Analysis of of apparently spontaneous yawing oscillations for a ship under way.

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0. Summary.

In this paper a mathematical model is presented to explain the phenomenon of course oscillations of a ship, sailing in a mean straight course with a constant mean speed.
With this model, periodic course changing can be simulated as a result of simultaneous periodic rolling and pitching and emphatically without the input of any rudder action.
The frequency spectrum of this course error appears to consist of two narrow bands around the frequencies $f_p + f_r$ and $f_p - f_r$, where f_p and f_r are the ship's natural pitching and rolling frequencies. On top of this, the course error is found to have a non zero time average.

1. Introduction.

Assume a ship of some 20.000 tons to be sailing in a choppy sea. The sea waves have a significant height of 4 to 5 meters and the mean wave direction is not coinciding with the longship's or thwartship's directions.

The mean state of the ship is assumed to be stationary, meaning that the mean speed and the mean course are constant in time. The wind waves are supposed to act as disturbances of this mean steady state , resulting in periodic movements of the ship's gravity centre G ,like heaving, i.e. movements in a vertical direction, and swaying, i.e. irregular movements in the beam direction.

As a lumped mass, the ship is assumed to be oscillating about the longship's and thwartship's axes through G . These oscil-lations are commonly known as rolling and pitching.

A last phenomenon that is known to occur is periodic yawing, i.e. periodic movements of the compass lubber mark with respect to the ship's compass, which is assumed to maintain its horizontal and azimuthal position.

This yawing is often caused by a disturbing couple around the ship's V_s-axis, exerted by the joint disturbing action of wind and waves upon the ship's hull.

This disturbance may result in a deviation of the ship's course from its mean value. In that case it is usually counteracted by a rudder angle.

If the ship is equipped with an automatic steering device, the result of these time changing vertical disturbances and the cor-responding rudder counter actions will be some kind of quasi-periodic course changing about a mean course.

There is however a second periodic yawing movement, consisting of periodic course changing with a rather high frequency of 5 to 10 periods per minute. These oscillations have been expe-rienced to occur i n t h e a b s e n c e o f a n y r u d-d e r a c t i o n . It is this particular movement, that this paper is concerned with.

The special and exceptional nature of this phenomenon is that, although it has been met with in practice, there is no plausible hydrodynamic explanation for it, like there is for periodic rolling and pitching.

A certain roll or pitch angle is always counteracted by some restoring moment, arising from the fact that the centre of buoy--ancy B is beyond the vertical line through the centre of gravity G .

However, if a course error is not being counteracted by a couple around the vertical axis, generated by a certain rudder angle, there is no first order explanation for the fact, that the ship turns back to her previous course.

In the following paragraph this apparent spontaneous yawing is modelled as a result of combining the oscillating rolling and pitching movements.

2. A mathematical model for spontaneous periodic yawing.

We first introduce some notations and conventions.

With the origin in G, we work with a right handed co-ordinate system, attached to the ship.

For the ship in a purely steady state, i.e. sailing on a plane water-surface with a constant speed, the X_s^+-direction is adopted as the horizontal direction of the ship's stem , the Y_s^+-direc--tion is the horizontal direction of the starboard beam and the V_s^+-direction is pointing vertically downward.

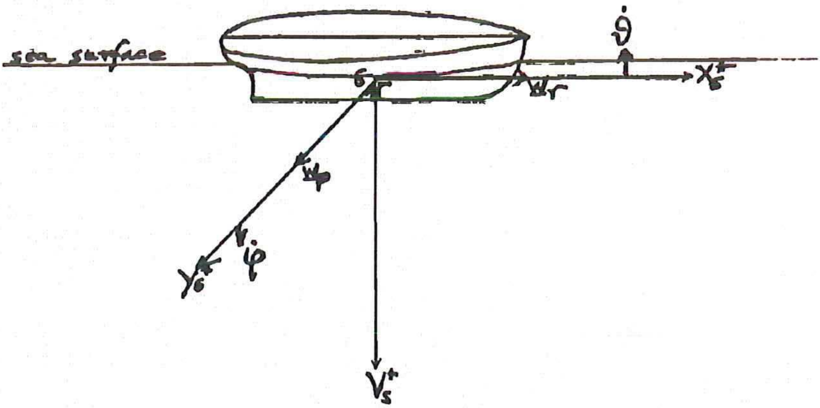

Figure 1

In this co-ordinate system a positive angular velocity vector $\underline{w}_r$, pointing into the X_s^+-direction, corresponds with a rolling to starboard. Analogously, a positive thwartship's angular velocity $\underline{w}_p$, pointing into the Y_s^+-direction, corresponds with an upward turn of the ship's stem.

The roll angle φ and the pitch angle θ are defined as the angles between the Y_s^+-axis - for φ - and the X_s^+-axis - for θ - with the horizontal plane. In consequence with the above mentioned sign conventions, a roll angle is positive, if the ship heels to starboard and a pitch angle is positive, if the stem direction is above the horizon.

Let us now consider figure 2.a on page 5, showing a sphere with radius R . The sphere's centre coincides with the ship's gravity centre G . TV is the vertical diameter.

The horizon on this sphere is the great circle NESW. GN points into the true North direction etc.

The ship has a positive pitch θ and a positive roll angle φ . A and B are the intersections of the sphere with the ship's momentary X_s^+- and Y_s^+-axes. We thus have

$$\text{arc } TA = \pi/2 - \theta \;, \quad \text{arc } AD = \theta \;,$$
$$\text{arc } TB = \pi/2 + \varphi \;, \quad \text{arc } FB = \varphi \;, \quad \text{arc } AB = \pi/2 \;.$$

In this skew state, the ship's course can be defined in two ways. The most common way is to define it as the azimuthal direction of the longship's axis GX_s^+ . This course is denoted as Ψ_a .

> According to international standard rules the ship's compass bowl should be mounted with the outer gimbal axis parallel to the GX_s^+-axis. This means, that the azimuthal course Ψ_a is the course, indicated on the - assumed to be horizontal - compass card by the longship's lubber mark.

From a hydrodynamical point of view however, it seems more logi--cal to conclude that, in the absence of any rudder action, the horizontal direction into which the ship moves, is the line of intersection of the longship's plane $X_s^+ G V_s^+$ with the horizon--tal plane. In figure 2.a this is the line GH .

In the sequel these courses will be called the azimuthal and the longship's course respectively, denoted as Ψ_a and Ψ_l .
In figure 2.a we see that Ψ_a = arc ND and Ψ_l = arc NH .

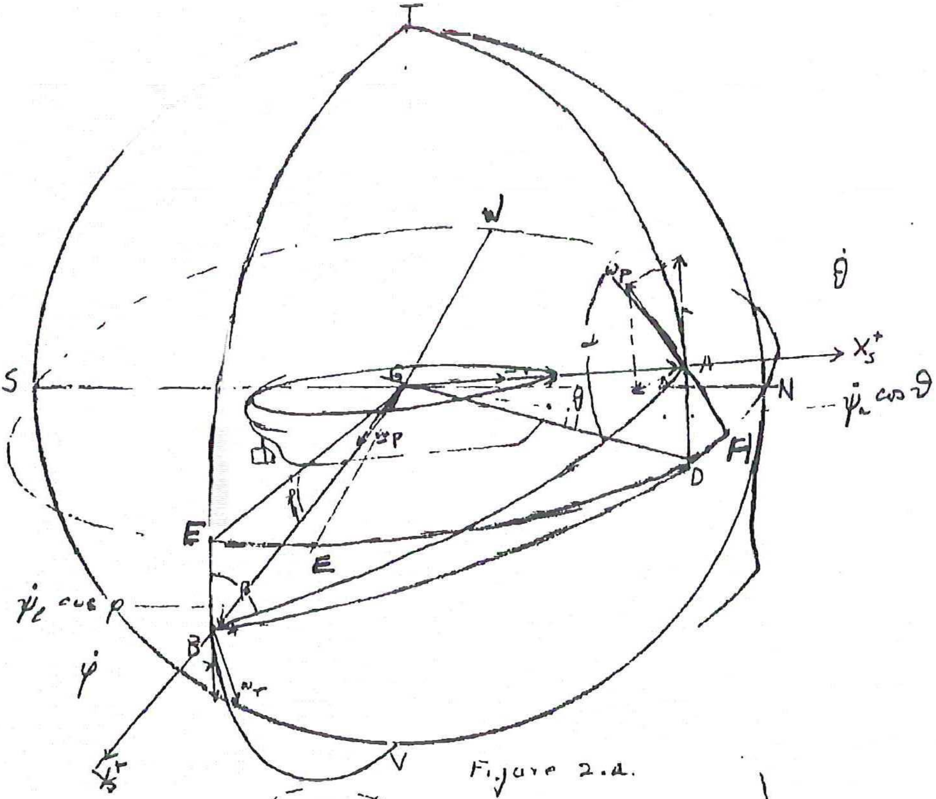

Figure 2.a.

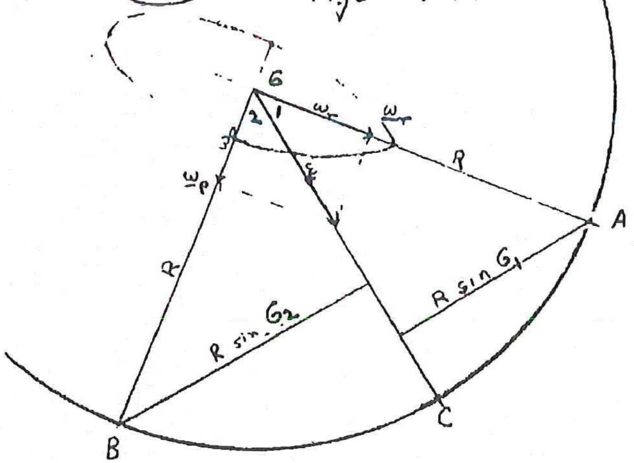

Figure 2.b

Since B is a pole to the great circle throgh A and H , we can conclude that arc BH = $\pi/2$.

With BF $\perp$ FH we now see that arc HF = $\pi/2$, so that

$$\text{arc } NF = \Psi_1 + \pi/2 \ .$$

This implies that, the time derivative of the longship's course Ψ_1 is equal to the azimuthal velocity of B .

We are now in a position to establish differential equations for the change in time of the angles θ , φ , Ψ_a and Ψ_1 .

The ship is assumed to be rotating about the X_s^+-axis with an angular velocity w_r and about the Y_s^+-axis with an angular velocity w_p .

Figure 2.b shows, how the ship is momentarily rotating about the line GC in the ship's X_s G Y_s-plane with the angular velo--city

$$w_t = (w_r^2 + w_p^2)^{\frac{1}{2}} \ .$$

The distance from A to this axis GC is R sin G_1 , so A has an upward linear velocity

$$v_A = w_t R \sin G_1 = R\, w_p \ .$$

For the arc velocity of A along the sphere we find

$$w_A = v_A / R = w_p \ .$$

A is moving upward in a direction, perpendicular to AB.

Analogously, we find that B has a downward arc velocity w_r , directed 90° from BA.

Denoting the time derivatives of θ , φ etc. as $\dot{\theta}$, $\dot{\varphi}$ etc. and putting $\angle$ BAT = α , $\angle$ ABT = β , we see that

$$\dot{\theta} = w_p \cos(\alpha - \pi/2) = w_p \sin\alpha \ ,$$

$$\dot{\varphi} = w_r \cos(\pi/2 - \beta) = w_r \sin\beta \ .$$

For the time derivatis of Ψ_a and Ψ_1 we find

$$\dot{\Psi}_a \cos\theta = w_p \sin(\alpha - \pi/2) \ ,$$

$$\dot{\Psi}_1 \cos\varphi = - w_r \sin(\pi/2 - \beta) \ ,$$

so

$$\dot{\Psi}_a = - w_p \cos\alpha / \cos\theta$$

$$\dot{\Psi}_1 = - w_r \cos\beta / \cos\varphi \ .$$

Applying the rule of cosines in triangle TAB, we find

$$\cos(\ \pi/2 + \varphi\ ) = \sin(\ \pi/2 - \theta\ )\cos\alpha\ ,$$

$$\cos(\ \pi/2 - \theta\ ) = \sin(\ \pi/2 + \varphi\ )\cos\beta\ .$$

We thus come to the differential equations

$$\dot{\theta} = w_p(1 - \sin^2\varphi/\cos^2\theta\ )^{\frac{1}{2}}\ , \qquad (2.1.a)$$

$$\dot{\varphi} = w_r(1 - \sin^2\theta\ /\cos^2\varphi\ )^{\frac{1}{2}}\ , \qquad (2.1.b)$$

$$\dot{\Psi}_a = w_p\sin\varphi/\ \cos^2\theta\ , \qquad (2.1.c)$$

$$\dot{\ }_1 = - w_r\sin\theta/\cos^2\varphi\ . \qquad (2.1.d)$$

Given the initial values of θ , φ , Ψ_a and Ψ_1 , we can now see, what happens to these quantities, if w_p and w_r are given as functions of time.

3. Simulation of the time behaviour of the azimuthal and the long--ship's courses on a rolling and pitching ship.

For the rolling and pitching velocities we adopt the time functions

$$w_r = A_r\cos(\ 2\pi f_r t)\ ,$$

$$w_p = A_p\cos(\ 2\pi f_p t)\ .$$

These time functions can be seen as the responses of the ship to disturbing inputs, caused by wind and swell waves. The responding frequencies f_r and f_p are known to be rather close to the ship's own rolling and pitching frequencies f_{rn} and f_{pn} . The small differences between f_r and f_{rn} and between f_p and f_{pn} are mainly due to the fact, that the basic second order differen--tial equations usually have a small damping coefficient.

The simultaneous set (2.1) was integrated numerically with Heun's predictor-corrector method with the initial conditions

$$\theta(0) = \varphi(0) = \Psi_a(0) = \Psi_1(0) = 0\ .$$

In a first example the author selected

$$A_r = A_p = \pi^2/108 \text{ radians per second,}$$

$$f_r = 1/12\ \sec^{-1}\ ,\quad f_p = 1/6\ \sec^{-1}\ .$$

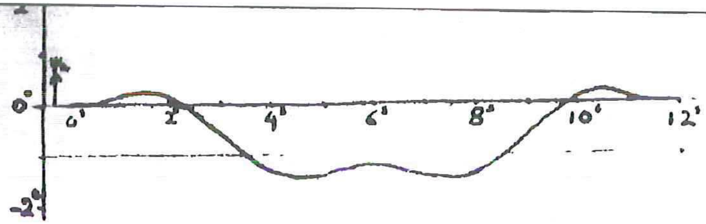

Figure 3. a
Graph of $\psi_a(\theta)$.

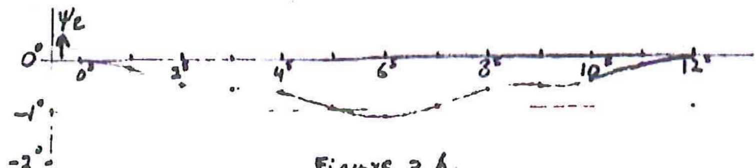

Figure 3. b.
Graph of $\psi_l(\theta)$

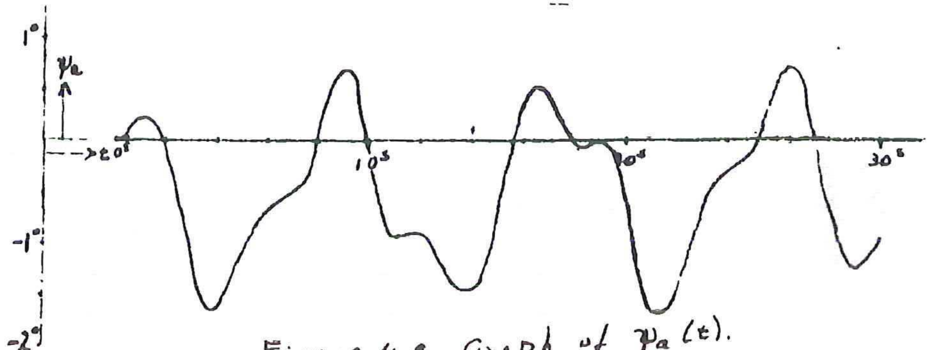

Figure 4 a Graph of $\psi_a(t)$.

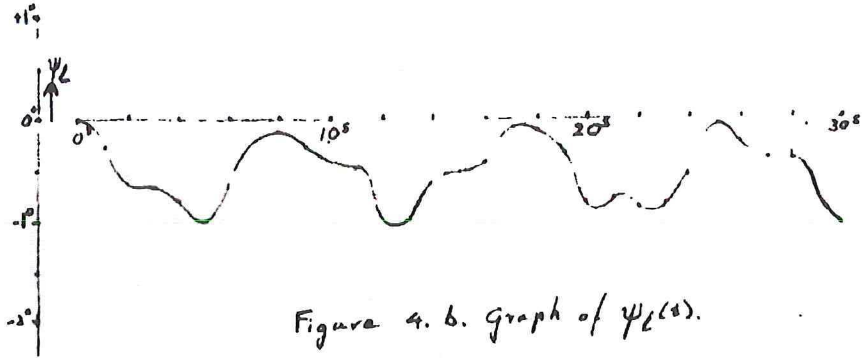

Figure 4. b. Graph of $\psi_l(t)$.

These values correspond with - discoupled - rolling and pitching amplitudes of 10^o and 5^o and with rolling and pitching periods of 12^{sec} and 6^{sec} respectively.

The resulting graphs of Ψ_a and Ψ_l are shown in figures 3.a&b.

The second example shows the effect of periodic pitching with 5^o amplitude and 5^{sec} period with rolling with 12^o amplitude and 12^{sec} period. The resulting graphs are shown in figures 4.a&b.

4. Analysis of the linearized model.

With $|\varphi| \lessapprox 0.2$ rad. and $|\theta| \lessapprox 0.1$ rad. the differential equations (2.1.a,b,c&d) can be linearized without a serious loss of accuracy.

Putting
$$\sin\varphi = \varphi + O(\varphi^3) ,$$
$$\sin\theta = \theta + O(\theta^3)$$
$$\cos\varphi = 1 + O(\varphi^2) ,$$
$$\cos\theta = 1 + O(\theta^2)$$

and omitting terms of $O(\varphi^2)$ and of $O(\theta^2)$, the set (2.1) can be reduced to

$$\dot{\theta} = w_p , \quad (4.1)$$

$$\dot{\varphi} = w_r , \quad (4.2)$$

$$\dot{\Psi}_a = w_p \varphi , \quad (4.3)$$

$$\dot{\Psi}_l = - w_r \theta . \quad (4.4)$$

Adopting
$$w_p = A_p \cos(2\pi f_p t) , \quad (4.5)$$
$$w_r = A_r \cos(2\pi f_r t) ,$$

and taking $\theta(0) = \varphi(0) = 0$, we find (4.6)

$$\varphi(t) = A_r \sin(2\pi f_r t)/(2\pi f_r) , \quad (4.7)$$

$$(t) = A_p \sin(2\pi f_p t)/(2\pi f_p) \quad (4.8)$$

Assuming zero initial conditions for Ψ_a and Ψ_l , we have, as a consequence of (4.3) to (4.8)

$$\Psi_a(t) = A_p A_r(-\cos(2\pi(f_p+f_r)t)/(f_p+f_r) + \cos(2\pi(f_p-f_r)t)/(f_p-f_r))/(8\pi^2 f_r) - A_p A_r/(4\pi^2(f_p^2 - f_r^2)) .$$

$$\Psi_1(t)=A_pA_r(\cos(2\pi(f_p+f_r)t)/(f_p+f_r)+\cos(2\pi(f_p-f_r)t)/(f_p-f_r))/8\pi^2 f_p)$$
$$- A_pA_r/(4\pi^2(f_p^2 - f_r^2)) .$$

In the first example this would amount to

$$\Psi_a(t) = - 0.29^o*\cos(90^o*t) + 0.87^o*\cos(30^o*t) - 0.58^o ,$$
$$\Psi_1(t) = 0.14^o*\cos(90^o*t) + 0.44^o*\cos(30^o*t) - 0.58^o ,$$

and in the second example this would lead to

$$\Psi_a(t) = - 0.37^o*\cos(102^o*t) + 0.90^o*\cos(42^o*t) - 0.53^o ,$$
$$\Psi_1(t) = 0.15^o*\cos(102^o*t) + 0.37^o*\cos(42^o*t) - 0.53^o .$$

These expressions give rise to the following conclusions:

(i) Both the azimuthal and the longship's courses have double periodic oscillations with frequencies equal to the sum and the difference of the pitching and rolling frequencies.

(ii) Both yawing movements have a non zero time average. In the assumed examples this time average of the course deviation from the steady course amounts to

$$\Delta\Psi = - A_pA_r/(4\pi^2(f_p^2 - f_r^2)) .$$