Functional Calculus via Transference, Double Operator Integrals and Applications

Functional Calculus via Transference, Double Operator Integrals and Applications

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Jan ROZENDAAL

Master of Science in Mathematics,
Universiteit Leiden, Nederland,
geboren te Rotterdam, Nederland.

Dit proefschrift is goedgekeurd door de promotores:

Prof. dr. B. de Pagter
Prof. dr. M.H.A. Haase

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Introduction

In a very general sense one may say that functional calculus theory studies the pairing of an operator A on some Banach space X and a function $f = f(z)$ of a variable z as an operator $f(A)$ on X. One would then like to derive properties of $f(A)$, for example norm bounds, from properties of the function f and the operator A. This thesis will be concerned with several instances of this problem.

Functional calculus theory arises in various contexts. The spectral theorem yields a beautiful functional calculus in which one can associate a bounded operator $f(A)$ with any normal operator A on a Hilbert space X and any bounded measurable function f on the spectrum of A. A similar theory exists on Banach spaces for scalar type operators, which correspond to diagonalizable matrices if X is a finite dimensional space. In these cases, the existence of a spectral measure allows for a natural definition of the functional calculus, and this calculus has many desirable properties.

A general bounded operator A on a Banach space does not have a spectral measure, and therefore the construction of a functional calculus for such operators should proceed in a different manner. The Riesz-Dunford functional calculus takes Cauchy's formula as a starting point, using that for a bounded operator A and λ not in the spectrum of A the definition of the operator $\frac{1}{\lambda - z}(A) = (\lambda - A)^{-1}$ is obvious. One then obtains a bounded operator $f(A)$ on X for each holomorphic function on a neighborhood of the spectrum of A, and the mapping $f \mapsto f(A)$ that arises from this procedure allows one to study a large class of operators associated with A, such as spectral projections corresponding to parts of the spectrum of A.

If A is an unbounded operator and f is holomorphic on a neighborhood of the spectrum of A and on a neighborhood of infinity, then one can use an extension of the Riesz-Dunford functional calculus to construct a functional calculus for A, see for instance [41]. However, since many interesting functions are not holomorphic on a neighborhood of infinity, this calculus is of limited use in applications.

An additional complication is that one is often interested in functions which have singularities. By this we mean that the function f need not in general be defined on a full neighborhood of the spectrum of A. For instance, in the functional calculus theory for generators of strongly continuous semigroups one often deals with an operator A with spectrum in the standard closed right half-plane and which intersects the imaginary axis, in particular at zero. In this case it is often unnatural to consider functions defined on regions which strictly contain zero, as this would restrict the number of interesting examples the theory applies to (think for instance of the fractional powers of an operator).

A functional calculus theory for a specific class of unbounded operators and functions with singularities at zero and infinity was developed by McIntosh and collaborators (see e.g. [88], [29]). This theory is now called the theory of H^∞-functional calculus, or simply H^∞-calculus (throughout, H^∞ denotes the Hardy space of bounded holomorphic functions on some domain). In the theory of H^∞-calculus problems arise that are not present in the Riesz-Dunford calculus for bounded operators. For example, for general bounded and holomorphic f one can only define $f(A)$ as an unbounded operator. The question for which bounded holomorphic functions f and operators A the operator $f(A)$ is bounded is still mostly unanswered in general.

It turns out that these obstacles make the theory of H^∞-calculus highly nontrivial, and many basic questions remain unanswered. If A is an operator such that $f(A)$ is bounded for all bounded holomorphic functions (on some domain) then A is said to have a *bounded* H^∞*-calculus*. It was shown early on by McIntosh that on Hilbert spaces, the boundedness of the H^∞-calculus for A is equivalent to the boundedness of certain square functions for A. The result in question deals with so-called sectorial operators, defined in Section 2.2.3. For the moment it suffices to note that a sectorial operator A of angle $\varphi \in (0,\pi)$ has spectrum contained in the closure of the sector S_φ with vertex at zero and opening angle 2φ which is symmetric around the positive real axis. If A is injective then A has a natural functional calculus that associates with functions in the class $\mathrm{H}^\infty(\mathrm{S}_\psi)$, for any $\psi \in (\varphi,\pi)$, an unbounded operator $f(A)$. By $\mathrm{H}^\infty_0(\mathrm{S}_\psi)$ we denote the subspace of $\mathrm{H}^\infty(\mathrm{S}_\psi)$ consisting of bounded holomorphic functions on S_ψ which decay polynomially at zero and infinity. The following then holds, cf. [88].

Theorem 1.1. *Let A be an injective sectorial operator of angle $\varphi \in (0,\pi)$ on a Hilbert space X. Then the following assertions are equivalent.*

- *For some $\psi \in (\varphi,\pi)$ and all $f \in \mathrm{H}^\infty(\mathrm{S}_\psi)$, $f(A)$ is bounded;*
- *For all $\psi \in (\varphi,\pi)$ and all $f \in \mathrm{H}^\infty(\mathrm{S}_\psi)$, $f(A)$ is bounded;*
- *For some $\psi \in (\varphi,\pi)$ and some nonzero $f \in \mathrm{H}^\infty_0(\mathrm{S}_\psi)$, there are constants $C_1, C_2 > 0$ such that*

$$C_1 \|x\| \leq \left(\int_0^\infty \|f(tA)x\|^2 \frac{\mathrm{d}t}{t} \right)^{1/2} \leq C_2 \|x\|$$

for all $x \in X$.

- *For all $\psi \in (\varphi, \pi)$ and all nonzero $f \in \mathrm{H}_0^\infty(\mathrm{S}_\psi)$, there are constants $C_1, C_2 > 0$ such that*

$$C_1 \|x\| \leq \left(\int_0^\infty \|f(tA)x\|^2 \frac{\mathrm{d}t}{t} \right)^{1/2} \leq C_2 \|x\| \tag{1.1}$$

for all $x \in X$.

Later, Cowling, Doust, McIntosh and Yagi generalized Theorem 1.1 to sectorial operators on general Banach spaces using weak square function estimates ([29]). The latter theory is useful in particular on L^p-spaces.

The square functions or quadratic estimates which occur in Theorem 1.1 come from harmonic analysis and go back to the classical Littlewood-Paley g-functions (see [115]). In fact, for A the square root of the negative Laplacian on $\mathbb{R}^n$ and $f(z) := \frac{z}{(1+z)^2}$, a change of variables shows that (1.1) is a generalization of the Littlewood-Paley g-function. The connection between square functions and functional calculus theory has been investigated by many authors, see for example [68], [75], [80] and [54]. It is a manifestation of the link between harmonic analysis and functional calculus theory which appears frequently throughout this thesis. In fact, one could say that it is one of the central themes of this work.

An instance of the link between functional calculus theory and harmonic analysis can also be found in the study of symmetric contraction semigroups on L^p-spaces. Let A be a positive operator (i.e. A is selfadjoint and the spectrum of A is contained in the nonnegative real numbers) on $\mathrm{L}^2(\Omega, \mu)$, where (Ω, μ) is a measure space. Let the operators $e^{-tA} = e^{-t\cdot}(A)$ be defined by the Borel functional calculus for A, and assume that $\left\|e^{-tA} f\right\|_p \leq \|f\|_p$ for all $f \in \mathrm{L}^p(\Omega, \mu) \cap \mathrm{L}^2(\Omega, \mu)$ and all $p \in [1, \infty]$. In this case $(e^{-tA})_{t \geq 0}$ extends to a consistent semigroup of bounded operators on $\mathrm{L}^p(\Omega, \mu)$ for all $p \in [1, \infty)$, and we say that $-A$ generates a *symmetric contraction semigroup*. Then A, considered as an operator on L^p, is a sectorial operator and one may ask whether A has a bounded H^∞-calculus. This question was considered by Stein in [115], who proved the first general theorem on functional calculus for symmetric contraction semigroups. Cowling extended this result in [28], from which we take the following theorem.

Theorem 1.2. *Let $-A$ be an injective generator of a symmetric contraction semigroup and let $p \in (1, \infty)$. Then A has a bounded H^∞-calculus on $\mathrm{L}^p(\Omega, \mu)$ for all $\psi \in (\pi|\frac{1}{p} - \frac{1}{2}|, \pi)$.*

The reason for stating this result here is, apart from its importance for functional calculus theory, the method of proof employed by Cowling. He used transference techniques of Coifman and Weiss (see [27]) to show that, if f is a bounded holomorphic function on the standard right half-plane such that the Fourier transform of the non-tangential limit of f on the imaginary axis is a Fourier multiplier (see section 2.4 for definitions), then $f(A)$ is bounded. He then used the Mikhlin Multiplier Theorem, Theorem 2.19 below, to deduce that A has a bounded $\mathrm{H}^\infty(\mathrm{S}_\psi)$-calculus for $\psi \in (\pi/2, \pi)$ and all $p \in (1, \infty)$. Finally, Stein interpolation yields Theorem 1.2. (It should be noted that the angle $\pi|\frac{1}{p} - \frac{1}{2}|$ in Theorem 1.2 is not optimal and that the optimal angle was determined recently in [25]).

The transference techniques developed by Coifman and Weiss in [27] (see also [26]) were influenced by work of Wiener in [123] and Calderón ([24], see also the survey [9]). Since then, they have been studied in e.g. [13] and [59], and applied to the theory of H^∞-calculus in [64], [58] and [59]. One of the key components in all these transference techniques is the idea of bounding the norm of an operator by relating it to another operator which is better understood, and then using bounds for the latter operator to bound the norm of the former. Usually the operators which are better understood come from harmonic analysis, for example as Fourier multipliers. This is the main technique, and simultaneously the central viewpoint on functional calculus theory, that one will find throughout this thesis.

If A is a normal operator on a Hilbert space, or more generally a scalar type operator on a Banach space (see Section 2.2.5 for definitions), then determining which functions f lead to bounded operators $f(A)$ is trivial: the spectral measure associated with A allows one to define in a natural way a bounded operator $f(A)$ for each bounded measurable function on the spectrum of A. However, there are still many nontrivial functional calculus questions that arise naturally. For instance, one may wonder for which unitarily invariant norms $\|\cdot\|$ and which continuous functions $f : \mathbb{R} \to \mathbb{R}$ an estimate of the form

$$\|f(A + B) - f(A)\| \leq C \|B\| \tag{1.2}$$

holds for all selfadjoint operators A and B on a Hilbert space H, with a constant $C \geq 0$ independent of A and B. Such questions arise when studying the interactions between atoms in a crystal ([82,83]) and also occur in scattering theory ([15,16]). Equivalently, one can consider

$$\|f(B) - f(A)\| \leq C \|B - A\| \tag{1.3}$$

for selfadjoint operators A and B.

If (1.3) holds then f is said to be *operator Lipschitz* with respect to $\|\cdot\|$, since (1.3) implies that f is Lipschitz as a mapping on the class of selfadjoint operators with respect to the norm $\|\cdot\|$. Determining when a function is operator Lipschitz with respect to a specific norm turns out to be highly nontrivial. Clearly (1.3) implies that f is Lipschitz (by letting $A := a \in \mathbb{R}$, $B := b \in \mathbb{R}$), but for which norms $\|\cdot\|$ are all Lipschitz functions operator Lipschitz? For which norms and specific important functions such as the absolute value function f does (1.3) hold? Answers to these questions can also be found by using tools from harmonic analysis. For example, when considering the operator norm $\|\cdot\|_{\mathcal{L}(H)}$ on a Hilbert space H, (1.3) was obtained by Peller in [95] for f in the Besov class $\dot{\mathrm{B}}^1_{\infty,1}(\mathbb{R})$ from Section 2.3.

Theorem 1.3. *Let H be a Hilbert space and $f \in \dot{\mathrm{B}}^1_{\infty,1}(\mathbb{R})$. Then there exists a constant $C \geq 0$ such that*

$$\|f(B) - f(A)\|_{\mathcal{L}(H)} \leq C \|B - A\|_{\mathcal{L}(H)}$$

for all selfadjoint $A, B \in \mathcal{L}(H)$.

To prove this result, Peller uses the technique of double operator integrals which goes back to Daleckiĭ and S. Kreĭn ([30]) and was developed extensively by Birman and Solomyak in a series of papers (see [17–20]). This technique views the difference $f(B) - f(A)$ as the image under a certain transformation of $B - A$. One then studies the associated transformation, and if the divided difference $\frac{f(y)-f(x)}{y-x}$ of f is sufficiently regular then one can bound the norm of this transformation to deduce the desired result. For example, Peller used the Littlewood-Paley decomposition of functions in $\dot{\mathrm{B}}^1_{\infty,1}(\mathbb{R})$ to show that the divided difference of f belongs to a class of functions (considered in Section 2.3) which have a specific integral representation. This integral representation ensures that the associated transformation is bounded with respect to $\|\cdot\|_{\mathcal{L}(H)}$, from which one deduces Theorem 1.3.

In this sense, the approach used to prove Theorem 1.3 is analogous to that of the transference techniques described above. To bound the norm of the difference $f(B) - f(A)$ one relates it to a better understood transformation. One then uses other techniques to bound the norm of this transformation to deduce the desired result. Moreover, the transformations which occur via the double operator integral technique are (continuous versions of) Schur multipliers. Since Schur multipliers can be viewed as noncommutative versions of Fourier multipliers, the analogy between transference techniques and the theory of double operator integration is even stronger.

Another link with harmonic analysis occurs when considering (1.3) with respect to other norms than the operator norm. It was proved by M. Kreĭn in [74] that, if $B - A$ is an element of the Schatten ideal $\mathcal{S}_1$ of trace-class operators, then $f(B) - f(A) \in \mathcal{S}_1$ for all $f \in \mathrm{C}^\infty_\mathrm{c}(\mathbb{R})$. Moreover, (1.9) holds with respect to the $\mathcal{S}_1$-norm. He also asked whether this result could be extended to all $f \in \mathrm{C}^1(\mathbb{R})$. One could then pose the same question for the Schatten ideal $\mathcal{S}_p$ for other values of $p \in [1, \infty]$.

Kreĭn's question has an affirmative answer for $p = 2$ but is false for $p = 1$ and $p = \infty$, as was shown by Farforovskaja in [46–48]. It was proved by Kato in [69] that the absolute value function f does not satisfy (1.3) with respect to the operator norm on an infinite dimensional Hilbert space. Later, it was proved by Davies [31] that for f the absolute value function, (1.3) holds with respect to the $\mathcal{S}_p$-norm if and only if $p \in (1, \infty)$. Finally, a complete answer to the question above was recently given by Potapov and Sukochev in [102], where they proved the following.

Theorem 1.4. *Let $p \in (1, \infty)$ and let $f : \mathbb{R} \to \mathbb{R}$ be Lipschitz. Then there exists a constant $C \geq 0$ such that*

$$\|f(B) - f(A)\|_{\mathcal{S}_p} \leq C \, \|B - A\|_{\mathcal{S}_p}$$

for all selfadjoint operators A and B on ℓ_2 such that $A - B \in \mathcal{S}_p$.

This result is proved by using double operator integral theory to relate the difference $f(B) - f(A)$ to the norm of a Schur multiplier, and then bounding the norm of this Schur multiplier. For the latter one uses vector-valued harmonic analysis, in particular the vector-valued Marcinkiewicz Multiplier Theorem due to Bourgain ([21]). Here one finds a very clear analogy with the pairing of transference methods and vector-valued Fourier analysis that was mentioned before, using as a vital ingredient that the Schatten p-classes are UMD spaces for $p \in (1, \infty)$. One possible proof of Theorem 1.4 even explicitly uses transference techniques.

The discussion above shows that many of the same principles that occur in the study of H^∞-calculus using transference principles apply also in the study of operator Lipschitz estimates using double operator integrals. It is the aim of this thesis to use transference methods and double operator integration theory to derive some new results concerning H^∞-calculus and operator Lipschitz estimates.

Applications of H^∞-calculus to semigroup theory can be found in various areas, for example in questions of maximal regularity (see e.g. [39] and [75]). In this thesis we shall mostly be interested in applications of H^∞-calculus to numerical analysis. Consider the abstract Cauchy problem

$$\begin{aligned} \tfrac{\mathrm{d}u}{\mathrm{d}t}(t) &= -Au(t) \qquad (t \geq 0) \\ u(0) &= x \end{aligned} \tag{1.4}$$

on a Banach space X. It is well-known that a unique mild solution to (1.4) exists for each initial value $x \in X$ (that is, (1.4) is well-posed) if and only if $-A$ generates a C_0-semigroup $(T(t))_{t\geq 0} \subseteq \mathcal{L}(X)$ of bounded operators on X. In this case the mild solution to (1.4) is given by $u(t) = T(t)x$ for all $t \geq 0$ and $x \in X$. If $-A$ generates a C_0-semigroup then A has a natural functional calculus for all bounded holomorphic functions on suitable half-planes, and $T(t) = e^{-tA}$ for all $t \geq 0$.

Even when (1.4) is well-posed, the semigroup $(T(t))_{t\geq 0}$ is often not given explicitly or is hard to work with analytically. Hence one would like to approximate the solution $u(t) = T(t)x$ at any time $t > 0$ by simpler functions. One possible way to do this is to approximate $T(t)$ by rational functions of A. In functional calculus terms this comes down to approximating $e^{-tA}x$ by $r_n(tA)x$ for a sequence $(r_n)_{n\in\mathbb{N}}$ of rational functions. In other words, one would like to determine when $\left\|e^{-tA}x - r_n(tA)x\right\|$ converges to zero as n tends to infinity. If, for some function norm $\|\cdot\|_F$ and a constant C (possibly depending on x), an estimate of the form

$$\left\|e^{-tA}x - r_n(tA)x\right\|_X \leq C \left\|e^{-t\cdot} - r_n(t\cdot)\right\|_F \tag{1.5}$$

holds for all $n \in \mathbb{N}$, and if $\left\|e^{-t\cdot} - r_n(t\cdot)\right\|_F \to 0$ as $n \to \infty$, then $r_n(tA)x$ converges to $e^{-tA}x$. Hence functional calculus estimates as in (1.5) can yield convergence of numerical approximation schemes for evolution equations. The classical result of Brenner and Thomée from [23] on convergence of rational approximation schemes is proved in this manner. More recently, a general functional calculus approach to the convergence of approximation schemes was set up in [53].

Most of these applications of functional calculus theory deal with the Hille-Phillips functional calculus (see Section 2.2 for the definition) and let F in (1.5) be the space of Laplace transforms of bounded measures. One then attempts to estimate the variation norm of the inverse Laplace transform of (a modified version of) $e^{-t\cdot} - r_n(t\cdot)$ and show that it converges to zero as $n \to \infty$. This is the most general approach possible, in the sense that the Hille-Phillips calculus applies to all generators of C_0-semigroups and an estimate of the form (1.5) for this F is the best that one can expect in general. However, for specific semigroups one might be able to obtain (1.5) for larger function spaces F and smaller norms $\|\cdot\|_F$, which then allows for a faster convergence rate of $\left\|e^{-tA}x - r_n(tA)x\right\|_X$ to zero. This approach was applied in [49] to generators of analytic semigroups with a bounded H^∞-calculus. Transference principles can also prove useful in this setting, as these allow one to obtain stronger functional calculus estimates for specific classes of semigroups using results from harmonic analysis.

In this thesis we shall use functional calculus theory to derive new results on convergence of approximation schemes. In particular, we shall use both the Hille-Phillips calculus to derive results valid for general bounded C_0-semigroups, and functional calculus estimates obtained using transference principles to improve the convergence rates for specific classes of semigroups.

Applications of the operator Lipschitz estimates in (1.3) can be found in matrix analysis, see [14]. Specifically, (1.2) shows that the functional calculus is stable under perturbations, with a constant independent of the size of the matrices involved.

Most of the research in this area has focused on the case of selfadjoint or normal matrices and unitarily invariant norms in (1.3). However, there are many interesting matrix norms that are not unitarily invariant. For example, the operator norm of an $n \times n$-matrix as an operator on $\mathbb{C}^n$ with the ℓ^p-norm for $p \neq 2$ and $n > 1$ is not unitarily invariant. In this case, when deriving operator Lipschitz estimates for diagonalizable matrices from those for normal matrices one gets a dependence of the constant C on the dimension n, as follows from the fact that the Banach-Mazur distance from ℓ_n^p to ℓ_n^2 tends to infinity as n tends to infinity.

The discussion above might lead one to think that estimates such as (1.2) and (1.3) for diagonalizable matrices and norms which are not unitarily invariant cannot be independent of the dimension. However, the double operator integral technique that has been useful in obtaining (1.3) in various cases relies mainly on the fact that the operators A and B in (1.3) have spectral measures. Just like a normal matrix, any diagonalizable matrix has a spectral measure. Therefore one could hope that the double operator integral technique can also prove useful when considering diagonalizable matrices and norms which are not unitarily invariant. In this thesis we shall show that this is indeed the case, and we shall derive an extension of Theorem 1.3 to general symmetric matrix norms and diagonalizable operators. This then yields dimension-independent perturbation inequalities such as (1.2) for diagonalizable matrices.

We now give a more detailed description of the contents of this thesis.

Part I: Preliminaries

In the first part of this thesis we collect some background material that is necessary for the understanding of the rest of the thesis. Most of this material is not new, and the parts which are new generally concern adaptations of existing concepts.

Preliminaries

We first introduce the basic notation and terminology that will be used throughout this thesis, after which we move on to discuss various functional calculi. In particular, we treat the Hille-Phillips calculus for generators of C_0-semigroups and C_0-groups. We also treat the half-plane type calculus for operators of half-plane type (of which generators of C_0-semigroups are the main example), the strip type calculus for strip type operators (of which generators of C_0-groups are the main example), the sectorial calculus for sectorial operators (of which generators of analytic semigroups are the main example) and briefly the parabola type calculus for generators of cosine functions. We discuss some of the basics of these calculi, such as the Convergence Lemma. We mention examples of operators which do not have a bounded H^∞-calculus, and give sufficient conditions for operators to have a bounded H^∞-calculus. Finally, we discuss the Borel functional calculus for scalar type operators defined by integration with respect to a spectral measure. These functional calculi are well known and much of this material can be found in, e.g. [45], [55] and [42].

We then introduce some of the function spaces which appear in this work. In particular, inhomogeneous Besov spaces will be essential in Chapter 4, whereas a particular homogeneous Besov space is important for Chapter 5.

We then discuss a class of functions $\mathfrak{A}$ which allow for a specific integral representation. This class will be important in Chapter 5, and has been studied for functions on the square of the real line in [32], [101]. We define it for functions on subsets of $\mathbb{C} \times \mathbb{C}$, and we discuss some of its basic properties.

We then treat Fourier multipliers on vector-valued spaces and some of their properties. We first consider Fourier multiplier operators on L^p-spaces and discuss the connection between Fourier multipliers and the geometry of the underlying space in the form of the UMD property and the Mikhlin Multiplier Theorem. This material is classical. We then move on to Fourier multipliers on vector-valued Besov spaces, which play a key role in Chapter 4. The most useful property of such multipliers is that one can obtain results about their boundedness regardless of the geometry of the underlying Banach space. This material is taken mostly from [51].

In the next section we consider several transference principles which link functional calculus with Fourier multiplier theory. In particular, we mention the transference principle by Berkson and Gillespie from [13] and the abstract transference principle from [59]. The latter we discuss in a specific, more concrete setting that will be sufficient for our purposes.

The notion of γ-boundedness is treated next. This notion was introduced by Kalton and Weis in [68] and is related to the more well known R-boundedness. It has been studied extensively since its introduction (see the survey [120]) and is known to allow one to transfer results that follow from Plancherel's Theorem on Hilbert spaces to general Banach spaces. In particular, this holds for Fourier multiplier results, which is why this notion is useful for us in Chapters 3 and 6. We consider the ideal property of the ideal of γ-radonifying operators and the γ-Multiplier Theorem. Moreover, we give two applications of the notion of γ-boundedness for H^∞-calculus on general Banach spaces.

Finally, we treat some basics of real interpolation spaces. These will mostly be used in Chapter 4, but will appear at several other places in this thesis as well. In particular, we use that vector-valued Besov spaces occur as interpolation spaces between vector-valued Sobolev spaces. This material is classical (see [12] and [86]).

Part II: Functional calculus using transference methods

In Part I of this thesis we present some new functional calculus results for (semi)group generators, obtained using transference principles.

We note that the transference approach has also been employed in [105] to derive functional calculus results for C_0-groups using the geometric notions of type and cotype of a Banach space.

Functional calculus for semigroup generators

In this chapter we consider functional calculus for semigroup generators. The study of H^∞-calculus for generators of general, not necessarily analytic, C_0-

semigroups is relatively new. Historically, H^∞-calculus has mostly been studied for sectorial operators and analytic semigroups. This theory allows for many elegant results, see for instance the theorems of McIntosh and Cowling discussed before.

In contrast, far less is known about H^∞-calculus for general semigroups. Many results about sectorial operators also apply to generators of semigroups, but often they yield bounded calculi on sectors bigger than the standard half-plane. Such statements are relatively useless for applications, due to the lack of interesting examples of functions which are bounded and holomorphic on these bigger sectors. For semigroup generators with spectrum in a half-plane, it is more natural to consider functional calculi for functions defined on half-planes. In particular, for $-A$ the generator of a uniformly bounded C_0-semigroup one would like to consider functions on half-planes which are slightly bigger than the standard right half-plane.

The desire to study functional calculus for functions defined on half-planes led in [7] to the definition of an operator of half-plane type, a notion which extends that of a generator of a C_0-semigroup. One can study functional calculus for such operators, and in [7] results were obtained about the boundedness of certain operators. Moreover, in [59] it was shown that generators of uniformly bounded C_0-semigroups allow for a bounded Besov-type functional calculus, a result similar to one obtained for generators of analytic semigroups by Vitse in [121].

To prove the results in [59] a general abstract transference principle was set up that will be used in this chapter as well. In [59] one can already see the interplay between harmonic analysis, the geometry of the underlying space and functional calculus theory that underlines this thesis. The results in [59] are most useful on Hilbert spaces or for γ-bounded semigroups, are still interesting on UMD spaces, and are of a more abstract nature on general Banach spaces.

In [59] the notion of an analytic multiplier algebra is introduced, a concept which allows one to elegantly capture results obtained from transference principles. The analytic multiplier algebra depends on both a parameter $p \in [1, \infty]$ and a Banach space X, and it is the algebra of bounded holomorphic functions which are $L^p(\mathbb{R}; X)$-Fourier multipliers on the boundary of their domain. By Plancherel's Theorem, the analytic multiplier algebra coincides with H^∞ for $p = 2$ and X a Hilbert space, but for general Banach spaces it is a smaller class. If X is a UMD space then various multiplier theorems, in particular the Mikhlin Multiplier Theorem, allow one to identify a large subclass of functions of the analytic multiplier algebra. For general Banach spaces, for instance L^1-spaces, it may occur that the analytic multiplier algebra consists of only the Laplace transforms of bounded measures.

Apart from the results about Besov-type functional calculi for semigroup generators mentioned above, significant results were obtained by Zwart in [125]. He showed that for $-A$ the generator of an exponentially stable C_0-semigroup $(T(t))_{t\in\mathbb{R}_+}$ on a separable Hilbert space and f a bounded holomorphic function defined on the standard right half-plane, $f(A)T(t)$ is bounded for each $t > 0$. It is this result which led to the present chapter of this thesis.

To prove his results Zwart used notions from systems theory, and although this approach to functional calculus theory was extended to a more general setting in [110], the method of proof in [125] does not appear to be easily extendable to UMD spaces or general Banach spaces. In this chapter we use the abstract transference principle in [59] to reprove some of the results in [125] and extend them to general Banach spaces.

In [125] it is shown that, for $-A$ the generator of an exponentially stable C_0-semigroup $(T(t))_{t\in\mathbb{R}_+}$ on a separable Hilbert space and f a bounded holomorphic function defined on the standard right half-plane, $\|f(A)T(t)\|$ grows at most like $t^{-1/2}$ as $t \downarrow 0$. In this chapter we show that on general Banach spaces in fact $\|f(A)T(t)\|$ grows at most logarithmically in t for f a function in the analytic multiplier algebra associated with X. In particular, this improves the results in [125] on Hilbert spaces. It should be noted that our result was recently shown to be sharp, in [111].

From the logarithmic growth mentioned above one can deduce the domain inclusion $\mathrm{D}(A^\alpha) \subseteq \mathrm{D}(f(A))$ for all $\alpha > 0$ and all functions in the analytic multiplier algebra. This in turn is equivalent to saying that $f(A)(1 + A)^{-\alpha}$ is bounded for all $\alpha > 0$ and all such f. This shows that the domain of $f(A)$ for functions in the analytic multiplier algebra is quite large, and also that $f(A)$ is a bounded operator for any f which decays exponentially at infinity. For generators of analytic semigroups such results are a simple consequence of the definition of the functional calculus, but for general semigroup generators they are new. That $f(A)T(t)$ is bounded for each $t > 0$ is again easy to deduce for generators of analytic semigroups, since the function $z \mapsto \mathrm{e}^{-tz}$ decays rapidly on sectors. Moreover, for analytic semigroups it has since been shown in [112] by more elementary means that the norm bound of $f(A)T(t)$ grows at most logarithmically in t as $t \downarrow 0$.

It was shown by Mubeen in [90] (see also [7]) that semigroup generators on Hilbert spaces allow for a so-called m-bounded H^∞-calculus. By this we mean that, if $-A$ generates a uniformly bounded C_0-semigroup on a Hilbert space, then $f^{(m)}(A)$ is bounded for each bounded holomorphic function f on a half-plane, where $f^{(m)}$ is the m-th derivative of f. Moreover, an estimate

$$\|f^{(m)}(A)\| \leq C\|f\|_\infty \tag{1.6}$$

holds with a constant C independent of f. In fact, it is shown that semigroup generators are characterized by (1.6), at least if one assumes that the constant C depends in a specific way on the size of the half-plane on which f is bounded and holomorphic. Moreover, for group generators the existence of an m-bounded H^∞-calculus for functions defined on strips is equivalent to the boundedness of the H^∞-calculus.

The method of proof in [90] and [7] relies on the underlying Hilbert space structure via Plancherel's Theorem, and it is not clear how one should extend the method to general Banach spaces. In this chapter we use a transference principle to reprove the results in [90] and extend them to general Banach spaces, using again the analytic multiplier algebra. This m-bounded calculus can be used to give an alternative proof of the fractional domain inclusion from above.

Apart from allowing for extensions to general Banach spaces via the analytic multiplier algebra, the transference principles we consider are also useful for extensions to γ-bounded semigroups. In particular, by factorizing operators via the space of γ-radonifying operators and using the ideal property of this space, we are able to extend the results on Hilbert spaces which were discussed above to γ-bounded semigroups on general Banach spaces.

The contents of this chapter are based on joint work with Markus Haase and have appeared in [61].

Functional calculus on real interpolation spaces for generators of C_0-groups

Although a semigroup generator on a Hilbert space need not have a bounded H^∞-calculus in general, each group generator on a Hilbert space has a bounded H^∞-calculus for functions on strips. For bounded groups this is classical, and for unbounded groups it was shown in [22]. On UMD spaces, it was shown in [64] that generators of uniformly bounded groups have a bounded H^∞-calculus on double sectors. These results can be proved using transference principles, and in [58] a transference principle for unbounded groups was developed that shows that unbounded groups on UMD spaces also have a specific bounded calculus for functions on strips.

As indicated before, the transference principles which we use throughout rely on the boundedness of certain Fourier multipliers to obtain functional calculus estimates. This approach therefore automatically seems to restrict one to considering Hilbert spaces or at least UMD spaces. In this chapter we show that one can in fact also obtain results on general Banach spaces for a large class of functions.

The approach that we use is as follows. Let $-\mathrm{i}A$ generate a C_0-group $(U(s))_{s\in\mathbb{R}}$ on a Banach space X. The classical transference principle of Berkson and Gillespie from [13], as well as the recent transference principle for unbounded groups in [58], rely on factorizing an operator U_μ of the form

$$U_\mu(x) := \int_{\mathbb{R}} U(s)x\,\mu(\mathrm{d}s) \qquad (x \in X) \tag{1.7}$$

via a convolution operator on $\mathrm{L}^p(\mathbb{R}; X)$ related to the measure μ. Then results about Fourier multipliers on $\mathrm{L}^p(\mathbb{R}; X)$ can be used to obtain norm bounds for U_μ.

It was shown in [51] that on X-valued Besov spaces, Fourier multiplier results hold that do not require a UMD assumption on the geometry of X. The results in [51] hold for operator-valued Fourier multipliers and depend on the Fourier type of X. Fourier type is a geometric condition which imposes a restriction on the generality of the space X. However, for scalar-valued multipliers no assumptions on the Fourier type of X, nor any other assumptions on X, are needed. In particular, a version of the Mikhlin Multiplier Theorem holds for Fourier multipliers on X-valued Besov spaces and general Banach spaces X.

Since it is well-known that Besov spaces are real interpolation spaces between L^p-spaces and Sobolev spaces, one can try to modify the transference principles mentioned above to factorize via X-valued Sobolev spaces. One is then naturally led to the domain $\mathrm{D}(A)$ of A (and the domains of other powers of A), and interpolation between the L^p-spaces and Sobolev spaces leads one to consider real interpolation spaces between X and $\mathrm{D}(A)$.

We use this approach to show that each group generator $-\mathrm{i}A$ on a general Banach space X has a bounded calculus on the real interpolation space $(X, \mathrm{D}(A))_{\theta,q}$ for each $\theta \in (0,1)$ and $q \in [1,\infty]$, for functions in the so-called analytic Mikhlin algebra H^∞_1. This algebra consists of all bounded and holomorphic functions f on strips which satisfy the condition (coming from the multiplier theorem in [51]) that $z \mapsto (1+|z|)|f'(z)|$ is bounded. Moreover, if the group generated by $-\mathrm{i}A$ is uniformly bounded then the constant bounding the H^∞_1-calculus is independent of the size of the strip. This result mirrors the analogous statement in [58] for group generators on UMD spaces, where the theorems are obtained for operators on X.

By considering the imaginary powers of a sectorial operator one can relate results about functional calculi for generators of groups to results about functional calculi for sectorial operators with bounded imaginary powers. In particular, as a consequence of our results we obtain the boundedness of the functional calculus for a new class of functions and operators A with bounded imaginary powers on a general Banach space X. This result is similar to a result obtained on UMD spaces in [58]. In our case restriction on the generality of the underlying space X is avoided by dealing with functional calculus on the real interpolation space $(X, \mathrm{D}(\log(A)))_{\theta,q}$, for $\theta \in (0,1)$ and $q \in [1,\infty]$.

In a similar manner, we deduce results for generators of cosine functions from the results for group generators.

This chapter is influenced by the results of Dore in [37] (see also [38] and [56]), who showed that the part of an invertible sectorial operator A on a Banach space X in the real interpolation space $(X, \mathrm{D}(A))_{\theta,q}$ has a bounded sectorial H^∞-calculus for all $\theta \in (0,1)$ and $q \in [1,\infty]$. Note that the operator A need not have a bounded calculus on X. The method of proof used in [38] is "elementary" in the sense that it relies on the definition of $f(A)$, for f a bounded holomorphic function of sufficient decay, via the Cauchy integral formula. This proof does not seem to apply to functions defined on strips.

However, in this chapter we show that transference techniques in fact yield nontrivial results for functions on strips.

We could have formulated our results in terms of analytic multiplier algebras, as in Chapter 3. Doing so would have led us to consider bounded holomorphic functions on strips whose restrictions to the boundary of the strip are Fourier multipliers on Besov spaces. For simplicity we have chosen to confine ourselves to considering the analytic Mikhlin algebra H_1^∞. Moreover, the value of such abstract multiplier algebras only arises when considering functions which do not satisfy the Mikhlin Multiplier Theorem but which satisfy other multiplier theorems.

The results in this chapter are based on joint work with Markus Haase (see [62]).

Part III: Double operator integrals and perturbation inequalities

Part II of this thesis is of a noncommutative nature. In this part we consider the technique of double operator integration and use it to derive perturbation inequalities for the functional calculus associated with scalar type operators on Banach spaces.

Operator Lipschitz functions on Banach spaces

We have already indicated that for normal operators on Hilbert spaces the questions considered before, about boundedness of various functional calculi, are trivial. The existence of a spectral measure E associated with a normal operator A on a Hilbert space H means that one can define a bounded operator $f(A)$ for each bounded measurable function $f : \sigma(A) \to \mathbb{C}$ by

$$f(A) := \int_{\sigma(A)} f(z)\,\mathrm{d}E(z). \tag{1.8}$$

Then $f \mapsto f(A)$ is a continuous algebra homomorphism from the space of bounded measurable functions on $\sigma(A)$, endowed with the supremum norm, to $\mathcal{L}(H)$.

However, in this theory new questions arise that are far from trivial to answer. For instance, under what conditions on f do bounds of the form

$$\|f(B) - f(A)\| \leq C\,\|B - A\| \tag{1.9}$$

hold for all selfadjoint operators A and B with respect to a given norm $\|\cdot\|$? Answers to this question have been obtained in [95, 96, 102] (see also Theorems 1.3 and 1.4) by combining harmonic analysis with the theory of double operator integration. In the theory of double operator integration one views the difference $f(B) - f(A)$ in (1.9) as a double integral with respect to the spectral measures E and F of A respectively B:

$$f(B) - f(A) = \int_{\mathbb{C}} \int_{\mathbb{C}} \psi_f(x,y)\, \mathrm{d}F(y)(B - A)\mathrm{d}E(x).$$

Here $\psi_f(x,y) = \frac{f(y)-f(x)}{y-x}$ is the divided difference of f. Then (1.9) can be obtained by studying the operator

$$S \mapsto \int_{\mathbb{C}} \int_{\mathbb{C}} \psi_f(x,y)\, \mathrm{d}F(y) S \mathrm{d}E(x) \tag{1.10}$$

and determining its boundedness with respect to various norms.

The technique of double operator integration is similar to the transference principles that were discussed before. To study the quantity one is interested in, in this case the difference $f(B) - f(A)$, one instead studies a transformation which is easier to understand. For transference principles this is done by factorizing an operator via a Fourier multiplier. In the theory of double operator integration the factorization is trivial, it merely consists of the change of viewpoint in studying (1.10) instead of $f(B) - f(A)$.

In this chapter we contribute some new results to the theory of double operator integration and perturbation inequalities. The viewpoint we take is the following: results about selfadjoint or normal operators on separable Hilbert spaces can be viewed as results about operators on ℓ^2. Not all Lipschitz functions are operator Lipschitz with respect to the operator norm on ℓ^2. In particular, (1.9) does not hold for f the absolute value function (see [69]). However, in this chapter we show that a similar inequality does hold for the absolute value function and for operators A on ℓ^p and B on ℓ^q for $p < q$.

In order to clarify what we mean by the previous statement, we rewrite (1.9) as

$$\|f(B)S - Sf(A)\| \leq C\,\|BS - SA\| \tag{1.11}$$

for S the identity operator. Note that (1.11) makes sense even for operators A and B defined on different spaces X and Y, if we let S be a bounded operator from X to Y. In the case where $X = Y$ and $A = B$, (1.11) yields a norm bound for the commutator of $f(A)$ and S in terms of the commutator of A and S. For this reason we will often refer to (1.11) as a commutator estimate.

A nontrivial difficulty in interpreting (1.11) for operators on ℓ^p and ℓ^q with $p \neq q$ is that at least one of the operators in question will not be defined on a Hilbert space. Hence we need to define $f(A)$ and $f(B)$ for operators on general Banach spaces. Since the absolute value function is not holomorphic one cannot use the Riesz-Dunford functional calculus for this. Moreover, the theory of double operator integration relies on the existence of a spectral measure for the underlying operators, and for general operators on a Banach space such a spectral measure is not available. It should be noted here that the double operator integration theory has been extended to the Banach space setting in [33]. However, these results are much weaker than in the Hilbert space setting.

One can identify an important class of examples for which a spectral measure does exist, the class of diagonalizable matrices. A diagonalizable matrix A has a spectral measure associated with it, namely the measure which associates with an eigenvalue λ of A the spectral projection corresponding to the eigenspace of λ. Dunford and collaborators studied (see [5,40,109]) more general operators on Banach spaces with a spectral measure, so-called spectral operators and scalar type operators. A scalar type operator A on a finite dimensional space is simply a diagonalizable matrix, and for f the absolute value function one can define $f(A)$ as in (1.8). In light of this discussion it seems natural to study (1.11) for scalar type operators on Banach spaces. In particular, we consider the class of scalar type operators which are diagonalizable with respect to an unconditional Schauder basis.

In this chapter we establish the following version of (1.11):

$$\|f(B)S - Sf(A)\|_{\mathcal{L}(\ell^p,\ell^q)} \leq C\,\|BS - SA\|_{\mathcal{L}(\ell^p,\ell^q)}\,. \tag{1.12}$$

Here f is the absolute value functions, $S \in \mathcal{L}(\ell^p, \ell^q)$ and A and B are diagonalizable operators on ℓ^p respectively ℓ^q for $p, q \in [1, \infty]$ with $p < q$, and A and B have real spectrum. The constant C in (1.12) in fact depends on A, B, p and q in the following sense:

$$C = C_{p,q} \inf \|U\|_{\mathcal{L}(\ell^p)}\|U^{-1}\|_{\mathcal{L}(\ell^p)}\|V\|_{\mathcal{L}(\ell^q)}\|V^{-1}\|_{\mathcal{L}(\ell^q)}, \tag{1.13}$$

where the infimum is taken over all $U \in \mathcal{L}(\ell^p)$ and $V \in \mathcal{L}(\ell^q)$ which diagonalize A respectively B, and $C_{p,q}$ is a constant depending only on p and q.

It might seem like this result is not the goal that we set out to achieve, which was to obtain (1.11) with a constant independent of A and B. However, when considering normal operators on ℓ_2 the constant C in (1.13) is in fact independent of A and B. Indeed, a normal matrix is diagonalizable by a unitary matrix, hence the infimum in (1.13) is a constant (in fact equal to 1) if A and B are normal matrices. This explains why the constants which appear in the classical results about (1.9) on Hilbert spaces do not depend on the operators A and B. In our setting one can also obtain constants independent of the underlying operators by restricting to diagonalizable operators for which the infimum in (1.13) is less than a prefixed value, as is already done implicitly on Hilbert spaces by considering only selfadjoint or normal operators, as opposed to all operators which are similar to a normal operator.

Commutator estimates for the absolute value function and operator ideals in $\mathcal{L}(H)$ have been studied in [36]. The proofs in [31,33,36] are based on Macaev's celebrated theorem (see [52]) or on the UMD-property of the reflexive Schatten von-Neumann ideals. In the presence of the UMD-property one can apply techniques from harmonic analysis, as we have discussed before. However, the spaces $\mathcal{L}(X,Y)$ are not UMD spaces, and therefore the techniques used in [31,33,36] do not apply. To study (5.2) for $X = \ell^p$ and $Y = \ell^q$, we use methods completely different from those of [31,33,36]. In this sense the results in the present chapter differ from earlier chapters, where the use of vector-valued Fourier analysis was a key ingredient. However, our analysis shows that one can still deduce nontrivial results in situations where the underlying space is not a UMD space, a philosophy which is also present in Chapter 4.

To obtain (1.12) we proceed in several steps. We first set up the general theory of double operator integration for scalar type operators on Banach spaces. We then establish a version of Theorem 1.3 for scalar type operators, which shows that for a large class of functions f one can obtain (1.11). Since the absolute value function f is not contained in this class, a more refined analysis is needed for this function. We relate estimates for (1.10) to estimates for so-called triangular truncation operators and thereby establish in our setting a connection which has already been observed for various spaces of operators (see [36,71]).

We then study the boundedness of triangular truncation operators on $\mathcal{L}(\ell^p, \ell^q)$ using properties of Schur multipliers on $\mathcal{L}(\ell^p, \ell^q)$, established by Bennett in [11]. In particular, we use that the classical triangular truncation operator is bounded on $\mathcal{L}(\ell^p, \ell^q)$ for $p < q$ (see [10]).

We also obtain results for operators on ℓ^p and ℓ^q with $p \geq q$, and we develop the theory of double operator integration in the setting of operator ideals. In particular, we show that each Lipschitz function is operator Lipschitz on the ideal of p-summing operators from $\ell^{p'}$ to ℓ^p.

The results that are obtained in this chapter specialize on finite dimensional spaces to results for diagonalizable matrices. In particular, we obtain (1.12) for diagonalizable matrices A and B with a constant independent of the size of the matrices. A particular case of this is the perturbation estimate

$$\|f(B) - f(A)\|_{\mathcal{L}(\ell^p_n, \ell^q_n)} \leq C \|B - A\|_{\mathcal{L}(\ell^p_n, \ell^q_n)}$$

for f the absolute value function and diagonalizable $n \times n$-matrices A and B, obtained by letting S be the identity matrix in (1.2).

The results in this chapter are based on joint work with Fedor Sukochev and Anna Tomskova (see [106]).

Part IV: Applications to numerical approximation methods

Part III of this thesis contains applications of the results in earlier chapters to numerical approximation methods. In particular, we prove convergence of a specific approximation method and determine the corresponding convergence rates.

Convergence of subdiagonal Padé approximations of C_0-semigroups

As indicated before, given a well-posed Cauchy problem

$$\begin{aligned} \frac{du}{dt}(t) &= -Au(t) \qquad (t \geq 0) \\ u(0) &= x \end{aligned} \tag{1.14}$$

on a Banach space X, one often wants to approximate the solution u of (1.14) by simpler expressions. A common way to do this is to use rational approximation: one takes a suitable sequence of rational functions $(r_n)_{n\in\mathbb{N}}$ and tries to approximate $u(t) = \mathrm{e}^{-tA}x$ by $r_n(tA)$ as $n \to \infty$. One would like to know when such an approximation is stable, meaning that $\|r_n(tA)x\|$ is uniformly bounded in n, and when it converges, by which we mean that $\|u(t) - r_n(tA)x\| \to 0$ as $n \to \infty$.

Assume that the solution u of (1.14) stays bounded for all initial values x, i.e. that the semigroup $(T(t))_{t\in\mathbb{R}_+}$ generated by $-A$ is uniformly bounded. Then, in order for there to be any chance of stability and convergence in general, $(r_n)_{n\in\mathbb{N}}$ should be a bounded sequence in $\mathrm{H}^\infty(\mathbb{C}_+)$ such that $r_n(tz) \to \mathrm{e}^{-tz}$ as $n \to \infty$ for all $z \in \mathbb{C}_+$ and $t \geq 0$. A common choice is to take a rational function r which approximates the exponential function to a fixed order around zero and to let $r_n(z) := r(\frac{z}{n})^n$ for $z \in \mathbb{C}_+$. For example, the classical result by Brenner and Thomée in [23] establishes convergence of approximation methods of this form.

A drawback of the method sketched above is that, for $r = p/q$ with p and q polynomials, $p(\frac{A}{n})^n$ and $q(\frac{A}{n})^{-n}$ need to be computed for large values of n. This can be time-consuming, already for A a finite matrix. Therefore, in [66] a method of rational approximation was proposed which does not require the computation of high powers of resolvents (see also [91]). This method was called *rational approximation without scaling and squaring* and relies on the partial fraction decomposition of a rational function $r = p/q$ with $\deg(p) \leq \deg(q)$ and with distinct poles to write $r(tA)$ as a linear combination of resolvents of A. If one can find a sequence $(r_n)_{n\in\mathbb{N}}$ such that each r_n is of this form and if $r_n(tA)x \to T(t)x$ as $n \to \infty$, then higher powers of resolvents of A are not needed to approximate the solution u of (1.14).

Hence in [66] the question was posed whether, for $-A$ the generator of a uniformly bounded C_0-semigroup $(T(t))_{t\in\mathbb{R}_+}$, one can find a sequence $(r_n)_{n\in\mathbb{N}}$ such that each $r_n = p_n/q_n$ has distinct poles and satisfies $\deg(p_n) \leq \deg(q_n)$, and such $r_n(tA)x \to T(t)x$ as $n \to \infty$ for each $x \in \mathrm{D}(A)$ and $t > 0$. Numerical experiments seemed to indicate that convergence should indeed hold, with rate $O(\frac{1}{\sqrt{n}})$.

The fact that convergence of such methods might not hold on all of X, and that the rates of convergence may depend on the subset of X which x belongs to, is classical (see [23]). From a functional calculus perspective this can be explained as follows. For a sequence $(r_n)_{n\in\mathbb{N}}$ of rational functions which is bounded in $\mathrm{H}^\infty(\mathbb{C}_+)$ and which satisfies $r_n(z) \to \mathrm{e}^{-z}$ for all $z \in \mathbb{C}_+$, convergence of r_n to e^{-z} in $\mathrm{H}^\infty(\mathbb{C}_+)$ (that is, uniform convergence on $\mathbb{C}_+$) generally will not hold. However, for appropriately chosen $(r_n)_{n\in\mathbb{N}}$ one can often find an $\alpha > 0$ such that

$$\sup_{z\in\mathbb{C}_+}\left|\frac{r_n(z)-\mathrm{e}^{-z}}{(1+z)^{\alpha}}\right|\to 0$$

as $n\to\infty$. Now, if A has a bounded $\mathrm{H}^\infty(\mathbb{C}_+)$-calculus then

$$\begin{aligned}\left\|r_n(tA)x-\mathrm{e}^{-tA}x\right\| &= \left\|(r_n(tA)-\mathrm{e}^{-tA})(1+A)^{-\alpha}(1+A^{\alpha})x\right\| \\ &\leq C\left\|\frac{r_n(t\cdot)-\mathrm{e}^{-t\cdot}}{(1+\cdot)^{-\alpha}}\right\|_{\mathrm{H}^\infty(\mathbb{C}_+)}\|(1+A^{\alpha})x\|\to 0\end{aligned}$$

as $n\to\infty$ for all $t\geq 0$ and $x\in\mathrm{D}(A^{\alpha})$. More generally, if

$$\left\|\frac{r_n(t\cdot)-\mathrm{e}^{-t\cdot}}{(1+\cdot)^{-\alpha}}\right\|_F\to 0 \tag{1.15}$$

for some function space norm $\|\cdot\|_F$ and if A has a bounded F-calculus, i.e. if

$$\|f(A)\|\leq C\,\|f\|_F$$

for all $f\in F$ with $C\geq 0$ independent of F, then convergence of $r_n(tA)x$ to $\mathrm{e}^{-tA}x$ as $n\to\infty$ follows in the same manner for $x\in\mathrm{D}(A^{\alpha})$. The rate of convergence then depends on the rate of convergence of $\frac{r_n(t\cdot)-\mathrm{e}^{-t\cdot}}{(1+\cdot)^{-\alpha}}$ to 0 in F.

This viewpoint has been used extensively, either explicitly or implicitly, in the past (see for instance [23] and [53]). Much of this research has focused on the case where $F=\mathrm{AM}_1(\mathbb{C}_+)$, the space of Laplace transforms of bounded measures with $\|f\|_F$ the variation norm of the pre-Laplace transform of $f\in F$. Scalar convergence results such as (1.15) obtained in this manner yield convergence of $r_n(tA)$ to $T(t)x$ on $\mathrm{D}(A^{\alpha})$ for general uniformly bounded semigroups, since any generator $-A$ of a uniformly bounded semigroup has a bounded $\mathrm{AM}_1(\mathbb{C}_+)$-calculus, by definition of the Hille-Phillips calculus.

Scalar convergence results of the form (1.15) for function spaces F which are larger than $\mathrm{AM}_1(\mathbb{C}_+)$ necessarily yield convergence of $r_n(tA)x$ to $T(t)x$ for a smaller class of generators A. However, for larger F it might be easier to obtain (1.15) and the convergence might occur with better rates. This is where the functional calculus theory considered in other chapters of this thesis proves to be useful.

In this chapter, by deriving (1.15) for $F=\mathrm{AM}_1(\mathbb{C}_+)$ and $(r_n)_{n\in\mathbb{N}}$ the sequence of subdiagonal Padé approximants, we answer the question posed in [66] in an affirmative manner: $r_n(tA)x\to T(t)x$ as $n\to\infty$ for all $t\in X$ and $x\in\mathrm{D}(A)$, with rate $O\left(\frac{1}{\sqrt{n}}\right)$ and locally uniformly in t. Using results from Chapter 3, we then improve the rates of convergence for generators of exponentially γ-stable semigroups, in particular for exponentially stable semigroups on Hilbert spaces. We also improve the results for generators of analytic semigroups and for operators with a bounded calculus for the class of bounded rational functions on $\mathbb{C}^+$, and we extend our results to obtain convergence on Favard spaces.

One can find applications of these results when considering inversion of the vector-valued Laplace transform. Let Y be a Banach space and let $X := C_{\mathrm{ub}}(\mathbb{R}_+;Y)$ be the space of X-valued uniformly continuous and bounded functions. Let $(T(t))_{t\in\mathbb{R}_+}$ be the left translation semigroup and $-A$ its generator. Then

$$((\lambda + A)^{-1}f)(0) = \int_0^\infty \mathrm{e}^{-\lambda t}(T(t)f)(0)\,\mathrm{d}t = \int_0^\infty \mathrm{e}^{-\lambda t} f(t)\,\mathrm{d}t = \widehat{f}(\lambda) \quad (1.16)$$

for $f \in X$ and $\lambda \in \mathbb{C}_+$, where $\widehat{f}$ is the Laplace transform of f. Since $(T(t)f)(0) = f(t)$ for all $t > 0$ and $f \in X$, convergence of linear combinations of $(\lambda + A)^{-1}f$ to $T(t)f$ implies the convergence of linear combinations of $\widehat{f}(\lambda)$ to $f(t)$. In other words, in this way one can numerically invert the Laplace transform $\widehat{f}$ of f using only knowledge of $\widehat{f}$. This is not the case for other numerical inversion formulas for the Laplace transform, which either make additional assumptions on f or require derivatives of $\widehat{f}$ in the computation (see [85, 117]).

This chapter is based on joint work with Moritz Egert and has appeared in [43].

Appendix A: Growth estimates

In this appendix we provide the proof of an estimate which is vital for Chapter 3, because it implies the logarithmic bound for the growth of the constant bounding $\|f(A)T(t)\|$ as $t \downarrow 0$. The estimate is proved using an adaptation of a lemma due to T. Hytönen in [59].

Appendix B: Estimates for Padé approximants

In this appendix we provide a technical analysis of the behavior of the subdiagonal Padé approximants. These results are essential for the proof of the main result in Chapter 6, but are quite technical and have been placed in an appendix to improve readability of the main text.

Part I

Preliminaries

2

Preliminaries

In this chapter we present the background knowledge which will be used throughout this thesis.

We treat the basic functional calculi that occur in this work, and we introduce some of the function spaces which occur frequently throughout. Then we treat Fourier multipliers on vector-valued L^p-spaces and Besov spaces, and link them to functional calculus theory via transference principles. We discuss the notions of γ-radonifying operators and γ-boundedness, and we give an overview of the basics of real interpolation spaces.

2.1 Notation and terminology

The natural numbers are $\mathbb{N} := \{1, 2, \ldots\}$ and we let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. We write $\mathbb{R}_+ := [0, \infty)$ for the nonnegative reals. The letters X and Y are used to denote Banach spaces over the complex number field $\mathbb{C}$. We write X^* for the dual of X. The space of bounded linear operators from X to Y is denoted by $\mathcal{L}(X, Y)$, and $\mathcal{L}(X) := \mathcal{L}(X, X)$. We identify the algebraic tensor product $X^* \otimes Y$ with the space of finite rank operators in $\mathcal{L}(X, Y)$ via $(x^* \otimes y)(x) := \langle x^*, x\rangle y$ for $x \in X$, $x^* \in X^*$ and $y \in Y$.

The *domain* $\mathrm{D}(A) \subseteq X$ of a closed unbounded operator A on a Banach space X is a Banach space when endowed with the norm

$$\|x\|_{\mathrm{D}(A)} := \|x\| + \|Ax\| \qquad (x \in \mathrm{D}(A)).$$

The *range* of A is denoted by $\mathrm{ran}(A)$, its *spectrum* by $\sigma(A)$, and the *resolvent set* is $\rho(A) := \mathbb{C} \setminus \sigma(A)$. The identity operator on X is denoted by I, and $R(z, A) := (z\mathrm{I} - A)^{-1} \in \mathcal{L}(X)$ is the *resolvent* of A at $z \in \rho(A)$.

The Borel σ-algebra on a Borel measurable subset $W \subseteq \mathbb{C}$ will be denoted by $\mathfrak{B}_W$, and $\mathfrak{B} := \mathfrak{B}_{\mathbb{C}}$. For measurable spaces (Ω_1, Σ_1) and (Ω_2, Σ_2) we denote by $\Sigma_1 \otimes \Sigma_2$ the σ-algebra on $\Omega_1 \times \Omega_2$ generated by all measurable rectangles $W_1 \times W_2$ with $W_1 \in \Sigma_1$ and $W_2 \in \Sigma_2$. If (Ω, Σ) is a measurable space then $\mathcal{B}(\Omega, \Sigma)$ is the space of all bounded Σ-measurable complex-valued functions on Ω, a Banach algebra with the supremum norm

$$\|f\|_{\mathcal{B}(\Omega,\Sigma)} := \sup_{\omega\in\Omega} |f(\omega)| \qquad (f \in \mathcal{B}(\Omega, \Sigma)).$$

We simply write $\mathcal{B}(\Omega) := \mathcal{B}(\Omega, \Sigma)$ and $\|f\|_\infty := \|f\|_{\mathcal{B}(\Omega,\Sigma)}$ when Σ respectively (Ω, Σ) are clear from the context.

If μ is a complex Borel measure on a measurable space (Ω, Σ) and X is a Banach space then a function $f : \Omega \to X$ is *μ-measurable* if there exists a sequence of X-valued simple functions converging to f μ-almost everywhere. For Banach spaces X and Y and a function $f : \Omega \to \mathcal{L}(X, Y)$, we say that f is *strongly measurable* if $\omega \mapsto f(\omega)x$ is a μ-measurable mapping $\Omega \to Y$ for each $x \in X$.

For $p \in [1, \infty]$, $\mathrm{L}^p(\mathbb{R}; X)$ is the Bochner space of equivalence classes of X-valued Lebesgue-measurable, p-integrable functions on $\mathbb{R}$. The Hölder conjugate of p is p', defined by $\frac{1}{p} + \frac{1}{p'} = 1$. The norm on $\mathrm{L}^p(\mathbb{R}; X)$ is usually denoted by $\|\cdot\|_p$. In the case $X = \mathbb{C}$ we simply write $\mathrm{L}^p(\mathbb{R}) := \mathrm{L}^p(\mathbb{R}; \mathbb{C})$.

For $p \in [1, \infty]$ and $m \in \mathbb{N}_0$, $\mathrm{W}^{m,p}(\mathbb{R}; X)$ is the Sobolev space of all $f \in \mathrm{L}^p(\mathbb{R}; X)$ which are m times weakly differentiable with $f^{(k)} \in \mathrm{L}^p(\mathbb{R}; X)$ for all $k \in \{0, 1, \dots, m\}$. We endow $\mathrm{W}^{m,p}(\mathbb{R}; X)$ with the norm

$$\|f\|_{\mathrm{W}^{m,p}(\mathbb{R};X)} := \|f\|_p + \|f^{(m)}\|_p \qquad (f \in \mathrm{W}^{m,p}(\mathbb{R}; X)).$$

We often write $\|\cdot\|_{m,p} = \|\cdot\|_{\mathrm{W}^{m,p}(\mathbb{R};X)}$, and in the case $X = \mathbb{C}$ we let $\mathrm{W}^{m,p}(\mathbb{R}) := \mathrm{W}^{m,p}(\mathbb{R}; \mathbb{C})$.

The space of uniformly continuous and bounded functions on $\mathbb{R}$ with values in a Banach space X is $\mathrm{C}_{\mathrm{ub}}(\mathbb{R}; X)$. For $m \in \mathbb{N}$, $\mathrm{C}^m_{\mathrm{ub}}(\mathbb{R}; X)$ consists of all $f \in \mathrm{C}_{\mathrm{ub}}(\mathbb{R}; X)$ which are m times differentiable with $f^{(k)} \in \mathrm{C}_{\mathrm{ub}}(\mathbb{R}; X)$ for all $k \in \{1, \dots, m\}$.

For $p \in [1, \infty]$, we denote by ℓ^p the space of all p-summable sequences $(x_k)_{k\in\mathbb{N}} \subset \mathbb{C}$, and by $\ell^p(\mathbb{Z})$ the space of all p-summable sequences $(x_k)_{k\in\mathbb{Z}} \subset \mathbb{C}$. Similarly, $\ell^p(\mathbb{N}_0)$ consists of the p-summable sequences $(x_k)_{k\in\mathbb{N}_0} \subset \mathbb{C}$.

For $p \in [1, \infty]$ we let $\mathcal{S}_p$ denote the *Schatten p-class* of compact operators $T \in \mathcal{L}(\ell^2)$ such that the sequence of singular values $(\lambda_n)_{n=1}^\infty$ of T is an element of ℓ^p, and we let

$$\|T\|_{\mathcal{S}_p} := \|(\lambda_n)_{n=1}^\infty\|_{\ell^p}.$$

For $\omega \in \mathbb{R}$ and $z \in \mathbb{C}$ we let $e_\omega(z) := \mathrm{e}^{\omega z}$. For $\Omega = \mathbb{R}$ or $\Omega = \mathbb{R}_+$, we denote by $\mathrm{M}(\Omega)$ the space of complex-valued Borel measures on Ω, and we write $\mathrm{M}_\omega(\Omega)$ for the distributions μ on Ω of the form $\mu(\mathrm{d}s) = \mathrm{e}^{\omega|s|}\nu(\mathrm{d}s)$ for some $\nu \in \mathrm{M}(\Omega)$. Then $\mathrm{M}_\omega(\Omega)$ is a Banach algebra under convolution with the norm

$$\|\mu\|_{\mathrm{M}_\omega(\Omega)} := \|e_{-\omega}\mu\|_{\mathrm{M}(\Omega)}.$$

For $\mu \in \mathrm{M}_\omega(\Omega)$ we let $\mathrm{supp}(\mu)$ be the support of $e_{-\omega}\mu$. A function g such that $e_{-\omega}g \in \mathrm{L}^1(\Omega)$ is usually identified with its associated measure $\mu \in \mathrm{M}_\omega(\Omega)$ given by $\mu(\mathrm{d}s) = g(s)\mathrm{d}s$.

For an open subset $\Omega \neq \emptyset$ of $\mathbb{C}$ we let $\mathrm{H}^\infty(\Omega)$ be the space of bounded holomorphic functions on Ω, a unital Banach algebra with respect to the norm

$$\|f\|_\infty := \|f\|_{\mathrm{H}^\infty(\Omega)} := \sup_{z\in\Omega} |f(z)| \qquad (f \in \mathrm{H}^\infty(\Omega)).$$

We shall mainly consider the case where Ω is equal to a right half-plane

$$\mathrm{R}_\omega := \{z \in \mathbb{C} \mid \mathrm{Re}(z) > \omega\}$$

for some $\omega \in \mathbb{R}$ (we write $\mathbb{C}_+ := \mathrm{R}_0$), or a strip of the form

$$\mathrm{St}_\omega := \{z \in \mathbb{C} \mid |\mathrm{Im}(z)| < \omega\}$$

for $\omega > 0$, with $\mathrm{St}_0 := \mathbb{R}$. At times we shall also let Ω be a sector

$$\mathrm{S}_\varphi := \{z \in \mathbb{C} \setminus \{0\} \mid |\arg z| < \varphi\} \tag{2.1}$$

for $\varphi \in (0, \pi)$, or a parabola

$$\Pi_\omega := \left\{z^2 \mid z \in \mathrm{St}_\omega\right\} \tag{2.2}$$

for $\omega \geq 0$.

For $\omega \in \mathbb{R}$ and $f \in \mathrm{H}^\infty(\mathrm{R}_\omega)$, we let $f(\omega + \mathrm{i}\cdot) \in \mathrm{L}^\infty(\mathbb{R})$ denote the *trace* of the holomorphic function f on the boundary $\partial\mathrm{R}_\omega = \omega + \mathrm{i}\mathbb{R}$, given by

$$f(\omega + \mathrm{i}s) := \lim_{\omega' \searrow \omega} f(\omega' + \mathrm{i}s) \tag{2.3}$$

for almost all $s \in \mathbb{R}$. Then $\|f(\omega + \mathrm{i}\cdot)\|_{\mathrm{L}^\infty(\mathbb{R})} = \|f\|_{\mathrm{H}^\infty(\mathrm{R}_\omega)}$ (see [104, Corollary 5.17]).

The *Schwartz class* $\mathcal{S}(\mathbb{R}; X)$ is the space of X-valued rapidly decreasing smooth functions on $\mathbb{R}$, and the space of X-valued *tempered distributions* is $\mathcal{S}'(\mathbb{R}; X)$. The *Fourier transform* of an X-valued tempered distribution $\Phi \in \mathcal{S}'(\mathbb{R}; X)$ is denoted by $\mathcal{F}\Phi$. For example, if $\mu \in \mathrm{M}_\omega(\mathbb{R})$ for $\omega > 0$ then $\mathcal{F}\mu \in \mathrm{H}^\infty(\mathrm{St}_\omega) \cap \mathrm{C}(\overline{\mathrm{St}_\omega})$ is given by

$$\mathcal{F}\mu(z) := \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}sz}\mu(\mathrm{d}s) \qquad (z \in \mathrm{St}_\omega).$$

For $\omega \in \mathbb{R}$ and $\mu \in \mathrm{M}_\omega(\mathbb{R}_+)$ we let $\widehat{\mu} \in \mathrm{H}^\infty(\mathrm{R}_\omega) \cap \mathrm{C}(\overline{\mathrm{R}_\omega})$,

$$\widehat{\mu}(z) := \int_0^\infty \mathrm{e}^{-zs}\,\mu(\mathrm{d}s) \qquad (z \in \mathrm{R}_\omega),$$

be the *Laplace-Stieltjes transform* of μ.

If μ is a positive measure on a measurable space (Ω, Σ) and $f : \Omega \to [0, \infty]$ is a function then we let

$$\overline{\int_\Omega} f(\omega)\,\mathrm{d}\mu(\omega) := \inf \int_\Omega g(\omega)\,\mathrm{d}\mu(\omega) \in [0, \infty],$$

where the infimum is taken over all measurable $g : \Omega \to [0, \infty]$ such that $g(\omega) \geq f(\omega)$ for $\omega \in \Omega$.

The indicator function of a subset W of a set Ω is denoted by $\mathbf{1}_W$. We will often identify functions defined on W with their extensions to Ω by setting them equal to zero off W, in particular if $\Omega = \mathbb{R}$ and $W = \mathbb{R}_+$.

For convenience we abbreviate the coordinate function $z \mapsto z$ simply by the letter z. Under this convention, $f = f(z)$ for a function f defined on some domain $\Omega \subseteq \mathbb{C}$.

Let X be a topological space and $f : X \to \mathbb{C}$, $g : X \to \mathbb{R}_+$, $x_0 \in X$. We write $f(x) \in O(g(x))$ as $x \to x_0$ if there exists a neighborhood $U \subseteq X$ of x_0 and a constant $C \geq 0$ such that $|f(x)| \leq Cg(x)$ for all $x \in U$.

We will occasionally use the abbreviation SOT for the strong operator topology.

2.2 Functional calculus

Here we summarize some of the basics of functional calculus theory for generators of operator (semi)groups. For more on operator (semi)groups see [45].

2.2.1 Semigroup generators

A C_0*-semigroup* $(T(t))_{t\in\mathbb{R}_+} \subseteq \mathcal{L}(X)$ is a strongly continuous representation of $(\mathbb{R}_+, +)$ on a Banach space X. Each C_0-semigroup $(T(t))_{t\in\mathbb{R}_+}$ is of *type* (M, ω) for some $M \geq 1$ and $\omega \in \mathbb{R}$, which means that $\|T(t)\| \leq M\mathrm{e}^{\omega t}$ for all $t \geq 0$. If the semigroup is of type $(M, 0)$ for some $M \geq 1$, then it is *uniformly bounded*, and the semigroup is *exponentially stable* if it is of type (M, ω) for some $\omega < 0$.

The *generator* of T is the unique closed operator $-A$ such that

$$(\lambda + A)^{-1}x = \int_0^\infty \mathrm{e}^{-\lambda t} T(t)x\,\mathrm{d}t \qquad (x \in X)$$

for $\mathrm{Re}(\lambda)$ large. The *Hille-Phillips (functional) calculus* for A is defined as follows. Fix $M \geq 1$ and $\omega_0 \in \mathbb{R}$ such that T is of type $(M, -\omega_0)$. For $\mu \in \mathrm{M}_{\omega_0}(\mathbb{R}_+)$ define $T_\mu \in \mathcal{L}(X)$ by

$$T_\mu x := \int_0^\infty T(t)x\,\mu(\mathrm{d}t) \qquad (x \in X). \tag{2.4}$$

For $f = \widehat{\mu}$ set $f(A) := T_\mu$. (This is allowed by the injectivity of the Laplace transform.) The mapping $f \mapsto f(A)$ is an algebra homomorphism. In a second step the definition of $f(A)$ is extended to a larger class of functions on R_{ω_0} via *regularization*:

$$f(A) := e(A)^{-1}(ef)(A)$$

if there exists $\mu \in \mathrm{M}_{\omega_0}(\mathbb{R}_+)$ such that $e = \widehat{\mu}$, such that $e(A)$ is injective and such that $ef = \widehat{\nu}$ for some $\nu \in \mathrm{M}_{\omega_0}(\mathbb{R}_+)$. Then $f(A)$ is a closed and (in general) unbounded operator on X and the definition of $f(A)$ is independent of the choice of regularizer e. The following lemma shows in particular that for $\omega < \omega_0$ the operator $f(A)$ is defined for all $f \in \mathrm{H}^\infty(\mathrm{R}_\omega)$ by virtue of the regularizer $e(z) = (z - \lambda)^{-1}$, where $\mathrm{Re}(\lambda) < \omega$.

Lemma 2.1. *Let $\alpha > \frac{1}{2}$, $\lambda \in \mathbb{C}$ and $\omega, \omega_0 \in \mathbb{R}$ with $\mathrm{Re}(\lambda) < \omega < \omega_0$. Let $f \in \mathrm{H}^\infty(\mathrm{R}_\omega)$. Then there exists $\mu \in \mathrm{M}_{\omega_0}(\mathbb{R}_+)$ with*

$$f(z)(z - \lambda)^{-\alpha} = \widehat{\mu}(z)$$

for all $z \in \mathrm{R}_{\omega_0}$.

Proof. After shifting we may suppose that $\omega = 0$. Set $h(z) := f(z)(z - \lambda)^{-\alpha}$ for $z \in \mathbb{C}_+$. Then $h(\mathrm{i} \cdot + a) \in \mathrm{L}^2(\mathbb{R})$ with

$$\|h(\mathrm{i} \cdot + a)\|^2_{\mathrm{L}^2(\mathbb{R})} \leq \int_{\mathbb{R}} \frac{|f(\mathrm{i}s + a)|^2}{|\mathrm{i}s + a - \lambda|^{2\alpha}} \, \mathrm{d}s \leq \|f\|^2_{\mathrm{H}^\infty(\mathbb{C}_+)} \int_{\mathbb{R}} \frac{1}{|\mathrm{i}s - \lambda|^{2\alpha}} \, \mathrm{d}s$$

for all $a > 0$, hence the Paley-Wiener Theorem ([104, Theorem 5.28]) implies that $h = \widehat{g}$ for some $g \in \mathrm{L}^2(\mathbb{R}_+)$. Now $\mu(\mathrm{d}s) := g(s)\mathrm{d}s$ defines $\mu \in \mathrm{M}_{\omega_0}(\mathbb{R}_+)$ as required. □

The Hille–Phillips calculus is an extension of the holomorphic functional calculus for operators of half-plane type. An operator A is *of half-plane type* $\omega_0 \in \mathbb{R}$ if $\sigma(A) \subseteq \overline{\mathrm{R}_{\omega_0}}$ with

$$\sup \{\|R(\lambda, A)\| \mid \lambda \in \mathbb{C} \setminus \mathrm{R}_\omega\} < \infty \qquad \text{for all } \omega < \omega_0.$$

For such an A, one can associate bounded operators $f(A) \in \mathcal{L}(X)$ with

$$f \in \mathcal{E}(\mathrm{R}_\omega) := \left\{ g \in \mathrm{H}^\infty(\mathrm{R}_\omega) \,\middle|\, g(z) \in O(|z|^{-\alpha}) \text{ for some } \alpha > 1 \text{ as } |z| \to \infty \right\}$$

for $\omega < \omega_0$. This is done using a Cauchy integral

$$f(A) := \frac{1}{2\pi \mathrm{i}} \int_{\partial \mathrm{R}_{\omega'}} f(z) R(z, A) \, \mathrm{d}z,$$

where $\partial \mathrm{R}_{\omega'}$ is the positively oriented boundary of $\mathrm{R}_{\omega'}$ for $\omega' \in (\omega, \omega_0)$. This procedure is independent of the choice of ω' by Cauchy's theorem, and yields an algebra homomorphism $\mathcal{E}(\mathrm{R}_\omega) \to \mathcal{L}(X)$. Just as for the Hille-Phillips calculus, the definition of $f(A)$ is extended to a larger class of functions by regularization:

$$f(A) := e(A)^{-1}(ef)(A)$$

if there exists $e \in \mathcal{E}(\mathrm{R}_\omega)$ with $e(A)$ injective and $ef \in \mathcal{E}(\mathrm{R}_\omega)$. This yields a closed unbounded operator $f(A)$ on X, and the definition of $f(A)$ is independent of the choice of the regularizer e. Each $f \in \mathrm{H}^\infty(\mathrm{R}_\omega)$ is regularizable by the function $(\lambda - z)^{-2}$ for $\mathrm{Re}(\lambda) < \omega$. It is straightforward to check that

$$f(A) + g(A) \subseteq (f+g)(A) \qquad \text{and} \qquad f(A)g(A) \subseteq (fg)(A) \tag{2.5}$$

if f and g are regularizable functions, with $\mathrm{D}(f(A)+g(A)) = \mathrm{D}(f(A)) \cap \mathrm{D}(g(A))$ and $\mathrm{D}(f(A)g(A)) = \mathrm{D}((fg)(A)) \cap \mathrm{D}(g(A))$. Hence equality holds in (2.5) if $g(A) \in \mathcal{L}(X)$.

If $-A$ generates a C_0-semigroup of type (M, ω_0) then A is an operator of half-plane type $-\omega_0$. Conversely, the following lemma from [7, Proposition 2.5] gives a functional calculus characterization of when an operator of half-plane type generates a C_0-semigroup.

Lemma 2.2. *Let A be an operator of half-plane type $\omega \in \mathbb{R}$ on a Banach space X. Then $-A$ generates a C_0-semigroup $(T(t))_{t\in\mathbb{R}_+} \subseteq \mathcal{L}(X)$ if and only if A is densely defined and if $e^{-tA} \in \mathcal{L}(X)$ for all $t \in [0,1]$, with $\sup_{t\in[0,1]} \|e^{-tA}\| < \infty$. In this case $T(t) = e^{-tA}$ for all $t \in \mathbb{R}_+$.*

If $-A$ generates a C_0-semigroup of type $(M, -\omega_0)$ then [7, Proposition 2.8] and [55, Proposition 3.3.2] imply that for $\omega < \omega_0$ and $f \in \mathrm{H}^\infty(\mathrm{R}_\omega)$ the definitions of $f(A)$ via the Hille-Phillips calculus and the half-plane calculus coincide.

For $-A$ the generator of a C_0-semigroup $(T(t))_{t\in\mathbb{R}_+} \subseteq \mathcal{L}(X)$ of type (M, ω_0) on a Banach space X, we will at times consider the scaled semigroup $(e^{\omega t}T(t))_{t\in\mathbb{R}_+} \subseteq \mathcal{L}(X)$ of type $(M, \omega_0 + \omega)$ for $\omega \in \mathbb{R}$. It is straightforward to check that this C_0-semigroup is generated by $-(A-\omega)$. The following lemma relates the functional calculi of A and $A - \omega$.

Lemma 2.3. *Let A be an operator of half-plane type $\omega_0 \in \mathbb{R}$ on a Banach space X, and let $\omega \in \mathbb{R}$. Then $A - \omega$ is of half-plane type $\omega_0 - \omega$, and*

$$f(\cdot + \omega)(A - \omega) = f(A)$$

for all $\omega' < \omega_0$ and $f \in \mathrm{H}^\infty(\mathrm{R}_{\omega'})$.

Proof. It is straightforward to check that $A - \omega$ is of half-plane type $\omega_0 - \omega$, since $R(\lambda, A) = R(\lambda - \omega, A - \omega)$ for $\lambda \in \rho(A)$. Let $\omega' < \omega_0$, $\nu \in (\omega', \omega_0)$ and $f \in \mathcal{E}(\mathrm{R}_{\omega'})$. Then $f(\cdot + \omega) \in \mathcal{E}(\mathrm{R}_{\omega' - \omega})$ and

$$\begin{aligned} f(\cdot+\omega)(A-\omega) &= \frac{1}{2\pi \mathrm{i}}\int_{\partial \mathrm{R}_{\nu-\omega}} f(z+\omega)R(z,A-\omega)\,\mathrm{d}z \\ &= \frac{1}{2\pi \mathrm{i}}\int_{\partial \mathrm{R}_{\nu}} f(z)R(z-\omega,A-\omega)\,\mathrm{d}z \\ &= \frac{1}{2\pi \mathrm{i}}\int_{\partial \mathrm{R}_{\nu}} f(z)R(z,A)\,\mathrm{d}z = f(A). \end{aligned}$$

For general $f \in \mathrm{H}^\infty(\mathrm{R}_{\omega'})$, let $e(z) := (\lambda - z)^{-2}$ for $\mathrm{Re}(\lambda) < \omega'$ and $z \in \mathrm{R}_{\omega'}$. Then $e, ef \in \mathcal{E}(\mathrm{R}_{\omega'})$ and $e(\cdot+\omega), (ef)(\cdot+\omega) \in \mathcal{E}(\mathrm{R}_{\omega'-\omega})$. By what has already been shown,

$$\begin{aligned} f(\cdot+\omega)(A-\omega) &= (e(\cdot+\omega)(A-w))^{-1}(ef)(\cdot+\omega)(A-\omega) \\ &= e(A)^{-1}ef(A) = f(A). \end{aligned}$$ □

The next lemma will be fundamental throughout this thesis. The proof is taken from [7, Theorem 3.1].

Lemma 2.4 (Convergence Lemma for operators of half-plane type). *Let A be an operator of half-plane type $\omega_0 \in \mathbb{R}$ on a Banach space X. Let $\omega < \omega_0$ and let $(f_j)_{j\in J} \subseteq \mathrm{H}^\infty(\mathrm{R}_\omega)$ be a net satisfying the following conditions:*

- $\sup\left\{|f_j(z)| \mid z \in \mathrm{R}_\omega, j \in J\right\} < \infty$;
- $f(z) := \lim_{j\in J} f_j(z)$ *exists for all* $z \in \mathrm{R}_\omega$.

Then $f \in \mathrm{H}^\infty(\mathrm{R}_\omega)$ and $f_j(A)x \to f(A)x$ for all $x \in \mathrm{D}(A^2)$.

Suppose moreover that A is densely defined and that $f_j(A) \in \mathcal{L}(X)$ for all $j \in J$ with $\sup_{j\in J}\left\|f_j(A)\right\| < \infty$. Then $f(A) \in \mathcal{L}(X)$, $f_j(A) \to f(A)$ strongly and

$$\|f(A)\| \le \limsup_{j\in J}\left\|f_j(A)\right\|.$$

Proof. Vitali's theorem ([3, Theorem A.5]) implies that $f \in \mathrm{H}^\infty(\mathrm{R}_w)$ and that $f_j(z) \to f(z)$ uniformly on compact subsets of R_ω. Let $\lambda < \omega < \omega' < \omega_0$. Then

$$\left(\frac{f_j(z)}{(\lambda-z)^2}\right)(A) = \lim_{n\to\infty}\frac{-1}{2\pi}\int_{-n}^{n}\frac{f_j(\omega'+\mathrm{i}s)}{(\lambda-\omega'-\mathrm{i}s)^2}R(\omega'+\mathrm{i}s)\,\mathrm{d}s$$

for each $j \in J$, where the limit is uniform in j. Since f_j converges to f uniformly on compacts, it is straightforward to show that $\left(f_j(z)(\lambda-z)^{-2}\right)(A)$ converges to $\left(f(z)(\lambda-z)^{-2}\right)(A)$ in norm. Hence

$$\begin{aligned} f_j(A)x &= \left(f_j(z)(\lambda-z)^{-2}\right)(A)(\lambda-A)^2x \\ &\to \left(f(z)(\lambda-z)^{-2}\right)(A)(\lambda-A)^2x = f(A)x \end{aligned}$$

for all $x \in \mathrm{D}(A^2)$. If A is densely defined, then $\mathrm{D}(A^2)$ is dense as well, and we deduce that $f(A) \in \mathcal{L}(X)$ and $\|f(A)\| \le \limsup_j \left\|f_j(A)\right\|$. The density of $\mathrm{D}(A^2)$ also implies that $f_j(A) \to f(A)$ strongly. □

Let A be an operator of half-plane type ω_0 and $\omega < \omega_0$. For a Banach algebra F of functions continuously embedded in $\mathrm{H}^\infty(\mathrm{R}_\omega)$, we say that A has a *bounded F-calculus* if there exists a constant $C \geq 0$ such that $f(A) \in \mathcal{L}(X)$ with

$$\|f(A)\|_{\mathcal{L}(X)} \leq C\,\|f\|_F \qquad \text{for all } f \in F. \tag{2.6}$$

The following result due to Le Merdy (see [79]) shows that on a Hilbert space the generator of a semigroup (that is even immediately compact) need not have a bounded H^∞-calculus.

Proposition 2.5. *Let H be a separable infinite dimensional Hilbert space. Then there exists an operator A on H such that $-A$ generates an exponentially stable C_0-semigroup $(T(t))_{t\in\mathbb{R}_+}$ such that $T(t)$ is compact for all $t > 0$ and such that A does not have a bounded $\mathrm{H}^\infty(\mathbb{C}_+)$-calculus.*

Finally, in Chapter 6 we will consider operators A of half-plane type 0 on a Banach space X which have a bounded $\mathcal{R}(\mathbb{C}_+)$-calculus, where $\mathcal{R}(\mathbb{C}_+)$ consists of all rational functions in $\mathrm{H}^\infty(\mathbb{C}_+)$. In other words, we assume that there exists a constant $C \geq 0$ such that

$$\|r(A)\|_{\mathcal{L}(X)} \leq C\,\|r\|_{\mathrm{H}^\infty(\mathbb{C}_+)} \tag{2.7}$$

for all $r \in \mathcal{R}(\mathbb{C}_+)$. By [55, Proposition F.3], the closure of $\mathcal{R}(\mathbb{C}_+)$ in $\mathrm{H}^\infty(\mathbb{C}_+)$ is the algebra

$$\mathcal{A}(\mathbb{C}_+) := \left\{ f \in \mathrm{H}^\infty(\mathbb{C}_+) \cap \mathrm{C}(\overline{\mathbb{C}_+}) \,\middle|\, \lim_{z\to\infty} f(z) \text{ exists} \right\}.$$

For $f \in \mathcal{A}(\mathbb{C}_+)$ set

$$f(A) := \lim_{n\to\infty} r_n(A),$$

where $(r_n)_{n\in\mathbb{N}} \subseteq \mathcal{R}(\mathbb{C}_+)$ is such that $\|r_n - f\|_{\mathrm{H}^\infty(\mathbb{C}_+)}$ as $n \to \infty$. This definition is independent of the choice of approximating sequence $(r_n)_{n\in\mathbb{N}} \subseteq \mathcal{R}(\mathbb{C}_+)$, and is the unique way in which to extend the definition of $f(A)$ to all $f \in \mathcal{A}(\mathbb{C}_+)$ so that

$$\|f(A)\|_{\mathcal{L}(X)} \leq C\,\|f\|_{\mathrm{H}^\infty(\mathbb{C}_+)}$$

holds for all $f \in \mathcal{A}(\mathbb{C}_+)$. Now $f \mapsto f(A)$ is a continuous algebra homomorphism $\mathcal{A}(\mathbb{C}_+) \to \mathcal{L}(X)$, and one can extend this calculus to all $f \in \mathrm{H}^\infty(\mathbb{C}_+) \cap \mathrm{C}(\overline{\mathbb{C}_+})$ by regularizing. Then (2.5) holds.

After Proposition 2.5, we choose to conclude with the positive result that (2.7) holds if $-A$ generates a contraction semigroup on a Hilbert space (see [55, Theorem 7.1.7]).

Proposition 2.6. *Let $-A$ generate a C_0-semigroup $(T(t))_{t\in\mathbb{R}_+} \subseteq \mathcal{L}(H)$ of type $(1, 0)$ on a Hilbert space H. Then A has a bounded $\mathcal{R}(\mathbb{C}_+)$-calculus. More precisely,* (2.7) *holds with $C = 1$.*

2.2.2 Group generators

A C_0-*group* $U = (U(s))_{s\in\mathbb{R}} \subseteq \mathcal{L}(X)$ is a strongly continuous representation of $(\mathbb{R}, +)$ on a Banach space X. For each C_0-group U the *group type* of U,

$$\theta(U) := \inf\left\{\omega \geq 0 \,\middle|\, \exists M \geq 1 : \|U(s)\| \leq M\mathrm{e}^{\omega|s|} \text{ for all } s \geq 0\right\}, \qquad (2.8)$$

is finite. A C_0-group $(U(s))_{s\in\mathbb{R}}$ is *(uniformly) bounded* if $\sup_{s\in\mathbb{R}} \|U(s)\| < \infty$.

The *generator* of a C_0-group $(U(s))_{s\in\mathbb{R}} \subseteq \mathcal{L}(X)$ is the unique closed operator $-\mathrm{i}A$ on X which generates the C_0-semigroup $(U(t))_{t\in\mathbb{R}_+}$. Such an A is a *strip type operator* of *height* $\omega_0 := \theta(U)$. This means that $\sigma(A) \subseteq \overline{\mathrm{St}_{\omega_0}}$ and

$$\sup_{\lambda\in\mathbb{C}\setminus\mathrm{St}_\omega} \|R(\lambda, A)\| < \infty \qquad \text{for all } \omega > \omega_0.$$

The *strip type functional calculus* for A is defined in a similar manner as for operators of half-plane type. Let

$$\mathcal{E}(\mathrm{St}_\omega) := \left\{g \in \mathrm{H}^\infty(\mathrm{St}_\omega) \,\middle|\, g(z) \in O(|z|^{-\alpha}) \text{ for some } \alpha > 1 \text{ as } |\mathrm{Re}(z)| \to \infty\right\}$$

for $\omega > \omega_0$. Associate an operator $f(A) \in \mathcal{L}(X)$ with $f \in \mathcal{E}(\mathrm{St}_\omega)$ by a Cauchy-type integral

$$f(A) := \frac{1}{2\pi\mathrm{i}} \int_{\partial\mathrm{St}_{\omega'}} f(z)R(z, A)\,\mathrm{d}z.$$

Here $\partial\mathrm{St}_{\omega'}$ is the positively oriented boundary of $\mathrm{St}_{\omega'}$ for $\omega' \in (\omega_0, \omega)$. This procedure is independent of the choice of ω' by Cauchy's theorem, and yields an algebra homomorphism $\mathcal{E}(\mathrm{St}_\omega) \to \mathcal{L}(X)$. The definition of $f(A)$ is extended to a larger class of functions by regularization as before:

$$f(A) := e(A)^{-1}(ef)(A)$$

if there exists $e \in \mathcal{E}(\mathrm{St}_\omega)$ with $e(A)$ injective and $ef \in \mathcal{E}(\mathrm{St}_\omega)$. This yields a closed unbounded operator $f(A)$ on X, and the definition of $f(A)$ is independent of the choice of the regularizer e. Each $f \in \mathrm{H}^\infty(\mathrm{St}_\omega)$ is regularizable by the function $z \mapsto (\lambda - z)^{-2}$ for $|\mathrm{Im}(\lambda)| > \omega$, and (2.5) holds.

Let $-\mathrm{i}A$ generate a C_0-group on a Banach space X. Then the Hille-Phillips functional calculus for A yields certain functions f that give rise to bounded operators $f(A)$. Fix $M \geq 1$ and $\omega \geq 0$ such that $\|U(s)\| \leq M\mathrm{e}^{\omega|s|}$ for all $s \in \mathbb{R}$. For $\mu \in \mathrm{M}_{-\omega}(\mathbb{R})$ define

$$U_\mu x := \int_{\mathbb{R}} U(s)x\,\mu(\mathrm{d}s) \qquad (x \in X). \qquad (2.9)$$

Then $\mu \mapsto U_\mu$ is an algebra homomorphism $\mathrm{M}_{-\omega}(\mathbb{R}) \to \mathcal{L}(X)$. The following lemma from [57, Lemma 2.2] shows that the Hille-Phillips calculus extends the strip type calculus for A.

Lemma 2.7. *Let X A and U be as above, and let $\omega' > \omega \geq 0$.*

a) For each $f \in \mathcal{E}(\mathrm{St}_{\omega'})$ there exists a $\mu \in \mathrm{M}_{-\omega}(\mathbb{R})$ such that $f = \mathcal{F}\mu$.
b) Let $\mu \in \mathrm{M}_{-\omega}(\mathbb{R})$ be such that $\mathcal{F}\mu$ extends to a regularizable function on $\mathrm{St}_{\omega'}$. Then $\mathcal{F}\mu(A) = U_\mu \in \mathcal{L}(X)$ and

$$\sup_{t \in \mathbb{R}} \|\mathcal{F}\mu(t + A)\| \leq M \|\mu\|_{\mathrm{M}_{-\omega}(\mathbb{R})} .$$

The next lemma from [55, Proposition 5.1.7] is the strip type version of Lemma 2.4. The proof is similar to that of Lemma 2.4.

Lemma 2.8 (Convergence Lemma for strip type operators). *Let A be a densely defined strip type operator of height ω_0 on a Banach space X. Let $\omega > \omega_0$ and let $(f_j)_{j \in J} \subseteq \mathrm{H}^\infty(\mathrm{St}_\omega)$ be a net satisfying the following conditions:*

- $\sup_{j \in J} \|f_j\|_{\mathrm{H}^\infty(\mathrm{St}_\omega)} < \infty$;
- $f(z) := \lim_j f_j(z)$ *exists for all* $z \in \mathrm{St}_\omega$;
- $\sup_{j \in J} \|f_j(A)\|_{\mathcal{L}(X)} < \infty$.

Then $f \in \mathrm{H}^\infty(\mathrm{St}_\omega)$, $f(A) \in \mathcal{L}(X)$, $f_j(A) \to f(A)$ strongly and

$$\|f(A)\| \leq \limsup_{j \in J} \|f_j(A)\| .$$

Let A be a strip type operator of height ω_0 and $\omega > \omega_0$. For a Banach algebra F of functions that is continuously embedded in $\mathrm{H}^\infty(\mathrm{St}_\omega)$, we say that A has a *bounded F-calculus* if there exists a constant $C \geq 0$ such that $f(A) \in \mathcal{L}(X)$ with

$$\|f(A)\|_{\mathcal{L}(X)} \leq C \|f\|_F \qquad \text{for all } f \in F.$$

In Proposition 2.5 we have seen that semigroup generators on Hilbert spaces do not have a bounded H^∞-calculus in general. For group generators the situation is different, as the following result by Boyadzhiev and de Laubenfels from [22] shows. We will sketch a proof of this result in Section 2.5.

Theorem 2.9. *Let $-\mathrm{i}A$ generate a C_0-group $(U(s))_{s \in \mathbb{R}} \subseteq \mathcal{L}(H)$ on a Hilbert space H. Then A has a bounded $\mathrm{H}^\infty(\mathrm{St}_\omega)$-calculus for all $\omega > \theta(U)$.*

If $(U(s))_{s \in \mathbb{R}}$ is uniformly bounded then the constant bounding the $\mathrm{H}^\infty(\mathrm{St}_\omega)$-calculus can be chosen independent of $\omega > 0$.

2.2.3 Sectorial operators

An operator A on a Banach space X is *sectorial* of *angle* $\varphi \in (0, \pi)$ if $\sigma(A) \subseteq \overline{\mathrm{S}_\varphi}$, where S_φ is as in (2.1), and

$$\sup \left\{ \|\lambda R(\lambda, A)\| \mid \lambda \in \mathbb{C} \setminus \overline{\mathrm{S}_\psi} \right\} < \infty$$

for all $\psi \in (\varphi, \pi)$. We say that A is sectorial of angle 0 if A is sectorial of angle φ for all $\varphi \in (0, \pi)$. If $-A$ generates a uniformly bounded C_0-semigroup $(T(t))_{t \in \mathbb{R}_+} \subseteq \mathcal{L}(X)$, then A is sectorial of angle $\frac{\pi}{2}$. We say that $(T(t))_{t \in \mathbb{R}_+}$ is *bounded analytic* if A is sectorial of angle $\varphi \in (0, \frac{\pi}{2})$. In this case $(T(t))_{t \in \mathbb{R}_+}$ extends to a holomorphic mapping $T : \mathrm{S}_{\pi/2-\varphi} \to \mathcal{L}(X)$ such that

- $T(\lambda)T(\mu) = T(\lambda + \mu)$ for all $\lambda, \mu \in \mathrm{S}_{\pi/2-\varphi}$;
- $\sup_{\lambda \in \mathrm{S}_\psi} \|T(\lambda)\|_{\mathcal{L}(X)} < \infty$ for all $\psi \in (0, \pi/2 - \varphi)$.

Conversely, each densely defined sectorial operator of angle $\varphi \in (0, \frac{\pi}{2})$ generates a uniformly bounded C_0-semigroup, which is then bounded analytic.

Sectorial operators admit a *sectorial functional calculus*, defined as follows. Let $\psi \in (0, \pi)$ and let $\mathrm{H}^\infty_0(\mathrm{S}_\psi)$ be the class of all $f \in \mathrm{H}^\infty(\mathrm{S}_\psi)$ for which there exist $C \geq 0$ and $s > 0$ such that

$$|f(z)| \leq C \min\{|z|^s, |z|^{-s}\} \qquad (z \in \mathrm{S}_\psi).$$

If A is a sectorial operator of angle $\varphi \in (0, \pi)$ and $f \in \mathrm{H}^\infty_0(\mathrm{S}_\psi)$ for $\psi \in (\varphi, \pi)$ then one can define $f(A) \in \mathcal{L}(X)$ as

$$f(A) := \frac{1}{2\pi \mathrm{i}} \int_{\partial \mathrm{S}_\nu} f(z) R(z, A) \, \mathrm{d}z,$$

where $\nu \in (\varphi, \psi)$ and $\partial \mathrm{S}_\nu$ is the positively oriented boundary of S_ν. This integral converges absolutely and is independent of the choice of ν, by Cauchy's theorem. Furthermore, define

$$g(A) := f(A) + c(\mathrm{I} + A)^{-1} + d$$

if g is of the form $g(\cdot) = f(\cdot) + c(1 + \cdot)^{-1} + d$ for $f \in \mathrm{H}^\infty_0(\mathrm{S}_\psi)$ and $c, d \in \mathbb{C}$. This definition is independent of the particular representation of g and yields an algebra homomorphism

$$\mathrm{H}^\infty_0(\mathrm{S}_\psi) \oplus \langle (1 + \cdot)^{-1} \rangle \oplus \langle \mathbf{1}_\mathbb{C} \rangle \to \mathcal{L}(X), \quad g \mapsto g(A).$$

In the same manner as before one extends this homomorphism by regularization to a larger class of functions on S_ψ, and (2.5) then holds. In particular, if A is injective, then each $f \in \mathrm{H}^\infty(\mathrm{S}_\psi)$ is regularizable by the function $z \mapsto z(z+1)^{-2}$.

For general A and $\alpha \in \mathbb{C}_+$, the function $z \mapsto z^\alpha$ is regularizable by $z \mapsto (1+z)^{-n}$, where $n > \mathrm{Re}(\alpha)$, and yields the *fractional power* A^α of A with domain $\mathrm{D}(A^\alpha)$. Clearly, $A^1 = A$ and $A^0 = \mathrm{I}$. Moreover, $A^{\alpha+\beta} = A^\alpha A^\beta$ for all $\alpha, \beta \in \mathbb{C}$ with $\mathrm{Re}(\alpha), \mathrm{Re}(\beta) > 0$ (see [55, Proposition 3.1.1]. This implies the domain inclusion

$$\mathrm{D}(A^\beta) \subseteq \mathrm{D}(A^\alpha) \tag{2.10}$$

for all $\alpha, \beta \in \mathbb{C}$ with $0 < \mathrm{Re}(\alpha) < \mathrm{Re}(\beta)$.

If $-A$ generates a uniformly bounded C_0-semigroup, then the Hille-Phillips calculus extends the holomorphic functional calculus for angles $\psi \in (\frac{\pi}{2}, \pi)$, see Lemma 3.3.1 and Proposition 3.3.2 in [55]. In particular, for $\alpha \in \mathbb{C}_+$ the fractional power A^α defined by the Hille-Phillips calculus yields the same operator as in the sectorial calculus.

Let $\psi \in (0, \pi)$ and let F be a Banach algebra of functions that is continuously embedded in $\mathrm{H}^\infty(\mathrm{S}_\psi)$. Similarly as for half-plane type operators and strip type operators, we say that a sectorial operator A of angle $\varphi \in (0, \psi)$ has a *bounded F-calculus* if there exists a constant $C \geq 0$ such that $f(A) \in \mathcal{L}(X)$ with

$$\|f(A)\|_{\mathcal{L}(X)} \leq C \|f\|_F \qquad \text{for all } f \in F.$$

If A is an injective sectorial operator of angle $\varphi \in (0, \pi)$ then $z \mapsto \log(z)$ is regularizable. An injective sectorial operator A has *bounded imaginary powers* if $-\mathrm{i}\log(A)$ is the generator of a C_0-group $(U(s))_{s\in\mathbb{R}} \subseteq \mathcal{L}(X)$ on X. Then $U(s) = A^{-\mathrm{i}s}$ for all $s \in \mathbb{R}$, and for $\psi \in [0, \pi)$ we write $A \in \mathrm{BIP}(X, \psi)$ if $\theta(U) \leq \psi$ (here $\theta(U)$ is the group type of U from (2.8)). We write $A \in \mathrm{BIP}(X)$ if $A \in \mathrm{BIP}(X, \psi)$ for some $\psi \in [0, \pi)$. If $A \in \mathrm{BIP}(X, \psi)$ then A is sectorial of angle ψ (see [55, Corollary 4.3.4]).

Proposition 2.5 showed that the generator of an exponentially stable semigroup on a Hilbert space need not have a bounded H^∞-calculus. That result in fact also yields an example of a sectorial operator without bounded imaginary powers. However, for historical reasons we state the following result which shows that on a Hilbert space, a sectorial operator of angle 0 need not have bounded imaginary powers. It is due to Baillon and Clément ([6]) and appeared around the same time as a similar example of McIntosh and Yagi ([89]).

Proposition 2.10. *Let H be a separable Hilbert space. Then there exists an injective sectorial operator A on H of angle 0 such that $A^{\mathrm{i}s} \notin \mathcal{L}(H)$ for all $s \in \mathbb{R} \setminus \{0\}$. In particular, $A \notin \mathrm{BIP}(X)$ and A does not have a bounded $\mathrm{H}^\infty(\mathrm{S}_\psi)$-calculus for any $\psi \in (0, \pi)$.*

We conclude by recalling that Theorem 1.1 gives a characterization of when an injective sectorial operator on a Hilbert space has a bounded H^∞-calculus, in terms of square function estimates. Moreover, Theorem 1.2 provides a large class of sectorial operators which have a bounded H^∞-calculus.

2.2.4 Generators of cosine functions

In this section we briefly discuss the basics of functional calculus theory for cosine function generators, as will be needed in Chapter 4. As the procedure here is analogous to that in previous sections, we do not provide details. For more information on functional calculus for generators of cosine functions see [60].

A *cosine function* $\mathrm{Cos} : \mathbb{R} \to \mathcal{L}(X)$ on a Banach space X is a strongly continuous mapping such that $\mathrm{Cos}(0) = I$ and

$$\mathrm{Cos}(t+s) + \mathrm{Cos}(t-s) = 2\mathrm{Cos}(t)\mathrm{Cos}(s)$$

for all $s, t \in \mathbb{R}$. Then

$$\theta(\mathrm{Cos}) = \inf\left\{\omega \geq 0 \,\middle|\, \exists M \geq 0 : \|\mathrm{Cos}(t)\| \leq M\mathrm{e}^{\omega|t|} \text{ for all } t \in \mathbb{R}\right\} < \infty.$$

The *generator* of a cosine function is the unique operator $-A$ on X that satisfies

$$\lambda R(\lambda^2, -A) = \int_0^\infty \mathrm{e}^{-\lambda t}\mathrm{Cos}(t)\,\mathrm{d}t$$

for $\lambda > \theta(\mathrm{Cos})$. The solution to the abstract Cauchy problem

$$u''(t) = Au, \qquad u(0) = x, \qquad u'(0) = 0$$

is then given by $u(t) = \mathrm{Cos}(t)x$ for $t \in \mathbb{R}$.

If $-A$ generates a cosine function, then A is an operator of *parabola type* $\omega = \theta(\mathrm{Cos})$. This means that $\sigma(A) \subseteq \overline{\Pi_\omega}$, where Π_ω is as in (2.2), and that for all $\omega' > \omega$ there exists $M_{\omega'} \geq 0$ such that

$$\|R(\lambda, A)\| \leq \frac{M_{\omega'}}{\sqrt{|\lambda|}\left(|\mathrm{Im}(\sqrt{\lambda})| - \omega'\right)}$$

for all $\lambda \notin \Pi_{\omega'}$. For such operators there is a natural functional calculus, constructed in the same way as for half-plane type, strip type and sectorial operators. In particular, if $f \in \mathrm{H}^\infty(\Pi_{\omega'})$ for $\omega' > \omega$ then $f(A)$ is defined as an unbounded operator on X.

The following proposition, a combination of [70, Theorem 2] and [57, Theorem 6.2], will be needed in Chapter 4.

Proposition 2.11. *Let $-A$ generate a cosine function $(\mathrm{Cos}(s))_{s\in\mathbb{R}} \subseteq \mathcal{L}(X)$ on a Banach space X. Then there is a unique subspace $V \subseteq X$ with* $\mathrm{D}(A) \subseteq V$ *such that the operator $-\mathrm{i}\mathcal{A}$, where*

$$\mathcal{A} := \mathrm{i}\begin{bmatrix} 0 & \mathrm{I}_V \\ -A & 0 \end{bmatrix}$$

with domain $\mathrm{D}(\mathcal{A}) := \mathrm{D}(A) \times V$, generates a C_0-group $(U(s))_{s\in\mathbb{R}}$ on $V \times X$. Moreover, $\theta(\mathrm{Cos}) = \theta(U)$.

The space V in Proposition 2.11 is called the *Kisyński space*.

2.2.5 Scalar type operators

In this section we summarize some of the basics of scalar type operators, taken from [42].

Let X be a Banach space, and recall that $\mathfrak{B}$ denotes the Borel σ-algebra on $\mathbb{C}$. A *spectral measure* on X is a map $E : \mathfrak{B} \to \mathcal{L}(X)$ such that the following hold:

- $E(\emptyset) = 0$ and $E(\mathbb{C}) = \mathrm{I}$;
- $E(W_1 \cap W_2) = E(W_1)E(W_2)$ for all $W_1, W_2 \in \mathfrak{B}$;
- $E(W_1 \cup W_2) = E(W_1) + E(W_2) - E(W_1)E(W_2)$ for all $W_1, W_2 \in \mathfrak{B}$;
- E is σ-additive in the strong operator topology.

These conditions imply that E is projection-valued. Moreover, by the Uniform Boundedness Principle there exists (see [42, Corollary XV.2.4]) a constant K such that

$$\|E(W)\|_{\mathcal{L}(X)} \leq K \qquad (W \in \mathfrak{B}). \tag{2.11}$$

An operator $A \in \mathcal{L}(X)$ is a *spectral operator* if there exists a spectral measure E on X such that $AE(W) = E(W)A$ and $\sigma(A, E(W)X) \subseteq \overline{W}$ for all $W \in \mathfrak{B}$, where $\sigma(A, E(W)X)$ denotes the spectrum of A in the space $E(W)X$. For a spectral operator A, we let $\nu(A)$ denote the minimal constant K occurring in (2.11) and call $\nu(A)$ the *spectral constant of* A. This is well-defined since the spectral measure E associated with A is unique, cf. [42, Corollary XV.3.8]. Moreover, E is supported on $\sigma(A)$ in the sense that $E(\sigma(A)) = \mathrm{I}$ ([42, Corollary XV.3.5]). Hence we can define an integral with respect to E of bounded Borel measurable functions on $\sigma(A)$, as follows. For $f = \sum_{j=1}^{n} \alpha_j \mathbf{1}_{W_j}$ a finite simple function with $\alpha_j \in \mathbb{C}$ and $W_j \subseteq \sigma(A)$ mutually disjoint Borel sets for $1 \leq j \leq n$, let

$$\int_{\sigma(A)} f \,\mathrm{d}E := \sum_{j=1}^{n} \alpha_j E(W_j). \tag{2.12}$$

This definition is independent of the representation of f, and

$$\begin{aligned}
\left\| \int_{\sigma(A)} f \,\mathrm{d}E \right\|_{\mathcal{L}(X)} &= \sup_{\|x\|_X = \|x^*\|_{X^*} = 1} \left| \sum_{j=1}^{n} \alpha_j \langle x^*, E(W_j)x \rangle \right| \\
&\leq \sup_j |\alpha_j| \sup_{\|x\|_X = \|x^*\|_{X^*} = 1} \|\langle x^*, E(\cdot)x \rangle\|_{\mathrm{M}(\mathbb{C})} \\
&\leq 4 \|f\|_{\mathcal{B}(\sigma(A))} \sup_{\|x\|_X = \|x^*\|_{X^*} = 1} \sup_{W \subseteq \sigma(A)} |\langle x^*, E(W)x \rangle| \\
&\leq 4\nu(A) \|f\|_{\mathcal{B}(\sigma(A))},
\end{aligned}$$

where $\|x^* E(\cdot)x\|_{\mathrm{M}(\mathbb{C})}$ is the variation norm of the measure $x^* E(\cdot)x$ on $\mathbb{C}$. Since the simple functions lie dense in $\mathcal{B}(\sigma(A))$, for general $f \in \mathcal{B}(\sigma(A))$ we may define

$$\int_{\sigma(A)} f \, \mathrm{d}E := \lim_{n\to\infty} \int_{\sigma(A)} f_n \, \mathrm{d}E \in \mathcal{L}(X)$$

if $(f_n)_{n\in\mathbb{N}} \subseteq \mathcal{B}(\sigma(A))$ is a sequence of simple functions with $\|f_n - f\|_\infty \to 0$ as $n \to \infty$. This definition is independent of the choice of approximating sequence and

$$\left\| \int_{\sigma(A)} f \, \mathrm{d}E \right\|_{\mathcal{L}(X)} \leq 4\nu(A) \, \|f\|_{\mathcal{B}(\sigma(A))} . \tag{2.13}$$

It is straightforward to check that

$$\int_{\sigma(A)} (\alpha f + g) \, \mathrm{d}E = \alpha \int_{\sigma(A)} f \, \mathrm{d}E + \int_{\sigma(A)} g \, \mathrm{d}E,$$
$$\int_{\sigma(A)} fg \, \mathrm{d}E = \left(\int_{\sigma(A)} f \, \mathrm{d}E \right) \left(\int_{\sigma(A)} g \, \mathrm{d}E \right)$$

for all $\alpha \in \mathbb{C}$ and simple $f, g \in \mathcal{B}(\sigma(A))$, and approximation extends these identities to general $f, g \in \mathcal{B}(\sigma(A))$. Moreover, $\int_{\sigma(A)} \mathbf{1} \, \mathrm{d}E = E(\sigma(A)) = \mathrm{I}$. Hence the map $f \mapsto \int_{\sigma(A)} f \, \mathrm{d}E$ is a continuous morphism $\mathcal{B}(\sigma(A)) \to \mathcal{L}(X)$ of unital Banach algebras. Since $\sigma(A)$ is compact, the identity function $\lambda \mapsto \lambda$ is bounded on $\sigma(A)$ and $\int_{\sigma(A)} \lambda \, \mathrm{d}E(\lambda) \in \mathcal{L}(X)$ is well defined.

Definition 2.12. *A* scalar type operator *is a spectral operator $A \in \mathcal{L}(X)$ with spectral measure E such that*

$$A = \int_{\sigma(A)} \lambda \, \mathrm{d}E(\lambda).$$

The class of scalar type operators on X is denoted by $\mathrm{Scal}(X)$.

For $A \in \mathrm{Scal}(X)$ with spectral measure E and $f \in \mathcal{B}(\sigma(A))$ we define

$$f(A) := \int_{\sigma(A)} f \, \mathrm{d}E. \tag{2.14}$$

Then, as remarked above, $f \mapsto f(A)$ is a continuous morphism $\mathcal{B}(\sigma(A)) \to \mathcal{L}(X)$ of unital Banach algebras with norm bounded by $4\nu(A)$. Note also that

$$\langle x^*, f(A)x \rangle = \int_{\sigma(A)} f(\lambda) \, \mathrm{d}\langle x^*, E(\lambda)x \rangle \tag{2.15}$$

for all $f \in \mathcal{B}(\sigma(A))$, $x \in X$ and $x^* \in X^*$. Indeed, for simple functions this follows from (2.12), and by taking limits one obtains (2.15) for general $f \in \mathcal{B}(\sigma(A))$.

Finally, we note that a normal operator A on a Hilbert space H is a scalar type operator with $\nu(A) = 1$, and in this case (2.13) improves to

$$\left\| \int_{\sigma(A)} f \, \mathrm{d}E \right\|_{\mathcal{L}(H)} \leq \|f\|_{\mathcal{B}(\sigma(A))} , \tag{2.16}$$

as is known from the Borel functional calculus for normal operators.

2.3 Function spaces

In this section we introduce some of the function spaces which will be used throughout this work.

2.3.1 Besov spaces

First we define the Besov spaces which will be used in Chapters 4 and 5. We combine material from [93], [118] and [1].

Let $\psi \in \mathrm{C}^\infty(\mathbb{R})$ be a nonnegative function with support in $[\frac{1}{2}, 2]$ such that

$$\sum_{k=-\infty}^{\infty} \psi(2^{-k}s) = 1 \qquad \text{for all } s \in (0,\infty).$$

For $k \in \mathbb{N}$ and $s \in \mathbb{R}$ let $\varphi_k(s) := \psi(2^{-k}|s|)$, and let $\varphi_0(s) := 1 - \sum_{k=1}^{\infty} \varphi_k(s)$. Let X be a Banach space and let $p, q \in [1,\infty]$ and $r \in \mathbb{R}$ be given. The *(inhomogeneous) Besov space* $\mathrm{B}^r_{p,q}(\mathbb{R};X)$ consists of all X-valued tempered distributions $f \in \mathcal{S}'(\mathbb{R};X)$ such that

$$\|f\|_{\mathrm{B}^r_{p,q}(\mathbb{R};X)} := \left\| \left(2^{kr} \|\mathcal{F}^{-1}(\varphi_k) * f\|_{\mathrm{L}^p(\mathbb{R};X)} \right)_{k\in\mathbb{N}_0} \right\|_{\ell^q(\mathbb{N}_0)} < \infty,$$

endowed with the norm $\|\cdot\|_{\mathrm{B}^r_{p,q}(\mathbb{R};X)}$. Then $\mathrm{B}^r_{p,q}(\mathbb{R};X)$ is a Banach space such that $\mathcal{S}(\mathbb{R};X) \subseteq \mathrm{B}^r_{p,q}(\mathbb{R};X)$, and a different choice of ψ leads to an equivalent norm on $\mathrm{B}^r_{p,q}(\mathbb{R};X)$.

The following continuous inclusions hold for each Banach space X, $m \in \mathbb{N}_0$ and $p \in [1,\infty)$:

$$\mathrm{B}^m_{p,1}(\mathbb{R};X) \subseteq \mathrm{W}^{m,p}(\mathbb{R};X) \subseteq \mathrm{B}^m_{p,\infty}(\mathbb{R};X), \tag{2.17}$$

where $\mathrm{W}^{m,p}(\mathbb{R};X)$ is the X-valued Sobolev space with paramaters m and p. These are proved in the same way as for $X = \mathbb{C}$, i.e. one proves the result for $m = 0$ directly and then uses that $f' \in \mathrm{B}^{s-1}_{p,q}(\mathbb{R};X)$ if and only if $f \in \mathrm{B}^s_{p,q}(\mathbb{R};X)$ for $s \in \mathbb{R}$ and $p, q \in [1,\infty]$. For details see [118, Proposition 2.5.7]. Similarly, the continuous inclusions

$$\mathrm{B}^m_{\infty,1}(\mathbb{R};X) \subseteq \mathrm{C}^m_{\mathrm{ub}}(\mathbb{R};X) \subseteq \mathrm{B}^m_{\infty,\infty}(\mathbb{R};X) \tag{2.18}$$

hold for each Banach space X and each $m \in \mathbb{N}_0$. The norm bounds of the inclusions in (2.17) and (2.18) do not depend on the underlying space X.

In the case $X = \mathbb{C}$ we also define the homogeneous Besov spaces. Let ψ be as before and let $\psi_k(s) := \psi(2^{-k}|s|)$ for $k \in \mathbb{Z}$ and $s \in \mathbb{R}$. For $p, q \in [1,\infty]$ and $r \in \mathbb{R}$, we let $\dot{\mathrm{B}}^r_{p,q}(\mathbb{R})$ be the space of all $f \in \mathcal{S}'(\mathbb{R})$ such that

$$\|f\|_{\dot{\mathrm{B}}^r_{p,q}(\mathbb{R})} := \left\| \left(2^{kr} \|\mathcal{F}^{-1}(\psi_k) * f\|_{\mathrm{L}^p(\mathbb{R})} \right)_{k\in\mathbb{Z}} \right\|_{\ell^q(\mathbb{Z})} < \infty,$$

modulo polynomials. More precisely, if $f \in \mathcal{S}'(\mathbb{R})$ is such that $\|f\|_{\dot{\mathrm{B}}^r_{p,q}(\mathbb{R})} < \infty$, then it admits a representation

$$f = \sum_{k \in \mathbb{Z}} \mathcal{F}^{-1}(\psi_k) * f + P$$

for some polynomial P (see [93, Chapter 3]). We let the *homogeneous Besov space* $\dot{\mathrm{B}}^r_{p,q}(\mathbb{R})$ consist of all those f for which $P = 0$, endowed with the norm $\|\cdot\|_{\dot{\mathrm{B}}^r_{p,q}(\mathbb{R})}$. Then $\dot{\mathrm{B}}^r_{p,q}(\mathbb{R})$ is a Banach space.

2.3.2 The class $\mathfrak{A}$

In Chapter 5 another function space will play a vital role. Let $W_1, W_2 \subseteq \mathbb{C}$ be Borel measurable subsets and let $\mathfrak{A}(W_1 \times W_2)$ be the class of Borel functions $\varphi : W_1 \times W_2 \to \mathbb{C}$ such that

$$\varphi(\lambda_1, \lambda_2) = \int_\Omega a_1(\lambda_1, \omega) a_2(\lambda_2, \omega)\, \mathrm{d}\mu(\omega) \tag{2.19}$$

for all $(\lambda_1, \lambda_2) \in W_1 \times W_2$, where (Ω, Σ, μ) is a finite measure space (with μ positive) and $a_1 \in \mathcal{B}(W_1 \times \Omega, \mathfrak{B}_{W_1} \otimes \Sigma)$, $a_2 \in \mathcal{B}(W_2 \times \Omega, \mathfrak{B}_{W_2} \otimes \Sigma)$ are bounded Borel measurable functions. For $\varphi \in \mathfrak{A}(W_1 \times W_2)$ let

$$\|\varphi\|_{\mathfrak{A}(W_1 \times W_2)} := \inf \int_\Omega \|a_1(\cdot, \omega)\|_{\mathcal{B}(W_1)} \|a_2(\cdot, \omega)\|_{\mathcal{B}(W_2)}\, \mathrm{d}\mu(\omega),$$

where the infimum runs over all possible representations[1] in (2.19). One can show that the map $\omega \mapsto \|a_1(\cdot, \omega)\|_{\mathcal{B}(W_1)} \|a_2(\cdot, \omega)\|_{\mathcal{B}(W_2)}$ is measurable by first considering simple a_1 and a_2 and then approximating general a_1, a_2 uniformly by simple functions.

Remark 2.13. The class $\mathfrak{A}(\sigma_1 \times \sigma_2)$ coincides with the class of functions $\varphi : W_1 \times W_2 \mapsto \mathbb{C}$ admitting the representation

$$\varphi(\lambda_1, \lambda_2) = \int_\Omega b_1(\lambda_1, \omega) b_2(\lambda_2, \omega)\, \mathrm{d}\nu(\omega) \tag{2.20}$$

for all $(\lambda_1, \lambda_2) \in W_1 \times W_2$, where (Ω, Σ, ν) is a measure space and $b_j : W_j \times \Omega \to \mathbb{C}$, for $j = 1, 2$, are such that $b_j(\cdot, \omega) : W_j \to \mathbb{C}$ is a bounded Borel function for $\omega \in \Omega$, with

$$\int_\Omega \|b_1(\cdot, \omega)\|_{\mathcal{B}(W_1)} \|b_2(\cdot, \omega)\|_{\mathcal{B}(W_2)}\, \mathrm{d}\nu(\omega) < \infty.$$

[1] Taking an infimum over all such representations might seem problematic from a set-theoretic viewpoint. The problem can be fixed by choosing an equivalence class of such a representation for each real number which can occur in the infimum.

Indeed, any $\varphi \in \mathfrak{A}(W_1 \times W_2)$ has a representation as in (2.20), and conversely any $\varphi : W_1 \times W_2 \to \mathbb{C}$ satisfying (2.20) also satisfies (2.19), with $a_j \in \mathcal{B}(W_j \times \Omega)$ defined by

$$a_j(\lambda_j, \omega) := \frac{b(\lambda_j, \omega)}{\|b(\cdot, \omega)\|_{\mathcal{B}(W_j)}}$$

for $j = 1, 2$, $\lambda_j \in \sigma_j$ and $\omega \in \Omega$, and with the finite measure μ given by $\mathrm{d}\mu(\omega) = \|b_1(\cdot, \omega)\|_{\mathcal{B}(W_1)} \|b_2(\cdot, \omega)\|_{\mathcal{B}(W_2)} \,\mathrm{d}\nu(\omega)$.

Lemma 2.14. *For all $W_1, W_2 \subseteq \mathbb{C}$ measurable, $\mathfrak{A}(W_1 \times W_2)$ is a unital Banach algebra which is contractively included in $\mathcal{B}(W_1 \times W_2)$.*

Proof. The proof is straightforward, we provide it for completeness.

Fix $W_1, W_2 \subseteq \mathbb{C}$ measurable and let $\varphi_1, \varphi_2 \in \mathfrak{A}(W_1 \times W_2)$. For $j = 1, 2$, let $(\Omega_j, \Sigma_j, \mu_j)$ be finite measure spaces and let $a_{1,j} \in \mathcal{B}(W_1 \times \Omega_j, \mathfrak{B}_{W_1} \otimes \Sigma_j)$, $a_{2,j} \in \mathcal{B}(W_2 \times \Omega_j, \mathfrak{B}_{W_2} \otimes \Sigma_j)$ be bounded Borel measurable functions such that

$$\varphi_j(\lambda_1, \lambda_2) = \int_{\Omega_j} a_{1,j}(\lambda_1, \omega) a_{2,j}(\lambda_2, \omega) \,\mathrm{d}\mu_j(\omega)$$

for all $(\lambda_1, \lambda_2) \in W_1 \times W_2$. Let (Ω, Σ, μ) be the direct sum measure space of $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ and define $a_1 \in \mathcal{B}(W_1 \times \Omega, \mathfrak{B}_{W_1} \otimes \Sigma)$ by

$$a_1(\lambda, \omega) := \begin{cases} a_{1,1}(\lambda, \omega) & \text{if } \omega \in \Omega_1 \\ a_{1,2}(\lambda, \omega) & \text{if } \omega \in \Omega_2 \end{cases}$$

for $\lambda \in W_1$. Define $a_2 \in \mathcal{B}(W_1 \times \Omega, \mathfrak{B}_{W_2} \otimes \Sigma)$ similarly. Then

$$\begin{aligned} &\varphi_1(\lambda_1, \lambda_2) + \varphi_2(\lambda_1, \lambda_2) \\ &= \int_{\Omega_1} a_{1,1}(\lambda_1, \omega) a_{2,1}(\lambda_2, \omega) \,\mathrm{d}\mu_1(\omega) + \int_{\Omega_2} a_{1,2}(\lambda_1, \omega) a_{2,2}(\lambda_2, \omega) \,\mathrm{d}\mu_2(\omega) \\ &= \int_{\Omega} a_1(\lambda_1, \omega) a_2(\lambda_2, \omega) \,\mathrm{d}\mu(\omega) \end{aligned}$$

for all $(\lambda_1, \lambda_2) \in W_1 \times W_2$, and

$$\begin{aligned} &\int_{\Omega} \|a_1(\cdot, \omega)\|_{\mathcal{B}(W_1)} \|a_2(\cdot, \omega)\|_{\mathcal{B}(W_2)} \,\mathrm{d}\mu(\omega) \\ &= \int_{\Omega_1} \|a_{1,1}(\cdot, \omega)\|_{\mathcal{B}(W_1)} \|a_{2,1}(\cdot, \omega)\|_{\mathcal{B}(W_2)} \,\mathrm{d}\mu_1(\omega) \\ &+ \int_{\Omega_2} \|a_{1,2}(\cdot, \omega)\|_{\mathcal{B}(W_1)} \|a_{2,2}(\cdot, \omega)\|_{\mathcal{B}(W_2)} \,\mathrm{d}\mu_2(\omega). \end{aligned}$$

Therefore $\varphi_1 + \varphi_2 \in \mathfrak{A}(W_1 \times W_2)$ with

$$\|\varphi_1 + \varphi_2\|_{\mathfrak{A}(W_1 \times W_2)} \le \|\varphi_1\|_{\mathfrak{A}(W_1 \times W_2)} + \|\varphi_2\|_{\mathfrak{A}(W_1 \times W_2)}.$$

To show that $\varphi_1\varphi_2 \in \mathfrak{A}(W_1 \times W_2)$, let (Ω, Σ, μ) be the product of the measure spaces $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$. Let

$$a_1(\lambda, (\omega_1, \omega_2)) := a_{1,1}(\lambda, \omega_1) a_{1,2}(\lambda, \omega_2)$$

for $\lambda \in W_1$, $(\omega_1, \omega_2) \in \Omega_1 \times \Omega_2$, and define a_2 similarly. Then

$$\begin{aligned}
&\varphi_1(\lambda_1, \lambda_2)\varphi_2(\lambda_1, \lambda_2) \\
&= \int_{\Omega_1} a_{1,1}(\lambda_1, \omega_1) a_{2,1}(\lambda_2, \omega_1) \, \mathrm{d}\mu_1(\omega_1) \int_{\Omega_2} a_{1,2}(\lambda_1, \omega_2) a_{2,2}(\lambda_2, \omega_2) \, \mathrm{d}\mu_2(\omega_2) \\
&= \int_{\Omega} a_1(\lambda_1, (\omega_1, \omega_2)) a_2(\lambda_2, (\omega_1, \omega_2) \, \mathrm{d}\mu((\omega_1, \omega_2))
\end{aligned}$$

for $(\lambda_1, \lambda_2) \in W_1 \times W_2$ and

$$\begin{aligned}
&\int_{\Omega} \|a_1(\cdot, (\omega_1, \omega_2))\|_{\mathcal{B}(W_1)} \|a_2(\cdot, (\omega_1, \omega_2))\|_{\mathcal{B}(W_2)} \, \mathrm{d}\mu((\omega_1, \omega_2)) \\
&\leq \int_{\Omega_1} \|a_{1,1}(\cdot, \omega_1)\|_{\mathcal{B}(W_1)} \|a_{1,2}(\cdot, \omega_1)\|_{\mathcal{B}(W_2)} \, \mathrm{d}\mu_1(\omega_1) \\
&\cdot \int_{\Omega_2} \|a_{1,2}(\cdot, \omega_2)\|_{\mathcal{B}(W_1)} \|a_{2,2}(\cdot, \omega_2)\|_{\mathcal{B}(W_2)} \, \mathrm{d}\mu_2(\omega_2).
\end{aligned}$$

If $\varphi \in \mathfrak{A}(W_1 \times W_2)$ and $\lambda \in \mathbb{C}$, then multiplying μ by λ in (2.19) shows that $\lambda\varphi \in \mathfrak{A}(W_1 \times W_2)$ with $\|\lambda\varphi\|_{\mathfrak{A}(W_1 \times W_2)} = |\lambda| \, \|\varphi\|_{\mathfrak{A}(W_1 \times W_2)}$. Hence we have shown that $\mathfrak{A}(W_1 \times W_2)$ is an algebra. Moreover,

$$|\varphi(\lambda_1, \lambda_2)| \leq \int_{\Omega} \|a_1(\cdot, \omega)\|_{\mathcal{B}(W_1)} \|a_2(\cdot, \omega)\|_{\mathcal{B}(W_2)} \, \mathrm{d}\mu(\omega)$$

for all $\varphi \in \mathfrak{A}(W_1 \times W_2)$ with representation (2.19) and $(\lambda_1, \lambda_2) \in W_1 \times W_2$. Taking the infimum over all representations as in (2.19) yields that $\mathfrak{A}(W_1 \times W_2)$ is contractively embedded in $\mathcal{B}(W_1 \times W_2)$, and now the considerations above imply that $\|\cdot\|_{\mathfrak{A}(W_1 \times W_2)}$ is a normed algebra.

To show that $\mathfrak{A}(W_1 \times W_2)$ is complete, let $(\varphi_n)_{n \in \mathbb{N}} \subseteq \mathfrak{A}(W_1 \times W_2)$ be a sequence in $\mathfrak{A}(W_1 \times W_2)$ with $\sum_{n=1}^{\infty} \|\varphi_n\|_{\mathfrak{A}(W_1 \times W_2)} < \infty$. For each $n \in \mathbb{N}$ let $(\Omega_n, \Sigma_n, \mu_n)$ and $a_{1,n} \in \mathcal{B}(W_1 \times \Omega_n, \mathfrak{B}_{W_1} \otimes \Sigma_n)$, $a_{2,n} \in \mathcal{B}(W_2 \times \Omega_n, \mathfrak{B}_{W_2} \otimes \Sigma_n)$ be such that

$$\varphi_n(\lambda_1, \lambda_2) = \int_{\Omega_n} a_{1,n}(\lambda_1, \omega) a_{2,n}(\lambda_2, \omega) \, \mathrm{d}\mu_n(\omega)$$

for all $(\lambda_1, \lambda_2) \in W_1 \times W_2$ and

$$\int_{\Omega_n} \|a_{1,n}(\cdot, \omega)\|_{\mathcal{B}(W_1)} \|a_{2,n}(\cdot, \omega)\|_{\mathcal{B}(W_2)} \, \mathrm{d}\mu_n(\omega) \leq \|\varphi_n\|_{\mathfrak{A}(W_1 \times W_2)} + 2^{-n}.$$

Let (Ω, Σ, μ) be the direct sum of the measure spaces $(\Omega_n, \Sigma_n, \mu_n)$ for $n \in \mathbb{N}$, and define $a_1 \in \mathcal{B}(W_1 \times \Omega, \mathfrak{B}_{W_1} \otimes \Sigma)$, $a_2 \in \mathcal{B}(W_2 \times \Omega, \mathfrak{B}_{W_2} \otimes \Sigma)$ by $a_1(\lambda, \omega) := a_{1,n}(\lambda, \omega)$ and $a_2(\lambda, \omega) := a_{2,n}(\lambda, \omega)$ if $\omega \in \Omega_n$. Then

$$\begin{aligned}&\int_\Omega \|a_1(\cdot,\omega)\|_{\mathcal{B}(W_1)}\|a_2(\cdot,\omega)\|_{\mathcal{B}(W_2)}\,\mathrm{d}\mu(\omega)\\&=\sum_{n=1}^\infty\int_{\Omega_n}\|a_{1,n}(\cdot,\omega)\|_{\mathcal{B}(W_1)}\|a_{2,n}(\cdot,\omega)\|_{\mathcal{B}(W_2)}\,\mathrm{d}\mu_n(\omega)<\infty\end{aligned}$$

and

$$\varphi(\lambda_1,\lambda_2):=\sum_{n=1}^\infty\varphi_n(\lambda_1,\lambda_2)=\int_\Omega a_1(\lambda_1,\omega)a_2(\lambda_2,\omega)\,\mathrm{d}\mu(\omega)$$

uniformly in $\mathcal{B}(\mathbb{C}^2)$. Hence $\varphi\in\mathfrak{A}(W_1\times W_2)$ and the series $\sum_{n=1}^\infty\varphi_n$ converges in $\mathfrak{A}(W_1\times W_2)$ to φ. We conclude that $\mathfrak{A}(W_1\times W_2)$ is complete. □

We now state sufficient conditions for a function to belong to $\mathfrak{A}$. The first is [101, Theorem 9] and will be used in Chapter 5.

Lemma 2.15. *Let $g\in\mathrm{W}^{1,2}(\mathbb{R})$ and let*

$$\psi_g(\lambda_1,\lambda_2):=\begin{cases}g(\log(\frac{\lambda_1}{\lambda_2}))\ \textit{if}\ \lambda_1,\lambda_2>0\\0\qquad\qquad\quad\ \textit{otherwise}\end{cases}.\tag{2.21}$$

Then $\psi_g\in\mathfrak{A}(\mathbb{R}^2)$ with $\|\psi_g\|_{\mathfrak{A}(\mathbb{R}^2)}\leq\sqrt{2}\,\|g\|_{\mathrm{W}^{1,2}(\mathbb{R})}$.

The second condition involves the homogeneous Besov space $\dot{\mathrm{B}}^1_{\infty,1}(\mathbb{R})$. For $f\in\dot{\mathrm{B}}^1_{\infty,1}(\mathbb{R})$, define $\psi_f:\mathbb{R}^2\to\mathbb{C}$ by

$$\psi_f(\lambda_1,\lambda_2):=\begin{cases}\frac{f(\lambda_2)-f(\lambda_1)}{\lambda_2-\lambda_1}\ \text{if}\ (\lambda_1,\lambda_2)\in\mathbb{R}^2\ \text{and}\ \lambda_1\neq\lambda_2\\f'(\lambda_1)\qquad\quad\ \text{if}\ \lambda_1=\lambda_2\in\mathbb{R}\end{cases}.$$

Lemma 2.16. *There exists a constant $C\geq0$ such that $\psi_f\in\mathfrak{A}(\mathbb{R}^2)$ for each $f\in\dot{\mathrm{B}}^1_{\infty,1}(\mathbb{R})$, with $\|\psi_f\|_{\mathfrak{A}(\mathbb{R}^2)}\leq C\|f\|_{\dot{\mathrm{B}}^1_{\infty,1}(\mathbb{R})}$.*

Proof. Let $f\in\dot{\mathrm{B}}^1_{\infty,1}(\mathbb{R})$. In [96, Theorem 2] (see also [97, p. 535]) it is shown that ψ_f has a representation

$$\psi_f(\lambda_1,\lambda_2)=\int_\Omega a_1(\lambda_1,\omega)a_2(\lambda_2,\omega)\,\mathrm{d}\mu(\omega)$$

for $(\lambda_1,\lambda_2)\in\mathbb{R}^2$, where (Ω,μ) is a measure space and a_1 and a_2 are measurable functions on $\mathbb{R}\times\Omega$ such that

$$\int_\Omega\|a_1(\cdot,\omega)\|_\infty\,\|a_2(\cdot,\omega)\|_\infty\,\mathrm{d}|\mu|(\omega)\leq C\,\|f\|_{\dot{\mathrm{B}}^1_{\infty,1}(\mathbb{R})}$$

for some constant $C\geq0$ independent of f. Now apply Remark 2.13. □

2.4 Fourier multipliers

In this section we collect some basic facts about Fourier multipliers, taken from [55, Appendix E], [1] and [51].

We note that, recently, in [107] new vector-valued Fourier multiplier theorems were proved using the geometric notions of type and cotype of a Banach space.

2.4.1 Fourier multipliers on L^p-spaces

First we consider Fourier multipliers on vector-valued L^p-spaces.

Let X be a Banach space and let $m \in \mathrm{L}^\infty(\mathbb{R}; \mathcal{L}(X))$ and $p \in [1, \infty]$. Then m is a *bounded* $\mathrm{L}^p(\mathbb{R}; X)$*-Fourier multiplier* if there exists a constant $C \geq 0$ such that

$$T_m(f) := \mathcal{F}^{-1}(m \cdot \mathcal{F}f) \in \mathrm{L}^p(\mathbb{R}; X) \quad \text{and} \quad \|T_m(f)\|_p \leq C \|f\|_p$$

for all $f \in \mathcal{S}(\mathbb{R}; X)$. In this case T_m extends uniquely to a bounded operator on $\mathrm{L}^p(\mathbb{R}; X)$ for $p < \infty$ and on $\mathrm{C}_0(\mathbb{R}; X)$ for $p = \infty$. We let $\|m\|_{\mathcal{M}_p(X)}$ be the norm of the operator T_m and let $\mathcal{M}_p(X)$ be the unital Banach algebra of all bounded $\mathrm{L}^p(\mathbb{R}; X)$-Fourier multipliers, endowed with the norm $\|\cdot\|_{\mathcal{M}_p(X)}$. Then $\mathcal{M}_p(X)$ is contractively embedded in $\mathrm{L}^\infty(\mathbb{R}; \mathcal{L}(X))$.

Each $\mu \in \mathrm{M}(\mathbb{R})$ yields a bounded $\mathrm{L}^p(\mathbb{R}; X)$-Fourier multiplier $\mathcal{F}\mu$ for all $p \in [1, \infty]$, with

$$T_{\mathcal{F}\mu}(f) = L_\mu(f) := \mu * f \tag{2.22}$$

for each $f \in \mathrm{L}^p(\mathbb{R}; X)$. Indeed, Young's inequality implies that

$$\|\mathcal{F}\mu\|_{\mathcal{M}_p(X)} \leq \|\mathcal{F}\mu\|_{\mathcal{M}_1(X)} = \|L_\mu\|_{\mathrm{L}^1(\mathbb{R};X)} = \|\mu\|_{\mathrm{M}(\mathbb{R})} = \|\mathcal{F}\mu\|_{\mathcal{M}_\infty(X)} \tag{2.23}$$

for all $p \in (1, \infty)$. Moreover, a scalar-valued function $m \in \mathrm{L}^\infty(\mathbb{R})$ is a bounded $\mathrm{L}^1(\mathbb{R}; X)$-Fourier multiplier if and only if $m = \mathcal{F}\mu$ for some $\mu \in \mathrm{M}(\mathbb{R})$, and this in turn is the case if and only if m is a bounded $\mathrm{L}^\infty(\mathbb{R}; X)$-Fourier multiplier.

A straightforward computation shows that $m * n \in \mathcal{M}_p(X)$ for each $m \in \mathcal{M}_p(X)$, $n \in \mathrm{L}^1(\mathbb{R})$ and $p \in [1, \infty]$, with

$$\|m * n\|_{\mathcal{M}_p(X)} \leq \|m\|_{\mathcal{M}_p(X)} \|n\|_{\mathrm{L}^1(\mathbb{R})} . \tag{2.24}$$

The following lemma from [55, Lemma E.4.1] will be used later on.

Lemma 2.17. *Let X be a Banach space and $p \in [1, \infty]$.*

i) Let $m \in \mathcal{M}_p(X)$, $a \in \mathbb{R}$ and $b \in \mathbb{R} \setminus \{0\}$. Then $e^{\mathrm{i}a\cdot} m(\cdot), m(b\cdot) \in \mathcal{M}_p(X)$, with

$$\left\| e^{\mathrm{i}a\cdot} m(\cdot) \right\|_{\mathcal{M}_p(X)} = \| m(b\cdot) \|_{\mathcal{M}_p(X)} = \| m \|_{\mathcal{M}_p(X)} . \tag{2.25}$$

ii) Let $m : \mathbb{R} \to \mathbb{C}$ and let $(m_n)_{n\in\mathbb{N}} \subseteq \mathcal{M}_p(X)$ be a $\|\cdot\|_{\mathcal{M}_p(X)}$-bounded sequence such that $m_n(s) \to m(s)$ for almost all $s \in \mathbb{R}$. Then $m \in \mathcal{M}_p(X)$ with $\|m\|_{\mathcal{M}_p(X)} \leq \limsup_{n\to\infty} \|m_n\|_{\mathcal{M}_p(X)}$.

Lemma 8.2.3 in [3] yields a useful class of bounded Fourier multipliers:

Proposition 2.18. *Let $m \in \mathrm{C}^1(\mathbb{R})$ be such that*

$$\max \left\{ \sup_{s\in\mathbb{R}} |s|^{\delta} |m(s)|, \ \sup_{s\in\mathbb{R}} |s|^{1+\delta} |m'(s)| \right\} < \infty$$

for some $\delta > 0$. Then $m \in \mathcal{M}_p(X)$ for each Banach space X and each $p \in [1, \infty]$.

2.4.2 UMD spaces

If X is a Hilbert space then Plancherel's Theorem yields

$$\mathcal{M}_2(X) = \mathrm{L}^\infty(\mathbb{R}; \mathcal{L}(X)) \qquad \text{with} \qquad \|m\|_{\mathcal{M}_2(X)} = \|m\|_{\mathrm{L}^\infty(\mathbb{R};\mathcal{L}(X))} \tag{2.26}$$

for all $m \in \mathcal{M}_2(X)$. If X is not a Hilbert space then such a simple description of $\mathcal{M}_p(X)$ for $p \in (1, \infty)$ is in general not known. In this section we discuss a class of spaces for which a useful sufficient condition for being an $\mathrm{L}^p(\mathbb{R}; X)$-Fourier multiplier is known.

A Banach space X is a *UMD space* if the function $s \mapsto \operatorname{sgn}(s)$ is a bounded $\mathrm{L}^2(X)$-Fourier multiplier. By (2.26), any Hilbert space is a UMD space, and so is the X-valued Bochner space $\mathrm{L}^p(\Omega, \mu; X)$ for (Ω, μ) a measure space, X UMD and $p \in (1, \infty)$. In particular, the scalar-valued L^p-spaces are UMD spaces for $p \in (1, \infty)$. Each UMD space is reflexive (see [100]).

On UMD spaces the following result, the vector-valued Mikhlin Multiplier Theorem, yields a large class of bounded scalar-valued Fourier multipliers. It shows that on UMD spaces one can let $\delta = 0$ in Lemma 2.18. For the following formulation see for instance [55, Theorem E.6.2].

Theorem 2.19. *Let X be a UMD space and $p \in (1, \infty)$. Then there exists a constant $C \geq 0$ such that the following holds. Let $m \in \mathrm{C}^1(\mathbb{R} \setminus \{0\})$ be such that*

$$\max \left\{ \|m\|_{\mathrm{L}^\infty(\mathbb{R})}, \sup_{s\in\mathbb{R}} |sm'(s)| \right\} < \infty.$$

Then $m \in \mathcal{M}_p(X)$ with

$$\|m\|_{\mathcal{M}_p(X)} \leq C \max \left\{ \|m\|_{\mathrm{L}^\infty(\mathbb{R})}, \sup_{s\in\mathbb{R}} |sm'(s)| \right\}.$$

To indicate the use of UMD spaces for functional calculus theory we present a result from [58] which is a version of Theorem 2.9 on UMD spaces. For $\omega > 0$ let

$$\mathrm{H}^\infty_{(1)}(\mathrm{St}_\omega) := \Big\{ f \in \mathrm{H}^\infty(\mathrm{St}_\omega) \,\Big|\, \sup_{z\in\mathrm{St}_\omega} |zf'(z)| < \infty \Big\} \tag{2.27}$$

be the *(homogeneous) analytic Mikhlin algebra* on St_ω, endowed with the norm

$$\|f\|_{\mathrm{H}^\infty_{(1)}(\mathrm{St}_\omega)} := \sup_{z\in\mathrm{St}_\omega} |f(z)| + |zf'(z)| \qquad (f \in \mathrm{H}^\infty_{(1)}(\mathrm{St}_\omega)). \tag{2.28}$$

The first statement in the following result is [58, Theorem 3.6]. The second statement follows in the same manner as the first by using a transference principle for bounded groups, as we will explain in Section 2.5.

Theorem 2.20. *Let $-\mathrm{i}A$ generate a C_0-group $(U(s))_{s\in\mathbb{R}} \subseteq \mathcal{L}(X)$ on a UMD space X. Then A has a bounded $\mathrm{H}^\infty_{(1)}(\mathrm{St}_\omega)$-calculus for all $\omega > \theta(U)$.*

If $(U(s))_{s\in\mathbb{R}}$ is uniformly bounded then the constant bounding the $\mathrm{H}^\infty_{(1)}(\mathrm{St}_\omega)$-calculus can be chosen independent of $\omega > 0$.

2.4.3 Fourier multipliers on Besov spaces

We now collect some facts about Fourier multipliers on (inhomogeneous) vector-valued Besov spaces, to be used in Chapter 4.

Let $m \in \mathrm{L}^\infty(\mathbb{R};\mathcal{L}(X))$, $p,q \in [1,\infty]$ and $r \in \mathbb{R}$. We say that m is a *bounded* $\mathrm{B}^r_{p,q}(\mathbb{R};X)$*-Fourier multiplier* if there is a unique bounded operator $T_m : \mathrm{B}^r_{p,q}(\mathbb{R};X) \to \mathrm{B}^r_{p,q}(\mathbb{R};X)$ such that

$$T_m(f) = \mathcal{F}^{-1}(m \cdot \mathcal{F}f) \tag{2.29}$$

for all $f \in \mathcal{S}(\mathbb{R};X)$. As in (2.22), each $\mu \in \mathrm{M}(\mathbb{R})$ induces a bounded $\mathrm{B}^r_{p,q}(\mathbb{R};X)$-Fourier multiplier $\mathcal{F}\mu$ for all $r \in \mathbb{R}$ and $p,q \in [1,\infty]$, with

$$T_{\mathcal{F}\mu}(f) = L_\mu(f) = \mu * f \qquad (f \in \mathcal{S}(\mathbb{R};X)).$$

The main results about $\mathrm{B}^r_{p,q}(\mathbb{R};X)$-Fourier multipliers that we will need are the following.

Theorem 2.21. *[51, Corollary 4.15] There exists a constant $C \geq 0$ such that the following holds. Let X be a Banach space, $p,q \in [1,\infty]$ and $r \in \mathbb{R}$. If $m : \mathbb{R} \to \mathbb{C}$ is such that $\varphi_k m \in \mathrm{B}^{1/2}_{2,1}(\mathbb{R};\mathbb{C})$ for all $k \in \mathbb{N}_0$, and*

$$M := \sup_{k\in\mathbb{N}_0} \inf_{a>0} \|(\varphi_k m)(a\cdot)\|_{\mathrm{B}^{1/2}_{2,1}(\mathbb{R};\mathbb{C})} < \infty,$$

then m is a bounded $\mathrm{B}^r_{p,q}(\mathbb{R};X)$-Fourier multiplier with $\|T_m\|_{\mathcal{L}(\mathrm{B}^r_{p,q}(\mathbb{R};X))} \leq CM$.

It should be noted that some of the results in [51] were improved in [65]. However, the actual improvements occur when considering Fourier multipliers on $\mathbb{R}^n$ for $n > 1$, hence for us the results in [51] suffice.

Corollary 2.22. *There exists a constant $C \geq 0$ such that for all Banach spaces X, $p, q \in [1, \infty]$, $r \in \mathbb{R}$ and all $m \in \mathrm{C}^1(\mathbb{R}; \mathbb{C})$ with*

$$N := \sup_{s \in \mathbb{R}} |m(s)| + (1 + |s|)|m'(s)| < \infty,$$

m is a bounded $\mathrm{B}^r_{p,q}(\mathbb{R}; X)$-Fourier multiplier with $\|T_m\|_{\mathcal{L}(\mathrm{B}^r_{p,q}(\mathbb{R};X))} \leq CN$.

Proof. This follows as in [51, Corollary 4.11]. See also [51, Remark 4.16]. □

Note that, for Fourier multipliers on Besov spaces, no geometric assumptions on X are needed, in contrast with the situation for $\mathrm{L}^p(\mathbb{R}; X)$-multipliers, cf. Theorem 2.19. Note however that singularities of m at zero are not allowed in Corollary 2.22, whereas in Theorem 2.19 m was allowed to have a singularity at zero. This is relevant for Chapter 4.

2.5 Transference principles

In this section we discuss some of the basic transference principles that will be used throughout this work.

We first state the classical transference principle by Berkson, Gillespie and Muhly from [13], for uniformly bounded groups. For $\mu \in \mathrm{M}(\mathbb{R})$ and $p \in [1, \infty]$, recall the definition of the convolution operator $L_\mu \in \mathcal{L}(\mathrm{L}^p(\mathbb{R}; X))$ from (2.22).

Proposition 2.23. *Let $(U(s))_{s \in \mathbb{R}} \subseteq \mathcal{L}(X)$ be a C_0-group on a Banach space X such that $M := \sup_{s \in \mathbb{R}} \|U(s)\| < \infty$. Let $p \in [1, \infty]$ and $\mu \in \mathrm{M}(\mathbb{R})$. Then*

$$\left\| \int_{\mathbb{R}} U(s)x \, \mu(\mathrm{d}s) \right\| \leq M^2 \|L_\mu\|_{\mathcal{L}(\mathrm{L}^p(\mathbb{R};X))} \|x\|$$

for all $x \in X$.

A transference principle for unbounded groups was established in [58, Theorem 3.2]. For $\omega \geq 0$ and $\mu \in \mathrm{M}_{-\omega}(\mathbb{R})$ let $\mu_\omega \in \mathrm{M}(\mathbb{R})$ be given by

$$\mu_\omega(\mathrm{d}s) := \cosh(\omega s)\mu(\mathrm{d}s). \tag{2.30}$$

Proposition 2.24. *Let $0 \leq \omega_0 < \omega$ and $p \in [1, \infty]$. Then there exists a constant $C > 0$ such that the following holds. Let $(U(s))_{s \in \mathbb{R}} \subseteq \mathcal{L}(X)$ be a C_0-group on a Banach space X such that $\|U(s)\| \leq M \cosh(\omega_0 s)$ for all $s \in \mathbb{R}$ and some $M \geq 1$, and let $\mu \in \mathrm{M}_{-\omega}(\mathbb{R})$. Then*

$$\left\| \int_{\mathbb{R}} U(s)x \, \mu(\mathrm{d}s) \right\| \leq CM^2 \|L_{\mu_\omega}\|_{\mathcal{L}(\mathrm{L}^p(\mathbb{R};X))} \|x\|$$

for all $x \in X$.

We will reprove Propositions 2.23 and 2.24 in Section 4.1. By combining Propositions 2.23 and 2.24 with (2.22) and (2.26), and using Lemma 2.8 as in [58, Theorem 3.6], one can now prove Theorem 2.9. Theorem 2.20 follows in the same manner, appealing to Theorem 2.19 instead of (2.26). In the same way one can obtain the result of Hieber and Prüss that the generator of a bounded group on a UMD space has a bounded H^∞-calculus on double sectors (see [64]).

In [59], an abstract transference principle for general sub-semigroups of locally compact groups was described, of which we present a specific case here. Let $S = \mathbb{R}$ or $S = \mathbb{R}_+$ and let $(T(s))_{s\in S} \subseteq \mathcal{L}(X)$ be a strongly continuous representation of S on a Banach space X such that $\|T(s)\| \leq M\mathrm{e}^{\omega|s|}$ for certain $M \geq 1$, $w \in \mathbb{R}$ and all $s \in S$. Let $\Phi(\mathbb{R}; X)$ and $\Psi(\mathbb{R}; X)$ be Banach spaces of X-valued Bochner measurable functions on $\mathbb{R}$. Let $q \in [1, \infty]$ and let $\varphi \in \mathrm{L}^q(S)$ and $\psi \in \mathrm{L}^{q'}(S)$. Recall that we extend functions and measures on $\mathbb{R}_+$ to $\mathbb{R}$ by setting them equal to zero off $\mathbb{R}_+$. Let

$$\eta(s) := \varphi * \psi(s) = \int_S \varphi(t)\psi(s-t)\,\mathrm{d}t$$

for $s \in S$. Suppose that $\mu \in \mathrm{M}(S)$ is such that $\eta\mu(\mathrm{d}s) := \eta(s)\mu(\mathrm{d}s)$ defines an element $\eta\mu \in \mathrm{M}_{-\omega}(S)$. Now define the following maps:

- $\iota : X \to \Psi(\mathbb{R}; X)$, $\iota x(s) := \psi(-s)T(-s)x$ for $x \in X$ and $s \in \mathbb{R}$;
- $L_\mu : \Psi(\mathbb{R}; X) \to \Phi(\mathbb{R}; X)$, $L_\mu(f) := \mu * f$ for $f \in \Psi(\mathbb{R}; X)$;
- $P : \Phi(\mathbb{R}; X) \to X$, $Pg := \int_S \varphi(t)T(t)g(t)\,\mathrm{d}t$ for $g \in \Phi(\mathbb{R}; X)$.

Recall that

$$T_\nu x := \int_S T(s)x\,\nu(\mathrm{d}s)$$

for $\nu \in \mathrm{M}_{-\omega}(S)$ and $x \in X$. The following is [59, Proposition 2.3]:

Proposition 2.25. *Suppose that the maps ι, L_μ and P are well-defined and bounded. Then the following diagram of bounded maps commutes:*

$$\begin{array}{ccc} \Psi(\mathbb{R}; X) & \xrightarrow{\;L_\mu\;} & \Phi(\mathbb{R}; X) \\ {\scriptstyle\iota}\uparrow & & \downarrow{\scriptstyle P} \\ X & \xrightarrow{\;T_{\eta\mu}\;} & X \end{array}$$

For $S = \mathbb{R}$, Proposition 2.25 yields a transference principle for C_0-groups that can be used to prove Propositions 2.23 and 2.24, as is demonstrated in [59, Section 2]. This will be used in Chapter 4 to yield new results. In Chapter 3 we use the case $S = \mathbb{R}_+$ to obtain a new transference principle for C_0-semigroups.

2.6 γ-Boundedness

In this section we discuss the notions of γ-radonifying operators and γ-boundedness that will be used in Chapters 3 and 6. Much of this material can be found in [120].

2.6.1 γ-Radonifying operators

Recall that a random variable $\gamma : \Omega \to \mathbb{C}$ on a probability space Ω is a *complex-valued standard Gaussian random variable* if $\gamma = \gamma_1 + \mathrm{i}\gamma_2$ for $\gamma_1, \gamma_2 : \Omega \to \mathbb{R}$ independent standard Gaussian random variables. Let H be a Hilbert space and X a Banach space. A linear operator $T : H \to X$ is *γ-summing* if

$$\|T\|_\gamma := \sup_F \Big(\mathbb{E} \Big\| \sum_{h \in F} \gamma_h T h \Big\|_X^2 \Big)^{1/2} < \infty,$$

where the supremum is taken over all finite orthonormal systems $F \subseteq H$ and where $(\gamma_h)_{h\in F}$ is an independent collection of complex-valued standard Gaussian random variables on some probability space. Endow

$$\gamma_\infty(H; X) := \{T : H \to X \mid T \text{ is } \gamma\text{-summing}\}$$

with the norm $\|\cdot\|_\gamma$ and let the space $\gamma(H; X)$ of all *γ-radonifying operators* be the closure in $\gamma_\infty(H; X)$ of the finite rank operators $H \otimes X$. The following proposition states one of the main properties of γ-summing and γ-radonifying operators. In particular, it shows that $\gamma_\infty(H; X)$ and $\gamma(H; X)$ are Banach ideals in $\mathcal{L}(H, X)$ as defined in Section 5.3. For a proof see [120, Theorem 6.2].

Proposition 2.26. *Let H, K be Hilbert spaces and X, Y Banach spaces. Let $R \in \mathcal{L}(X, Y)$, $S \in \gamma_\infty(H; X)$ and $T \in \mathcal{L}(K; H)$. Then $RST \in \gamma_\infty(K; Y)$ with*

$$\|RST\|_\gamma \le \|R\|_{\mathcal{L}(X,Y)} \|S\|_\gamma \|T\|_{\mathcal{L}(K;H)}.$$

If $S \in \gamma(H; X)$ then $RST \in \gamma(K; Y)$.

For a measure space (Ω, μ), let $\gamma_2(\Omega; X)$ be the space of all $f \in \mathrm{L}^2(\Omega; X)$ such that $J_f \in \gamma(\mathrm{L}^2(\Omega); X)$, where $J_f : \mathrm{L}^2(\Omega) \to X$ is given by

$$J_f(g) := \int_\Omega g \cdot f \,\mathrm{d}\mu \tag{2.31}$$

for $g \in \mathrm{L}^2(\Omega)$. Endow $\gamma_2(\Omega; X)$ with the norm[2] $\|f\|_{\gamma_2(\Omega;X)} := \|J_f\|_\gamma$. Then $\gamma_2(\Omega; X)$ is a space of X-valued functions, and any finite rank operator $T \in$

[2] This definition of the norm on $\gamma_2(\Omega; X)$ is different from that in [54], where the quantity $\|f\|_{\mathrm{L}^2(\mathbb{R};X)} + \|J_f\|_\gamma$ is considered. The present definition will be more useful for us.

$L^2(\Omega) \otimes X$ induces an element of $\gamma_2(\Omega; X)$. Since the finite rank operators lie dense in $\gamma(L^2(\Omega); X)$, but in general not every $T \in \gamma(L^2(\Omega); X)$ satisfies $T = J_f$ for some $f \in L^2(\Omega; X)$, $\gamma_2(\Omega; X)$ is not a Banach space in general. Nevertheless, it will be a useful space for us later on.

The following corollary of Proposition 2.26 is of fundamental importance. For $\mu \in M(\mathbb{R})$, recall the definition of the convolution operator $L_\mu \in \mathcal{L}(L^2(\mathbb{R}; X))$ from (2.22).

Corollary 2.27. *Let $\mu \in M(\mathbb{R})$. Then $L_\mu \in \mathcal{L}(\gamma_2(\mathbb{R}; X))$ with*

$$\|L_\mu\|_{\mathcal{L}(\gamma_2(\mathbb{R};X))} \leq \|\mathcal{F}\mu\|_{L^\infty(\mathbb{R})} .$$

Proof. Let $f \in L^2(\mathbb{R}; X)$. Then $\mu * f \in L^2(\mathbb{R}; X)$ and $J_{\mu * f}(g) = J_f(\widetilde{\mu * \widetilde{g}})$ for each $g \in L^2(\mathbb{R})$. Here $\widetilde{h}(s) = h(-s)$ for $h \in L^2(\mathbb{R})$ and $s \in \mathbb{R}$. Since

$$\|\widetilde{\mu * \widetilde{g}}\|_{L^2(\mathbb{R})} \leq \|\mathcal{F}\mu\|_{L^\infty(\mathbb{R})} \|g\|_{L^2(\mathbb{R})}$$

by Plancherel's Theorem, Proposition 2.26 yields the desired result. □

Corollary 6.3 from [59] yields a useful class of functions in $\gamma_2(\Omega; X)$ for $\Omega = (a, b) \subseteq \mathbb{R}$ (we use the convention $\infty \cdot 0 = 0$):

Lemma 2.28. *Let X be a Banach space, $(a, b) \subseteq \mathbb{R}$, $u \in W^{1,1}_{loc}((a, b); X)$ and $\varphi : (a, b) \to \mathbb{C}$. Suppose that one of the following conditions is satisfied:*

- $\|\varphi\|_{L^2(a,b)} \|u(a)\|_X < \infty$ *and* $\int_a^b \|\varphi\|_{L^2(s,b)} \|u'(s)\|_X \, ds < \infty$;
- $\|\varphi\|_{L^2(a,b)} \|u(b)\|_X < \infty$ *and* $\int_a^b \|\varphi\|_{L^2(a,s)} \|u'(s)\|_X \, ds < \infty$;

Then $\varphi \cdot u \in \gamma_2((a, b); X)$.

A collection $\mathcal{T} \subseteq \mathcal{L}(X)$ is *γ-bounded* if there exists a constant $C \geq 0$ such that

$$\Big\| \sum_{T \in \mathcal{T}'} \gamma_T T x_T \Big\|_{L^2(\Omega;X)} \leq C \Big\| \sum_{T \in \mathcal{T}'} \gamma_T x_T \Big\|_{L^2(\Omega;X)}$$

for all finite subsets $\mathcal{T}' \subseteq \mathcal{T}$, sequences $(x_T)_{T \in \mathcal{T}'} \subseteq X$ and independent complex-valued standard Gaussian random variables $(\gamma_T)_{T \in \mathcal{T}'}$ on some probability space Ω. The smallest such C is the *γ-bound* of $\mathcal{T}$ and is denoted by $[\![\mathcal{T}]\!]^\gamma$. Every γ-bounded collection is uniformly bounded with supremum bound less than or equal to the γ-bound, and the converse holds if X is a Hilbert space.

A basic result concerning γ-boundedness (as well as boundedness with respect to more general random variables, although we will not discuss such generalizations here) is the Kahane contraction principle from [67]:

Lemma 2.29. *Let $p \in [1,\infty]$, $n \in \mathbb{N}$ and let $(\gamma_k)_{k=1}^n$ be a sequence of independent complex-valued standard Gaussian random variables on some probability space Ω. Let $\lambda_1,\dots,\lambda_n \in \mathbb{C}$ and $x_1,\dots,x_n \in X$. Then*

$$\Big\| \sum_{k=1}^n \lambda_k \gamma_k x_k \Big\|_{L^p(\Omega;X)} \le \max_{1\le k\le n} |\lambda_k| \, \Big\| \sum_{k=1}^n \gamma_k x_k \Big\|_{L^p(\Omega;X)}.$$

Another important result involving γ-boundedness is the *γ-Multiplier Theorem*. We state a version that is tailored to our purposes. Given a Banach space Y, a function $g : \mathbb{R} \to Y$ is *piecewise* $W^{1,\infty}$ if $g \in W^{1,\infty}(\mathbb{R} \setminus \{a_1,\dots,a_n\}; Y)$ for some finite set $\{a_1,\dots,a_n\} \subseteq \mathbb{R}$.

Theorem 2.30 (γ-Multiplier Theorem). *Let X and Y be Banach spaces and $T : \mathbb{R} \to \mathcal{L}(X,Y)$ a strongly measurable mapping such that*

$$\mathcal{T} := \{T(s) \mid s \in \mathbb{R}\}$$

is γ-bounded. Suppose furthermore that there exists a dense subset $D \subseteq X$ such that $s \mapsto T(s)x$ is piecewise $W^{1,\infty}$ for all $x \in D$. Then the multiplication operator

$$\mathcal{M}_T : L^2(\mathbb{R}) \otimes X \to L^2(\mathbb{R};Y) \qquad \mathcal{M}_T(f\otimes x) = f(\cdot)T(\cdot)x$$

defines a bounded operator $\mathcal{M}_T \in \mathcal{L}(\gamma_2(\mathbb{R};X))$ with $\|\mathcal{M}_T\|_{\mathcal{L}(\gamma_2(\mathbb{R};X))} \le [\![\mathcal{T}]\!]^\gamma$.

Proof. Let $f \in \gamma_2(\mathbb{R};X)$. That $J_{\mathcal{M}_T(f)} \in \gamma_\infty(L^2(\mathbb{R});Y)$ with $\left\|J_{\mathcal{M}_T f}\right\|_\gamma \le [\![\mathcal{T}]\!]^\gamma$ is the content of [120, Theorem 5.2]. To see that in fact $J_{\mathcal{M}_T f} \in \gamma(L^2(\mathbb{R});Y)$ we use an approximation argument. Let $x \in D$ and $f \in C_c(\mathbb{R})$. Let $a_1,\dots,a_n \in \mathbb{R}$ be such that $s \mapsto T(s)x \in W^{1,\infty}(\mathbb{R}\setminus\{a_1,\dots,a_n\};Y)$, and set $a_0 := -\infty$, $a_{n+1} := \infty$. Note that

$$\int_{a_j}^{a_{j+1}} \|f\|_{L^2(s,a_{j+1})} \left\|T(s)'x\right\| \, \mathrm{d}s < \infty$$

for all $j \in \{1,\dots,n\}$. Furthermore,

$$\int_{-\infty}^{a_1} \|f\|_{L^2(-\infty,s)} \left\|T(s)'x\right\| \, \mathrm{d}s < \infty.$$

Lemma 2.28 yields $(\mathbf{1}_{(a_j,a_{j+1})}f)(\cdot)T(\cdot)x \in \gamma_2(\mathbb{R};Y)$ for all $0 \le j \le n$, hence $f(\cdot)T(\cdot)x \in \gamma_2(\mathbb{R};Y)$. Now, $C_c(\mathbb{R}) \otimes D$ is dense in $L^2(\mathbb{R}) \otimes X$, which in turn is dense in $\gamma(L^2(\mathbb{R});X)$. For a general $f \in \gamma_2(\mathbb{R};X)$, approximate J_f by elements of $C_c(\mathbb{R}) \otimes D$ to obtain $J_{\mathcal{M}_T f} \in \gamma(L^2(\mathbb{R});Y)$. □

2.6.2 γ-Bounded (semi)groups

To illustrate the use of the concept of γ-boundedness for functional calculus theory, we state the following important result due to Kalton and Weis [68]. Note that it extends Proposition 2.9 to general Banach spaces.

Theorem 2.31. *Let* $-\mathrm{i}A$ *generate a* C_0*-group* $(U(s))_{s\in\mathbb{R}} \subseteq \mathcal{L}(X)$ *on a Banach space* X*, and suppose that there exists an* $\omega \in \mathbb{R}_+$ *such that*

$$\left\{ e^{-\omega|s|}U(s) \,\middle|\, s \in \mathbb{R} \right\} \subseteq \mathcal{L}(X)$$

is γ*-bounded. Then* A *has a bounded* $\mathrm{H}^\infty(\mathrm{St}_{\omega'})$*-calculus for each* $\omega' > \omega$.

We note that, under the assumption that X has the so-called *property* (α), the converse implication in Theorem 2.31 also holds, although not necessarily for the same ω (for details see [68]).

We note for later reference the following result due to Le Merdy (see [81]), which links γ-bounded C_0-groups to the scalar type operators from Section 2.12.

Proposition 2.32. *Let* $-\mathrm{i}A$ *generate a* C_0*-group* $(U(s))_{s\in\mathbb{R}} \subseteq \mathcal{L}(X)$ *on a Banach space* X*. Then* $\{U(s) \mid s \in \mathbb{R}\} \subseteq \mathcal{L}(X)$ *is* γ*-bounded if and only if* A *is a scalar type operator.*

We now present some terminology for C_0-semigroups which will be useful later on. Note that a C_0-semigroup $(T(t))_{t\in\mathbb{R}_+}$ is of type (M,ω) (as defined in Section 2.2.1), if and only if $\sup\left\{\left\|e^{-\omega t}T(t)\right\| \mid t \in \mathbb{R}_+\right\} \leq M$. This motivates the following definition. We say that a C_0-semigroup $T = (T(t))_{t\in\mathbb{R}_+}$ is of *γ-type* $(M,\omega) \in [1,\infty) \times \mathbb{R}$ if $[\![\mathrm{e}^{-\omega t}T(t) \mid t \in \mathbb{R}_+]\!]^\gamma \leq M$. The *exponential γ-bound* of T is defined as

$$\omega_\gamma(T) := \inf\{\omega \in \mathbb{R} \mid T \text{ is of } \gamma\text{-type } (M,\omega)\} \in [-\infty,\infty],$$

and T is *exponentially γ-stable* if $\omega_\gamma(T) < 0$. Clearly type and γ-type coincide for semigroups on Hilbert spaces.

In Chapter 6 we will use the following corollary of Theorem 2.31. Note that it applies in particular to an exponentially stable semigroup $(T(t))_{t\in\mathbb{R}_+}$ on a Hilbert space for which all the operators $T(t)$, $t > 0$, are invertible.

Corollary 2.33. *Let* $-A$ *be the generator of an exponentially* γ*-stable* C_0*-semigroup* $(T(t))_{t\in\mathbb{R}_+}$ *on a Banach space* X*. Suppose that* $T(t)$ *is invertible for each* $t \in \mathbb{R}_+$*, and that* $\{e^{-w_0 t}T(t)^{-1} \mid t > 0\} \subseteq \mathcal{L}(X)$ *is* γ*-bounded for some* $\omega_0 \in \mathbb{R}_+$*. Then* A *has a bounded* $\mathrm{H}^\infty(\mathbb{C}_+)$*-calculus.*

Proof. Let $\omega_1 > 0$ and $M \geq 1$ be such that $(T(t))_{t\in\mathbb{R}_+}$ is of type $(M,-\omega_1)$, and set $\omega_2 := \frac{\omega_0}{2} + \omega_1 \in \mathbb{R}_+$. Let $B := -\mathrm{i}(A - \omega_2)$. Then $-\mathrm{i}B$ generates the C_0-group $(U(s))_{s\in\mathbb{R}} \subseteq \mathcal{L}(X)$ given by $U(s) = e^{\omega_2 s}T(s)$ for $s \in \mathbb{R}_+$, and $U(s) = e^{\omega_2 s}T(-s)^{-1}$ for $s < 0$. Let $\omega := \frac{\omega_0}{2}$. Then

$$\{e^{-ws}U(s) \mid s \in \mathbb{R}_+\} = \{e^{w_1 t}T(t) \mid t \in \mathbb{R}_+\}$$

is γ-bounded by assumption. By Lemma 2.29, so is

$$\{e^{ws}U(s) \mid s < 0\} = \{e^{-(\omega_0+w_1)t}T(t)^{-1} \mid t > 0\}.$$

Hence Theorem 2.31 yields a constant $C \geq 0$ such that

$$\|f(\omega_2 + \mathrm{i}\,\cdot)(B)\|_{\mathcal{L}(X)} \leq C\,\|f(\omega_2 + \mathrm{i}\,\cdot)\|_{\mathrm{H}^\infty(\mathrm{St}_{\omega_2})} \leq C\,\|f\|_{\mathrm{H}^\infty(\mathbb{C}_+)} \tag{2.32}$$

for all $f \in \mathrm{H}^\infty(\mathbb{C}_+)$. In the same manner as in Lemma 2.3, one can show that $f(\omega_2 + \mathrm{i}\,\cdot)(B) = f(A)$ for each $f \in \mathrm{H}^\infty(\mathbb{C}_+)$. Hence (2.32) concludes the proof. □

2.7 Interpolation spaces

In this section we give a summary of some aspects of interpolation theory that will be used in Chapter 4.

2.7.1 Real interpolation spaces

If X and Y are Banach spaces that are embedded continuously into a Hausdorff topological vector space Z, then we call (X, Y) an *interpolation couple*. We let

$$K(t,z) := \inf\{\|x\|_X + t\,\|y\|_Y \mid x \in X, y \in Y, x + y = z\}$$

for $t > 0$ and $z \in X + Y \subseteq Z$. The *real interpolation space* of X and Y with parameters $\theta \in [0,1]$ and $q \in [1,\infty]$ is

$$(X,Y)_{\theta,q} := \left\{z \in X + Y \mid [t \mapsto t^{-\theta}K(t,z)] \in \mathrm{L}^q((0,\infty), \mathrm{d}t/t)\right\}, \tag{2.33}$$

a Banach space when equipped with the norm

$$\|z\|_{(X,Y)_{\theta,q}} := \left\|t \mapsto t^{-\theta}K(t,z)\right\|_{\mathrm{L}^q((0,\infty),\,\mathrm{d}t/t)} \qquad (z \in (X,Y)_{\theta,q}).$$

If $T : X + Y \to X + Y$ restricts to a bounded operator on X and a bounded operator on Y then

$$\|T\|_{\mathcal{L}((X,Y)_{\theta,q})} \leq \|T\|_{\mathcal{L}(X)}^{1-\theta}\,\|T\|_{\mathcal{L}(Y)}^{\theta} \tag{2.34}$$

for all $\theta \in (0,1)$ and $q \in [1,\infty]$ (see [12, Theorem 3.1.2]).

By [12, Theorem 3.4.1], the following holds for any interpolation couple (X, Y) and $\theta \in (0,1)$, $q \in [1,\infty]$:

$$(Y,X)_{\theta,q} = (X,Y)_{1-\theta,q} \tag{2.35}$$

with equality of norms.

In the case where Y is continuously embedded in X, the real interpolation spaces are nested, cf. [86, Proposition 1.1.4]:

$$(X, Y)_{\theta_2,p} \subseteq (X, Y)_{\theta_1,q} \tag{2.36}$$

for $0 < \theta_1 < \theta_2 < 1$ and all $p, q \in [1, \infty]$.

In Chapter 4 we will need the following version of the *Reiteration Theorem*. For a proof see e.g. [12, Theorem 3.5.3].

Theorem 2.34. *Let (X, Y) be an interpolation couple and let $q, q_1, q_2 \in [1, \infty]$ and $\theta, \theta_1, \theta_2 \in (0, 1)$ with $\theta_1 \neq \theta_2$. Then*

$$((X, Y)_{\theta_1,q_1}, (X, Y)_{\theta_2,q_2})_{\theta,q} = (X, Y)_{(1-\theta)\theta_1+\theta\theta_2,q}$$

with equivalent norms.

An important class of real interpolation spaces is the class of Besov spaces from Section 2.3.1. For $m \in \mathbb{N}$ and $p \in [1, \infty]$, recall the definition of the Sobolev space $\mathrm{W}^{m,p}(\mathbb{R}; X)$ from Section 2.1.

Lemma 2.35. *Let $\theta \in (0, 1)$, $p \in [1, \infty)$, $q \in [1, \infty]$ and $m \in \mathbb{N}$. Then there exists a constant $C > 0$ such that, for each Banach space X,*

$$(\mathrm{L}^p(\mathbb{R}; X), \mathrm{W}^{m,p}(\mathbb{R}; X))_{\theta,q} = \mathrm{B}^{m\theta}_{p,q}(\mathbb{R}; X)$$

with

$$\frac{1}{C} \|f\|_{\mathrm{B}^{m\theta}_{p,q}(\mathbb{R};X)} \leq \|f\|_{(\mathrm{L}^p(\mathbb{R};X),\mathrm{W}^{m,p}(\mathbb{R};X))_{\theta,q}} \leq C \|f\|_{\mathrm{B}^{m\theta}_{p,q}(\mathbb{R};X)}$$

for each $f \in \mathrm{B}^{m\theta}_{p,q}(\mathbb{R}; X)$.

Similarly, $(\mathrm{C}_{ub}(\mathbb{R}; X), \mathrm{C}^m_{ub}(\mathbb{R}; X))_{\theta,q} = \mathrm{B}^{m\theta}_{\infty,q}(\mathbb{R}; X)$ with equivalent norms, and the constant describing the equivalence of the norms is independent of X.

Proof. The proof is the same as that of [118, Theorem 2.5.7] in the case $X = \mathbb{C}$. One first uses that

$$\mathrm{B}^{m\theta}_{p,q}(\mathbb{R}; X) = (\mathrm{B}^0_{p,1}(\mathbb{R}; X), \mathrm{B}^m_{p,1}(\mathbb{R}; X))_{\theta,q} = (\mathrm{B}^0_{p,\infty}(\mathbb{R}; X), \mathrm{B}^m_{p,\infty}(\mathbb{R}; X))_{\theta,q},$$

with equivalent norms (where the constant of equivalence does not depend on X). This is shown just as in [118, Theorem 2.4.2]. Then the inclusions

$$\begin{aligned}\mathrm{B}^{m\theta}_{p,q}(\mathbb{R}; X) = (\mathrm{B}^0_{p,1}(\mathbb{R}; X), \mathrm{B}^m_{p,1}(\mathbb{R}; X))_{\theta,q} &\subseteq (\mathrm{L}^p(\mathbb{R}; X), \mathrm{W}^{m,p}(\mathbb{R}; X))_{\theta,q} \\ &\subseteq (\mathrm{B}^0_{p,\infty}(\mathbb{R}; X), \mathrm{B}^m_{p,\infty}(\mathbb{R}; X))_{\theta,q} = \mathrm{B}^{m\theta}_{p,q}(\mathbb{R}; X)\end{aligned}$$

follow from (2.17). The proof of the second statement is analogous. □

2.7.2 Functional calculus on interpolation spaces

We now discuss functional calculus for operators on interpolation spaces. We will deal with strip type operators and generators of C_0-groups, because of the applications of the theory in this section to Chapter 4. However, much of the material in this section is equally valid for the other types of operators that were considered in Section 2.2, and the proofs are analogous. This will be used at times in Chapter 4.

We mainly consider interpolation spaces for the couple $(X, \mathrm{D}(A))$, where A is a closed operator on X. We write

$$\mathrm{D}_A(\theta, q) := (X, \mathrm{D}(A))_{\theta,q} \tag{2.37}$$

and

$$\|x\|_{\theta,q} := \|x\|_{\mathrm{D}_A(\theta,q)} \qquad (x \in \mathrm{D}_A(\theta, q)).$$

If $-A$ generates a C_0-semigroup on X, the following inclusions hold for $\alpha, \beta, \gamma \in \mathbb{C}$ with $\mathrm{Re}(\gamma) < \mathrm{Re}(\beta) < \mathrm{Re}(\alpha)$ and $\mathrm{Re}(\gamma) > 0$ or $\gamma = 0$, by [55, Corollary 6.6.3]:

$$(\mathrm{D}(A^\gamma), \mathrm{D}(A^\alpha))_{\theta,1} \subseteq \mathrm{D}(A^\beta) \subseteq (\mathrm{D}(A^\gamma), \mathrm{D}(A^\alpha)_{\theta,\infty}, \tag{2.38}$$

where $\theta \in (0, 1)$ satisfies $\mathrm{Re}(\beta) = (1 - \theta)\,\mathrm{Re}(\gamma) + \theta\,\mathrm{Re}(\alpha)$.

For an operator B on X and a continuously embedded $Y \hookrightarrow X$, the operator B_Y on Y that satisfies $B_Y y = By$ for elements in its domain

$$\mathrm{D}(B_Y) := \{y \in \mathrm{D}(B) \cap Y \mid By \in Y\}$$

is the *part* of B in Y. If $Y = \mathrm{D}_A(\theta, q)$ for $\theta \in (0, 1)$ and $q \in [1, \infty]$ then we write

$$B_{\theta,q} := B_{\mathrm{D}_A(\theta,q)}. \tag{2.39}$$

Now let A be a strip type operator as in Section 2.2.2. The following lemma shows, in particular, that the functional calculi for A and $A_{\theta,q}$ are compatible.

Lemma 2.36. *Let A be a strip type operator of height $\omega_0 \geq 0$ on a Banach space X and let $\theta \in (0, 1)$, $q \in [1, \infty]$ and $m, n \in \mathbb{N}_0$. Let $Y := (\mathrm{D}(A^m), \mathrm{D}(A^n))_{\theta,q}$.*

a) The part A_Y of A in Y is a strip type operator of height ω_0. Furthermore, each $f : \mathrm{St}_\omega \to \mathbb{C}$ with $\omega > \omega_0$ which is regularizable in the calculus for A is regularizable in the calculus for A_Y, and $f(A_Y) = f(A)_Y$.

b) If $-\mathrm{i}A$ generates a C_0-group $(U(s))_{s\in\mathbb{R}}$ on X and $q < \infty$, then $-\mathrm{i}A_Y$ generates the C_0-group $(U(s)_Y)_{s\in\mathbb{R}}$. In particular, $\mathrm{D}(A_Y)$ is dense in Y.

Proof. *a*) First note that, for all $k \in \mathbb{N}_0$ and $\lambda \in \rho(A)$, $R(\lambda, A)$ leaves $\mathrm{D}(A^k)$ invariant with $\|R(\lambda, A)\|_{\mathcal{L}(\mathrm{D}(A^k))} \leq \|R(\lambda, A)\|_{\mathcal{L}(X)}$. By (2.34), $R(\lambda, A)$ leaves Y invariant with

$$\|R(\lambda, A)\|_{\mathcal{L}(Y)} \leq \|R(\lambda, A)\|_{\mathcal{L}(X)} . \tag{2.40}$$

By [55, Proposition A.2.8], $\sigma(A_Y) \subseteq \sigma(A)$ and $R(\lambda, A_Y) = R(\lambda, A)_Y$ for all $\lambda \in \rho(A)$. Hence (2.40) yields the first statement. Let $\omega > \omega_0$ and $f \in \mathcal{E}(\mathrm{St}_\omega)$ be given. Then

$$f(A_Y)y = \frac{1}{2\pi \mathrm{i}} \int_\Gamma f(z)R(z, A_Y)y \, \mathrm{d}z = \frac{1}{2\pi \mathrm{i}} \int_\Gamma f(z)R(z, A)y \, \mathrm{d}z = f(A)y$$

for some contour Γ and all $y \in Y$. Now let $f : \mathrm{St}_\omega \to \mathbb{C}$ be a function which is regularizable in the calculus for A and let e be a regularizer for f. Then e is a regularizer for f in the calculus for A_Y, since $e(A_Y) = e(A)_Y$ is injective. The rest follows by regularization.

b) By (2.34), $\|U(s)_Y\| \leq \|U(s)\|$ for all $s \in \mathbb{R}$. Hence $(U(s)_Y)_{s \in \mathbb{R}}$ is locally bounded. Since it is strongly continuous on the subset $\mathrm{D}(A^{\max(n,m)}) \subseteq Y$, which is dense by [86, Proposition 1.2.5], it is strongly continuous on Y. By [45, p. 60], $-\mathrm{i}A_Y$ is its generator. □

Remark 2.37. It follows from part b) of Lemma 2.36 that, for $x \in \mathrm{D}_A(\theta, q)$ with $q < \infty$,

$$U_\mu x = \int_{\mathbb{R}} U(s)x \, \mu(\mathrm{d}s) \tag{2.41}$$

exists as an integral of a $\mathrm{D}_A(\theta, q)$-valued function. Even though in general $(U(s))_{s \in \mathbb{R}}$ is not strongly continuous on $\mathrm{D}_A(\theta, \infty)$, for $x \in \mathrm{D}_A(\theta, \infty)$ (2.41) exists as an integral of an X-valued function. Since $\mathrm{D}_A(\theta, q)$ is continuously embedded in X for all $\theta \in (0, 1)$ and $q \in [1, \infty]$, the value of (2.41) does not depend on the space in which we consider $s \mapsto U(s)x$. Hence we regularly will not specify in which space we consider (2.41).

We conclude with an important link between real interpolation spaces and functional calculus theory provided by a theorem of Dore (see [37]). It is the first instance of a theme that will be further investigated in Chapter 4, namely that the functional calculus properties of an operator may improve upon restriction to a real interpolation space.

Theorem 2.38. *Let A be an invertible sectorial operator of angle $\varphi \in (0, \pi)$ on a Banach space X, and let $\theta \in (0, 1)$ and $q \in [1, \infty]$. Then $A_{\theta,q}$ has a bounded $\mathrm{H}^\infty(\mathrm{S}_\psi)$-calculus on $(X, \mathrm{D}(A))_{\theta,q}$ for all $\psi \in (\varphi, \pi)$.*

Versions of this result for operators which are not invertible can be found in [38] and [55, Chapter 6]. Note that, by Proposition 2.10, there exists a sectorial operator A of angle 0 on a Hilbert space X which does not have a bounded $\mathrm{H}^\infty(\mathrm{S}_\psi)$-calculus on X for any $\psi \in (0, \pi)$.

Part II

Functional calculus using transference methods

3

Functional calculus for semigroup generators

We have seen in Proposition 2.5 that not every semigroup generator has a bounded H^∞-calculus, even if the underlying Banach space is a Hilbert space. However, by Proposition 2.9 group generators on Hilbert spaces do have a bounded H^∞-calculus, and this follows from the fact that group generators allow for transference principles as in Propositions 2.23 and 2.24. In the present chapter we derive a transference principle for operators T_μ as in (2.4) in the case where μ has support away from zero. This allows us to show that, although a semigroup generator $-A$ does not in general have a bounded H^∞-calculus, there is a large class of functions f for which $f(A)$ is bounded. In particular, on Hilbert spaces we obtain the following result. (See Section 3.3 for the definition of a strong m-bounded calculus.)

Theorem 3.1. *Let $-A$ be the generator of a uniformly bounded C_0-semigroup $(T(t))_{t\in\mathbb{R}_+}$ of type $(M,0)$ on a Hilbert space H. Then the following assertions hold.*

a) *For $\omega < 0$ and $f \in \mathrm{H}^\infty(\mathrm{R}_\omega)$ one has $f(A)T(\tau) \in \mathcal{L}(H)$ with*

$$\|f(A)T(\tau)\| \leq c(\tau)M^2 \|f\|_{\mathrm{H}^\infty(\mathrm{R}_\omega)}, \tag{3.1}$$

where $c(\tau) \in O(|\log(\tau)|)$ as $\tau \searrow 0$, and $c(\tau) \in O(1)$ as $\tau \to \infty$.

b) *For $\omega < 0 < \alpha$ and $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda < 0$ there is $C \geq 0$ such that*

$$\left\|f(A)(A-\lambda)^{-\alpha}\right\| \leq CM^2 \|f\|_{\mathrm{H}^\infty(\mathrm{R}_\omega)} \tag{3.2}$$

for all $f \in \mathrm{H}^\infty(\mathrm{R}_\omega)$. In particular, $\mathrm{D}(A^\alpha) \subseteq \mathrm{D}(f(A))$.

c) *A has a strong m-bounded H^∞-calculus of type 0 for each $m \in \mathbb{N}$.*

When X is a UMD space one can derive similar results, stated in Section 3.4.2. Our results generalize to arbitrary Banach spaces by using (subalgebras of) the analytic $\mathrm{L}^p(\mathbb{R};X)$-Fourier multiplier algebra from (3.3). However, they are useful only if the underlying Banach space has a geometry that allows for nontrivial Fourier multiplier operators. In Section 3.5 we take a different approach, in the spirit of Theorem 2.31, and extend the Hilbert space results to general Banach spaces by replacing the assumption of boundedness of the semigroup by its γ-boundedness. In particular, Theorem 3.1 holds true for γ-bounded semigroups on arbitrary Banach spaces with M being the γ-bound of the semigroup.

In Section 3.1 we introduce the analytic multiplier algebras which will be used throughout this chapter, and we study some of their properties. In Section 3.2 we derive a transference principle for semigroup generators and measures with support away from zero, and apply it to deduce results about boundedness of certain functional calculi. In Section 3.3 we study m-bounded functional calculus, and show that each semigroup generator has an m-bounded analytic multiplier calculus. In Section 3.4 we specialize the results in the preceding sections to Hilbert and UMD spaces. In Section 3.5 we study γ-bounded semigroups and apply our results to their generators.

As noted in Section 2.2.1, $-A$ generates a C_0-semigroup $(T(t))_{t\in\mathbb{R}_+}$ of type (M,ω) if and only if $-(A+\omega)$ generates the semigroup $(\mathrm{e}^{-\omega t}T(t))_{t\in\mathbb{R}_+}$ of type $(M,0)$. By Lemma 2.3, the functional calculi for A and $A+\omega$ are linked by the composition rule "$f(A+\omega) = f(\omega+z)(A)$". Therefore, in this chapter we shall almost exclusively consider uniformly bounded semigroups; all results carry over to general semigroups by shifting.

3.1 Multiplier algebras

In this section we define the analytic multiplier algebras that occur frequently in this chapter, and we derive some of their basic properties.

Let X be a Banach space and $p \in [1,\infty]$, and recall the definition of the space $\mathcal{M}_p(X)$ of all bounded $\mathrm{L}^p(\mathbb{R};X)$-Fourier multipliers from Section 2.4. For $\omega \in \mathbb{R}$ and $p \in [1,\infty]$ we let

$$\mathrm{AM}_p^X(\mathrm{R}_\omega) := \{f \in \mathrm{H}^\infty(\mathrm{R}_\omega) \mid f(\omega+\mathrm{i}\cdot) \in \mathcal{M}_p(X)\} \tag{3.3}$$

be the *analytic* $\mathrm{L}^p(\mathbb{R};X)$*-Fourier multiplier algebra* on R_ω, endowed the norm

$$\|f\|_{\mathrm{AM}_p^X} := \|f\|_{\mathrm{AM}_p^X(\mathrm{R}_\omega)} := \|f(\omega+\mathrm{i}\cdot)\|_{\mathcal{M}_p(X)}.$$

Here $f(\omega+\mathrm{i}\cdot) \in \mathrm{L}^\infty(\mathbb{R})$ is the trace of f from (2.3). To simplify notation we sometimes omit reference to the Banach space X and write $\mathrm{AM}_p(\mathrm{R}_\omega)$ instead of $\mathrm{AM}_p^X(\mathrm{R}_\omega)$ whenever it is convenient.

The space $\mathrm{AM}_p^X(\mathrm{R}_\omega)$ is a unital Banach algebra. Since $\mathcal{M}_p(X)$ is contractively embedded in $\mathrm{L}^\infty(\mathbb{R};\mathcal{L}(X))$, $\mathrm{AM}_p^X(\mathrm{R}_\omega)$ is contractively embedded in $\mathrm{H}^\infty(\mathrm{R}_\omega)$. Moreover, $\mathrm{AM}_1^X(\mathrm{R}_\omega) = \mathrm{AM}_\infty^X(\mathrm{R}_\omega)$ is contractively embedded in $\mathrm{AM}_p^X(\mathrm{R}_\omega)$ for all $p \in (1,\infty)$, as follows from (2.23) and the fact that any scalar-valued $m \in \mathcal{M}_1(X)$ satisfies $m = \mathcal{F}\mu$ for some $\mu \in \mathrm{M}(\mathbb{R})$.

For our main results we need a few lemmas about the analytic multiplier algebra.

Lemma 3.2. *For every Banach space X, all* $\omega \in \mathbb{R}$ *and* $p \in [1,\infty]$,

$$\mathrm{AM}_p^X(\mathrm{R}_\omega) = \Big\{ f \in \mathrm{H}^\infty(\mathrm{R}_\omega) \Big| \sup_{\omega' > \omega} \|f(\omega' + \mathrm{i}\cdot)\|_{\mathcal{M}_p(X)} < \infty \Big\}$$

with $\|f\|_{\mathrm{AM}_p^X(\mathrm{R}_\omega)} = \sup_{\omega'>\omega} \|f(\omega' + \mathrm{i}\cdot)\|_{\mathcal{M}_p(X)}$ *for all* $f \in \mathrm{AM}_p^X(\mathrm{R}_\omega)$.

Proof. Let $\omega \in \mathbb{R}$, $p \in [1,\infty]$ and $f \in \mathrm{AM}_p(\mathrm{R}_\omega)$. For all $\omega' > \omega$ and $s \in \mathbb{R}$,

$$f(\omega' + \mathrm{i}s) = \frac{\omega' - \omega}{\pi} \int_{\mathbb{R}} \frac{f(\omega - \mathrm{i}r)}{(s-r)^2 + (\omega' - \omega)^2} \, \mathrm{d}r$$

by [104, Theorem 5.18]. The right-hand side is the convolution of $f(\omega - i\cdot)$ and the Poisson kernel given by $P_{\omega'-\omega}(r) := \frac{\omega'-\omega}{\pi(r^2+(\omega'-\omega)^2)}$ for $r \in \mathbb{R}$. It is straightforward to check that $\|P_{\omega'-\omega}\|_{\mathrm{L}^1(\mathbb{R})} = 1$, so (2.24) yields

$$\|f(\omega' + \mathrm{i}\cdot)\|_{\mathcal{M}_p(X)} \leq \|f(\omega - \mathrm{i}\cdot)\|_{\mathcal{M}_p(X)} = \|f\|_{\mathrm{AM}_p^X(\mathrm{R}_\omega)}.$$

The converse follows from (2.3) and Lemma 2.17 (*ii*). □

For $\mu \in \mathrm{M}(\mathbb{R})$ and $p \in [1,\infty]$ recall the definition of the convolution operator $L_\mu \in \mathcal{L}(\mathrm{L}^p(\mathbb{R}; X))$ from (2.22).

Lemma 3.3. *For each* $\omega \in \mathbb{R}$ *the Laplace transform induces an isometric algebra isomorphism from* $\mathrm{M}_\omega(\mathbb{R}_+)$ *onto* $\mathrm{AM}_1^{\mathbb{C}}(\mathrm{R}_\omega) = \mathrm{AM}_1^X(\mathrm{R}_\omega)$. *Moreover,*

$$\|\widehat{\mu}\|_{\mathrm{AM}_p^X(\mathrm{R}_\omega)} = \|L_{e_{-\omega}\mu}\|_{\mathcal{L}(\mathrm{L}^p(X))}$$

for all $\mu \in \mathrm{M}_\omega(\mathbb{R}_+)$, $p \in [1,\infty]$.

Proof. The mappings $\mu \mapsto e_{-\omega}\mu$ and $f \mapsto f(\cdot + \omega)$ are isometric algebra isomorphisms $\mathrm{M}_\omega(\mathbb{R}_+) \to \mathrm{M}(\mathbb{R}_+)$ and $\mathrm{AM}_p(\mathrm{R}_\omega) \to \mathrm{AM}_p(\mathbb{C}_+)$ respectively. Hence it suffices to let $\omega = 0$. If $\mu \in \mathrm{M}(\mathbb{R}_+)$ and $f = \widehat{\mu} \in \mathrm{H}^\infty(\mathbb{C}_+)$ then $f(\mathrm{i}\cdot) = \mathcal{F}\mu(\cdot)$. Therefore (2.22) and (2.23) imply that $f(\mathrm{i}\cdot) \in \mathcal{M}_1(X)$ with $\|f(\mathrm{i}\cdot)\|_{\mathcal{M}_1(X)} = \|\mu\|_{\mathrm{M}(\mathbb{R}_+)}$, and that $\|f(\mathrm{i}\cdot)\|_{\mathcal{M}_p(X)} = \|L_\mu\|_{\mathcal{L}(\mathrm{L}^p(X))}$ for $p \in [1,\infty]$. Conversely, if $f \in \mathrm{AM}_1(\mathbb{C}_+)$ then $f(\mathrm{i}\cdot) = \mathcal{F}\mu$ for some $\mu \in \mathrm{M}(\mathbb{R})$. An application of Liouville's theorem shows that $\mathrm{supp}(\mu) \subseteq \mathbb{R}_+$, hence $f = \widehat{\mu}$. □

Lemma 3.3 implies that Lemma 2.1 can be reformulated as follows, using the analytic multiplier algebra.

Lemma 3.4. *Let* $\alpha > \frac{1}{2}$, $\lambda \in \mathbb{C}$ *and* $\omega, \omega_0 \in \mathbb{R}$ *with* $\mathrm{Re}(\lambda) < \omega < \omega_0$. *Then*

$$f(z)(z-\lambda)^{-\alpha} \in \mathrm{AM}_1(\mathrm{R}_{\omega_0}) \quad \textit{for all } f \in \mathrm{H}^\infty(\mathrm{R}_\omega).$$

3.2 Transference and functional calculus for semigroups

In this section we establish a transference principle for C_0-semigroups and measures with support away from zero. We then apply this transference principle to obtain the main functional calculus result of this chapter.

3.2.1 Transference for measures with support away from zero

Define the function $\eta : (0,\infty) \times (0,\infty) \times [1,\infty] \to \mathbb{R}_+$ by

$$\eta(\alpha, t, q) := \inf \left\{ \|\psi\|_q \|\varphi\|_{q'} \mid \psi * \varphi \equiv e_{-\alpha} \text{ on } [t,\infty) \right\}. \tag{3.4}$$

Here the infimum is taken over all $\psi \in \mathrm{L}^q(\mathbb{R}_+)$ and $\varphi \in \mathrm{L}^{q'}(\mathbb{R}_+)$ with $\psi * \varphi(s) = \mathrm{e}^{-\alpha s}$ for all $s \in [t,\infty)$. This set is not empty: choose for instance $\psi := \mathbf{1}_{[0,t]} e_{-\alpha}$ and $\varphi := \frac{1}{t} e_{-\alpha}$. By Lemma A.2,

$$\eta(\alpha, t, q) \in O(|\log(\alpha t)|) \qquad \text{as } \alpha t \to 0,$$

for $q \in (1,\infty)$. We now derive the main transference principle of this chapter.

Proposition 3.5. *Let $(T(t))_{t\in\mathbb{R}_+}$ be a C_0-semigroup of type $(M, 0)$ on a Banach space X. Let $p \in [1,\infty]$, $\tau, \omega > 0$ and $\mu \in \mathrm{M}_{-\omega}(\mathbb{R}_+)$ with $\operatorname{supp}(\mu) \subseteq [\tau,\infty)$. Then*

$$\left\| T_\mu \right\|_{\mathcal{L}(X)} \leq M^2 \eta(\omega, \tau, p) \left\| L_{e_\omega \mu} \right\|_{\mathcal{L}(\mathrm{L}^p(\mathbb{R};X))}. \tag{3.5}$$

Proof. Let $\psi \in \mathrm{L}^p(\mathbb{R}_+)$ and $\varphi \in \mathrm{L}^{p'}(\mathbb{R}_+)$ be such that $\psi * \varphi \equiv e_{-\omega}$ on $[\tau,\infty)$. Define $\iota : X \to \mathrm{L}^p(\mathbb{R}; X)$ by $\iota x(s) := \psi(-s)T(-s)x$ for $s \leq 0$ and $\iota x(s) := 0$ for $s > 0$. Clearly

$$\|\iota x\|_{\mathrm{L}^p(\mathbb{R};X)} \leq M \|\psi\|_p \|x\| \qquad (x \in X), \tag{3.6}$$

so ι is well-defined and bounded. Moreover, let $P : \mathrm{L}^p(\mathbb{R}; X) \to X$ be given by

$$Pf := \int_0^\infty \varphi(t)T(t)f(t)\,\mathrm{d}t \qquad (f \in \mathrm{L}^p(\mathbb{R}; X)).$$

By Hölder's inequality,

$$\|Pf\| \leq M \|\varphi\|_{p'} \|f\|_{\mathrm{L}^p(\mathbb{R};X)} \qquad (f \in \mathrm{L}^p(\mathbb{R}; X)), \tag{3.7}$$

so P is also well-defined and bounded. Finally, $L_{e_\omega \mu}$ is a bounded operator on $\mathrm{L}^p(\mathbb{R}; X)$, by Young's inequality. Letting $\Psi(\mathbb{R}; X) = \Phi(\mathbb{R}; X) = \mathrm{L}^p(\mathbb{R}; X)$ and using that $(\psi * \varphi)e_\omega \mu = \mu$, Proposition 2.25 yields the commutative diagram

$$\begin{array}{ccc} \mathrm{L}^p(\mathbb{R};X) & \xrightarrow{L_{e_\omega\mu}} & \mathrm{L}^p(\mathbb{R};X) \\ \iota\big\uparrow & & \big\downarrow P \\ X & \xrightarrow{\quad T_\mu \quad} & X \end{array}$$

Estimating $\|T_\mu\|$ through this factorization and using (3.6) and (3.7) yields

$$\|T_\mu\| \leq M^2 \|\psi\|_p \left\|L_{e_\omega\mu}\right\|_{\mathcal{L}(\mathrm{L}^p(\mathbb{R};X))} \|\varphi\|_{p'}\,.$$

Finally, taking the infimum over all such ψ and φ yields (3.5). □

3.2.2 Functional calculus for functions of exponential decay

Define, for a Banach space X, $\omega \in \mathbb{R}$, $p \in [1,\infty]$ and $\tau > 0$, the space

$$\mathrm{AM}^X_{p,\tau}(\mathrm{R}_\omega) := \left\{ f \in \mathrm{AM}^X_p(\mathrm{R}_\omega) \mid f(z) \in O(\mathrm{e}^{-\tau\,\mathrm{Re}(z)}) \text{ as } |z| \to \infty \right\},$$

endowed with the norm of $\mathrm{AM}^X_p(\mathrm{R}_\omega)$.

Lemma 3.6. *For every Banach space X, $\omega \in \mathbb{R}$, $p \in [1,\infty]$ and $\tau > 0$*

$$\mathrm{AM}^X_{p,\tau}(\mathrm{R}_\omega) = \mathrm{AM}^X_p(\mathrm{R}_\omega) \cap e_{-\tau}\mathrm{H}^\infty(\mathrm{R}_\omega) = e_{-\tau}\mathrm{AM}^X_p(\mathrm{R}_\omega). \tag{3.8}$$

In particular, $\mathrm{AM}^X_{p,\tau}(\mathrm{R}_\omega)$ is a closed ideal in $\mathrm{AM}^X_p(\mathrm{R}_\omega)$.

Proof. The first equality in (3.8) is clear, as is the inclusion $e_{-\tau}\mathrm{AM}_p(\mathrm{R}_\omega) \subseteq \mathrm{AM}_{p,\tau}(\mathrm{R}_\omega)$. Conversely, if $f \in \mathrm{AM}_p(\mathrm{R}_\omega) \cap e_{-\tau}\mathrm{H}^\infty(\mathrm{R}_\omega)$ then $e_\tau f \in \mathrm{AM}_p(\mathrm{R}_\omega)$ since

$$\|\mathrm{e}^{\tau(\omega+\mathrm{i}\cdot)} f(\omega + i\cdot)\|_{\mathcal{M}_p(X)} = \mathrm{e}^{\tau\omega}\|f(\omega + i\cdot)\|_{\mathcal{M}_p(X)},$$

by Lemma 2.17 (i).

Now suppose that $(f_n)_{n\in\mathbb{N}} \subseteq \mathrm{AM}_{p,\tau}(\mathrm{R}_\omega)$ converges to $f \in \mathrm{AM}_p(\mathrm{R}_\omega)$. The maximum principle for holomorphic functions implies

$$\|e_\tau f_n\|_{\mathrm{H}^\infty(\mathrm{R}_\omega)} = \mathrm{e}^{\tau\omega} \|f_n\|_{\mathrm{H}^\infty(\mathrm{R}_\omega)}\,,$$

hence $(e_\tau f_n)_{n\in\mathbb{N}}$ is Cauchy in $\mathrm{H}^\infty(\mathrm{R}_\omega)$. Since it converges pointwise to $e_\tau f$, (3.8) implies $f \in \mathrm{AM}_{p,\tau}(\mathrm{R}_\omega)$. □

We are now ready to prove the main result of this section. Note that the union of the ideals $\mathrm{AM}^X_{p,\tau}(\mathrm{R}_\omega)$ for $\tau > 0$ is dense in $\mathrm{AM}^X_p(\mathrm{R}_\omega)$ with respect to pointwise and bounded convergence of sequences. If there were a single constant independent of τ bounding the $\mathrm{AM}^X_{p,\tau}(\mathrm{R}_\omega)$-calculus for all τ, Lemma 2.4 would imply that A has a bounded $\mathrm{AM}^X_p(\mathrm{R}_\omega)$-calculus, but this is false in general, by Proposition 2.5. Moroever, it was shown recently in [111] that the logarithmic bound in the following result is sharp in general.

Theorem 3.7. *For each $p \in (1,\infty)$ there exists a constant $c_p \geq 0$ such that the following holds. Let $-A$ generate a C_0-semigroup $(T(t))_{t\in\mathbb{R}_+}$ of type $(M,0)$ on a Banach space X and let $\tau,\omega > 0$. Then $f(A) \in \mathcal{L}(X)$ and*

$$\|f(A)\| \leq \begin{cases} c_p\, M^2|\log(\omega\tau)|\ \|f\|_{\mathrm{AM}_p^X} & \text{if } \omega\tau \leq \min(\frac{1}{p},\frac{1}{p'}),\\ 2M^2\mathrm{e}^{-\omega\tau}\ \|f\|_{\mathrm{AM}_p^X} & \text{if } \omega\tau > \min(\frac{1}{p},\frac{1}{p'}) \end{cases}$$

for all $f \in \mathrm{AM}^X_{p,\tau}(\mathrm{R}_{-\omega})$. In particular, A has a bounded $\mathrm{AM}^X_{p,\tau}(\mathrm{R}_{-\omega})$-calculus.

Proof. First consider $f \in \mathrm{AM}_{1,\tau}(\mathrm{R}_{-\omega})$. Let $\delta_\tau \in \mathrm{M}_{-\omega}(\mathbb{R}_+)$ be the unit point mass at τ. By Lemmas 3.6 and 3.3 there exists a $\mu \in \mathrm{M}_{-\omega}(\mathbb{R}_+)$ such that $f = e_{-\tau}\widehat{\mu} = \widehat{\delta_\tau * \mu}$. Since $\delta_\tau * \mu \in \mathrm{M}_{-\omega}(\mathbb{R}_+)$ with $\operatorname{supp}(\delta_\tau * \mu) \subseteq [\tau,\infty)$, Proposition 3.5 and Lemma 3.3 yield

$$\|f(A)\| \leq M^2\eta(\omega,\tau,p)\,\|f\|_{\mathrm{AM}_p^X}. \tag{3.9}$$

Now suppose that $f \in \mathrm{AM}_{p,\tau}(\mathrm{R}_{-\omega})$ is arbitrary. For $\epsilon > 0$, $k \in \mathbb{N}$ and $z \in \mathrm{R}_{-\omega}$ set $g_k(z) := \frac{k}{z-\omega+k}$ and $f_{k,\epsilon}(z) := f(z+\epsilon)g_k(z+\epsilon)$. Lemma 3.4 yields $f_{k,\epsilon} \in \mathrm{AM}_{1,\tau}(\mathrm{R}_{-\omega})$, hence, by what we have already shown,

$$\left\|f_{k,\epsilon}(A)\right\| \leq M^2\eta(\omega,\tau,p)\,\left\|f_{k,\epsilon}\right\|_{\mathrm{AM}_p^X}.$$

The inclusion $\mathrm{AM}_1(\mathrm{R}_{-\omega}) \subseteq \mathrm{AM}_p(\mathrm{R}_{-\omega})$ is contractive, so Lemma 3.3 implies that $g_k \in \mathrm{AM}_p(\mathrm{R}_{-\omega})$ with

$$\|g_k\|_{\mathrm{AM}_p^X} \leq \|g_k\|_{\mathrm{AM}_1^X} = k\,\|e_{-k}\|_{\mathrm{L}^1(\mathbb{R}_+)} = 1.$$

Combining this with Lemma 3.2 yields

$$\begin{aligned}\left\|f_{k,\epsilon}\right\|_{\mathrm{AM}_p^X} &\leq \|f(\cdot+\epsilon)\|_{\mathrm{AM}_p^X}\,\|g_k(\cdot+\epsilon)\|_{\mathrm{AM}_p^X}\\ &\leq \|f\|_{\mathrm{AM}_p^X}.\end{aligned}$$

In particular, $\sup_{k,\epsilon}\left\|f_{k,\epsilon}\right\|_\infty < \infty$ and $\sup_{k,\epsilon}\left\|f_{k,\epsilon}(A)\right\| < \infty$. Lemma 2.4 implies that $f(A) \in \mathcal{L}(X)$ satisfies (3.9), and Lemma A.2 concludes the proof. □

Remark 3.8. Because $\mathrm{AM}_1(\mathrm{R}_{-\omega}) = \mathrm{AM}_\infty(\mathrm{R}_{-\omega})$ is contractively embedded in $\mathrm{AM}_p(\mathrm{R}_{-\omega})$, Theorem 3.7 also holds for $p = 1$ and $p = \infty$. However, A trivially has a bounded AM_1-calculus and a bounded AM_∞-calculus, by Lemma 3.3 and the definition of Hille-Phillips calculus.

Note that the exponential decay of $|f(z)|$ is only required as the real part of z tends to infinity. If $|f(z)|$ decays exponentially as $|z| \to \infty$ the result is not interesting. Indeed, Lemma 3.4 then implies that $f \in \mathrm{AM}_1$ and therefore that $f(A) \in \mathcal{L}(X)$.

We can equivalently formulate Theorem 3.7 as a statement about composition with semigroup operators.

Corollary 3.9. *Under the assumptions of Theorem 3.7, $f(A)T(\tau) \in \mathcal{L}(X)$ and*

$$\|f(A)T(\tau)\| \leq \begin{cases} c_p\, M^2 |\log(\omega\tau)|\, \mathrm{e}^{\omega\tau}\, \|f\|_{\mathrm{AM}_p^X} & \text{if } \omega\tau \leq \min(\frac{1}{p}, \frac{1}{p'}), \\ 2M^2\, \|f\|_{\mathrm{AM}_p^X} & \text{if } \omega\tau > \min(\frac{1}{p}, \frac{1}{p'}) \end{cases}$$

for all $f \in \mathrm{AM}_p^X(\mathrm{R}_{-\omega})$.

Proof. Note that $f(A)T(\tau) = (e_{-\tau}f)(A)$ and

$$\|e_{-\tau}f\|_{\mathrm{AM}_p^X(\mathrm{R}_{-\omega})} = \left\| e^{-\tau(-\omega+\mathrm{i}\cdot)} f(-\omega + \mathrm{i}\cdot) \right\|_{\mathcal{M}_p(X)} = \mathrm{e}^{\omega\tau}\, \|f\|_{\mathrm{AM}_p^X(\mathrm{R}_{-\omega})},$$

by Lemma 2.17 (i). Hence Theorem 3.7 yields the result. □

3.2.3 Additional results for semigroup generators

In this section we discuss some additional results that can be derived from Theorem 3.7.

As a first corollary of Theorem 3.7 we obtain a sufficient condition for a semigroup generator to have a bounded AM_p^X-calculus.

Corollary 3.10. *Let $-A$ generate a uniformly bounded C_0-semigroup $(T(t))_{t\in\mathbb{R}_+}$ on X with*

$$\bigcup_{\tau>0} \mathrm{ran}(T(\tau)) = X.$$

Then A has a bounded $\mathrm{AM}_p^X(\mathrm{R}_\omega)$-calculus for all $\omega < 0$, $p \in [1, \infty]$.

Proof. Using Corollary 3.9, $f(A)T(\tau) \in \mathcal{L}(X)$ implies $\mathrm{ran}(T(\tau)) \subseteq \mathrm{D}(f(A))$. Therefore $f(A) \in \mathcal{L}(X)$ for each $f \in \mathrm{AM}_p^X(\mathrm{R}_\omega)$ and the map $\mathrm{AM}_p^X(\mathrm{R}_\omega) \to \mathcal{L}(X)$, $f \mapsto f(A)$, is well-defined and linear. Suppose that $f \in \mathrm{AM}_p^X(\mathrm{R}_\omega)$, $(f_k)_{k\in\mathbb{N}} \subseteq \mathrm{AM}_p^X(\mathrm{R}_\omega)$ and $T \in \mathcal{L}(X)$ are such that $f_k \to f$ in $\mathrm{AM}_p^X(\mathrm{R}_\omega)$ and $f_k(A) \to T$ in $\mathcal{L}(X)$ as $k \to \infty$. By Lemma 2.4, $f_k(A)x \to f(A)x$ for each $x \in \mathrm{D}(A^2)$. Since $\mathrm{D}(A^2)$ is dense in X, $f(A) = T$. Now the closed graph theorem yields (2.6). □

Theorem 3.11. *Let $p \in (1, \infty)$, $\omega > 0$ and $\alpha, \lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) < 0 < \mathrm{Re}(\alpha)$. There exists a constant $C = C(p, \alpha, \lambda, \omega) \geq 0$ such that the following holds. Let $-A$ generate a C_0-semigroup $(T(t))_{t\in\mathbb{R}_+}$ of type $(M, 0)$ on a Banach space X. Then $\mathrm{D}((A - \lambda)^\alpha) \subseteq \mathrm{D}(f(A))$ and*

$$\|f(A)(A - \lambda)^{-\alpha}\| \leq CM^2\, \|f\|_{\mathrm{AM}_p^X(\mathrm{R}_{-\omega})}$$

for all $f \in \mathrm{AM}_p^X(\mathrm{R}_{-\omega})$.

Proof. $-(A-\lambda)$ generates the exponentially stable semigroup $(\mathrm{e}^{\lambda t}T(t))_{t\in\mathbb{R}_+}$. So Corollary 3.3.6 in [55] allows us to write

$$(A-\lambda)^{-\alpha}x = \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}\mathrm{e}^{\lambda t}T(t)x\,\mathrm{d}t \qquad (x\in X).$$

Fix $f\in \mathrm{AM}_p(\mathrm{R}_{-\omega})$ and set $a := \frac{1}{\omega}\min\left\{\frac{1}{p},\frac{1}{p'}\right\}$. By Corollary 3.9,

$$\int_0^\infty t^{\mathrm{Re}(\alpha)-1}\mathrm{e}^{\mathrm{Re}(\lambda)t}\,\|f(A)T(t)x\|\,\mathrm{d}t \le CM^2\,\|f\|_{\mathrm{AM}_p^X}\,\|x\| < \infty \tag{3.10}$$

for all $x\in X$, where

$$C = c_p\int_0^a t^{\mathrm{Re}(\alpha)-1}|\log(\omega t)|\mathrm{e}^{(\mathrm{Re}(\lambda)+\omega)t}\,\mathrm{d}t + 2\int_a^\infty t^{\mathrm{Re}(\alpha)-1}\mathrm{e}^{\mathrm{Re}(\lambda)t}\,\mathrm{d}t$$

is independent of f, M, and x. Since $f(A)$ is a closed operator, this implies that $(A-\lambda)^{-\alpha}$ maps into $\mathrm{D}(f(A))$ with

$$f(A)(A-\lambda)^{-\alpha}x = \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}\mathrm{e}^{\lambda t}f(A)T(t)x\,\mathrm{d}t \tag{3.11}$$

for all $x\in X$. Applying (3.10) to (3.11) concludes the proof. □

Remark 3.12. Theorem 3.11 shows that for each analytic multiplier function f the domain $\mathrm{D}(f(A))$ is relatively large, it contains the real interpolation spaces $(X,\mathrm{D}(A))_{\theta,q}$ for all $\theta\in(0,1)$ and $q\in[1,\infty]$. This follows from (2.36) and (2.38).

Remark 3.13. We can describe the range of $f(A)(A-\lambda)^{-\alpha}$ in Theorem 3.11 more explicitly. In fact,

$$\mathrm{ran}(f(A)(A-\lambda)^{-\alpha}) \subseteq \mathrm{D}\left((A-\lambda)^{\beta}\right)$$

for all $\mathrm{Re}(\beta) < \mathrm{Re}(\alpha)$. Indeed, this follows if we show $\mathrm{ran}(A-\lambda)^{-\alpha} \subseteq \mathrm{D}((A-\lambda)^{\beta}f(A))$, and [55, Theorem 1.3.2] implies

$$\mathrm{D}((A-\lambda)^{\beta}f(A)) = \mathrm{D}(f(A))\cap\mathrm{D}\left([(z-\lambda)^{\beta}f(z)](A)\right).$$

The inclusion $\mathrm{ran}((A-\lambda)^{-\alpha})\subseteq\mathrm{D}(f(A))$ follows from Theorem 3.11. Since

$$[(z-\lambda)^{\beta}f(z)](A)(A-\lambda)^{-\alpha} = [(z-\lambda)^{\beta-\alpha}f(z)](A) = f(A)(A-\lambda)^{\beta-\alpha},$$

the same holds for the inclusion $\mathrm{ran}((A-\lambda)^{-\alpha})\subseteq\mathrm{D}\left([(z-\lambda)^{\beta}f(z)](A)\right)$.

3.3 *m*-Bounded functional calculus

In this section we describe another transference principle for semigroups, one that provides estimates for the norms of operators of the form $f^{(m)}(A)$ for f an analytic multiplier function and $f^{(m)}$ its m-th derivative, $m \in \mathbb{N}$. We use terminology from Section 5 of [7]. Moreover, we recall our notational simplification $\mathrm{AM}_p(\mathrm{R}_\omega) := \mathrm{AM}_p^X(\mathrm{R}_\omega)$.

Let $\omega < \omega_0$ be real numbers. An operator A of half-plane type ω_0 on a Banach space X has an *m-bounded* $\mathrm{AM}_p^X(\mathrm{R}_\omega)$*-calculus* if there exists $C \geq 0$ such that $f^{(m)}(A) \in \mathcal{L}(X)$ with

$$\|f^{(m)}(A)\|_{\mathcal{L}(X)} \leq C\|f\|_{\mathrm{AM}_p^X} \qquad \text{for all } f \in \mathrm{AM}_p^X(\mathrm{R}_\omega).$$

This is well defined since the Cauchy integral formula implies that $f^{(m)}$ is bounded on every half-plane $\mathrm{R}_{\omega'}$ with $\omega' > \omega$:

$$|f^{(m)}(z)| \leq \|f\|_{\mathrm{H}^\infty(\mathrm{R}_\omega)} \frac{m!}{2\pi} \int_{\delta R_\omega} \frac{1}{|y - \omega'|^{m+1}} \mathrm{d}|y| \qquad (z \in \mathrm{R}_{\omega'}).$$

We say that A has a *strong m-bounded* AM_p^X*-calculus of type* ω_0 if A has an m-bounded $\mathrm{AM}_p^X(\mathrm{R}_\omega)$-calculus for *every* $\omega < \omega_0$ and if for some $C \geq 0$ one has

$$\|f^{(m)}(A)\|_{\mathcal{L}(X)} \leq \frac{C}{(\omega_0 - \omega)^m} \|f\|_{\mathrm{AM}_p^X(\mathrm{R}_\omega)} \tag{3.12}$$

for all $f \in \mathrm{AM}_p^X(\mathrm{R}_\omega)$ and $\omega < \omega_0$.

Lemma 3.14. *Let A be an operator of half-plane type $\omega_0 \in \mathbb{R}$ on a Banach space X, and let $p \in [1, \infty]$ and $m \in \mathbb{N}$. If A has a strong m-bounded AM_p^X-calculus of type ω_0, then A has a strong n-bounded AM_p^X-calculus of type ω_0 for all $n > m$.*

Proof. Let $\omega < \alpha < \beta < \omega_0$, $f \in \mathrm{AM}_p(\mathrm{R}_\omega)$ and $n \in \mathbb{N}$. Then

$$\begin{aligned} f^{(n)}(\beta + \mathrm{i}s) &= \frac{n!}{2\pi\mathrm{i}} \int_{\mathbb{R}} \frac{f(\alpha + \mathrm{i}r)}{(\alpha + \mathrm{i}r - (\beta + \mathrm{i}s))^{n+1}} \,\mathrm{d}r \\ &= \frac{n!}{2\pi\mathrm{i}} \Big(f(\alpha + \mathrm{i}\cdot) * (\alpha - \beta - \mathrm{i}\cdot)^{-n-1} \Big)(s) \end{aligned}$$

for all $s \in \mathbb{R}$, by the Cauchy integral formula. Hence, using Lemma 3.2 and (2.24),

$$\begin{aligned} \|f^{(n)}(\beta + \mathrm{i}\cdot)\|_{\mathcal{M}_p(X)} &\leq \frac{n!}{2\pi} \|(\alpha - \beta - \mathrm{i}\cdot)^{-n-1}\|_{\mathrm{L}^1(\mathbb{R})} \|f(\alpha + \mathrm{i}\cdot)\|_{\mathcal{M}_p(X)} \\ &\leq \frac{C}{(\beta - \alpha)^n} \|f\|_{\mathrm{AM}_p(\mathrm{R}_\omega)} \end{aligned}$$

for some $C = C(n) \geq 0$ independent of f, β, α and ω. Letting α tend to ω yields

$$\|f^{(n)}\|_{\mathrm{AM}_p(\mathrm{R}_\beta)} = \|f^{(n)}(\beta + \mathrm{i}\cdot)\|_{\mathcal{M}_p(X)} \leq \frac{C}{(\beta - \omega)^n}\|f\|_{\mathrm{AM}_p(\mathrm{R}_\omega)}. \tag{3.13}$$

Now let $n > m$. Applying (3.13) with $n - m$ in place of n shows that $f^{(n-m)} \in \mathrm{AM}_p(\mathrm{R}_\beta)$ with

$$\begin{aligned}\|f^{(n)}(A)\|_{\mathcal{L}(X)} &\leq \frac{C'}{(\omega_0 - \beta)^m}\|f^{(n-m)}\|_{\mathrm{AM}_p(\mathrm{R}_\beta)} \\ &\leq \frac{CC'}{(\omega_0 - \beta)^m(\beta - \omega)^{n-m}}\|f\|_{\mathrm{AM}_p(\mathrm{R}_\omega)}.\end{aligned}$$

Finally, letting $\beta = \frac{1}{2}(\omega + \omega_0)$,

$$\|f^{(n)}(A)\|_{\mathcal{L}(X)} \leq \frac{C''}{(\omega_0 - \omega)^n}\|f\|_{\mathrm{AM}_p(\mathrm{R}_\omega)}$$

for some $C'' \geq 0$ independent of f and ω. □

For the transference principle in Proposition 3.5 it is essential that the support of $\mu \in \mathrm{M}_\omega(\mathbb{R}_+)$ is contained in some interval $[\tau, \infty)$ with $\tau > 0$. In general one cannot expect to find such a transference principle for arbitrary μ, as this would allow one to prove that semigroup generators have a bounded analytic multiplier calculus. But this is false in general, cf. Proposition 2.5. However, if we let $t\mu$ be given by $(t\mu)(\mathrm{d}t) := t\mu(\mathrm{d}t)$ then we can deduce the following transference principle. We use the conventions $1/\infty := 0$, $\infty^0 := 1$.

Proposition 3.15. *Let $-A$ be the generator of a C_0-semigroup $(T(t))_{t\in\mathbb{R}_+}$ of type $(M, 0)$ on a Banach space X. Let $p \in [1, \infty]$, $\omega < 0$ and $\mu \in \mathrm{M}_\omega(\mathbb{R}_+)$. Then*

$$\left\|T_{t\mu}\right\| \leq \frac{M^2}{|\omega|}p^{-1/p}(p')^{-1/p'}\left\|L_{e_{-\omega}\mu}\right\|_{\mathcal{L}(\mathrm{L}^p(\mathbb{R};X))}.$$

Proof. As in the proof of Proposition 3.5, define $\iota : X \to \mathrm{L}^p(\mathbb{R}; X)$ by $\iota x(s) := \mathrm{e}^{-\omega s}T(-s)x$ for $s \leq 0$, while $\iota x(s) := 0$ if $s > 0$. Then

$$\|\iota x\|_{\mathrm{L}^p(\mathbb{R};X)} \leq M(|\omega|p)^{-1/p}\|x\| \qquad (x \in X), \tag{3.14}$$

so ι is well-defined and bounded. Moreover, let $P : \mathrm{L}^p(\mathbb{R}; X) \to X$ be given by

$$Pf := \int_0^\infty \mathrm{e}^{\omega t}T(t)f(t)\,\mathrm{d}t \qquad (f \in \mathrm{L}^p(\mathbb{R}; X)).$$

By Hölder's inequality,

$$\|Pf\| \leq M(|\omega|p')^{-1/p'} \|f\|_{\mathrm{L}^p(\mathbb{R};X)} \qquad (f \in \mathrm{L}^p(\mathbb{R};X)), \tag{3.15}$$

so P is also well-defined and bounded. Moreover, $L_{e_{-\omega}\mu}$ is a bounded operator on $\mathrm{L}^p(\mathbb{R};X)$, by Young's inequality. Letting $\Psi(\mathbb{R};X) = \Phi(\mathbb{R};X) = \mathrm{L}^p(\mathbb{R};X)$ and using

$$(\mathbf{1}_{\mathbb{R}_+}e_\omega * \mathbf{1}_{\mathbb{R}_+}e_\omega)(t)e_{-\omega}(t)\mu(\mathrm{d}t) = t\mu(\mathrm{d}t),$$

Proposition 2.25 yields the commutative diagram

$$\begin{array}{ccc} \mathrm{L}^p(\mathbb{R};X) & \xrightarrow{L_{e_{-\omega}\mu}} & \mathrm{L}^p(\mathbb{R};X) \\ \iota \uparrow & & \downarrow P \\ X & \xrightarrow{\quad T_{t\mu} \quad} & X \end{array}$$

Finally, estimate the norm of $T_{t\mu}$ through this factorization, and combine (3.14) and (3.15) to conclude the proof. □

We are now ready to prove our main result on m-bounded functional calculus, a generalization of [7, Theorem 7.1] to arbitrary Banach spaces. The idea for the proof of the implication (ii) ⇒ (i) comes from [7, Theorem 6.4].

Theorem 3.16. *Let A be a densely defined operator of half-plane type 0 on a Banach space X. Then the following assertions are equivalent:*

(i) *$-A$ is the generator of a uniformly bounded C_0-semigroup on X.*
(ii) *A has a strong m-bounded AM^X_p-calculus of type 0 for some/all $p \in [1, \infty]$ and some/all $m \in \mathbb{N}$.*

In particular, if $-A$ generates a uniformly bounded C_0-semigroup then A has an m-bounded $\mathrm{AM}^X_p(\mathrm{R}_\omega)$-calculus for all $\omega < 0$, $p \in [1, \infty]$ and $m \in \mathbb{N}$.

Proof. (i) ⇒ (ii) By Lemma 3.14 it suffices to let $m = 1$. We proceed along the same lines as in the proof of Theorem 3.7. Let $(T(t))_{t\in\mathbb{R}_+} \subseteq \mathcal{L}(X)$ be the semigroup generated by $-A$ and fix $\omega < 0$, $p \in [1, \infty]$ and $f \in \mathrm{AM}_p(\mathrm{R}_\omega)$. Define the functions $f_{k,\epsilon} := f(\cdot + \epsilon)g_k(\cdot + \epsilon)$ for $k \in \mathbb{N}$ and $\epsilon > 0$, where $g_k(z) := \frac{k}{z-\omega+k}$ for $z \in \mathrm{R}_\omega$. Then $f_{k,\epsilon} \in \mathrm{AM}_1(\mathrm{R}_\omega)$ by Lemma 3.4, and Lemma 3.3 yields $\mu_{k,\epsilon} \in \mathrm{M}_\omega(\mathbb{R}_+)$ with $f_{k,\epsilon} = \widehat{\mu_{k,\epsilon}}$. Now

$$\begin{aligned} f'_{k,\epsilon}(z) &= \lim_{h\to 0} \frac{f_{k,\epsilon}(z+h) - f_{k,\epsilon}(z)}{h} = \lim_{h\to 0} \int_0^\infty \frac{\mathrm{e}^{-(z+h)t} - \mathrm{e}^{-zt}}{h}\, \mu_{k,\epsilon}(\mathrm{d}t) \\ &= -\int_0^\infty t\mathrm{e}^{-zt}\, \mu_{k,\epsilon}(\mathrm{d}t) = -\widehat{t\mu_{k,\epsilon}}(z) \end{aligned}$$

for $z \in \mathrm{R}_\omega$, by the Dominated Convergence Theorem. Hence $f'_{k,\epsilon}(A) = -T_{t\mu_{k,\epsilon}}$, and Proposition 3.15 and Lemma 3.3 imply

$$\|f'_{k,\epsilon}(A)\|_{\mathcal{L}(X)} \leq \frac{M^2}{|\omega|}\, p^{-1/p}(p')^{-1/p'}\|f_{k,\epsilon}\|_{\mathrm{AM}_p^X(\mathrm{R}_\omega)}.$$

By Lemma 3.2 and because the inclusion $\mathrm{AM}_1^X(\mathrm{R}_\omega) \subseteq \mathrm{AM}_p^X(\mathrm{R}_\omega)$ is contractive,

$$\begin{aligned}\|f_{k,\epsilon}\|_{\mathrm{AM}_p(\mathrm{R}_\omega)} &\leq \|f(\cdot+\epsilon)\|_{\mathrm{AM}_p(\mathrm{R}_\omega)}\|g_k(\cdot+\epsilon)\|_{\mathrm{AM}_p(\mathrm{R}_\omega)}\\ &\leq \|f\|_{\mathrm{AM}_p(\mathrm{R}_\omega)}\|g_k\|_{\mathrm{AM}_1(\mathrm{R}_\omega)} = \|f\|_{\mathrm{AM}_p(\mathrm{R}_\omega)}.\end{aligned}$$

In particular, the $f_{k,\epsilon}$ are uniformly bounded on R_ω. By the Cauchy integral formula, the derivatives $f'_{k,\epsilon}$ are uniformly bounded on $\mathrm{R}_{\omega'}$ for each $\omega' \in (\omega, 0)$. Since $f'_{k,\epsilon}(z) \to f'(z)$ for all $z \in \mathrm{R}_{\omega'}$ as $k \to \infty$, $\epsilon \to 0$, the Convergence Lemma yields $f'(A) \in \mathcal{L}(X)$ with

$$\|f'(A)\|_{\mathcal{L}(X)} \leq \frac{M^2}{|\omega|}\, p^{-1/p}(p')^{-1/p'}\|f\|_{\mathrm{AM}_p^X(\mathrm{R}_\omega)},$$

which is (3.12) for $m = 1$.

For (ii) $\Rightarrow$ (i) assume that A has a strong m-bounded AM_p-calculus of type 0 for some $p \in [1, \infty]$ and some $m \in \mathbb{N}$. Then

$$e_{-t} \in \mathrm{AM}_1(\mathrm{R}_\omega) \subseteq \mathrm{AM}_p(\mathrm{R}_\omega)$$

for all $t > 0$ and $\omega < 0$, with

$$\|e_{-t}\|_{\mathrm{AM}_p(\mathrm{R}_\omega)} \leq \|e_{-t}\|_{\mathrm{AM}_1(\mathrm{R}_\omega)} = \mathrm{e}^{-t\omega}.$$

Now $(e_{-t})^{(m)} = (-t)^m e_{-t}$ implies that $e^{-tA} \in \mathcal{L}(X)$ with

$$t^m\|\mathrm{e}^{-tA}\|_{\mathcal{L}(X)} \leq \frac{C}{|\omega|^m}\mathrm{e}^{-t\omega}.$$

Letting $\omega := -\frac{1}{t}$ and using Lemma 2.2 yields the required statement. □

3.4 Semigroups on Hilbert and UMD spaces

In this section we apply the results from previous sections, which involve the abstract analytic multiplier algebra, to C_0-semigroups on Hilbert spaces and UMD spaces.

3.4.1 Semigroups on Hilbert spaces

If $X = H$ is a Hilbert space then (2.26) implies that $\mathrm{AM}_2^H = \mathrm{H}^\infty$ with equality of norms. Hence the theory in Section 3.2 specializes to the following result, yielding a) and b) of Theorem 3.1.

Corollary 3.17. *Let $-A$ generate a C_0-semigroup $(T(t))_{t\in\mathbb{R}_+}$ of type $(M,0)$ on a Hilbert space H. Then the following assertions hold.*

a) *There exists a universal constant $c \geq 0$ such that the following holds. Let $\tau, \omega > 0$ and $f \in e_{-\tau}\mathrm{H}^\infty(\mathrm{R}_{-\omega})$. Then $f(A) \in \mathcal{L}(H)$ and*

$$\|f(A)\| \leq \begin{cases} c\, M^2 |\log(\omega\tau)| \; \|f\|_{\mathrm{H}^\infty(\mathrm{R}_{-\omega})} & \text{if } \omega\tau \leq \frac{1}{2}, \\ 2M^2 \mathrm{e}^{-\omega\tau} \, \|f\|_{\mathrm{H}^\infty(\mathrm{R}_{-\omega})} & \text{if } \omega\tau > \frac{1}{2} \end{cases}$$

Moreover, $f(A)T(\tau) \in \mathcal{L}(H)$ with

$$\|f(A)T(\tau)\| \leq \begin{cases} c\, M^2 |\log(\omega\tau)| \mathrm{e}^{\omega\tau} \, \|f\|_{\mathrm{H}^\infty(\mathrm{R}_{-\omega})} & \text{if } \omega\tau \leq \frac{1}{2}, \\ 2M^2 \, \|f\|_{\mathrm{H}^\infty(\mathrm{R}_{-\omega})} & \text{if } \omega\tau > \frac{1}{2} \end{cases}$$

for all $f \in \mathrm{H}^\infty(\mathrm{R}_{-\omega})$.

b) *If*

$$\bigcup\nolimits_{\tau>0} \mathrm{ran}(T(\tau)) = H,$$

then A has a bounded $\mathrm{H}^\infty(\mathrm{R}_\omega)$-calculus for all $\omega < 0$.

c) *For $\omega < 0$ and $\alpha, \lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) < 0 < \mathrm{Re}(\alpha)$ there is $C = C(\alpha, \lambda, \omega) \geq 0$ such that*

$$\left\| f(A)(A-\lambda)^{-\alpha} \right\| \leq CM^2 \, \|f\|_{\mathrm{H}^\infty(\mathrm{R}_\omega)}$$

for all $f \in \mathrm{H}^\infty(\mathrm{R}_\omega)$. In particular, $\mathrm{D}(A^\alpha) \subseteq \mathrm{D}(f(A))$.

Part c) shows that, even though semigroup generators on Hilbert spaces do not have a bounded H^∞-calculus in general, each function f that decays with polynomial rate $\alpha > 0$ at infinity yields a bounded operator $f(A)$. For $\alpha > \frac{1}{2}$ this is already covered by Lemma 3.4, but for $\alpha \in (0, \frac{1}{2}]$ it appears to be new.

Remark 3.18. Part c) of Corollary 3.10 implies the stability of certain numerical methods. Let $-A$ generate an exponentially stable semigroup $(T(t))_{t\geq 0}$ on a Hilbert space, let $r \in \mathrm{H}^\infty(\mathbb{C}_+)$ be such that $\|r\|_{\mathrm{H}^\infty(\mathbb{C}_+)} \leq 1$, and let $\alpha, h > 0$. Then

$$\sup \left\{ \|r(hA)^n x\| \mid n \in \mathbb{N}, x \in \mathrm{D}(A^\alpha), \|A^\alpha x\| \leq 1 \right\} < \infty \tag{3.16}$$

follows from c) in Corollary 3.10 after shifting the generator. Elements of the form $r^n(hA)x$ are often used in numerical methods to approximate the solution of the abstract Cauchy problem associated to $-A$ with initial value x, and (3.16) shows that such approximations are stable whenever the semigroup is exponentially stable on a Hilbert space and if $x \in \mathrm{D}(A^\alpha)$ for some $\alpha > 0$.

Theorem 3.16 specializes to the following result, which contains part c) of Theorem 3.1. This result also follows from [7, Corollary 6.5 and (7.1)].

Corollary 3.19. *Let A be a densely defined operator of half-plane type* 0 *on a Hilbert space H. Then the following assertions are equivalent:*

(i) *$-A$ is the generator of a bounded C_0-semigroup on H.*
(ii) *A has a strong m-bounded H^∞-calculus of type* 0 *for some/all $m \in \mathbb{N}$.*

In particular, if $-A$ generates a uniformly bounded C_0-semigroup then A has an m-bounded $\mathrm{H}^\infty(\mathrm{R}_\omega)$-calculus for all $\omega < 0$ and $m \in \mathbb{N}$.

3.4.2 Semigroups on UMD spaces

For $\omega \in \mathbb{R}$ let

$$\mathrm{H}_1^\infty(\mathrm{R}_\omega) := \{f \in \mathrm{H}^\infty(\mathrm{R}_\omega) \mid (z-\omega)f'(z) \in \mathrm{H}^\infty(\mathrm{R}_\omega)\}$$

be the *analytic Mikhlin algebra* on R_ω, a Banach algebra endowed with the norm

$$\|f\|_{\mathrm{H}_1^\infty} = \|f\|_{\mathrm{H}_1^\infty(\mathrm{R}_\omega)} := \sup_{z\in \mathrm{R}_\omega} |f(z)| + |(z-\omega)f'(z)| \qquad (f \in \mathrm{H}_1^\infty(\mathrm{R}_\omega)).$$

Theorem 2.19 yields the continuous inclusion

$$\mathrm{H}_1^\infty(\mathrm{R}_\omega) \hookrightarrow \mathrm{AM}_p^X(\mathrm{R}_\omega)$$

for each $p \in (1,\infty)$, if X is a UMD space. Combining this with Theorems 3.7 and 3.16 and Corollaries 3.9 and 3.10 proves the following theorem.

Theorem 3.20. *Let $-A$ generate a C_0-semigroup $(T(t))_{t\in\mathbb{R}_+}$ of type $(M,0)$ on a UMD space X. Then the following assertions hold.*

a) *For each $p \in (1,\infty)$ there exists a constant $c_p = c(p,X) \geq 0$ such that the following holds. Let $\tau,\omega > 0$. Then $f(A) \in \mathcal{L}(X)$ with*

$$\|f(A)\| \leq \begin{cases} c_p\, M^2 |\log(\omega\tau)|\; \|f\|_{\mathrm{H}_1^\infty(\mathrm{R}_\omega)} & \textit{if } \omega\tau \leq \min\left\{\frac{1}{p},\frac{1}{p'}\right\}, \\ 2c_p M^2 \mathrm{e}^{-\omega\tau}\; \|f\|_{\mathrm{H}_1^\infty(\mathrm{R}_\omega)} & \textit{if } \omega\tau > \min\left\{\frac{1}{p},\frac{1}{p'}\right\} \end{cases}$$

for all $f \in \mathrm{H}_1^\infty(\mathrm{R}_{-\omega}) \cap e_{-\tau}\mathrm{H}^\infty(\mathrm{R}_{-\omega})$, and $f(A)T(\tau) \in \mathcal{L}(X)$ with

$$\|f(A)T(\tau)\| \leq \begin{cases} c_p\, M^2 |\log(\omega\tau)| \mathrm{e}^{\omega\tau}\; \|f\|_{\mathrm{H}_1^\infty(\mathrm{R}_\omega)} & \textit{if } \omega\tau \leq \min\left\{\frac{1}{p},\frac{1}{p'}\right\}, \\ 2c_p M^2\; \|f\|_{\mathrm{H}_1^\infty(\mathrm{R}_\omega)} & \textit{if } \omega\tau > \min\left\{\frac{1}{p},\frac{1}{p'}\right\} \end{cases}$$

for all $f \in \mathrm{H}_1^\infty(\mathrm{R}_{-\omega})$.

b) *If*

$$\bigcup_{\tau>0} \operatorname{ran}(T(\tau)) = X,$$

then A has a bounded $\mathrm{H}^{\infty}_{1}(\mathrm{R}_{\omega})$-calculus for all $\omega < 0$.

c) *A has a strong m-bounded H^{∞}_{1}-calculus of type 0 for all $m \in \mathbb{N}$.*

Remark 3.21. Theorem 3.11 yields the domain inclusion $\mathrm{D}(A^{\alpha}) \subseteq \mathrm{D}(f(A))$ for all $\alpha \in \mathbb{C}_{+}$, $\omega < 0$ and $f \in \mathrm{H}^{\infty}_{1}(\mathrm{R}_{\omega})$, on a UMD space X. However, this inclusion in fact holds true on a general Banach space X. Indeed, for $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) < 0$, Proposition 2.18 implies $\frac{f(z)}{(\lambda-z)^{\alpha}} \in \mathrm{AM}_{1}(\mathbb{C}_{+})$, hence $f(A)(\lambda - A)^{-\alpha} \in \mathcal{L}(X)$ and $\mathrm{D}(A^{\alpha}) \subseteq \mathrm{D}(f(A))$. The estimate

$$\left\|f(A)(\lambda - A)^{-\alpha}\right\| \leq C \left\|f\right\|_{\mathrm{H}^{\infty}_{1}(\mathrm{R}_{\omega})} \tag{3.17}$$

then follows from Lemma 2.4. Indeed, the Convergence Lemma implies that the map $f \mapsto f(A)(\lambda - A)^{-\alpha}$ is a closed operator, hence the closed graph theorem yields (3.17).

Remark 3.22. To apply Theorem 3.20 one can use the continuous inclusion

$$\mathrm{H}^{\infty}(\mathrm{R}_{\omega} \cup (S_{\varphi} + a)) \subseteq \mathrm{H}^{\infty}_{1}(\mathrm{R}_{\omega'}) \tag{3.18}$$

for $\omega' > \omega$, $a \in \mathbb{R}$ and $\varphi \in (\pi/2, \pi]$. Here $\mathrm{R}_{\omega} \cup (S_{\varphi} + a)$ is the union of R_{ω} and the translated sector $S_{\varphi} + a$. Indeed, it suffices to consider $a < 0$ in (3.18). Then for each $f \in \mathrm{H}^{\infty}(\mathrm{R}_{\omega} \cup (S_{\varphi} + a))$ and $z \in \mathrm{R}_{\omega'}$ the Cauchy integral formula yields

$$|zf'(z)| \leq \frac{1}{2\pi} \int_{\Gamma} \frac{|zf(y)|}{|y - z|^{2}} \mathrm{d}|y| \tag{3.19}$$

for Γ the boundary of $\mathrm{R}_{\omega''} \cup S_{\varphi}$, for $\omega'' \in (\omega, \omega')$. To obtain (3.18) from this, split the integral in (3.19) into two parts, corresponding to the part of Γ on $\partial\mathrm{R}_{\omega''}$ respectively ∂S_{φ}. For the first part, which is bounded, use that $\sup_{z \in \mathrm{R}_{\omega'}} |f'(z)| \leq \|f\|_{\mathrm{H}^{\infty}(\mathrm{R}_{\omega})}$. For the second part use a rescaling argument (see also [55, Lemma 8.2.6]).

3.5 γ-Bounded semigroups

The geometry of the underlying Banach space X played an essential role in the results of Sections 3.2 and 3.3 in the form of properties of the analytic multiplier algebra AM^{X}_{p}. Indeed, in order to identify nontrivial functions in AM^{X}_{p} one needs a geometric assumption on X, for instance that it is a Hilbert or a UMD space. In this section we take a different approach and make additional assumptions on the semigroup instead of the underlying space. We show that if the semigroup in question is γ-bounded then one can recover the Hilbert space results on an arbitrary Banach space X.

The following theorem generalizes part a) of Corollary 3.17. Recall that

$$e_{-\tau}\mathrm{H}^\infty(\mathrm{R}_\omega) = \left\{ f \in \mathrm{H}^\infty(\mathrm{R}_\omega) \mid f(z) \in O(\mathrm{e}^{-\tau \operatorname{Re}(z)}) \text{ as } |z| \to \infty \right\}$$

for $\tau > 0, \omega \in \mathbb{R}$.

Theorem 3.23. *There exists a universal constant $c \geq 0$ such that the following holds. Let $-A$ generate a C_0-semigroup $(T(t))_{t\in\mathbb{R}_+} \subseteq \mathcal{L}(X)$ of γ-type $(M,0)$ on a Banach space X, and let $\tau, \omega > 0$. Then $f(A) \in \mathcal{L}(X)$ with*

$$\|f(A)\| \leq \begin{cases} c\, M^2 |\log(\omega\tau)| \; \|f\|_\infty & \text{if } \omega\tau \leq \frac{1}{2} \\ 2M^2 \mathrm{e}^{-\omega\tau} \|f\|_\infty & \text{if } \omega\tau > \frac{1}{2} \end{cases} \tag{3.20}$$

for all $f \in e_{-\tau}\mathrm{H}^\infty(\mathrm{R}_{-\omega})$.

In particular, A has a bounded $e_{-\tau}\mathrm{H}^\infty(\mathrm{R}_{-\omega})$-calculus.

Proof. We first show that the estimate (3.5) in Proposition 3.5 can be refined to

$$\left\|T_\mu\right\| \leq M^2 \eta(\omega, \tau, 2) \left\|L_{e_\omega \mu}\right\|_{\mathcal{L}(\gamma_2(\mathbb{R};X))} \tag{3.21}$$

for $\mu \in \mathrm{M}_{-\omega}(\mathbb{R}_+)$ with $\operatorname{supp}(\mu) \subseteq [\tau, \infty)$. To this end, let $\psi, \varphi \in \mathrm{L}^2(\mathbb{R}_+)$ be such that $\psi * \varphi \equiv e_{-\omega}$ on $[\tau,\infty)$, and define $\iota : X \to \gamma_2(\mathbb{R}; X)$ and $P : \gamma_2(\mathbb{R}; X) \to X$ by

$$\begin{aligned} \iota x(s) &:= \psi(-s)T(-s)x \qquad (x \in X, s \in \mathbb{R}), \\ Pg &:= \int_0^\infty \varphi(t)T(t)g(t)\,\mathrm{d}t \qquad (g \in \gamma_2(\mathbb{R}; X)). \end{aligned}$$

Note that $s \mapsto T(-s)x$ is piecewise $W^{1,\infty}$ for all x in the dense subset $\mathrm{D}(A) \subseteq X$ and that

$$\psi(-\cdot) \otimes x \in \mathrm{L}^2(-\infty, 0) \otimes X \subseteq \gamma_2(\mathbb{R}; X).$$

Theorem 2.30 now implies that ι is well-defined and bounded, with

$$\|\iota x\|_{\gamma_2(\mathbb{R};X)} \leq M \|\psi(-\cdot) \otimes x\|_{\mathrm{L}^2(-\infty,0)} = M \|\psi\|_{\mathrm{L}^2(\mathbb{R}_+)} \|x\|_X \tag{3.22}$$

for $x \in X$. As for P, write

$$Pg = \int_0^\infty \varphi(t)T(t)g(t)\,\mathrm{d}t = J_{Tg}(\varphi) \qquad (g \in \gamma_2(\mathbb{R}; X)),$$

where J_{Tg} is as in (2.31). Now use Theorem 2.30 once again to see that $Tg \in \gamma_2(\mathbb{R}; X)$, with

$$\|Pg\|_X \leq \left\|J_{Tg}\right\|_\gamma \|\varphi\|_{L^2(\mathbb{R}_+)} \leq M \|\varphi\|_{L^2(\mathbb{R}_+)} \|g\|_{\gamma_2(\mathbb{R};X)} \tag{3.23}$$

for each $g \in \gamma_2(\mathbb{R}; X)$.

Hence ι and P are well-defined and bounded maps, and Proposition 2.25 yields the commutative diagram

$$\begin{array}{ccc} \gamma_2(\mathbb{R}; X) & \xrightarrow{L_{e_\omega \mu}} & \gamma_2(\mathbb{R}; X) \\ \iota \Big\uparrow & & \Big\downarrow P \\ X & \xrightarrow{\quad T_\mu \quad} & X \end{array}$$

Finally, estimating the norm of T_μ through this factorization, taking the infimum over all ψ and φ and using (3.22) and (3.23) yields (3.21).

Now one uses that

$$\left\|L_{e_\omega \mu}\right\|_{\mathcal{L}(\gamma(\mathbb{R};X))} \leq \|\widehat{e_\omega \mu}\|_{H^\infty(\mathbb{C}_+)} = \|\widehat{\mu}\|_{H^\infty(R_{-\omega})}$$

by 2.27, to obtain

$$\|f(A)\| \leq M^2 \eta(\omega, \tau, 2) \|f\|_{H^\infty(R_{-\omega})}$$

if $f = \widehat{\mu}$ for some $\mu \in M_{-\omega}(\mathbb{R}_+)$ with $\operatorname{supp}(\mu) \subseteq [\tau, \infty)$. For a general $f \in e_{-\tau} H^\infty(R_\omega)$, define $g_k(z) := \frac{k}{z-\omega+k}$ and $f_{k,\epsilon}(z) := f(z+\epsilon) g_k(z+\epsilon)$ for $\epsilon > 0$ $k \in \mathbb{N}$ and $z \in R_{-\omega}$. Then Lemmas 3.4 and 3.6 imply that $f_{k,\epsilon} = \widehat{\mu}$ for some $\mu \in M_{-\omega}(R_+)$ with $\operatorname{supp}(\mu) \subseteq [\tau, \infty)$. Hence, by what we have already shown,

$$\left\|f_{k,\epsilon}(A)\right\| \leq M^2 \eta(\omega, \tau, p) \left\|f_{k,\epsilon}\right\|_{H^\infty(R_{-\omega})}.$$

Moreover,

$$\left\|f_{k,\epsilon}\right\|_{H^\infty(R_{-\omega})} \leq \|f(\cdot + \epsilon)\|_{H^\infty(R_{-\omega})} \|g_k(\cdot + \epsilon)\|_{H^\infty(R_{-\omega})} \leq \|f\|_{H^\infty(R_{-\omega})}.$$

Now Lemma 2.4 implies that $f(A) \in \mathcal{L}(X)$ and that $f(A)$ satisfies (3.20), and finally Lemma A.2 concludes the proof. □

Corollary 3.24. *Let $-A$ generate a γ-bounded C_0-semigroup $(T(t))_{t \in \mathbb{R}_+} \subseteq \mathcal{L}(X)$ of γ-type $(M, 0)$ on a Banach space X. Then the following assertions hold.*

a) *There exists a universal constant $c \geq 0$ such that the following holds. Let $\tau, \omega > 0$ and $f \in H^\infty(R_{-\omega})$. Then $f(A)T(\tau) \in \mathcal{L}(H)$ with*

$$\|f(A)T(\tau)\| \leq \begin{cases} c\, M^2 |\log(\omega\tau)| e^{\omega\tau} \|f\|_{H^\infty(R_{-\omega})} & \text{if } \omega\tau \leq \frac{1}{2}, \\ 2M^2 \|f\|_{H^\infty(R_{-\omega})} & \text{if } \omega\tau > \frac{1}{2} \end{cases}$$

for all $f \in H^\infty(R_{-\omega})$.

b) *If*

$$\bigcup_{\tau>0} \operatorname{ran}(T(\tau)) = H,$$

then A has a bounded $\mathrm{H}^\infty(\mathrm{R}_\omega)$-calculus for all $\omega < 0$.

c) *For $\omega < 0$ and $\alpha, \lambda \in \mathbb{C}$ with $\operatorname{Re}(\lambda) < 0 < \operatorname{Re}(\alpha)$ there exists a $C = C(\alpha, \lambda, \omega) \geq 0$ such that*

$$\left\| f(A)(A-\lambda)^{-\alpha} \right\| \leq CM^2 \left\| f \right\|_{\mathrm{H}^\infty(\mathrm{R}_\omega)}$$

for all $f \in \mathrm{H}^\infty(\mathrm{R}_\omega)$. In particular, $\mathrm{D}(A^\alpha) \subseteq \mathrm{D}(f(A))$.

Proof. Part a) follows from Theorem (3.23) just as Corollary 3.9 followed from Theorem 3.7, using that $\|e_{-\tau} f\|_{\mathrm{H}^\infty(\mathrm{R}_{-\omega})} = \mathrm{e}^{\omega\tau} \|f\|_{\mathrm{H}^\infty(\mathrm{R}_{-\omega})}$.

Part b) follows from part a) in the same way as Corollary 3.10 followed from Corollary 3.9, by noting that the map $\mathrm{H}^\infty(\mathrm{R}_{-\omega}) \to \mathcal{L}(X)$, $f \mapsto f(A)$, is well-defined and linear, and by using the Closed Graph Theorem and Lemma 2.4.

For part c), Corollary 3.3.6 in [55] yields

$$(A-\lambda)^{-\alpha} x = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} \mathrm{e}^{\lambda t} T(t) x \, \mathrm{d}t \qquad (x \in X),$$

since $-(A-\lambda)$ generates the exponentially stable semigroup $(\mathrm{e}^{\lambda t} T(t))_{t \in \mathbb{R}_+}$. Fix $f \in \mathrm{H}^\infty(\mathrm{R}_{-\omega})$ and set $a := \frac{1}{2\omega}$. By part a),

$$\int_0^\infty t^{\operatorname{Re}(\alpha)-1} \mathrm{e}^{\operatorname{Re}(\lambda)t} \left\| f(A) T(t) x \right\| \mathrm{d}t \leq CM^2 \left\| f \right\|_{\mathrm{H}^\infty(\mathrm{R}_{-\omega})} \left\| x \right\| < \infty \tag{3.24}$$

for all $x \in X$, where

$$C = c \int_0^a t^{\operatorname{Re}(\alpha)-1} |\log(\omega t)| \mathrm{e}^{(\operatorname{Re}(\lambda)+\omega)t} \, \mathrm{d}t + 2 \int_a^\infty t^{\operatorname{Re}(\alpha)-1} \mathrm{e}^{\operatorname{Re}(\lambda)t} \, \mathrm{d}t$$

is independent of f, M, and x. Since $f(A)$ is a closed operator, this implies that $(A-\lambda)^{-\alpha}$ maps into $\mathrm{D}(f(A))$ with

$$f(A)(A-\lambda)^{-\alpha} x = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} \mathrm{e}^{\lambda t} f(A) T(t) x \, \mathrm{d}t$$

for all $x \in X$. Applying (3.24) to this expression concludes the proof of c). □

The following result will be used in Section 6.2.3.

Corollary 3.25. *Let $-A$ generate a C_0-semigroup $(T(t))_{t \in \mathbb{R}_+} \subseteq \mathcal{L}(X)$ of γ-type $(M, -\omega)$ for $M \geq 1$ and $\omega > 0$, and let $\beta > 0$. Then there is a constant $C = C(\omega, \beta) \geq 0$ such that $f(A)A^{-\beta} \in \mathcal{L}(X)$ with*

$$\|f(A)A^{-\beta}\|_{\mathcal{L}(X)} \leq CM^2 \|f\|_{\mathrm{H}^\infty(\mathbb{C}_+)}$$

for all $f \in \mathrm{H}^\infty(\mathbb{C}_+)$.

Proof. Let $f \in \mathrm{H}^\infty(\mathbb{C}_+)$ and note that $-(A-\omega)$ generates the semigroup $(e^{\omega t}T(t))_{t\in\mathbb{R}_+}$ of γ-type $(M, 0)$. Moreover, $f(\cdot+\omega) \in \mathrm{H}^\infty(\mathrm{R}_{-\omega})$ with $f(\cdot+\omega)(A-\omega) = f(A)$, by Lemma 2.3. Hence Corollary 3.24 c) yields a constant $C = C(\omega, \beta) \geq 0$ such that

$$\begin{aligned}\|f(A)A^{-\beta}\|_{\mathcal{L}(X)} &= \|f(\cdot+\omega)(A-\omega)(A-\omega-(-\omega))^{-\beta}\|_{\mathcal{L}(X)} \\ &\leq CM^2\|f(\cdot+\omega)\|_{\mathrm{H}^\infty(\mathrm{R}_{-\omega})} \leq CM^2\|f\|_{\mathrm{H}^\infty(\mathbb{C}_+)}. \qquad \square\end{aligned}$$

Theorem 3.16 can also be extended to a γ-version:

Theorem 3.26. *Let $-A$ generate a C_0-semigroup $(T(t))_{t\in\mathbb{R}_+} \subseteq \mathcal{L}(X)$ of γ-type $(M, 0)$ on a Banach space X. Then A has a strong m-bounded H^∞-calculus of type 0 for all $m \in \mathbb{N}$. In particular,*

$$\|f'(A)\|_{\mathcal{L}(X)} \leq \frac{M^2}{2|\omega|}\|f\|_{\mathrm{H}^\infty(\mathrm{R}_{-\omega})} \tag{3.25}$$

for each $\omega > 0$ and $f \in \mathrm{H}^\infty(\mathrm{R}_{-\omega})$.

Proof. By replacing AM_p^X by H^∞ in the proof of Lemma 3.14, it suffices to let $m = 1$. Let $\omega > 0$. We first obtain (3.25) for $f = \widehat{\mu}$, where $\mu \in \mathrm{M}_{-\omega}(\mathbb{R}_+)$. A modification of the proof of Proposition 3.15, along the same lines as in the proof of Theorem 3.23, shows that the maps $\iota : X \to \gamma_2(\mathbb{R}; X)$ and $P : \gamma_2(\mathbb{R}; X) \to X$, given by

$$\begin{aligned}\iota x(s) &:= \mathrm{e}^{-\omega s}T(-s)x && (x \in X, s \in \mathbb{R}), \\ Pf &:= \int_0^\infty \mathrm{e}^{\omega t}T(t)f(t)\,\mathrm{d}t && (f \in \gamma_2(\mathbb{R}; X)),\end{aligned}$$

are well-defined and bounded. More precisely, one obtains

$$\|\iota x\|_{\gamma_2(\mathbb{R};X)} \leq M(2|\omega|)^{-1/2}\,\|x\|_X \tag{3.26}$$

for $x \in X$, and

$$\|Pf\|_X \leq M(2|\omega|)^{-1/2}\,\|f\|_{\gamma_2(\mathbb{R};X)} \tag{3.27}$$

for $f \in \gamma_2(\mathbb{R}; X)$. By Lemma 2.27,

$$\left\|L_{e_{-\omega}\mu}\right\|_{\mathcal{L}(\gamma(\mathbb{R};X))} \leq \|\widehat{e_\omega\mu}\|_{\mathrm{H}^\infty(\mathbb{C}_+)} = \|\widehat{\mu}\|_{\mathrm{H}^\infty(\mathrm{R}_{-\omega})}. \tag{3.28}$$

Letting $\Psi(\mathbb{R}; X) = \Phi(\mathbb{R}; X) = \gamma_2(\mathbb{R}; X)$ in Proposition 2.25, and using that

$$(\mathbf{1}_{\mathbb{R}_+}e_\omega * \mathbf{1}_{\mathbb{R}_+}e_\omega)(t)e_{-\omega}(t)\mu(\mathrm{d}t) = t\mu(\mathrm{d}t),$$

we obtain the commutative diagram

$$\begin{array}{ccc} \gamma_2(\mathbb{R};X) & \xrightarrow{L_{e_{-\omega}\mu}} & \gamma_2(\mathbb{R};X) \\ \uparrow \iota & & \downarrow P \\ X & \xrightarrow{\quad T_{t\mu} \quad} & X \end{array}$$

As in Theorem 3.16, $f'(z) = \widehat{t\mu}(z)$ for all $z \in \mathrm{R}_{-\omega}$. Hence estimating the norm of $T_{t\mu} = f'(A)$ through the above factorization, and using (3.26), (3.27) and (3.28) one obtains (3.25) if $f = \widehat{\mu}$ for $\mu \in \mathrm{M}_{-\omega}(\mathbb{R}_+)$.

For general $f \in \mathrm{H}^\infty(\mathrm{R}_w)$ define the functions $f_{k,\epsilon} := f(\cdot + \epsilon)g_k(\cdot + \epsilon)$ for $k \in \mathbb{N}$ and $\epsilon > 0$, where $g_k(z) := \frac{k}{z-\omega+k}$ for $z \in \mathrm{R}_{-\omega}$. Lemma 3.4 and Lemma 3.3 yield $\mu_{k,\epsilon} \in \mathrm{M}_\omega(\mathbb{R}_+)$ with $f_{k,\epsilon} = \widehat{\mu_{k,\epsilon}}$ for each $k \in \mathbb{N}$ and $\epsilon > 0$. We have shown above that

$$\|f'_{k,\epsilon}(A)\|_{\mathcal{L}(X)} \leq \frac{M^2}{2|\omega|}\|f_{k,\epsilon}\|_{\mathrm{H}^\infty(\mathrm{R}_\omega)} \leq \frac{M^2}{2|\omega|}\|f\|_{\mathrm{H}^\infty(\mathrm{R}_\omega)},$$

where we used that $\|f_{k,\epsilon}\|_{\mathrm{H}^\infty(\mathrm{R}_\omega)} \leq \|f\|_{\mathrm{H}^\infty(\mathrm{R}_\omega)}$ for all $k \in \mathbb{N}$ and $\epsilon > 0$. In particular, the $f_{k,\epsilon}$ are uniformly bounded on $\mathrm{R}_{-\omega}$. By the Cauchy integral formula, the derivatives $f'_{k,\epsilon}$ are uniformly bounded on $\mathrm{R}_{\omega'}$ for each $\omega' \in (-\omega, 0)$. Since $f'_{k,\epsilon}(z) \to f'(z)$ for all $z \in \mathrm{R}_{\omega'}$ as $k \to \infty$, $\epsilon \to 0$, Lemma 2.4 yields $f'(A) \in \mathcal{L}(X)$ with

$$\|f'(A)\|_{\mathcal{L}(X)} \leq \frac{M^2}{2|\omega|}\|f\|_{\mathrm{H}^\infty(\mathrm{R}_{-\omega})},$$

which is (3.12) for $m = 1$. □

4

Functional calculus on real interpolation spaces for generators of C_0-groups

In Theorems 2.9 and 2.20 we have seen that group generators have a bounded calculus for a class of functions which may depend on the underlying space. Given what we encountered in Chapter 3, where the results depended heavily on the geometry of the underlying space, this is perhaps not very surprising. Moreover, just like the results in Chapter 3, Theorems 2.9 and 2.20 can be proved using transference principles.

Although Theorems 2.9 and 2.20 are powerful statements, geometric assumptions on the underlying space restrict the generality of the results. In particular, Hilbert and UMD spaces are reflexive. Therefore the transference approach which we used so far does not yield interesting results for groups of operators on non-reflexive spaces such as $\mathrm{C}(K)$-spaces or L^1-spaces.

In this chapter we take a different approach and consider transference principles on interpolation spaces. We have seen in Theorem 2.38 that the functional calculus properties of an operator can improve upon restriction to interpolation spaces. However, unlike in Theorem 2.38 we are interested in the strip type functional calculus considered in Section 2.2.2, which is a more natural and useful functional calculus for group generators.

The reason that the transference principles in Propositions 2.23 and 2.24 do not directly yield anything interesting for C_0-groups on spaces X which are not UMD is that no results about $\mathrm{L}^p(\mathbb{R}; X)$-Fourier multipliers are available in this context. However, we have seen in Theorem 2.21 that $\mathrm{B}^r_{p,q}(\mathbb{R}; X)$-Fourier multiplier results exist that do not depend on the geometry of X. We have also seen in Lemma 2.35 that Besov spaces are obtained from real interpolation between L^p and Sobolev spaces, and this fits well into the setting of a transference principle on interpolation spaces.

In this chapter we consider transference principles on the real interpolation space $\mathrm{D}_A(\theta, q)$ from (2.37). In particular, in Proposition 4.4 we establish the following interpolation version of the classical transference principle from 2.23.

Proposition 4.1. *Let $\theta \in (0,1)$, $p \in [1,\infty)$ and $q \in [1,\infty]$. Then there exists a constant $C \geq 0$ such that the following holds. Let $-\mathrm{i}A$ generate a C_0-group $(U(s))_{s\in\mathbb{R}} \subseteq \mathcal{L}(X)$ on a Banach space X with $M := \sup_{s\in\mathbb{R}} \|U(s)\| < \infty$, and let $\mu \in \mathrm{M}(\mathbb{R})$. Then*

$$\left\| \int_{\mathbb{R}} U(s)x\,\mu(\mathrm{d}s) \right\|_{\theta,q} \leq CM^2 \|L_\mu\|_{\mathcal{L}(\mathrm{B}^{\theta}_{p,q}(\mathbb{R};X))} \|x\|_{\theta,q}$$

for all $\mu \in \mathrm{M}(\mathbb{R})$ and $x \in \mathrm{D}_A(\theta,q)$.

We also establish an interpolation version of the transference principle for unbounded groups from Proposition 2.24.

We then combine these transference principles with Theorem 2.21 to derive a functional calculus result for the part $A_{\theta,q}$ of A in $\mathrm{D}_A(\theta,q)$ from (2.39). To this end, for each $\omega > 0$ define the *(inhomogeneous) analytic Mikhlin algebra*

$$\mathrm{H}^\infty_1(\mathrm{St}_\omega) := \left\{ f \in \mathrm{H}^\infty(\mathrm{St}_\omega) \;\middle|\; \sup_{z\in\mathrm{St}_\omega} (1+|z|)\,|f'(z)| < \infty \right\} \tag{4.1}$$

on St_ω, endowed with the norm

$$\|f\|_{\mathrm{H}^\infty_1(\mathrm{St}_\omega)} := \sup_{z\in\mathrm{St}_\omega} |f(z)| + (1+|z|)\,|f'(z)| \qquad (f \in \mathrm{H}^\infty_1(\mathrm{St}_\omega)). \tag{4.2}$$

It is straightforward to show that, for all $\omega > 0$, $\mathrm{H}^\infty_1(\mathrm{St}_\omega)$ is equal to the homogenous analytic Mikhlin algebra $\mathrm{H}^\infty_{(1)}(\mathrm{St}_\omega)$ from (2.27), with equivalent norms. However, (4.2) is more natural in the setting of transference principles on inhomogeneous Besov spaces, since Fourier multiplier results on such spaces require an inhomogeneous condition at zero. Moreover, the norm equivalence of $\mathrm{H}^\infty_1(\mathrm{St}_\omega)$ and $\mathrm{H}^\infty_{(1)}(\mathrm{St}_\omega)$ fails as $\omega \downarrow 0$, which means that the distinction between the two spaces is relevant for uniformly bounded groups. See also Remarks 4.9 and 4.11.

The main functional calculus result of this chapter is the following version of Theorems 2.9 and 2.20 on interpolation spaces. For a proof see Theorem 4.10.

Theorem 4.2. *Let $-\mathrm{i}A$ generate a C_0-group $(U(s))_{s\in\mathbb{R}} \subseteq \mathcal{L}(X)$ on a Banach space X, and let $\theta \in (0,1)$, $q \in [1,\infty]$. Then $A_{\theta,q}$ has a bounded $\mathrm{H}^\infty_1(\mathrm{St}_\omega)$-calculus on $\mathrm{D}_A(\theta,q)$ for all $\omega > \theta(U)$.*

If $(U(s))_{s\in\mathbb{R}}$ is uniformly bounded then the constant bounding the $\mathrm{H}^\infty_1(\mathrm{St}_\omega)$-calculus can be chosen independent of $\omega > 0$.

Theorem 4.2 shows that, just as in Theorem 2.38, the functional calculus properties of an operator can improve upon restriction to a real interpolation space.

An important class of C_0-groups is given by the groups of imaginary powers of a sectorial operator with bounded imaginary powers. For a sectorial operator A with bounded imaginary powers one can use Theorem 4.2 to obtain a specific bounded calculus on real interpolation spaces between the underlying space X and the domain of the logarithm of A. This result complements Theorem 2.38.

In a similar manner, one can deduce results about convergence of principal value integrals and functional calculus for generators of cosine functions from Theorem 4.2.

In Section 4.1 we establish transference principles on interpolation spaces, and in Section 4.2 we prove Theorem 4.2. Section 4.3 contains additional results that can be derived from this.

4.1 Transference principles on real interpolation spaces

In this section we derive versions of Propositions 2.23 and 2.24 on interpolation spaces.

4.1.1 Unbounded groups

We first establish an interpolation version of the transference principle for unbounded groups from Proposition 2.24. For $\omega \geq 0$ and $\mu \in \mathrm{M}_{-\omega}(\mathbb{R})$, recall the definition of the measure $\mu_\omega \in \mathrm{M}(\mathbb{R})$ from (2.30).

Proposition 4.3. *Let* $0 \leq \omega_0 < \omega$, $\theta \in (0,1)$, $p \in [1,\infty)$ *and* $q \in [1,\infty]$. *Then there exists a constant* $C \geq 0$ *such that the following holds. Let* $-\mathrm{i}A$ *generate a* C_0*-group* $(U(s))_{s\in\mathbb{R}} \subseteq \mathcal{L}(X)$ *on a Banach space* X *such that* $\|U(s)\|_{\mathcal{L}(X)} \leq M\cosh(\omega_0 s)$ *for all* $s \in \mathbb{R}$ *and some* $M \geq 1$, *and let* $\mu \in \mathrm{M}_{-\omega}(\mathbb{R})$. *Then*

$$\left\| \int_{\mathbb{R}} U(s)x\, \mu(\mathrm{d}s) \right\|_{\theta,q} \leq CM^2 \left\| L_{\mu_\omega} \right\|_{\mathcal{L}(\mathrm{B}^{\theta}_{p,q}(\mathbb{R};X))} \|x\|_{\theta,q}$$

for all $x \in \mathrm{D}_A(\theta,q)$.

Proof. Let U_μ be as in (2.9). By Proposition 2.25, we can factorize $U_\mu \in \mathcal{L}(X)$ as $U_\mu = P \circ L_{\mu_\omega} \circ \iota$, where

- $\iota : X \to \mathrm{L}^p(\mathbb{R};X)$ is given by

$$\iota x(s) := \psi(-s)U(-s)x \qquad (x \in X, s \in \mathbb{R}),$$

  with

$$\psi(s) := \frac{1}{\cosh(2\omega s)} \qquad (s \in \mathbb{R}).$$

- $P : \mathrm{L}^p(\mathbb{R}; X) \to X$ is given by

$$Pf := \int_{\mathbb{R}} \varphi(s)U(s)f(s)\,\mathrm{d}s \qquad (f \in \mathrm{L}^p(\mathbb{R}; X)),$$

  with

$$\varphi(s) := \frac{\sqrt{8}\omega}{\pi}\frac{\cosh(\omega s)}{\cosh(2\omega s)} \qquad (s \in \mathbb{R}).$$

Here we use that $\psi * \varphi(s) = \frac{1}{\cosh(\omega s)}$ for $s \in \mathbb{R}$, as can be seen by taking Fourier transforms (see [58, Theorem 3.2]). By Hölder's inequality,

$$\|\iota\|_{\mathcal{L}(X,\mathrm{L}^p(\mathbb{R};X))} \le M \|\psi \cosh(\omega_0 \cdot)\|_p, \tag{4.3}$$

$$\|P\|_{\mathcal{L}(\mathrm{L}^p(\mathbb{R};X),X)} \le M \|\varphi \cosh(\omega_0 \cdot)\|_{p'}. \tag{4.4}$$

We claim that $\iota : \mathrm{D}(A) \to \mathrm{W}^{1,p}(\mathbb{R}; X)$ and $P : \mathrm{W}^{1,p}(\mathbb{R}; X) \to \mathrm{D}(A)$ are well-defined and bounded. To prove this claim, let $x \in \mathrm{D}(A)$. Then $\iota x \in \mathrm{C}^1(\mathbb{R}; X)$ with

$$\begin{aligned}(\iota x)'(s) &= -\psi'(-s)U(-s)x + \mathrm{i}\psi(-s)U(-s)Ax \\ &= -2\omega\frac{\tanh(2\omega s)}{\cosh(2\omega s)}U(-s)x + \mathrm{i}\frac{1}{\cosh(2\omega s)}U(-s)Ax\end{aligned}$$

for all $s \in \mathbb{R}$. Hence $(\iota x)' \in \mathrm{L}^p(\mathbb{R}; X)$ with

$$\|(\iota x)'\|_p \le 2\omega M \|\tanh\|_{\mathrm{L}^\infty(\mathbb{R})} \left\|\frac{\cosh(\omega_0\cdot)}{\cosh(2\omega\cdot)}\right\|_p \|x\|_X + M \left\|\frac{\cosh(\omega_0\cdot)}{\cosh(2\omega\cdot)}\right\|_p \|Ax\|_X.$$

Combining this with (4.3) implies that $\iota x \in \mathrm{W}^{1,p}(\mathbb{R}; X)$ with

$$\|\iota x\|_{1,p} \le M(2\omega \|\tanh\|_{\mathrm{L}^\infty(\mathbb{R})} + 1) \left\|\frac{\cosh(\omega_0\cdot)}{\cosh(2\omega\cdot)}\right\|_p \|x\|_{\mathrm{D}(A)}. \tag{4.5}$$

This shows that $\iota : \mathrm{D}(A) \to \mathrm{W}^{1,p}(\mathbb{R}; X)$ is bounded. To prove the claim for P, fix $f \in \mathcal{S}(X)$ and note that

$$\frac{1}{h}(U(h) - \mathrm{I})Pf = \int_{\mathbb{R}} U(s)\frac{\varphi(s-h)f(s-h) - \varphi(s)f(s)}{h}\,\mathrm{d}s$$

for $h > 0$. The latter expression converges to $-\int_{\mathbb{R}} U(s)(\varphi f)'(s)\,\mathrm{d}s \in X$ as $h \to 0$, by the dominated convergence theorem. Hence $Pf \in \mathrm{D}(A)$ with

$$APf = \lim_{h\to 0}\frac{1}{h}(U(h) - \mathrm{I})Pf = -\int_{\mathbb{R}} U(s)(\varphi'(s)f(s) + \varphi(s)f'(s))\,\mathrm{d}s.$$

Another application of Hölder's inequality yields

$$\|APf\|_X \leq M \left\|\varphi' \cosh(\omega_0 \cdot)\right\|_{p'} \|f\|_p + M \|\varphi \cosh(\omega_0 \cdot)\|_{p'} \left\|f'\right\|_p .$$

Combining this with (4.4) implies

$$\|Pf\|_{\mathrm{D}(A)} \leq M \left(\|\varphi \cosh(\omega_0 \cdot)\|_{p'} + \left\|\varphi' \cosh(\omega_0 \cdot)\right\|_{p'} \right) \|f\|_{1,p} . \tag{4.6}$$

As $\mathcal{S}(X)$ is dense in $\mathrm{W}^{1,p}(\mathbb{R}; X)$, $P : \mathrm{W}^{1,p}(\mathbb{R}; X) \to \mathrm{D}(A)$ is bounded.

Since $L_{\mu_\omega} \in \mathcal{L}(\mathrm{W}^{1,p}(\mathbb{R}; X))$, we can factorize $U_\mu \in \mathcal{L}(\mathrm{D}(A))$ as $U_\mu = P \circ L_{\mu_\omega} \circ \iota$ via bounded maps through $\mathrm{W}^{1,p}(\mathbb{R}; X)$. Applying the real interpolation method with parameters θ and q to the two factorizations of U_μ, through $\mathrm{L}^p(\mathbb{R}; X)$ respectively $\mathrm{W}^{1,p}(\mathbb{R}; X)$, yields the commutative diagram of bounded maps

$$\begin{array}{ccc} (\mathrm{L}^p(\mathbb{R}; X), \mathrm{W}^{1,p}(\mathbb{R}; X))_{\theta,q} & \xrightarrow{L_{\mu_\omega}} & (\mathrm{L}^p(\mathbb{R}; X), \mathrm{W}^{1,p}(\mathbb{R}; X))_{\theta,q} \\ \iota \uparrow & & \downarrow P \\ \mathrm{D}_A(\theta, q) & \xrightarrow{\quad U_\mu \quad} & \mathrm{D}_A(\theta, q) \end{array}$$

Finally, estimate the norms of ι and P in this diagram by applying (2.34) to (4.3) and (4.5) respectively (4.4) and (4.6). This yields

$$\|U_\mu\|_{\mathcal{L}(\mathrm{D}_A(\theta,q))} \leq C' M^2 \|L_\mu\|_{\mathcal{L}((\mathrm{L}^p(\mathbb{R};X), \mathrm{W}^{1,p}(\mathbb{R};X))_{\theta,q})} \tag{4.7}$$

for a constant $C' \geq 0$ independent of μ. Now Lemma 2.35 concludes the proof. □

4.1.2 Bounded groups

In this section we establish a version of Proposition 2.23 on interpolation spaces, already stated in the introduction to this chapter as Proposition 4.1. In the proof we use the convention $1/\infty := 0$.

Proposition 4.4. *Let $\theta \in (0, 1)$, $p \in [1, \infty)$ and $q \in [1, \infty]$. Then there exists a constant $C \geq 0$ such that the following holds. Let $-\mathrm{i}A$ generate a C_0-group $(U(s))_{s \in \mathbb{R}} \subseteq \mathcal{L}(X)$ on a Banach space X with $M := \sup_{s \in \mathbb{R}} \|U(s)\| < \infty$, and let $\mu \in \mathrm{M}(\mathbb{R})$. Then*

$$\left\| \int_{\mathbb{R}} U(s) x \, \mu(\mathrm{d}s) \right\|_{\theta,q} \leq C M^2 \|L_\mu\|_{\mathcal{L}(\mathrm{B}^\theta_{p,q}(\mathbb{R};X))} \|x\|_{\theta,q} \tag{4.8}$$

for all $\mu \in \mathrm{M}(\mathbb{R})$ and $x \in \mathrm{D}_A(\theta, q)$.

Proof. First note that it suffices to establish (4.8) for measures with compact support. Indeed, approximating by measures with compact support then extends (4.8) to all $\mu \in \mathrm{M}(\mathbb{R})$. So fix $N > 0$ and suppose that $\mathrm{supp}(\mu) \subseteq$

$[-N,N]$. We will factorize U_μ using Proposition 2.25. To this end, let $\rho \in \mathrm{C}^\infty(\mathbb{R})$ be defined by

$$\rho(s) := \begin{cases} c_1 \exp\left(\frac{1}{s^2-1}\right) & |s| < 1 \\ 0 & |s| \geq 1 \end{cases},$$

where $c_1 \geq 0$ is such that $\int_{\mathbb{R}} \rho(s)\,\mathrm{d}s = 1$. Fix $\alpha, \beta > 0$ and define $\sigma(s) := \frac{1}{\alpha}\rho\left(\frac{s}{\alpha}\right)$ for $s \in \mathbb{R}$, and let

$$\psi := \sigma * \mathbf{1}_{[-(N+3\alpha+\beta),N+3\alpha+\beta]} \qquad \text{and} \qquad \varphi := \frac{1}{2(\alpha+\beta)}\sigma * \mathbf{1}_{[-(\alpha+\beta),\alpha+\beta]}.$$

Then $\psi, \varphi \in \mathrm{C}^\infty(\mathbb{R})$ are such that $\operatorname{supp}(\varphi) \subseteq [-(2\alpha+\beta), 2\alpha+\beta]$,

$$\psi \equiv 1 \text{ on } [-(2\alpha+N+\beta), 2\alpha+N+\beta] \quad \text{and} \quad \int_{-(2\alpha+\beta)}^{2\alpha+\beta} \varphi(s)\,\mathrm{d}s = 1.$$

Hence $\psi * \varphi \equiv 1$ on $[-N,N]$. Let $\iota : X \to \mathrm{L}^p(\mathbb{R};X)$ be given by

$$\iota x(s) := \psi(-s)U(-s)x \qquad (x \in X, s \in \mathbb{R}),$$

and $P : \mathrm{L}^p(\mathbb{R};X) \to X$ by

$$Pf := \int_{\mathbb{R}} \varphi(s)U(s)f(s)\,\mathrm{d}s \qquad (f \in \mathrm{L}^p(\mathbb{R};X)).$$

Proposition 2.25 yields the factorization $U_\mu = P \circ L_\mu \circ \iota$, where we use that $(\psi * \varphi)\mu = \mu$. By Hölder's inequality,

$$\|\iota\|_{\mathcal{L}(X,\mathrm{L}^p(\mathbb{R};X))} \leq M\|\psi\|_p \quad \text{and} \quad \|P\|_{\mathcal{L}(\mathrm{L}^p(\mathbb{R};X),X)} \leq M\|\varphi\|_{p'} \tag{4.9}$$

Moreover, $\iota : \mathrm{D}(A) \to \mathrm{W}^{1,p}(\mathbb{R};X)$ and $P : \mathrm{W}^{1,p}(\mathbb{R};X) \to \mathrm{D}(A)$ are bounded with

$$\|\iota\|_{\mathcal{L}(\mathrm{D}(A),\mathrm{W}^{1,p}(\mathbb{R};X))} \leq M\|\psi\|_{1,p} \quad \text{and} \quad \|P\|_{\mathcal{L}(\mathrm{W}^{1,p}(\mathbb{R};X),\mathrm{D}(A))} \leq M\|\varphi\|_{1,p'}. \tag{4.10}$$

This follows by arguments analogous to those in the proof of Proposition 4.3. Applying the real interpolation method with parameters θ and q to the two factorizations of U_μ, through $\mathrm{L}^p(\mathbb{R};X)$ and $\mathrm{W}^{1,p}(\mathbb{R};X)$, produces the commutative diagram of bounded maps

$$\begin{array}{ccc} (\mathrm{L}^p(\mathbb{R};X),\mathrm{W}^{1,p}(\mathbb{R};X))_{\theta,q} & \xrightarrow{L_\mu} & (\mathrm{L}^p(\mathbb{R};X),\mathrm{W}^{1,p}(\mathbb{R};X))_{\theta,q} \\ \iota\uparrow & & \downarrow P \\ \mathrm{D}_A(\theta,q) & \xrightarrow{\quad U_\mu \quad} & \mathrm{D}_A(\theta,q) \end{array}$$

Use (2.34) on (4.9) and (4.10) to estimate the norms of ι and P in this factorization as $\|\iota\| \leq M\,\|\psi\|_{1,p}$ and $\|P\| \leq M\,\|\varphi\|_{1,p'}$. This yields

$$\|U_\mu\|_{\mathcal{L}(\mathrm{D}_A(\theta,q))} \leq M^2 \|\psi\|_{1,p}\|\varphi\|_{1,p'}\|L_\mu\|_{\mathcal{L}((\mathrm{L}^p(\mathbb{R};X),\mathrm{W}^{1,p}(\mathbb{R};X))_{\theta,q})}. \tag{4.11}$$

To determine $\|\psi\|_{1,p}$ and $\|\varphi\|_{1,p'}$ note that

$$\begin{aligned}\|\psi\|_p &\leq \|\sigma\|_1\|\mathbf{1}_{[-(N+3\alpha+\beta),N+3\alpha+\beta]}\|_p = (2(N+3\alpha+\beta))^{1/p},\\ \|\varphi\|_{p'} &\leq \frac{1}{2(\alpha+\beta)}\|\sigma\|_1\|\mathbf{1}_{[-(\alpha+\beta),\alpha+\beta]}\|_{p'} = (2(\alpha+\beta))^{-1/p},\end{aligned}$$

by Young's inequality. Since σ is an even function that is decreasing on $[0,\alpha]$ and supported on $[-\alpha,\alpha]$, its derivative satisfies

$$\|\sigma'\|_1 = -2\int_0^\alpha \sigma'(s)\,\mathrm{d}s = 2(\sigma(0)-\sigma(\alpha)) = \frac{2\rho(0)}{\alpha}.$$

Let $c_2 := 2\rho(0)$. Another application of Young's inequality yields

$$\begin{aligned}\|\psi'\|_p &\leq \|\sigma'\|_1\|\mathbf{1}_{[-(N+3\alpha+\beta),N+3\alpha+\beta]}\|_p = \frac{c_2}{\alpha}(2(N+3\alpha+\beta))^{1/p},\\ \|\varphi'\|_p &\leq \frac{1}{2(\alpha+\beta)}\|\sigma'\|_1\|\mathbf{1}_{[-(\alpha+\beta),\alpha+\beta]}\|_p = \frac{c_2}{\alpha}(2(\alpha+\beta))^{-1/p}.\end{aligned}$$

Hence (4.11) becomes

$$\|U_\mu\|_{\mathcal{L}(\mathrm{D}_A(\theta,q))} \leq M^2\left(1+\frac{c_2}{\alpha}\right)^2\left(\frac{N+3\alpha+\beta}{\alpha+\beta}\right)^{1/p}\|L_\mu\|_{\mathcal{L}((\mathrm{L}^p(\mathbb{R};X),\mathrm{W}^{1,p}(\mathbb{R};X))_{\theta,q})}$$

and taking the infimum over α and β yields

$$\|U_\mu\|_{\mathcal{L}(\mathrm{D}_A(\theta,q))} \leq M^2\,\|L_\mu\|_{\mathcal{L}((\mathrm{L}^p(\mathbb{R};X),\mathrm{W}^{1,p}(\mathbb{R};X))_{\theta,q})}. \tag{4.12}$$

Lemma 2.35 now establishes (4.8) and concludes the proof. □

Remark 4.5. Note that the constant C in Proposition 4.4 comes only from the equivalence of the norms on $(\mathrm{L}^p(\mathbb{R};X),\mathrm{W}^{1,p}(\mathbb{R};X))_{\theta,q}$ and $\mathrm{B}^\theta_{p,q}(\mathbb{R};X)$, whereas in Proposition 4.3 a constant is present which is inherent to the transference method.

Remark 4.6. Let $p \in [1,\infty)$ and let $(U(s))_{s\in\mathbb{R}} \subseteq \mathcal{L}(\mathrm{L}^p(\mathbb{R}))$ be the left translation group given by $U(s)f(t) := f(t+s)$ for $f \in \mathrm{L}^p(\mathbb{R})$, $s \in \mathbb{R}$ and almost all $t \in \mathbb{R}$. Then $(U(s)))_{s\in\mathbb{R}}$ is generated by $-\mathrm{i}A$, where $Af := \mathrm{i}f'$ for $f \in \mathrm{D}(A) = \mathrm{W}^{1,p}(\mathbb{R})$. Hence $\mathrm{D}_A(\theta,q) = (\mathrm{L}^p(\mathbb{R}),\mathrm{W}^{1,p}(\mathbb{R}))_{\theta,q}$ for $\theta \in (0,1)$ and $q \in [1,\infty]$. Moreover, for $\mu \in \mathrm{M}(\mathbb{R})$ and $f \in \mathrm{L}^p(\mathbb{R})$,

$$\int_{\mathbb{R}} U(s)f\,\mathrm{d}\mu(s) = \mu * f = L_\mu(f).$$

Hence, with U_μ as in (2.9),

$$\|U_\mu\|_{\mathcal{L}(\mathrm{D}_A(\theta,q))} = \|L_\mu\|_{\mathcal{L}((\mathrm{L}^p(\mathbb{R}),\mathrm{W}^{1,p}(\mathbb{R}))_{\theta,q})}.$$

This shows that (4.12) is sharp in general, up to possibly a change of constant. By Lemma 2.35, the same holds for (4.8).

Proposition 4.4 and Corollary 2.22 combine to yield the following result.

Corollary 4.7. *Let $\theta \in (0,1)$ and $q \in [1,\infty]$. Then there exists a constant $C \geq 0$ such that the following holds. Let $-\mathrm{i}A$ generate a C_0-group $(U(s))_{s\in\mathbb{R}} \subseteq \mathcal{L}(X)$ on a Banach space X with $M := \sup_{s\in\mathbb{R}} \|U(s)\| < \infty$, and let $\mu \in \mathrm{M}(\mathbb{R})$ be such that $\mathcal{F}\mu \in \mathrm{C}^1(\mathbb{R})$ with $\sup_{s\in\mathbb{R}}(1+|s|)\,|(\mathcal{F}\mu)'(s)| < \infty$. Then*

$$\left\|\int_{\mathbb{R}} U(s)x\,\mu(\mathrm{d}s)\right\|_{\theta,q} \leq CM^2\,\|x\|_{\theta,q} \sup_{s\in\mathbb{R}} \Big(|\mathcal{F}\mu(s)| + (1+|s|)\,|(\mathcal{F}\mu)'(s)|\Big)$$

for all $x \in \mathrm{D}_A(\theta,q)$.

Remark 4.8. For Corollary 4.7 we used Corollary 2.22, but there are other ways to verify the conditions of Proposition 2.21, for instance Hörmander-type assumptions, cf. [51, pp. 47-49]. These then yield functional calculus results for other function norms than in Corollary 4.7.

Remark 4.9. If X is a UMD space then Proposition 2.23 and Theorem 2.19 yield an estimate

$$\left\|\int_{\mathbb{R}} U(s)x\,\mu(\mathrm{d}s)\right\|_{X} \leq CM^2\,\|x\|_X \sup_{s\in\mathbb{R}} \Big(|\mathcal{F}\mu(s)| + |s(\mathcal{F}\mu)'(s)|\Big)$$

for all $x \in X$. Corollary 4.7 then follows from (2.34), and in fact in this case $(\mathcal{F}\mu)'$ need not be bounded near zero. However, the inhomogeneity of the Besov space $\mathrm{B}^r_{p,q}(\mathbb{R};X)$ implies that for general Banach spaces in Corollary 4.7 a condition at zero on the multiplier is needed to deal with the term $\varphi_0 m$ in Proposition 2.21.

4.2 Functional calculus results for groups on interpolation spaces

We now use the theory established in the previous sections to prove the main functional calculus result of this chapter, Theorem 4.2. Recall the definition of the (inhomogeneous) analytic Mikhlin algebra $\mathrm{H}^\infty_1(\mathrm{St}_\omega)$ from (4.1).

Theorem 4.10. *Let $-\mathrm{i}A$ generate a C_0-group $(U(s))_{s\in\mathbb{R}} \subseteq \mathcal{L}(X)$ on a Banach space X and let $\theta \in (0,1)$, $q \in [1,\infty]$ and $\omega > \theta(U)$. Then there exists a constant $C \geq 0$ such that $f(A_{\theta,q}) \in \mathcal{L}(\mathrm{D}_A(\theta,q))$ with*

$$\left\|f(A_{\theta,q})\right\|_{\mathcal{L}(\mathrm{D}_A(\theta,q))} \leq C \left\|f\right\|_{\mathrm{H}^\infty_1(\mathrm{St}_\omega)}$$

for all $f \in \mathrm{H}^\infty_1(\mathrm{St}_\omega)$. *If* $(U(s))_{s\in\mathbb{R}}$ *is uniformly bounded then* C *can be chosen independent of* $\omega > 0$.

Proof. First consider $f \in \mathrm{H}^\infty_1(\mathrm{St}_\omega) \cap \mathcal{E}(\mathrm{St}_\omega)$ and fix $\alpha \in (\theta(U), \omega)$ and $p \in [1,\infty)$. By Lemma 2.7 there exists a $\mu \in \mathrm{M}_{-\alpha}(\mathbb{R})$ such that $f = \mathcal{F}\mu$. Let μ_α be as in (2.30). By Lemmas 2.7 and 2.36 and Proposition 4.3,

$$\left\|f(A_{\theta,q})\right\| = \left\|(U_\mu)_{\theta,q}\right\| \leq C_1 \left\|L_{\mu_\alpha}\right\|_{\mathcal{L}(\mathrm{B}^\theta_{p,q}(\mathbb{R};X))} = C_1 \left\|T_{\mathcal{F}\mu_\alpha}\right\|_{\mathcal{L}(\mathrm{B}^\theta_{p,q}(\mathbb{R};X))} \tag{4.13}$$

for some constant $C_1 \geq 0$, where $T_{\mathcal{F}\mu_\alpha}$ is as in (2.29). Since

$$\mathcal{F}\mu_\alpha(s) = \frac{f(s+i\alpha)+f(s-i\alpha)}{2} \qquad (s \in \mathbb{R}),$$

Corollary 2.22 yields a constant $C_2 \geq 0$ such that

$$\|f(A_{\theta,q})\| \leq C_2 \sup_{s\in\mathbb{R}} \Big(|\mathcal{F}\mu_\alpha(s)| + (1+|s|)|(\mathcal{F}\mu_\alpha)'(s)|\Big) \leq C_2\|f\|_{\mathrm{H}^\infty_1(\mathrm{St}_\omega)}. \tag{4.14}$$

For general $f \in \mathrm{H}^\infty_1(\mathrm{St}_\omega)$ first assume that $q < \infty$. By part b) of Lemma 2.36, $\mathrm{D}(A_{\theta,q})$ is dense in $\mathrm{D}_A(\theta,q)$. Let $\tau_k(z) := -k^2(ik-z)^{-2}$ for $k \in \mathbb{N}$ with $k > \omega$ and $z \in \mathrm{St}_\omega$. Then $\tau_k, f\tau_k \in \mathrm{H}^\infty_1(\mathrm{St}_\omega) \cap \mathcal{E}(\mathrm{St}_\omega)$,

$$\sup_k \|f\tau_k\|_{\mathrm{H}^\infty_1(\mathrm{St}_\omega)} \leq \|f\|_{\mathrm{H}^\infty_1(\mathrm{St}_\omega)} \sup_k \|\tau_k\|_{\mathrm{H}^\infty_1(\mathrm{St}_\omega)} < \infty$$

and $f\tau_k(z) \to f(z)$ as $k \to \infty$, for all $z \in \mathrm{St}_\omega$. Now (4.14) yields

$$\left\|f\tau_k(A_{\theta,q})\right\| \leq C_2 \left\|f\tau_k\right\|_{\mathrm{H}^\infty_1(\mathrm{St}_\omega)} \leq C \left\|f\right\|_{\mathrm{H}^\infty_1(\mathrm{St}_\omega)}$$

for some $C \geq 0$. Hence Lemma 2.8 implies $f(A) \in \mathcal{L}(X)$ and

$$\left\|f(A_{\theta,q})\right\| \leq C \left\|f\right\|_{\mathrm{H}^\infty_1(\mathrm{St}_\omega)}. \tag{4.15}$$

Finally, for $q = \infty$ Theorem 2.34 yields

$$\mathrm{D}_A(\theta,\infty) = (\mathrm{D}_A(\theta_1,1), \mathrm{D}_A(\theta_2,1))_{\theta_3,\infty}$$

with equivalence of norms, where $\theta_1, \theta_2, \theta_3 \in (0,1)$ are such that $\theta_1 \neq \theta_2$ and $\theta_1(1-\theta_3) + \theta_2\theta_3 = \theta$. Combining (3.9) and (2.34) concludes the proof of the first statement.

In the case where $(U(s))_{s\in\mathbb{R}}$ is uniformly bounded, use Proposition 4.4 instead of 4.3 in (4.13) to obtain

$$\left\|f(A_{\theta,q})\right\| \leq C_1 \left\|T_{\mathcal{F}\mu}\right\|_{\mathcal{L}(\mathrm{B}^\theta_{p,q}(\mathbb{R};X))}$$

for all $f \in \mathrm{H}^\infty_1(\mathrm{St}_\omega) \cap \mathcal{E}(\mathrm{St}_\omega)$ and some constant $C_1 \geq 0$ independent of ω. The rest of the proof is the same as before. □

Remark 4.11. If X is a UMD space then Theorem 4.10 follows from Theorem 2.20 by interpolation. Moreover, in this case one seems to obtain a stronger result since the term $\sup_{z\in \mathrm{St}_\omega}|f'(z)|$ which occurs in $\|f\|_{\mathrm{H}^\infty_1(\mathrm{St}_\omega)}$ does not appear in the norm

$$\|f\|_{\mathrm{H}^\infty_{(1)}(\mathrm{St}_\omega)} = \sup_{z\in \mathrm{St}_\omega} |f(z)| + |zf'(z)|$$

of $\mathrm{H}^\infty_{(1)}(\mathrm{St}_\omega)$. However, the norms $\|\cdot\|_{\mathrm{H}^\infty_{(1)}(\mathrm{St}_\omega)}$ and $\|\cdot\|_{\mathrm{H}^\infty_1(\mathrm{St}_\omega)}$ are equivalent, since $0 \in \mathrm{St}_\omega$ for all $\omega > 0$. So for generators of unbounded groups, Theorem 2.20 does not yield an essentially better estimate than Theorem 4.10 on $\mathrm{D}_A(\theta, q)$. This is different for generators of uniformly bounded groups, since the norm equivalence of $\sup_{z\in \mathrm{St}_\omega}|f(z)| + |zf'(z)|$ and $\|f\|_{\mathrm{H}^\infty_1(\mathrm{St}_\omega)}$ fails as $\omega \downarrow 0$. For generators of uniformly bounded groups on UMD spaces, Theorem 2.20 yields a strictly stronger result on $\mathrm{D}_A(\theta, q)$ than Theorem 4.10.

Remark 4.12. Let $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > \omega$. By (2.38), $\mathrm{D}((\lambda - \mathrm{i}A)^\alpha) \subseteq \mathrm{D}_A(\alpha, \infty)$ for each $\alpha \in (0, 1)$. Hence Theorem 4.10 yields $f(A)(\lambda - \mathrm{i}A)^{-\alpha} \in \mathcal{L}(X)$ for all $\omega > \theta(U)$, $f \in \mathrm{H}^\infty_1(\mathrm{St}_\omega)$ and $\alpha > 0$. However, this already follows from Proposition 2.18, in a similar manner as indicated in Remark 3.21. Moreover, using arguments as in Remark 3.13, Proposition 2.18 implies that $f(A) : \mathrm{D}_A(\theta, q) \to \mathrm{D}_A(\theta', q')$ is bounded for all $\theta' < \theta$ and $q, q' \in [1, \infty]$. The improvement that Theorem 4.10 provides lies in going from $\theta' < \theta$ to $\theta' = \theta$.

Remark 4.13. As already noted in Remark 4.8, in the proof above we could have used Fourier multiplier results on Besov spaces other than Corollary 2.22. These lead to statements about the boundedness of functional calculi for other function algebras.

For $\psi \in (0, \pi/2)$ and $\omega > 0$ let

$$\Sigma_\psi := \mathrm{S}_\psi \cup -\mathrm{S}_\psi,$$

where S_ψ is as in (2.1), and

$$\mathrm{V}_{\psi,\omega} := \mathrm{St}_\omega \cup \Sigma_\psi.$$

The next lemma follows from [58, Lemma 4.5], using that $\mathrm{H}^\infty_1(\mathrm{St}_\omega) = \mathrm{H}^\infty_{(1)}(\mathrm{St}_\omega)$ with equivalent norms for each $\omega > 0$.

Lemma 4.14. *Let $\omega > \omega' > 0$ and $\psi \in (0, \pi/2)$. Then $\mathrm{H}^\infty(\mathrm{V}_{\omega,\psi})$ is continuously embedded in $\mathrm{H}^\infty_1(\mathrm{St}_{\omega'})$.*

Corollary 4.15. *Let $-\mathrm{i}A$ generate a C_0-group $(U(s))_{s\in\mathbb{R}} \subseteq \mathcal{L}(X)$ on a Banach space X and let $\theta \in (0, 1)$ and $q \in [1, \infty]$. Then $A_{\theta,q}$ has a bounded $\mathrm{H}^\infty(\mathrm{V}_{\omega,\psi})$-calculus for all $\omega > \theta(U)$ and $\psi \in (0, \pi/2)$.*

So far we have considered functional calculus on interpolation spaces for the couple $(X, \mathrm{D}(A))$. The next corollary extends our results to other interpolation couples.

Corollary 4.16. *Let $-\mathrm{i}A$ generate a C_0-group $(U(s))_{s\in\mathbb{R}} \subseteq \mathcal{L}(X)$ on a Banach space X and let $\theta \in (0,1)$, $q \in [1,\infty]$ and $m, n \in \mathbb{N}_0$ with $m \neq n$. Then the part of A in $(\mathrm{D}(A^m), \mathrm{D}(A^n))_{\theta,q}$ has a bounded $\mathrm{H}^\infty_1(\mathrm{St}_\omega)$-calculus for all $\omega > \theta(U)$.*

If $(U(s))_{s\in\mathbb{R}}$ is uniformly bounded then the constant bounding the $\mathrm{H}^\infty_1(\mathrm{St}_\omega)$-calculus can be chosen independent of $\omega > 0$.

Proof. By (2.35) we may assume that $m < n$. Using the similarity transform $R(\lambda, A)^m : X \to \mathrm{D}(A^m)$, it suffices to let $m = 0$. Suppose that $n\theta \notin \mathbb{N}$. By Lemma 3.1.3 and Proposition 3.1.8 in [86],

$$(X, \mathrm{D}(A^n))_{\theta,q} = (\mathrm{D}(A^k), \mathrm{D}(A^{k+1}))_{\theta',q}$$

for some $k \in \mathbb{N}_0$ and $\theta' \in (0,1)$. Another similarity transform shows that we can let $k = 0$. Now Theorem 4.10 yields the statement.

If $k := n\theta \in \mathbb{N}$, then Theorem 2.34 yields

$$(X, \mathrm{D}(A^n))_{\theta,q} = \Big((\mathrm{D}(A^{k-1}), \mathrm{D}(A^k))_{1/2,q}, (\mathrm{D}(A^k), \mathrm{D}(A^{k+1}))_{1/2,q}\Big)_{1/2,q}.$$

By what we have already shown and by (2.34), this concludes the proof. □

4.3 Additional results on interpolation spaces

We now consider several results which follow from Theorem 4.10. Corollary 4.16 can be applied in this section to yield results for other interpolation couples.

4.3.1 Principal value integrals

We first state a proposition about the convergence of certain principal value integrals. If $g \in \mathrm{L}^1[-1,1]$ is an even function then by $\mathrm{PV} - g(s)/s$ we mean the distribution defined by

$$\langle \mathrm{PV} - g(s)/s, \varphi\rangle := \lim_{\epsilon\searrow 0} \int_{\epsilon\leq|s|\leq 1} g(s)\varphi(s)\frac{\mathrm{d}s}{s} = \int_0^1 g(s)\frac{\varphi(s)-\varphi(-s)}{s}\mathrm{d}s$$

for $\varphi \in \mathrm{C}^\infty_c(\mathbb{R})$. By $\mathrm{BV}[-1,1]$ we denote the functions g of bounded variation $\mathrm{Var}_{[-1,1]}(g)$ on $[-1,1]$.

Lemma 4.17. *Let $g \in \mathrm{BV}[-1,1]$ be an even function and set $f := \mathcal{F}(\mathrm{PV} - g(s)/s)$. Then $f \in \mathrm{H}^\infty_1(\mathrm{St}_\omega)$ for all $\omega > 0$, with*

$$\|f\|_{\mathrm{H}^\infty_1(\mathrm{St}_\omega)} \leq C(\mathrm{Var}_{[-1,1]}(g) + g(1)) \tag{4.16}$$

for a constant $C = C(\omega) \geq 0$ independent of g.

Proof. By [58, Lemma 4.3], $f \in \mathrm{H}^\infty_{(1)}(\mathrm{St}_\omega)$ for each $\omega > 0$ and (4.16) holds with respect to the $\mathrm{H}^\infty_{(1)}(\mathrm{St}_\omega)$-norm. But as noted in Remark 4.11, $\mathrm{H}^\infty_1(\mathrm{St}_\omega) = \mathrm{H}^\infty_{(1)}(\mathrm{St}_\omega)$ with equivalent norms. □

The following is an interpolation version of [58, Theorem 4.4] on general Banach spaces.

Proposition 4.18. *Let $-\mathrm{i}A$ generate a C_0-group $(U(s))_{s\in\mathbb{R}} \subseteq \mathcal{L}(X)$ on a Banach space X. Let $g \in \mathrm{BV}[-1,1]$ be an even function and set $f := \mathcal{F}(\mathrm{PV} - g(s)/s)$. Then $f(A_{\theta,q}) \in \mathcal{L}(\mathrm{D}_A(\theta,q))$ for all $\theta \in (0,1)$, $q \in [1,\infty]$, and*

$$f(A)x = \lim_{\epsilon\searrow 0} \int_{\epsilon\leq|s|\leq 1} g(s)U(s)x \,\frac{\mathrm{d}s}{s} \tag{4.17}$$

for all $\theta \in (0,1)$, $q \in [1,\infty)$ and $x \in \mathrm{D}_A(\theta,q)$.

Proof. By Lemma 4.17, $f \in \mathrm{H}^\infty_1(\mathrm{St}_\omega)$ for all $\omega > 0$. Theorem 4.10 now yields the first statement.

Let $q < \infty$, and for $\epsilon > 0$ let $g_\epsilon := (\mathbf{1}_{[-1,1]} - \mathbf{1}_{(-\epsilon,\epsilon)})g$ and $f_\epsilon := \mathcal{F}(\mathrm{PV} - g_\epsilon(s)/s)$. Then

$$\sup_{\epsilon>0} \mathrm{Var}_{[-1,1]}(g_\epsilon) + g_\epsilon(1) < \infty$$

and $f_\epsilon(z) \to f(z)$ as $\epsilon \downarrow 0$ for $z \in \mathbb{C}$. Moreover, $\sup_{\epsilon>0} \|f\|_{\mathrm{H}^\infty_1(\mathrm{St}_\omega)} < \infty$ by Lemma 4.17. Hence Theorem 4.10 and Lemma 2.8 conclude the proof. □

4.3.2 Results for sectorial operators and cosine functions

We now apply the results from previous sections to functional calculus theory for sectorial operators and generators of cosine functions.

For $\psi \in (0,\pi)$ define $\mathrm{H}^\infty_{\log}(\mathrm{S}_\psi)$ to be the unital Banach algebra of all $f \in \mathrm{H}^\infty(\mathrm{S}_\psi)$ for which

$$\|f\|_{\mathrm{H}^\infty_{\log}(\mathrm{S}_\psi)} := \sup_{z\in\mathrm{S}_\psi} |f(z)| + (1 + |\log(z)|)|zf'(z)| < \infty,$$

endowed with the norm $\|\cdot\|_{\mathrm{H}^\infty_{\log}(\mathrm{S}_\psi)}$. Recall the definition of a sectorial operator with bounded imaginary powers from Section 2.2.3.

Proposition 4.19. *Let X be a Banach space and $A \in \mathrm{BIP}(X,\varphi)$ with $\varphi < \pi$. Let $\theta \in (0,1)$ and $q \in [1,\infty]$. Set $Y := (X, \mathrm{D}(\log(A)))_{\theta,q}$. Then A_Y has a bounded $\mathrm{H}^\infty_{\log}(\mathrm{S}_\psi)$-calculus on Y for all $\psi \in (\varphi,\pi)$.*

If $\sup_{s\in\mathbb{R}} \|A^{\mathrm{i}s}\| < \infty$ then the constant bounding the $\mathrm{H}^\infty_{\log}(\mathrm{S}_\psi)$-calculus can be chosen independent of $\psi > 0$.

Proof. Let $\psi \in (\varphi, \pi)$ be given and note that $f \mapsto f \circ \log$ is an isometric algebra isomorphism $\mathrm{H}^\infty_1(\mathrm{St}_\psi) \to \mathrm{H}^\infty_{\log}(\mathrm{S}_\psi)$. By Lemma 2.36 as well as Theorem 4.2.4 and Proposition 6.1.2 from [55],

$$f(\log(A)_Y) = f(\log(A))_Y = (f \circ \log)(A)_Y = (f \circ \log)(A_Y)$$

for all $f \in \mathrm{H}^\infty_1(\mathrm{St}_\psi)$. Now Theorem 4.10 concludes the proof. □

Remark 4.20. Let A be an injective sectoral operator of angle $\varphi \in (0, \pi)$, and let $\alpha > 0$, $\theta \in (0,1)$ and $q \in [1,\infty]$. By [55, Corollary 6.6.3], a special case of which is Theorem 2.38, the part of A in $(X, \mathrm{D}(A^\alpha) \cap \mathrm{ran}(A^\alpha))_{\theta,q}$ has a bounded $\mathrm{H}^\infty(\mathrm{S}_\psi)$-calculus for all $\psi \in (\varphi, \pi)$. By (2.38), and because $\log(A)A^{\alpha\theta}(1+A)^{-2\alpha\theta} \in \mathcal{L}(X)$ (by definition of the sectorial calculus for A),

$$(X, \mathrm{D}(A^\alpha) \cap \mathrm{ran}(A^\alpha))_{\theta,q} \subseteq (X, \mathrm{D}(A^\alpha))_{\theta,q} \subseteq \mathrm{D}(A^{\alpha\theta}) \subseteq \mathrm{D}(\log(A)),$$

and in general $\mathrm{D}(\log(A))$ is strictly included in $(X, \mathrm{D}(\log(A)))_{\theta',q'}$ for all $\theta' \in (0,1)$ and $q' \in [1,\infty]$. Hence Theorem 2.38 does not imply Proposition 4.19.

We now apply Theorem 4.10 to the generators of cosine functions considered in Section 2.2.4. For $\omega > 0$ let

$$\mathrm{H}^\infty_1(\Pi_\omega) := \left\{ f \in \mathrm{H}^\infty(\Pi_\omega) \,\middle|\, \|f\|_{\mathrm{H}^\infty_1(\Pi_\omega)} := \sup_{z \in \Pi_\omega} |f(z)| + (1+|z|)|f'(z)| < \infty \right\}$$

be the *(inhomogeneous) analytic Mikhlin algebra* on Π_ω, a Banach algebra endowed with the norm $\|\cdot\|_{\mathrm{H}^\infty_1(\Pi_\omega)}$.

In the following result we use that a version of Lemma 2.36 holds for operators of parabola type. This is proved in the same manner as Lemma 2.36.

Proposition 4.21. *Let $-A$ generate a cosine function $(\mathrm{Cos}(s))_{s\in\mathbb{R}} \subseteq \mathcal{L}(X)$ on a Banach space X and let $\theta \in (0,1)$, $q \in [1,\infty]$. Then the part $A_{\theta,q}$ of A in $\mathrm{D}_A(\theta, q)$ has a bounded $\mathrm{H}^\infty_1(\Pi_\omega)$-calculus for all $\omega > \theta(\mathrm{Cos})$.*

If $\sup_{s\in\mathbb{R}} \|\mathrm{Cos}(s)\| < \infty$ then the constant bounding the $\mathrm{H}^\infty_1(\Pi_\omega)$-calculus can be chosen independent of $\omega > 0$.

Proof. Let $V \subseteq X$ be the Kisyński space from Proposition 2.11, and let

$$\mathcal{A} := \mathrm{i} \begin{bmatrix} 0 & \mathrm{I}_V \\ -A & 0 \end{bmatrix},$$

with domain $\mathrm{D}(\mathcal{A}) := \mathrm{D}(A) \times V$, be such that $-\mathrm{i}\mathcal{A}$ generates the C_0-group $(U(s))_{s\in\mathbb{R}}$ on $V \times X$ with $\theta(\mathrm{Cos}) = \theta(U)$.

Note that

$$\mathcal{A}^2 := \begin{bmatrix} A_V & 0 \\ 0 & A \end{bmatrix}$$

with domain $\mathrm{D}(\mathcal{A}^2) = \mathrm{D}(A_V) \times \mathrm{D}(A)$. By [86, Proposition 3.1.4],

$$\mathrm{D}(A) \times V \in K_{1/2}(V \times X, \mathrm{D}(A_V) \times \mathrm{D}(A)) \cap J_{1/2}(V \times X, \mathrm{D}(A_V) \times \mathrm{D}(A)),$$

where the classes $K_{1/2}$ and $J_{1/2}$ are as in [86, Definition 1.3.1]. Inspecting the second component yields

$$V \in K_{1/2}(X, \mathrm{D}(A)) \cap J_{1/2}(X, \mathrm{D}(A)).$$

Now [86, Theorem 1.3.5] yields

$$\begin{aligned} Y := (V \times X, \mathrm{D}(\mathcal{A}))_{\theta,q}(V \times X, \mathrm{D}(A) \times V)_{\theta,q} &= (V, \mathrm{D}(A))_{\theta,q} \times (X, V)_{\theta,q} \\ = \mathrm{D}_A\left(\tfrac{1+\theta}{2}, q\right) \times \mathrm{D}_A\left(\tfrac{\theta}{2}, q\right). \end{aligned}$$

Let $\omega > \theta(\mathrm{Cos})$. Then $f \in \mathrm{H}^\infty(\Pi_\omega)$ is an element of $\mathrm{H}^\infty_1(\Pi_\omega)$ if and only if $g(z) := f(z^2)$ defines an element of $\mathrm{H}^\infty_1(\mathrm{St}_\omega)$, with $\|g\|_{\mathrm{H}^\infty_1(\mathrm{St}_\omega)} \leq 4\,\|f\|_{\mathrm{H}^\infty_1(\Pi_\omega)}$. Moreover, it is straightforward to see that $f(A_V) \oplus f(A) = g(\mathcal{A})$ and

$$g(\mathcal{A}_Y) = g(\mathcal{A})_Y = (f(A_V) \oplus f(A))_Y = f(A_{(1+\theta)/2,q}) \oplus f(A_{\theta/2,q})$$

for all $f \in \mathrm{H}^\infty_1(\Pi_\omega)$, by what we have already shown. Theorem 4.10 and the Reiteration Theorem, Theorem 2.34, conclude the proof. □

Part III

Double operator integrals and perturbation inequalities

5

Operator Lipschitz functions on Banach spaces

Up until now we have investigated the functional calculus properties of generators of strongly continuous (semi)groups. For such operators one is usually most interested in determining whether they have a bounded H^∞-calculus, and Chapters 3 and 4 were dedicated to determining boundedness of various functional calculi for (semi)group generators. We now leave this setting behind and consider the scalar type operators from Section 2.2.5. As we have seen, these operators have a very rich functional calculus theory associated with them which renders questions about the boundedness of various functional calculi trivial. Instead, we will consider operator Lipschitz estimates

$$\|f(B) - f(A)\|_{\mathcal{L}(X)} \leq C\,\|B - A\|_{\mathcal{L}(X)} \tag{5.1}$$

for a bounded Borel function $f \in \mathcal{B}(\mathbb{C})$ and scalar type operators $A, B \in \mathcal{L}(X)$, with a constant $C \geq 0$ independent of A and B. More generally, we study commutator estimates of the form

$$\|f(B)S - Sf(A)\|_{\mathcal{L}(X,Y)} \leq C\,\|BS - SA\|_{\mathcal{L}(X,Y)} \tag{5.2}$$

for Banach spaces X and Y, scalar type operators $A \in \mathcal{L}(X)$ and $B \in \mathcal{L}(Y)$, and $S \in \mathcal{L}(X, Y)$.

We have seen in Theorem 1.3 that (5.1) was established by Peller for self-adjoint operators on a Hilbert space and $f \in \dot{\mathrm{B}}^1_{\infty,1}(\mathbb{R})$. In this chapter we extend Theorem 1.3 to scalar type operators on general Banach spaces. More generally, (5.2) holds for $f \in \dot{\mathrm{B}}^1_{\infty,1}(\mathbb{R})$ and for all $S \in \mathcal{L}(X, Y)$ (see Corollary 5.12).

If f is the absolute value function then $f \notin \dot{\mathrm{B}}^1_{\infty,1}(\mathbb{R})$ and the results mentioned above do not apply. Moreover, the techniques which we use to obtain (5.1) for $f \in \dot{\mathrm{B}}^1_{\infty,1}(\mathbb{R})$, and which involve the class $\mathfrak{A}$ from Section 2.3.2, cannot be applied to the absolute value function (see Remark 5.43). Because of the importance of the absolute value function for matrix analysis and perturbation theory (see [14, Sections VII.5 and X.2]), we study (5.2) for this function in the case $X = \ell^p$ and $Y = \ell^q$ with $p, q \in [1, \infty]$.

It was shown by Kato in [69] that the absolute value function does not satisfy (5.1) for $X = \ell^2$. An earlier example of McIntosh [87] showed the failure of the commutator estimate (5.2) for this function in the case $X = Y = \ell^2$. Nonetheless, in this chapter we establish a version of (5.2) for $X = \ell^p$ and $Y = \ell^q$ with $p < q$. More precisely, in Theorem 5.28 we prove the following result. For the definition of a diagonalizable operator see Section 5.3.

Theorem 5.1. *Let $p, q \in [1, \infty]$ with $p < q$, and let $f(t) := |t|$ for $t \in \mathbb{R}$. Let $A \in \mathcal{L}(\ell^p)$ and $B \in \mathcal{L}(\ell^q)$ (where ℓ^∞ should be replaced by c_0) be diagonalizable operators with real spectrum. Then*

$$\|f(B)S - Sf(A)\|_{\mathcal{L}(\ell^p,\ell^q)} \leq C_{A,B,p,q} \|BS - SA\|_{\mathcal{L}(\ell^p,\ell^q)} \tag{5.3}$$

for all $S \in \mathcal{L}(\ell^p, \ell^q)$, where

$$C_{A,B,p,q} = C_{p,q} \inf \|U\|_{\mathcal{L}(\ell^p)} \|U^{-1}\|_{\mathcal{L}(\ell^p)} \|V\|_{\mathcal{L}(\ell^q)} \|V^{-1}\|_{\mathcal{L}(\ell^q)} \tag{5.4}$$

for a constant $C_{p,q} \geq 0$ depending only on p and q, and where the infimum is taken over all $U \in \mathcal{L}(\ell^p)$ and $V \in \mathcal{L}(\ell^q)$ which diagonalize A and B, respectively.

If $p = 1$ or $q = \infty$ then (5.3) holds for each Lipschitz function f with $C_{p,q} = \|f\|_{\mathrm{Lip}}$, where

$$\|f\|_{\mathrm{Lip}} := \sup_{\substack{z_1, z_2 \in \mathbb{C} \\ z_1 \neq z_2}} \frac{|f(z_1) - f(z_2)|}{|z_1 - z_2|}. \tag{5.5}$$

In fact, (5.3) holds for $p = q = 1$ and $p = q = \infty$ (with ℓ^∞ replaced by c_0) and therefore (5.1) holds on ℓ^1 and c_0, for each Lipschitz function f (see Theorem 5.29).

We also obtain results for $p \geq q$. In particular, for $p = q = 2$ we prove (see Corollary 5.35) that for each $\epsilon \in (0, 1]$ there exists a constant $C \geq 0$ such that the following holds. Let $A, B \in \mathcal{L}(\ell^2)$ be compact selfadjoint operators, and let $U, V \in \mathcal{L}(\ell^2)$ be unitaries such that

$$UAU^{-1} = \sum_{j=1}^{\infty} \lambda_j \mathcal{P}_j \qquad \text{and} \qquad VBV^{-1} = \sum_{j=1}^{\infty} \mu_j \mathcal{P}_j,$$

where $(\lambda_j)_{j=1}^\infty$ and $(\mu_j)_{j=1}^\infty$ are sequences of real numbers and $\mathcal{P}_j \in \mathcal{L}(\ell^2)$, for $j \in \mathbb{N}$, is the j-th standard basis projection. Then

$$\||B| - |A|\|_{\mathcal{L}(\ell^2)} \leq C \min(\|V(B-A)U^{-1}\|_{\mathcal{L}(\ell^2,\ell^{2-\epsilon})}, \|V(B-A)U^{-1}\|_{\mathcal{L}(\ell^{2+\epsilon},\ell^2)}) \tag{5.6}$$

where the right-hand side equals infinity if $V(B-A)U^{-1} \notin \mathcal{L}(\ell^2, \ell^{2-\epsilon}) \cup \mathcal{L}(\ell^{2+\epsilon}, \ell^2)$.

The results stated here for the absolute value function in fact extend to a larger class of functions. This is briefly mentioned in Remark 5.36.

The constants in our results, for example the constant $C_{A,B,p,q}$ in Theorem 5.1, depend on A and B via the infimum in (5.4). This quantity is independent of the norms of A and B, and to obtain constants in (5.3) which do not depend on A and B in any way one merely has to restrict to operators with a sufficiently bounded spectral or diagonalizability constant. This is already done implicitly on Hilbert spaces by considering normal operators, for which this quantity is equal to 1. For example, in (5.6) the constant C does not depend on A or B in any way. Our results therefore truly extend the known estimates on Hilbert spaces, the main difference between Hilbert spaces and general Banach spaces being that on Hilbert spaces one has a large and easily identifiable class of operators which are diagonalizable by an isometry.

Throughout this chapter we in fact study the commutator estimate in (5.2) in the more general form

$$\|f(B)S - Sf(A)\|_{\mathcal{I}} \leq C\,\|BS - SA\|_{\mathcal{I}}\,, \tag{5.7}$$

where $\mathcal{I}$ is a Banach ideal in $\mathcal{L}(X,Y)$ (for the definition of a Banach ideal see Section 5.1). For example, in Corollary 5.12 we extend Theorem 1.3 to a general Banach ideal $\mathcal{I}$ in $\mathcal{L}(X)$ with the strong convex compactness property (for the definitions of this property see Section 5.1), with respect to the norm $\|\cdot\|_{\mathcal{I}}$.

We also present (see Theorem 5.39) an example of a Banach ideal $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ in $\mathcal{L}(\ell^{p'}, \ell^p)$, for $p \in [1,\infty)$ and $\frac{1}{p} + \frac{1}{p'} = 1$ (with ℓ^∞ replaced by c_0), namely the ideal of p-summing operators, such that any Lipschitz function f (in particular, the absolute value function) satisfies (5.7).

One of the main motivations for the work in this chapter is that any diagonalizable matrix is a scalar type operator. Hence the results in this chapter, e.g. (5.3), (5.6) and (5.7) hold for diagonalizable matrices A and B with a constant independent of the size of the matrix. It should be noted that, by Proposition 2.32, the results in this chapter also apply to generators of γ-bounded C_0-groups on general Banach spaces.

In Section 5.1 we study the convex compactness property which plays a major role when studying (5.7). In Section 5.2 we set up the theory of double operator integration for scalar type operators on Banach spaces and extend Theorem 1.3 to this setting. We then study the absolute value function and relate (5.7), for f the absolute value function, to estimates for triangular truncation operators. In Section 5.4 we study these triangular truncation operators on $\mathcal{L}(\ell^p, \ell^q)$ and prove Theorem 5.1. We then study (5.7) for the ideal of p-summing operators from $\ell^{p'}$ to ℓ^p, and in the final section we apply our results to finite dimensional spaces to obtain dimension-independent inequalities for diagonalizable matrices.

5.1 The strong convex compactness property

In this section we discuss a property which will be used extensively in this chapter.

First we provide a lemma about approximation by finite rank operators. Recall that a Banach space X has the *bounded approximation property* if there exists $M \geq 1$ such that, for each $K \subseteq X$ compact and $\epsilon > 0$, there exists $S \in X^* \otimes X$ with $\|S\|_{\mathcal{L}(X)} \leq M$ and $\sup_{x \in K} \|Sx - x\|_X < \epsilon$.

Lemma 5.2. *Let X and Y be Banach spaces such that X is separable and either X or Y has the bounded approximation property. Then each $T \in \mathcal{L}(X, Y)$ is the SOT-limit of a norm bounded sequence of finite rank operators.*

Proof. Fix $T \in \mathcal{L}(X, Y)$. By [84, Proposition 1.e.14] there exists a norm bounded net $(T_j)_{j \in J} \subseteq X^* \otimes Y$ having T as its SOT-limit. It is straightforward to see that the strong operator topology is metrizable on bounded subsets of $\mathcal{L}(X, Y)$ by

$$d(S_1, S_2) := \sum_{k=1}^{\infty} 2^{-k} \|S_1 x_k - S_2 x_k\|_Y \qquad (S_1, S_2 \in \mathcal{L}(X, Y)),$$

where $\{x_k\}_{k \in \mathbb{N}} \subseteq X$ is a countable subset that is dense in the unit ball of X. Hence there exists a subsequence of $(T_j)_{j \in J}$ with T as its SOT-limit. □

Let X and Y be Banach spaces and let Z be a Banach space which is continuously embedded in $\mathcal{L}(X, Y)$. Following [122] (in the case where Z is a subspace of $\mathcal{L}(X, Y)$), we say that Z has the *strong convex compactness property* if the following holds. For any finite measure space (Ω, Σ, μ) and any strongly measurable and bounded $f : \Omega \to Z$, the operator $T \in \mathcal{L}(X, Y)$ defined by

$$Tx := \int_\Omega f(\omega) x \, \mathrm{d}\mu(\omega) \qquad (x \in X), \tag{5.8}$$

belongs to Z with $\|T\|_Z \leq \overline{\int_\Omega} \|f(\omega)\|_Z \, \mathrm{d}\mu(\omega)$. By the Pettis Measurability Theorem, any separable Z has this property. Indeed, if Z is separable then combining Propositions 1.9 and 1.10 in [119] shows that any strongly measurable $f : \Omega \to Z$ is μ-measurable as a map to Z. If f is bounded as well, then (5.8) defines an element of Z with

$$\|T\|_Z \leq \int_\Omega \|f(\omega)\|_Z \, \mathrm{d}\mu(\omega).$$

It is shown in [122] and [108] that the subspaces of compact and weakly compact operators in $\mathcal{L}(X, Y)$ have the strong convex compactness property, but not all subspaces of $\mathcal{L}(X, Y)$ do.

Lemma 5.3. *Let X and Y be separable Banach spaces and Z a Banach space continuously embedded in $\mathcal{L}(X,Y)$. If $B_Z := \{z \in Z \mid \|z\|_Z \leq 1\}$ is SOT-closed in $\mathcal{L}(X,Y)$, then Z has the strong convex compactness property.*

Proof. The proof follows that of [4, Lemma 3.5]. First we show that B_Z is a Polish space in the strong operator topology. As in the proof of Lemma 5.2, bounded subsets of $\mathcal{L}(X,Y)$ are SOT-metrizable. The finite rank operators are SOT-dense in $\mathcal{L}(X,Y)$, hence $\mathcal{L}(X,Y)$ is SOT-separable. Therefore B_Z is SOT-separable and metrizable. By assumption, B_Z is complete.

Now let (Ω,μ) be a finite measure space and let $f : \Omega \to Z$ be bounded and strongly measurable. Without loss of generality, we may assume that $f(\Omega) \subseteq B_Z$ and that μ is a probability measure. For each $y^* \in Y^*$ and $x \in X$, the mapping $B_Z \to [0,\infty)$, $T \mapsto |\langle y^*, Tx\rangle|$, is continuous. The collection of all these mappings, for $y^* \in Y^*$ and $x \in X$, separates the points of B_Z. Moreover, $\omega \mapsto |\langle y^*, f(\omega)x\rangle|$ is a measurable mapping $\Omega \to [0,\infty)$ for each $y^* \in Y^*$ and $x \in X$. By [119, Propositions 1.9 and 1.10], f is the μ-almost everywhere SOT-limit of a sequence of B_Z-valued simple functions $(f_k)_{k=1}^{\infty}$. Let $T_k := \int_\Omega f_k \,\mathrm{d}\mu \in B_Z$ for $k \in \mathbb{N}$. By the dominated convergence theorem, $T_k(x) \to T(x) := \int_\Omega f(\omega)x \,\mathrm{d}\mu(\omega)$ as $k \to \infty$, for all $x \in X$. Since B_Z is SOT-closed by the assumption, we conclude that $T \in B_Z$.

Now let $g : \Omega \to [0,\infty)$ be measurable such that $1 \geq g(\omega) \geq \|f(\omega)\|_Z$ for $\omega \in \Omega$, and define $h(\omega) := \frac{f(\omega)}{g(\omega)}$ and $\mathrm{d}\nu(\omega) := \frac{g(\omega)}{\int_\Omega g(\eta)\,\mathrm{d}\mu(\eta)}\mathrm{d}\mu(\omega)$ for $\omega \in \Omega$ (where we let $\frac{0}{0} := 1$). By what we have shown above, $x \mapsto \int_\Omega h(\omega)x \,\mathrm{d}\nu(\omega)$ defines an element of B_Z. Since $Tx = \int_\Omega f(\omega)x\,\mathrm{d}\mu(\omega) = \int_\Omega g(\omega)\,\mathrm{d}\mu(\omega) \int_\Omega h(\omega)x\,\mathrm{d}\nu(\omega)$, we obtain $\|T\|_Z \leq \int_\Omega g(\omega)\,\mathrm{d}\mu(\omega)$, as remained to be shown. □

Remark 5.4. Note that the converse implication does not hold. Indeed, if X is a Hilbert space (or more generally, a Banach space with the metric approximation property) then the finite rank operators of norm less than or equal to 1 are SOT-dense in the unit ball of $\mathcal{L}(X)$. Therefore the compact operators of norm less than or equal to 1 are not SOT-closed in $\mathcal{L}(X)$ if X is infinite dimensional. However, by [122, Theorem 1.3], the space of compact operators on X has the strong convex compactness property.

Let X and Y be Banach spaces and $\mathcal{I}$ a Banach space which is continuously embedded in $\mathcal{L}(X,Y)$. We say that $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ is a *Banach ideal* in $\mathcal{L}(X,Y)$ if

- For all $R \in \mathcal{L}(Y)$, $S \in \mathcal{I}$ and $T \in \mathcal{L}(X)$, $RST \in \mathcal{I}$ with $\|RST\|_{\mathcal{I}} \leq \|R\|_{\mathcal{L}(Y)} \|S\|_{\mathcal{I}} \|T\|_{\mathcal{L}(X)}$;
- $X^* \otimes Y \subseteq \mathcal{I}$ with $\|x^* \otimes y\|_{\mathcal{I}} = \|x^*\|_{X^*} \|y\|_Y$ for all $x^* \in X^*$ and $y \in Y$.

By Lemma 5.3 and [34, Proposition 17.21] (using that the SOT and weak operator topology closures of a convex set coincide), for separable X and Y, any *maximal* Banach ideal (for the definition see e.g. [99]) in $\mathcal{L}(X,Y)$ has the strong convex compactness property. This includes a large class of operator ideals, such as the ideal of absolutely p-summing operators, the ideal of integral operators, etc (see [34, p. 203]).

5.2 Double operator integrals and Lipschitz estimates

In this section we extend the theory of double operator integration to scalar type operators on Banach spaces, and use this theory to obtain commutator and Lipschitz estimates for scalar type operators.

5.2.1 Double operator integrals

Fix Banach spaces X and Y, scalar type operators $A \in \mathrm{Scal}(X)$ and $B \in \mathrm{Scal}(Y)$ with spectral measures E respectively F, and $\varphi \in \mathfrak{A}(\sigma(A) \times \sigma(B))$ (recall the definition of the class $\mathfrak{A}$ from Section 2.3). Let a representation as in (2.19) for φ be given:

$$\varphi(\lambda_1, \lambda_2) = \int_\Omega a_1(\lambda_1, \omega) a_2(\lambda_2, \omega)\, \mathrm{d}\mu(\omega) \tag{5.9}$$

for all $(\lambda_1, \lambda_2) \in \sigma(A) \times \sigma(B)$, where (Ω, Σ, μ) is a finite measure space and $a_1 \in \mathcal{B}(\sigma(A) \times \Omega, \mathfrak{B}_{\sigma(A)} \otimes \Sigma)$, $a_2 \in \mathcal{B}(\sigma(B) \times \Omega, \mathfrak{B}_{\sigma(B)} \otimes \Sigma)$ are bounded Borel measurable functions. For $\omega \in \Omega$, let $a_1(A, \omega) := a_1(\cdot, \omega)(A) \in \mathcal{L}(X)$ and $a_2(B, \omega) := a_2(\cdot, \omega)(B) \in \mathcal{L}(Y)$ be defined by the functional calculus for A respectively B from Section 2.2.5.

Lemma 5.5. *Let $S \in \mathcal{L}(X, Y)$ have separable range. Then, for each $x \in X$, $\omega \mapsto a_2(B, \omega) S a_1(A, \omega) x$ is a weakly measurable map $\Omega \to Y$.*

Proof. Fix $x \in X$. If $a_1 = \mathbf{1}_W$ for some $W \subseteq \sigma(A) \times \Omega$ then it is straightforward to show that $\langle x^*, a_1(A, \cdot) x \rangle$ is measurable for each $x^* \in X^*$. As S has separable range, $S a_1(A, \cdot) x$ is μ-measurable by the Pettis Measurability Theorem. If a_2 is an indicator function as well, the same argument shows that $a_2(B, \cdot) y$ is weakly measurable for each $y \in Y$. General arguments, approximating $S a_1(A, \cdot) x$ by simple functions, show that $a_2(B, \cdot) S a_1(A, \cdot) x$ is weakly measurable. By linearity this extends to simple a_1 and a_2, and for general a_1 and a_2 let $(f_k)_{k \in \mathbb{N}}$, $(g_k)_{k \in \mathbb{N}}$ be sequences of simple functions such that $a_1 = \lim_{k \to \infty} f_k$ and $a_2 = \lim_{k \to \infty} g_k$ uniformly. Then $a_1(A, \omega) = \lim_{k \to \infty} f_k(A)$ and $a_2(B, \omega) = \lim_{k \to \infty} g_k(B)$ in the operator norm, for each $\omega \in \Omega$. The desired measurability now follows. □

Now suppose that Y is separable, that $\mathcal{I}$ is a Banach ideal in $\mathcal{L}(X, Y)$ and let $S \in \mathcal{L}(X, Y)$. By (2.13),

$$\|a_2(B, \omega) S a_1(A, \omega)\|_{\mathcal{I}} \leq 16\, \nu(A) \nu(B)\, \|S\|_{\mathcal{I}}\, \|a_1(\cdot, \omega)\|_{\mathcal{B}(\sigma(A))} \|a_2(\cdot, \omega)\|_{\mathcal{B}(\sigma(B))} \tag{5.10}$$

for $w \in \Omega$. Since $\mathcal{I}$ is continuously embedded in $\mathcal{L}(X,Y)$, by the Pettis Measurability Theorem, Lemma 5.5 and (5.10) we can define the *double operator integral*

$$T_{\varphi}^{A,B}(S)x := \int_{\Omega} a_2(B,\omega)Sa_1(A,\omega)x\,\mathrm{d}\mu(\omega) \in Y \qquad (x \in X). \tag{5.11}$$

Throughout, we will use T_φ for $T_\varphi^{A,B}$ when the operators A and B are clear from the context.

Proposition 5.6. *Let X and Y be separable Banach spaces such that X or Y has the bounded approximation property, and let $A \in \mathrm{Scal}(X)$, $B \in \mathrm{Scal}(Y)$, and $\varphi \in \mathfrak{A}(\sigma(A) \times \sigma(B))$. Let $\mathcal{I}$ be a Banach ideal in $\mathcal{L}(X,Y)$ with the strong convex compactness property. Then* (5.11) *defines an operator $T_\varphi^{A,B} \in \mathcal{L}(\mathcal{I})$ which is independent of the choice of representation of φ in* (2.19)*, with*

$$\|T_\varphi^{A,B}\|_{\mathcal{L}(\mathcal{I})} \leq 16\,\nu(A)\nu(B)\|\varphi\|_{\mathfrak{A}(\sigma(A)\times\sigma(B))}. \tag{5.12}$$

Proof. By (5.10) and the strong convex compactness property, $T_\varphi(S) \in \mathcal{L}(\mathcal{I})$ for all $S \in \mathcal{I}$, with

$$\big\|T_\varphi(S)\big\|_{\mathcal{I}} \leq 16\,\nu(A)\nu(B)\,\|S\|_{\mathcal{I}} \int_\Omega \|a_1(\cdot,\omega)\|_{\mathcal{B}(\sigma(A))}\|a_2(\cdot,\omega)\|_{\mathcal{B}(\sigma(B))}\,\mathrm{d}\mu(\omega).$$

Clearly T_φ is linear, hence the result follows if we establish that T_φ is independent of the representation of φ. For this it suffices to let $\varphi \equiv 0$. Now, first consider $S = x^* \otimes y$ for $x^* \in X^*$ and $y \in Y$, and let $x \in X$, $y^* \in Y^*$ and $w \in \Omega$. Recall that E and F are the spectral measures of A and B, respectively. Then

$$\begin{aligned}
&\langle y^*, a_2(B,\omega)Sa_1(A,\omega)x\rangle \\
&= \int_{\sigma(B)} a_2(\eta,\omega)\,\mathrm{d}\langle y^*, F(\eta)Sa_1(A,\omega)x\rangle \\
&= \int_{\sigma(B)} a_2(\eta,\omega)\langle x^*, a_1(A,\omega)x\rangle \mathrm{d}\langle y^*, F(\eta)y\rangle \\
&= \int_{\sigma(B)}\int_{\sigma(A)} a_1(\lambda,\omega)a_2(\eta,\omega)\mathrm{d}\langle x^*, E(\lambda)x\rangle \mathrm{d}\langle y^*, F(\eta)y\rangle
\end{aligned}$$

by (2.15). Now Fubini's Theorem and the assumption on φ yield

$$\begin{aligned}
&\langle y^*, T_\varphi(S)x\rangle \\
&= \int_\Omega \langle y^*, a_2(B,\omega)Sa_1(A,\omega)x\rangle \mathrm{d}\mu(\omega) \\
&= \int_\Omega\int_{\sigma(B)}\int_{\sigma(A)} a_1(\lambda,\omega)a_2(\eta,\omega)\mathrm{d}\langle x^*, E(\lambda)x\rangle \mathrm{d}\langle y^*, F(\eta)y\rangle \mathrm{d}\mu(\omega) \\
&= \int_{\sigma(B)}\int_{\sigma(A)}\int_\Omega a_1(\lambda,\omega)a_2(\eta,\omega)\mathrm{d}\mu(\omega)\mathrm{d}\langle x^*, E(\lambda)x\rangle \mathrm{d}\langle y^*, F(\eta)y\rangle
\end{aligned}$$

$$= \int_{\sigma(B)} \int_{\sigma(A)} \varphi(\lambda, \eta) \mathrm{d}\langle x^*, E(\lambda)x\rangle \mathrm{d}\langle y^*, F(\eta)y\rangle = 0.$$

By linearity, $T_\varphi(S) = 0$ for all $S \in X^* \otimes Y$. By Lemma 5.2, a general element $S \in \mathcal{I}$ is the SOT-limit of a bounded (in $\mathcal{L}(X,Y)$) sequence $(S_n)_{n\in\mathbb{N}} \subseteq X^* \otimes Y$. Hence for each $x \in X$ there exists a constant $C \geq 0$ such that, for all $n \in \mathbb{N}$,

$$\begin{aligned} &\int_\Omega \|a_2(B,\omega)S_n a_1(A,\omega)x\|_Y \,\mathrm{d}\mu(\omega) \\ &\leq 16\,\nu(A)\nu(B)\,\|S_n\|_{\mathcal{L}(X,Y)}\,\|x\|_X \int_\Omega \|a_1(\cdot,\omega)\|_{\mathcal{B}(\sigma(A))}\|a_2(\cdot,\omega)\|_{\mathcal{B}(\sigma(B))}\,\mathrm{d}\mu(\omega) \\ &\leq C \int_\Omega \|a_1(\cdot,\omega)\|_{\mathcal{B}(\sigma(A))}\|a_2(\cdot,\omega)\|_{\mathcal{B}(\sigma(B))}\,\mathrm{d}\mu(\omega) < \infty, \end{aligned}$$

where we have used (2.13). Now the dominated convergence theorem shows that $T_\varphi(S)x = \lim_{n\to\infty} T_\varphi(S_n)x = 0$ for all $x \in X$, which implies that T_φ is independent of the representation of φ and concludes the proof. □

If A and B are normal operators on separable Hilbert spaces X and Y, then (5.12) improves to

$$\|T_\varphi^{A,B}\|_{\mathcal{L}(\mathcal{I})} \leq \|\varphi\|_{\mathfrak{A}(\sigma(A)\times\sigma(B))} \tag{5.13}$$

by appealing to (2.16) instead of (2.13) in (5.10).

Remark 5.7. Let H be an infinite dimensional separable Hilbert space and $\mathcal{S}_2$ the ideal of Hilbert-Schmidt operators on H. There is a natural definition (see [20]) of a double operator integral $\mathcal{T}_\varphi^{A,B} \in \mathcal{L}(\mathcal{S}_2)$ for all $\varphi \in \mathcal{B}(\mathbb{C}^2)$ and normal operators $A, B \in \mathcal{L}(H)$, such that $\mathcal{T}_\varphi^{A,B} = T_\varphi^{A,B}$ if $\varphi \in \mathfrak{A}(\sigma(A) \times \sigma(B))$. One could wonder whether Proposition 5.6 can be extended to a larger class of functions than $\mathfrak{A}(\sigma(A) \times \sigma(B))$ using an extension of the definition of $T_\varphi^{A,B}$ in (5.11) which coincides with $\mathcal{T}_\varphi^{A,B}$ on $\mathcal{S}_2$. But it follows from [95, Theorem 1] (see also Remark 2.13) that $\mathcal{T}_\varphi^{A,B}$ extends to a bounded operator on $\mathcal{I} = \mathcal{L}(H)$ if and only if $\varphi \in \mathfrak{A}(\sigma(A) \times \sigma(B))$. Hence Proposition 5.6 cannot be extended to a larger function class than $\mathfrak{A}(\sigma(A) \times \sigma(B))$ in general. However, for specific Banach ideals, e.g. ideals with the UMD property, results have been obtained for larger classes of functions [33, 102].

Remark 5.8. The assumption in Proposition 5.6 that X or Y has the bounded approximation property is only used, via Lemma 5.2, to ensure that each $S \in I$ is the SOT-limit of a bounded (in $\mathcal{L}(X,Y)$) net of finite rank operators. Clearly this is true for general Banach spaces X and Y if I is the closure in $\mathcal{L}(X,Y)$ of $X^* \otimes Y$. In [78] the authors consider an assumption on X and I, called condition c_λ^*, which guarantees that each $S \in I$ is the SOT-limit of a bounded net of finite rank operators. It is shown in [78] that for certain nontrivial ideals this condition is strictly weaker than the bounded approximation property. In the results throughout the paper where we assume that X has the bounded approximation property, one may assume instead that X satisfies condition c_λ^* for I for some $\lambda \geq 1$.

5.2.2 Commutator and Lipschitz estimates

For $W_1, W_2 \subseteq \mathbb{C}$ Borel, let $p_1 : W_1 \times W_2 \to W_1$ and $p_2 : W_1 \times W_2 \to W_2$ be the coordinate projections given by $p_1(\lambda_1, \lambda_2) := \lambda_1$, $p_2(\lambda_1, \lambda_2) := \lambda_2$ for $(\lambda_1, \lambda_2) \in W_1 \times W_2$. Note that $f \circ p_1, f \circ p_2 \in \mathfrak{A}(W_1 \times W_2)$ for all $W_1, W_2 \subseteq \mathbb{C}$ Borel and $f \in \mathcal{B}(W_1 \cup W_2)$. For selfadjoint operators A and B on a Hilbert space and for a Schatten von Neumann ideal $\mathcal{I}$, the following lemma is [101, Lemma 3].

Lemma 5.9. *Under the assumptions of Proposition 5.6, the following hold:*

1. *The map $\varphi \mapsto T_\varphi^{A,B}$ is a morphism $\mathfrak{A}(\sigma(A) \times \sigma(B)) \to \mathcal{L}(\mathcal{I})$ of unital Banach algebras.*
2. *Let $f \in \mathcal{B}(\sigma(A) \cup \sigma(B))$ and $S \in \mathcal{L}(X,Y)$. Then $T_{f\circ p_1}(S) = Sf(A)$ and $T_{f\circ p_2}(S) = f(B)S$. In particular, $T_{p_1}(S) = SA$ and $T_{p_2}(S) = BS$.*

Proof. The structure of the proof is the same as that of [101, Lemma 3]. Linearity in (1) is straightforward. Fix $\varphi_1, \varphi_2 \in \mathfrak{A}(\sigma(A) \times \sigma(B))$ with representations as in (2.19), with corresponding measure spaces (Ω_j, μ_j) and bounded Borel functions $a_{1,j} \in \mathcal{B}(\sigma(A) \times \Omega_j)$ and $a_{2,j} \in \mathcal{B}(\sigma(B) \times \Omega_j)$ for $j \in \{1, 2\}$. Then $\varphi := \varphi_1\varphi_2$ also has a representation as in (2.19), with $\Omega = \Omega_1 \times \Omega_2$, $\mu = \mu_1 \times \mu_2$ the product measure and $a_1 = a_{1,1}a_{1,2}$, $a_2 = a_{2,1}a_{2,2}$. By multiplicativity of the functional calculus for A,

$$a_1(A, (\omega_1, \omega_2)) = \big(a_{1,1}(\cdot, \omega_1)a_{1,2}(\cdot, \omega_2)\big)(A) = a_{1,1}(A, \omega_1)a_{1,2}(A, \omega_2)$$

for all $(\omega_1, \omega_2) \in \Omega$, and similarly for $a_2(B, (\omega_1, \omega_2))$. Applying this to (5.11) yields

$$\begin{aligned} T_\varphi(S)x &= \int_\Omega a_2(B, \omega)Sa_1(A, \omega)x \, \mathrm{d}\mu(\omega) \\ &= \int_{\Omega_1} a_{2,1}(B, \omega_1)T_{\varphi_2}(S)a_{1,1}(A, \omega_1)x \, \mathrm{d}\mu_1(\omega_1) \\ &= T_{\varphi_1}(T_{\varphi_2}(S))x \end{aligned}$$

for all $S \in \mathcal{I}$ and $x \in X$, which proves (1). Part (2) follows from (5.11) and the fact that T_φ is independent of the representation of φ. □

For $f : \sigma(A) \cup \sigma(B) \to \mathbb{C}$ define

$$\varphi_f(\lambda_1, \lambda_2) := \frac{f(\lambda_2) - f(\lambda_1)}{\lambda_2 - \lambda_1} \tag{5.14}$$

for $(\lambda_1, \lambda_2) \in \sigma(A) \times \sigma(B)$ with $\lambda_1 \neq \lambda_2$.

Theorem 5.10. *Let X and Y be separable Banach spaces such that X or Y has the bounded approximation property, and let $\mathcal{I}$ be a Banach ideal in $\mathcal{L}(X,Y)$ with the strong convex compactness property. Let $A \in$ Scal(X) and $B \in$ Scal(Y), and let*

$f \in \mathcal{B}(\sigma(A) \cup \sigma(B))$ be such that φ_f extends to an element of $\mathfrak{A}(\sigma(A) \times \sigma(B))$. Then

$$\|f(B)S - Sf(A)\|_{\mathcal{I}} \leq 16\,\nu(A)\nu(B)\|\varphi_f\|_{\mathfrak{A}(\sigma(A)\times\sigma(B))}\,\|BS - SA\|_{\mathcal{I}} \tag{5.15}$$

for all $S \in \mathcal{L}(X, Y)$ such that $BS - SA \in \mathcal{I}$.

In particular, if $X = Y$ and $B - A \in \mathcal{I}$,

$$\|f(B) - f(A)\|_{\mathcal{I}} \leq 16\,\nu(A)\nu(B)\|\varphi_f\|_{\mathfrak{A}(\sigma(A)\times\sigma(B))}\,\|B - A\|_{\mathcal{I}}.$$

Proof. Note that $(p_2 - p_1)\varphi_f = f \circ p_2 - f \circ p_1$. By Lemma 5.9,

$$\begin{aligned} f(B)S - Sf(A) &= T_{f\circ p_2}(S) - T_{f\circ p_1}(S) = T_{(p_2-p_1)\varphi_f}(S) \\ &= T_{p_2\varphi_f}(S) - T_{p_1\varphi_f}(S) = T_{\varphi_f}(T_{p_2}(S) - T_{p_1}(S)) \\ &= T_{\varphi_f}(BS - SA) \end{aligned}$$

for each $S \in \mathcal{I}$. Proposition 5.6 now concludes the proof. □

Letting X and Y be Hilbert spaces and A and B normal operators, we generalize results from [20, 101] to all Banach ideals with the strong convex compactness property. As mentioned in Section 5.1, this includes all separable ideals and the so-called maximal operator ideals, which in turn is a large class of ideals containing the absolutely (p, q)-summing operators, the integral operators, and more [34, p. 203]. Note that, for normal operators, we can improve the estimate in (5.15) by appealing to (5.13) instead of (5.12).

Corollary 5.11. *Let $A \in \mathcal{L}(X)$ and $B \in \mathcal{L}(Y)$ be normal operators on separable Hilbert spaces X and Y. Let $\mathcal{I}$ be a Banach ideal in $\mathcal{L}(X, Y)$ with the strong convex compactness property, and let $f \in \mathcal{B}(\sigma(A) \cup \sigma(B))$ be such that φ_f extends to an element of $\mathfrak{A}(\sigma(A) \times \sigma(B))$. Then*

$$\|f(B)S - Sf(A)\|_{\mathcal{I}} \leq \|\varphi_f\|_{\mathfrak{A}(\sigma(A)\times\sigma(B))}\,\|BS - SA\|_{\mathcal{I}}$$

for all $S \in \mathcal{L}(X, Y)$ such that $BS - SA \in \mathcal{I}$. In particular, if $X = Y$ and $B - A \in \mathcal{I}$,

$$\|f(B) - f(A)\|_{\mathcal{I}} \leq \|\varphi_f\|_{\mathfrak{A}(\sigma(A)\times\sigma(B))}\,\|B - A\|_{\mathcal{I}}.$$

Combining Theorem 5.10 with Lemma 2.16 yields the following, a generalization of [96, Theorem 4].

Corollary 5.12. *There exists a universal constant $C \geq 0$ such that the following holds. Let X and Y be separable Banach spaces such that X or Y has the bounded approximation property, and let $\mathcal{I}$ be a Banach ideal in $\mathcal{L}(X, Y)$ with the strong convex compactness property. Let $f \in \dot{\mathrm{B}}^1_{\infty,1}(\mathbb{R})$, and let $A \in \mathrm{Scal}(X)$ and $B \in \mathrm{Scal}(Y)$ be such that $\sigma(A) \cup \sigma(B) \subseteq \mathbb{R}$. Then*

$$\|f(B)S - Sf(A)\|_{\mathcal{I}} \leq C\nu(A)\nu(B) \|f\|_{\dot{B}^1_{\infty,1}(\mathbb{R})} \|BS - SA\|_{\mathcal{I}} \tag{5.16}$$

for all $S \in \mathcal{L}(X,Y)$ such that $BS - SA \in \mathcal{I}$. In particular, if $X = Y$ and $B - A \in \mathcal{I}$,

$$\|f(B) - f(A)\|_{\mathcal{I}} \leq C\nu(A)\nu(B) \|f\|_{\dot{B}^1_{\infty,1}(\mathbb{R})} \|B - A\|_{\mathcal{I}} .$$

In the case where the Banach ideal $\mathcal{I}$ is the space $\mathcal{L}(X,Y)$ of all bounded operators from X to Y, we obtain the following corollary, an extension of Theorem 1.3 to scalar type operators on Banach spaces.

Corollary 5.13. *There exists a universal constant $C \geq 0$ such that the following holds. Let X and Y be separable Banach spaces such that either X or Y has the bounded approximation property. Let $f \in \dot{B}^1_{\infty,1}(\mathbb{R})$, and let $A, B \in \mathrm{Scal}(X)$ be such that $\sigma(A) \cup \sigma(B) \subseteq \mathbb{R}$. Then*

$$\|f(B)S - Sf(A)\|_{\mathcal{L}(X,Y)} \leq C\nu(A)\nu(B) \|f\|_{\dot{B}^1_{\infty,1}(\mathbb{R})} \|BS - SA\|_{\mathcal{L}(X,Y)} \tag{5.17}$$

for all $S \in \mathcal{L}(X,Y)$. In particular, if $X = Y$ then

$$\|f(B) - f(A)\|_{\mathcal{L}(X)} \leq C\nu(A)\nu(B) \|f\|_{\dot{B}^1_{\infty,1}(\mathbb{R})} \|B - A\|_{\mathcal{L}(X)} .$$

Remark 5.14. Corollaries 5.12 and 5.13 yield global estimates, in the sense that (5.16) and (5.17) hold for all scalar type operators A and B with real spectrum, and the constant in the estimate depends on A and B only through their spectral constants $\nu(A)$ and $\nu(B)$. Local estimates follow if $f \in \mathcal{B}(\mathbb{R})$ is contained in the Besov class locally. More precisely, given scalar type operators $A \in \mathrm{Scal}(X)$ and $B \in \mathrm{Scal}(Y)$ with real spectrum, suppose there exists $g \in \dot{B}^1_{\infty,1}(\mathbb{R})$ with $g(s) = f(s)$ for all $s \in \sigma(A) \cup \sigma(B)$. Then (with notation as in Corollary 5.12), Theorem 5.10 yields

$$\|f(B)S - Sf(A)\|_{\mathcal{I}} \leq C\nu(A)\nu(B) \|g\|_{\dot{B}^1_{\infty,1}(\mathbb{R})} \|BS - SA\|_{\mathcal{I}} \tag{5.18}$$

for all $S \in \mathcal{L}(X,Y)$ such that $BS - SA \in \mathcal{I}$.

5.3 Spaces with an unconditional basis

In this section we prove some results for specific scalar type operators, namely operators which are diagonalizable with respect to an unconditional Schauder basis. These results will be used in later sections. In this section we assume all spaces to be infinite dimensional, but the results and proofs carry over directly to finite dimensional spaces. This will be used in Section 5.6.

5.3.1 Diagonalizable operators

A sequence $(e_j)_{j=1}^{\infty} \subseteq X$ in a Banach space X is said to be a *Schauder basis* if, for each $x \in X$, there exists a unique sequence $(x_j)_{j=1}^{\infty} \subseteq \mathbb{C}$ such that

$$x = \sum_{j=1}^{\infty} x_j e_j,$$

where the sum converges in the norm of X. A Schauder basis $(e_j)_{j=1}^{\infty} \subseteq X$ is *unconditional* if, for each sequence $(x_j)_{j=1}^{\infty} \subseteq \mathbb{C}$ such that $\sum_{j=1}^{\infty} x_j e_j$ converges in X and for each permutation π of $\mathbb{N}$, $\sum_{j=1}^{\infty} x_{\pi(j)} e_j$ converges in X.

Let X be a Banach space with an unconditional Schauder basis $(e_j)_{j=1}^{\infty} \subseteq X$. For $j \in \mathbb{N}$, let $\mathcal{P}_j \in \mathcal{L}(X)$ be the projection given by $\mathcal{P}_j(x) := x_j e_j$ for all $x = \sum_{k=1}^{\infty} x_k e_k \in X$.

Assumption 5.15. *We assume in this section that* $\left\| \sum_{j \in N} \mathcal{P}_j \right\|_{\mathcal{L}(X)} = 1$ *for all nonempty* $N \subseteq \mathbb{N}$. *This condition is satisfied in the examples we consider in later sections, and simplifies the presentation. For general bases one merely gets additional constants in the results.*

An operator $A \in \mathcal{L}(X)$ is *diagonalizable* (with respect to $(e_j)_{j=1}^{\infty}$) if there exists $U \in \mathcal{L}(X)$ invertible and a sequence $(\lambda_j)_{j=1}^{\infty} \in \ell^{\infty}$ of complex numbers such that

$$UAU^{-1}x = \sum_{j=1}^{\infty} \lambda_j \mathcal{P}_j x \qquad (x \in X), \tag{5.19}$$

where the series converges since $(e_k)_{k=1}^{\infty}$ is unconditional (see [113, Lemma 16.1]). In this case A is a scalar type operator, with point spectrum equal to $\{\lambda_j\}_{j=1}^{\infty}$, $\sigma(A) = \overline{\{\lambda_j\}_{j=1}^{\infty}}$ and spectral measure E given by

$$E(\sigma) = \sum_{\lambda_j \in W} U^{-1} \mathcal{P}_j U \tag{5.20}$$

for $W \subseteq \mathbb{C}$ Borel. The set of all diagonalizable operators on X is denoted by $\mathcal{L}_{\mathrm{d}}(X)$. We do not explicitly mention the basis $(e_j)_{j=1}^{\infty}$ with respect to which an operator is diagonalizable, since this basis will be fixed throughout. Often we write $A \in \mathcal{L}_{\mathrm{d}}(X, (\lambda_j)_{j=1}^{\infty}, U)$ in order to identify the operator U and the sequence $(\lambda_j)_{j=1}^{\infty}$ from above (note that the set $\mathcal{L}_d(X, (\lambda_j)_{j=1}^{\infty}, U)$ consists of at most one element). For $A \in \mathcal{L}_{\mathrm{d}}(X, (\lambda_j)_{j=1}^{\infty}, U)$ and $f \in \mathcal{B}(\mathbb{C})$, it follows from (2.14) that

$$f(A) = U^{-1}\Big(\sum_{j=1}^{\infty} f(\lambda_j) \mathcal{P}_j \Big) U. \tag{5.21}$$

Since any Banach space with a Schauder basis is separable and has the bounded approximation property, we can apply the results from the previous section to diagonalizable operators, and we obtain for instance the following.

Corollary 5.16. *There exists a universal constant $C \geq 0$ such that the following holds. Let X and Y be Banach spaces with unconditional Schauder bases, and let $\mathcal{I}$ be a Banach ideal in $\mathcal{L}(X,Y)$ with the strong convex compactness property. Let $f \in \dot{\mathrm{B}}^1_{\infty,1}(\mathbb{R})$, and let $A \in \mathcal{L}_{\mathrm{d}}(X)$ and $B \in \mathcal{L}_{\mathrm{d}}(Y)$ be such that $\sigma(A) \cup \sigma(B) \subseteq \mathbb{R}$. Then*

$$\|f(B)S - Sf(A)\|_{\mathcal{I}} \leq C\nu(A)\nu(B)\,\|f\|_{\dot{\mathrm{B}}^1_{\infty,1}(\mathbb{R})}\|BS - SA\|_{\mathcal{I}}$$

for all $S \in \mathcal{L}(X,Y)$ such that $BS - SA \in \mathcal{I}$. In particular, if $X = Y$ and $B - A \in \mathcal{I}$,

$$\|f(B) - f(A)\|_{\mathcal{I}} \leq C\nu(A)\nu(B)\,\|f\|_{\dot{\mathrm{B}}^1_{\infty,1}(\mathbb{R})}\|B - A\|_{\mathcal{I}}\,.$$

Since this result does not apply to the absolute value function (and neither does the more general Theorem 5.10), and because of the importance of the absolute value function, we will now study Lipschitz estimates for more general functions.

Let Y be a Banach space with an unconditional Schauder basis $(f_k)_{k=1}^{\infty} \subseteq Y$, and let the projections $\mathcal{Q}_k \in \mathcal{L}(Y)$ be given by $\mathcal{Q}_k(y) := y_k f_k$ for all $y = \sum_{l=1}^{\infty} y_l f_l \in Y$ and $k \in \mathbb{N}$. Let $\lambda = (\lambda_j)_{j=1}^{\infty}$ and $\mu = (\mu_k)_{k=1}^{\infty}$ be sequences of complex numbers, and let $\varphi : \mathbb{C}^2 \to \mathbb{C}$. For $n \in \mathbb{N}$, define $T^{\lambda,\mu}_{\varphi,n} \in \mathcal{L}(\mathcal{L}(X,Y))$ by

$$T^{\lambda,\mu}_{\varphi,n}(S) := \sum_{j,k=1}^{n} \varphi(\lambda_j,\mu_k)\mathcal{Q}_k S\mathcal{P}_j \qquad (S \in \mathcal{L}(X,Y)). \tag{5.22}$$

Note that $T^{\lambda,\mu}_{\varphi,n} \in \mathcal{L}(\mathcal{I})$ for each Banach ideal $\mathcal{I}$ in $\mathcal{L}(X,Y)$.

For $f \in \mathcal{B}(\mathbb{C})$ extend the divided difference φ_f from (5.14), given by $\varphi_f(\lambda_1,\lambda_2) := \frac{f(\lambda_2)-f(\lambda_1)}{\lambda_2-\lambda_1}$ for $(\lambda_1,\lambda_2) \in \mathbb{C}^2$ with $\lambda_1 \neq \lambda_2$, to a function $\varphi_f : \mathbb{C}^2 \to \mathbb{C}$.

Lemma 5.17. *Let X and Y be Banach spaces with unconditional Schauder bases, and let $\mathcal{I}$ be a Banach ideal in $\mathcal{L}(X,Y)$. Let $\lambda = (\lambda_j)_{j=1}^{\infty}$ and $\mu = (\mu_k)_{k=1}^{\infty}$ be sequences of complex numbers, and let $A \in \mathcal{L}_{\mathrm{d}}(X,\lambda,U)$, $B \in \mathcal{L}_{\mathrm{d}}(Y,\mu,V)$, $f \in \mathcal{B}(\mathbb{C})$ and $n \in \mathbb{N}$. Then*

$$\|f(B)S_n - S_n f(A)\|_{\mathcal{I}} \leq \|U\|_{\mathcal{L}(X)}\|V^{-1}\|_{\mathcal{L}(Y)}\|T^{\lambda,\mu}_{\varphi_f,n}(V(BS - SA)U^{-1})\|_{\mathcal{I}}$$

for all $S \in \mathcal{L}(X,Y)$ such that $BS - SA \in \mathcal{I}$, where

$$S_n := \sum_{j,k=1}^{n} V^{-1}\mathcal{Q}_k VSU^{-1}\mathcal{P}_j U.$$

Proof. Let $S \in \mathcal{I}$ be such that $BS - SA \in \mathcal{I}$. For the duration of the proof write $P_j := U^{-1}\mathcal{P}_j U \in \mathcal{L}(X)$ and $Q_k := V^{-1}\mathcal{Q}_k V \in \mathcal{L}(Y)$ for $j, k \in \mathbb{N}$. By (5.21), and using that $P_j P_k = 0$ and $Q_j Q_k = 0$ for $j \neq k$,

$$\begin{aligned}
f(B)S_n - S_n f(A) &= \sum_{k=1}^{\infty} f(\mu_k) Q_k \Big(\sum_{i,l=1}^{n} Q_l S P_i \Big) - \sum_{j=1}^{\infty} f(\lambda_j) \Big(\sum_{i,l=1}^{n} Q_l S P_i \Big) P_j \\
&= \sum_{j,k=1}^{n} (f(\mu_k) - f(\lambda_j)) Q_k S P_j \\
&= \sum_{j,k=1}^{n} \sum_{\mu_k \neq \lambda_j} \frac{f(\mu_k) - f(\lambda_j)}{\mu_k - \lambda_j} (\mu_k Q_k S P_j - \lambda_j Q_k S P_j) \\
&= \sum_{j,k=1}^{n} \varphi_f(\lambda_j, \mu_k) Q_k \Big(\Big(\sum_{l=1}^{\infty} \mu_l Q_l \Big) S - S \Big(\sum_{i=1}^{\infty} \lambda_i P_i \Big) \Big) P_j \\
&= \sum_{j,k=1}^{n} \varphi_f(\lambda_j, \mu_k) Q_k (BS - SA) P_j \\
&= V^{-1} T_{\varphi_f}^{A,B} (V(BS - SA)U^{-1}) U.
\end{aligned}$$

Now use the ideal property of $\mathcal{I}$ to conclude the proof. □

For $A \in \mathcal{L}_{\mathrm{d}}(X, \lambda, U)$ define

$$K_A := \inf \Big\{ \|U\|_{\mathcal{L}(X)} \|U^{-1}\|_{\mathcal{L}(X)} \Big| \exists \lambda \text{ such that } A \in \mathcal{L}_{\mathrm{d}}(X, \lambda, U) \Big\}. \quad (5.23)$$

We call K_A the *diagonalizability constant of* A. Using the unconditionality of the Schauder basis of X and Assumption 5.15, one can show that K_A does not depend on the specific ordering of the sequence λ. Since the sequence λ is, up to ordering, uniquely determined by A (it is the point spectrum of A), K_A only depends on A. Moreover, by Assumption 5.15 and (5.20), $\|E(W)\|_{\mathcal{L}(X)} \leq \|U^{-1}\|_{\mathcal{L}(X)} \|U\|_{\mathcal{L}(X)}$ for all $W \subseteq \mathbb{C}$ Borel and $U \in \mathcal{L}(X)$ such that $A \in \mathcal{L}_{\mathrm{d}}(X, \lambda, U)$, where E is the spectral measure of A. Hence

$$\nu(A) \leq K_A, \quad (5.24)$$

where $\nu(A)$ is the spectral constant of A from Section 2.2.5.

We now derive commutator estimates for A and B in the operator norm, under a boundedness assumption which will be verified for specific X and Y in later sections.

Proposition 5.18. *Let X and Y be Banach spaces with unconditional Schauder bases, $A \in \mathcal{L}_{\mathrm{d}}(X, \lambda, U)$, $B \in \mathcal{L}_{\mathrm{d}}(Y, \mu, V)$ and $f \in \mathcal{B}(\mathbb{C})$. Suppose that*

$$C := \sup_{n \in \mathbb{N}} \Big\| T_{\varphi_f, n}^{\lambda, \mu} \Big\|_{\mathcal{L}(\mathcal{L}(X,Y))} < \infty. \quad (5.25)$$

Then

$$\|f(B)S - Sf(A)\|_{\mathcal{L}(X,Y)} \leq CK_A K_B \|BS - SA\|_{\mathcal{L}(X,Y)}$$

for all $S \in \mathcal{L}(X, Y)$.

Proof. Let $S \in \mathcal{L}(X, Y)$ and for $n \in \mathbb{N}$ let $S_n \in \mathcal{L}(X, Y)$ be as in Lemma 5.17. It is straightforward to show that, for each $x \in X$, $S_n x \to Sx$ as $n \to \infty$. Hence $f(B)S_n x - S_n f(A)x \to f(B)Sx - Sf(A)x$ as $n \to \infty$, for each $x \in X$. Lemma 5.17 and (5.25) now yield

$$\begin{aligned}\|f(B)S - Sf(A)\|_{\mathcal{L}(X,Y)} &\leq \limsup_{n\to\infty} \|f(B)S_n - S_n f(A)\|_{\mathcal{L}(X,Y)} \\ &\leq C\|U\|\|V^{-1}\|\|V(BS - SA)U^{-1}\|_{\mathcal{L}(X,Y)} \\ &\leq C\|U\|\|U^{-1}\|\|V\|\|V^{-1}\| \, \|BS - SA\|_{\mathcal{L}(X,Y)}.\end{aligned}$$

Taking the infimum over U and V concludes the proof. □

Remark 5.19. Proposition 5.18 also holds for more general Banach ideals in $\mathcal{L}(X, Y)$. Indeed, let $\mathcal{I}$ be a Banach ideal in $\mathcal{L}(X, Y)$ with the property that, if $(S_m)_{m=1}^{\infty} \subseteq \mathcal{I}$ is an $\mathcal{I}$-bounded sequence which SOT-converges to some $S \in \mathcal{L}(X, Y)$ as $m \to \infty$, then $S \in \mathcal{I}$ with $\|S\|_{\mathcal{I}} \leq \limsup_{m\to\infty} \|S_m\|_{\mathcal{I}}$. If

$$C := \sup_{n\in\mathbb{N}} \left\| T^{\lambda,\mu}_{\varphi_f,n} \right\|_{\mathcal{L}(\mathcal{I})} < \infty$$

then the proof of Proposition 5.18 shows that

$$\|f(B)S - Sf(A)\|_{\mathcal{I}} \leq CK_A K_B \|BS - SA\|_{\mathcal{I}}$$

for all $S \in \mathcal{L}(X, Y)$ such that $BS - SA \in \mathcal{I}$.

5.3.2 Estimates for the absolute value function

It is known that operator Lipschitz estimates for the absolute value function are related to estimates for so-called triangular truncation operators. For example, in [71] and [36] it was shown that the boundedness of the standard triangular truncation on many spaces of operators is equivalent to Lipschitz estimates for the absolute value function. We prove that triangular truncation operators are connected to Lipschitz estimates for the absolute value function in our setting as well. We do so by relating the assumption in (5.25) to triangular truncation operators associated to sequences. We will then bound the norms of these operators in later sections for specific X and Y.

Let $\lambda = (\lambda_j)_{j=1}^{\infty}$ and $\mu = (\mu_k)_{k=1}^{\infty}$ be sequences of real numbers, and let X, Y, $(\mathcal{P}_j)_{j=1}^{\infty}$ and $(\mathcal{Q}_k)_{k=1}^{\infty}$ be as before. For $n \in \mathbb{N}$ define $T^{\lambda,\mu}_{\Delta,n} \in \mathcal{L}(\mathcal{L}(X, Y))$ by

$$T^{\lambda,\mu}_{\Delta,n}(S) := \sum_{j,k=1}^{n} \sum_{\mu_k \le \lambda_j} \mathcal{Q}_k S \mathcal{P}_j \qquad (S \in \mathcal{L}(X,Y)). \tag{5.26}$$

We call $T^{A,B}_{\Delta}$ the *triangular truncation associated with* λ *and* μ.

For $f(t) := |t|$, $t \in \mathbb{R}$, define $\varphi_f : \mathbb{C}^2 \to \mathbb{C}$ by

$$\varphi_f(\lambda_1,\lambda_2) := \begin{cases} \frac{|\lambda_1|-|\lambda_2|}{\lambda_1-\lambda_2} & \text{if } \lambda_1 \neq \lambda_2 \\ 1 & \text{otherwise} \end{cases}.$$

The following result relates the norm of $T^{\lambda,\mu}_{\varphi_f,n}$ to that of $T^{\lambda,\mu}_{\Delta,n}$.

Proposition 5.20. *There exists a universal constant $C \ge 0$ such that the following holds. Let X and Y be Banach spaces with unconditional Schauder bases and let $\mathcal{I}$ be a Banach ideal in $\mathcal{L}(X,Y)$ with the strong convex compactness property. Let λ and μ be bounded sequences of real numbers. Let $f(t) := |t|$ for $t \in \mathbb{R}$. Then*

$$\|T^{\lambda,\mu}_{\varphi_f,n}(S)\|_{\mathcal{I}} \le C\Big(\|S\|_{\mathcal{I}} + \|T^{\lambda,\mu}_{\Delta,n}(S)\|_{\mathcal{I}}\Big)$$

for all $n \in \mathbb{N}$ and $S \in \mathcal{I}$. In particular, if $\sup_{n\in\mathbb{N}} \|T^{\lambda,\mu}_{\Delta,n}(S)\|_{\mathcal{L}(\mathcal{L}(X,Y))} < \infty$ *then* (5.25) *holds.*

Proof. Let $n \in \mathbb{N}$ and $S \in \mathcal{I}$, and write $\lambda = (\lambda_j)_{j=1}^{\infty}$ and $\mu = (\mu_k)_{k=1}^{\infty}$. Throughout the proof we will only consider λ_j and μ_k for $1 \le j,k \le n$, but to simplify the presentation we will not mention this explicitly. We can decompose $T^{\lambda,\mu}_{\varphi_f,n}(S)$ as

$$\begin{aligned} T^{\lambda,\mu}_{\varphi_f,n}(S) = \sum_{\lambda_k,\mu_k \ge 0} \mathcal{Q}_k S \mathcal{P}_j - \sum_{\mu_k<0<\lambda_j} \frac{\mu_k+\lambda_j}{\mu_k-\lambda_j} \mathcal{Q}_k S \mathcal{P}_j + \\ \sum_{\lambda_j<0<\mu_k} \frac{\mu_k+\lambda_j}{\mu_k-\lambda_j} \mathcal{Q}_k S \mathcal{P}_j - \sum_{\lambda_k,\mu_k \le 0} \mathcal{Q}_k S \mathcal{P}_j + \sum_{\lambda_k,\mu_k=0} \mathcal{Q}_k S \mathcal{P}_j. \end{aligned}$$

Note that some of these terms may be zero. By the ideal property of $\mathcal{I}$ and Assumption 5.15,

$$\Big\| \sum_{\lambda_j,\mu_k \ge 0} \mathcal{Q}_k S \mathcal{P}_j \Big\|_{\mathcal{I}} \le \Big\| \sum_{\mu_k \ge 0} \mathcal{Q}_k \Big\|_{\mathcal{L}(Y)} \|S\|_{\mathcal{I}} \Big\| \sum_{\lambda_j \ge 0} \mathcal{P}_j \Big\|_{\mathcal{L}(X)} \le \|S\|_{\mathcal{I}}. \tag{5.27}$$

Similarly, $\Big\|\sum_{\lambda_k,\mu_k\le 0} \mathcal{Q}_k S \mathcal{P}_j\Big\|_{\mathcal{I}}$ and $\Big\|\sum_{\lambda_k,\mu_k=0} \mathcal{Q}_k S \mathcal{P}_j\Big\|_{\mathcal{I}}$ are each bounded by $\|S\|_{\mathcal{I}}$. To bound the other terms it is sufficient to show that

$$\Big\| \sum_{\lambda_j,\mu_k>0} \frac{\mu_k-\lambda_j}{\mu_k+\lambda_j} \mathcal{Q}_k S \mathcal{P}_j \Big\|_{\mathcal{I}} \le C' \Big(\|S\|_{\mathcal{I}} + \Big\| T^{\lambda,\mu}_{\Delta,n}(S) \Big\|_{\mathcal{I}} \Big)$$

for some universal constant $C' \geq 0$. Indeed, replacing λ by $-\lambda$ and μ by $-\mu$ then yields the desired conclusion. Let

$$\Phi(S) := \sum_{\lambda_j,\mu_k>0} \frac{\mu_k - \lambda_j}{\mu_k + \lambda_j} \mathcal{Q}_k S \mathcal{P}_j,$$

and define $g \in \mathrm{W}^{1,2}(\mathbb{R})$ by $g(t) := \frac{2}{e^{|t|}+1}$ for $t \in \mathbb{R}$. Note that $\Phi(S)$ is equal to

$$\sum_{0<\mu_k\leq\lambda_j} \left(g\left(\log\tfrac{\lambda_j}{\mu_k}\right) - 1\right) \mathcal{Q}_k S \mathcal{P}_j + \sum_{0<\lambda_j<\mu_k} \left(1 - g\left(\log\tfrac{\lambda_j}{\mu_k}\right)\right) \mathcal{Q}_k S \mathcal{P}_j.$$

Now let $\psi_g : \mathbb{R}^2 \to \mathbb{C}$ be as in (2.21), and let $A := \sum_{j=1}^{\infty} \lambda_j \mathcal{P}_j \in \mathcal{L}(X)$ and $B := \sum_{k=1}^{\infty} \mu_k \mathcal{Q}_k \in \mathcal{L}(Y)$. Let $T_{\psi_g}^{A,B}$ be as in (5.11). One can check that

$$\begin{aligned}\Phi(S) =& T_{\psi_g}^{A,B}(T_{\Delta,n}^{\lambda,\mu}(S)) - \sum_{\lambda_j,\mu_k>0} \mathcal{Q}_k T_{\Delta,n}^{\lambda,\mu}(S)\mathcal{P}_j \\ &+ \sum_{\lambda_j,\mu_k>0} \mathcal{Q}_k(S - T_{\Delta,n}^{\lambda,\mu}(S))\mathcal{P}_j - T_{\psi_g}^{A,B}(S - T_{\Delta,n}^{\lambda,\mu}(S)).\end{aligned}$$

Any Banach space with a Schauder basis is separable and has the bounded approximation property, hence Lemma 2.15 and Proposition 5.6 yield

$$\left\|T_{\psi_g}^{A,B}(T_{\Delta,n}^{\lambda,\mu}(S))\right\|_{\mathcal{I}} \leq 16\sqrt{2}\,\nu(A)\nu(B)\,\|g\|_{\mathrm{W}^{1,2}(\mathbb{R})}\left\|T_{\Delta,n}^{\lambda,\mu}(S))\right\|_{\mathcal{I}}.$$

By (5.24), $\nu(A) = \nu(B) = 1$. Similarly,

$$\left\|T_{\psi_g}^{A,B}(S - T_{\Delta,n}^{\lambda,\mu}(S))\right\|_{\mathcal{I}} \leq 16\sqrt{2}\,\|g\|_{\mathrm{W}^{1,2}(\mathbb{R})}\left(\left\|S\right\|_{\mathcal{I}} + \left\|T_{\Delta,n}^{\lambda,\mu}(S))\right\|_{\mathcal{I}}\right).$$

By the same arguments as in (5.27),

$$\begin{aligned}\Big\|\sum_{\lambda_j,\mu_k>0} \mathcal{Q}_k T_{\Delta,n}^{\lambda,\mu}(S)\mathcal{P}_j\Big\|_{\mathcal{I}} + \Big\|\sum_{\lambda_j,\mu_k>0} \mathcal{Q}_k(S - T_{\Delta,n}^{\lambda,\mu}(S))\mathcal{P}_j\Big\| \\ \leq 2\left\|S\right\|_{\mathcal{I}} + \left\|T_{\Delta,n}^{\lambda,\mu}(S)\right\|_{\mathcal{I}}.\end{aligned}$$

Combining all these estimates yields

$$\|\Phi(S)\|_{\mathcal{I}} \leq \left(2 + 32\sqrt{2}\,\|g\|_{\mathrm{W}^{1,2}(\mathbb{R})}\right)\left(\left\|S\right\|_{\mathcal{I}} + \left\|T_{\Delta,n}^{\lambda,\mu}(S)\right\|_{\mathcal{I}}\right),$$

as desired. □

5.4 The absolute value function on $\mathcal{L}(\ell^p, \ell^q)$

In this section we study the absolute value function on the space $\mathcal{L}(\ell^p, \ell^q)$. We obtain the commutator estimate (5.2) for the absolute value function and $X = \ell^p$ and $Y = \ell^q$ with $p < q$, and we obtain (5.1) for each Lipschitz function and $X = \mathcal{L}(\ell^1)$ or $X = \mathcal{L}(c_0)$. We also obtain results for $p \geq q$.

The key idea of the proof is entirely different from the techniques used in [31], [33], [36] and [71], which are based on the fact that the reflexive Schatten von Neumann ideals are UMD spaces, a property which $\mathcal{L}(\ell^p, \ell^q)$ obviously does not have ($\mathcal{L}(\ell^p, \ell^q)$ is not reflexive). Instead, we prove our results by relating estimates for the operators from (5.26) to the standard triangular truncation operator, defined in (5.28) below. For this we use the theory of Schur multipliers on $\mathcal{L}(\ell^p, \ell^q)$ developed in [11]. We then appeal to results from [10] about the boundedness of the standard triangular truncation on $\mathcal{L}(\ell^p, \ell^q)$.

5.4.1 Schur multipliers

For $p \in [1, \infty)$, let $(e_j)_{j=1}^{\infty}$ be the standard Schauder basis of ℓ^p, with the corresponding projections $\mathcal{P}_j(x) := x_j e_j$ for $x = \sum_{k=1}^{\infty} x_k e_k$ and $j \in \mathbb{N}$. We consider this basis and the corresponding projections on all ℓ^p-spaces simultaneously, for simplicity of notation. Note that Assumption 5.15 is satisfied for this basis. For $q \in [1, \infty]$, any operator $S \in \mathcal{L}(\ell^p, \ell^q)$ can be represented by an infinite matrix $\tilde{S} = (s_{jk})_{j,k=1}^{\infty}$, where $s_{jk} := \mathcal{P}_j(S(e_k))$ for $j, k \in \mathbb{N}$. For an infinite matrix $M = (m_{jk})_{j,k=1}^{\infty}$ the product $M * \tilde{S} := (m_{jk} s_{jk})$ is the *Schur product* of the matrices M and $\tilde{S}$. The matrix M is a *Schur multiplier* if the mapping $\tilde{S} \mapsto M * \tilde{S}$ is a bounded operator on $\mathcal{L}(\ell^p, \ell^q)$. Throughout, we identify Schur multipliers with their corresponding operators.

The notion of a Schur multiplier is a discrete version of a double operator integral (for details see e.g. [103, 114]). Schur multipliers on the space $\mathcal{L}(\ell^p, \ell^q)$ are also called (p, q)-multipliers. We denote by $\mathcal{M}(p, q)$ the Banach space of (p, q)-multipliers with the norm

$$\|M\|_{(p,q)} := \sup \left\{ \|M * \tilde{S}\|_{\mathcal{L}(\ell^p, \ell^q)} \,\middle|\, \|S\|_{\mathcal{L}(\ell^p, \ell^q)} \leq 1 \right\}.$$

Remark 5.21. We also consider (p, q)-multipliers M for $p = \infty$ and $q \in [1, \infty]$. Any operator $S \in \mathcal{L}(c_0, \ell^q)$ corresponds to an infinite matrix $\tilde{S} = (s_{jk})_{j,k=1}^{\infty}$, and M is said to be an (∞, q)-multiplier if the mapping $S \mapsto M * \tilde{S}$ is a bounded operator on $\mathcal{L}(c_0, \ell^q)$. We define the Banach space $\mathcal{M}(\infty, q)$ in the obvious way. Often we do not explicitly distinguish the case $p = \infty$ from $1 \leq p < \infty$, but the reader should keep in mind that for $p = \infty$ the space ℓ^p should be replaced by c_0.

Remark 5.22. It is straightforward to see that $\|M\|_{(p,q)} \geq \sup_{j,k \in \mathbb{N}} |m_{j,k}|$ for all $p, q \in [1, \infty]$ and $M = (m_{jk})_{j,k=1}^{\infty} \in \mathcal{M}(p, q)$.

For $p, q \in [1, \infty]$ and $S \in \mathcal{L}(\ell^p, \ell^q)$, define

$$\mathcal{T}_\Delta(S) := \sum_{k \leq j} \mathcal{P}_k S \mathcal{P}_j, \tag{5.28}$$

which is a well-defined element of $\mathcal{L}(\ell^r, \ell^s)$ for suitable $r, s \in [1, \infty]$ by Proposition 5.23 below. The operator $\mathcal{T}_\Delta$ is the *(standard) triangular truncation* (see [76]). This operator can be identified with the following Schur multiplier. Let $T'_\Delta = (t'_{jk})_{j,k=1}^\infty$ be a matrix given by $t'_{jk} = 1$ for $k \leq j$ and $t'_{jk} = 0$ otherwise. It is clear that $\mathcal{T}_\Delta$ extends to a bounded linear operator on $\mathcal{L}(\ell^p, \ell^q)$ if and only if T'_Δ is a (p, q)-multiplier. For $n \in \mathbb{N}$ and $r, s \in [1, \infty]$ we will consider the operators $\mathcal{T}_{\Delta,n} \in \mathcal{L}(\mathcal{L}(\ell^p, \ell^q), \mathcal{L}(\ell^r, \ell^s))$, given by

$$\mathcal{T}_{\Delta,n}(S) := \sum_{1 \leq k \leq j \leq n} \mathcal{P}_k S \mathcal{P}_j \qquad (S \in \mathcal{L}(\ell^p, \ell^q).$$

The dependence of the (p, q)-norm of $\mathcal{T}_\Delta$ on the indices p and q was determined in [10] and [76] (see also [116]), and is as follows.

Proposition 5.23. *Let $p, q \in [1, \infty]$. Then the following statements hold.*

(i) *[10, Theorem 5.1] If $p < q$, $1 = p = q$ or $p = q = \infty$, then $\mathcal{T}_\Delta \in \mathcal{M}(p, q)$.*

(ii) *[76, Proposition 1.2] If $1 \neq p \geq q \neq \infty$, then there is a constant $C > 0$ such that*

$$\|\mathcal{T}_{\Delta,n}\|_{\mathcal{L}(\mathcal{L}(\ell^p, \ell^q))} \geq C \ln n$$

for all $n \in \mathbb{N}$.

(iii) *[10, Theorem 5.2] If $1 \neq p \geq q \neq \infty$, then for each $s > q$ and $r < p$,*

$$\mathcal{T}_\Delta : \mathcal{L}(\ell^p, \ell^q) \to \mathcal{L}(\ell^p, \ell^s) \quad \textit{and} \quad \mathcal{T}_\Delta : \mathcal{L}(\ell^p, \ell^q) \to \mathcal{L}(\ell^r, \ell^q)$$

are bounded.

Remark 5.24. In Proposition 5.23 (i), a stronger statement holds if $p = 1$ or $q = \infty$. Then, for $M = (m_{jk})_{j,k=1}^\infty$ a matrix, $M \in \mathcal{M}(p, q)$ if and only if $\sup_{j,k \in \mathbb{N}} |m_{jk}| < \infty$, in which case $\|M\|_{(p,q)} = \sup_{j,k \in \mathbb{N}} |m_{jk}|$. This follows immediately from the well-known identities (see [11, p. 605, (2) and (3)])

$$\|S\|_{\mathcal{L}(\ell^1, \ell^q)} = \sup_{k \in \mathbb{N}} \Big(\sum_{j=1}^\infty |s_{jk}|^q \Big)^{1/q}$$

for $q \in [1, \infty)$ and $S = (s_{jk})_{j,k=1}^\infty \in \mathcal{L}(\ell^1, \ell^q)$, and

$$\|S\|_{\mathcal{L}(\ell^p, \ell^\infty)} = \sup_{j \in \mathbb{N}} \Big(\sum_{k=1}^\infty |s_{jk}|^{p'} \Big)^{1/p'}$$

for $p \in [1, \infty]$ and $S = (s_{jk})_{j,k=1}^\infty \in \mathcal{L}(\ell^p, \ell^\infty)$ (with the obvious modification for $p = 1$).

We will also need the following result, a generalization of [11, Theorem 4.1]. For a matrix $M = (m_{jk})_{j,k=1}^{\infty}$, let $\widetilde{M} = (\widetilde{m}_{jk})_{j,k=1}^{\infty}$ be obtained from M by repeating the first column, i.e. $\widetilde{m}_{j1} = m_{j1}$ and $\widetilde{m}_{jk} = m_{j(k-1)}$ for $j \in \mathbb{N}$ and $k \geq 2$.

Proposition 5.25. *Let $p, q, r, s \in [1, \infty]$ with $r \leq p$. Let $M = (m_{jk})_{j,k=1}^{\infty}$ be such that $S \mapsto M * S$ is a bounded mapping $\mathcal{L}(\ell^p, \ell^q) \to \mathcal{L}(\ell^r, \ell^s)$. Then $S \mapsto \widetilde{M} * S$ is also a bounded mapping $\mathcal{L}(\ell^p, \ell^q) \to \mathcal{L}(\ell^r, \ell^s)$, with*

$$\|M\|_{\mathcal{L}(\mathcal{L}(\ell^p,\ell^q),\mathcal{L}(\ell^r,\ell^s))} = \|\widetilde{M}\|_{\mathcal{L}(\mathcal{L}(\ell^p,\ell^q),\mathcal{L}(\ell^r,\ell^s))}.$$

In particular, if $M \in \mathcal{M}(p,q)$ then $\widetilde{M} \in \mathcal{M}(p,q)$ with $\|M\|_{(p,q)} = \|\widetilde{M}\|_{(p,q)}$.

Proof. For any $S = (s_{j,k})_{j,k=1}^{\infty} \in \mathcal{L}(\ell^p, \ell^q)$, the matrix $\hat{S} = (\hat{s}_{j,k})_{j,k=1}^{\infty}$ given by $\hat{s}_{j1} = 0$ and $\hat{s}_{jk} = s_{jk}$ for $j \in \mathbb{N}, k \geq 2$ satisfies $\hat{S} \in \mathcal{L}(\ell^p, \ell^q)$ with $\|\hat{S}\|_{\mathcal{L}(\ell^p,\ell^q)} \leq \|S\|_{\mathcal{L}(\ell^p,\ell^q)}$ and $\widetilde{M} * \hat{S} \in \mathcal{L}(\ell^r, \ell^s)$ with $\left\|\widetilde{M} * \hat{S}\right\|_{\mathcal{L}(\ell^r,\ell^s)} = \|M * S\|_{\mathcal{L}(\ell^r,\ell^s)}$. Hence $\left\|\widetilde{M}\right\| \geq \|M\|$.

For the converse inequality, let $S \in \mathcal{L}(\ell^p, \ell^q)$ and $x = (x_k)_{k=1}^{\infty} \in \ell^r$. Define $\hat{x} = (\hat{x}_k)_{k=1}^{\infty}$ by $\hat{x}_1 := (|x_1|^r + |x_2|^r)^{1/r}$, $\hat{x}_k := x_{k+1}$ for $k \geq 2$. Then $\hat{x} \in \ell^r$ with $\|\hat{x}\|_{\ell^r} = \|x\|_{\ell^r}$. Define also $T = (t_{jk})_{j,k=1}^{\infty}$ by $t_{j1}\hat{x}_1 = s_{j1}x_1 + s_{j2}x_2$ if $\hat{x}_1 \neq 0$, and $t_{j1} = 0$ otherwise, and $t_{jk} = s_{j(k+1)}$ for $j \in \mathbb{N}$, $k \geq 2$. We claim that $T \in \mathcal{L}(\ell^p, \ell^q)$ with $\|T\|_{\mathcal{L}(\ell^p,\ell^q)} \leq \|S\|_{\mathcal{L}(\ell^p,\ell^q)}$. If this is true, we obtain

$$\begin{aligned}\|(\widetilde{M} * S)x\|_{\ell^s}^s &= \sum_{j=1}^{\infty} |\sum_{k=1}^{\infty} \widetilde{m}_{jk} s_{jk} x_k|^s = \sum_{j=1}^{\infty} |\sum_{k=1}^{\infty} m_{jk} t_{jk} \hat{x}_k|^s \\ &\leq \|M\|^s \|T\|_{\mathcal{L}(\ell^p,\ell^q)}^s \|\hat{x}\|_{\ell^r}^s \leq \|M\| \|S|_{\mathcal{L}(\ell^p,\ell^q)} \|x\|_{\ell^r},\end{aligned}$$

as was to be shown. Hence it only remains to prove the claim.

Let $y = (y_k)_{k=1}^{\infty} \in \ell^p$ and define $\hat{y} = (\hat{y}_k)_{k=1}^{\infty}$ by $\hat{x}_1\hat{y}_1 = x_1y_1$, $\hat{x}_1\hat{y}_2 = x_2y_1$ if $\hat{x}_1 \neq 0$, and $\hat{y}_1 = \hat{y}_2 = 0$ otherwise, and $\hat{y}_k = y_{k-1}$ for $k \geq 3$. Then $\hat{y} \in \ell^p$ with

$$\|\hat{y}\|_{\ell^p}^p = \sum_{k=1}^{\infty} |\hat{y}_k|^p \leq \Big(\frac{|x_1|^p + |x_2|^p}{(|x_1|^r + |x_2|^r)^{p/r}} |y_1|^p + \sum_{k=2}^{\infty} |y_k|^p \Big) \leq \|y\|_{\ell^p}^p,$$

where we have used that $\frac{|x_1|^p + |x_2|^p}{(|x_1|^r + |x_2|^r)^{p/r}} \leq 1$ since $p \geq r$. Hence we obtain

$$\begin{aligned}\|Ty\|_{\ell^q}^q &= \sum_{j=1}^{\infty} |\sum_{k=1}^{\infty} t_{jk} y_k|^q = \sum_{j=1}^{\infty} |\sum_{k=1}^{\infty} s_{jk} \hat{y}_k|^q \\ &\leq \|S\|_{\mathcal{L}(\ell^p,\ell^q)}^q \|\hat{y}\|_{\ell^p}^q \leq \|S\|_{\mathcal{L}(\ell^p,\ell^q)}^q \|y\|_{\ell^p}^q .\end{aligned}$$

Therefore $\|T\|_{\mathcal{L}(\ell^p,\ell^q)} \leq \|S\|_{\mathcal{L}(\ell^p,\ell^q)}$ and the claim is proved. □

Remark 5.26. For a matrix $M = (m_{jk})_{j,k=1}^{\infty}$, let $M' = (m'_{jk})_{j,k=1}^{\infty}$, with $m'_{jk} = m_{kj}$ for $j, k \in \mathbb{N}$, be the transpose of M. Let $S \in \mathcal{L}(\ell^{q'}, \ell^{p'})$, $S = (s_{jk})_{j,k=1}^{\infty}$ and let $S' = (s'_{jk})_{j,k=1}^{\infty}$ be the transpose of S. Then $S' \in \mathcal{L}(\ell^p, \ell^q)$ with $\|S'\|_{\mathcal{L}(\ell^p,\ell^q)} = \|S\|_{\mathcal{L}(\ell^{q'},\ell^{p'})}$. Let $y \in \ell^{s'}$. Then

$$\begin{aligned}
\|(M' * S)y\|_{\ell^{r'}} &= \sup_{\|x\|_{\ell^r} \le 1} |\langle (M' * S)y, x\rangle| = \sup_{\|x\|_{\ell^r} \le 1} \Big| \sum_{j=1}^{\infty} \Big(\sum_{k=1}^{\infty} m'_{jk} s_{jk} y_k \Big) x_j \Big| \\
&= \sup_{\|x\|_{\ell^r} \le 1} \Big| \sum_{k=1}^{\infty} \Big(\sum_{j=1}^{\infty} m_{kj} s'_{kj} x_j \Big) y_k \Big| \le \|M * S'\|_{\mathcal{L}(\ell^r,\ell^s)} \|y\|_{\ell^{s'}} \\
&\le \|M\|_{\mathcal{L}(\mathcal{L}(\ell^p,\ell^q),\, \mathcal{L}(\ell^r,\ell^s))} \|S'\|_{\mathcal{L}(\ell^p,\ell^q)} \|y\|_{\ell^{s'}} \\
&= \|M\|_{\mathcal{L}(\mathcal{L}(\ell^p,\ell^q),\, \mathcal{L}(\ell^r,\ell^s))} \|S\|_{\mathcal{L}(\ell^{q'},\ell^{p'})} \|y\|_{\ell^{s'}},
\end{aligned}$$

hence $\|M'\|_{\mathcal{L}(\mathcal{L}(\ell^{q'},\ell^{p'}),\, \mathcal{L}(\ell^{s'},\ell^{r'}))} \le \|M\|_{\mathcal{L}(\mathcal{L}(\ell^p,\ell^q),\, \mathcal{L}(\ell^r,\ell^s))}$. Taking transposes again yields $\|M'\|_{\mathcal{L}(\mathcal{L}(\ell^{q'},\ell^{p'}),\, \mathcal{L}(\ell^{s'},\ell^{r'}))} = \|M\|_{\mathcal{L}(\mathcal{L}(\ell^p,\ell^q),\, \mathcal{L}(\ell^r,\ell^s))}$.

Hence Proposition 5.25 implies that row repetitions do not change the $\mathcal{L}(\mathcal{L}(\ell^p, \ell^q), \mathcal{L}(\ell^r, \ell^s))$-norm of a matrix if $s \le q$. Moreover, since $\|S\|_{\mathcal{L}(\ell^p,\ell^q)}$ is invariant under permutations of the columns and rows of $S \in \mathcal{L}(\ell^p, \ell^q)$, rearrangements of the rows and columns of $M \in \mathcal{L}(\mathcal{L}(\ell^p, \ell^q), \mathcal{L}(\ell^r, \ell^s))$ leave $\|M\|_{\mathcal{L}(\mathcal{L}(\ell^p,\ell^q),\, \mathcal{L}(\ell^r,\ell^s))}$ invariant.

The following lemma is crucial to our main results.

Lemma 5.27. *Let $p, q, r, s \in [1, \infty]$ with $r \le p$ and $s \le q$. Let $\lambda = (\lambda_j)_{j=1}^{\infty}$ and $\mu = (\mu_k)_{k=1}^{\infty}$ be sequences of real numbers. Then*

$$\|T_{\Delta,n}^{\lambda,\mu}\|_{\mathcal{L}(\mathcal{L}(\ell^p,\ell^q),\, \mathcal{L}(\ell^r,\ell^s))} \le \|\mathcal{T}_{\Delta,n}\|_{\mathcal{L}(\mathcal{L}(\ell^p,\ell^q),\, \mathcal{L}(\ell^r,\ell^s))}$$

for all $n \in \mathbb{N}$.

Proof. Note that $T_{\Delta,n}^{\lambda,\mu}(S) = M * S$ for all $S \in \mathcal{L}(\ell^p, \ell^q)$, where $M = (m_{jk})_{j,k=1}^{\infty}$ is such that $m_{jk} = 1$ if $1 \le j, k \le n$ and $\mu_k \le \lambda_j$, and $m_{jk} = 0$ otherwise. We show that $\|M\|_{\mathcal{L}(\mathcal{L}(\ell^p,\ell^q),\, \mathcal{L}(\ell^r,\ell^s))} \le \|\mathcal{T}_{\Delta,n}\|_{\mathcal{L}(\mathcal{L}(\ell^p,\ell^q),\, \mathcal{L}(\ell^r,\ell^s))}$. Assume that M is nonzero, otherwise the statement is trivial. By Remark 5.26, rearrangement of the rows and columns of M does not change its norm. Hence we may assume that $(\lambda_j)_{j=1}^{n}$ and $(\mu_k)_{k=1}^{n}$ are decreasing. Now M has the property that if $m_{jk} = 1$ then $m_{il} = 1$ for all $i \le j$ and $k \le l \le m_2$. By Proposition 5.25 and Remark 5.26, we may omit repeated rows and columns of M, and doing this repeatedly reduces M to $\mathcal{T}_{\Delta,N}$ for some $1 \le N \le n$. Noting that $\|\mathcal{T}_{\Delta,N}\|_{\mathcal{L}(\mathcal{L}(\ell^p,\ell^q),\, \mathcal{L}(\ell^r,\ell^s))} \le \|\mathcal{T}_{\Delta,n}\|_{\mathcal{L}(\mathcal{L}(\ell^p,\ell^q),\, \mathcal{L}(\ell^r,\ell^s))}$ concludes the proof. □

5.4.2 The case $p < q$

We now combine the theory from the previous sections to deduce our main result, Theorem 5.1.

Theorem 5.28. *Let $p, q \in [1, \infty]$ with $p < q$, and let $f(t) := |t|$ for $t \in \mathbb{R}$. Then there exists a constant $C \geq 0$ such that the following holds (where ℓ^∞ should be replaced by* c_0*). Let $A \in \mathcal{L}_{\mathrm{d}}(\ell^p)$ and $B \in \mathcal{L}_{\mathrm{d}}(\ell^q)$ have real spectrum. Then*

$$\|f(B)S - Sf(A)\|_{\mathcal{L}(\ell^p,\ell^q)} \leq CK_A K_B \|BS - SA\|_{\mathcal{L}(\ell^p,\ell^q)} \tag{5.29}$$

for all $S \in \mathcal{L}(\ell^p, \ell^q)$.

Proof. Simply combine Propositions 5.18 and 5.20 and Lemma 5.27 with Proposition 5.23 (i), using that $\|\mathcal{T}_{\Delta,n}\|_{(p,q)} \leq \|\mathcal{T}_\Delta\|_{(p,q)}$ for all $n \in \mathbb{N}$. □

We can deduce a stronger statement if $p = 1$ or $q = \infty$ in Theorem 5.28. For $f : \mathbb{C} \to \mathbb{C}$ a Lipschitz function, recall the definition of $\|f\|_{\mathrm{Lip}}$ from (5.5). Moreover, let $\varphi_f : \mathbb{C}^2 \to \mathbb{C}$ be given by

$$\varphi_f(\lambda_1, \lambda_2) := \begin{cases} \frac{f(\lambda_1) - f(\lambda_2)}{\lambda_1 - \lambda_2} & \text{if } \lambda_1 \neq \lambda_2 \\ 0 & \text{otherwise} \end{cases}. \tag{5.30}$$

Theorem 5.29. *Let $p, q \in [1, \infty]$ with $p = 1$ or $q = \infty$ (with ℓ^∞ replaced by* c_0*). Let $A \in \mathcal{L}_{\mathrm{d}}(\ell^p)$ and $B \in \mathcal{L}_{\mathrm{d}}(\ell^q)$, and let $f : \mathbb{C} \to \mathbb{C}$ be Lipschitz. Then*

$$\|f(B)S - Sf(A)\|_{\mathcal{L}(\ell^p,\ell^q)} \leq K_A K_B \|f\|_{\mathrm{Lip}} \|BS - SA\|_{\mathcal{L}(\ell^p,\ell^q)} \tag{5.31}$$

for all $S \in \mathcal{L}(\ell^p, \ell^q)$. In particular, for $p = q = 1$,

$$\|f(B) - f(A)\|_{\mathcal{L}(\ell^1)} \leq K_A K_B \|f\|_{\mathrm{Lip}} \|B - A\|_{\mathcal{L}(\ell^1)},$$

and for $p = q = \infty$,

$$\|f(B) - f(A)\|_{\mathcal{L}(c_0)} \leq K_A K_B \|f\|_{\mathrm{Lip}} \|B - A\|_{\mathcal{L}(c_0)}.$$

Proof. Let $\lambda = (\lambda_j)_{j=1}^\infty$ and $\mu = (\mu_k)_{k=1}^\infty$ be sequences such that $A \in \mathcal{L}_{\mathrm{d}}(\ell^p, \lambda, U)$ and $B \in \mathcal{L}_{\mathrm{d}}(\ell^q, \mu, V)$ for certain $U \in \mathcal{L}(\ell^p)$ and $V \in \mathcal{L}(\ell^q)$. By Proposition 5.18, it suffices to prove that $\sup_{n\in\mathbb{N}} \|T^{\lambda,\mu}_{\varphi_f,n}\|_{\mathcal{L}(\mathcal{L}(\ell^p,\ell^q))} \leq \|f\|_{\mathrm{Lip}}$. Fix $n \in \mathbb{N}$ and note that $T^{\lambda,\mu}_{\varphi_f,n}(S) = M * S$ for all $S \in \mathcal{L}(\ell^p, \ell^q)$, where $M = (m_{jk})_{j,k=1}^\infty$ is the matrix given by $m_{jk} = \varphi_f(\lambda_j, \mu_k)$ for $1 \leq j,k \leq n$, and $m_{jk} = 0$ otherwise. Then

$$\sup_{j,k\in\mathbb{N}} |m_{jk}| \leq \sup_{j,k\in\mathbb{N}} |\varphi_f(\lambda_j, \mu_k)| \leq \|f\|_{\mathrm{Lip}}.$$

Remark 5.24 now concludes the proof. □

Remark 5.30. Theorem 5.29 shows that each Lipschitz function f is operator Lipschitz on ℓ^1 and c_0, in the following sense. For fixed $M \geq 1$ and $f : \mathbb{C} \to \mathbb{C}$ Lipschitz, there exists a constant $C \geq 0$ such that

$$\|f(B) - f(A)\|_{\mathcal{L}(\ell^1)} \leq C \|B - A\|_{\mathcal{L}(\ell^1)}$$

for all $A, B \in \mathcal{L}_{\mathrm{d}}(\ell^1)$ such that $K_A, K_B \leq M$, and C is independent of A and B. Similarly for c_0.

For $p < q$ an analogous statement holds. By considering $A, f(A) \in \mathcal{L}(\ell^p)$ and $B, f(B) \in \mathcal{L}(\ell^q)$ as operators from ℓ^p to ℓ^q, and by letting S be the inclusion mapping $\ell^p \hookrightarrow \ell^q$ in Theorems 5.28 and 5.29, one can suggestively write

$$\|f(B) - f(A)\|_{\mathcal{L}(\ell^p,\ell^q)} \leq C \|B - A\|_{\mathcal{L}(\ell^p,\ell^q)},$$

for all $A \in \mathcal{L}_{\mathrm{d}}(\ell^p)$ and $B \in \mathcal{L}_{\mathrm{d}}(\ell^q)$ with $K_A, K_B \leq M$. Here f is the absolute value function for general $p < q$ in $[1, \infty]$ and any Lipschitz function if $p = 1$ or $q = \infty$.

This remark also applies to Corollaries 5.31 and 5.32 below.

In the case of Theorems 5.28 and 5.29 where $p = 2$ or $q = 2$, we can apply our results to compact normal operators. By the spectral theorem, any compact normal operator $A \in \mathcal{L}(\ell^2)$ has an orthonormal basis of eigenvectors, and therefore $A \in \mathcal{L}_{\mathrm{d}}(\ell^2, \lambda, U)$ for some sequence λ of real numbers and an isometry $U \in \mathcal{L}(\ell^2)$. Thus Theorems 5.28 and 5.29 yield the following corollaries.

Corollary 5.31. *Let $p \in (1, 2)$. Then there exists a constant $C \geq 0$ such that the following holds. Let $A \in \mathcal{L}_{\mathrm{d}}(\ell^p)$ have real spectrum and let $B \in \mathcal{L}(\ell^2)$ be compact and selfadjoint. Then*

$$\|f(B)S - Sf(A)\|_{\mathcal{L}(\ell^p,\ell^2)} \leq CK_A \|BS - SA\|_{\mathcal{L}(\ell^p,\ell^2)}$$

for all $S \in \mathcal{L}(\ell^p, \ell^2)$, where $f(t) := |t|$ for $t \in \mathbb{R}$. Moreover,

$$\|f(B)S - Sf(A)\|_{\mathcal{L}(\ell^1,\ell^2)} \leq K_A \|f\|_{\mathrm{Lip}} \|BS - SA\|_{\mathcal{L}(\ell^1,\ell^2)}$$

for each $A \in \mathcal{L}_{\mathrm{d}}(\ell^1)$ and $S \in \mathcal{L}(\ell^1, \ell^2)$, each compact and normal $B \in \mathcal{L}(\ell^2)$ and each Lipschitz function $f : \mathbb{C} \to \mathbb{C}$.

Corollary 5.32. *Let $q \in (2, \infty)$. Then there exists a constant $C \geq 0$ such that the following holds. Let $A \in \mathcal{L}(\ell^2)$ be compact and selfadjoint, and let $B \in \mathcal{L}_{\mathrm{d}}(\ell^q)$ have real spectrum. Then*

$$\|f(B)S - Sf(A)\|_{\mathcal{L}(\ell^2,\ell^q)} \leq CK_B \|BS - SA\|_{\mathcal{L}(\ell^2,\ell^q)}$$

for all $S \in \mathcal{L}(\ell^2, \ell^q)$, where $f(t) := |t|$ for $t \in \mathbb{R}$. Moreover,

$$\|f(B)S - Sf(A)\|_{\mathcal{L}(\ell^2,c_0)} \leq K_B \|f\|_{\mathrm{Lip}} \|BS - SA\|_{\mathcal{L}(\ell^2,c_0)}$$

for each compact and normal $A \in \mathcal{L}_{\mathrm{d}}(\ell^2)$, each $B \in \mathcal{L}_{\mathrm{d}}(c_0)$ and $S \in \mathcal{L}(\ell^2, c_0)$, and each Lipschitz function $f : \mathbb{C} \to \mathbb{C}$.

5.4.3 The case $p \geq q$

We now examine the absolute value function f on $\mathcal{L}(\ell^p, \ell^q)$ for $p \geq q$, and obtain the following result.

Proposition 5.33. *Let $p, q \in (1, \infty]$ with $p \geq q$. Then for each $s < q$ there exists a constant $C \geq 0$ such that the following holds (where ℓ^∞ should be replaced by c_0). Let $A \in \mathcal{L}_{\mathrm{d}}(\ell^p, \lambda, U)$ and $B \in \mathcal{L}_{\mathrm{d}}(\ell^q, \mu, V)$ have real spectrum, and let $S \in \mathcal{L}(\ell^p, \ell^q)$ be such that $V(BS - SA)U^{-1} \in \mathcal{L}(\ell^p, \ell^s)$. Then*

$$\|f(B)S - Sf(A)\|_{\mathcal{L}(\ell^p,\ell^q)} \leq C\|U\|_{\mathcal{L}(\ell^p)}\|V^{-1}\|_{\mathcal{L}(\ell^q)}\|V(BS - SA)U^{-1}\|_{\mathcal{L}(\ell^p,\ell^s)}.$$

In particular, if $p = q$ and $V(B - A)U^{-1} \in \mathcal{L}(\ell^p, \ell^s)$, then

$$\|f(B) - f(A)\|_{\mathcal{L}(\ell^p)} \leq C\|U\|_{\mathcal{L}(\ell^p)}\|V^{-1}\|_{\mathcal{L}(\ell^p)}\|V(B - A)U^{-1}\|_{\mathcal{L}(\ell^p,\ell^s)}.$$

Proof. Let $R := V(BS - SA)U^{-1}$. With notation as in Lemma 5.17,

$$\|f(B)S_n - S_nf(A)\|_{\mathcal{L}(\ell^p,\ell^q)} \leq \|U\|_{\mathcal{L}(\ell^p)}\|V^{-1}\|_{\mathcal{L}(\ell^q)}\|T^{\lambda,\mu}_{\varphi_f,n}(R)\|_{\mathcal{L}(\ell^p,\ell^q)}$$

for each $n \in \mathbb{N}$. Proposition 5.20, Lemma 5.27 (with $p = r$ and with q and s interchanged) and Proposition 5.23 (*iii*) (with q and s interchanged) yield a constant $C' \geq 0$ such that

$$\|T^{\lambda,\mu}_{\varphi_f,n}(R)\|_{\mathcal{L}(\ell^p,\ell^q)} \leq C'(\|R\|_{\mathcal{L}(\ell^p,\ell^q)} + \|R\|_{\mathcal{L}(\ell^p,\ell^s)}).$$

Since $\mathcal{L}(\ell^p, \ell^s) \hookrightarrow \mathcal{L}(\ell^p, \ell^q)$ contractively,

$$\|f(B)S_n - S_nf(A)\|_{\mathcal{L}(\ell^p,\ell^q)} \leq C\|U\|_{\mathcal{L}(\ell^p)}\|V^{-1}\|_{\mathcal{L}(\ell^q)}\|V(BS - SA)U^{-1}\|_{\mathcal{L}(\ell^p,\ell^s)}$$

for all $n \in \mathbb{N}$, where $C := 2C'$. Finally, as in the proof of Proposition 5.18, one lets n tend to infinity to conclude the proof. □

In the same way, appealing to the second part of Proposition 5.23 (*iii*), one deduces the following result.

Proposition 5.34. *Let $p, q \in [1, \infty)$ with $p \geq q$. Then for each $r > p$ there exists a constant $C \geq 0$ such that the following holds (where ℓ^∞ should be replaced by c_0). Let $A \in \mathcal{L}_{\mathrm{d}}(\ell^p, \lambda, U)$ and $B \in \mathcal{L}_{\mathrm{d}}(\ell^q, \mu, V)$ have real spectrum, and let $S \in \mathcal{L}(\ell^p, \ell^q)$ be such that $V(BS - SA)U^{-1} \in \mathcal{L}(\ell^r, \ell^q)$. Then*

$$\|f(B)S - Sf(A)\|_{\mathcal{L}(\ell^p,\ell^q)} \leq C\|U\|_{\mathcal{L}(\ell^p)}\|V^{-1}\|_{\mathcal{L}(\ell^q)}\|V(BS - SA)U^{-1}\|_{\mathcal{L}(\ell^r,\ell^q)}.$$

In particular, if $p = q$ and $V(B - A)U^{-1} \in \mathcal{L}(\ell^r, \ell^q)$, then

$$\|f(B) - f(A)\|_{\mathcal{L}(\ell^p)} \leq C\|U\|_{\mathcal{L}(\ell^p)}\|V^{-1}\|_{\mathcal{L}(\ell^p)}\|V(B - A)U^{-1}\|_{\mathcal{L}(\ell^r,\ell^q)}.$$

We single out the case where $p = q = 2$. Here we write $f(A) = |A|$ for a normal operator $A \in \mathcal{L}(\ell^2)$, since then $f(A)$ is equal to $|A| := \sqrt{A^*A}$. Note that the following result, stated before as (5.6), applies in particular to compact selfadjoint operators. For simplicity of the presentation we only consider $\epsilon \in (0,1]$.

Corollary 5.35. *For each $\epsilon \in (0,1]$ there exists a constant $C \geq 0$ such that the following holds. Let $A \in \mathcal{L}_{\mathrm{d}}(\ell^2, \lambda, U)$ and $B \in \mathcal{L}_{\mathrm{d}}(\ell^2, \mu, V)$ be selfadjoint, with U and V unitaries, and let $S \in \mathcal{L}(\ell^2)$. If $V(BS - SA)U^{-1} \in \mathcal{L}(\ell^2, \ell^{2-\epsilon})$, then*

$$\||B|S - S|A|\|_{\mathcal{L}(\ell^2)} \leq C\|V(BS - SA)U^{-1}\|_{\mathcal{L}(\ell^2, \ell^{2-\epsilon})}$$

and if $V(BS - SA)U^{-1} \in \mathcal{L}(\ell^{2+\epsilon}, \ell^2)$ then

$$\||B|S - S|A|\|_{\mathcal{L}(\ell^2)} \leq C\|V(BS - SA)U^{-1}\|_{\mathcal{L}(\ell^{2+\epsilon}, \ell^2)}.$$

In particular, if $V(B - A)U^{-1} \in \mathcal{L}(\ell^2, \ell^{2-\epsilon})$, then

$$\||B| - |A|\|_{\mathcal{L}(\ell^2)} \leq C\|V(B - A)U^{-1}\|_{\mathcal{L}(\ell^2, \ell^{2-\epsilon})}$$

and if $V(B - A)U^{-1} \in \mathcal{L}(\ell^{2+\epsilon}, \ell^2)$, then

$$\||B| - |A|\|_{\mathcal{L}(\ell^2)} \leq C\|V(B - A)U^{-1}\|_{\mathcal{L}(\ell^{2+\epsilon}, \ell^2)}.$$

Remark 5.36. Let $\mathcal{J}$ be the class of all $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(t) = at + b + \int_{-\infty}^{t} (t - s)\,\mathrm{d}\mu(s) \tag{5.32}$$

for all $t \in \mathbb{R}$, where $a, b \in \mathbb{R}$ and μ is a signed measure of compact support. This class is introduced by Davies in [31, p. 156], and he states that $f : \mathbb{R} \to \mathbb{R}$ satisfies (5.32) for a positive μ if and only if f is convex and linear for large $|t|$. The results in this section for f the absolute value function can be extended to all $f \in \mathcal{J}$, in the same way as in [31, Theorem 17].

5.5 Lipschitz estimates on the ideal of p-summing operators

Let H be a separable infinite dimensional Hilbert space. It was shown in [2] that a matrix $M = (m_{jk})_{j,k=1}^{\infty}$ is a Schur multiplier on the Hilbert-Schmidt class $\mathcal{S}_2 \subseteq \mathcal{L}(H)$ if and only if $\sup_{j,k} |m_{jk}| < \infty$. By [94], $\mathcal{S}_2$ coincides with the Banach ideal $\Pi_p(H)$ of all p-summing operators (see the definition below) for all $p \in [1, \infty)$. Hence a matrix $M = (m_{jk})_{j,k=1}^{\infty}$ is a Schur multiplier on $\Pi_p(H)$ if and only if $\sup_{j,k} |m_{jk}| < \infty$. In Corollary 5.38 below we show that the same statement is true for the Banach ideal $\Pi_p(\ell^{p'}, \ell^p)$ in $\mathcal{L}(\ell^{p'}, \ell^p)$, for $p \in [1, \infty)$.

As a corollary we obtain operator Lipschitz estimates on $\Pi_p(\ell^{p'}, \ell^p)$ for each Lipschitz function f on $\mathbb{C}$.

Let X and Y be Banach spaces and $1 \leq p < \infty$. An operator $S : X \to Y$ is *p-absolutely summing* if there exists a constant C such that for each $n \in \mathbb{N}$ and each collection $\{x_j\}_{j=1}^n \subseteq X$,

$$\Big(\sum_{j=1}^n \|S(x_j)\|_Y^p\Big)^{\frac{1}{p}} \leq C \sup_{\|x^*\|_{X^*} \leq 1} \Big(\sum_{j=1}^n |\langle x^*, x_j\rangle|^p\Big)^{\frac{1}{p}}. \tag{5.33}$$

The smallest such constant is denoted by π_p, and $\Pi_p(X,Y)$ is the space of p-absolutely summing operators from X to Y. We let $\Pi_p(X) := \Pi_p(X,X)$. By Propositions 2.3, 2.4 and 2.6 in [35], $(\Pi_p(X,Y), \pi_p(\cdot))$ is a Banach ideal in $\mathcal{L}(X,Y)$.

Below we consider p-absolutely summing operators from $\ell^{p'}$ to ℓ^p. We first present the following result.

Lemma 5.37. *Let $p \in [1,\infty)$ and $S = (s_{jk})_{j,k=1}^\infty$. Then $S \in \Pi_p(\ell^{p'}, \ell^p)$ (with ℓ^∞ replaced by c_0) if and only if*

$$c_p := \Big(\sum_{j=1}^\infty \sum_{k=1}^\infty |s_{jk}|^p\Big)^{\frac{1}{p}} < \infty.$$

In this case, $\pi_p(S) = c_p$.

Proof. It follows from [35, Example 2.11] that, if $c_p < \infty$ for $p \in (1,\infty)$, then $S \in \Pi_p(\ell^{p'}, \ell^p)$ with $\pi_p(S) \leq c_p$. An inspection of the proof of [35, Example 2.11] shows that this statement in fact also holds for $p = 1$. For the converse, let $n \in \mathbb{N}$ and let $x_j := e_j \in \ell^{p'}$ for $1 \leq j \leq n$. By (5.33) (with $X = \ell^{p'}$ and $Y = \ell^p$),

$$\Big(\sum_{k=1}^n \sum_{j=1}^\infty |s_{jk}|^p\Big)^{\frac{1}{p}} \leq \pi_p(S).$$

Letting n tend to infinity concludes the proof. □

For the following corollary of Lemma 5.37, recall that a matrix M is said to be a Schur multiplier on a subspace $\mathcal{I} \subseteq \mathcal{L}(\ell^p, \ell^q)$ if $S \mapsto M * S$ is a bounded map on $\mathcal{I}$. Recall also the definition of the standard triangular truncation $\mathcal{T}_\Delta$ from (5.28).

Corollary 5.38. *Let $p \in [1,\infty)$ and let $M = (m_{jk})_{j,k=1}^\infty$ be a matrix. Then M is a Schur multiplier on $\Pi_p(\ell^{p'}, \ell^p)$ (with ℓ^∞ replaced by c_0) if and only if $\sup_{j,k\in\mathbb{N}} |m_{jk}| < \infty$. In this case,*

$$\|M\|_{\mathcal{L}(\Pi_p(\ell^{p'}, \ell^p))} = \sup_{j,k\in\mathbb{N}} |m_{jk}|.$$

In particular, $\mathcal{T}_\Delta \in \mathcal{L}(\Pi_p(\ell^{p'}, \ell^p))$ with $\|\mathcal{T}_\Delta\|_{\mathcal{L}(\Pi_p(\ell^{p'}, \ell^p))} = 1$.

Observe that $\mathcal{T}_\Delta \notin \mathcal{L}(\mathcal{L}(\ell^{p'}, \ell^p))$ if $p' \geq p$, by Proposition 5.23 (ii). Nevertheless, $\mathcal{T}_\Delta$ is bounded on the ideal $\Pi_p(\ell^{p'}, \ell^p) \subset \mathcal{L}(\ell^{p'}, \ell^p)$ for all $p \in [1, \infty)$.

We now prove our main result concerning commutator estimates on $\Pi_p(\ell^{p'}, \ell^p)$.

Theorem 5.39. *Let $p \in [1, \infty)$, $A \in \mathcal{L}_{\mathrm{d}}(\ell^{p'})$ (with ℓ^∞ replaced by c_0) and $B \in \mathcal{L}_{\mathrm{d}}(\ell^p)$. Let $f : \mathbb{C} \to \mathbb{C}$ be Lipschitz. Then*

$$\pi_p(f(B)S - Sf(A)) \leq K_A K_B \, \|f\|_{\mathrm{Lip}} \, \pi_p(BS - SA) \tag{5.34}$$

for all $S \in \mathcal{L}(\ell^{p'}, \ell^p)$ such that $BS - SA \in \Pi_p(\ell^{p'}, \ell^p)$.

Proof. Let $A \in \mathcal{L}_{\mathrm{d}}(\ell^{p'}, \lambda, U)$ and $B \in \mathcal{L}_{\mathrm{d}}(\ell^p, \mu, V)$ for certain $\lambda = (\lambda_j)_{j=1}^\infty$, $\mu = (\mu_k)_{k=1}^\infty$, $U \in \mathcal{L}(\ell^{p'})$ and $V \in \mathcal{L}(\ell^p)$. If $(S_m)_{m=1}^\infty \subseteq \Pi_p(\ell^{p'}, \ell^p)$ is a π_p-bounded sequence which SOT-converges to $S \in \mathcal{L}(X, Y)$, then $S \in \Pi_p(\ell^{p'}, \ell^p)$ with $\pi_p(S) \leq \limsup_{m \to \infty} \pi_p(S_m)$, by (5.33). Hence, by Remark 5.19, it suffices to prove that $\sup_{n \in \mathbb{N}} \|T^{\lambda,\mu}_{\varphi_f, n}\|_{\mathcal{L}(\Pi_p(\ell^{p'}, \ell^p))} \leq \|f\|_{\mathrm{Lip}}$, where φ_f is as in (5.30). This is done as in the proof of Theorem 5.29, using Corollary 5.38 instead of Remark 5.24. □

5.6 Matrix inequalities

In this section we apply the theory developed in Sections 5.2, 5.3 and 5.4 to finite dimensional spaces, and in doing so we obtain Lipschitz estimates which are independent of the dimension of the underlying space. The dimension-independent estimates that follow from the results in Section 5.5 can be obtained in the same manner.

5.6.1 Finite dimensional spaces

Let $n \in \mathbb{N}$ and let X be an n-dimensional Banach space with basis $\{e_1, \ldots, e_n\}$ and the corresponding basis projections $\mathcal{P}_k \in \mathcal{L}(X)$ for $1 \leq k \leq n$. Recall that an operator $A \in \mathcal{L}(X)$ is diagonalizable if there exists $U \in \mathcal{L}(X)$ invertible such that

$$UAU^{-1} = \sum_{k=1}^{n} \lambda_k \mathcal{P}_k$$

for some $(\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$. We then write $A \in \mathcal{L}_{\mathrm{d}}(X, (\lambda_j)_{j=1}^n, U)$. Recall also the definition of spectral and scalar type operators from Section 2.2.5.

Lemma 5.40. *Let $A \in \mathcal{L}(X)$. Then A is a spectral operator, and A is a scalar type operator if and only if A is diagonalizable. If $A \in \mathcal{L}_{\mathrm{d}}(X, (\lambda_j)_{j=1}^n, U)$ then the spectral measure E of A is given by $E(W) = 0$ if $W \cap \sigma(A) = \emptyset$, and $E(\{\lambda\}) = \sum_{\lambda_j = \lambda} U^{-1}\mathcal{P}_j U$ for $\lambda \in \sigma(A)$.*

Proof. It was already remarked in Section 5.3 that any diagonalizable operator is a scalar type operator, with spectral measure as specified. By [42, Theorem XV.4.5], an operator $T \in \mathcal{L}(Y)$ on an arbitrary Banach space Y is a spectral operator if and only if $T = S + N$ for a commuting scalar type operator $S \in \mathcal{L}(Y)$ and a generalized nilpotent operator $N \in \mathcal{L}(Y)$, and this decomposition is unique. The Jordan decomposition for matrices yields a commuting diagonalizable S and a nilpotent N such that $A = S + N$, hence A is a spectral operator. If A is a scalar type operator, then the Jordan decomposition for matrices yields a commuting diagonalizable S and a nilpotent N such that $A = S + N$. Since $A = A + 0$ and $A = S + N$ are two decompositions of A as a sum of a commuting scalar-type operator and a nilpotent operator, the uniqueness of such a decomposition [42, Theorem XV.4.5] implies that $N = 0$ and that $A = S$ is diagonalizable. □

Let Y be a finite dimensional Banach space. As in [14], a norm $\|\cdot\|$ on $\mathcal{L}(X, Y)$ is said to be *symmetric* if

- $\|RST\| \leq \|R\|_{\mathcal{L}(Y)} \|S\| \|T\|_{\mathcal{L}(X)}$ for all $R \in \mathcal{L}(Y)$, $S \in \mathcal{L}(X, Y)$ and $T \in \mathcal{L}(X)$;
- $\|x^* \otimes y\| = \|x^*\|_{X^*} \|y\|_Y$ for all $x^* \in X^*$ and $y \in Y$.

Clearly $(\mathcal{L}(X, Y), \|\cdot\|)$ is a Banach ideal in $\mathcal{L}(X, Y)$ in the sense of Section 5.1 if and only if $\|\cdot\|$ is symmetric. Note that, for $A \in \mathcal{L}_{\mathrm{d}}(X, (\lambda_j)_{j=1}^n, U)$,

$$f(A) = U^{-1}\Big(\sum_{k=1}^{n} f(\lambda_k)\mathcal{P}_k\Big)U,$$

as in (5.21). Let $\mathfrak{A} := \mathfrak{A}(\mathbb{C} \times \mathbb{C})$ be as in Section 2.3, and for $f \in \mathcal{B}(\mathbb{C})$ and $(\lambda_1, \lambda_2) \in \mathbb{C}^2$ with $\lambda_1 \neq \lambda_2$ let $\varphi_f(\lambda_1, \lambda_2) := \frac{f(\lambda_2) - f(\lambda_1)}{\lambda_2 - \lambda_1}$, as in (5.14). The following corollary of Theorem 5.10 extends results for selfadjoint operators and unitarily invariant norms (see e.g. [71] and [14, Chapter X]) to diagonalizable operators and symmetric norms. Note that a symmetric norm on $\mathcal{L}(X, Y)$ need not be unitarily invariant.

Proposition 5.41. *Let $f \in \mathcal{B}(\mathbb{C})$ be such that φ_f extends to an element of $\mathfrak{A}$. Let X and Y be finite dimensional Banach spaces, let $\|\cdot\|$ be a symmetric norm on $\mathcal{L}(X, Y)$, and let $A \in \mathcal{L}_{\mathrm{d}}(X)$ and $B \in \mathcal{L}_{\mathrm{d}}(Y)$. Then*

$$\|f(B)S - Sf(A)\| \leq 16\,\nu(A)\nu(B)\|\varphi_f\|_{\mathfrak{A}}\|BS - SA\|$$

for all $S \in \mathcal{L}(X, Y)$. In particular, if $X = Y$,

$$\|f(B) - f(A)\| \leq 16\,\nu(A)\nu(B)\|\varphi_f\|_{\mathfrak{A}}\|B - A\|. \tag{5.35}$$

Corollary 5.42. *There exists a universal constant $C \geq 0$ such that the following holds. Let X and Y be finite dimensional Banach spaces and $\|\cdot\|$ a symmetric norm* *on* $\mathcal{L}(X,Y)$. *Let* $f \in \dot{\mathrm{B}}^1_{\infty,1}(\mathbb{R})$, *and let* $A \in \mathcal{L}_{\mathrm{d}}(X)$ *and* $B \in \mathcal{L}_{\mathrm{d}}(Y)$ *be such that* $\mathrm{sp}(A) \cup \mathrm{sp}(B) \subseteq \mathbb{R}$. *Then*

$$\|f(B)S - Sf(A)\| \leq C\,\nu(A)\nu(B)\|f\|_{\dot{\mathrm{B}}^1_{\infty,1}(\mathbb{R})}\|BS - SA\|$$

for all $S \in \mathcal{L}(X,Y)$. *In particular, if* $X = Y$,

$$\|f(B) - f(A)\| \leq C\,\nu(A)\nu(B)\|f\|_{\dot{\mathrm{B}}^1_{\infty,1}(\mathbb{R})}\|B - A\|.$$

Remark 5.43. Let $W_1, W_2 \subset \mathbb{C}$ be finite sets. Then any $\varphi : W_1 \times W_2 \to \mathbb{C}$ belongs to $\mathfrak{A}(W_1 \times W_2)$. Indeed, one can find a representation as in (2.19) by letting Ω be finite and solving a system of linear equations. Therefore Theorem 5.10 yields an estimate

$$\|f(B)S - Sf(A)\| \leq 16\,\nu(A)\nu(B)\|\varphi_f\|_{\mathfrak{A}(\sigma(A)\times\sigma(B))}\|BS - SA\|$$

as in (5.15) for all $f \in \mathcal{B}(\mathbb{C})$. This might lead one to think that the assumption in Theorem 5.41 that φ_f extends to an element of $\mathfrak{A}$ is not really necessary. However, for general $f \in \mathcal{B}(\mathbb{C})$ the norm $\|\varphi_f\|_{\mathfrak{A}(\sigma(A)\times\sigma(B))}$ may blow up as the number of points in $\sigma(A)$ and $\sigma(B)$ grows to infinity. Indeed, for $f \in \mathcal{B}(\mathbb{C})$ the absolute value function and $\|\cdot\|$ the operator norm, a dimension-independent estimate as in (5.35) does not hold for all selfadjoint operators on all finite dimensional Hilbert spaces [14, (X.25)]. Hence φ_f does not extend to an element of $\mathfrak{A}$, and one cannot expect to obtain Theorem 5.41 for all bounded Borel functions on $\mathbb{C}$.

5.6.2 The absolute value function

We now apply our results for the absolute value function to finite dimensional spaces. First note that Lemma 5.17 and Proposition 5.20 relate commutator estimates for general symmetric norms to triangular truncation operators.

For $n \in \mathbb{N}$ and $p \in [1,\infty)$ let ℓ^n_p denote $\mathbb{C}^n$ with the p-norm

$$\|(x_1,\ldots,x_n)\|_p := \Big(\sum_{j=1}^{n}|x_j|^p\Big)^{1/p} \qquad \big((x_1,\ldots,x_n) \in \mathbb{C}^n\big),$$

and let ℓ^n_∞ be $\mathbb{C}^n$ with the norm

$$\|(x_1,\ldots,x_n)\|_\infty := \max_{1\leq j\leq n}|x_j| \qquad \big((x_1,\ldots,x_n) \in \mathbb{C}^n\big).$$

Applying Theorem 5.28 with $S : \mathbb{C}^n \to \mathbb{C}^n$ the identity operator immediately yields the following result. It shows that, although the operator Lipschitz estimate

$$\|f(B)-f(A)\|_{\mathcal{L}(\ell_p^n,\ell_q^n)} \leq C\,\|B-A\|_{\mathcal{L}(\ell_p^n,\ell_q^n)}$$

does not hold with a constant independent of the dimension n for f the absolute value function, $p=q=2$ and all selfadjoint operators on ℓ_2^n, one can nevertheless obtain such estimates for $p<q$, $p=q=1$ or $p=q=\infty$ by considering diagonalizable operators A and B for which $K_A, K_B \leq M$, for some fixed $M \geq 1$. For A a diagonalizable operator, recall the definition of K_A from (5.23).

Theorem 5.44. *Let $f(t) := t$ for $t \in \mathbb{R}$, and let $p,q \in [1,\infty]$ with $p<q$, $p=q=1$ or $p=q=\infty$. Then there exists a constant $C \geq 0$ such that the following holds. Let $n \in \mathbb{N}$ and let $A \in \mathcal{L}_{\mathrm{d}}(\ell_p^n)$ and $B \in \mathcal{L}_{\mathrm{d}}(\ell_q^n)$ have real spectrum. Then*

$$\|f(B)S-Sf(A)\|_{\mathcal{L}(\ell_p^n,\ell_q^n)} \leq CK_AK_B\,\|BS-SA\|_{\mathcal{L}(\ell_p^n,\ell_q^n)}$$

for all $S \in \mathcal{L}(\ell_p^n,\ell_q^n)$. In particular,

$$\|f(B)-f(A)\|_{\mathcal{L}(\ell_p^n,\ell_q^n)} \leq CK_AK_B\,\|B-A\|_{\mathcal{L}(\ell_p^n,\ell_q^n)}\,.$$

Theorem 5.29 shows that, for $p=1$ or $q=\infty$, Theorem 5.44 extends to all Lipschitz functions $f:\mathbb{C}\to\mathbb{C}$, with $C=\|f\|_{\mathrm{Lip}}$. Corollaries 5.31 and 5.32 imply that for $p=2$ or $q=2$ and A or B selfadjoint, the estimate in Theorem 5.44 simplifies.

For $p \geq q$, Propositions 5.33 and 5.34 yield dimension-independent estimates. We state the estimates which follow from Proposition 5.33, the analogous estimates which follow from Proposition 5.34 should be obvious.

Proposition 5.45. *Let $p,q \in (1,\infty]$ with $p \geq q$. For each $s<q$ there exists a constant $C \geq 0$ such that the following holds. Let $n \in \mathbb{N}$, and let $A \in \mathcal{L}_{\mathrm{d}}(\ell_p^n,\lambda,U)$ and $B \in \mathcal{L}_{\mathrm{d}}(\ell_q^n,\mu,V)$ have real spectrum. Then*

$$\|f(B)S-Sf(A)\|_{\mathcal{L}(\ell_p^n,\ell_q^n)} \leq C\|U\|_{\mathcal{L}(\ell_p^n)}\|V^{-1}\|_{\mathcal{L}(\ell_q^n)}\|V(BS-SA)U\|_{\mathcal{L}(\ell_p^n,\ell_s^n)}$$

for all $S \in \mathcal{L}(\ell_p^n,\ell_q^n)$. In particular,

$$\|f(B)-f(A)\|_{\mathcal{L}(\ell_p^n,\ell_q^n)} \leq C\|U\|_{\mathcal{L}(\ell_p^n)}\|V^{-1}\|_{\mathcal{L}(\ell_q^n)}\|V(B-A)U\|_{\mathcal{L}(\ell_p^n,\ell_s^n)}.$$

In the case $p=q=2$, Corollary 5.35 implies the following. Again we only consider $\epsilon \in (0,1]$, for simplicity, but the result extends in an obvious manner to other $\epsilon>0$. We write $f(A)=|A|=\sqrt{A^*A}$ for a normal operator A on ℓ_2^n.

Corollary 5.46. *For each $\epsilon \in (0,1]$ there exists a constant $C \geq 0$ such that the following holds. Let $n \in \mathbb{N}$. Then*

$$\||B|-|A|\|_{\mathcal{L}(\ell_n^2)} \leq C\min(\|V(B-A)U^{-1}\|_{\mathcal{L}(\ell_n^2,\ell_n^{2-\epsilon})},\|V(B-A)U^{-1}\|_{\mathcal{L}(\ell_n^{2+\epsilon},\ell_n^2)})$$

for all $A \in \mathcal{L}_{\mathrm{d}}(\ell_n^2,\lambda,U)$ and $B \in \mathcal{L}_{\mathrm{d}}(\ell_n^2,\mu,V)$ selfadjoint, with U and V unitaries.

Finally note that, under the assumptions of Corollary 5.46,

$$\||B| - |A|\|_{\mathcal{L}(\ell_n^2)} \leq C\|B - A\|_{\mathcal{L}(\ell_n^2)} \min(\|V\|_{\mathcal{L}(\ell_n^2, \ell_n^{2-\epsilon})}, \|U^{-1}\|_{\mathcal{L}(\ell_n^{2+\epsilon}, \ell_n^2)}).$$

Part IV

Applications to numerical approximation methods

6

Convergence of subdiagonal Padé approximations of C_0-semigroups

In [66, Rem. 3] the question was raised, whether there are complex numbers $\lambda_{n,m}$ and $b_{n,m}$ such that any uniformly bounded C_0-semigroup $(T(t))_{t\in\mathbb{R}_+}$ with generator $-A$ on a Banach space X can be approximated in the strong sense on the domain $\mathrm{D}(A)$ of A by sums of the form

$$\frac{b_{n,1}}{t}\left(\frac{\lambda_{n,1}}{t}+A\right)^{-1}+\cdots+\frac{b_{n,m_n}}{t}\left(\frac{\lambda_{n,m_n}}{t}+A\right)^{-1} \qquad (t>0),$$

as $n\to\infty$, locally uniformly in $t\in\mathbb{R}_+$ (see also [91]). According to [91] such a method is called rational approximation without scaling and squaring, because of the absence of both the successive squaring of the resolvent and the scaling of the generator by $\frac{1}{n}$ that is common to other approximation methods for the choice $n=2^k$, $k\in\mathbb{N}$, see e.g. [23]. Recently, the rational approximation method above has been used to provide new powerful inversion formulas for the vector-valued Laplace transform (see [66, 91]).

Suppose $(r_n)_{n\in\mathbb{N}}$ is a sequence of rational functions such that the degree of the numerator of r_n is less than the degree of its denominator and such that each r_n has pairwise distinct poles $\lambda_{n,m}$ which all lie in the open right half-plane $\mathbb{C}_+$. Developing r_n into partial fractions, there are complex numbers $b_{n,m}$ such that

$$r_n(z)=\frac{b_{n,1}}{\lambda_{n,1}-z}+\cdots+\frac{b_{n,m_n}}{\lambda_{n,m_n}-z} \qquad (z\in\mathbb{C}\setminus\{\lambda_{n,1},\ldots,\lambda_{n,m_n}\}).$$

If $-A$ generates a uniformly bounded C_0-semigroup, the open right half-plane belongs to the resolvent set of $-tA$ for any $t\in\mathbb{R}_+$. By the Hille-Phillips calculus,

$$r_n(-tA)=\frac{b_{n,1}}{t}\left(\frac{\lambda_{n,1}}{t}+A\right)^{-1}+\cdots+\frac{b_{n,m_n}}{t}\left(\frac{\lambda_{n,m_n}}{t}+A\right)^{-1}.$$

Hence the stated problem is solved provided one can prove that there exists a sequence $(r_n)_{n\in\mathbb{N}}$ of rational functions satisfying the properties above and such that, for any generator $-A$ of a uniformly bounded semigroup $(T(t))_{t\in\mathbb{R}_+}$ on a Banach space X,

$$r_n(-tA)x \overset{n\to\infty}{\longrightarrow} T(t)x \tag{6.1}$$

for all $x \in \mathrm{D}(A)$, locally uniformly in $t \in \mathbb{R}_+$.

For bounded generators $-A$ this was shown in [124]. There, the author takes r_n to be the n-th subdiagonal Padé approximation to the exponential function and uses a refinement of an error estimate shown in [91]. For this choice of r_n numerical experiments in [124] indicate that (6.1) holds with rate $\mathcal{O}(\frac{1}{\sqrt{n}})$ for each (unbounded) generator $-A$ of a uniformly bounded C_0-semigroup.

The main result of this chapter is as follows. For $n \in \mathbb{N}$, let r_n be the n-th subdiagonal Padé approximation to the exponential function, defined in Section 6.1.

Theorem 6.1. *Let $\alpha > \frac{1}{2}$. Then there exists a constant $C = C(\alpha) \geq 0$ such that the following holds. Let $(T(t))_{t\in\mathbb{R}_+}$ be a C_0-semigroup of type $(M, 0)$ on a Banach space X, with generator $-A$, and let $x \in \mathrm{D}(A^\alpha)$. Then*

$$\|r_n(-tA)x - T(t)x\| \leq CM\, t^\alpha (n+1)^{-\alpha+\frac{1}{2}}\, \|A^\alpha x\|$$

for all $t \geq 0$ and $n \in \mathbb{N}$ with $n \geq \alpha - \frac{1}{2}$.

For a proof of this result see Theorem 6.9. Theorem 6.1 shows that (6.1) holds with rate $\mathcal{O}(n^{-\alpha+\frac{1}{2}})$ for $x \in \mathrm{D}(A^\alpha)$, locally uniformly in $t \in \mathbb{R}_+$. In particular, the choice $\alpha = 1$ yields the rate suggested by the numerical experiments in [124]. In this case the approximation method without scaling and squaring converges with the same rate as is known for the classical scaling and squaring methods due to Brenner and Thomée [23].

Improvements on Theorem 6.1 are established in the following cases:

(i) The semigroup T is bounded analytic. In this case (6.1) holds on $\mathrm{D}(A^\alpha)$ for each $\alpha > 0$ with rate $\mathcal{O}(n^{-\alpha})$, locally uniformly in $t \in \mathbb{R}_+$. See Theorem 6.12.

(ii) The semigroup T is exponentially γ-stable, as defined in Section 2.6. Then, by results from Chapter 3, for each $\alpha > 0$, (6.1) holds with rate $\bigcap_{a<\alpha} \mathcal{O}(n^{-a})$ on $\mathrm{D}(A^\alpha)$, locally uniformly in $t \in \mathbb{R}_+$. This holds in particular for any exponentially stable C_0-semigroup on a Hilbert space. See Theorem 6.14.

(iii) The operator A has a bounded $\mathcal{R}(\mathbb{C}_+)$-calculus, where $\mathcal{R}(\mathbb{C}_+)$ is the space of bounded rational functions $r \in \mathrm{H}^\infty(\mathbb{C}_+)$ with no poles in $\overline{\mathbb{C}_+}$. In this case (6.1) holds for each $\alpha > 0$ with rate $\mathcal{O}(n^{-\alpha})$ on $\mathrm{D}(A^\alpha)$, locally uniformly in t. In addition, one has local uniform convergence in t on the whole space X. This applies in particular if T is (similar to) a contraction semigroup on a Hilbert space. See Theorem 6.16.

This chapter is organized as follows. Section 6.1 provides some of the basics on Padé approximation and states several lemmas that are crucial for Section 6.2, where the results stated above are proved. To improve readability, the technical lemmas from Section 6.1 are proved in Appendix B. Extensions of the main results to intermediate spaces such as Favard spaces can be found in Section 6.3. Finally, Section 6.4 discusses some applications of our results to the inversion of the vector-valued Laplace transform.

6.1 Padé approximation

Let $n \in \mathbb{N}$. Set

$$\begin{aligned} P_n(z) &= \sum_{j=0}^{n} \frac{(2n+1-j)!n!}{(2n+1)!j!(n-j)!} z^j, \\ Q_n(z) &= \sum_{j=0}^{n+1} \frac{(2n+1-j)!(n+1)!}{(2n+1)!j!(n+1-j)!} (-z)^j \end{aligned} \tag{6.2}$$

for $z \in \mathbb{C}$, and define $r_n := \frac{P_n}{Q_n}$. Then r_n is said to be the n-th *subdiagonal Padé approximant* to the exponential function. Cf. [63, Thm. 3.11], P_n and Q_n are the unique polynomials of degree n and $n+1$ such that $P_n(0) = Q_n(0) = 1$ and

$$|r_n(z) - \mathrm{e}^z| \leq C|z|^{2n+2}$$

for $z \in \mathbb{C}$ in a neighborhood of 0. It was observed by Perron [98, Sect. 75] that

$$r_n(z) - \mathrm{e}^z = \frac{(-1)^{n+2}}{Q_n(z)} \frac{1}{(2n+1)!} z^{2n+2} \mathrm{e}^z \int_0^1 s^n (1-s)^{n+1} \mathrm{e}^{-sz} \, \mathrm{d}s \tag{6.3}$$

for all $z \in \mathbb{C}$ such that $Q_n(z) \neq 0$. By [44, Cor. 3.2], r_n is *$\mathcal{A}$-stable*, which means that r_n is holomorphic in a neighborhood of the closed left halfplane $\mathbb{C} \setminus \mathbb{C}_+$ and $|r_n(z)| \leq 1$ for all $z \in \mathbb{C} \setminus \mathbb{C}_+$. The polynomial Q_n has pairwise distinct roots [63, Thm. 4.11] and, combining Corollaries 1.1 and 3.7 in [44], it follows that these roots are contained in $\mathbb{C}_+$.

The following proposition is proved in Appendix B, as Proposition B.2.

Proposition 6.2. *Let $n \in \mathbb{N}$ and $z \in \overline{\mathbb{C}_+}$. Then*

$$|r_n(-z) - \mathrm{e}^{-z}| \leq \frac{1}{2} \left(\frac{n!}{(2n+1)!} \right)^2 |z|^{2n+2}$$

and

$$|r_n'(-z) - \mathrm{e}^{-z}| \leq \left(\frac{n!}{(2n+1)!} \right)^2 \left(\frac{4}{5} |z|^{2n+2} + (n+1)|z|^{2n+1} \right).$$

For $n \in \mathbb{N}$ and $\alpha > 0$, define $f_{n,\alpha} : \overline{\mathbb{C}_+} \setminus \{0\} \to \mathbb{C}$ by

$$f_{n,\alpha}(z) := \frac{r_n(-z) - \mathrm{e}^{-z}}{z^\alpha} \qquad (z \in \overline{\mathbb{C}_+} \setminus \{0\}). \tag{6.4}$$

Corollary 6.3. *Let $n \in \mathbb{N}$ and $\alpha \in (0, 2n+2)$. Then*

$$|f_{n,\alpha}(z)| \leq 2 \left(\frac{n!}{(2n+1)!} \right)^{\frac{\alpha}{n+1}}$$

for all $z \in \overline{\mathbb{C}_+} \setminus \{0\}$.

Proof. Since r_n is $\mathcal{A}$-stable,

$$|f_{n,\alpha}(z)| = \left| \frac{r_n(-z) - \mathrm{e}^{-z}}{z^\alpha} \right| \leq \frac{2}{|z|^\alpha}$$

for all $z \in \overline{\mathbb{C}_+} \setminus \{0\}$. Hence Proposition 6.2 yields

$$\sup_{z \in \overline{\mathbb{C}_+} \setminus \{0\}} |f_{n,\alpha}(z)| \leq \sup_{z \in \overline{\mathbb{C}_+} \setminus \{0\}} \min \left\{ \frac{2}{|z|^\alpha}, \frac{1}{2} \left(\frac{n!}{(2n+1)!} \right)^2 |z|^{2n+2-\alpha} \right\}. \tag{6.5}$$

Note that $|z|^{-\alpha}$ is a strictly decreasing function and $|z|^{2n+2-\alpha}$ a strictly increasing function of $|z| \in (0, \infty)$. Hence, the supremum on the right-hand side of (6.5) is actually a maximum that is attained at the value of $|z|$ for which

$$\frac{2}{|z|^\alpha} = \frac{1}{2} \left(\frac{n!}{(2n+1)!} \right)^2 |z|^{2n+2-\alpha},$$

that is, for

$$|z| = \left(\frac{2(2n+1)!}{n!} \right)^{\frac{1}{n+1}}.$$

Inserting such a z in (6.5) concludes the proof. □

We collect the following two straightforward lemmas for later use.

Lemma 6.4. *Let $u, v, U, V > 0$ and $w \in \left[0, \left(\frac{V}{U}\right)^{\frac{1}{u+v}}\right]$. Then*

$$\int_w^\infty \min\left\{Ur^u, Vr^{-v}\right\} \frac{\mathrm{d}r}{r} = V \left(\frac{U}{V} \right)^{\frac{v}{u+v}} \frac{u+v}{uv} - \frac{U}{u} w^u.$$

Proof. Let $r_0 \geq 0$ be such that $Ur_0^u = Vr_0^{-v}$ and note that $r_0 \geq w$. Hence

$$\int_w^\infty \min\left\{Ur^u, Vr^{-v}\right\} \frac{\mathrm{d}r}{r} = \int_w^{r_0} Ur^u \frac{\mathrm{d}r}{r} + \int_{r_0}^\infty Vr^{-v} \frac{\mathrm{d}r}{r}.$$

Calculating the integrals on the right-hand side and simplifying yields the claim. □

Lemma 6.5. *Let $n \in \mathbb{N}$. Then*

$$\left(\frac{n!}{(2n+1)!}\right)^{\frac{1}{n+1}} \leq \frac{1}{n+1}.$$

Proof. Simply note that

$$\left(\frac{n!}{(2n+1)!}\right)^{\frac{1}{n+1}} (n+1) = \left(\frac{n!(n+1)^{n+1}}{(2n+1)!}\right)^{\frac{1}{n+1}} \leq 1. \qquad \square$$

For the proof of the main result of this chapter, the following L^2-estimates for the restriction of $f_{n,\alpha}$ to the imaginary axis are crucial. This proposition is proved in Appendix B as Proposition B.5.

Proposition 6.6. *Let $n \in \mathbb{N}$ and $\alpha \in (\frac{1}{2}, n+\frac{1}{2}]$. Then*

$$\|f_{n,\alpha}(\mathrm{i}\,\cdot)\|_2 \leq \frac{4}{\sqrt{2\alpha-1}}(n+1)^{-\alpha+\frac{1}{2}}$$

and

$$\left\|(f_{n,\alpha}(\mathrm{i}\,\cdot))'\right\|_2 \leq \left(\frac{8\alpha}{(2\alpha+1)^{3/2}} + \frac{13^\alpha}{10^\alpha}\sqrt{\frac{5^{2\alpha}}{6\cdot 13^{2\alpha}} + \frac{360}{13(2\alpha-1)}}\right)(n+1)^{-\alpha+\frac{1}{2}}.$$

6.2 Convergence of Padé approximations

In this section we prove the results stated in the introduction to this chapter. In particular, we prove Theorem 6.1. Then we improve this theorem for analytic semigroups, exponentially γ-stable semigroups, and semigroups whose generator has a bounded calculus for the class of bounded rational functions on $\mathbb{C}_+$.

6.2.1 Uniformly bounded semigroups

To prove Theorem 6.1 we require the following lemma.

Lemma 6.7. *Let $f \in \mathrm{H}^\infty(\mathbb{C}_+)$ be such that $f(\mathrm{i}\cdot) \in \mathrm{W}^{1,2}(\mathbb{R})$. Then there exists a $g \in \mathrm{L}^1(\mathbb{R}_+)$ with $f(z) = \hat{g}(z)$ for all $z \in \mathbb{C}_+$, and*

$$\|g\|_{\mathrm{L}^1(\mathbb{R}_+)} \leq \frac{1}{\sqrt{2}} \|f(\mathrm{i}\,\cdot)\|^{\frac{1}{2}}_{\mathrm{L}^2(\mathbb{R})} \left\|(f(\mathrm{i}\,\cdot))'\right\|^{\frac{1}{2}}_{\mathrm{L}^2(\mathbb{R})}.$$

Proof. The Paley-Wiener Theorem ([104, Theorem 5.28]) yields a $g \in \mathrm{L}^2(\mathbb{R}_+)$ such that $f(z) = \hat{g}(z)$ for all $z \in \mathbb{C}_+$. Let $h(s) = f(\mathrm{i}s)$ for $s \in \mathbb{R}$. Then $\mathcal{F}^{-1}h$ is the extension of g to the whole real axis by setting g equal to zero off $\mathbb{R}_+$. Plancherel's Theorem implies that $\mathcal{F}^{-1}h \in \mathrm{L}^2(\mathbb{R})$ with

$$[\xi \mapsto -\mathrm{i}\xi(\mathcal{F}^{-1}h)(\xi) = (\mathcal{F}^{-1}h')(\xi)] \in \mathrm{L}^2(\mathbb{R}).$$

Therefore Carlson's inequality [8, p.175] yields $\mathcal{F}^{-1}h \in \mathrm{L}^1(\mathbb{R})$ with

$$\|\mathcal{F}^{-1}h\|_{\mathrm{L}^1(\mathbb{R})} \leq \sqrt{\pi}\|\mathcal{F}^{-1}h\|^{\frac{1}{2}}_{\mathrm{L}^2(\mathbb{R})}\|\mathcal{F}^{-1}h'\|^{\frac{1}{2}}_{\mathrm{L}^2(\mathbb{R})}.$$

Thus, by definition of h and by Plancherel's Theorem,

$$\|g\|_{\mathrm{L}^1(\mathbb{R}_+)} = \|\mathcal{F}^{-1}(f(\mathrm{i}\,\cdot))\|_{\mathrm{L}^1(\mathbb{R})} \leq \frac{1}{\sqrt{2}}\|f(\mathrm{i}\,\cdot)\|^{\frac{1}{2}}_{\mathrm{L}^2(\mathbb{R})}\|(f(\mathrm{i}\,\cdot))'\|^{\frac{1}{2}}_{\mathrm{L}^2(\mathbb{R})}. \qquad \square$$

Remark 6.8. Lemma 6.7 is very similar to Lemmas 1 and 2 in [23]. However, we use that $\mathrm{supp}(\mathcal{F}^{-1}h) \subseteq \mathbb{R}_+$. This results in a constant smaller by a factor of $\frac{1}{2}$ in Carlson's inequality and consequently also in the L^1-estimate for g.

We are now ready to prove Theorem 6.1, with an explicit estimate for the constant involved.

Theorem 6.9. *Let $(T(t))_{t\in\mathbb{R}_+}$ be a C_0-semigroup of type $(M,0)$ on a Banach space X, with generator $-A$. Let $\alpha > \frac{1}{2}$ and $x \in \mathrm{D}(A^\alpha)$. Then*

$$\|r_n(-tA)x - T(t)x\| \leq C(\alpha)Mt^\alpha(n+1)^{-\alpha+\frac{1}{2}}\,\|A^\alpha x\| \tag{6.6}$$

for all $t \in \mathbb{R}_+$ and $n \in \mathbb{N}$ with $n \geq \alpha - \frac{1}{2}$. Here,

$$C(\alpha) := \frac{\sqrt{2}}{(2\alpha-1)^{1/4}}\sqrt{\frac{8\alpha}{(2\alpha+1)^{3/2}} + \frac{13^\alpha}{10^\alpha}\sqrt{\frac{5^{2\alpha}}{6\cdot 13^{2\alpha}} + \frac{360}{13(2\alpha-1)}}}. \tag{6.7}$$

In particular, for each $\alpha > \frac{1}{2}$ the sequence $(r_n(-tA))_{n\in\mathbb{N}}$ converges strongly on $\mathrm{D}(A^\alpha)$ and locally uniformly in $t \in \mathbb{R}_+$ to $T(t)$ with rate $\mathcal{O}(n^{-\alpha+\frac{1}{2}})$.

Proof. Fix $t \in \mathbb{R}_+$ and $n \in \mathbb{N}$ with $n \geq \alpha - \frac{1}{2}$. Since r_n is $\mathcal{A}$-stable and $r_n = \frac{P_n}{Q_n}$ with $\deg(Q_n) = n+1 > n = \deg(P_n)$, it follows from Lemma 6.7 that $r_n(-t\cdot)$ is the Laplace transform of a bounded measure. Note also that

$e^{-t\cdot}$ is the Laplace transform of the unit point mass at t. Hence $r_n(-tA) = r_n(-t\,\cdot)(A)$ and $\mathrm{e}^{-t\,\cdot}(A) = T(t)$ are well-defined in the Hille-Phillips calculus for A. Set

$$f(z) := t^\alpha f_{n,\alpha}(tz) = \frac{r_n(-tz) - \mathrm{e}^{-tz}}{z^\alpha} \qquad (z \in \overline{\mathbb{C}_+} \setminus \{0\})$$

with $f_{n,\alpha}$ as in (6.4). Then $f \in \mathrm{H}^\infty(\mathbb{C}_+) \cap \mathrm{C}(\overline{\mathbb{C}_+})$, by Corollary 6.3, and $f(z) \in \mathcal{O}(|z|^{-\alpha})$ as $|z| \to \infty$, since r_n is $\mathcal{A}$-stable. Moreover, $f(\mathrm{i}\,\cdot) \in \mathrm{W}^{1,2}(\mathbb{R})$ by Proposition 6.6. Thus Lemma 6.7 yields a function $g \in \mathrm{L}^1(\mathbb{R}_+)$ such that $f = \hat{g}$ and

$$\begin{aligned} \|g\|_{\mathrm{L}^1(\mathbb{R}_+)} &\leq \frac{1}{\sqrt{2}} \|f(\mathrm{i}\,\cdot)\|_{\mathrm{L}^2(\mathbb{R})}^{\frac{1}{2}} \left\|(f(\mathrm{i}\,\cdot))'\right\|_{\mathrm{L}^2(\mathbb{R})}^{\frac{1}{2}} \\ &= \frac{t^\alpha}{\sqrt{2}} \|f_{n,\alpha}(\mathrm{i}\,\cdot)\|_{\mathrm{L}^2(\mathbb{R})}^{\frac{1}{2}} \left\|(f_{n,\alpha}(\mathrm{i}\,\cdot))'\right\|_{\mathrm{L}^2(\mathbb{R})}^{\frac{1}{2}}. \end{aligned}$$

By the definition of the Hille-Phillips calculus,

$$(r_n(-tA) - T(t))x = f(A)A^\alpha x = \hat{g}(A)A^\alpha x = \int_0^\infty g(s)T(s)A^\alpha x \,\mathrm{d}s. \tag{6.8}$$

Hence, by taking norms,

$$\|r_n(-tA)x - T(t)x\| \leq \frac{Mt^\alpha}{\sqrt{2}} \|f_{n,\alpha}(\mathrm{i}\,\cdot)\|_{\mathrm{L}^2(\mathbb{R})}^{\frac{1}{2}} \left\|(f_{n,\alpha}(\mathrm{i}\,\cdot))'\right\|_{\mathrm{L}^2(\mathbb{R})}^{\frac{1}{2}} \|A^\alpha x\|.$$

The conclusion follows by using Proposition 6.6 for the right-hand side of this inequality. □

Remark 6.10. For applications to numerical analysis, one might inquire about the size of $C(\alpha)$ for $\alpha = 1$ and other small values. It is easily checked that

$$C(1) \leq 4.10, \quad C(2) \leq 2.76, \quad C(3) \leq 2.41, \quad C(4) \leq 2.28.$$

Remark 6.11. Suppose that, in the setting of Theorem 6.9, $T = (T(t))_{t\in\mathbb{R}_+}$ is exponentially stable. Then, depending on the type of T, it may be possible to provide a sharper upper bound for the approximation error. More precisely, let T be of type $(M, -\omega)$ for some $\omega > 0$. With f, $f_{n,\alpha}$ and g as in the proof of Theorem 6.9, Lemma 6.7 and Plancherel's Theorem yield

$$\|g\|_{\mathrm{L}^2(\mathbb{R}_+)} = \frac{1}{\sqrt{2\pi}} \|f(\mathrm{i}\,\cdot)\|_{\mathrm{L}^2(\mathbb{R})} = \frac{t^{\alpha-\frac{1}{2}}}{\sqrt{2\pi}} \|f_{n,\alpha}(\mathrm{i}\,\cdot)\|_{\mathrm{L}^2(\mathbb{R})}.$$

Taking norms in (6.8) and applying Hölder's inequality yields

$$\begin{aligned} \|r_n(-tA)x - T(t)x\| &\leq M \|ge_{-\omega}\|_{\mathrm{L}^1(\mathbb{R})} \\ &\leq \frac{M}{\sqrt{4\pi\omega}} \frac{4}{\sqrt{2\alpha-1}} t^{\alpha-\frac{1}{2}} (n+1)^{-\alpha+\frac{1}{2}} \|A^\alpha x\| \end{aligned}$$

for all $\alpha > \frac{1}{2}$, $n \geq \alpha - \frac{1}{2}$, $t \in \mathbb{R}_+$ and $x \in \mathrm{D}(A^\alpha)$, which is a sharper estimate than (6.6) for sufficiently large $\omega > 0$.

6.2.2 Analytic semigroups

For A a sectorial operator of angle $\varphi \in (0, \pi)$ on a Banach space X, and $\psi \in (\varphi, \pi]$, let

$$M(A, \psi) := \sup_{\lambda \in \mathbb{C} \setminus \mathrm{S}_\psi} \|\lambda R(\lambda, A)\| < \infty. \tag{6.9}$$

Note that $M_A := M(A, \pi) = \sup_{t>0} \left\|t(t+A)^{-1}\right\|$ and that

$$M(A, \pi) = \inf_{\psi \in (\varphi, \pi)} M(A, \psi), \tag{6.10}$$

by [55, Remark 2.1.2].

For bounded analytic semigroups we deduce the following improvement of Theorem 6.9.

Theorem 6.12. *Let A be a densely defined sectorial operator of angle $\varphi \in (0, \frac{\pi}{2})$ on a Banach space X and let $(T(t))_{t \in \mathbb{R}_+} \subseteq \mathcal{L}(X)$ be the bounded analytic C_0-semigroup generated by $-A$. Let $\alpha > 0$ and $x \in \mathrm{D}(A^\alpha)$. Then*

$$\|r_n(-tA)x - T(t)x\| \leq \frac{4M_A}{\alpha\pi} t^\alpha (n+1)^{-\alpha} \|A^\alpha x\|$$

for all $t \in \mathbb{R}_+$ and $n \in \mathbb{N}$ such that $n \geq \alpha - 1$.

In particular, for each $\alpha > 0$ the sequence $(r_n(-tA))_{n \in \mathbb{N}}$ converges strongly on $\mathrm{D}(A^\alpha)$ and locally uniformly in $t \in \mathbb{R}_+$ to $T(t)$ with rate $\mathcal{O}(n^{-\alpha})$.

Proof. Fix $n \geq \alpha - 1$. Observe that for $t = 0$ the statement is trivial, while for $t > 0$ the operator tA is also sectorial of angle φ with $M_A = M_{tA}$. Hence it suffices to consider $t = 1$.

Let $\nu \in (\varphi, \frac{\pi}{2})$ and let $f_{n,\alpha}$ be as in (6.4). Proposition 6.2 and the $\mathcal{A}$-stability of r_n yield

$$|f_{n,\alpha}(z)| \leq \min\left\{\frac{1}{2}\left(\frac{n!}{(2n+1)!}\right)^2 |z|^{2n+2-\alpha}, 2|z|^{-\alpha}\right\} \tag{6.11}$$

for $z \in \mathbb{C}_+$, so that $f_{n,\alpha} \in \mathrm{H}_0^\infty(\mathrm{S}_\nu)$. By the holomorphic functional calculus for sectorial operators,

$$(r_n(-A) - T(1))x = f_{n,\alpha}(A)A^\alpha x = \frac{1}{2\pi \mathrm{i}} \int_{\partial \mathrm{S}_\psi} f_{n,\alpha}(z) R(z, A) A^\alpha x \, \mathrm{d}z$$

for $\psi \in (\nu, \frac{\pi}{2})$. By taking operator norms and estimating the integrand on the right-hand side by means of (6.9) and (6.11), $\|(r_n(-A) - T(1))\, x\|$ is bounded from above by

$$\frac{M(A,\psi)}{\pi} \|A^\alpha x\| \int_0^\infty \min\left\{\frac{1}{2}\left(\frac{n!}{(2n+1)!}\right)^2 r^{2n+2-\alpha}, 2r^{-\alpha}\right\} \frac{\mathrm{d}r}{r}.$$

By Lemma 6.4,

$$\begin{aligned}\|(r_n(-A) - T(1))x\| &\leq \frac{4M(A,\psi)}{\alpha\pi}\frac{n+1}{2n+2-\alpha}\left(\frac{n!}{2(2n+1)!}\right)^{\frac{\alpha}{n+1}}\|A^\alpha x\| \\ &\leq \frac{4M(A,\psi)}{\alpha\pi}\left(\frac{n!}{(2n+1)!}\right)^{\frac{\alpha}{n+1}}\|A^\alpha x\|.\end{aligned}$$

The last term can be estimated by means of Lemma 6.5. This leads to

$$\|(r_n(-A) - T(1))x\| \leq \frac{4M(A,\psi)}{\alpha\pi}(n+1)^{-\alpha}\|A^\alpha x\|.$$

Taking the infimum over all $\psi \in (\nu, \frac{\pi}{2})$ and using (6.10) yields the required estimate for $t = 1$, and concludes the proof. □

Remark 6.13. In the situation of Theorem 6.12 the scaling and squaring methods associated to a fixed subdiagonal Padé approximant converge strongly on X and even in $\mathcal{L}(X)$, see [77, Thm. 4.4]. Whether this is true for the method without scaling and squaring as well is an open problem. In Section 6.2.4 we prove strong convergence on X if A has a bounded calculus for the collection of bounded rational functions on $\mathbb{C}_+$.

6.2.3 Exponentially γ-stable semigroups

In this section we improve Theorem 6.9 for the exponentially γ-stable C_0-semigroups from Section 2.6. Recall the definition of γ-type from Section 2.6. Note that the following result applies in particular to exponentially stable C_0-semigroups on Hilbert spaces.

Theorem 6.14 (Convergence for exponentially γ-stable semigroups). *Let $-A$ generate a C_0-semigroup $T = (T(t))_{t\in\mathbb{R}_+}$ of γ-type $(M, -\omega)$, for $M \geq 1$ and $\omega > 0$, on a Banach space X. Let $\alpha > 0$, $a \in (0, \alpha)$ and $x \in \mathrm{D}(A^\alpha)$ be given. Then there is a constant $C = C(M, \omega, \alpha - a)$ such that*

$$\|r_n(-tA)x - T(t)x\| \leq Ct^a(n+1)^{-a}\|A^\alpha x\|$$

for all $t \in \mathbb{R}_+$ and all $n \in \mathbb{N}$ such that $n > \frac{a}{2} - 1$.

In particular, for each $\alpha > 0$ the sequence $(r_n(-tA))_{n\in\mathbb{N}}$ converges strongly on $\mathrm{D}(A^\alpha)$ and locally uniformly in $t \in \mathbb{R}_+$ to $T(t)$ with rate $\bigcap_{a<\alpha}\mathcal{O}(n^{-a})$.

Proof. The proof is very similar to that of Theorem 6.9, appealing to Corollary 3.25 instead of Lemma 6.7. Fix $t \in \mathbb{R}_+$ and $n > \frac{a}{2} - 1$, and let

$$f(z) := t^a f_{n,a}(tz) = \frac{r_n(-tz) - \mathrm{e}^{-tz}}{z^a} \qquad (z \in \overline{\mathbb{C}_+} \setminus \{0\}).$$

Then Corollary 3.25 yields

$$\left\|(r_n(-tA) - T(t))A^{-\alpha}\right\| = \left\|f(A)A^{a-\alpha}\right\| \leq C(M,\omega,\alpha - a)\, \|f\|_{\mathrm{H}^\infty(\mathbb{C}_+)} .$$

Lemma 6.3 and Lemma 6.5 imply

$$\|f\|_{\mathrm{H}^\infty(\mathbb{C}_+)} = \sup_{z\in\mathbb{C}_+} t^a \left|\frac{r_n(-tz) - \mathrm{e}^{-tz}}{(tz)^a}\right| \leq 2t^a \left(\frac{n!}{(2n+1)!}\right)^{\frac{a}{n+1}} \leq 2t^a(n+1)^{-a}.$$

Combining these two estimates yields

$$\|(r_n(-tA) - T(t))x\| = \left\|(r_n(-tA) - T(t))A^{-\alpha}A^\alpha x\right\| \leq Ct^a(n+1)^{-a}\, \|A^\alpha x\|$$

for some constant $C = C(M,\omega,\alpha - a) \geq 0$. □

Remark 6.15. Theorem 6.14 yields convergence rate $\mathcal{O}(n^{-a})$ on $\mathrm{D}(A^\alpha)$ for arbitrary $a < \alpha$ but does not make any statement concerning the limit case $a = \alpha$. If, in addition to the hypotheses from Theorem 6.14, the operators $T(t)$, for $t > 0$, are invertible and that the collection $\left\{e^{-\omega_0 t}T(t)^{-1} \mid t > 0\right\}$ is γ-bounded for some $\omega_0 \in \mathbb{R}$. Then Corollary 2.33 implies that A has a bounded $\mathrm{H}^\infty(\mathbb{C}_+)$-calculus. Hence, Theorem 6.16 from the next section implies convergence of order $\mathcal{O}(n^{-\alpha})$ on $\mathrm{D}(A^\alpha)$ and even strong convergence on the whole space X. This remark applies in particular to exponentially stable semigroups T on a Hilbert space for which all the operators $T(t)$, $t \in \mathbb{R}_+$, are invertible.

6.2.4 Generators with a bounded $\mathcal{R}(\mathbb{C}_+)$-calculus

Let $-A$ be the generator of a uniformly bounded semigroup $T = (T(t))_{t\in\mathbb{R}_+}$ on a Banach space X. Up until now, in the results of this chapter we have obtained

$$r_n(-tA)x \overset{n\to\infty}{\longrightarrow} T(t)x$$

only for x belonging to some proper subspace of X, at least if A is unbounded. In this section we show that this convergence can be extended to all $x \in X$ if A has a bounded $\mathcal{R}(\mathbb{C}_+)$-calculus, where $\mathcal{R}(\mathbb{C}_+)$ is the space of all bounded rational functions on $\mathbb{C}_+$. In other words, we assume that (2.7) holds:

$$\|r(A)\|_{\mathcal{L}(X)} \leq C\, \|r\|_{\mathrm{H}^\infty(\mathbb{C}_+)}$$

for all $r \in \mathcal{R}(\mathbb{C}_+)$. We call the smallest constant C in this inequality the H^∞-*bound* of A. For such A the following improvement of Theorem 6.9 holds. We note that this result for instance applies to generators of contraction semigroups on Hilbert spaces (with $C = 1$), by Proposition 2.6, to generators of bounded analytic semigroups which satisfy the square function estimates in Theorem 1.1, and to the generators of symmetric contraction semigroups from Theorem 1.2.

Theorem 6.16. *Let $-A$ generate a uniformly bounded C_0-semigroup $(T(t))_{t\in\mathbb{R}_+}$ on a Banach space X and suppose that A has a bounded $\mathcal{R}(\mathbb{C}_+)$-calculus with H^∞-bound C. Let $\alpha > 0$ and $x \in \mathrm{D}(A^\alpha)$ be given. Then*

$$\|r_n(-tA)x - T(t)x\| \leq 2Ct^\alpha(n+1)^{-\alpha}\,\|A^\alpha x\| \tag{6.12}$$

for all $t \in \mathbb{R}_+$ and all $n \in \mathbb{N}$ such that $n > \frac{\alpha}{2} - 1$.

In particular, for each $\alpha > 0$ the sequence $(r_n(-tA))_{n\in\mathbb{N}}$ converges to $T(t)$ strongly on $\mathrm{D}(A^\alpha)$ and locally uniformly in $t \in \mathbb{R}_+$ to $T(t)$ with rate $\mathcal{O}(n^{-\alpha})$. Moreover, $(r_n(-tA))_{n\in\mathbb{N}}$ converges strongly on X and locally uniformly in $t \in \mathbb{R}_+$ to $T(t)$.

Proof. As noted in Section 2.2.1, under the present assumptions one can extend the functional calculus for A to all functions $f \in \mathrm{H}^\infty(\mathbb{C}_+) \cap \mathrm{C}(\overline{\mathbb{C}_+})$ by taking uniform limits of rational functions (see [55, Proposition F.3]) and then regularizing. Then

$$\|f(A)\|_{\mathcal{L}(X)} \leq C\,\|f\|_{\mathrm{H}^\infty(\mathbb{C}_+)} \tag{6.13}$$

for all $f \in \mathcal{A}(\mathbb{C}_+)$, where $\mathcal{A}(\mathbb{C}_+)$ consists of all $f \in \mathrm{H}^\infty(\mathbb{C}_+) \cap \mathrm{C}(\overline{\mathbb{C}_+})$ for which $\lim_{z\to\infty} f(z)$ exists.

Fix $t \in \mathbb{R}_+$ and $n > \frac{\alpha}{2} - 1$ and let

$$f(z) := \frac{r_n(-tz) - \mathrm{e}^{-tz}}{z^\alpha} \qquad (z \in \overline{\mathbb{C}_+} \setminus \{0\}).$$

Then $f \in \mathcal{A}(\mathbb{C}_+)$ by Lemma 6.2 and by $\mathcal{A}$-stability of the r_n. Hence,

$$\|r_n(-tA)x - T(t)x\| = \|f(A)A^\alpha x\| \leq C\,\|f\|_{\mathrm{H}^\infty(\mathbb{C}_+)}\,\|A^\alpha x\|,$$

and (6.12) follows by estimating $\|f\|_{\mathrm{H}^\infty(\mathbb{C}_+)}$ using Lemma 6.3 and Lemma 6.5.

Finally, we prove that $(r_n(-tA))_{n\in\mathbb{N}}$ converges strongly to $T(t)$ on X, locally uniformly in $t \in \mathbb{R}_+$. To this end note that, for K a bounded subset of $\mathbb{R}_+$, the family

$$\{r_n(-tA) - T(t) \mid n \in \mathbb{N}, t \in K\} \subseteq \mathcal{L}(X)$$

is bounded due to (6.13). Since $r_n(-tA)$ converges to $T(t)$ strongly on $\mathrm{D}(A)$ and uniformly in $t \in K$ as $n \to \infty$, the density of $\mathrm{D}(A)$ in X implies that this convergence extends to all of X. □

6.3 Extension to other intermediate spaces

In this section the results from the previous sections are extended to other classes of intermediate spaces. Throughout, let $-A$ be the generator of a uniformly bounded C_0-semigroup $T = (T(t))_{t\in\mathbb{R}_+}$ on a Banach space X. For $k \in \mathbb{N}$ the k-th *Favard space* is

$$\mathrm{F}_k := \Big\{ x \in \mathrm{D}(A^{k-1}) \mid L(A^{k-1}x) := \limsup_{t\downarrow 0} \tfrac{1}{t}\|T(t)A^{k-1}x - A^{k-1}x\| < \infty \Big\}.$$

Then $\mathrm{D}(A^k) \subseteq \mathrm{F}_k$ and for non-reflexive Banach spaces this inclusion can be strict, see e.g. Section 6.4 below. It is therefore noteworthy that the convergence results from the previous sections immediately extend from $\mathrm{D}(A^k)$ to F_k upon replacing $\|A^k x\|$ by $L(A^{k-1}x)$. This is due to the following lemma from [73, Prop. 1].

Lemma 6.17. *Let $S \in \mathcal{L}(X)$ and suppose there exist $k \in \mathbb{N}$ and $C \geq 0$ such that $\|Sx\| \leq C\|A^k x\|$ for all $x \in \mathrm{D}(A^k)$. Then $\|Sx\| \leq CL(A^{k-1}x)$ for all $x \in \mathrm{F}_k$.*

Proof. Approximate $x \in \mathrm{F}_k$ by elements

$$x_t := \frac{1}{t}\int_0^t T(s)x\,\mathrm{d}s \in \mathrm{D}(A^k) \qquad (t > 0)$$

and note that

$$\|Sx_t\| \leq C\|A^k x_t\| = C\|\tfrac{1}{t}(T(t)A^{k-1}x - A^{k-1}x)\| \qquad (t > 0).$$

The conclusion follows by passing to the limit superior as $t \downarrow 0$. □

Using (2.10) and (2.38) as well as [55, Proposition B.3.5], the convergence results in this chapter carry over to the domains of certain complex fractional powers, as well as to real interpolation spaces. This includes Favard spaces of non-integer order, as discussed in [72, Section 3.3].

6.4 Application to the inversion of the Laplace transform

Following an idea from [91], we show how the results in this chapter can be used to obtain inversion formulas for the vector-valued Laplace transform, with precise error-estimates.

For X a Banach space and $k \in \mathbb{N}$, denote by $\mathrm{C}^{k,1}_{\mathrm{ub}}(\mathbb{R}_+; X)$ the space of all functions $f \in \mathrm{C}^k_{\mathrm{ub}}(\mathbb{R}_+; X)$ for which $f^{(k)}$ is Lipschitz. For $f : \mathbb{R}_+ \to X$ define

$$L(f) := \limsup_{t\downarrow 0} \frac{1}{t}\|f(t+\cdot) - f\|_\infty \in [0, \infty].$$

On $\mathrm{C}_{\mathrm{ub}}(\mathbb{R}_+; X)$ we consider the differential operator $Af := -f'$ with maximal domain $\mathrm{C}^1_{\mathrm{ub}}(\mathbb{R}_+; X)$. Then $-A$ generates the strongly continuous left translation semigroup $T_l = (T_l(t))_{t\in\mathbb{R}_+}$, where $(T_l(t)f)(\cdot) = f(t + \cdot)$. By definition, the associated Favard spaces are given by

$$\mathrm{F}_k = \{ f \in \mathrm{C}^{k-1}_{\mathrm{ub}}(\mathbb{R}_+; X) \mid L(f^{(k-1)}) < \infty \} = \mathrm{C}^{k-1,1}_{\mathrm{ub}}(\mathbb{R}_+; X) \qquad (k \in \mathbb{N}).$$

Moreover, T_l is of type $(1, 0)$ and if $f \in \mathrm{C}_{\mathrm{ub}}(\mathbb{R}_+; X)$ and $\lambda \in \mathbb{C}_+$, then

$$((\lambda + A)^{-1} f)(0) = \int_0^\infty \mathrm{e}^{-\lambda t} (T_l(t) f)(0) \,\mathrm{d}t = \int_0^\infty \mathrm{e}^{-\lambda t} f(t) \,\mathrm{d}t = \hat{f}(\lambda),$$

with $\hat{f} : \mathbb{C}_+ \to X$ the (vector-valued) Laplace transform of f. Let

$$r_n(z) = \frac{b_{n,1}}{\lambda_{n,1} - z} + \cdots + \frac{b_{n,n+1}}{\lambda_{n,n+1} - z} \qquad (z \in \mathbb{C} \setminus \{\lambda_{n,1}, \ldots, \lambda_{n,n+1}\})$$

be the partial fraction decomposition of the n-th subdiagonal Padé approximation. Applying Theorem 6.9 to T_l and evaluating at zero yields the following result for all $f \in \mathrm{C}^k_{\mathrm{ub}}(\mathbb{R}_+; X)$. Lemma 6.17 then extends it to all $f \in \mathrm{C}^{k-1,1}_{\mathrm{ub}}(\mathbb{R}_+; X)$.

Corollary 6.18. *Let X be a Banach space and $f \in \mathrm{C}^{k-1,1}_{ub}(\mathbb{R}_+; X)$ for some $k \in \mathbb{N}$. Let $t > 0$ and $n \in \mathbb{N}$ such that $n \geq k - \frac{1}{2}$. Then*

$$\Big\| \sum_{j=1}^{n+1} \frac{b_{n,j}}{t} \hat{f}\left(\frac{\lambda_{n,j}}{t}\right) - f(t) \Big\|_X \leq C(k) t^k (n+1)^{-k+\frac{1}{2}} L(f^{(k-1)}),$$

where $C(k)$ is as in (6.7). *In particular, $\sum_{j=1}^{n+1} \frac{b_{n,j}}{t} \hat{f}\left(\frac{\lambda_{n,j}}{t}\right)$ converges locally uniformly in t to $f(t)$ with rate $\mathcal{O}(n^{-k+\frac{1}{2}})$.*

Remark 6.19. The Laplace inversion formula from Corollary 6.18 actually converges for any $f \in \mathrm{C}_{\mathrm{ub}}(\mathbb{R}_+; X)$ that is α-Hölder continuous for some $\alpha \in (\frac{1}{2}, 1)$ with rate depending on α. This follows again from Theorem 6.9, since such an f is contained in the real interpolation space $(\mathrm{C}_{\mathrm{ub}}(\mathbb{R}_+; X), \mathrm{D}(A))_{\alpha,\infty}$, which continuously embeds into $\mathrm{D}(A^a)$ for any $a < \alpha$, as follows from (2.36) and (2.38).

Remark 6.20. It should be noted that Corollary 6.18 provides a Laplace inversion formula that does not require any knowledge of derivatives of $\hat{f}$ and only uses finite sums as approximants, compare with e.g. [66], [3]. Moreover, $C(k)$ can be computed explicitly (see Remark 6.10).

A

Growth estimates

In this appendix we examine the function $\eta : (0,\infty) \times (0,\infty) \times [1,\infty] \to \mathbb{R}_+$ from (3.4) given by

$$\eta(\alpha, t, q) := \inf \left\{ \|\psi\|_q \, \|\varphi\|_{q'} \mid \psi * \varphi \equiv e_{-\alpha} \text{ on } [t,\infty) \right\}.$$

We will need the following lemma. It is contained in the proof of [59, Lemma A.1] and is due to T. Hytönen.

Lemma A.1. *Let $\theta \in (0,1)$. Then there exist sequences $(\beta_j)_j, (\beta'_j)_j \subseteq \mathbb{R}_+$ such that $\beta_j = O((1+j)^{-\theta})$ and $\beta'_j = O((1+j)^{\theta-1})$ as $j \to \infty$, and such that*

$$\psi_0 := \sum_{j=0}^{\infty} \beta_j \mathbf{1}_{(j,j+1)} \qquad \text{and} \qquad \varphi_0 := \sum_{j=0}^{\infty} \beta'_j \mathbf{1}_{(j,j+1)}$$

satisfy

$$(\psi_0 * \varphi_0)(s) = \begin{cases} s, & s \in [0,1) \\ 1, & s \geq 1 \end{cases}$$

In this appendix we will use the notation $f \lesssim g$ for real-valued functions $f, g : Z \to \mathbb{R}$ on some set Z to indicate that there exists a constant $c \geq 0$ such that $f(z) \leq c g(z)$ for all $z \in Z$.

Lemma A.2. *For each $q \in (1,\infty)$ there exist constants $c_q, d_q \geq 0$ such that*

$$d_q |\log(\alpha t)| \leq \eta(\alpha, t, q) \leq c_q |\log(\alpha t)| \tag{A.1}$$

if $\alpha t \leq \min\left\{\frac{1}{q}, \frac{1}{q'}\right\}$. If $\alpha t > \min\left\{\frac{1}{q}, \frac{1}{q'}\right\}$ then

$$\mathrm{e}^{-\alpha t} \leq \eta(\alpha, t, q) \leq 2\mathrm{e}^{-\alpha t}. \tag{A.2}$$

Proof. First note that $\eta(\alpha,t,q) = \eta(\alpha t,1,q) = \eta(1,\alpha t,q)$ for all α, t and q. Indeed, for $\psi \in \mathrm{L}^q(\mathbb{R}_+)$, $\varphi \in \mathrm{L}^{q'}(\mathbb{R}_+)$ with $\psi * \varphi \equiv e_{-\alpha}$ on $[1,\infty)$ define $\psi_t(s) := \frac{1}{t^{1/q}}\psi(s/t)$ and $\varphi_t(s) := \frac{1}{t^{1/q'}}\varphi(s/t)$ for $s \geq 0$. Then

$$\psi_t * \varphi_t(r) = \int_0^\infty \psi\left(\tfrac{r-s}{t}\right)\varphi\left(\tfrac{s}{t}\right)\tfrac{\mathrm{d}s}{t} = \psi * \varphi\left(\tfrac{r}{t}\right)$$

for all $r \geq 0$, so $\psi_t * \varphi_t \equiv e_{-\alpha}$ on $[t,\infty)$. Moreover,

$$\|\psi_t\|_q^q = \int_0^\infty |\psi(\tfrac{s}{t})|^q\,\tfrac{\mathrm{d}s}{t} = \int_0^\infty |\psi(s)|^q\,\mathrm{d}s = \|\psi\|_q^q,$$

and similarly $\|\varphi_t\|_{q'} = \|\varphi\|_{q'}$. Hence $\eta(\alpha,t,q) \leq \eta(\alpha t,1,q)$. Considering $\psi_{1/t}$ and $\varphi_{1/t}$ yields $\eta(\alpha,t,q) = \eta(\alpha t,1,q)$. The other equality follows immediately. Hence, to prove any of the inequalities in (A.1) or (A.2), we can assume either that $\alpha = 1$ or that $t = 1$ (but not both).

For the left-hand inequalities, we assume that $\alpha = 1$ and we first consider the left-hand inequality of (A.1). Let $t < 1$ and $\psi \in \mathrm{L}^q(\mathbb{R}_+)$, $\varphi \in \mathrm{L}^{q'}(\mathbb{R}_+)$ such that $\psi * \varphi \equiv e_{-1}$ on $[t,\infty)$. Then

$$\begin{aligned}
|\log(t)| = -\log(t) &= \int_t^1 \frac{\mathrm{d}s}{s} \leq \mathrm{e}\int_t^1 \mathrm{e}^{-s}\frac{\mathrm{d}s}{s} = \mathrm{e}\int_t^1 |\psi * \varphi(s)|\frac{\mathrm{d}s}{s} \\
&\leq \mathrm{e}\int_t^1 \int_0^s |\psi(s-r)|\cdot|\varphi(r)|\,\mathrm{d}r\,\frac{ds}{s} \\
&\leq \mathrm{e}\int_0^\infty \int_r^\infty \frac{|\varphi(s-r)|}{s}\,\mathrm{d}s\,|\psi(r)|\,\mathrm{d}r \\
&= \mathrm{e}\int_0^\infty \int_0^\infty \frac{|\psi(r)||\varphi(s)|}{s+r}\,\mathrm{d}s\,dr \leq \frac{\mathrm{e}\pi}{\sin(\pi/q)}\|\psi\|_q\|\varphi\|_{q'},
\end{aligned}$$

where we used Hilbert's absolute inequality [50, Theorem 5.10.1]. It follows that

$$\eta(1,t,q) \geq \frac{\sin(\pi/q)}{\mathrm{e}\pi}|\log(t)|.$$

For the left-hand inequality of (A.2), we assume that $\alpha = 1$ and let $t > 0$ be arbitrary. Then

$$\mathrm{e}^{-t} = (\psi * \varphi)(t) \leq \int_0^t |\psi(t-s)||\varphi(s)|\,\mathrm{d}s \leq \|\psi\|_q\|\varphi\|_{q'}$$

by Hölder's inequality, hence $\mathrm{e}^{-t} \leq \eta(1,t,q)$.

For the right-hand inequalities in (A.1) and (A.2), we assume that $t = 1$ and first consider the right-hand inequality in (A.1) for $\alpha \leq \min\left\{\frac{1}{q}, \frac{1}{q'}\right\}$. Lemma A.1 (with $\theta = 1/q$) yields

$$(\psi_0 * \varphi_0)(s) = \begin{cases} s, & s \in [0,1) \\ 1, & s \geq 1 \end{cases},$$

where

$$\psi_0 := \sum_{j=0}^{\infty} \beta_j \mathbf{1}_{(j,j+1)} \qquad \text{and} \qquad \varphi_0 := \sum_{j=0}^{\infty} \beta'_j \mathbf{1}_{(j,j+1)}$$

for sequences $(\beta_j)_j$ and $(\beta'_j)_j$ of positive scalars such that $\beta_j = O((1+j)^{-1/q})$ and $\beta'_j = O((1+j)^{-1/q'})$ as $j \to \infty$. Let $\psi := e_{-\alpha}\psi_0$ and $\varphi := e_{-\alpha}\varphi_0$. Then $\psi * \varphi \equiv e_{-\alpha}$ on $[1,\infty)$ and

$$\begin{aligned} \|\psi\|_q^q = \|e_{-\alpha}\psi_0\|_q^q &= \sum_{j=0}^{\infty} \beta_j^q \int_j^{j+1} \mathrm{e}^{-\alpha q s}\, \mathrm{d}s \lesssim \sum_{j=0}^{\infty} \frac{\mathrm{e}^{-\alpha q j}}{1+j} \\ &\leq 1 + \int_0^{\infty} \frac{\mathrm{e}^{-\alpha q s}}{1+s}\, \mathrm{d}s = 1 + \mathrm{e}^{\alpha q} \int_{\alpha q}^{\infty} \frac{\mathrm{e}^{-s}}{s}\, \mathrm{d}s. \end{aligned}$$

The constant in the first inequality depends only on q. Since $\alpha q \leq 1$,

$$\begin{aligned} \|\psi\|_q^q &\lesssim 1 + \mathrm{e}^{\alpha q} \left(\int_{\alpha q}^{1} \frac{\mathrm{e}^{-s}}{s}\, \mathrm{d}s + \int_1^{\infty} \frac{\mathrm{e}^{-s}}{s}\, \mathrm{d}s \right) \leq 1 + \int_{\alpha q}^{1} \frac{1}{s}\, \mathrm{d}s + \mathrm{e}^{\alpha q} \int_1^{\infty} \mathrm{e}^{-s}\, \mathrm{d}s \\ &= 1 - \log(\alpha q) + \mathrm{e}^{\alpha q - 1} \leq \log\left(\tfrac{1}{\alpha}\right) + 2. \end{aligned}$$

Moreover, $\frac{1}{\alpha} \geq q > 1$ hence $\log\left(\frac{1}{\alpha}\right) \geq \log(q) > 0$ and

$$\log\left(\tfrac{1}{\alpha}\right) + 2 \leq \left(1 + \frac{2}{\log(q)}\right) \log\left(\tfrac{1}{\alpha}\right).$$

Therefore

$$\|\psi\|_q \lesssim \log\left(\tfrac{1}{\alpha}\right)^{1/q} = |\log(\alpha)|^{1/q},$$

for a constant depending only on q. In a similar manner we deduce

$$\|\varphi\|_{q'} \lesssim |\log(\alpha)|^{1/q'}$$

for a constant depending only on q' (and thus on q). This yields (A.1).

For the right-hand side of (A.2) we assume that $t = 1$ and, without loss of generality (since $\eta(\alpha,t,q) = \eta(\alpha,t,q')$), that $\alpha > \frac{1}{q}$. Let $\varphi := \mathbf{1}_{[0,1]} e_{\alpha(q-1)}$ and $\psi := \frac{\alpha q}{\mathrm{e}^{\alpha q}-1} \mathbf{1}_{\mathbb{R}_+} e_{-\alpha}$. Then

$$\psi * \varphi(r) = \frac{\alpha q}{\mathrm{e}^{\alpha q} - 1} \int_0^1 \mathrm{e}^{\alpha(q-1)s} \mathrm{e}^{-\alpha(r-s)}\, \mathrm{d}s = \mathrm{e}^{-\alpha r}$$

for $r \geq 1$. Hence

$$\eta(\alpha, 1, q) \leq \|\psi\|_q \|\varphi\|_{q'} = \frac{\alpha q}{e^{\alpha q} - 1} \left(\int_0^\infty e^{-\alpha q s} \, ds \right)^{1/q} \left(\int_0^1 e^{\alpha(q-1)q' s} \, ds \right)^{1/q'}$$
$$= \frac{(\alpha q)^{(q-1)/q}}{e^{\alpha q} - 1} \left(\int_0^1 e^{\alpha q s} \, ds \right)^{\frac{q-1}{q}} = (e^{\alpha q} - 1)^{-1/q} \leq 2^{1/q} e^{-\alpha} \leq 2e^{-\alpha},$$

where we have used the assumption $\alpha > \frac{1}{q}$ in the penultimate inequality. □

B

Estimates for Padé approximants

In this appendix we prove several technical results from Section 6.1.

The following lemma, proved recently in [92], is essential for the results in this appendix. For $n \in \mathbb{N}$, recall the definitions of Q_n and r_n from Section 6.1.

Lemma B.1. *Let $n \in \mathbb{N}$ and $t \in \mathbb{R}$. Then*

$$|Q_n(\mathrm{i}t)| \geq 1, \quad \left|\frac{Q_n'(\mathrm{i}t)}{Q_n(\mathrm{i}t)}\right| \leq 1, \quad \textit{and} \quad |r_n'(\mathrm{i}t)| \leq 2.$$

We use this lemma to prove Proposition 6.2 from the main text.

Proposition B.2. *Let $n \in \mathbb{N}$ and $z \in \overline{\mathbb{C}_+}$. Then*

$$|r_n(-z) - \mathrm{e}^{-z}| \leq \frac{1}{2}\left(\frac{n!}{(2n+1)!}\right)^2 |z|^{2n+2}$$

and

$$|r_n'(-z) - \mathrm{e}^{-z}| \leq \left(\frac{n!}{(2n+1)!}\right)^2 \left(\frac{4}{5}|z|^{2n+2} + (n+1)|z|^{2n+1}\right).$$

Proof. By standard properties of Euler's beta function,

$$\int_0^1 s^n(1-s)^{n+1}\mathrm{d}s = \frac{n!(n+1)!}{(2n+2)!}.$$

Hence (6.3) yields

$$\begin{aligned}
|r_n(-z) - \mathrm{e}^{-z}| &\leq \frac{1}{|Q_n(-z)|}\frac{1}{(2n+1)!}|z|^{2n+2}\int_0^1 s^n(1-s)^{n+1}|\mathrm{e}^{(s-1)z}|\,\mathrm{d}s \\
&\leq \frac{1}{|Q_n(-z)|}\frac{1}{(2n+1)!}|z|^{2n+2}\frac{n!(n+1)!}{(2n+2)!}.
\end{aligned}$$

As Q_n is a polynomial having all its roots in $\mathbb{C}_+$, Lemma B.1 and the maximum principle for holomorphic functions imply $\frac{1}{|Q_n(-z)|} \leq 1$. Hence

$$|r_n(-z) - \mathrm{e}^{-z}| \leq \frac{n!(n+1)!}{(2n+1)!(2n+2)!}|z|^{2n+2} = \frac{1}{2}\left(\frac{n!}{(2n+1)!}\right)^2 |z|^{2n+2},$$

which is the first statement that was to be proved. Now differentiate (6.3) with respect to z and write

$$r_n'(-z) - \mathrm{e}^{-z} = T_{n,1}(z) + T_{n,2}(z) + T_{n,3}(z), \tag{B.1}$$

where

$$\begin{aligned}
T_{n,1}(z) &= \frac{(-1)^{n+3}Q_n'(-z)}{Q_n(-z)^2}\frac{(-z)^{2n+2}}{(2n+1)!}\int_0^1 s^n(1-s)^{n+1}\mathrm{e}^{(s-1)z}\,\mathrm{d}s \\
&= -\frac{Q_n'(-z)}{Q_n(-z)}(r_n(-z) - \mathrm{e}^{-z}), \\
T_{n,2}(z) &= \frac{(-1)^{n+2}}{Q_n(-z)}\frac{(2n+2)(-z)^{2n+1}}{(2n+1)!}\int_0^1 s^n(1-s)^{n+1}\mathrm{e}^{(s-1)z}\,\mathrm{d}s \\
&= -\frac{2n+2}{z}(r_n(-z) - \mathrm{e}^{-z}), \quad \text{and} \\
T_{n,3}(z) &= \frac{(-1)^{n+2}}{Q_n(-z)}\frac{(-z)^{2n+2}}{(2n+1)!}\int_0^1 s^n(1-s)^{n+2}\mathrm{e}^{(s-1)z}\,\mathrm{d}s.
\end{aligned}$$

We estimate these three terms separately. As all roots of Q_n lie in $\mathbb{C}_+$, Lemma B.1 and the maximum principle for holomorphic functions yield $\left\|\frac{Q_n'(-\cdot)}{Q_n(-\cdot)}\right\|_{\mathrm{H}^\infty(\mathbb{C}_+)} \leq 1$. The first part of this lemma then implies

$$|T_{n,1}(z)| \leq \frac{1}{2}\left(\frac{n!}{(2n+1)!}\right)^2 |z|^{2n+2} \tag{B.2}$$

and

$$|T_{n,2}(z)| \leq (n+1)\left(\frac{n!}{(2n+1)!}\right)^2 |z|^{2n+1}. \tag{B.3}$$

Finally, note that the only difference between $T_{n,3}(z)$ and $r_n(-z) - \mathrm{e}^{-z}$ is an additional factor $(1-s)$ under the respective integral sign. By the same arguments as in the proof of the first assertion of this lemma, taking into account that $\frac{n+2}{2n+3} \leq \frac{3}{5}$,

$$|T_{n,3}(z)| \leq \frac{n!(n+2)!}{(2n+1)!(2n+3)!}|z|^{2n+2} \leq \frac{3}{10}\left(\frac{n!}{(2n+1)!}\right)^2 |z|^{2n+2}. \tag{B.4}$$

The proof is concluded by combining (B.1) with (B.2), (B.3) and (B.4). □

We now recall Lemmas 6.4 and 6.5.

Lemma B.3. *Let $u, v, U, V > 0$ and $w \in \left[0, \left(\frac{V}{U}\right)^{\frac{1}{u+v}}\right]$. Then*

$$\int_w^\infty \min\left\{Ur^u, Vr^{-v}\right\} \frac{\mathrm{d}r}{r} = V\left(\frac{U}{V}\right)^{\frac{v}{u+v}} \frac{u+v}{uv} - \frac{U}{u}w^u.$$

Lemma B.4. *Let $n \in \mathbb{N}$. Then*

$$\left(\frac{n!}{(2n+1)!}\right)^{\frac{1}{n+1}} \leq \frac{1}{n+1}.$$

We now prove Proposition 6.6. For $n \in \mathbb{N}$ and $\alpha \in (0, \infty)$, recall the function $f_{n,\alpha} : \overline{\mathbb{C}_+} \setminus \{0\} \to \mathbb{C}$,

$$f_{n,\alpha}(z) = \frac{r_n(-z) - \mathrm{e}^{-z}}{z^\alpha} \qquad (z \in \overline{\mathbb{C}_+} \setminus \{0\}),$$

from (6.4).

Proposition B.5. *Let $n \in \mathbb{N}$ and $\alpha \in (\frac{1}{2}, n + \frac{1}{2}]$. Then*

$$\|f_{n,\alpha}(\mathrm{i}\,\cdot)\|_2 \leq \frac{4}{\sqrt{2\alpha - 1}}(n+1)^{-\alpha+\frac{1}{2}}$$

and

$$\left\|(f_{n,\alpha}(\mathrm{i}\,\cdot))'\right\|_2 \leq \left(\frac{8\alpha}{(2\alpha+1)^{3/2}} + \frac{13^\alpha}{10^\alpha}\sqrt{\frac{5^{2\alpha}}{6 \cdot 13^{2\alpha}} + \frac{360}{13(2\alpha-1)}}\right)(n+1)^{-\alpha+\frac{1}{2}}.$$

Proof. First, we prove the estimate for $\|f_{n,\alpha}(\mathrm{i}\,\cdot)\|_2$. By Proposition B.2 and the $\mathcal{A}$-stability of r_n,

$$|f_{n,\alpha}(\mathrm{i}t)| \leq \min\left\{\frac{1}{2}\left(\frac{n!}{(2n+1)!}\right)^2 |t|^{2n+2-\alpha}, 2|t|^{-\alpha}\right\}$$

for all $t \in \mathbb{R} \setminus \{0\}$. Hence

$$\|f_{n,\alpha}(\mathrm{i}\,\cdot)\|_2^2 \leq 2\int_0^\infty \min\left\{\frac{1}{4}\left(\frac{n!}{(2n+1)!}\right)^4 |t|^{4n+5-2\alpha}, 4|t|^{-(2\alpha-1)}\right\} \frac{\mathrm{d}t}{t}.$$

By Lemma B.3,

$$\|f_{n,\alpha}(\mathrm{i}\,\cdot)\|_2^2 \leq \frac{1}{2\alpha-1}\frac{32(n+1)}{4n+5-2\alpha}\left(\frac{n!}{2(2n+1)!}\right)^{\frac{2\alpha-1}{n+1}}.$$

Now, Lemma B.4 and $\alpha \leq n + \frac{3}{2}$ imply

$$\|f_{n,\alpha}(\mathrm{i}\,\cdot)\|_2^2 \leq \frac{16}{2\alpha - 1}\left(\frac{n!}{2(2n+1)!}\right)^{\frac{2\alpha-1}{n+1}} \leq \frac{16}{2\alpha-1}(n+1)^{-2\alpha+1},$$

which proves the first statement.

For the proof of the second statement we write

$$\frac{\mathrm{d}}{\mathrm{d}t} f_{n,\alpha}(\mathrm{i}t) = -\mathrm{i}\frac{r_n'(-\mathrm{i}t) - \mathrm{e}^{-\mathrm{i}t}}{(\mathrm{i}t)^{\alpha}} - \mathrm{i}\alpha\frac{r_n(-\mathrm{i}t) - \mathrm{e}^{-\mathrm{i}t}}{(\mathrm{i}t)^{\alpha+1}} =: -\mathrm{i}g_{n,\alpha}(t) - \mathrm{i}\alpha h_{n,\alpha}(t) \tag{B.5}$$

for $t \neq 0$. Note that $h_{n,\alpha}(\cdot) = f_{n,\alpha+1}(\mathrm{i}\,\cdot)$. Since we have only used $\alpha \leq n + \frac{3}{2}$ to prove the first part of this lemma, we can apply the first part with $\alpha + 1$ in place of α to find

$$\|h_{n,\alpha}\|_2 \leq \frac{4}{\sqrt{2\alpha+1}}(n+1)^{-\alpha-\frac{1}{2}} \leq \frac{8}{(2\alpha+1)^{3/2}}(n+1)^{-\alpha+\frac{1}{2}}, \tag{B.6}$$

where the second step is valid since $\alpha \in (\frac{1}{2}, n + \frac{1}{2}]$.

As for $g_{n,\alpha}$, combining Lemma B.1 and Proposition B.2 yields

$$|g_{n,\alpha}(t)| \leq \min\left\{\left(\frac{n!}{(2n+1)!}\right)^2\left(\frac{4}{5}|t|^{2n+2-\alpha} + (n+1)|t|^{2n+1-\alpha}\right), 3|t|^{-\alpha}\right\} \mathrm{d}t$$

for $t \neq 0$. Let $K := \left(\frac{n!}{(2n+1)!}\right)^2$ and $L := n + \frac{9}{5}$. As $|t|^{2n+2-\alpha} \leq |t|^{2n+1-\alpha}$ if and only if $|t| \leq 1$,

$$\|g_{n,\alpha}\|_2^2 \leq 2\int_0^1 K^2L^2t^{4n+2-2\alpha}\,\mathrm{d}t + 2\int_1^{\infty} \min\left\{K^2L^2t^{4n+5-2\alpha}, 9t^{-(2\alpha-1)}\right\}\frac{\mathrm{d}t}{t}. \tag{B.7}$$

Using Lemma B.4, $\alpha \leq n + \frac{1}{2}$ and $\frac{n+\frac{9}{5}}{n+1} \leq \frac{7}{5}$ in sequence yields

$$\frac{K^2L^2}{3(n+1)} \leq \frac{\left(n+\frac{9}{5}\right)^2}{3(n+1)^{4n+5}} \leq \frac{2}{3(n+1)^{4\alpha+1}} \leq \frac{1}{3\cdot 2^{2\alpha+1}(n+1)^{2\alpha-1}}. \tag{B.8}$$

In particular, $9K^{-2}L^{-2} \geq \frac{9}{2}(n+1)^{4\alpha} \geq 1$, which in turn allows us to compute the second integral in (B.7) by means of Lemma B.3. This yields

$$\|g_{n,\alpha}\|_{\mathrm{L}^2(\mathbb{R})}^2 \leq \frac{2K^2L^2}{4n+3-2\alpha} + \frac{18}{2\alpha-1}\left(\frac{KL}{3}\right)^{\frac{2\alpha-1}{2n+2}}\frac{4n+4}{4n+5-2\alpha} - \frac{2K^2\mathrm{L}^2}{4n+5-2\alpha}.$$

Simplifying this expression, using $\alpha \in (\frac{1}{2}, n + \frac{1}{2}]$ and (B.8), we can estimate $\|g_{n,\alpha}\|_2^2$ from above by

$$\frac{K^2L^2}{3(n+1)} + \frac{36}{2\alpha-1}\left(\frac{KL}{3}\right)^{\frac{2\alpha-1}{2n+2}} \leq \frac{1}{3\cdot 2^{2\alpha+1}} n^{-2\alpha+1} + \frac{36}{2\alpha-1}\left(\frac{KL}{3}\right)^{\frac{2\alpha-1}{2n+2}}.$$

For the second term in this inequality, Lemma B.4 and the inequality $n + \frac{9}{5} \leq \left(\frac{13}{10}\right)^{2n+2}$, which can easily be verified by induction, yield

$$\left(\frac{KL}{3}\right)^{\frac{1}{2n+2}} = \left(\frac{n!}{\sqrt{3}(2n+1)!}\right)^{\frac{1}{n+1}} \left(n + \frac{9}{5}\right)^{\frac{1}{2n+2}} \leq \frac{13}{10}\frac{1}{n+1}.$$

Hence

$$\begin{aligned} \|g_{n,\alpha}\|_2^2 &\leq \frac{1}{3\cdot 2^{2\alpha+1}}(n+1)^{-2\alpha+1} + \frac{36}{2\alpha-1}\left(\frac{13}{10}\right)^{2\alpha-1}(n+1)^{-2\alpha+1} \\ &= \frac{13^{2\alpha}}{10^{2\alpha}}\left(\frac{5^{2\alpha}}{6\cdot 13^{2\alpha}} + \frac{360}{13(2\alpha-1)}\right)(n+1)^{-2\alpha+1}. \end{aligned}$$

The conclusion follows by combining the estimate above, (B.6) and (B.5). □

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Index

List of symbols

General mathematics

$\mathbb{N} = \{1, 2, 3, \dots\}$
$\mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$
$\mathbb{Z}$, integers
$\mathbb{R}$, real numbers
$\mathbb{R}_+ = [0, \infty)$
$\mathbb{C}$, complex numbers
$f^{(m)}$, m-th derivative of f
$O(\cdot)$, p. 26
$\mathcal{F}\mu$, p. 25
$\widehat{\mu}$, p. 25

Banach spaces

X, Banach space
X^*, dual of X
$\langle \cdot, \cdot \rangle$, duality bracket
H, Hilbert space
$\mathcal{L}(X, Y)$, p. 23
$\mathcal{L}(X) = \mathcal{L}(X, X)$
$X^* \otimes Y$, p. 23
$\mathrm{Scal}(X)$, p. 37
$\mathcal{L}_\mathrm{d}(X)$, p. 106
$\mathcal{L}_\mathrm{d}(X, (\lambda_j)_{j=1}^\infty, U)$, p. 106
$(X, Y)_{\theta,q}$, p. 52
ℓ^p, p. 24
$\mathcal{S}_p$, p. 24
$\Pi_p(X, Y)$, p. 120

Operators

I, identity operator
A, operator
$\mathrm{D}(A)$, p. 23
$\mathrm{D}_A(\theta, q) = (X, \mathrm{D}(A))_{\theta,q}$
$\mathrm{ran}(A)$, range of A
$\sigma(A)$, spectrum of A
$\rho(A) = \mathbb{C} \setminus \sigma(A)$
$R(z, A) = (z\mathrm{I} - A)^{-1}$
B_Y, p. 54
$B_{\theta,q} = B_{\mathrm{D}_A(\theta,q)}$
$\mathrm{BIP}(X, \psi)$, p. 34
T_μ, p. 26
U_μ, p. 31
T_m, p. 43/45
$T_\varphi^{A,B}$, p. 101
$T_{\varphi,n}^{\lambda,\mu}$, p. 107

Function spaces

Measure theory

Functions

Norms

Constants

Miscellaneous

Summary

Functional calculus via transference, double operator integrals and applications

This thesis is dedicated to the study of several aspects of the theory of functional calculus. This theory considers the combination of an operator A and a function $f(z)$ of a variable z, resulting in an operator $f(A)$. One then attempts to study properties of the operator $f(A)$ in terms of properties of the operator A and the function f.

A classical example of a functional calculus is the calculus for diagonalizable matrices. This calculus is based on the idea that for a diagonal matrix $D = (d_{ij})_{i,j=1}^{n}$ and a function f, $f(D)$ should be defined by simply applying f to the entries of D. In other words, $f(D) := (f(d_{ij}))_{i,j=1}^{n}$. Then, for a matrix A and an invertible matrix U such that $UAU^{-1} = D$ has diagonal form, one defines $f(A) := U^{-1}f(D)U$. Despite its relatively straightforward construction, many nontrivial questions arise when studying this calculus.

Recently, functional calculus theory has also proven useful when studying partial differential equations from a functional analytic perspective. The functional analytic viewpoint on a large class of partial differential equation leads to an equation of the form

$$\frac{\mathrm{d}u}{\mathrm{d}t} = Au, \qquad u(0) = x \tag{.1}$$

on an infinite-dimensional space X. Formally, (.1) has the solution $u(t) = \mathrm{e}^{tA}x$. Functional calculus theory allows one to make this formal intuition precise and provides a convenient framework for studying the differential operator A in (.1) as well as operators related to A.

In this thesis both the functional calculus for diagonalizable matrices and the calculus for the operator A in (.1) are studied. Using transference principles and double operator integrals, we link the theory of functional calculus

to the area of harmonic analysis. Then we use theorems from harmonic analysis to deduce new results in functional calculus theory.

We also apply functional calculus theory to a problem in the theory of the numerical approximation of the solutions of (.1). Since the solution $e^{tA}x$ of (.1) is usually hard to deal with analytically, one tries to approximate it by simpler expressions, for instance by $r_n(A)x$ for $(r_n)_{n=1}^{\infty}$ a sequence of rational functions. One would then like to know whether this approximation converges, and functional calculus theory is a useful tool with which one can deal with this question.

A more detailed description of the contents of this thesis is as follows.

In Part I we treat some of the basic tools which will be used throughout the thesis. These include: the basics of the theory of functional calculus, some function space theory and preliminaries on vector-valued harmonic analysis, as well as the transference principles which link these three topics together.

In Part II we study the functional calculus theory associated with (.1). We obtain new links between harmonic analysis and the functional calculus for the operator A in (.1). In Chapter 3 this allows us to deduce properties of $f(A)$ for a wide class of functions f. This class of functions depends heavily on geometrical aspects of the underlying space, and therefore so do the results which we obtain. By contrast, in Chapter 4 we study $f(A)$ for a class of functions f which does not depend on geometrical properties of the underlying space. However, here the results are only valid when considering (.1) for initial values in so-called interpolation spaces.

In Part III we study the functional calculus for diagonalizable matrices. For diagonalizable matrices A and B we determine how properties of $f(B) - f(A)$ relate to properties of $B - A$. In particular, we study the inequality

$$\|f(B) - f(A)\| \leq C \|B - A\|$$

for various norms $\|\cdot\|$ and functions f, where the constant C is independent of A and B. Again we relate a question in functional calculus theory to harmonic analysis, this time using the technique of double operator integration.

In Part IV we apply functional calculus theory to the study of numerical approximation methods for the solutions to (.1). In particular, we consider a recently proposed numerical approximation method for (.1). We show that this approximation method converges for a large set of initial values and determine the corresponding rates of convergence. Then, using the theory from earlier chapters, we improve these rates of convergence for specific classes of operators.

Finally, two appendices contain results which are used in Chapters 3 and 6. These results are of a technical nature and have been placed in appendices to improve the readability of the main text.

Samenvatting

Functional calculus via transference, double operator integrals and applications

Dit proefschrift is gewijd aan de studie van verscheidene aspecten van de theorie van functionaalcalculus. Deze theorie behandelt de combinatie van een operator A en een functie $f(z)$ van een variabele z, met als resultaat een operator $f(A)$. Men probeert dan eigenschappen van de operator $f(A)$ te bestuderen in termen van eigenschappen van de operator A en de functie f.

Een klassiek voorbeeld van een functionaalcalculus is de calculus voor diagonalizeerbare matrices. Deze calculus is gebaseerd op het idee dat, voor een diagonaalmatrix $D = (d_{ij})_{i,j=1}^{n}$ en een functie f, $f(D)$ gedefinieerd dient te worden door simpelweg f toe te passen op de coëfficienten van D. Oftewel, $f(D) := (f(d_{ij}))_{i,j=1}^{n}$. Voor een matrix A en een inverteerbare matrix U zodanig dat $UAU^{-1} = D$ een diagonaalmatrix is, definieert men dan $f(A) := U^{-1}f(D)U$. Ondanks de relatieve eenvoud van deze constructie komen er veel niet-triviale vragen naar voren bij het bestuderen van deze calculus.

Recentelijk heeft functionaalcalculus zich ook bewezen als een nuttig gereedschap bij het bestuderen van partiële differentiaalvergelijkingen vanuit een functionaalanalytisch perspectief. De functionaalanalytische aanpak van een grote klasse partiële differentiaalvergelijkingen leidt tot een vergelijking van de vorm

$$\frac{\mathrm{d}u}{\mathrm{d}t} = Au, \qquad u(0) = x \tag{.2}$$

op een oneindig-dimensionale ruimte X. Formeel heeft (.2) als oplossing $u(t) = \mathrm{e}^{tA}x$. De theorie van functionaalcalculus stelt ons in staat om deze

formele intuïtie precies te maken, en vormt ook een handig kader waarbinnen de operator A in (.2), alsmede operatoren gerelateerd aan A, bestudeerd kan worden.

In dit proefschrift worden zowel de functionaalcalculus voor diagonalizeerbare matrices als de calculus voor de operator A in (.2) bestudeerd. Met behulp van zogenaamde 'transference principles' en dubbele operatorintegralen koppelen we de theorie van functionaalcalculus aan de theorie van harmonische analyse. Vervolgens gebruiken we stellingen uit de harmonische analyse om nieuwe resultaten in de theorie van functionaalcalculus af te leiden.

We passen functionaalcalculus ook toe op een probleem in de theorie van de numerieke benadering van de oplossingen van (.2). Aangezien het over het algemeen moeilijk is om de oplossing $e^{tA}x$ van (.2) analytisch te behandelen probeert men deze oplossing vaak te benaderen met simpelere grootheden, bijvoorbeeld met $r_n(A)x$ voor $(r_n)_{n=1}^{\infty}$ een rij van rationale functies. Het is dan van belang om te weten of deze benadering convergeert, en de theorie van functionaalcalculus is een nuttig gereedschap voor het beantwoorden van deze vraag.

Een meer gedetailleerde beschrijving van de inhoud van dit proefschrift is als volgt.

In Deel I behandelen we enkele van de basisbegrippen die in de rest van het proefschrift voorkomen. Hieronder vallen: de basis van de theorie van functionaalcalculus, aspecten van de theorie van functieruimtes en de beginselen van de theorie van vectorwaardige harmonische analyse, alsmede de transference principles die deze drie gebieden met elkaar verbinden.

In Deel II bestuderen we de functionaalcalculus behorende bij (.2). We leiden nieuwe connecties af tussen harmonische analyse en de functionaalcalculus voor de operator A in (.2). In Hoofdstuk 3 stelt dit ons in staat om eigenschappen van $f(A)$ af te leiden voor een grote klasse functies f. Deze klassie van functies is sterk afhankelijk van meetkundige aspecten van de onderliggende ruimte, en daarmee zijn de verkregen resultaten dat ook. Daarentegen bestuderen we in Hoofdstuk 4 $f(A)$ voor een klasse functies f die niet afhangt van de onderliggende ruimte. Maar de betreffende resultaten zijn alleen van toepassing op beginwaarden van (.2) in zogenaamde interpolatieruimtes.

In Deel III bestuderen we de functionaalcalculus voor diagonalizeerbare matrices. Voor diagonalizeerbare matrices A en B bepalen we hoe eigenschappen van $f(B) - f(A)$ afhangen van eigenschappen van $B - A$. In het bijzonder bestuderen we de ongelijkheid

$$\|f(B) - f(A)\| \leq C \|B - A\|$$

voor verscheidene normen $\|\cdot\|$ en functies f, waarbij de constante C onafhankelijk is van A en B. Wederom brengen we een probleem in de functionaalcalculus in verband met de harmonische analyse, ditmaal met behulp van de techniek van dubbele operatorintegratie.

In Deel IV passen we de theorie van functionaalcalculus toe op de studie van numerieke benaderingsmethoden voor de oplossingen van (.2). In het bijzonder beschouwen we een recentelijk voorgestelde numerieke benaderingsmethode voor (.2). We laten zien dat deze benaderingsmethode convergeert voor een grote klasse beginvoorwaarden, en we bepalen de bijbehorende convergentiesnelheden. Dan verbeteren we deze convergentiesnelheden voor specifieke klasses operatoren, gebruikmakend van de theorie uit eerdere hoofdstukken.

Tenslotte bevatten twee appendices resultaten die gebruikt worden in Hoofdstukken 3 en 6. Deze resultaten zijn van een technische aard en zijn in appendices geplaatst om de leesbaarheid van de hoofdtekst te bevorderen.

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Delft, April 2015 *Jan Rozendaal*

Curriculum Vitae

Jan Rozendaal was born on the 4th of August, 1984, in Rotterdam, The Netherlands. In 2003 he completed his secondary education at the Erasmiaans Gymnasium in Rotterdam. He then began studying Physics at Leiden University, switching later to the study of Mathematics. From 2006 to 2010 he was a member of the Royal Student Rowing Society (KSRV) Njord. In 2011 he obtained his MSc. degree in Mathematics 'cum laude'. He then commenced his PhD research under supervision of dr. M. Haase and Prof. dr. B. de Pagter at Delft University of Technology. Part of this research was carried out during a six-month stay at the University of New South Wales in Sydney, Australia.

List of Publications

1. M. Egert and J. Rozendaal. Convergence of subdiagonal Padé approximations of C_0-semigroups, *J. Evol. Equ.*, 13: 875-895, 2013.
2. M. Haase and J. Rozendaal. Functional calculus for semigroup generators via transference, *J. Funct. Anal.*, 265: 3345-3368, 2013.
3. M. Haase and J. Rozendaal. Functional calculus on real interpolation spaces for generators of C_0-groups. Online at `http://arxiv.org/abs/1409.6101`, 18 pages, 2014. To appear in Mathematische Nachrichten.
4. M. de Jeu and J. Rozendaal. Disintegrating positive group representations on L^p-spaces into order indecomposables. Online at `http://arxiv.org/abs/1502.00755`, 17 pages, 2015.
5. J. Rozendaal. Functional calculus for C_0-groups using (co)type. Online at `http://arxiv.org/abs/1508.02036`, 25 pages, 2015.
6. J. Rozendaal, F. Sukochev, and A. Tomskova. Operator Lipschitz functions on Banach spaces. Online at `http://arxiv.org/abs/1501.03267`, 30 pages, 2015.
7. J. Rozendaal, M. Veraar. Fourier multiplier theorems involving (co)type, and stability theorems. In preparation, 2015.