11}$, $\bar{d}_4$ and $\bar{d}_7$ are defined in Appendix C of Ref. [2].

APPENDIX B COEFFICIENTS OF EQUATIONS (5)

$$\bar{\alpha}_{s1} = 1 + \frac{3}{8}\,\varepsilon[\bar{A}^2\cos^2\chi_1 + 2\delta_{n,\ell}(\delta_2+\hat{\delta}_2)\,\bar{A}\,\cos\chi_1 + \delta_{n,\ell}(\delta_2+\hat{\delta}_2)^2]$$

$$\begin{aligned}\bar{\alpha}_{s2} = {} & 2\gamma_s - \frac{3}{4}\,\varepsilon\Omega_s\bar{A}^2\sin\chi_1[\cos\chi_1 + \delta_{n,\ell}(\delta_2+\hat{\delta}_2)] + \frac{3}{4}\,\gamma_s\varepsilon\bar{A}[\bar{A}\cos^2\chi_1 + \\ & + 2\delta_{n,\ell}(\delta_2+\hat{\delta}_2)\cos\chi_1 + \delta_{n,\ell}(\delta_2+\hat{\delta}_2)^2]\end{aligned}$$

$$\begin{aligned}\bar{\alpha}_{s3} = {} & 1 + \frac{3}{8}\,\varepsilon[\Omega_s^2\bar{A}^2(\sin^2\chi_1 - 2\cos^2\chi_1) + \Omega_s^2\bar{B}^2(\cos^2\chi_2 - \sin^2\chi_2) + \\ & - 2\Omega_s^2\bar{A}\delta_{n,\ell}(\delta_2+\hat{\delta}_2)\cos\chi_1] - \frac{3}{4}\,\gamma_s\varepsilon\Omega_s[\bar{A}^2\sin(2\chi_1) - \frac{1}{2}\,\bar{B}^2\sin(2\chi_2) + \\ & - 2\bar{A}\delta_{n,\ell}(\delta_2+\hat{\delta}_2)\sin\chi_1] + 2\alpha_{s1}\bar{A}\cos\chi_1 + 3\alpha_{s3}\bar{A}^2\cos^2\chi_1 + \alpha_{s4}\bar{B}^2\sin^2\chi_2 + \\ & + \alpha_{s5}[4\bar{A}^3\cos^3\chi_1 + \bar{B}^2\sin^2\chi_2 + 2\bar{A}\bar{B}^2\sin^2\chi_2\cos\chi_1] + \\ & + 2\alpha_{s6}[2\bar{A}^3\cos^3\chi_1 + \bar{B}^2\sin^2\chi_2 + 2\bar{A}\bar{B}^2\sin^2\chi_2\cos\chi_1] + \\ & + \alpha_{s7}[5\bar{A}^4\cos^4\chi_1 + 6\bar{A}^2\bar{B}^2\sin^2\chi_2\cos^2\chi_1 + \bar{B}^4\sin^4\chi_2]\end{aligned}$$

$$\bar{\alpha}_{s4} = \frac{3}{8}\,\varepsilon\bar{B}[\bar{A}\cos\chi_1\sin\chi_2 + \delta_{n,\ell}(\delta_2+\hat{\delta}_2)\sin\chi_2]$$

$$\begin{aligned}\bar{\alpha}_{s5} = {} & \frac{3}{4}\,\gamma_s\varepsilon\bar{B}[\bar{A}\cos\chi_1\sin\chi_2 + \delta_{n,\ell}(\delta_2+\hat{\delta}_2)\sin\chi_2] + \frac{3}{4}\,\varepsilon\Omega_s\bar{B}[\bar{A}\cos\chi_1\cos\chi_2 + \\ & + \delta_{n,\ell}(\delta_2+\hat{\delta}_2)\cos\chi_2]\end{aligned}$$

$$\begin{aligned}\bar{\alpha}_{s6} = {} & -\frac{3}{8}\,\varepsilon\Omega_s^2\,\bar{B}\sin\chi_2[\bar{A}\cos\chi_1 + \delta_{n,\ell}(\delta_2+\hat{\delta}_2)] + \frac{3}{4}\,\varepsilon\gamma_s\Omega_s\bar{B}\cos\chi_2[\bar{A}\cos\chi_1 + \\ & + \delta_{n,\ell}(\delta_2+\hat{\delta}_2)] + 2\alpha_{s2}\bar{B}\sin\chi_2 + 2\alpha_{s4}\bar{A}\bar{B}\cos\chi_1\sin\chi_2 + 2\alpha_{s5}\bar{A}^2\bar{B}\cos^2\chi_1\sin\chi_2 + \\ & + 4\alpha_{s6}[\bar{B}^3\sin^3\chi_2 + \bar{A}^2\bar{B}\cos^2\chi_1\sin\chi_2] + 4\alpha_{s7}[\bar{A}^3\bar{B}\cos^3\chi_1\sin\chi_2 + \\ & + \bar{A}\bar{B}^3\sin^3\chi_2\cos\chi_1]\end{aligned}$$

$$\bar{\beta}_{s1} = 1 + \frac{3}{8}\varepsilon\bar{B}^2\sin^2\chi_2$$

$$\bar{\beta}_{s2} = 2\gamma_s + \frac{3}{8}\varepsilon\Omega_s\bar{B}^2\sin(2\chi_2) + \frac{3}{4}\gamma_s\varepsilon\bar{B}^2\sin\chi_2$$

$$\begin{aligned}\bar{\beta}_{s3} = {} & \beta_7/\beta_2 + \frac{3}{8}\varepsilon[\Omega_s^2\bar{A}^2(\sin^2\chi_1 - \cos^2\chi_1) + \Omega_s^2\bar{B}^2(\cos^2\chi_2 - \sin^2\chi_2) + \\ & - \Omega_s^2\bar{A}\delta_{n,\ell}(\delta_2+\hat{\delta}_2)\cos\chi_1] + \\ & - \frac{3}{4}\,\gamma_s\varepsilon\Omega_s[\frac{1}{2}\,\bar{A}^2\sin(2\chi_1) - \bar{B}^2\sin(2\chi_2) + \bar{A}\delta_{n,\ell}(\delta_2+\hat{\delta}_2)\sin\chi_1] + \beta_{s2}\bar{A}\cos\chi_1 + \\ & + \beta_{s3}\,\bar{A}^2\cos^2\chi_1 + 3\beta_{s4}\,\bar{B}^2\sin^2\chi_2 + \beta_{s5}[\bar{A}^3\cos^3\chi_1 + 3\bar{A}\bar{B}^2\cos\chi_1\sin^2\chi_2] + \\ & + \beta_{s6}[\bar{A}^4\cos^4\chi_1 + 5\bar{B}^4\sin^4\chi_2 + 6\bar{A}^2\bar{B}^2\cos^2\chi_1\sin^2\chi_2]\end{aligned}$$

APPENDIX C DERIVATION OF EQUATIONS (5)

The details of the process is demonstrated here by deriving the equations (5-1) from equation (1-1).

Substituting equations (3) into equation(1-1) and keeping only the first order terms of the perturbations in the resulting equation, one obtains

$$f_1(\Omega_s,\gamma_s,\bar{A},\bar{B},F_{sD}) + f_2(\Omega_s,\gamma_s,\zeta_1,\zeta_2,\eta_1,\eta_2,\bar{A},\bar{B}) = 0 \qquad \text{(B-1)}$$

where

$$\begin{aligned} f_1 = & -\Omega_s^2\bar{A}\cos\chi_1 + 2\gamma_s(-\Omega_s\bar{A}\sin\chi_1) + \bar{A}\cos\chi_1 + \frac{3}{8}\,\varepsilon[\Omega_s^2\bar{A}^2\sin^2\chi_1 - \Omega_s^2\bar{A}^2\cos^2\chi_1 + \\ & + \Omega_s^2\bar{B}^2\cos^2\chi_2 - \Omega_s^2\bar{B}^2\sin^2\chi_2 + \delta_{n,\ell}(\delta_2+\hat{\delta}_2)(-\Omega_s^2\bar{A}\cos\chi_1)][\bar{A}\cos\chi_1 + \\ & + \delta_{n,\ell}(\delta_2+\hat{\delta}_2)] + \\ & + \frac{3}{4}\gamma_s\varepsilon[-\Omega_s\bar{A}^2\sin\chi_1\cos\chi_1 + \Omega_s\bar{B}^2\sin\chi_2\cos\chi_2 + \delta_{n,\ell}(\delta_2+\hat{\delta}_2)(-\Omega_s\bar{A}\sin\chi_1)] \\ & [\bar{A}\cos\chi_1 + \delta_{n,\ell}(\delta_2+\hat{\delta}_2)] + \alpha_{s1}\bar{A}^2\cos^2\chi_1 + \alpha_{s2}\bar{B}^2\sin^2\chi_2 + \alpha_{s3}\bar{A}^3\cos^3\chi_1 + \\ & + \alpha_{s4}\bar{A}\bar{B}^2\sin^2\chi_2\cos\chi_1 + \alpha_{s5}[\bar{A}^4\cos^4\chi_1 + \bar{A}^2\bar{B}^2\cos^2\chi_1\sin^2\chi_2] + \alpha_{s6}[\bar{A}^4\cos^4\chi_1 + \\ & + \bar{B}^4\sin^4\chi_2 + 2\bar{A}^2\bar{B}^2\cos^2\chi_1\sin^2\chi_2] + \alpha_{s7}[\bar{A}^4\cos^4\chi_1 + \bar{B}^4\sin^4\chi_2 + \\ & + 2\bar{A}^2\bar{B}^2\cos^2\chi_1\sin^2\chi_2]\bar{B}\sin\chi_2 - F_{sD}\cos\Omega\tau \end{aligned} \qquad \text{(B-2)}$$

and

$$f_2 = \bar{\alpha}_{s1}\frac{d^2\zeta}{d\tau^2} + \bar{\alpha}_{s2}\frac{d\zeta}{d\tau} + \bar{\alpha}_{s3}\,\zeta + \bar{\alpha}_{s4}\frac{d^2\eta}{d\tau^2} + \bar{\alpha}_{s5}\frac{d\eta}{d\tau} + \bar{\alpha}_{s6}\,\eta \qquad \text{(B-3)}$$

It is evidence that the function f_1 is identical to zero since equations (3) are its solution. Only f_2 remains. The equation (5-2) can be obtained using a similar procedure.

APPENDIX D DERIVATION OF EQUATION (4)

The purpose of this appendix is to describe the details of the derivation of equation (9).

Substituting expressions for the derivatives (7) into the equations (5) yields two coupled equations:

$$\bar{\alpha}_{s1}\{-\zeta_1\Omega_s^2\cos\chi_1 - \zeta_2\Omega_s^2\sin\chi_1 - \frac{d\zeta_1}{d\tau}\Omega_s\sin\chi_1 + \frac{d\zeta_2}{d\tau}\Omega_s\cos\chi_1\} +$$
$$+ \bar{\alpha}_{s2}\{-\zeta_1\Omega_s\sin\chi_1 + \zeta_2\Omega_s\cos\chi_1\} + \bar{\alpha}_{s3}\{\zeta_1\cos\chi_1 + \zeta_2\sin\chi_1\} +$$
$$+ \bar{\alpha}_{s4}\{\frac{d\eta_1}{d\tau}\Omega_s\cos\chi_2 - \frac{d\eta_2}{d\tau}\Omega_s\sin\chi_2 - \eta_1\Omega_s^2\sin\chi_2 - \eta_2\Omega_s^2\cos\chi_2\} +$$
$$+ \bar{\alpha}_{s5}\{\eta_1\Omega_s\cos\chi_2 - \eta_2\Omega_s\sin\chi_2\} + \bar{\alpha}_{s6}\{\eta_1\sin\chi_2 + \eta_2\cos\chi_2\} = 0 \qquad \text{(D-1)}$$

$$\bar{\beta}_{s1}\{\frac{d\eta_1}{d\tau}\Omega_s\cos\chi_2 - \frac{d\eta_2}{d\tau}\Omega_s\sin\chi_2 - \eta_1\Omega_s^2\sin\chi_2 - \eta_2\Omega_s^2\cos\chi_2\} +$$
$$+ \bar{\beta}_{s2}\{\eta_1\Omega_s\cos\chi_2 - \eta_2\Omega_s\sin\chi_2\} + \bar{\beta}_{s3}\{\eta_1\sin\chi_2 + \eta_2\cos\chi_2\} +$$
$$+ \bar{\beta}_{s4}\{-\zeta_1\Omega_s^2\cos\chi_1 - \zeta_2\Omega_s^2\sin\chi_1 - \frac{d\zeta_1}{d\tau}\Omega_s\sin\chi_1 + \frac{d\zeta_2}{d\tau}\Omega_s\cos\chi_1\} +$$
$$+ \bar{\beta}_{s5}\{-\zeta_1\Omega_s\sin\chi_1 + \zeta_2\Omega_s\cos\chi_1\} + \bar{\beta}_{s6}\{\zeta_1\cos\chi_1 + \zeta_2\sin\chi_1\} = 0 \qquad \text{(D-2)}$$

Equation (D-1) is multiplied by $\sin\chi_1$ and is added to the auxiliary condition (8-1) after the latter has been multiplied by $(-\bar{\alpha}_{s1}\Omega_s\cos\chi_1)$. This procedure yields the following equation:

$$-\bar{\alpha}_{s1}\frac{d\zeta_1}{d\tau}\Omega_s + \bar{\alpha}_{s4}\frac{d\eta_1}{d\tau}\Omega_s\sin\chi_1\cos\chi_2 - \bar{\alpha}_{s4}\frac{d\eta_2}{d\tau}\Omega_s\sin\chi_1\sin\chi_2$$
$$= \{\frac{1}{2}\bar{\alpha}_{s1}\Omega_s^2\sin 2\chi_1 + \bar{\alpha}_{s2}\Omega_s\sin^2\chi_1 - \frac{1}{2}\bar{\alpha}_{s3}\sin 2\chi_1\}\zeta_1 +$$
$$+ \{\bar{\alpha}_{s1}\Omega_s^2\sin^2\chi_1 - \frac{1}{2}\bar{\alpha}_{s2}\Omega_s\sin 2\chi_1 + \bar{\alpha}_{s3}\sin^2\chi_1\}\zeta_2 +$$
$$+ \{\bar{\alpha}_{s4}\Omega_s^2\sin\chi_2\sin\chi_1 - \bar{\alpha}_{s5}\Omega_s\sin\chi_1\cos\chi_2 - \bar{\alpha}_{s6}\sin\chi_2\sin\chi_1\}\eta_1 +$$
$$+ \{\bar{\alpha}_{s4}\Omega_s^2\sin\chi_1\cos\chi_2 + \bar{\alpha}_{s5}\Omega_s\sin\chi_1\sin\chi_2 - \bar{\alpha}_{s6}\sin\chi_1\cos\chi_2\}\eta_2 \qquad \text{(D-3)}$$

In this state of the analysis, the coefficients $\bar{\alpha}_{s1}$, $\bar{\alpha}_{s2}$, ..., $\bar{\alpha}_{s6}$ are substituted into equation (D-3) and then the resulting equation is "averaged" by integrating τ from 0 to 2π. This procedure yields

$$m_{11}\frac{d\bar{\zeta}_1}{d\tau} + m_{12}\frac{d\bar{\zeta}_2}{d\tau} + m_{13}\frac{d\bar{\eta}_1}{d\tau} + m_{14}\frac{d\bar{\eta}_2}{d\tau} = n_{11}\bar{\zeta}_1 + n_{12}\bar{\zeta}_2 + n_{13}\bar{\eta}_1 + n_{14}\bar{\eta}_2 \qquad \text{(D-4)}$$

Similarly, if both sides of equation (D-1) are multiplied by $\cos\chi_1$ and the equation resulted is added to the auxiliary condition (8-1) after the latter has been multiplied by $(-\bar{\alpha}_{s1}\sin\chi_1)$, and then the average procedure mentioned above is used again, one can obtain a second equation as

$$m_{21}\frac{d\bar{\zeta}_1}{d\tau} + m_{22}\frac{d\bar{\zeta}_2}{d\tau} + m_{23}\frac{d\bar{\eta}_1}{d\tau} + m_{24}\frac{d\bar{\eta}_2}{d\tau} = n_{21}\bar{\zeta}_1 + n_{22}\bar{\zeta}_2 + n_{23}\bar{\eta}_1 + n_{24}\bar{\eta}_2 \qquad \text{(D-5)}$$

Applying the same procedure to equation (D-2) yields another two equations.

$$m_{31}\frac{d\bar{\zeta}_1}{d\tau} + m_{32}\frac{d\bar{\zeta}_2}{d\tau} + m_{33}\frac{d\bar{\eta}_1}{d\tau} + m_{34}\frac{d\bar{\eta}_2}{d\tau} = n_{31}\bar{\zeta}_1 + n_{32}\bar{\zeta}_2 + n_{33}\bar{\eta}_1 + n_{34}\bar{\eta}_2 \qquad \text{(D-6)}$$

and

$$m_{41}\frac{d\bar{\zeta}_1}{d\tau} + m_{42}\frac{d\bar{\zeta}_2}{d\tau} + m_{43}\frac{d\bar{\eta}_1}{d\tau} + m_{44}\frac{d\bar{\eta}_2}{d\tau} = n_{41}\bar{\zeta}_1 + n_{42}\bar{\zeta}_2 + n_{43}\bar{\eta}_1 + n_{44}\bar{\eta}_2 \qquad \text{(D-7)}$$

where $\bar{\zeta}_1$, $\bar{\zeta}_2$, $\bar{\eta}_1$ and $\bar{\eta}_2$ are average values of ζ_1, ζ_2, η_1 and η_2 respectively and m_{ij} are coefficients defined in Appendix E.

These four linear differential equations can be put in matrix form as follows:

$$[M]\{\phi\} = [N]\{\frac{d\phi}{d\tau}\} \qquad \text{(D-8)}$$

APPENDIX E

$$m_{11} = -\Omega_s\{1 + \frac{3}{16}\varepsilon\bar{A}^2 + \frac{3}{8}\varepsilon\delta_{n,\ell}(\delta_2+\hat{\delta}_2)^2\}$$
$$m_{12} = m_{21} = 0$$
$$m_{13} = \frac{3}{64}\varepsilon\Omega_s\bar{A}\bar{B}\cos 2\bar{\Delta}$$
$$m_{14} = m_{13}$$
$$m_{22} = -m_{11}$$
$$m_{23} = -\frac{3}{64}\varepsilon\Omega_s\bar{A}\bar{B}\sin 2\bar{\Delta}$$
$$m_{24} = -\frac{3}{32}\varepsilon\Omega_s\bar{A}\bar{B}(1-\frac{1}{2}\cos 2\bar{\Delta})$$
$$m_{31} = -m_{13}$$
$$m_{32} = m_{23}$$

$$m_{33} = \Omega_s\{1 + \frac{3}{16}\varepsilon\bar{B}^2\}$$
$$m_{34} = m_{43} = 0$$
$$m_{41} = -m_{23}$$
$$m_{42} = -m_{24}$$
$$m_{44} = -m_{33}$$

$$n_{11} = \Omega_s\gamma_s\{1 + \frac{3}{8}\varepsilon[-\frac{3}{4}\bar{A}^2 + \frac{1}{4}\bar{B}^2\cos 2\bar{\Delta} + \delta_{n,\ell}(\delta_2+\hat{\delta}_2)^2]\} - \frac{1}{8}\bar{B}^2\{\frac{3}{4}\varepsilon\Omega_s^2 + \alpha_{s4} + $$
$$+ \alpha_{s7}[3\bar{A}^2 - \bar{B}^2]\}\sin 2\bar{\Delta}$$
$$n_{12} = \frac{3}{8}\bar{A}^2\{\alpha_{s3} - \frac{1}{8}\varepsilon\Omega_s^2\} - \frac{3}{32}\varepsilon\Omega_s^2\bar{B}^2\cos 2\bar{\Delta} + \frac{3}{32}\varepsilon\gamma_s\Omega_s\bar{B}^2\sin 2\bar{\Delta} +$$
$$+ \frac{1}{4}\alpha_{s4}\bar{B}^2(1+\frac{1}{2}\cos 2\bar{\Delta}) + \frac{1}{16}\alpha_{s7}\{5\bar{A}^4 + 6\bar{A}^2\bar{B}^2 + \bar{B}^4(3 + 2\cos 2\bar{\Delta}) +$$
$$- \Omega_s^2\{\frac{1}{2} + \frac{3}{32}\varepsilon\bar{A}^2(1-\frac{1}{2}\cos^2\bar{\Delta}) + \frac{3}{16}\varepsilon\delta_{n,\ell}(\delta_2+\hat{\delta}_2)^2\} + \frac{1}{2}$$
$$n_{13} = -\frac{1}{4}\bar{A}\bar{B}\{\alpha_{s4} + \alpha_{s7}[\bar{A}^2 + 2\bar{B}^2] - \frac{3}{4}\varepsilon\Omega_s^2\}\sin 2\bar{\Delta} + \frac{3}{16}\varepsilon\gamma_s\Omega_s\bar{A}\bar{B}\cos 2\Delta$$
$$n_{14} = \frac{1}{4}\bar{A}\bar{B}\{\alpha_{s4} + \alpha_{s7}[\bar{A}^2 + \bar{B}^2] - \frac{3}{4}\varepsilon\Omega_s^2\}\cos 2\bar{\Delta} + \frac{3}{16}\varepsilon\gamma_s\Omega_s\bar{A}\bar{B}\sin 2\Delta$$
$$n_{21} = \frac{9}{16}\bar{A}^2\{\alpha_{s3} + \frac{1}{4}\varepsilon\Omega_s^2\} - \frac{3}{32}\varepsilon\Omega_s^2\bar{B}^2\cos 2\bar{\Delta} - \frac{3}{32}\varepsilon\Omega_s\gamma_s\bar{B}^2\sin 2\bar{\Delta} +$$
$$+ \frac{1}{4}\alpha_{s4}\bar{B}^2(1-\frac{1}{2}\cos 2\bar{\Delta}) + \frac{1}{16}\alpha_{s7}\{25\bar{A}^4 + 12\bar{A}^2\bar{B}^2(\frac{3}{2} - \cos 2\bar{\Delta})\} + 2\bar{B}^4(\frac{3}{2} - \cos 2\bar{\Delta}) +$$
$$+ \Omega_s^2\{\frac{1}{2} + \frac{9}{64}\varepsilon\bar{A}^2 + \frac{3}{16}\varepsilon\delta_{n,\ell}(\delta_2+\hat{\delta}_2)^2\} + \frac{1}{2}$$

$$n_{22} = -\Omega_s\gamma_s\{1 + \frac{3}{8}\varepsilon[\frac{5}{4}\bar{A}^2 - \frac{1}{4}\bar{B}^2\cos 2\bar{\Delta} + \delta_{n,\ell}(\delta_2+\hat{\delta}_2)^2\} +$$
$$+ \frac{1}{8}\bar{B}^2\{\frac{3}{4}\varepsilon\Omega_s^2 - \alpha_{s4} - \alpha_{s7}[3\bar{A}^2 + \bar{B}^2]\}\sin 2\Delta$$
$$n_{23} = -\frac{1}{2}\bar{A}\bar{B}\{\alpha_{s4}(1-\frac{1}{2}\cos 2\bar{\Delta}) + \alpha_{s7}[\bar{A}^2 + \bar{B}^2](\frac{3}{2} - \cos 2\bar{\Delta}) + \frac{3}{8}\varepsilon\Omega_s^2\cos 2\bar{\Delta}\} +$$
$$+ \frac{3}{16}\varepsilon\gamma_s\Omega_s\bar{A}\bar{B}\sin 2\Delta$$
$$n_{24} = \frac{1}{4}\bar{A}\bar{B}\{\alpha_{s4} + \alpha_{s7}[2\bar{A}^2 + \bar{B}^2] - \frac{3}{4}\varepsilon\Omega_s^2\}\sin 2\bar{\Delta} - \frac{3}{16}\varepsilon\gamma_s\Omega_s\bar{A}\bar{B}\cos 2\Delta$$
$$n_{31} = \frac{1}{4}\bar{A}\bar{B}\{\beta_{s3} + \beta_{s6}[2\bar{A}^2 + \bar{B}^2] - \frac{3}{4}\varepsilon\Omega_s^2\}\sin 2\bar{\Delta} + \frac{3}{16}\varepsilon\gamma_s\Omega_s\bar{A}\bar{B}\cos 2\Delta$$
$$n_{32} = -\bar{A}\bar{B}\{\beta_{s5} + \frac{1}{4}\beta_{s6}[\bar{A}^2 + \bar{B}^2] - \frac{3}{16}\varepsilon\Omega_s^2\}\cos 2\bar{\Delta} + \frac{3}{16}\varepsilon\gamma_s\Omega_s\bar{A}\bar{B}\sin 2\Delta$$
$$n_{33} = \frac{1}{8}\bar{A}^2\{\beta_{s3} + \beta_{s6}[\bar{A}^2+3\bar{B}^2] - \frac{3}{4}\varepsilon\Omega_s^2\}\sin 2\bar{\Delta} + \frac{3}{32}\varepsilon\gamma_s\Omega_s\bar{A}^2\cos 2\Delta +$$
$$- \frac{9}{32}\varepsilon\gamma_s\Omega_s\bar{B}^2 - \gamma_s\Omega_s$$
$$n_{34} = \frac{1}{2}\Omega_s^2\{1 + \frac{3}{16}\varepsilon[\bar{B}^2 + \bar{A}^2\cos 2\bar{\Delta}]\} + \frac{3}{32}\varepsilon\gamma_s\Omega_s\bar{A}^2\sin 2\Delta - \frac{1}{4}\beta_{s3}\bar{A}^2(1+\frac{1}{2}\cos 2\bar{\Delta}) +$$
$$- \frac{1}{8}\beta_{s6}[\bar{A}^4(\frac{3}{2} + \cos 2\bar{\Delta}) + \frac{5}{2}\bar{B}^4 + 3\bar{A}^2\bar{B}^2] - \frac{1}{2}\beta_{s1} - \frac{3}{8}\beta_{s4}\bar{B}^2$$
$$n_{41} = -\frac{3}{16}\varepsilon\bar{A}\bar{B}\{\Omega_s^2\cos 2\bar{\Delta} + \gamma_s\Omega_s\sin 2\bar{\Delta}\} - \frac{1}{2}\beta_{s3}\bar{A}\bar{B}(1-\frac{1}{2}\cos 2\bar{\Delta}) +$$
$$- \frac{1}{2}\beta_{s6}\bar{A}\bar{B}[\bar{A}^2 + \bar{B}^2](\frac{3}{2} - \cos 2\bar{\Delta})$$
$$n_{42} = -\frac{3}{16}\varepsilon\bar{A}\bar{B}\{\Omega_s^2\sin 2\bar{\Delta} - \gamma_s\Omega_s\cos 2\bar{\Delta}\} + \frac{1}{4}\beta_{s3}\bar{A}\bar{B}\sin 2\bar{\Delta} + \frac{1}{4}\beta_{s6}\bar{A}\bar{B}[\bar{A}^2 + 2\bar{B}^2]\sin 2\bar{\Delta}$$
$$n_{43} = \frac{1}{2}\Omega_s^2\{1 + \frac{3}{16}\varepsilon[3\bar{B}^2 - \bar{A}^2\cos 2\bar{\Delta}]\} - \frac{3}{32}\varepsilon\gamma_s\Omega_s\bar{A}^2\sin 2\bar{\Delta} - \frac{1}{4}\beta_{s3}\bar{A}^2(1-\frac{1}{2}\cos 2\bar{\Delta}) +$$
$$- \frac{1}{8}\beta_{s6}[\bar{A}^4(\frac{3}{2} - \cos 2\bar{\Delta}) + \frac{25}{2}\bar{B}^4 + 6\bar{A}^2\bar{B}^2(\frac{3}{2} - \cos 2\bar{\Delta})] - \frac{1}{2}\beta_{s1} - \frac{9}{8}\beta_{s4}\bar{B}^2$$
$$n_{44} = \frac{1}{8}A^2\{\beta_{s3} + \beta_{s6}[\bar{A}^2 + 3\bar{B}^2] - \frac{3}{4}\varepsilon\Omega_s^2\}\sin 2\bar{\Delta} + \frac{3}{32}\varepsilon\gamma_s\Omega_s\bar{A}^2\cos 2\bar{\Delta} - \frac{3}{32}\varepsilon\gamma_s\Omega_s\bar{B}^2 +$$
$$+ \gamma_s\Omega_s$$

Notice that in the case of $\bar{A} \neq 0$ and $\bar{B} \neq 0$, namely, the case of coupled mode response $\sin 2\bar{\Delta}$ and $\cos 2\bar{\Delta}$ are those defined in [1], which are repeated here for convenience:

$$\sin 2\bar{\Delta} = -\gamma\Omega\{2[\beta_6\Omega^2 - \beta_8 - 2\beta_{10}(2\bar{B}^2 + \bar{A}^2)] + \beta_6[\beta_7 - \Omega^2 + \bar{B}^2(\beta_9 - \beta_8) +$$
$$+ 2\beta_8\bar{A}^2 + \beta_{10}(\bar{B}^4 + 4\bar{A}^2\bar{B}^2 + 3\bar{A}^4)]\}/S_d$$
$$\cos 2\bar{\Delta} = \{\Omega^2\gamma^2\beta_6(2 + \beta_6\bar{B}^2) + [\Omega^2(1 + \beta_6\bar{B}^2) - \beta_7 - \beta_9\bar{B}^2 - 2\beta_8\bar{A}^2$$
$$- \beta_{10}(5\bar{B}^4 + 6\bar{A}^2\bar{B}^2 + 3\bar{A}^4)][\beta_6\Omega^2 - \beta_8 - 2\beta_{10}(\bar{A}^2 + \bar{B}^2)]\}/S_d$$

$$S_d = \bar{A}^2\{(\beta_6\gamma\Omega)^2 + [\beta_6\Omega^2 - \beta_8 - 2\beta_{10}(\bar{A}^2 + \bar{B}^2)]^2 +$$
$$- 2\beta_{10}\bar{A}^2\bar{B}^2[\beta_6\Omega^2 - \beta_8 - 2\beta_{10}(\bar{A}^2 + \bar{B}^2)]\}$$

But for the case of $\bar{A} \neq 0$ and $\bar{B} = 0$, namely the case of single mode response the $\sin 2\bar{\Delta}$ and $\cos 2\bar{\Delta}$ must be replaced by $\sin\bar{\phi}$ and $\cos\bar{\phi}$ respectively.

$$\sin\bar{\phi} = \{-\Omega^2\bar{A}[1 + \beta_1\bar{A}^2 + 2\beta_1\delta_{n,\ell}(\delta_2+\hat{\delta}_2)^2] + \beta_2\bar{A} + \beta_3\bar{A}^3 + 5\beta_5\bar{A}^5\}/\bar{F}_D$$
$$\cos\bar{\phi} = -\Omega\gamma[2\bar{A} + \beta_1\bar{A}^3 + 4\beta_1\bar{A}\delta_{n,\ell}(\delta_2+\hat{\delta}_2)^2]/\bar{F}_D$$

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