Shells and arches

Developing a new method to calculate shells and arches through graphic statics

Master thesis

by

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CHAPTER 1 – INTRODUCTION

1.1 Background

Shell structures have been around for ages. The dome of the Pantheon for instance, built around the year 120, is one of the oldest examples of a shell structure. It is only in the recent history however that shell structures would take a more complex form. The shells by Heinz Isler for instance, or the shells by Candela are freeform shells from the 20th and 21st century (Figure 1 (b) and Figure 2).

Figure 1 (a) Interior of the Pantheon, Rome, painted by Panini (1734).(b) L'Oceanogràfic designed by Felix Candela, 2003 (Gabaldón, 2010)

Figure 2 Heinz Isler, 1968, Laboratory and research facility for the Gips union (Töffpix, 2011)

Shell structures are usually calculated using the Finite Element Method (FEM). In this method, a structure is split up in small parts and the forces working on each part are calculated, including the forces working in between parts. Using this method, all the forces in the structure can be determined.

Using FEM for shell structures has two main disadvantages. The most important disadvantage is that it does not give any insight in the mechanics of shell structures in general. The calculations give numerical results, which can tell us something about a specific shell structure, and whether or not the structure will fail, but does not show the connection between the geometry and the structural performance, making it a kind of black-box method. Because of the lack of knowledge on this connection, designing a shell structure can only be done intuitively, by trial and error, which makes designing shells a difficult task.

The second problem is the efficiency of this method. These calculations can only be done for a finished structural design. If the design turns out not to be strong enough or is suspected to be inefficient in material use, a new design has to be made, which can then be checked, after which the cycle may need to be repeated. This can make the design process very time-consuming for more complicated designs, like the Candela shell for instance. Because a finished design is needed in order to get the calculation results, the feedback on the structural performance is slow which can result in an inefficient design process.

These problems can be solved when a new direct graphical method is found to calculate shells. To illustrate why this can help, the calculation of a simple truss structure can be taken as an example, for instance the one in Figure 3. The truss can be calculated by FEM (upper figure). To calculate the structure this way, everything about the structure has to be known, the type of material the material properties and the geometry of the material, so it should be a finished design. Also a mesh to subdivide the structure needs to be made. The calculations will output numbers on where the normal forces will be the biggest, which bars are in compression and which are in tension etcetera. However it does not show why some bars are in tension and others in compression or why some normal forces are bigger than others.

When this truss is calculated using graphic statics (the lower figure), the normal forces can be calculated without knowing the material properties and the thickness of the structure. Using this method, it will also give more insight on why some bars are in tension and others are in compression.

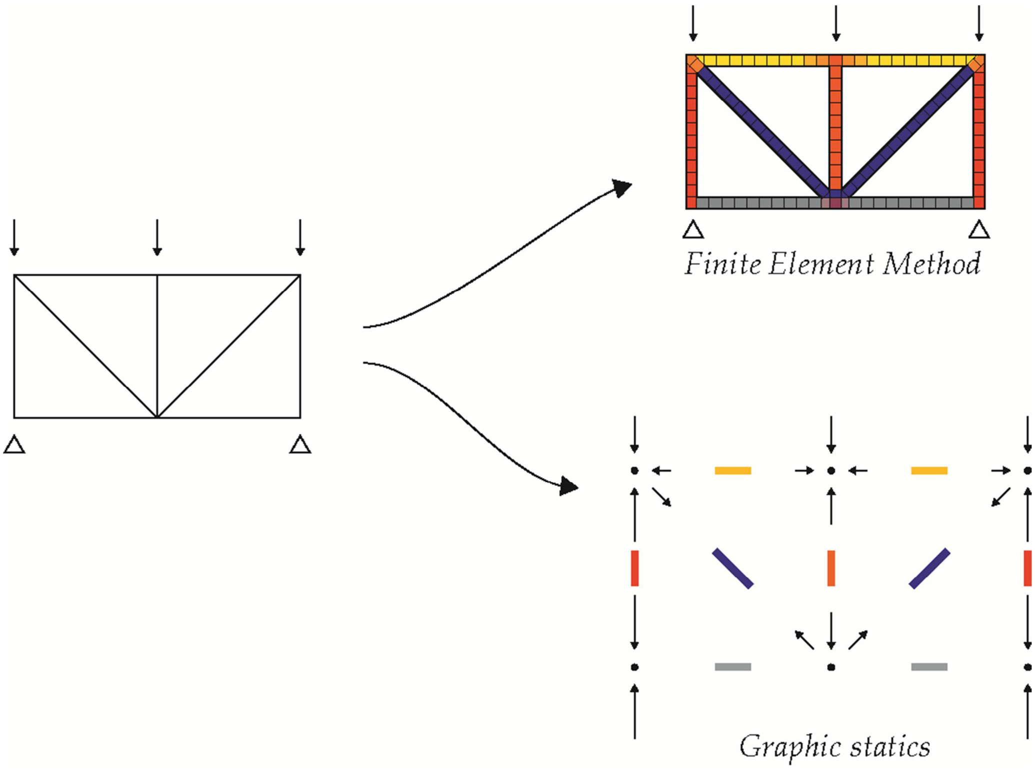


Figure 3 A simple truss calculated through the Finite Element Method (upper figure) and through graphic statics (lower figure)

Both the calculation methods have their strengths and their weaknesses. The FEM is a method which is very useful when a design is finished and a structural engineer wants to make sure that the structure is strong enough. The graphic statics method however is more useful at the beginning of the design process, being able to give insight in the way the truss works and why it works that way, without knowing the exact material properties yet.

When for shells a similar method is found, using graphic statics, it is expected to show the link between the structure and its performance in a similar way.

1.2 Current state of research

Some research is done already on this subject by several graduating students at the TU Delft, focusing on a way to calculate shell structures using graphic statics. The general aim of this research is to (i) find a graphical method to calculate these shells and to (ii) model these methods in a computational program linked to a 3D visualization, which results in a tool that can be used early in the design process and gives direct feedback on the structural performance of a shell. Some results of this research are discussed in chapter 2.

Part of this research focuses on a simple graphical method of calculating arches, with the idea that once a simple and accurate method is found for arches, this can be translated into a method for calculating shells. This is not a simple translation since a shell has some properties that an arch does not have (the ability to carry hoop forces for instance). However, since an arch is the two-dimensional equivalent of a shell, it is likely that structural concepts occurring in arches can be translated to shells.

A method to calculate arches in a graphical way is already found in the minimum complementary energy method (section 2.4). This is an iterative method which uses the thrust line to find the stresses in an arch. However, this method is not a solution for aforementioned problem because it does not give the possibility to calculate arches or shells in a direct way and does not provide a direct link between the structure and its structural performance.

The most recent research by van Dijk (2014) proposes a new method to calculate arches (the equal area method) which is further explained in section 2.5. This method is not proven yet, but the results of the calculations using this method look promising.

1.3 Problem statement

Calculating a shell using the Finite Element Method does not give insight in the relation between the geometry and the structural performance. An input can generate an output, but why shells transport forces the way they do isn't explained through this method.

Another problem with the current method of calculating a shell is that it results in a time-consuming design process because it only can be done on a finished structural design.

This problem is already partially solved for arches, using the minimum energy method, which provides a graphical way to calculate arches. It is only known how to use this method in an iterative way. So for a structure, several thrust lines can be generated and checked whether it is the correct one, until the correct solution is found. It is a method which (given enough calculation power) can find the solution. The problem with this method is that it is an indirect method which does not directly show the relation between geometry and structural performance.

A more direct method is desired for two reasons. The most important reason is that a direct method will make the relation between the structure and the results more visible. It can easier be deducted which factors influence the calculation results the most.

Another reason is that a direct method will cost less calculation power and can maybe even be done by hand, making it an easier to use method early in the design process.

1.4 Objective

The aim of this research is to find a method to calculate shell structures in a graphical way, so that the relation between the geometry and the structural performance is preserved.

To be able to get direct feedback on changes in geometry, a computational algorithm will be designed and modelled in a 3D visualization program.

In order to solve this problem for shells, it needs to be solved for arches first. Starting point of the research will be the equal area method, which provides a promising hypothesis on how this works for arches. When this method is proven, the step to shells can be made.

1.5 Research question

The problem statement and objectives stated in previous paragraphs lead to the research question:

How can the structural performance of a shell structure be calculated in such a way that the relation between the geometry and the structural performance is shown?

To answer this questions, the sub questions that need to be answered are:

- *Can the method of equal areas be proven for arches?*
- *Which methods can be made applicable to shell structures?*
- *How can this calculation method be translated into a computational algorithm and modelled in a 3D visualization program?*

To get started on this subject, a literature study is done, starting with the question:

What methods can be used to calculate shell and arch structures?

1.6 Relevance

Scientific relevance

Currently there is not a lot of insight in the mechanics behind shell and arch structures. This research aims to give more insight in these mechanics.

Societal relevance

This research aims to provide in a tool for designers which gives them insight in the structural performance of a shell structure earlier in the design process. This will lead to a less time-consuming design process, but also to a more direct feedback on the design changes. It will probably lead to more efficient structural design, in which less material can be used for a similar performance.

More on the relevance of the results will be discussed in section 4.2

1.7 Approach and methodology

Literature study

The whole research will be done within the field of structural mechanics. For this reason, the literature to be studied is mainly in the field of structural mechanics. To get the research started, a literature study on several subjects needs to be done. Part of these subjects are studied already. The following subjects will be studied:

- Force density method
- Complementary energy method
- Graphic statics in arches

Method development

From the literature study hypotheses will emerge. From these hypotheses one or several methods to calculate arch structures will be developed. Once a method for arches is proven to work, the step to shell structures can be made.

Design computational algorithm

The found method will be translated into a computational algorithm and modelled in a 3D visualization tool. To model this, the 3D program Rhino will be used, with the Grasshopper-plugin. This should be a tool which can be used early in the design process, to give more insight in the structural performance of an arch or shell.

Validate method

The computational algorithm will be compared to FEM calculations for several case studies. Differences in results from these calculations will show whether or not the method is valid.

CHAPTER 2 - THEORY

Before starting the method development, a literature study was done to get insight in the current state of research. During method development some more subjects came up that needed to be studied. The results of these studies are summarized in this chapter. It mostly deals with theory regarding the calculation of beams and shells. First of all, a method developed by Calladine for calculating shells is summarized. The second and third section give an overview of methods of designing arches, using a chain line to construct a thrust line. It explains what a thrust line and its force polygon are, and how it relates to an arch and its loads. The fourth section explains a method to determine which thrust line is the correct one and the fifth section summarizes a graduation report by van Dijk (2014) which is partially the starting point for this research. The sixth and seventh paragraph summarize some insights and some research done alongside the development of the calculation method. Lopshits equation, a way to calculate the area of any polygon, is introduced and related to the calculation of arch structures. The final section of this chapter discusses the topics of research and how they relate to the calculation method to be developed.

2.1 Calculating shells through the split in surfaces

To be able to calculate a shell, C.R. Calladine (1977) describes an element of a shallow shell which is conceptually split into two elements, one element which can only stretch and one which can only bend. The stretching element is similar to for instance a bar network with hinges only, it can only transport normal forces. For the bending element there is no well-known equivalent, but it can be visualized as the beam shown in figure Figure 4(d), which shows a beam that can only transport bending moments.

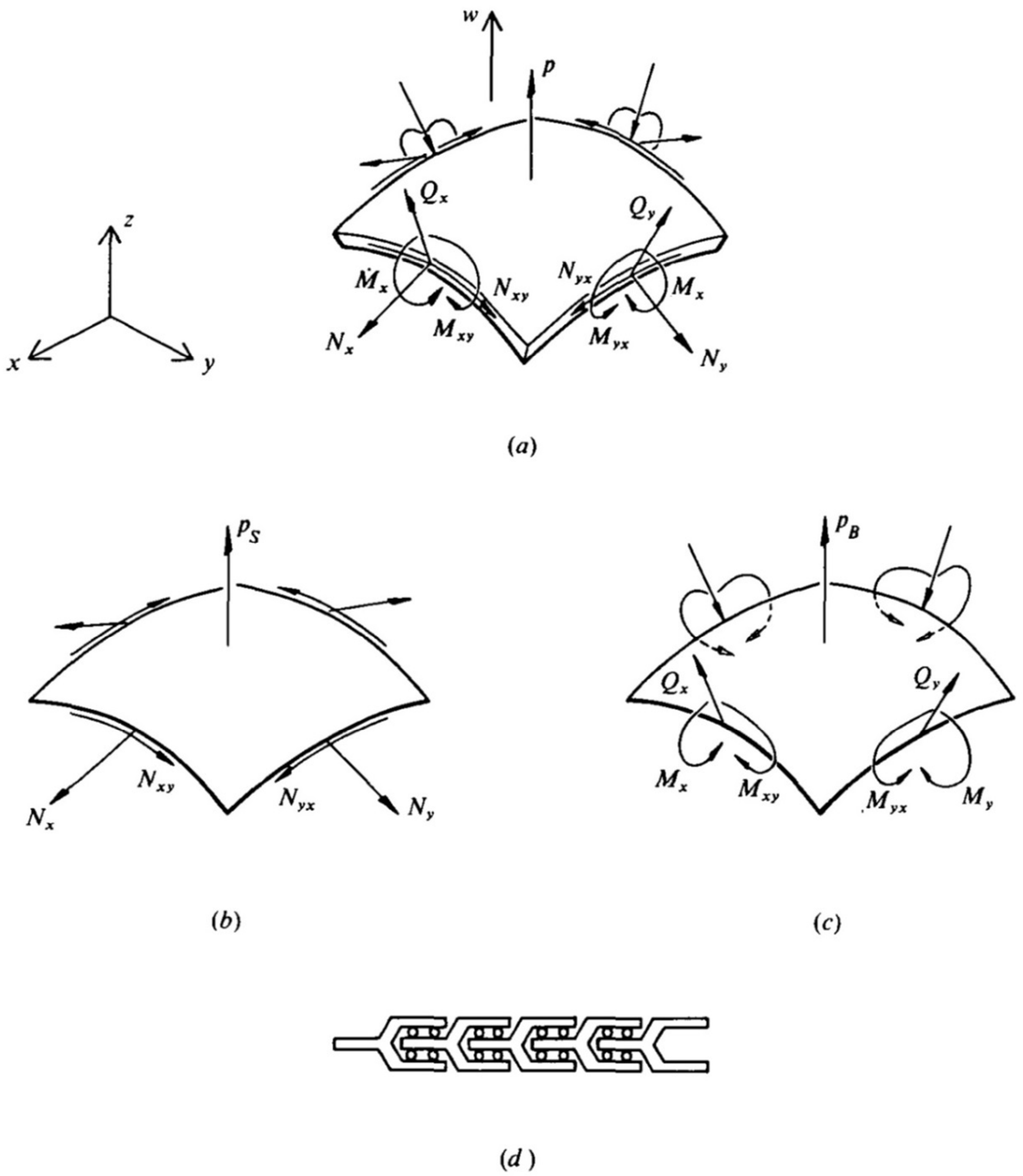


Figure 4 A representation of Calladine's split of a surface (a) into a stretching (b) and bending (c) surface. (d) shows a 2-dimensional equivalent of a bending surface. (Calladine, 1977)

Figure 4 (a)-(c) visualize this conceptual split. Surface (a) is the actual shell surface with all stress resultants. Figure 4(b) shows the stretching surface (S-surface), with all the stress resultants working in plane. The external load is represented by a pressure p. Figure 4(c)shows the bending surface (B-surface) with the bending and twisting stress resultants and the shear stress resultants working out of plane.

These two surfaces are separated so the equilibrium equations can be written out separately. To relate these two surfaces together, it is stated that the sum of the pressure of each surface should equal the pressure in the non-split surface. So:

$p = p_S + p_B$

The two surfaces are also related through the equation

$g_S = g_B$

which means that the geometry of the S-surface should always be equal to the geometry of the B-surface. This means that the deformations of the two surfaces should be the same, because in reality both are one surface.

Pavlovic (1984) made a scheme to use this theory to solve an element of a shell (Figure 5). In this scheme, a value for p_S is chosen and with this value, the deformation is calculated. The new geometry g_S is set equal to g_B and from g_B the value for p_B is calculated. If the resulting p_B and the chosen p_S added up together are equal to p, the solution is reached. If not, the value for p_S should be changed and the cycle repeated until the correct solution is found. So the solution for each element in this case can be found through trial and error.

So the ratio between p_B and p_S is unknown. Once it is known how the ratio between these two is determined for each part of the structure, it is possible to make a relation between the geometry and the structural mechanics of a shell.

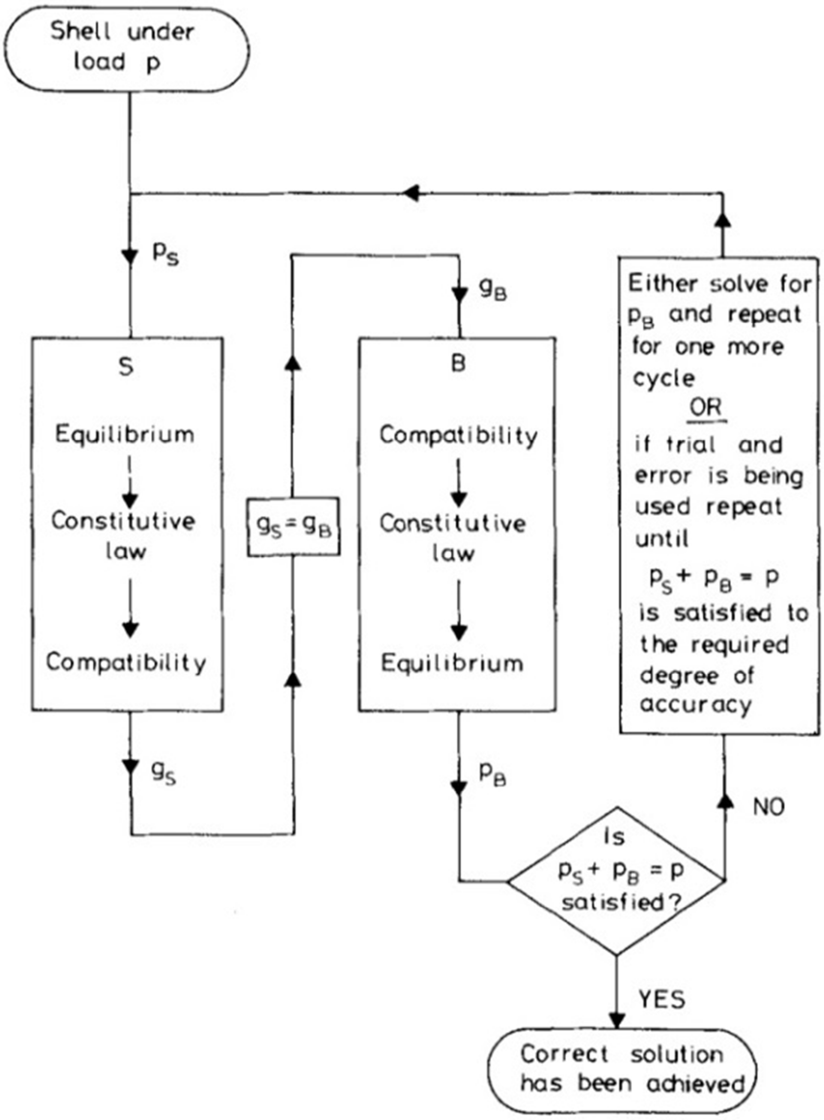


Figure 5 The flowchart by Pavlovic, used to determine the split in bending and normal forces in a shell structure (Pavlovic, 1984)

2.2 Arches and thrust lines

A thrust line is the line in which a 2-dimensional structure (arch or beam) with a given load (own weight for instance) can make equilibrium using compression only. This line can be found for masonry arches using the analogy of the chain models as shown in Figure 6. The loads resulting from the own weight are discretized in a point load and projected on a chain (a). This chain line inverted in (a) shows the thrust line, the line along which all the masonry blocks make equilibrium through compression only.

In (c) all the forces working on a masonry block are drawn. To prove that these make equilibrium, a force polygon can be drawn, using the head-tail method (d). In this method all the forces working on the block are joined together, head to tail. If the forces make a closed polygon, it means that the sum of the vectors is equal to 0, proving that the block is in equilibrium. Since two of the three forces on block 6 have their reaction forces on other blocks (5 and 7 in this case), all the closed force polygons of the different blocks can be joined together, resulting in the force polygon in (b). This force polygon and the thrust line in (a) are each other's reciprocal figures, which means that if one of them is changed, the other one will change accordingly.

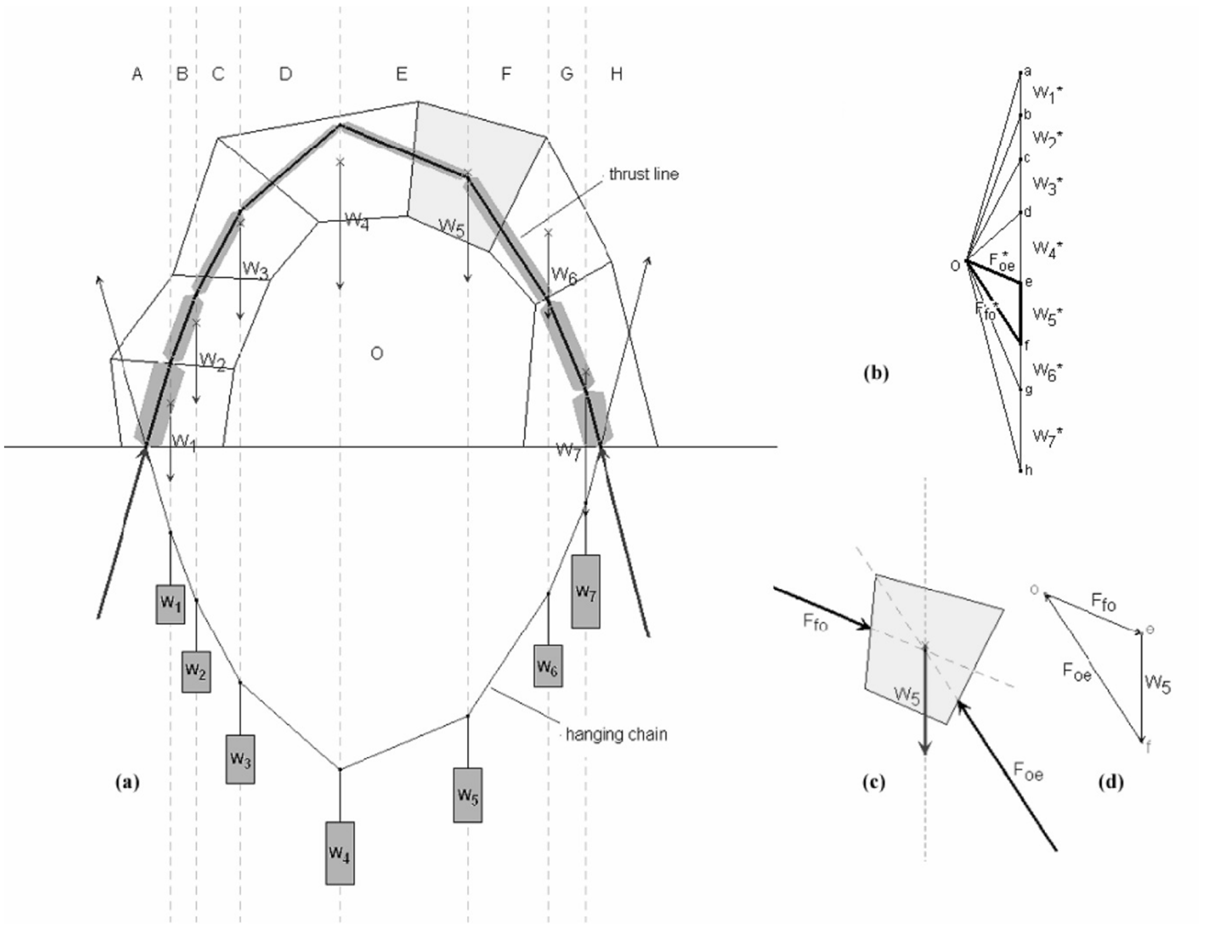


Figure 6 A thrust line in masonry blocks. (Block, 2006)

This means that other correct thrust lines can be drawn for the same load, simply by moving point O (called the polar coordinate) for instance closer to the own weight lines. If the point moves closer, the thrust line will become steeper (see Figure 7). So for each load an infinite amount of thrust lines can be drawn.

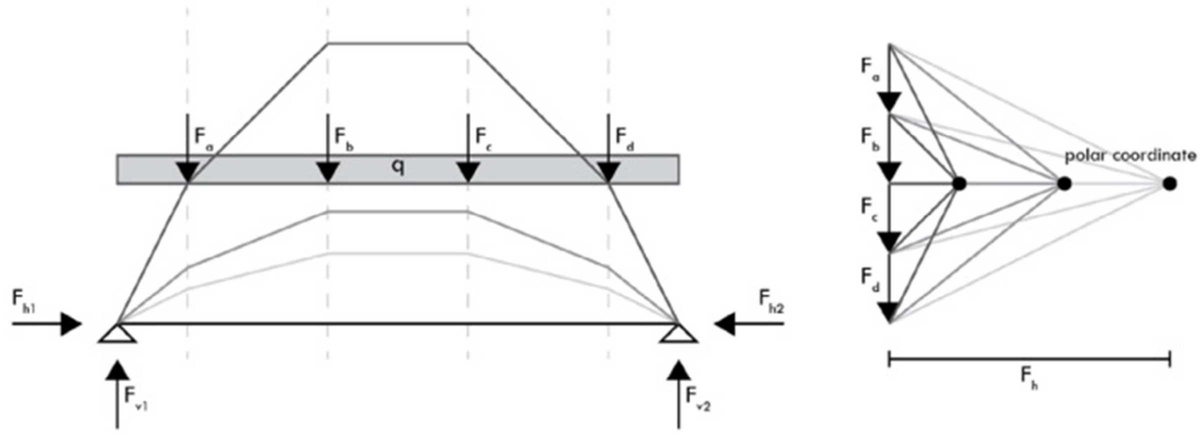


Figure 7 A set of loads can give an infinite number of thrust lines by moving the polar coordinate to the left or right. (van den Dool, 2012)

As long as the line lies within the masonry blocks, the blocks will make equilibrium through compression. Since these blocks can't handle tension or bending moments, the line has to lie within the structure, otherwise the structure will fail. However, when a structure can handle bending moments, a structure made out of reinforced concrete for instance, the line of thrust can lie outside the material without the structure failing. In this case the thrust line does not represent the actual transport of loads through normal forces anymore, because it is a combination of normal forces and bending moments.

Consider the example in Figure 8 to see what a line of thrust outside the material represents. The force in the thrust line in point A can be found in the reciprocal grid as F_A. So the force F_A can be drawn on the thrust line. (a) This force can be translated onto the structure by decomposing F_A in two forces, one shear force and one normal force, and adding a bending moment equal to the eccentricity e times F_N. (c) This can be done for the whole structure resulting in a combination of bending moment, shear force and normal force. In this case again, an infinite amount of thrust lines can be drawn. For both the thrust line in- and outside the structure, section 2.4 will explain which one the correct one is.

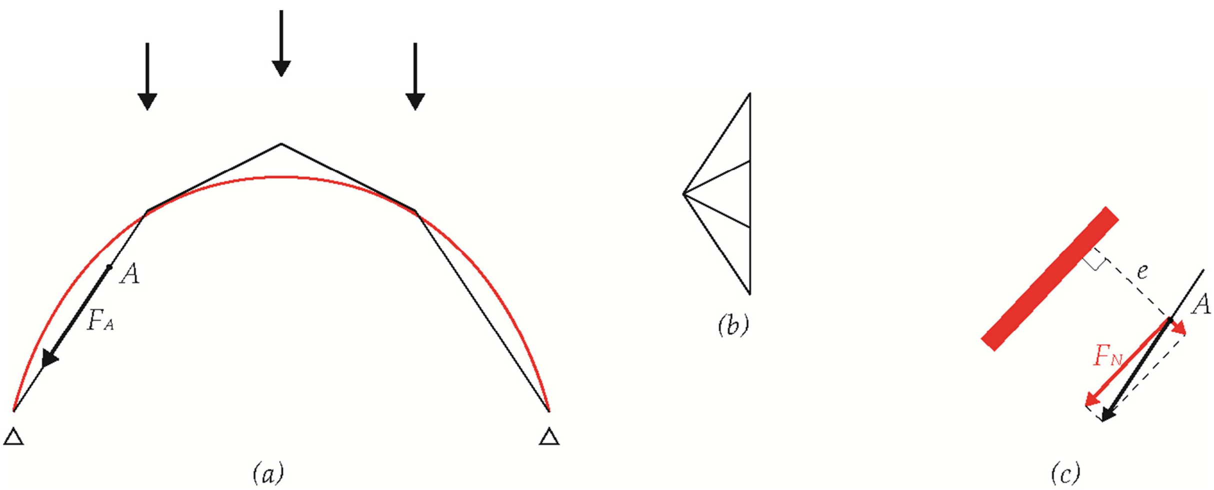


Figure 8 Projecting the force in a thrust line outside the material.

The thrust line has its 3D equivalent for plates and shells in the thrust network or thrust surface (Figure 9). The thrust network also has a reciprocal figure as can be seen in Figure 10. More on this thrust network and how to calculate it can be found in the graduation report of Tiggeler (2009) and the research of Block (2007).

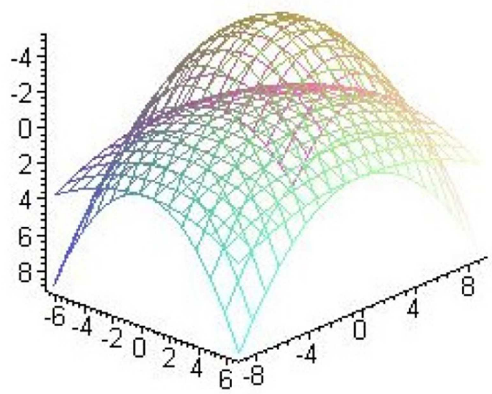


Figure 9 The 3D equivalent of a thrust line: a thrust surface (Tiggeler, 2009)

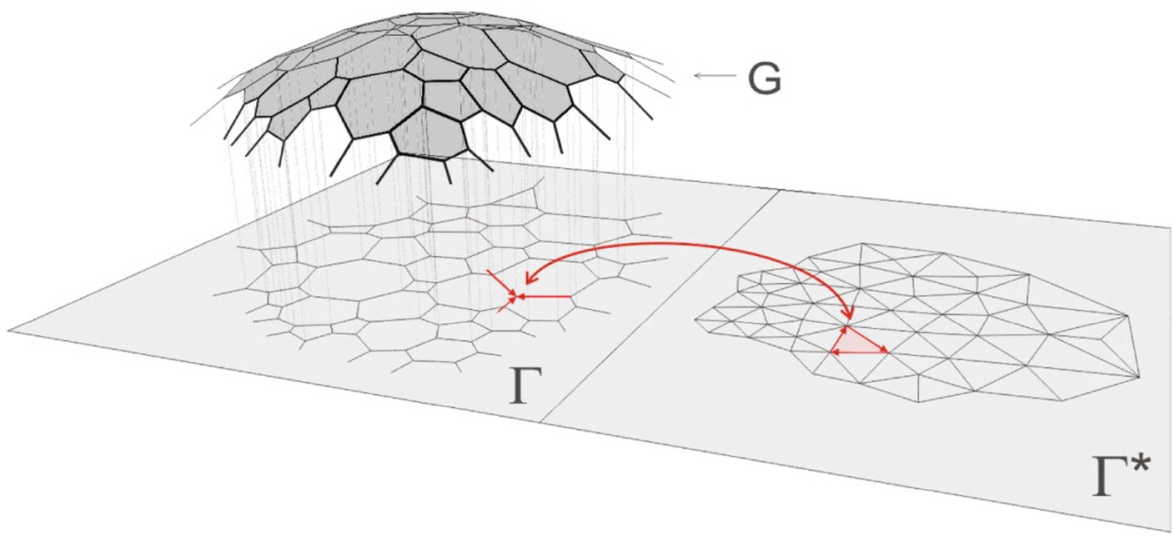


Figure 10 The reciprocal grid of the thrust surface (Block, 2009)

2.3 Uneven supports

In the previous paragraph, only situations with supports in a horizontal line (or loads perpendicular to the line between the two supports) are shown. The reciprocal figure can be used for uneven supports as well. Consider a situation as in Figure 11, four loads and two supports (a) are drawn, one of them being positioned lower than the other. For the force polygon the loads are drawn and a random position is chosen for the polar coordinate (b). If the corresponding thrust line is drawn, chances are that the thrust line will not fit to the supports. (c) To find the situations for which the thrust line will fit, the following steps need to be taken: draw a line between the first support and the end of the non-fitting thrust line (the blue line in c). Draw a line parallel to this one through the first random chosen polar coordinate and mark the position where this line intersects the loads in the force polygon drawing. Draw a line through the two supports, called the closing line (the red line in c). Finally draw a line parallel to the closing line through the point marked in the force polygon (d).

Any point on this line chosen as polar coordinate will result in a thrust line which begins and ends in the two supports (e/f). The angle of this closing line will be used in this research and will be called θ_c. (Beranek, 1980)

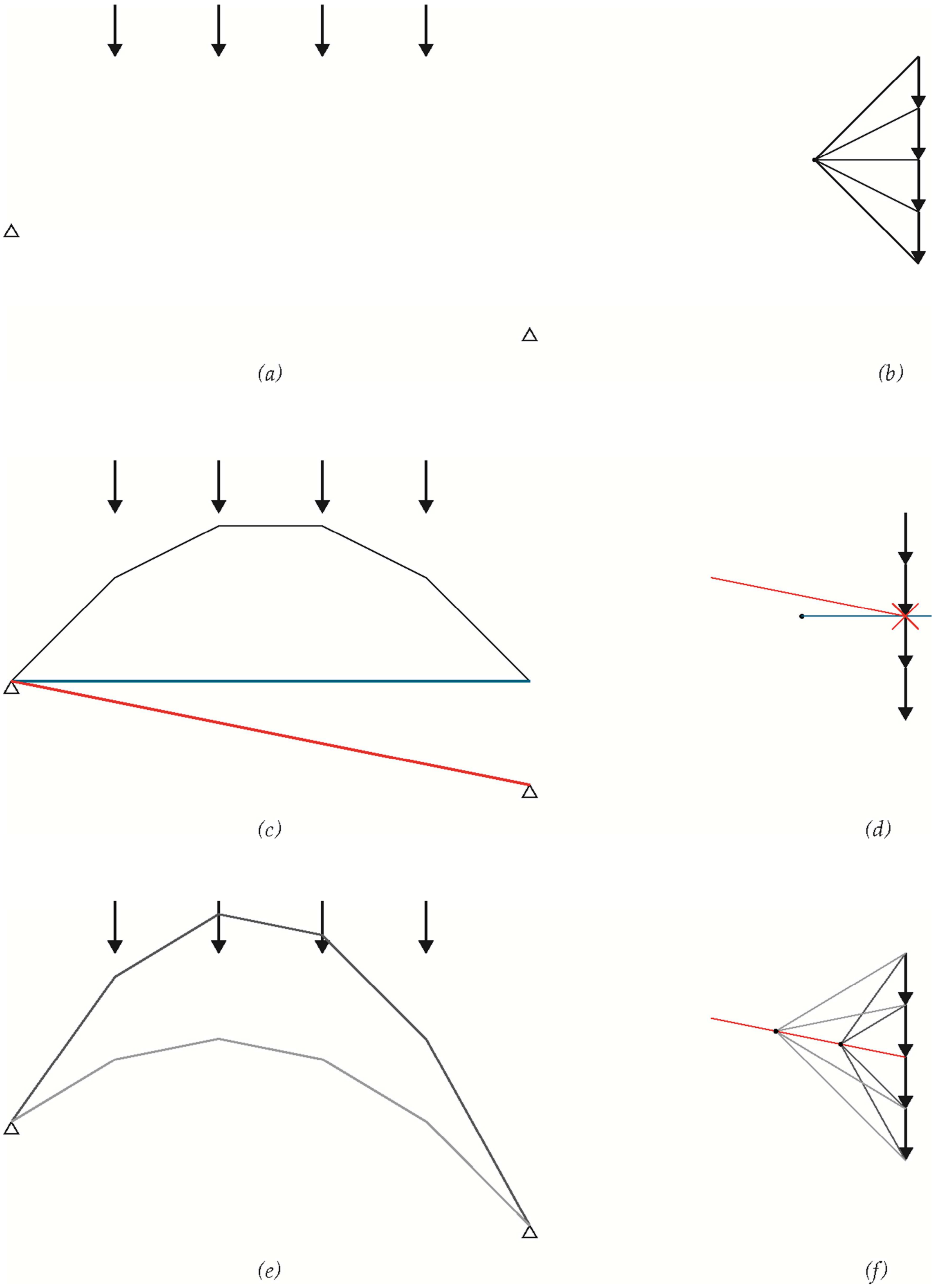


Figure 11 Determining and using the closing line

2.4 Complementary energy method

As shown in the previous sections, each set of loads can generate an infinite number of thrust lines. To determine which the correct one is, the complementary energy method can be used. This section will give a brief explanation of this method. For more examples and a more extensive explanation of the rewriting of the equations shown in this section, see the paper by Borgart and Liem (2011) and the graduation report by van Dijk (2014).

The complementary energy method is based on the premise that nature will always strive for the situation containing the least energy. Even though complementary energy is something conceptual, it does not exist in nature, when simplified it can be considered to be the energy needed to deform a certain structure. Forces can theoretically be transported through a structure in several ways, but in reality it will always be the case which contains the least deformation energy. So the correct thrust line for a structure, the one which will occur in reality, will always be the one with the least complementary energy. If the structure is made from a single homogenous material, the correct thrust line will also be the one which causes the structure to deform the least.

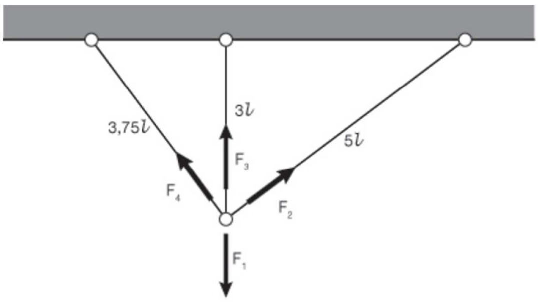
When a normal force works on a single bar the stress in the bar will cause it to strain. The work needed for this deformation is called strain energy. From the strain energy the complementary energy can be calculated by:

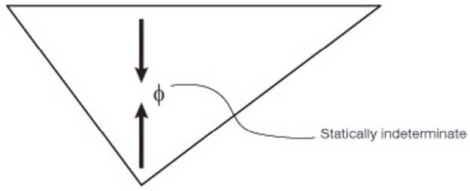
$$E_{c;N} = \frac{1}{2}\frac{N^2 l}{EA}$$

The complementary energy due to bending moments can be calculated by:

$$E_{c;M} = \frac{1}{2}\frac{M^2 l}{EI}$$

This complementary energy can be used to determine the flow of forces through a statically indeterminate structure. A simple example is shown in Figure 12, in which a statically indeterminate structure can be seen. The structure consists of three bars, three supports and one load called F_1 is applied. To calculate the forces in this structure, one force is replaced by a force with a value ϕ. Using ϕ, all the normal forces can be calculated as shown in the table. From the normal force and the length of the bars, E_c can be calculated as well, with ϕ as the only unknown. The three values for E_c added together is the total complementary energy. Taking the derivative of this total and setting it equal to zero will result in the ϕ for which the complementary energy is the lowest. Since ϕ is now known, all normal forces can be calculated.





member	N_i	l_i	E_c
1	$0{,}8(F-\phi)$	3,75l	$1{,}2\frac{l}{EA}(F-\phi)^2$
2	$0{,}6(F-\phi)$	5l	$0{,}9\frac{l}{EA}(F-\phi)^2$
3	ϕ	3l	$1{,}2\frac{l}{EA}\phi^2$

Figure 12 Calculating a statically indeterminate structure by finding the least complementary energy (Liem, 2011)

This principle can be applied to thrust lines as well. For a certain set of loads and a structure, the correct thrust line can be found by determining the one with the least complementary energy.

Since we are not interested in the actual energy but only in the situation in which the complementary energy is minimal and we consider the Young's modulus E to be equal throughout the structure, the equations for the complementary energy can be simplified to:

$$E_{c;N} = N^2 l$$

$$E_{c;M} = \frac{12}{t^2} M^2 l$$

Both these energies together will give the total complementary energy. Finding the thrust line for a structure containing the lowest total energy will result in the correct flow of forces through the structure.

2.5 Graphic statics in arches

In the graduation report by van Dijk (2014), an iterative graphical way to calculate beams and arches is described, and an iterative tool is made in Grasshopper to calculate arches in a quick and easy way.

The calculation method is summarized in Figure 13. The first step is to draw an arch, which can be irregular shaped and with uneven supports. The program discretizes both the projected load and the load due to own weight of the arch. This discretized load can lead to an infinite number of force polygons and their corresponding thrust lines as shown in Figure 7. From a thrust line and the structure together, the complementary energy is calculated as shown in the previous paragraph. With the height of the thrust line as a variable, the solution with the least complementary energy can be found, simply by changing the horizontal support until the lowest value is found.

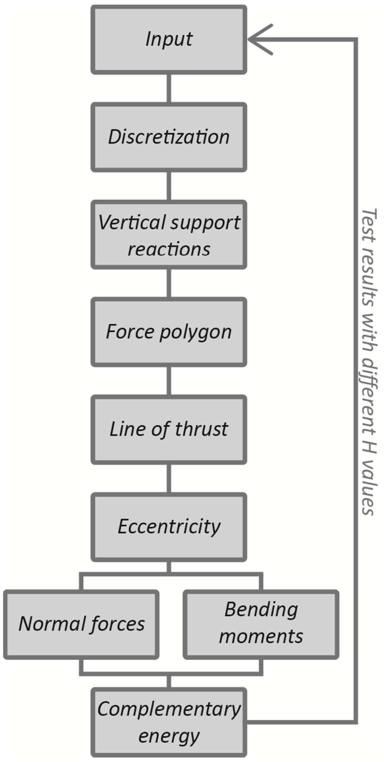


Figure 13 Calculating a thrust line by changing the horizontal support reaction (van Dijk, 2014)

In this report, the discovery is described that the area under the correct line of thrust equals the area under the structure (Figure 14). This leads to a shorter iteration loop (Figure 15) for calculating the correct thrust line. Even though there is not a mathematical proof for these areas being equal yet, this reports shows some arches (Figure 16) for which the forces are calculated using three different methods: the method of equal areas, the Finite Element Method and the method using the lowest complementary energy. This leads to results with a maximum deviation of 4%. These deviations are attributed to different ways of discretizing loads in different calculation methods.

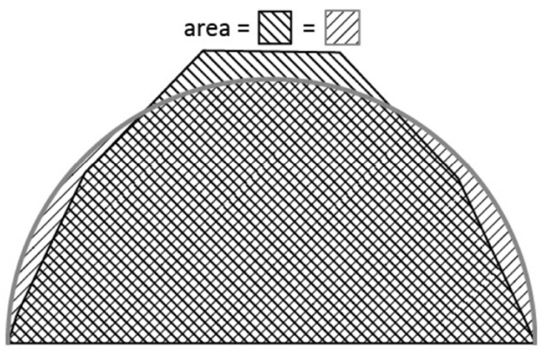


Figure 14 The area under the structure (grey) equals the area under the thrust line (black). (van Dijk, 2014)

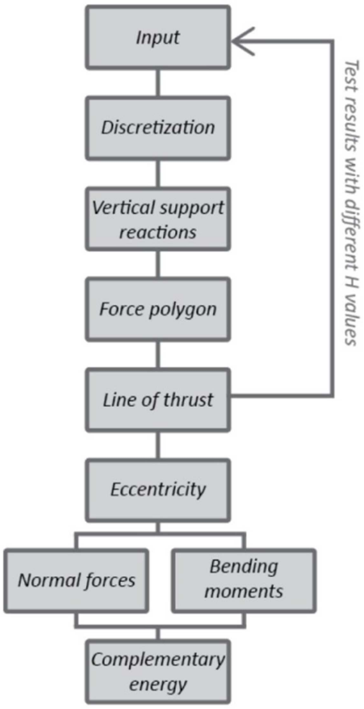


Figure 15 The shorter iteration loop uses the method of equal areas (van Dijk, 2014)

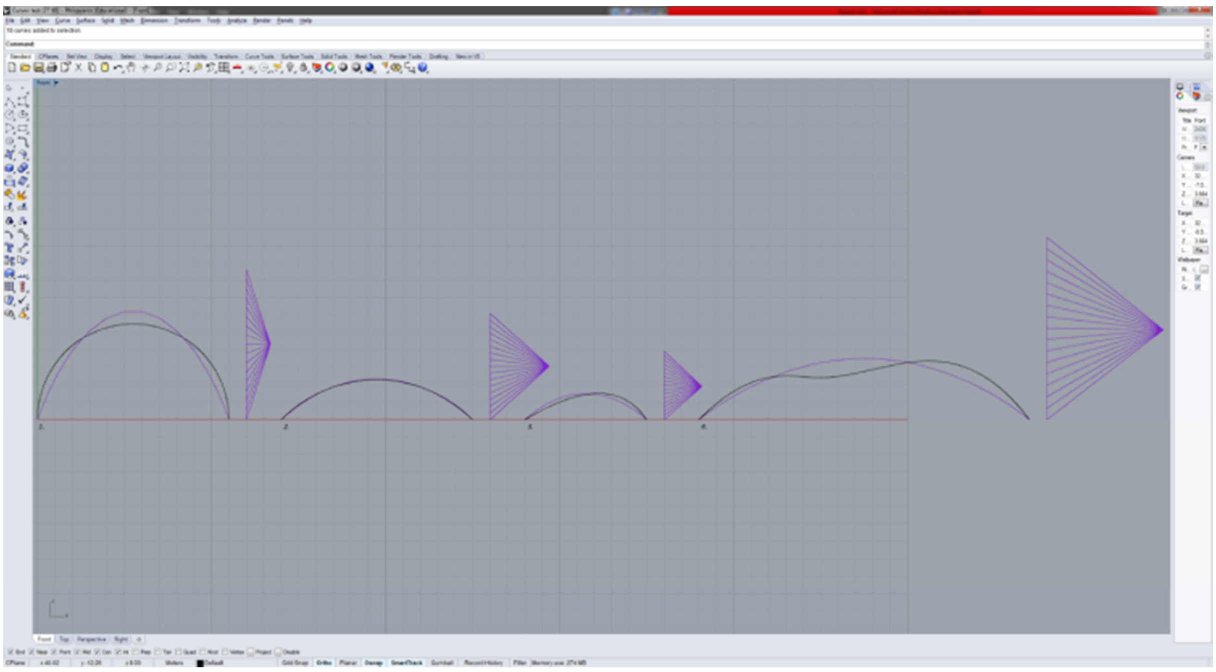


Figure 16 Arches for which the method of equal areas was tested (van Dijk, 2014)

2.6 Force density

Using the thrust line and its reciprocal figure, the force polygon, some observations are made on how these work together. This section summarizes these observation, not to prove a theory, but to make the reader familiar with how these relate to each other. In this section the force density is introduced as well, a ratio between the force in a bar and the length of that bar.

If we consider a force polygon as shown in Figure 17 corresponding to a structure, the force F and the length l of a bar can be drawn in one figure. If both are projected onto a horizontal line, the ratio between F and l will stay the same, so:

$$\frac{F}{l} = \frac{F'}{l'}$$

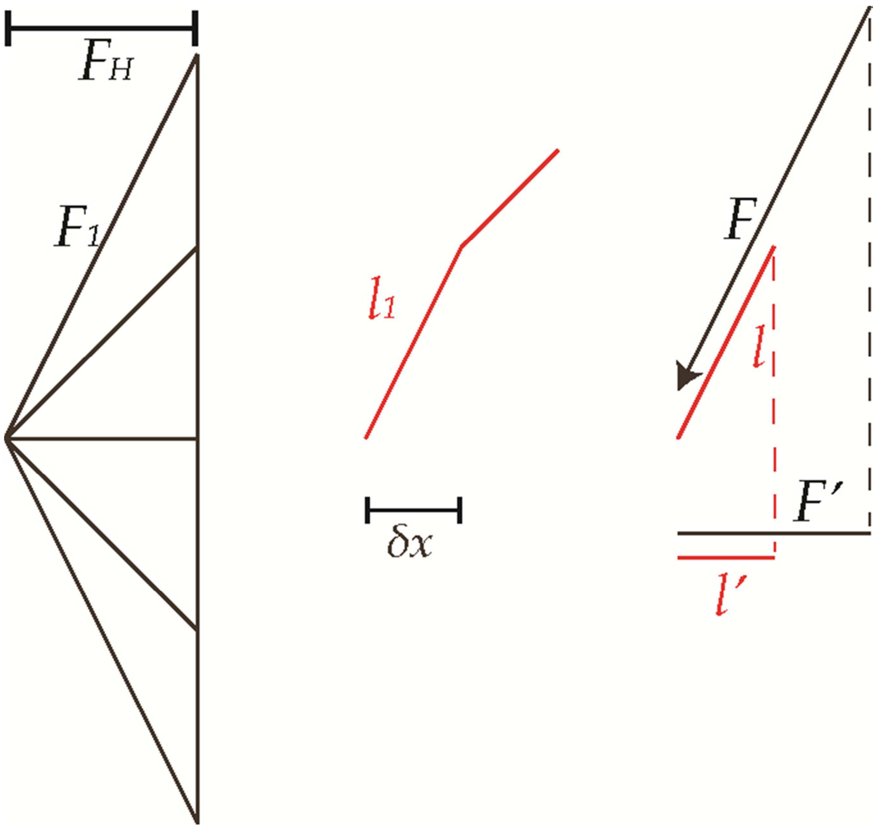


Figure 17 Relation between force polygon and structure

The horizontal projection of each force in the force polygon equals F_H as can be seen in the force polygon. The horizontal projection of each bar in the structure equals δx. The force density (*FD*) equals the force divided by the length of a bar, so:

$$FD = \frac{F_n}{l_n} = \frac{F_H}{\delta x}$$

In a similar way can be shown that:

$$FD = \frac{F_V}{\delta h}$$

This explains why if the polar coordinate of a reciprocal grid is moved horizontally, the FD will change in a similar way. If F_H doubles, FD doubles as well. This also explains why the FD in each bar is the same when a constant δx is chosen. The F_H for each bar is equal resulting in an equal FD.

Figure 18 shows a force and a bar length in one figure. For the F, the F_V stays the same, for the l, δx stays the same. F_H is changed by steps of 1. If $\delta x = 2$, this figure shows that the FD will increase by ½.

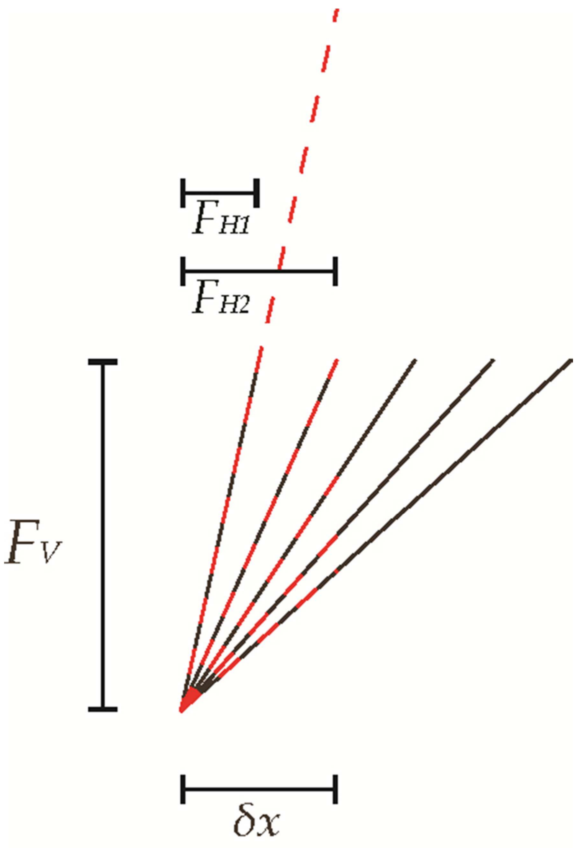


Figure 18 The force in a bar and the length of that bar in one figure

2.7 Calculating the area of a polygon

Part of this research deals with trying to prove that the area under the structure equals the area under the correct thrust line. In this process Lopshits (1956) way to calculate the area of any force polygon is used. This paragraph will explain how this method can be used.

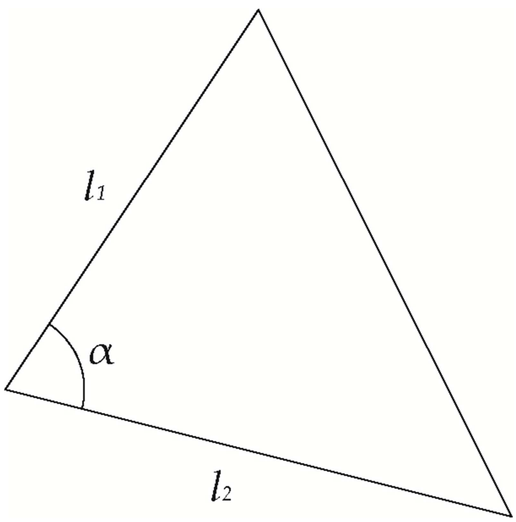


Figure 19 Calculating the area of a triangle

This method calculates the area by using only the length of each segment and the inner angle of each two segments. The area of for example the triangle in Figure 19 can be calculated by:

$$A = \frac{1}{2} l_1 l_2 \sin \alpha$$

To understand the way the Lopshits equation works, the following concept needs to be understood. Consider any parallelogram, the one in Figure 20 (a) for instance. Two points on the same place on two opposing sides are moved (the drawing may seem like a 3D representation, it is a 2D drawing though), creating two new parallelograms (b). The area of the two parallelograms together (2 and 3) equal the area of the first parallelogram (c). This can be seen if we look at the areas marked as A, A' and B and B' in (d). The area removed at the bottom is added at the top of the figure, resulting in the same area. This goes for the triangles marked in (e) as well, since each triangle equals half the parallelogram.

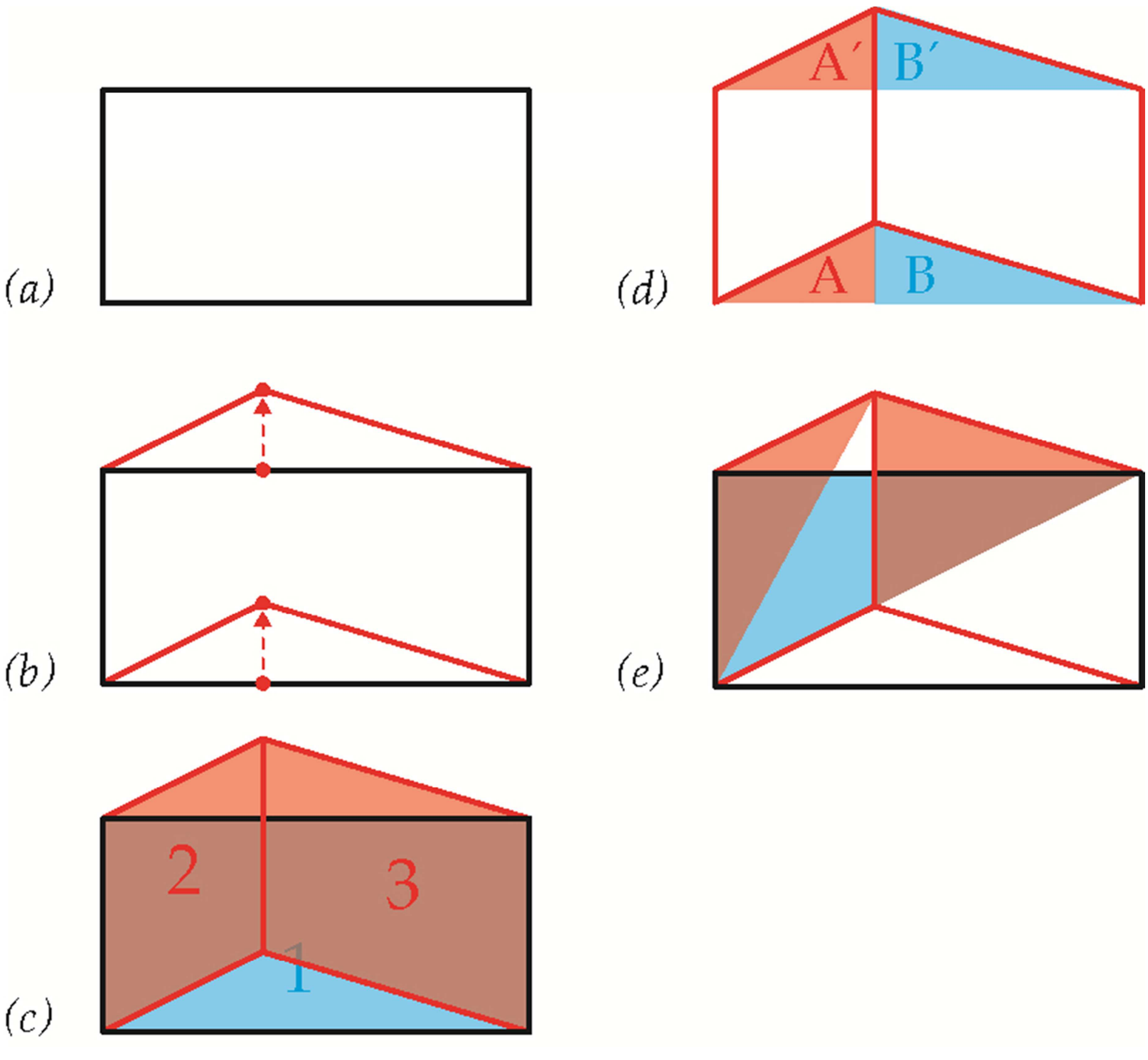


Figure 20 The area of rectangle 1 equals the areas of the parallelograms 2 and 3 together

Consider as an example for the Lopshits calculation the polygon drawn in Figure 21, which consists of five points. The coordinates of the points are unknown, only the lengths of l_1 to l_5 are known and the angles α_1 to α_5. To calculate the total, Lopshits adds up the areas shown in Figure 22:

$$A_1A_2A_3A_4A_5 = A_1A_2A_3 + A_1A_3A_4 + A_1A_4A_5$$

The first step is quite simple since all the variables are known:

$$A_1A_2A_3 = \frac{1}{2} l_1 l_2 \sin \alpha_1$$

For the second step, calculating $A_1A_3A_4$, Lopshits translates the figure (or at least the points needed for this calculation) over l_3, creating several parallelograms (Figure 24). Using the theory above, we can see that:

$$A_1A_3A_4 = A_1A_2B_1 + A_2A_3B_2$$

The same is done for the final triangle. Even though the final triangle can be calculated by using l_4, l_5, and α_5, in this example it will be calculated using Lopshits equation for consistency. In this case the figure is translated over l_4. Since this is not the biggest triangle of the ones used (like triangle 1 in Figure 20), not all the triangles need to be added, one needs to be subtracted. An easy way to determine this is to check in which direction the points rotate from low to high, A to B. The areas of the triangles with the same direction as the triangle to be calculated need to be added (Figure 25), the other one subtracted. This results in:

$$A_1A_4A_5 = -A_1A_2C_1 + A_2A_3C_2 + A_3A_4C_3$$

Combining these equation we see that:

$$A_1A_2A_3A_4A_5 = A_1A_2A_3 + A_1A_2B_1 + A_2A_3B_2 + -A_1A_2C_1 + A_2A_3C_2 + A_3A_4C_3$$

To calculate the total area, only the angles shown in Figure 26 needs to be known. These can be calculated using the known angles.

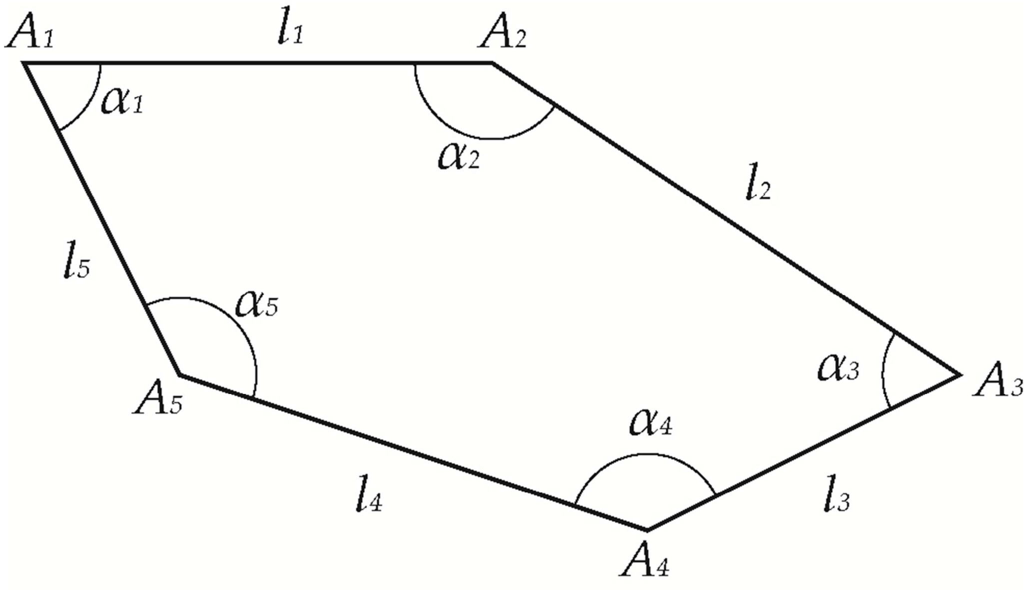


Figure 21 A random chosen polygon with only length and inner angels known

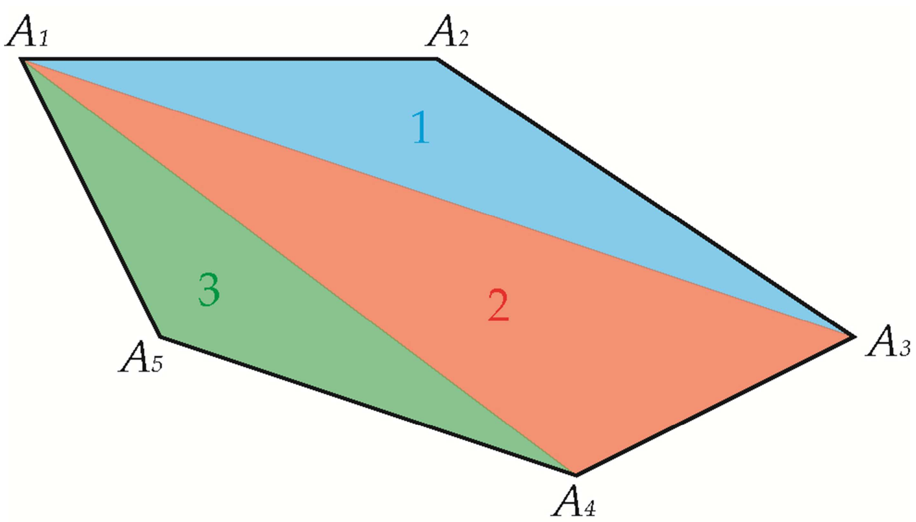


Figure 22 The polygon is split up in triangles

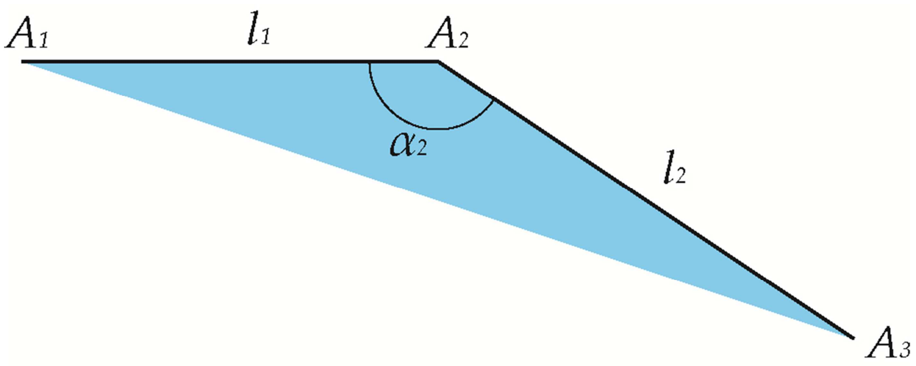


Figure 23 Calculating $A_1A_2A_3$

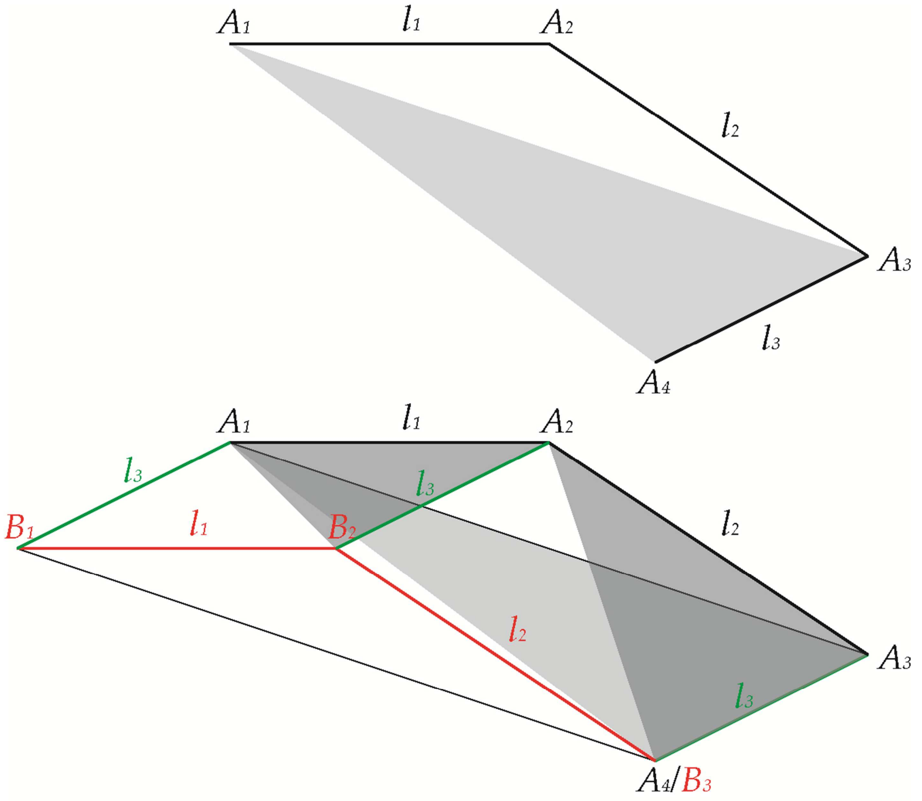


Figure 24 Calculating $A_1A_3A_4$

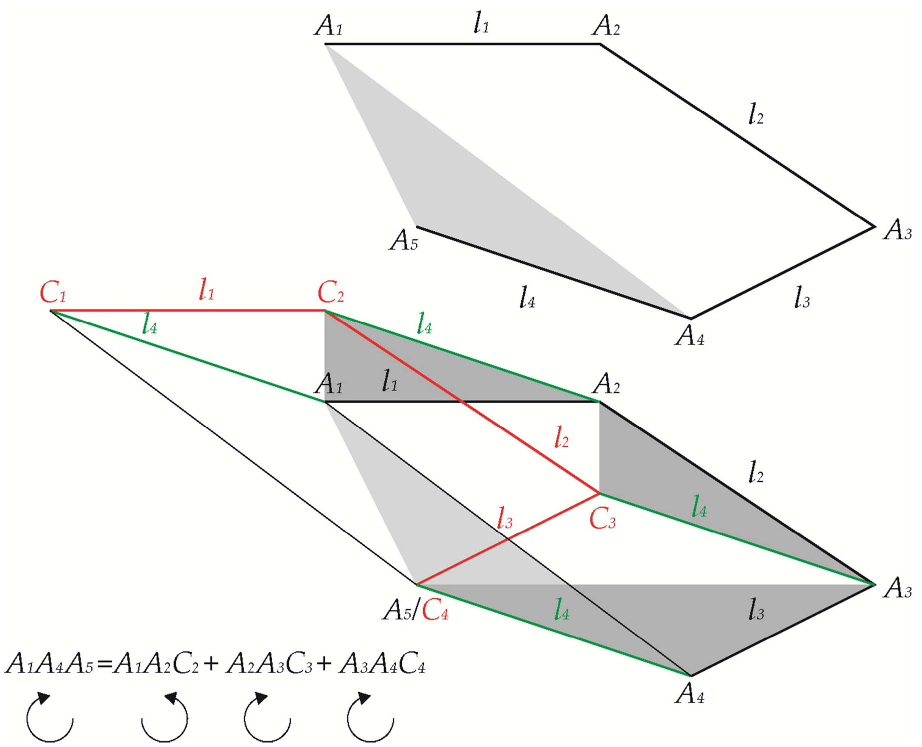


Figure 25 Calculating $A_1A_4A_5$

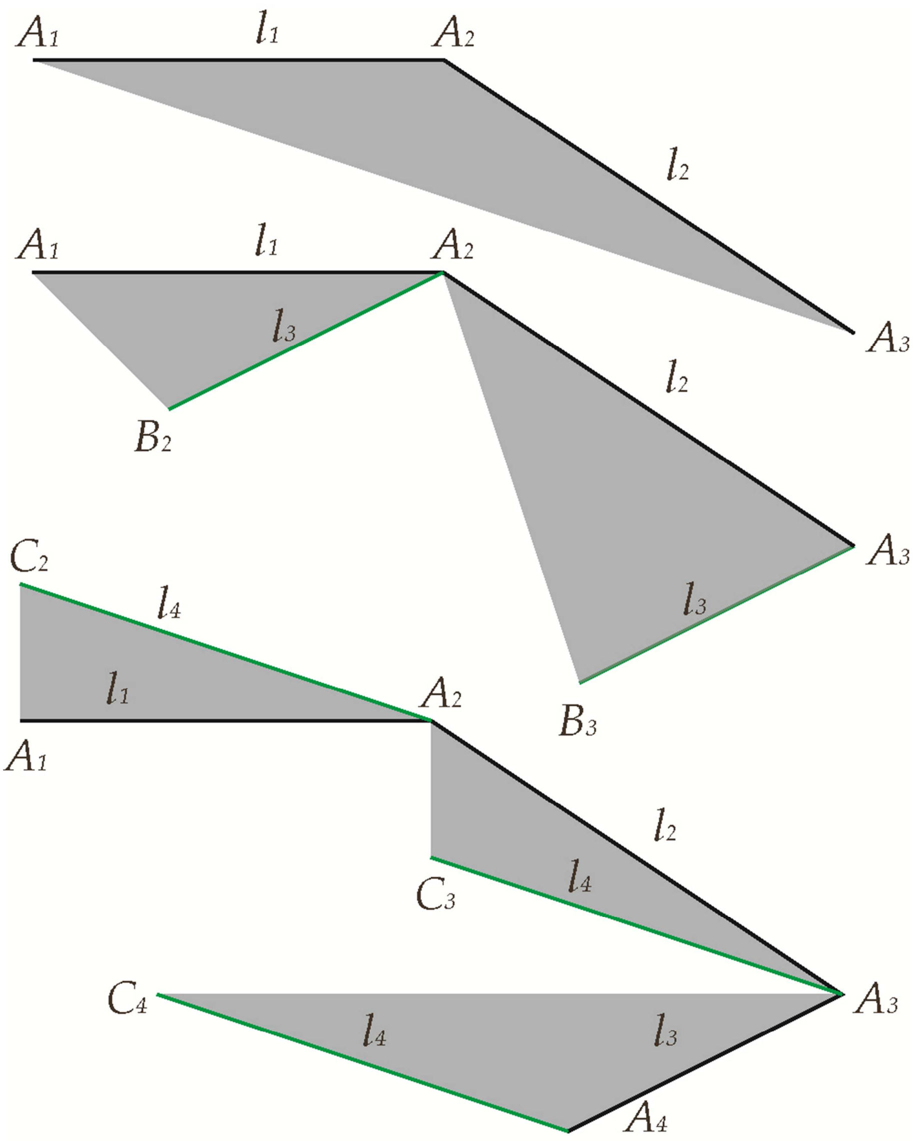


Figure 26 All angles and lengths needed to calculate the total surface of the polygon

2.7.1 Lopshits for thrust lines

In Figure 27 a simple thrust line is drawn and Lopshits is used to calculate the area. In this case not the inner angles are known but only the differences between the slopes is known and used to calculate the area. The first triangle can simply be calculated by:

$$A_1 = \frac{1}{2} l_1 l_2 \sin(180 - \beta_1)$$

For the second triangle the two red triangles in Figure 28 are calculated. The lengths are all known. The figure shows that the first inner angle equals 180 - β_1 - β_2. In a similar way all the areas can be calculated:

$$A_2 = \frac{1}{2} l_1 l_3 \sin(180 - \beta_1 - \beta_2) + \frac{1}{2} l_2 l_3 \sin(180 - \beta_2)$$

$$A_3 = \frac{1}{2} l_1 l_4 \sin(180 - \beta_1 - \beta_2 - \beta_3) + \frac{1}{2} l_2 l_4 \sin(180 - \beta_2 - \beta_3) + \frac{1}{2} l_3 l_4 \sin(180 - \beta_3)$$

$$A_4 = \frac{1}{2} l_1 l_5 \sin(180 - \beta_1 - \beta_2 - \beta_3 - \beta_4) + \frac{1}{2} l_2 l_5 \sin(180 - \beta_2 - \beta_3 - \beta_4) + \frac{1}{2} l_3 l_5 \sin(180 - \beta_3 - \beta_4) + \frac{1}{2} l_4 l_5 \sin(180 - \beta_4)$$

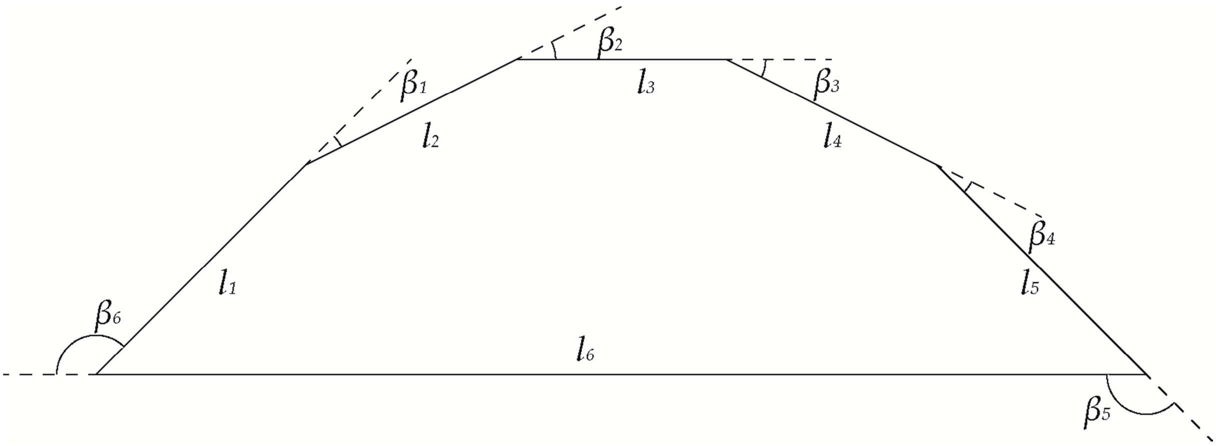


Figure 27 A thrust line with the lengths and change in slope known

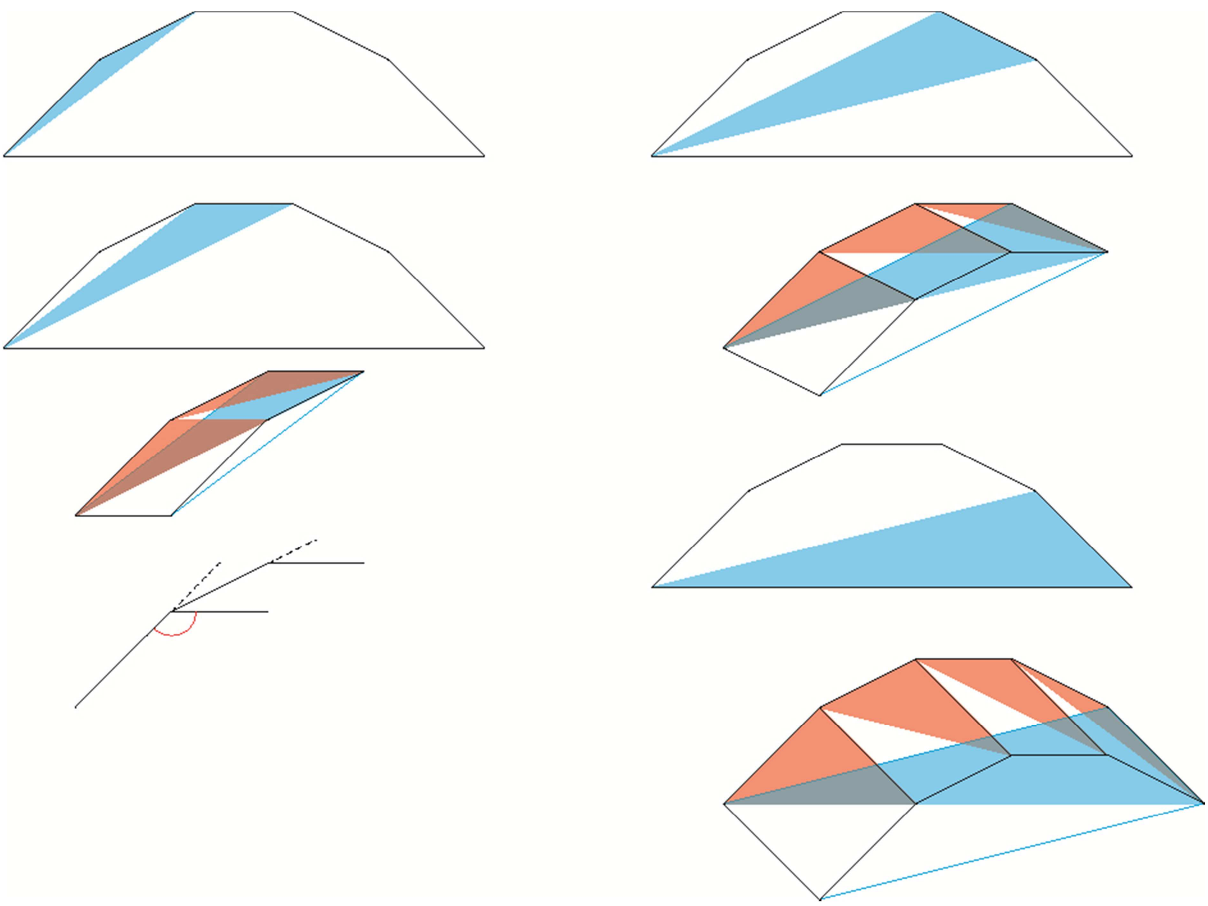


Figure 28 Calculating the area of thrust line with Lopshits equation

2.7.2 Relating the angles of a thrust line

If in a force polygon the polar coordinate moves, the thrust line changes in a uniform way. The angles of all the bars change in a uniform way as well. This shows that they are all related to each other. The attempts to relate the angles which are described in chapter 3 use a way of relating them through F_H. Another way in which this relation can be uses is within the Lopshits theory. This theory uses the outer angle α or the inner angle β. These angles are of course related:

$$\beta_n = 180 - \alpha_n$$

This section shows how the angles between two forces in the force polygon can be seen as the outer angles in the thrust line.

In Figure 29 can be seen how these are related. The β angles in the force polygon can be seen in the thrust line as well. β_1 and β_6 can be found in the force polygon if we extend the horizontal supports (or the closing line in the force polygon). This line is not an actual force in this direction but it shows where this angle can be found.

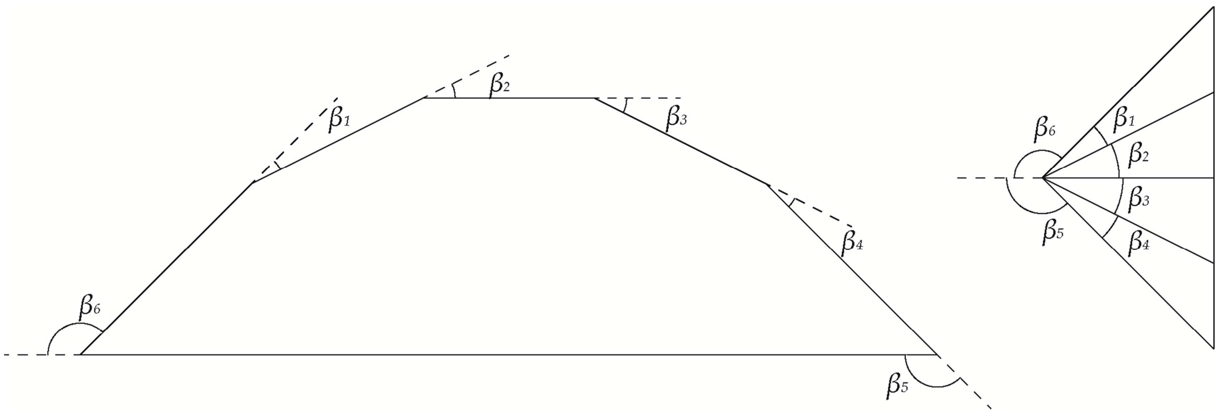


Figure 29 The outer angles of a thrust line equal the angles in the force polygon

2.8 Conclusions

The aim of this chapter is to determine which methods can be used to calculate arches and shells. To understand the methods, the relation between arches and their thrust lines is described.

The only new theory described is the method using equal areas (section 2.5). This method looks very promising, the first calculations suggest the method works. So if the method can be proven it will provide a simple and easy to understand method to calculate thrust lines. If it can be translated to shells, it can make shell calculating simpler as well.

The second method described is the complementary energy method, an iterative method to calculate arches and shells. This method works but does not give a direct result and does not provide a direct link between the structure and the calculation results. If this iterative method is written out in an analytical way and the resulting equation is rewritten for finding the lowest value it offers a direct method of calculating arches. It is expected that this method can then be translated to shells as well.

If the equal area method is correct as well, both methods can be linked together. If the complementary energy method is written out analytically, it should result in the same as the area method. Rewriting one method into the other will prove the validity of the equal area method. The starting point can be either one of these methods.

Since the aim of this research is to find a simple method to directly calculate arches and shells, the most logical next step is to investigate the equal area method and try to prove it.

CHAPTER 3 - CALCULATING ARCHES THROUGH A DIRECT METHOD

This chapter deals with the main aim of the research: finding a new method to calculate arches. In section 2.4, a method is described to calculate the correct thrust line for a certain load in an iterative way, the complementary energy method. Section 2.5 describes how van Dijk (2014) suggests a new way of calculating the correct thrust line using the equal area method. The first step in this chapter is to connect these two methods with each other. If the minimum energy method, which can be considered a valid method, is translated into the equal area method, this method can be considered proven. Proving this method is the most logical first step because it is a plausible method, since the calculation results are equal to FEM results. If it is a valid method it provides a simple to use graphical method.

Proving this method can have two starting points, either by translating the complementary energy method into something related to the areas of the structure and thrust line or by describing the equal area method and see how this can be translated into something related to the complementary energy method. These two approaches are described in the first two sections. From the equal area method an equation is derived which allows for a direct calculation of the thrust line instead of an iterative one. This equation is translated into a tool which calculates the thrust line from a structure as input using the equal area method. However, when testing this tool by comparing it to other calculation methods, the results are slightly off, causing the first doubts about this method.

The third section is a more extensive comparison of different calculation methods, including the equal area method, all tested for a wider variety in topology of structures. From these tests, the conclusion is drawn that the theory of equal areas does not always give the correct result.

During the research aimed at proving the equal area method, a new idea emerged: the idea that minimizing the energy resulting from a bending moment will give the correct thrust line. This idea is further explored in section 3.4. Using this idea, a new method is developed to calculate the correct thrust line, first for a simple 2-bar structure, after that for a more generic one. This section also tests this new method for accuracy and discusses why this method does work.

3.1 MINIMIZING COMPLEMENTARY ENERGY FROM NORMAL FORCES

The minimum complementary energy method is the method to calculate the thrust line. The equal area method is a quick and simple method to calculate the thrust line as well but this method is not proven yet. The first step in this research is to describe in an analytical way how the solution with the minimum complementary energy can be found. If this equation is rewritten to an equation which outputs the minimum complementary energy, it will provide a direct method to calculate the thrust line.

The expectation is that the result of this can be related to the equal area method, and will thus prove this method. If this method is proven, the result will be a simpler method of determining the thrust line. If the complementary energy method however can not be related to the equal area method, the rewritten and minimized equation will still provide a correct method of calculating the thrust line.

Since describing the energy in a structure mathematically is a very extensive exercise, it is first done for a simplified situation. The first analytical description of the energy in a structure is done for normal forces only. Also, the starting point is not a structure which needs to be calculated, but just a set of loads and two supports. Once the simpler situation is described, the method can be expanded to more extensive situation, including a predefined structure and also including energy due to bending moments.

First of all, the starting point of this research will be discussed, for instance which variables are known and which are unknown. After that, the aim of this part of the research is described, followed by a section in which an equation is derived using θ_n as the only known variable. Since this means that there are still multiple variables, the θ of each bar, the next paragraph expresses all the different angles in one single angle, θ_1. The final section discusses the results.

3.1.1 STARTING SITUATION

The aim is to calculate a thrust line for a given load and supports. These supports are not necessarily even. See for instance Figure 30, two supports and four loads and their positions together define the starting situation. For the reciprocal figure, only the loads are known, which make up the line at the most right of the figure. As described in section 2.3, the closing line can be determined as well and drawn in this partial reciprocal figure. The result is a line along which the polar coordinate can move to make a thrust line which touches both the supports. One point on this line represents the solution containing the least complementary energy.

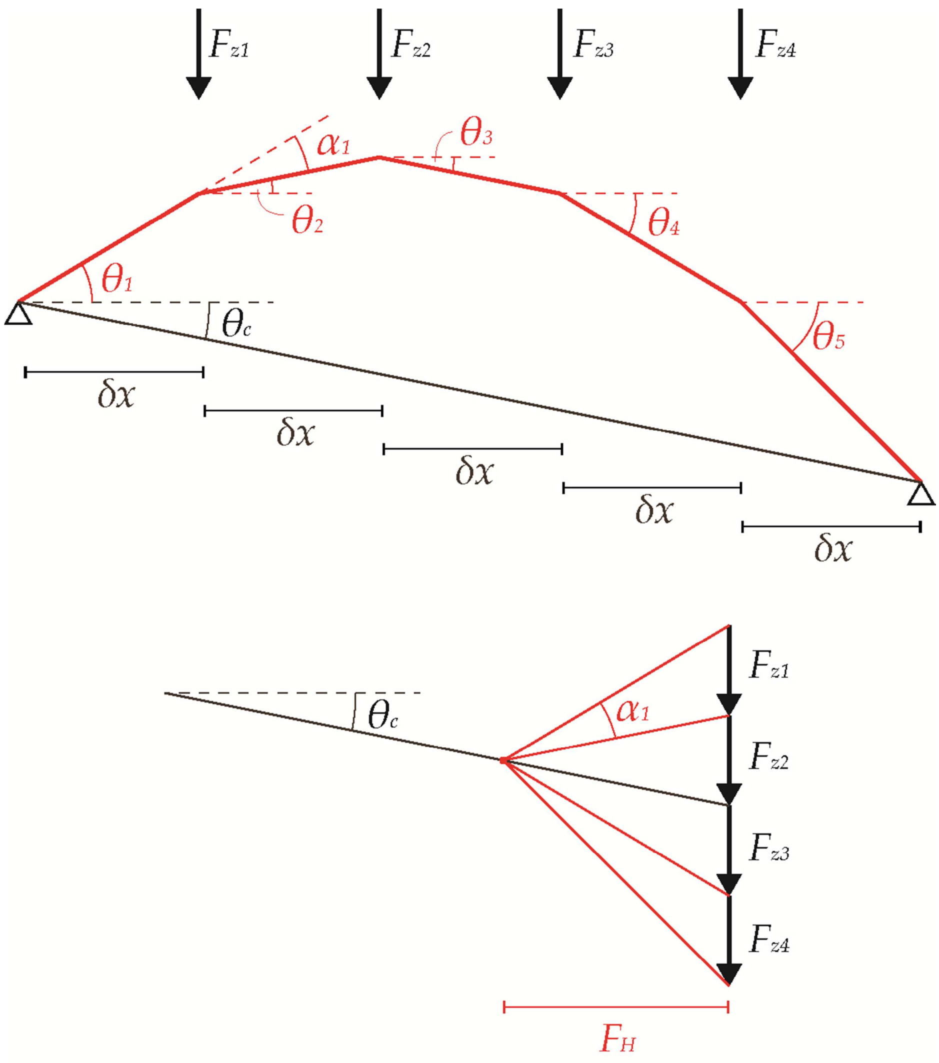


Figure 30 The initial situation in black, with the unknown values in red

In this situation, some variables (which are all dependent on each other) are unknown, others are known.

Unknown variables	*Known variables*
F_H	Fz_1 - Fz_n
$\boldsymbol{\alpha}$	$\delta x_1 - \delta x_n$
$\boldsymbol{\theta_1}$	θ_c
$\boldsymbol{\theta_n}$	

3.1.2 Aim

In this part of the research a structure is not introduced yet. The aim is to find the thrust line which can transport a certain set of load in the most efficient way. This question can be answered by finding the thrust line with the least complementary energy due to normal forces.

The complementary energy for each bar can be calculated by:

$$E_{c;N} = N^2 \cdot l$$

Moving the polar coordinate closer to the loads will result in a small normal force and a big bar length. Moving it further away will do the opposite. So in this case we are looking for the best trade-off between *N* and *l*. Complicating factor is that this trade-off will probably be different for each bar of the structure. But since the shape of all bars are related together through the reciprocal grid, there is probably a way to relate them in an analytical way.

The most logical way for calculating the correct thrust line in a direct way is by setting one variable, which defines where the polar coordinate is placed. With only this one variable the complementary energy must be calculated. Once an equation is found with only one variable, the derivative can be taken from this equation and can be set equal to zero. The solution for this equation will result in the correct thrust line.

3.1.3 Energy dependent on the angles

This section explains how the complementary energy can be calculated with only the slopes of the bars (θ) as variables. Figure 31 is a part of a polar figure in which only F_n, F_{n+1} and Fz_n are shown. In this drawing, the slope of the forces and the difference between the two slopes (α_n) are visible. F_{n+1} is extended and a new line called A is introduced. The upper angle of this triangle is called β_n. Since the two opposite angles are equal and the two right angles are equal, the angle underneath α_n is equal to β_n. From this drawing follows:

$$F_n = \frac{A}{\sin \alpha_n}$$

$$A = \cos \beta_n \cdot F_{zn}$$

$$\beta_n = \theta_n - \alpha_n$$

Since α_n is the difference between two slopes

$$\alpha_n = \theta_n - \theta_{n+1}$$

These equations combined give the equation for the force in a member expressed in the slope of that member, the member next to it:

$$F_n = \frac{\cos \theta_{n+1} \cdot F_{zn}}{\sin(\theta_n - \theta_{n+1})}$$

The length of a member can be expressed in the slope and the δx:

$$l_n = \frac{\delta x_n}{\cos \theta_n}$$

Substituting these equations in the simplified equation for the complementary energy as described in section 2.4 gives:

$$E_{c;Nn} = {N_n}^2 \cdot l_n$$

$$E_{c;Nn} = \left(\frac{\cos\ \theta_{n+1} \cdot F_{zn}}{\sin(\theta_n - \theta_{n+1})}\right)^2 \cdot \frac{\delta x_n}{\cos\theta_n}$$

$$= \frac{cos^2\theta_{n+1} \cdot F_{zn}{}^2}{sin^2(\theta_n - \theta_{n+1})} \cdot \frac{\delta x_n}{\cos\theta_n}$$

Which means for the complementary energy in the whole structure:

$$E_{c;N} = \sum \frac{cos^2\theta_{n+1} \cdot F_{zn}{}^2}{sin^2(\theta_n - \theta_{n+1})} \cdot \frac{\delta x_n}{\cos\theta_n}$$

The result is an equation which uses the loads and distance between the loads as input (F_z and δx) and has the θ as a variable.

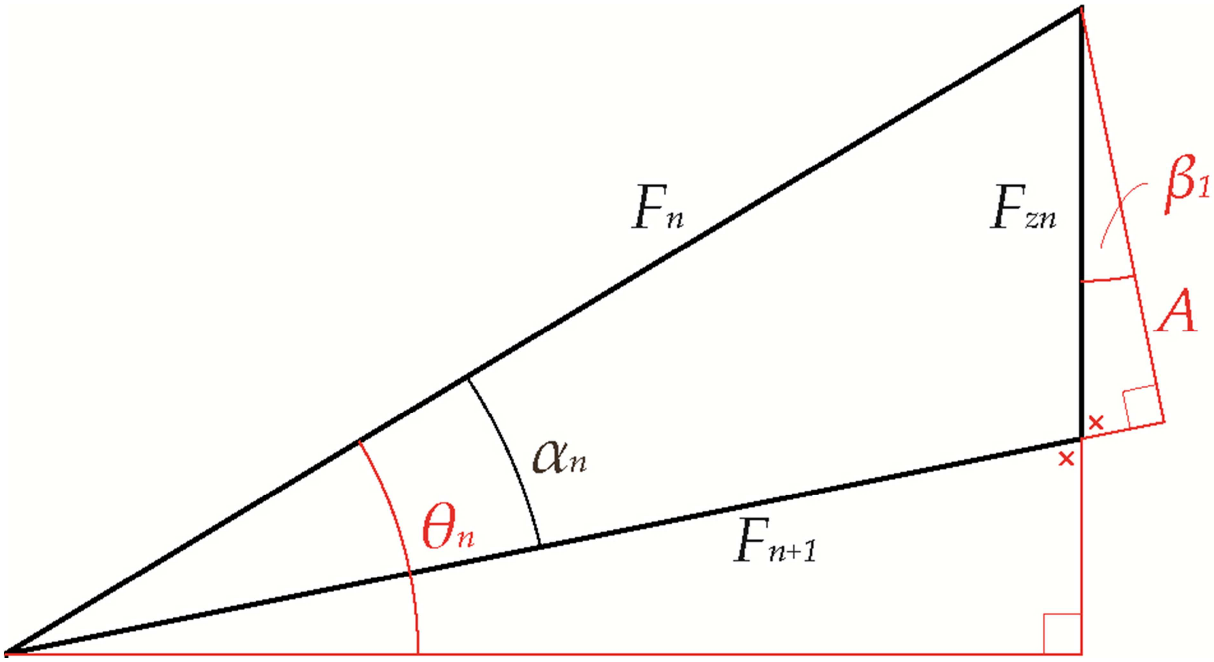


Figure 31 A part of a polar figure

This equation is tested in Excel for a simple situation and a situation with uneven supports. The result is the same result as a calculation by hand, showing that the equation derived is correct and can be used to calculate the complementary energy due to normal forces. The file can be obtained from the CD and is named 3.1.3_minEcN.

The next step would be taking the derivative of this function and setting it equal to zero to find the situation with the least complementary energy. However, there are two problems which need to be solved in order to be able to take the derivative. The first one being the fact that there are (for each member) two variables, θ_n and θ_{n+1}. The second problem is the fact that for the whole structure, there are as many variables as there are members, from θ_1 up to θ_n. For taking the derivative, only one unknown variable is allowed. The polar figure shows that changing the polar coordinate affects all the slopes, so all the slopes are related to each other and supposedly can be expressed in one variable. This way the energy can be calculated with only one variable, which should lead to a direct method of calculating the correct thrust line. The next section describes an attempt to change this equation into having only one variable

3.1.4 Relating all angles

The first attempt was to use θ_1 as the only variable. In the initial situation (Figure 30), θ_1 determines the length and position of F_1. From this the position of the polar coordinate follows which results in all the slopes of all the forces. This means that the energy of the whole structure can be expressed in θ_1 (or in any other θ).

In Figure 32 can be seen how the horizontal support reaction F_H can be calculated from the known values. In this figure the values F_{vA} and F_{vB} are introduced, which are only temporarily needed.

$$F_H = \frac{F_{vA}}{\tan\theta_1}$$

$$F_H = \frac{F_{vB}}{\tan\theta_c}$$

And if we define F_{vn} as the vertical distance from the point where the closing line intersects the loads to the point where F_n crosses the loads:

$$F_{v1} = F_{vA} + F_{vB} = (\tan\theta_1 + \tan\theta_c) \cdot F_H$$

$$\tan\theta_n = \frac{F_{Vn}}{F_H} = F_{Vn} \cdot \frac{\tan\theta_1 + \tan\theta_c}{F_{v1}}$$

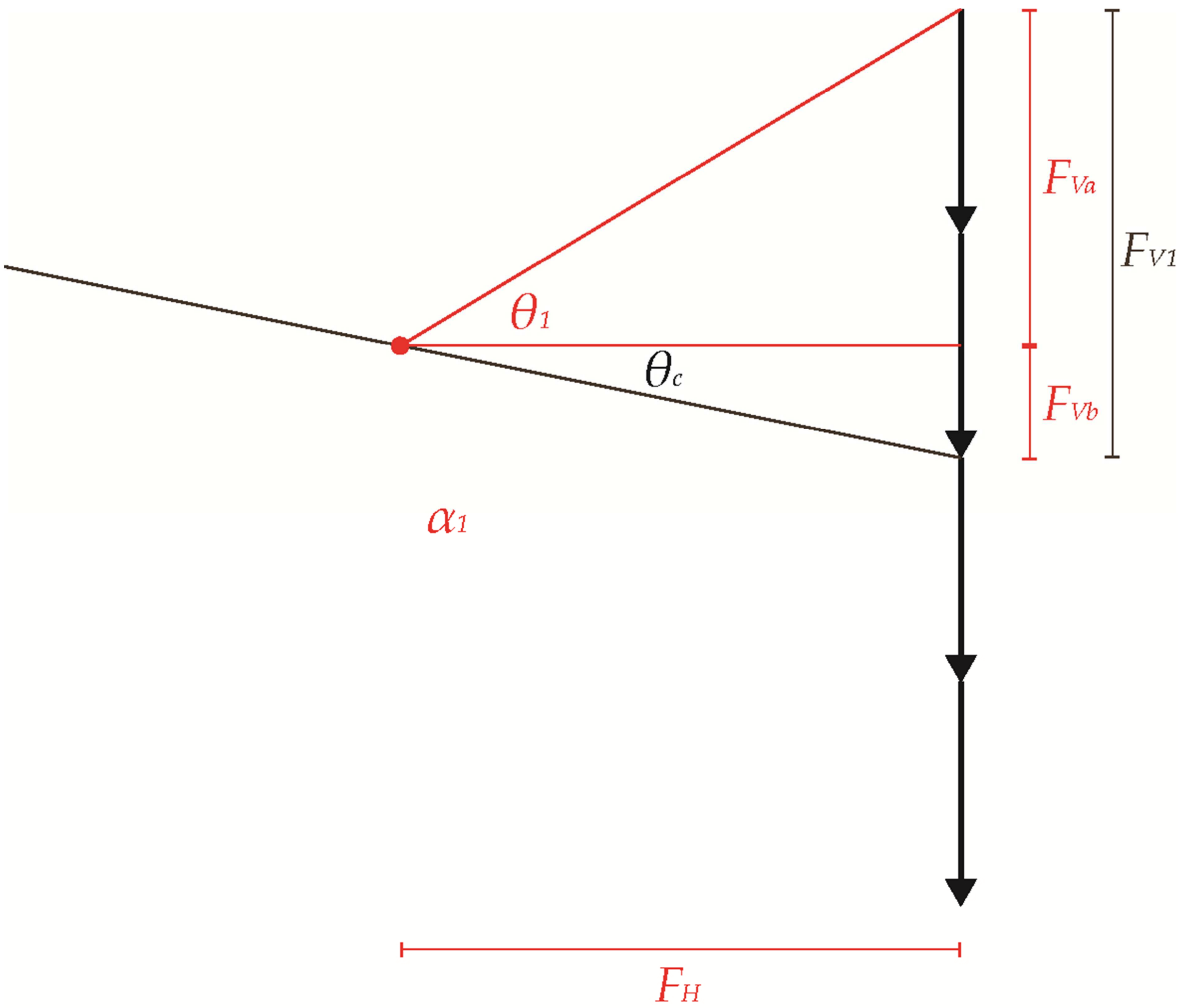


Figure 32 Calculating all angles from θ_1

3.1.5 Conclusions

A method is found to calculate the energy in a certain thrust line, depending on all the angles of the structure bars. It is obvious that these angles are all related, it can be seen from the way the structure changes with a change in the force polygon. If one angle is set, there is only one possibility of what the whole thrust line could look like. This relation is described in the previous section. These two section provide two equations:

$$E_c = \sum \frac{cos^2\theta_{n+1} \cdot {F_{zn}}^2}{sin^2(\theta_n - \theta_{n+1})} \cdot \frac{\delta x_n}{\cos\theta_n}$$

$$\tan\theta_n = \frac{F_{Vn}}{F_H} = F_{Vn} \cdot \frac{\tan\theta_1 + \tan\theta_c}{F_{v1}}$$

So once θ_n is now expressed in known values, the θ-values in the E_c equation can be substituted resulting in a complete equation for the complementary energy of a thrust line. The next step would be taking the derivative of this new equation and setting it equal to zero. This should result in a direct method of calculating the minimum complementary energy. However it turns out that this equation a very extensive one and not very insightful as well. And since this equation still excludes energy due to bending moments, because a structure is not predetermined, this method seems far off from the initial goal: finding a direct and insightful method. For this reason, problem is approached from a different angle: the equal area method. The next section starts with describing the equal area method to see if it can be related to the complementary energy method.

3.2 Calculating arches using areas of thrust line and structure

Since minimizing the normal energy didn't lead to a simple method of calculating the thrust line, another approach is applied. This time it is based on the presumption described in chapter 2.5, that the thrust line with an equal area to the area under the structure is the thrust line containing the least complementary energy. The idea is that if both areas are expressed in a mathematical way and related together, the result might be an equation which is a new starting point from which the equal area theory can be proven.

The first section shows how a direct method of calculating the thrust line with an equal area is derived. In the second section is described how this is translated into a Grasshopper algorithm. This algorithm is compared with other calculations in the third section, showing that the results do not match. The final section discusses what the explanation can be why the results do not match.

3.2.1 Calculating the area under a structure

The area under the structure can be represented by the rectangles shown in Figure 33. All the areas can be calculated by

$$A_{str} =$$

$$\delta x_1 \cdot \frac{1}{2} \cdot \delta y_1$$

$$+ \delta x_2 \left(\delta y_1 + \frac{1}{2} \cdot \delta y_2 \right)$$

$$+ \delta x_3 \left(\delta y_1 + \delta y_2 + \frac{1}{2} \cdot \delta y_3 \right)$$

$$+ \delta x_4 \left(\delta y_1 + \delta y_2 + \delta y_3 + \frac{1}{2} \cdot \delta y_4 \right)$$

$$+ \delta x_5 \left(\delta y_1 + \delta y_2 + \delta y_3 + \delta y_4 + \frac{1}{2} \cdot \delta y_5 \right)$$

or more general:

$$A_{str} =$$

$$\delta x_1 \cdot \frac{1}{2} \cdot \delta y_1$$

$$+ \delta x_2 \left(\delta y_1 + \frac{1}{2} \cdot \delta y_2 \right)$$

$$+ \delta x_3 \left(\delta y_1 + \delta y_2 + \frac{1}{2} \cdot \delta y_3 \right)$$

$$+ \ldots$$

$$+ \delta x_n \left(\delta y_1 + \ldots + \delta y_{n-1} + \frac{1}{2} \cdot \delta y_n \right)$$

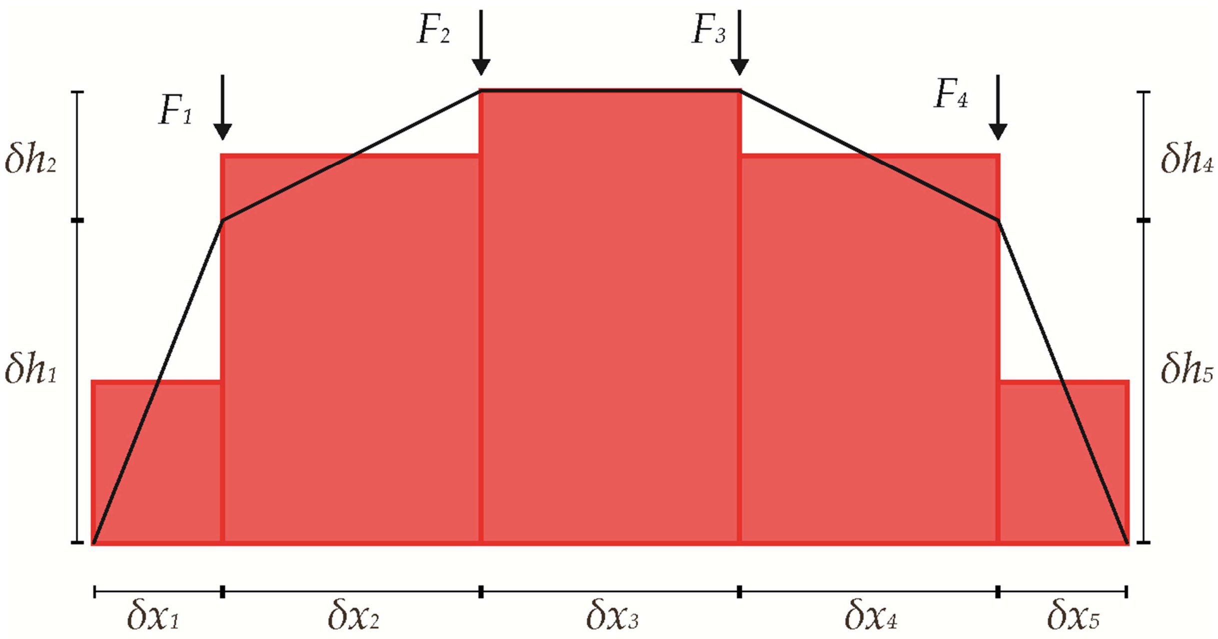


Figure 33 Calculating the area of a structure

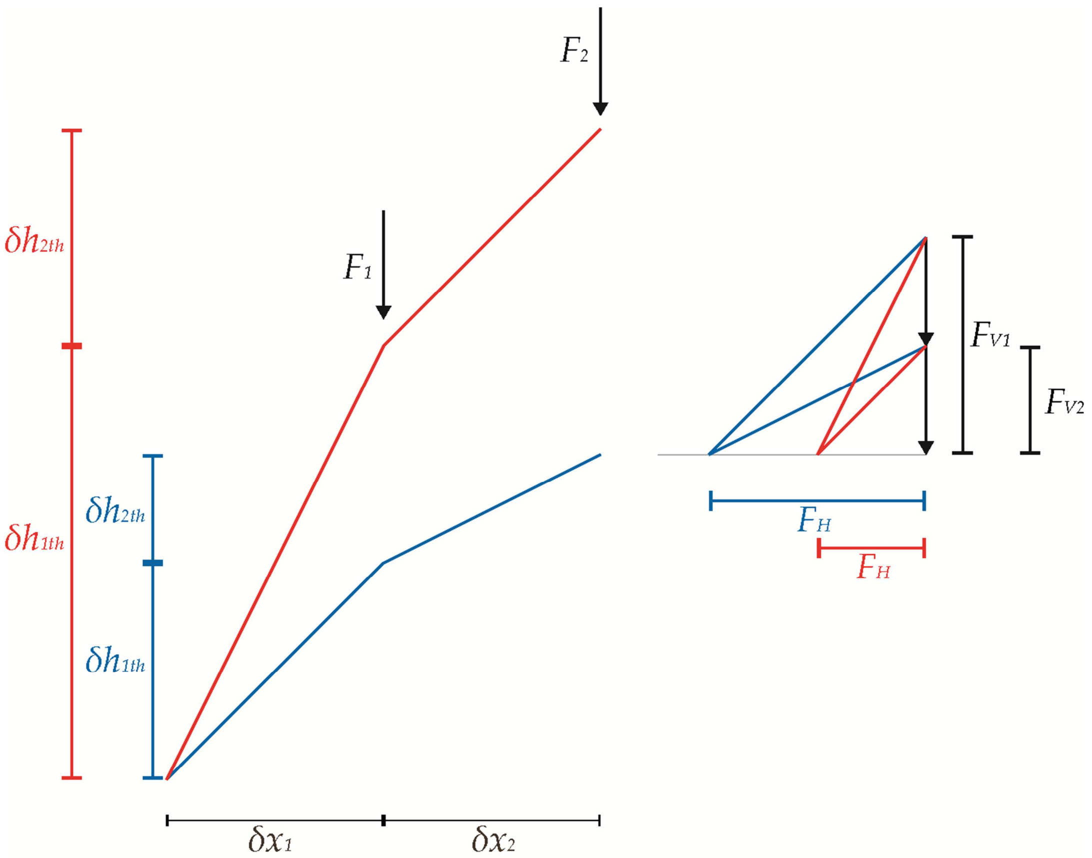


Figure 34 The force polygon related to the thrust line

In Figure 34 is shown that the area under the thrust line depends on the variable F_H. The other variables, such as F_{Vn} are determined by the loads, which are not variable. $\delta y_{th\,n}$ is variable as well but this variable can be expressed in F_H:

$\frac{F_{Vn}}{F_H} = \frac{\delta y_{th\,n}}{\delta x_n}$

which means that:

$$\delta y_{th\,n} = \frac{F_{Vn} \cdot \delta x_n}{F_H}$$

The area under the thrust surface can be calculated in a similar way as the area under the structure. δy is substituted using the equation above.

$$\begin{aligned}
A_{th} = & \\
& \delta x_1 \cdot \frac{1}{2} \cdot \frac{F_{V1} \cdot \delta x_1}{F_H} \\
& + \delta x_2 \left(\frac{F_{V1} \cdot \delta x_1}{F_H} + \frac{1}{2} \cdot \frac{F_{V2} \cdot \delta x_2}{F_H} \right) \\
& + \delta x_3 \left(\frac{F_{V1} \cdot \delta x_1}{F_H} + \frac{F_{V2} \cdot \delta x_2}{F_H} + \frac{1}{2} \cdot \frac{F_{V3} \cdot \delta x_3}{F_H} \right) \\
& + \ldots \\
& + \delta x_n \left(\frac{F_{V1} \cdot \delta x_1}{F_H} + \ldots + \frac{F_{V(n-1)} \cdot \delta x_{n-1}}{F_H} + \frac{1}{2} \cdot \frac{F_{Vn} \cdot \delta x_n}{F_H} \right)
\end{aligned}$$

Since every term contains the factor $\frac{1}{H}$ the equation can be rewritten as:

$$\begin{aligned}
F_H = \frac{1}{A_{th}} \cdot \Bigg(& \delta x_1 \cdot \frac{1}{2} \cdot F_{V1} \cdot \delta x_1 \ + \ \delta x_2 \left(F_{V1} \cdot \delta x_1 + \frac{1}{2} \cdot F_{V2} \cdot \delta x_2 \right) + \cdots \\
& + \ \delta x_n \left(F_{V1} \cdot \delta x_1 + \ldots + F_{V(n-1)} \cdot \delta x_{n-1} + \frac{1}{2} \cdot F_{Vn} \cdot \delta x_n \right) \Bigg)
\end{aligned}$$

Since we assume $A_{th} = A_{str}$, the equation for the area of the structure can be substituted, resulting in:

$$F_H = \frac{\delta x_1 \cdot \frac{1}{2} \cdot F_{V1} \cdot \delta x_1 + \ \delta x_2 \left(F_{V1} \cdot \delta x_1 + \frac{1}{2} \cdot F_{V2} \cdot \delta x_2 \right) + \ldots + \delta x_n \left(F_{V1} \cdot \delta x_1 + \ldots + F_{V(n-1)} \cdot \delta x_{n-1} + \frac{1}{2} \cdot F_{Vn} \cdot \delta x_n \right)}{\delta x_1 \cdot \frac{1}{2} \cdot \delta y_1 + \ \delta x_2 \left(\delta y_1 + \frac{1}{2} \cdot \delta y_2 \right) + \ \delta x_3 \left(\delta y_1 + \delta y_2 + \frac{1}{2} \cdot \delta y_3 \right) + \ \ldots + \ \delta x_n \left(\delta y_1 + \ldots + \delta y_{n-1} + \frac{1}{2} \cdot \delta y_n \right)}$$

With this equation, the correct F_H can be calculated from a given structure resulting in the correct thrust line.

3.2.2 Generating the thrust line

The method above gives a direct way to calculate the F_H of a certain structure. Until now it was only possible to do this in an iterative way as described in section 2.5. This method is written in Grasshopper, a Rhino-plugin. Figure 35 shows the input and the output of the component. The actual algorithm is summarized in Figure 36. As a first step, a curve is drawn and set as input of for the algorithm. This curve needs to be drawn in the XY-plane and needs to start at the origin of the grid. The end points are marked as supports. For two situations the load is determined. In one situation a q-load is projected onto the surface. The other situation has its own weight as input. The algorithm allows choosing one of these. In the next step the load is discretized in point loads. This discretization is based on the chosen δx or the segment length, depending on whether it is projected q-load or the own weight of the structure. In the next step the area of the structure is determined, using a Grasshopper function. After that, F_H is calculated by the equation which is shown in the previous section. With a known F_H the force polygon is drawn and the thrust line is generated. The Grasshopper file can be obtained from the CD and has the file name 3.2.2_eqA.

The result of this algorithm is a 2D program in which any 2-dimensional structure can be drawn which will instantly output the thrust line with the same area. Since a thrust line is now known, which holds all the information for calculating a certain structure, this algorithm can quite easily be extended with functions like outputting the bending moment diagram for instance, or any other diagram. Also, a function can be added which allows for assigning material and section properties to the structure and outputting deformations of the structure. But before options like this are added, the newly designed algorithm is compared with other calculations to verify whether it outputs the correct results. These calculations are described in the next section.

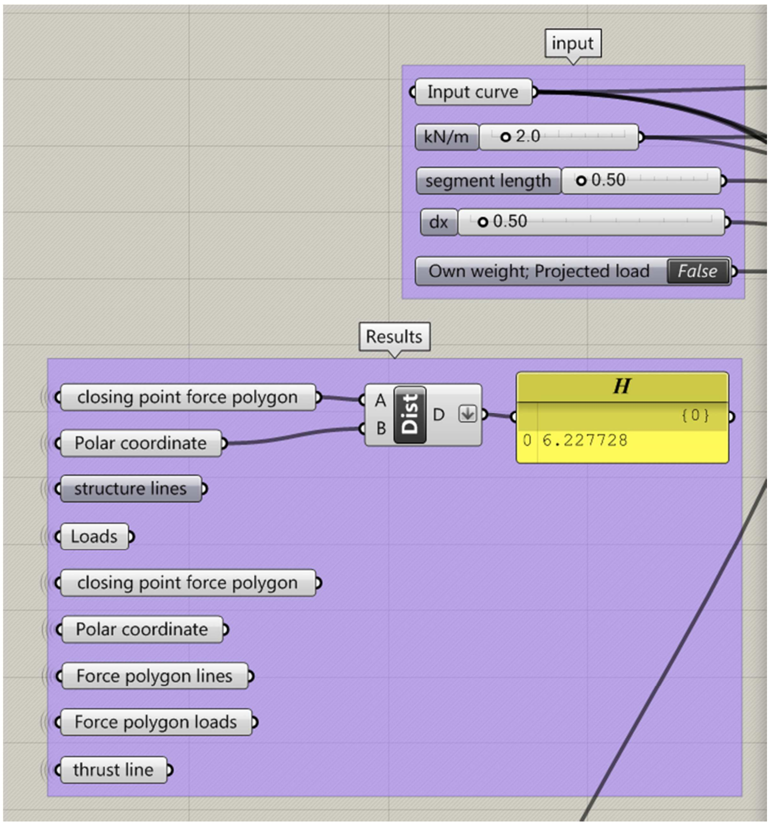


Figure 35 The in- and output of the Grasshopper algorithm

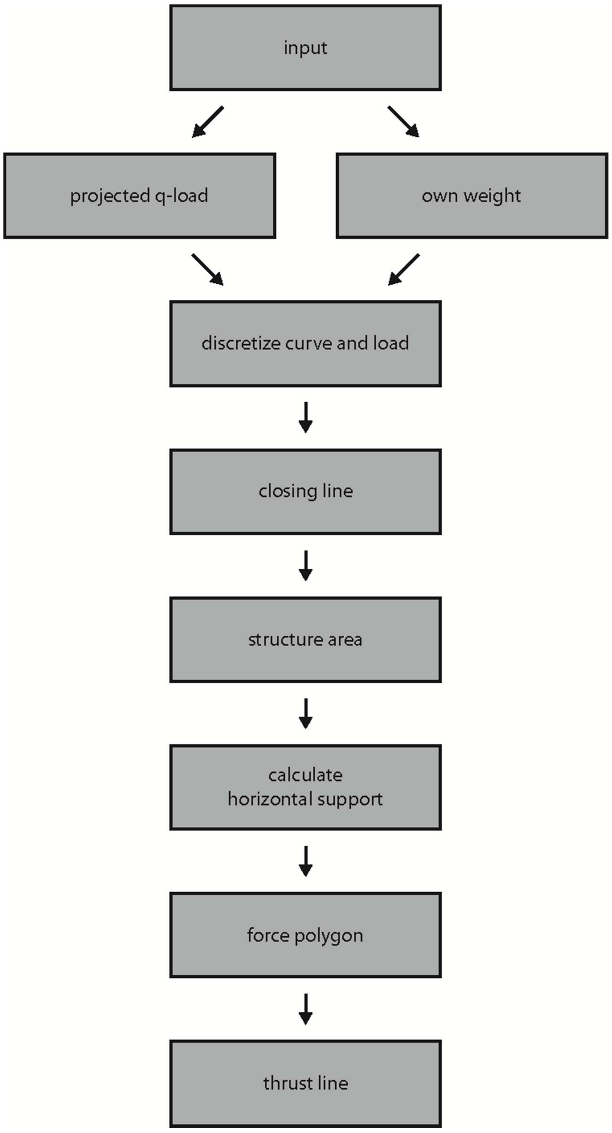


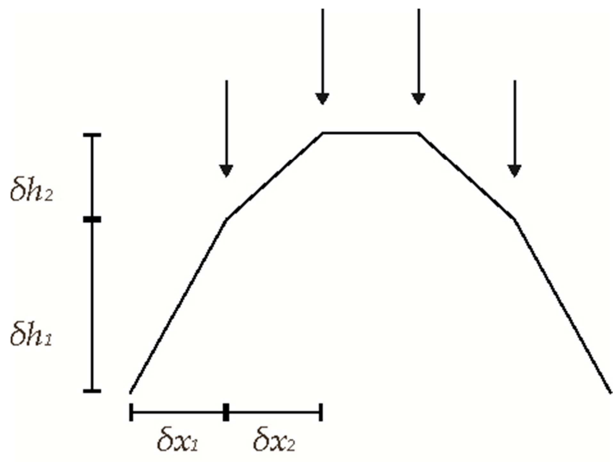
Figure 36 The algorithm summarized

3.2.3 Comparing to complementary energy method

To verify whether or not this equation works, three calculations were done. These calculations only deal with the question whether or not this equation can predict the thrust line containing the lowest complementary energy in a linear way. The control calculations are done in a non-linear way. The energy of a certain structure combined with a thrust line is calculated in excel, based on a randomly chosen F_H. The value for F_H is changed until the lowest complementary energy is found. Subsequently the direct method as described in the previous section is applied to the same structure. If these calculations lead to the same result for F_H, this equation can be considered a valid method to determine the thrust line with the least complementary energy. The calculations in this section can be found in a spreadsheet on the CD with the file name 3.2.3_eqA_minEc.

Calculation 1: a thrust line as structure

For the first calculation, a structure and loading is used from which is known that the thrust line will coincide with the structure (Figure 37). This way, bending moments don't play a role in the calculation, making it a first step in verifying the validity of this method.



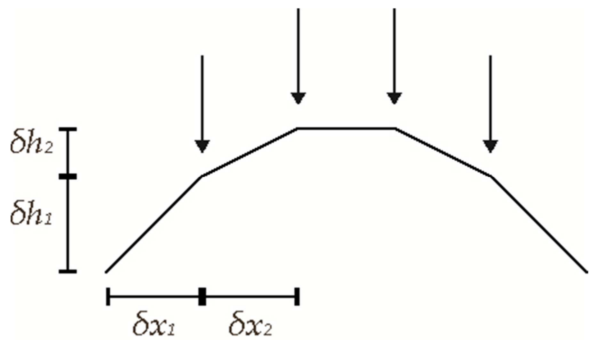
$method$	F_H
min E	2,22
eq A	2,22

Figure 37 A structure with coincides with a thrust line

First the energy is calculated by using $E_{c;N} = N^2 \cdot l$ to calculate the energy for each bar. Bending moments are not included in this calculations because they don't occur in this structure. The energy of these bars are added together for the total complementary energy of the structure. The value for F_H is changed until the lowest energy is found. This shows that, according to this method, the F_H is 2,22, making the thrust line coincide with the structure. The equal area method results in the same value.

Calculation 2: a symmetrical structure

For the second verification, a random structure and loading is chosen, expecting the thrust line to be outside the structure (Figure 38).



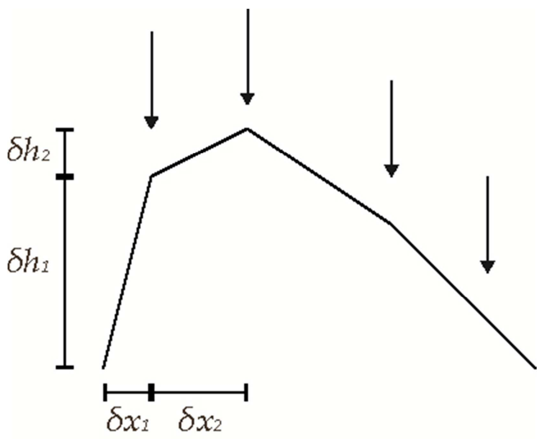
$method$	F_H
min E	4,00
eq A	4,00

Figure 38 A symmetrical structure with a thrust line outside the material

In section 2.2 is described how, using the thrust line outside the material, the forces can be projected on the structure by adding the eccentricity of the forces to the equation. This is needed for the non-linear calculations, used to check the new method. Since this is not easily done in excel, an approximation is done. The bending moments should be projected on the point perpendicular to the thrust line. This results in a change in bending moment throughout the bar. To keep the calculations simple, for each bar the bending moment is calculated from the thrust line within the same δx. The eccentricity is calculated from the thrust line halfway each bar. This way the results will lose accuracy but it is a simple first calculation to see whether the results are promising. Once the forces and bending moments are determined, a check can be done. Using $E_{c;N,M} = \left(N^2 + \frac{12(N \cdot e)^2}{t^2}\right) l$ the complementary energy due to normal forces and bending moments is calculated. The overall thickness is considered to be 0,1 m. The results are shown in the table next to the figure.

Calculation 3: an irregular shaped arch

The third calculation is conducted in a way similar to the second. The only difference is that this structure is more irregular shaped (Figure 39). The results of the direct calculation method gives a F_H of 1,89 N. The table below shows that the indirect method results in a F_H of 1,70 N, making this result a little bit off.



method	$\boldsymbol{F_H}$
min E	1,70
eq A	1,89

Figure 39 A irregular shaped structure

3.2.4 Conclusions

The aim of this section was to describe both the complementary energy method and the equal area method in a mathematical way and relate them together. Both methods are described in a mathematical way but could not be related together. However, the equal area method is now described in an equation which makes it possible to be applied in a direct way, instead of an iterative way. Turning this equation into a Grasshopper tool allowed for easy testing this method. The previous section compares the equal area method with the complementary energy method. The first two calculations give a very accurate result, the third is slightly off. There is at this point in the research no simple explanation for these differences, resulting in doubts about the validity of the equal area method. To get more clarity on the validity of the equal area method, the next section describes a broader comparison of different calculation methods and more structures.

During the derivation of the equal area method, attempts were made to connect both methods through the Lopshits equation (section 2.7), however, when the doubts about the validity of this method arose, this path of research was not continued.

3.3 Comparing different methods

To verify whether or not the equal area method works, for a broader range of structures different calculation methods are compared. In the previous section, only the complementary energy method and the equal area method are compared. To eliminate the possibility that the complementary energy method is not valid, a third method is added, the Finite Element Method. The different calculation methods are described in the first section.

The structures which are tested include the structures from the previous section, this time also calculated with the Finite Element Method. Next to these, two more structures are added, which are both tested for distributed loads as well. The tested structures and their results can be seen in the second section. The first four figures show two structures with point loads and distributed loads. The first structure is a symmetrical arch which is chosen so the method can be tested for a simple situation. The second structure is chosen as a more asymmetrical one, with a bigger deviation from the thrust lines. This way the method can also be tested for situations with bigger bending moments.

3.3.1 Methods used

FEM

The Finite Element Method calculations are done using GSA Suite. This method is a reliable method which is used in current practice. It is used as a method from which we can be sure that it is correct. The results of both other methods are compared with this method to see how well they perform.

Minimum E_c

The method using the minimum E_c is for the point loads calculated in an excel file as described in section 3.2.3. Through trial and error (so in an iterative way) the situation with the lowest complementary energy is found. The excel files used in this calculations can be found on the CD and are called 3.3.1_eqA_minEc_1 and 3.3.1_eqA_minEc_2.

Since this excel file doesn't allow for q-loads to be entered, the load case with the q-load is calculated using a grasshopper algorithm which is an adaptation of an algorithm created by van Dijk (2014). A new algorithm was made because the old one did not give correct results, possibly due to updates in the Grasshopper software. The algorithm is based on the flowchart in Figure 13. The algorithm was tested by comparing it to FEM results, giving the same results. The calculations are done through an iterative method, by changing F_H until the lowest complementary energy is found. (Grasshopper file on CD is called 3.3.1_minE)

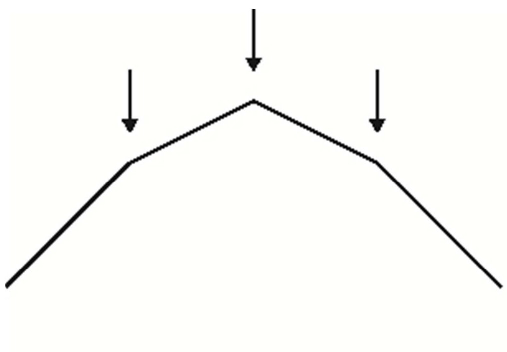
Equal areas

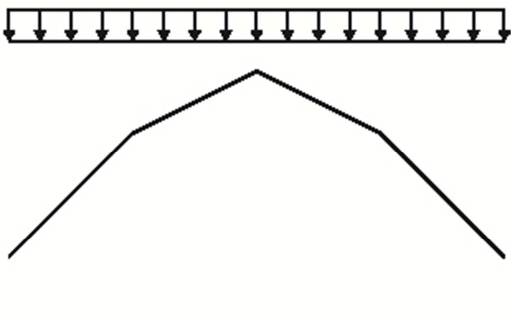
The equal area method is calculated for the q-load using the equation derived in section 3.2.1. This is done in Grasshopper using the file described in section 3.2.2

The case with the point loads is calculated with the same excel files which are used for the minimum E_c method. The situation with equal areas is found in an iterative way.

3.3.2 Results

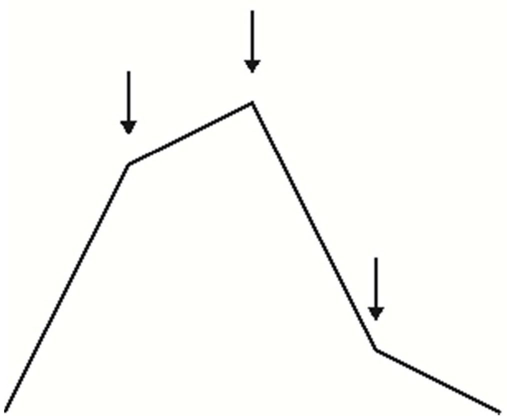
The results are shown in the tables below. All the minimum E_c calculations and FEM calculations give a very good result. For the first shape, the equal area method gives a slight difference of only 1,1% and 2,3%. The second shape gives bigger differences of 11,7% and 13,1%.

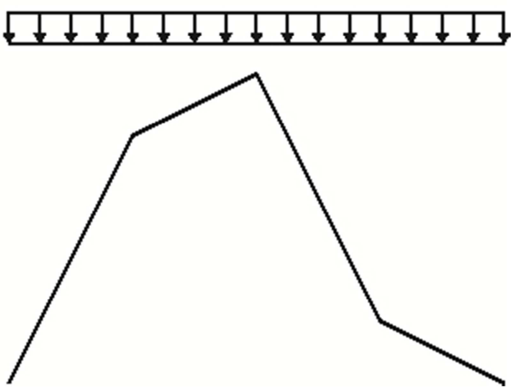




method	**H**
equal A	2,86
min. E_c	2,83
FEM	2,83

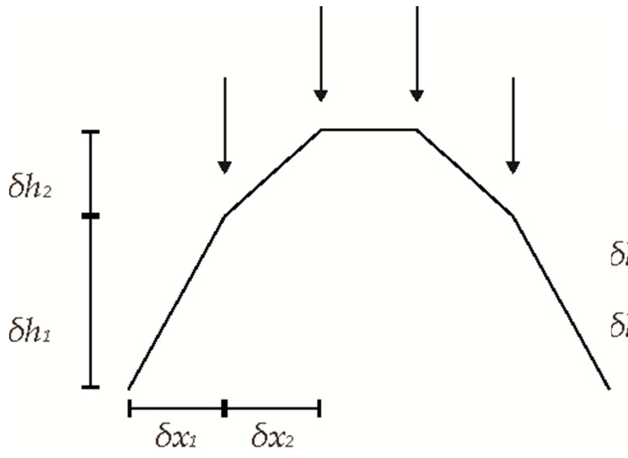
method	**H**
equal A	12,19
min. E_c	11,96
FEM	11,96

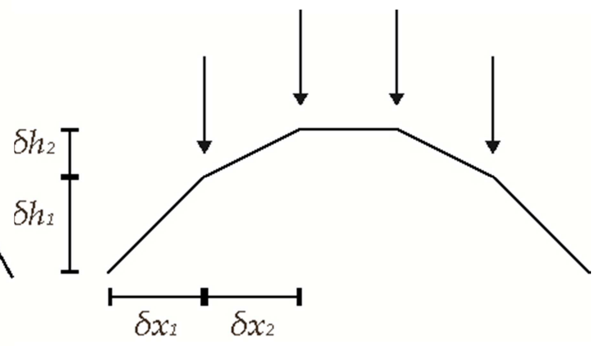




method	**H**
equal A	2,00
min. E_c	1,79
FEM	1,79

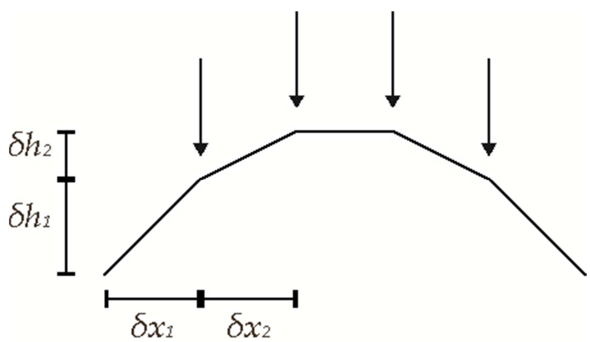
method	**H**
equal A	8,53
min. E_c	7,55
FEM	7,54





method	**H**
equal A	2,22
min. E_c	2,22
FEM	2,22

method	**H**
equal A	4,00
min. E_c	4,00
FEM	3,99



method	**H**
equal A	1,89
min. E_c	1,70
FEM	1,70

3.3.3 Conclusions

The differences, especially for the second shape, are too big to be explained by discretization errors or rounding errors. From these calculations can be concluded that the method of equal areas actually does not work for calculating the correct thrust line. The reason why it seems to work can be explained with the help of Figure 40. The correct thrust line is the one with the lowest complementary energy. This energy is dependent on the bending moment and the normal force.

$$E_c = N^2 l + \frac{12}{t^2} M^2 l$$

Since the thickness *t* of the structure will often be quite small in comparison to the width of the total structure and thus the $\sum l$ as well, the bending moment will have a much bigger influence on the total energy then the normal force. The bending moment can be calculated by:

$$M = e_V \cdot H$$

The figure shows that for symmetrical arches with not too irregular geometry, the thrust line calculated by equal areas (shown in red) lies very close to the thrust line calculated through minimum energy (shown in green), because the eccentricity *e* is very small for equal areas. So as long as the arches to be calculated are chosen quite conventional, (the arch to the right) the equal area method will result in small errors. The structure to the left shows that for more irregular shapes the lines are further apart. The figure to the right is one of the arches also used by van Dijk (2014) to support the equal area method. More of the arches from that research are included in appendix II. This shows how it could be that the equal area method seemed so promising, even though it turned out not to be correct. The arches calculated were all quite symmetrical (even though not completely symmetrical), resulting in very small differences in thrust lines.

Figure 40 Structures (in black) with both the thrust line calculated using equal areas (red) and minimum energy (green). In the right figure, both lines coincide.

3.4 Minimizing complementary energy from bending moment

Since the method of equal areas does not give the correct thrust line, a new simple method needs to be found. When trying to prove the equal area method for a simple situation, a new idea emerged on how a method for calculating the correct thrust line can be developed. This method is based on the idea that the thrust line with a minimum total energy is approximately the same as the thrust line with a minimum bending energy. The reason for this is the equation for the total energy as introduced in chapter 2.4:

$$E_c = \left(N^2 + \frac{12}{t^2}M^2\right)l$$

Since the thickness t of the arch or shell will be quite small in relation to the length l, and the thickness is squared as well, the bending moment will often be of a much bigger influence on the total than the normal force will be.

This section explains how this proposed method works and why it works that way. In the first subsection will be discussed in more detail how this idea conceptually works. Before the actual derivation can be explained, the second subsection will explain in more detail than the equation above how the bending energy can be calculated. The third subsection is a description of a proof written to show that for a certain situation, the thrust line with a minimum bending energy is the same thrust line as the one having an equal area to the structure. This proof was written when trying to prove the method of equal areas. It is however included in this chapter because it also illustrates a way of finding the situation with the minimum bending energy. The fourth section shows how this proof can be translated into a method of minimizing bending energy for a more complex situation, giving a more general method. The following two subsections compare the results of the developed method to FEM calculations to show that the equation gives correct results for normal situations and also explore the boundaries of this method by determining the accuracy for different thicknesses. Finally the possible applications of this method in practice are discussed.

3.4.1 Omitting complementary energy due to normal forces

When the complementary energy in a structure is calculated to find the correct thrust line, we are only looking for the situation in which the energy is the lowest. Because of this, factors like the Young's modulus are omitted of the equation already, they do not influence which thrust line gives the lowest value since these values are equal for the whole structure. The normal forces however do influence the thrust line, and so do the bending moments and the length of the structure. This results in the equation for the complementary energy:

$$E_c = N^2 l + \frac{12}{t^2}M^2 l$$

It does not actually return the correct value for the complementary energy since *EA* and *EI* are omitted, but the lowest value will be found for the same *N, M, t* and *l*.

The thickness t is for arches and shells usually quite low, also in comparison to beams because of the very nature of arches and shells: that they mainly transport normal forces. If the thickness t is low, the factor $\frac{12}{t^2}$ will be big, giving the part with the bending moment a bigger influence on the total E_c than the bending moment part. If t is small enough, the bending moment will have such a big influence on the outcome that the normal force will not be relevant anymore. In the following sections, this idea is further explored by omitting the N^2l from the equation and see how accurate the results are.

To understand this concept it can also be seen in another way: if we consider the complementary energy to be the energy stored in a structure due to deformation of the structure, in arches and shells, the deformation due to bending moments is so big in comparison to deformation due to normal forces that the deformation due to normal forces can be ignored.

3.4.2 Calculating the energy due to bending moment

The energy due to bending moments can be calculated, as mentioned before, by $E_{C;M} = \frac{12}{t^2}M^2l$. There are more steps involved to get from a drawing with a force polygon, a thrust line and a structure to the energy from a bending moment in a bar. This is somewhat more complex than the energy due to the normal forces, especially because the bending moment changes throughout a bar and the normal forces doesn't. Consider for instance the situation in Figure 41. A part of a structure is shown together with a part of the corresponding thrust line. The force drawn in this thrust line can usually be obtained by the corresponding force polygon which is not shown in this case. There are several different ways of determining the bending moment in a certain point of the structure, from which three will be discussed here.

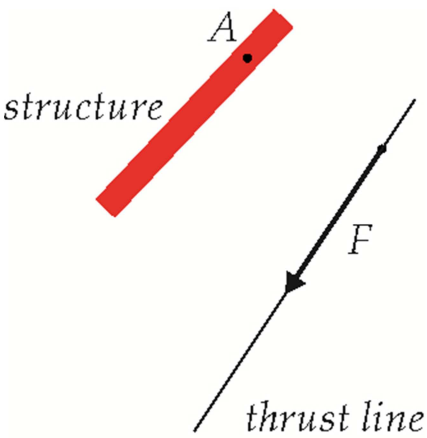


Figure 41 Thrust line outside the material

The first method (Figure 42.a) is to draw a line perpendicular to the force in the thrust line through point A. The bending moment equals the length of the eccentricity *e* times the force in the thrust line. The advantage of this method is that the force can be directly obtained from the force polygon.

$$M = e_{\perp F} \cdot F$$

The second method (Figure 42.b) shows a line perpendicular to the structure from point A to the thrust line. The force needs to be decomposed into one perpendicular to the eccentricity and one parallel to the eccentricity, resulting in:

$$M = e_{\perp str} \cdot F_N$$

In the third method (Figure 42.c) the eccentricity is chosen vertically. The corresponding component of the force is the horizontal component, which can be obtained from the force polygon as well.

$M = e_V \cdot F_H$

The advantage of this method is that for a vertical load, the force needed to calculate this bending moment will be equal throughout the whole structure.

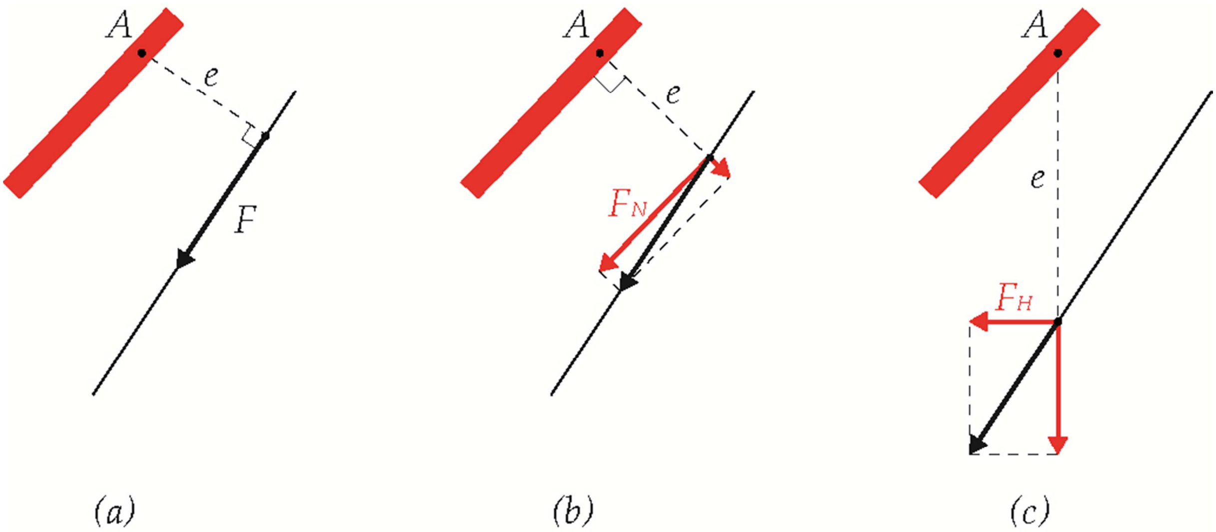


Figure 42 Three different ways to calculate the bending moment from the thrust line

To calculate the energy for a structure, the example of Figure 43 is used. The structure, force polygon and thrust line are shown. The first step is calculating the e_V, the vertical difference between the thrust line and the structure. This difference is plotted in Figure 43.b. Figure 43.c shows the bending moment throughout the structure which equals the first plot times F_H. Since these graphs show the bending moment changing over δx, and the change over the length is needed to calculate the energy, the x-axis is changed to the length of each bar, resulting in Figure 43.d. To get a better understanding of what this step does, one can also consider this as being the bending moment diagram, which is usually drawn directly on the structure, in which the structure is taken as the x-axis (Figure 44). In the next step, the graph with the bending moments is squared, since the energy consists of the bending moment squared times the length. This result in the Figure 43.e in which the area between the x-axis and the graph equals the total $E_{c;M}$.

This method will be used in the following sections to find an analytical way for calculating the minimum complementary bending energy.

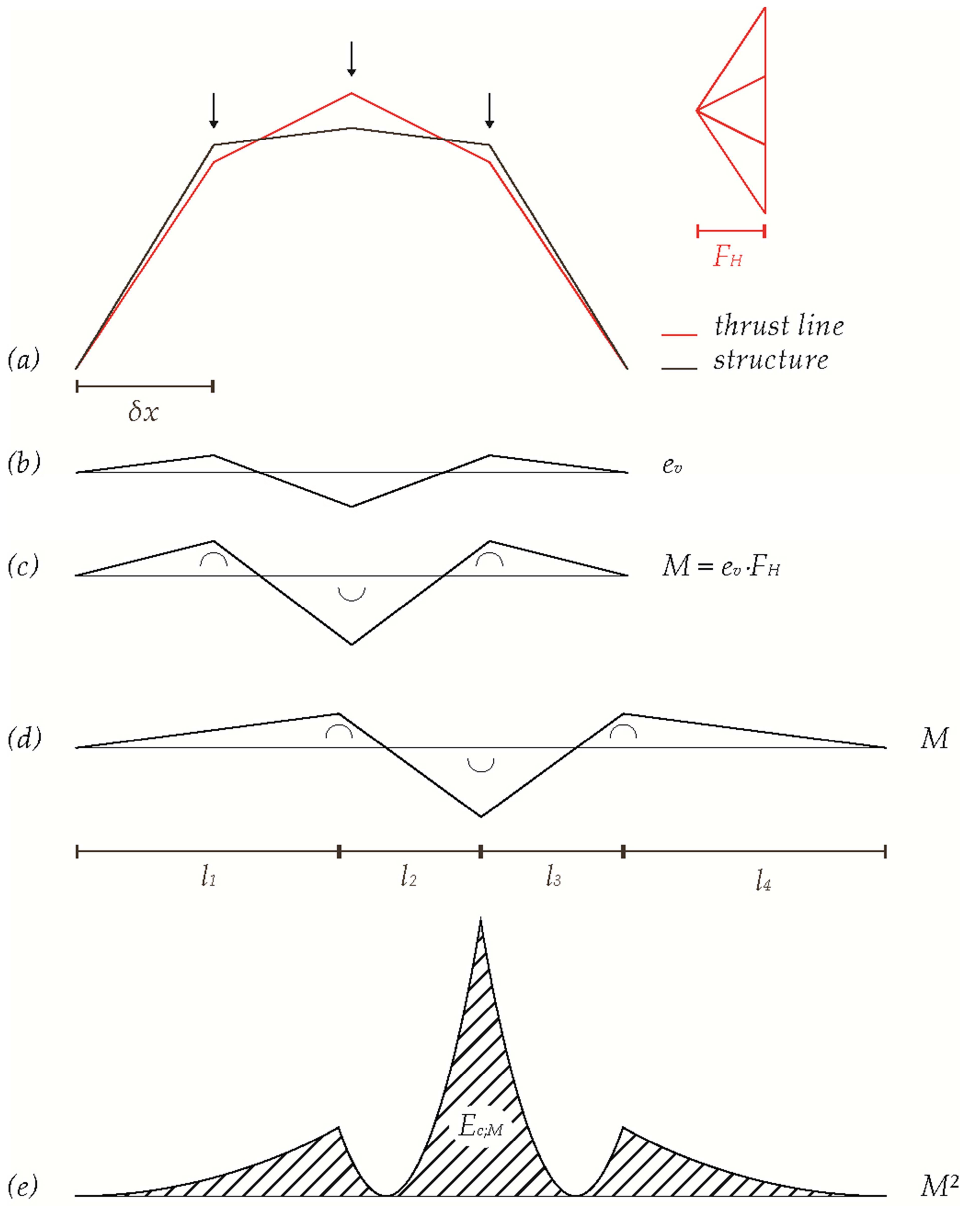


Figure 43 Calculating the bending energy of a structure

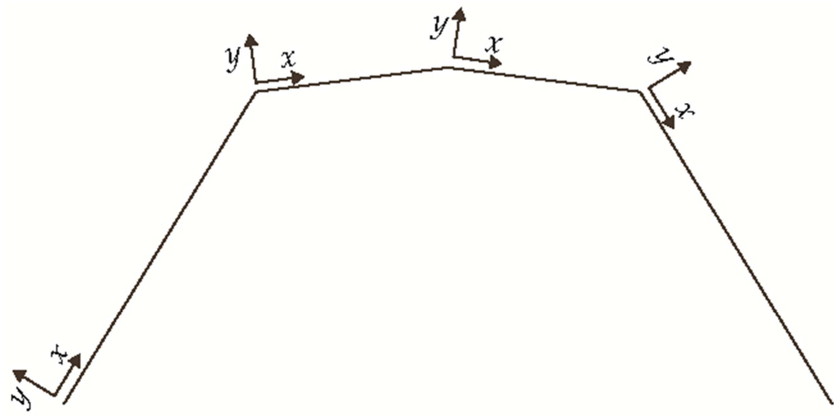


Figure 44 The structure is used as the x-axis

3.4.3 Proof of equal areas

In an attempt to prove in an analytical way that a thrust line with an area equal to the structure will be the line with the lowest $E_{c;M}$, the proof is written for a very simple situation (Figure 45), one load, two supports, two bars. In red, the force polygon and the thrust line are drawn. It is obvious that the lowest energy in this case will be the case where the areas are equal, since the thrust line will, for a correct F_H be equal to the structure, resulting in the bending moment being zero. So the result of this proof is not very surprising, however it is a basis on which a more extended proof is written later.

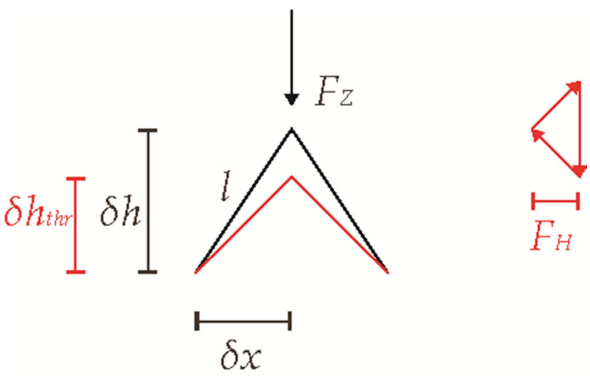


Figure 45 A structure consisting of two bars and one point load

The area of this structure can be calculated by $\delta h \cdot \delta x$. The area of the thrust line by $\delta h_{thr} \cdot \delta x$. So to prove that both areas are equal the following needs to be proven:

$$\delta h \cdot \delta x = \delta h_{thr} \cdot \delta x$$

The height of the thrust line is dependent on the F_H and can be described as:

$$\delta h_{thr} = \frac{1}{2}\frac{F_z}{F_H}\delta x$$

Since $E_{c;N}$ is not included in this calculation, $\frac{12}{t^2}$ does not influence the position of the lowest point of the complementary energy, it can be omitted from the calculations.

$$E_{c;M} = M^2 l$$

The vertical eccentricity at any point in the structure can be calculated by:

$e_V = h - h_{thr}$

The left half of the structure can be described as

$$h = \frac{\delta h}{\delta x} x \text{ for domain } [0, \delta x]$$

Since this can be applied to the left half of the thrust line as well, the vertical eccentricity can be calculated

$$e_V = \frac{\delta h - \delta h_{thr}}{\delta x} x \text{ for domain } [0, \delta x]$$

If the length of the bar is taken as x-axis instead of δx, the following equation applies:

$$e_V = \frac{\delta h - \delta h_{thr}}{l} x \text{ for domain } [0, l]$$

Substituting the equation for δh_{thr} and rewriting the equation results in

$$e_V = \left(\delta h - \frac{1}{2}\frac{F_z}{F_H}\delta x\right)\frac{x}{l}$$

Multiplying this with F_H gives the equation for the bending moment:

$$M = \left(F_H \delta h - \frac{1}{2} F_z \delta x\right)\frac{x}{l}$$

$$M^2 = \left(F_H \delta h - \frac{1}{2} F_z \delta x\right)^2 \frac{x^2}{l^2} \text{ for domain } [0, l]$$

The M^2 is the equation which represents a graph similar to Figure 43.e for the left bar of the structure. Integrating this function over *x* for domain $[0, l]$ will give the complementary energy for this part.

$$E_{c;M} = \int_0^l \left(F_H\, \delta h - \frac{1}{2} F_z\, \delta x\right)^2 \frac{1}{l^2} x^2\ \delta l$$

$$E_{c;M} = \frac{1}{3}\left(F_H\, \delta h - \frac{1}{2} F_z\, \delta x\right)^2 \frac{1}{l^2} l^3$$

$$= \frac{1}{3}\left(F_H\, \delta h - \frac{1}{2} F_z\, \delta x\right)\left(F_H\, \delta h - \frac{1}{2} F_z\, \delta x\right) l$$

$$= \frac{1}{3}\left({F_H}^2 \delta h^2 - F_z\, F_H\, \delta x\, \delta h + \frac{1}{4}{F_z}^2 \delta x^2\right) l$$

$$= \frac{1}{3}\delta h^2 l\, {F_H}^2 - \frac{1}{3} F_z\, \delta x\, \delta h\, l\, F_H + \frac{1}{12}{F_z}^2 \delta x^2 l$$

Taking the derivative of this equation with F_H as the only variable, and setting that equal to zero should result in the minimum $E_{c;M}$.

$$E'_{c;M} = \frac{2}{3}\delta h^2 l\, F_H - \frac{1}{3}F_z\, \delta x\, \delta h\, l$$

$$\frac{2}{3}\delta h^2 l\, F_H - \frac{1}{3}F_z\, \delta x\, \delta h\, l = 0$$

$$\frac{2}{3}\delta h^2 l\, F_H = \frac{1}{3}F_z\, \delta x\, \delta h\, l$$

$$2\, \delta h\, F_H = F_z\, \delta x$$

$$\delta h = \frac{1}{2}\frac{F_z}{F_H}\, \delta x$$

Substituting the equation for δh_{thr} shows:

$$\delta h = \delta h_{thr}$$

This shows that through calculating the bending moment and setting the derivative of that bending moment equal to zero, it can be proven that these two areas are equal when the bending moment is minimal.

During the making of this proof, the first doubts arose about the equal area method. For this reason no attempt was made to prove this for a more complex structure. However this method is expanded in the next paragraph, not to prove the areas to be equal, but to find another method through which the complementary energy can be calculated in a direct way.

3.4.4 Minimizing the energy due to bending for a three-bar structure

The approach described in the previous section will be applied to a more advanced and slightly more general structure in this section. This time however the aim is not to prove the areas being equal, the method is only used to provide a direct method of calculating the structure with the minimum bending energy. The structure calculated is shown in Figure 46. This time the structure can be non-symmetrical (if δh_2 does not equal zero) so it can have bending moments.

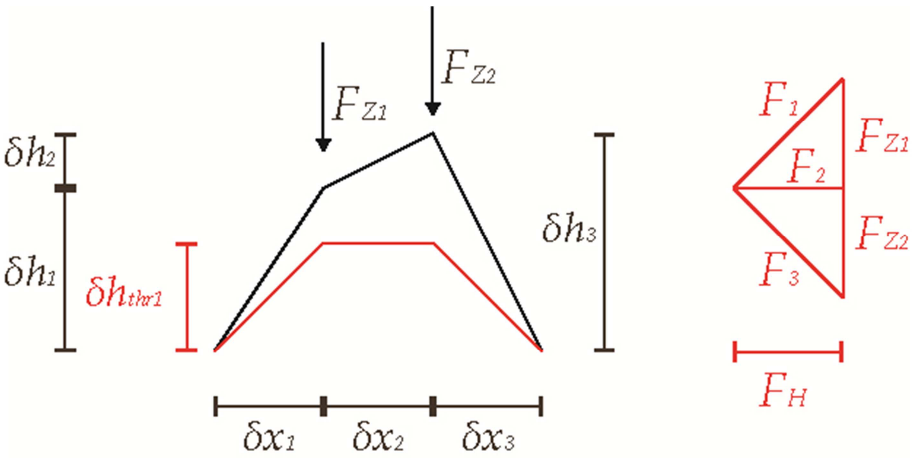


Figure 46 A structure consisting of three bars (in black) with the force polygon an thrust line (red)

To make the calculation simpler, some values are set equal to each other:

$$\delta x_1 = \delta x_2 = \delta x_3 = \delta x$$

$\delta x_{thr1} = \delta x_{thr2} = \delta x_{thr3} = \delta x$

$F_{Z1} = F_{Z2} = F_Z$

$F_1 = F_3$

$\delta h_{thr1} = -\delta h_{thr3}$

$\delta x_{thr2} = 0$

The height of δh_{thr1} can be calculated by

$$\delta h_{thr1} = \frac{F_z}{F_H}\delta x$$

Just like in the previous section, the bending energy can be calculated by

$$E_{c;M} = M^2 l$$

Again, the e_V is calculated for each bar.

$$e_{V1} = \frac{\delta h_1 - \delta h_{thr1}}{l_1}x \text{ for domain } [0, l_1]$$

$$e_{V2} = \frac{\delta h_2}{l_2}x + \delta h_1 - \delta h_{thr1} \text{ for domain } [0, l_2]$$

$$e_{V3} = \frac{\delta h_3 + \delta h_{thr1}}{l_3}x + \delta h_1 + \delta h_2 - \delta h_{thr1} \text{ for domain } [0, l_3]$$

This can be written as

$$e_{V1} = \frac{\delta h_1}{l_1}x - \frac{F_z}{F_H}\frac{\delta x}{l_1}x \text{ for domain } [0, l_1]$$

$$e_{V2} = \frac{\delta h_2}{l_2}x + \delta h_1 - \frac{F_z}{F_H}\delta x \text{ for domain } [0, l_2]$$

$$e_{V3} = \frac{\delta h_3}{l_3}x + \frac{F_z}{F_H}\frac{\delta x}{l_3}x + \delta h_1 + \delta h_2 - \frac{F_z}{F_H}\delta x \text{ for domain } [0, l_3]$$

These eccentricities are multiplied by F_H and then squared to get the M^2. Rewriting this equation in the form of $ax^2 + bx + c$ results in

$$M_1{}^2 = F_H{}^2\left(\frac{\delta h_1}{l_1} - \frac{F_z}{F_H}\frac{\delta x}{l_1}\right)^2 x^2 \text{ for domain } [0, l_1]$$

$$\begin{aligned} M_2{}^2 = F_H{}^2\frac{\delta h_2{}^2}{l_2{}^2}x^2 + 2\frac{\delta h_2}{l_2}F_H(F_H\,\delta h_1 - F_z\,\delta x)x + \delta h_1{}^2 F_H{}^2 + F_Z{}^2\delta x^2 \\ - 2\,\delta h_1\,F_Z\,\delta x\,F_H \text{ for domain } [0, l_2] \end{aligned}$$

$$M_3{}^2 = F_H{}^2\left(\frac{\delta h_3}{l_3}+\frac{F_z}{F_H}\frac{\delta x}{l_3}\right)^2 x^2 + 2\,F_H{}^2\left(\frac{\delta h_3}{l_3}+\frac{F_z}{F_H}\frac{\delta x}{l_3}\right)\left(\delta h_1 + \delta h_2 - \frac{F_z}{F_H}\delta x\right)x + \delta h_1{}^2 F_H{}^2$$
$$+ 2\,\delta h_1\,\delta h_2\,F_H{}^2 - 2\,F_H\,F_z\,\delta x\,\delta h_1 + \delta h_2{}^2 F_H{}^2 - 2\,F_H\,F_z\,\delta x\,\delta h_2$$
$$+ \delta x^2 F_Z{}^2 \text{ for domain } [0, l_3]$$

Calculating these areas, as shown in Figure 43 can be done by integrating these equations. This results in the bending energy.

$$E_{c;M1} = \int_0^l M_1{}^2\ \delta l = \frac{1}{3}F_H{}^2\left(\frac{\delta h_1}{l_1}-\frac{F_z}{F_H}\frac{\delta x}{l_1}\right)^2 l_1{}^3$$

$$E_{c;M2} = \int_0^l M_2{}^2\ \delta l$$
$$= \frac{1}{3}F_H{}^2\delta h_2{}^2\ l_2 + \delta h_2\,F_H(F_H\,\delta h_1 - F_z\,\delta x)l_2 + \delta h_1{}^2 F_H{}^2 l_2 + F_Z{}^2\delta x^2 l_2$$
$$- 2\,\delta h_1\,F_Z\,\delta x\,F_H\,l_2$$

$$E_{c;M3} = \int_0^l M_3{}^2\ \delta l$$
$$= \frac{1}{3}F_H{}^2\left(\frac{\delta h_3}{l_3}+\frac{F_z}{F_H}\frac{\delta x}{l_3}\right)^2 l_3{}^3 + F_H{}^2\left(\frac{\delta h_3}{l_3}+\frac{F_z}{F_H}\frac{\delta x}{l_3}\right)\left(\delta h_1 + \delta h_2 - \frac{F_z}{F_H}\delta x\right)l_3{}^2$$
$$+ \delta h_1{}^2 F_H{}^2 l_3 + 2\,\delta h_1\,\delta h_2\,F_H{}^2 l_3 - 2\,F_H\,F_z\,\delta x\,\delta h_1\,l_3 + \delta h_2{}^2 F_H{}^2 l_3$$
$$- 2\,F_H\,F_z\,\delta x\,\delta h_2\,l_3 + \delta x^2 F_Z{}^2 l_3$$

These equations are rewritten in the form of $aF_H{}^2 + bF_H + c$ from which c is omitted since these terms do not influence the position of the minimum bending energy

$$E_{c;M1} = \frac{1}{3}\delta h_1{}^2 l_1 F_H{}^2 - \frac{2}{3}\delta h_1\,F_z\,\delta x\,l_1 F_H + c$$

$$E_{c;M2} = \left(\frac{1}{3}\delta h_2{}^2 + \delta h_1\delta h_2 + \delta h_1{}^2\right)\ l_2\,F_H{}^2 - (2\,\delta h_1 + \delta h_2)\,F_z\,\delta x\,l_2 F_H + c$$

$$E_{c;M3} = \left(\frac{1}{3}\delta h_3{}^2 + 2\,\delta h_1\delta h_2 + \delta h_1{}^2 + \delta h_2{}^2 + \delta h_1\delta h_3 + \delta h_2\delta h_3\right)\ l_3\,F_H{}^2$$
$$- \left(\delta h_1 + \delta h_2 + \frac{1}{3}\delta h_3\right)\ F_z\,\delta x\,l_3 F_H + c$$

Taking the derivative of the sum of these equations and setting that equal to zero and solving it will result in the point for the least complementary energy.

$$E'_{c;M} = \frac{2}{3}\delta h_1{}^2 l_1 F_H - \frac{2}{3}\delta h_1\,F_z\,\delta x\,l_1 + \left(\frac{2}{3}\delta h_2{}^2 + 2\,\delta h_1\delta h_2 + 2\,\delta h_1{}^2\right)\ l_2\,F_H$$
$$- (2\,\delta h_1 + \delta h_2)\,F_z\,\delta x\,l_2$$
$$+ 2\left(\frac{1}{3}\delta h_3{}^2 + 2\,\delta h_1\delta h_2 + \delta h_1{}^2 + \delta h_2{}^2 + \delta h_1\delta h_3 + \delta h_2\delta h_3\right)\ l_3\,F_H$$
$$- \left(\delta h_1 + \delta h_2 + \frac{1}{3}\delta h_3\right)\ F_z\,\delta x\,l_3 = 0$$

Rewriting this gives

$$F_H\left(\frac{2}{3}\delta {h_1}^2 l_1 + \frac{2}{3}\delta {h_2}^2 l_2 + 2\,\delta h_1 \delta h_2 l_2 + 2\,\delta {h_1}^2 l_2 + \frac{2}{3}\delta {h_3}^2 l_3 + 4\,\delta h_1 \delta h_2 l_3 + 2\,\delta {h_1}^2 l_3 + 2\,\delta {h_2}^2 l_3 + 2\,\delta h_1 \delta h_3 l_3 + 2\,\delta h_2 \delta h_3 l_3\right) = F_z\,\delta x\left(\frac{2}{3}\delta h_1 l_1 + 2\,\delta h_1 l_2 + \delta h_2 l_2 + \delta h_1 l_3 + \delta h_2 l_3 + \frac{1}{3}\delta h_3 l_3\right)$$

$$F_H = \frac{F_z\,\delta x\left(\frac{2}{3}\delta h_1 l_1 + 2\,\delta h_1 l_2 + \delta h_2 l_2 + \delta h_1 l_3 + \delta h_2 l_3 + \frac{1}{3}\delta h_3 l_3\right)}{\frac{2}{3}\delta {h_1}^2 l_1 + \frac{2}{3}\delta {h_2}^2 l_2 + 2\,\delta h_1 \delta h_2 l_2 + 2\,\delta {h_1}^2 l_2 + \frac{2}{3}\delta {h_3}^2 l_3 + 4\,\delta h_1 \delta h_2 l_3 + 2\,\delta {h_1}^2 l_3 + 2\,\delta {h_2}^2 l_3 + 2\,\delta h_1 \delta h_3 l_3 + 2\,\delta h_2 \delta h_3}$$

In this way, the situation with the minimum bending energy can be calculated directly.

3.4.5 Comparing equation to FEM

With the equation described in the previous section, the correct thrust line can be directly calculated from the structural properties combined with the loads. To check the equation, the results are compared with the results obtained from the Finite Element Method. Again GSA is used for the FEM calculations. The structures calculated with the equation can be applied to any thickness of the structure. For the FEM calculations, a thickness of 0,2 m is chosen for a span of 6m. The results are shown in the tables below the figures. The differences that occur are very small, all less than 0,10%. These calculations show that the derived equation is valid to find the situation with the least complementary energy due to bending moments, which is nearly the same as the one with the least total complementary energy.

Both the GSA and excel files can be found on the CD.

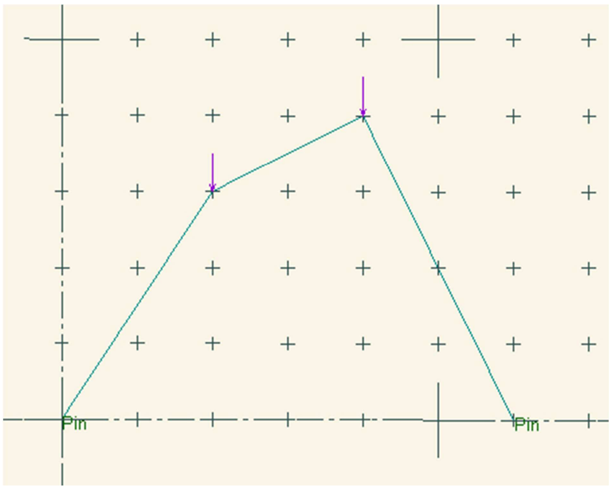


Figure 47

	F_H	
FEM	1117	
min. $E_{c;M}$	1118	0,09 %

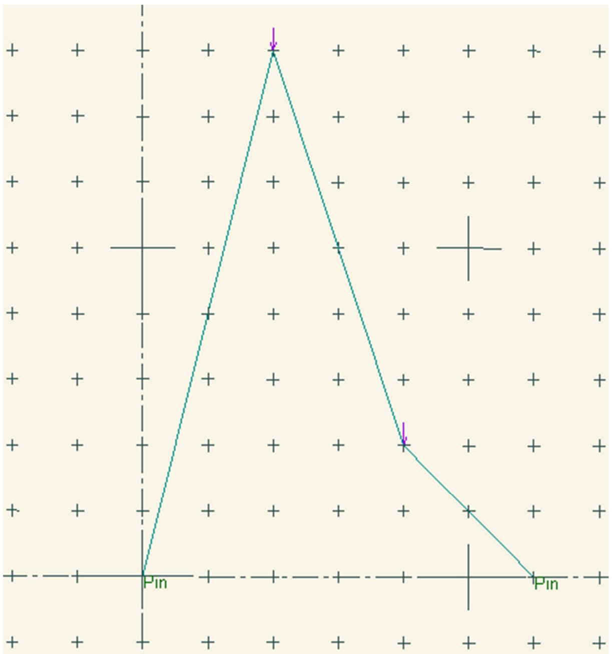


Figure 48

	F_H	
FEM	311	
min. $E_{c;M}$	311	0,00 %

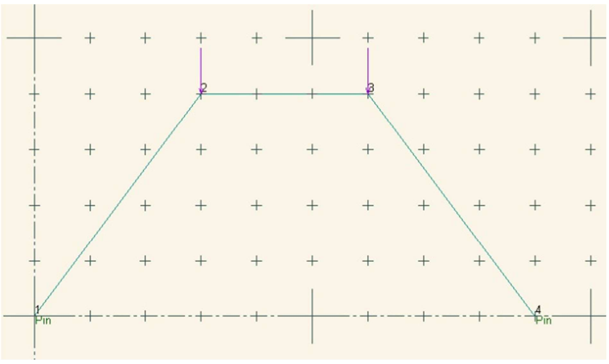


Figure 49

	F_H	
FEM	2249	
min. $E_{c;M}$	2250	0,04 %

3.4.6 Boundaries of the method

As discussed in section 3.4.2, this method is a simplification since the normal forces are ignored when calculating the correct thrust line. This can be done because the thickness *t* is assumed to be very small. The bigger the thickness is, the less accurate the method will be. This section shows to what extent the method is accurate. Figure 50 shows four structures, each with the same shape, only the thickness is changed. Since the minimum $E_{c;M}$ method does not take thickness into account, the resulting thrust line is for all the shapes the same, the one with a F_H of 1118 N. When the calculations are done in GSA using FEM however, the results change with a changing thickness. The results of this can be seen in the table. This shows that with a thickness of 0,5 m (which is quite big for a span of 6 m), the difference is still lower than 0,40%.

The GSA and excel files of these calculations can be obtained from the CD.

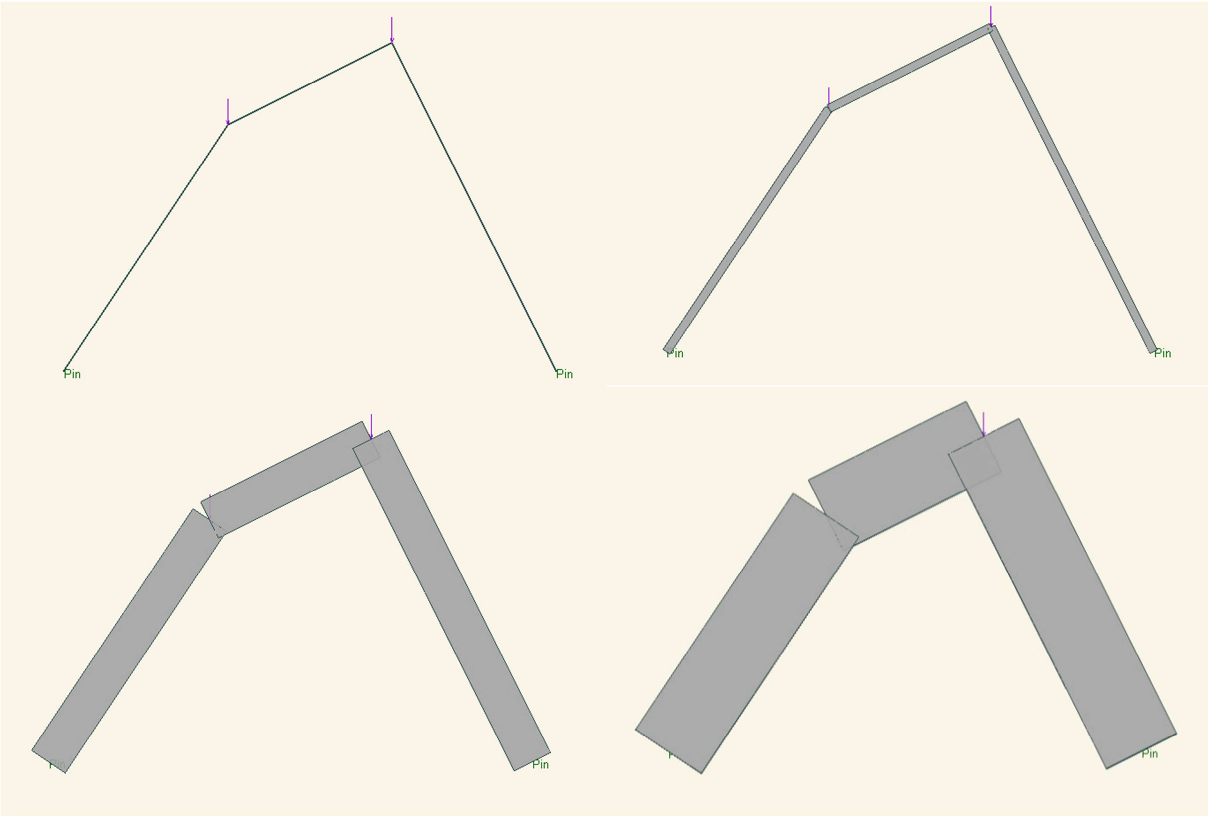


Figure 50 Four structures with a thickness varying from 0,01 m up to 1,00 m. The span of each structure is 6 m.

Direct method	F_H	
t undefined	1118	

FEM	F_H	
t = 0,01	1118	0,00%
t = 0,1	1118	0,00%
t = 0,5	1114	0,36%
t = 1,00	1102	1,45%

3.4.7 Applications of the method

This method can be used in several different ways. Not only does it give insight in the way arches carry loads, it has advantages for the design of arches as well: it can be used as a calculation method and it can be used as a guideline during design.

A graphic calculation method

Not only gives this method a way to calculate arches in a direct way through an equation as derived in section 3.4.4, it can also function as a graphic method. Consider for instance the structure and the loads as given in Figure 51 (a).

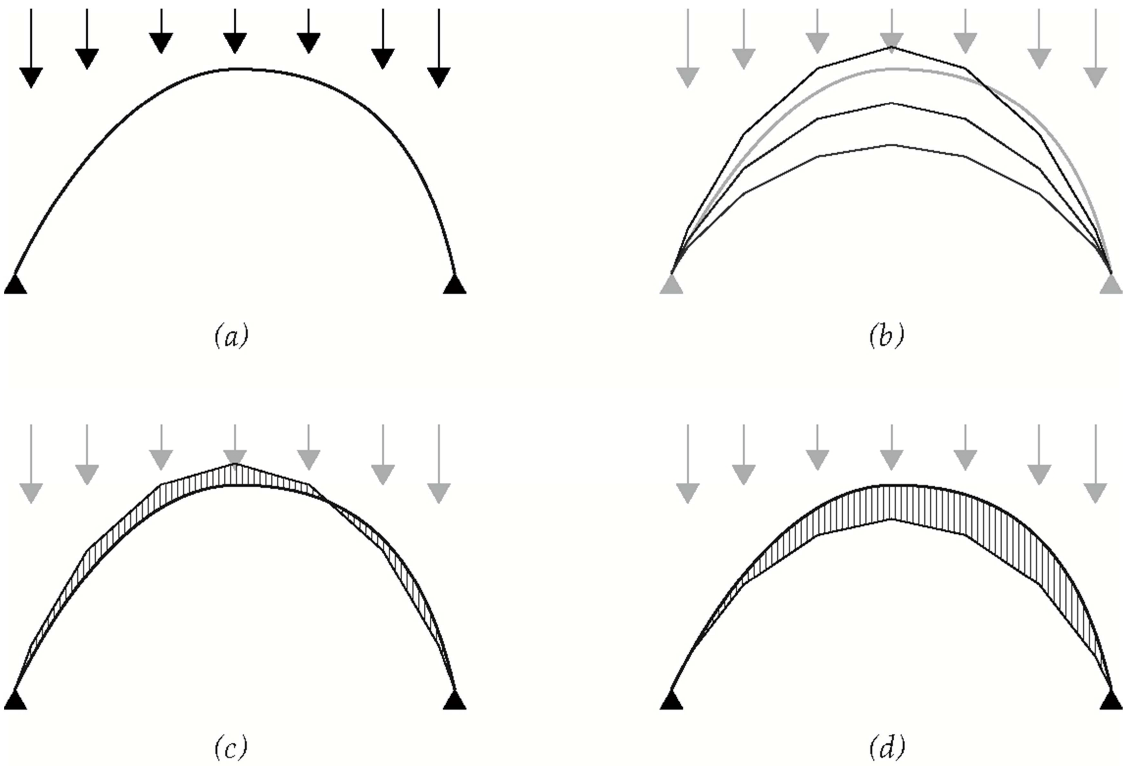


Figure 51 Determining which thrust line is the correct one

From these loads, several different thrust lines can be drawn (b). The correct thrust line can be calculated by minimizing $E_{c;M}$, which is calculated by:

$$E_{c;M} = {e_V}^2 \cdot {F_H}^2 \cdot l$$

Since the length is a property of the structure, which is already determined in this case, minimizing the energy is a trade-off of the vertical eccentricity and the horizontal component of the reaction force. This means that the situation for the lowest energy will be found by making ${e_V}^2 \cdot {F_H}^2$ as small as possible**.** Figure 51 (c) and (d) show how the thrust lines can be compared, each line has a total e_V equal to the areas between the thrust line and each line has a F_H, which is bigger for lower thrust lines and smaller for higher thrust lines. The line with the lowest combination of these two is the correct one.

Determining which line is the correct one can give a rough indication of the forces in the arch, already in the early phase of the design.

A partially graphic calculation method

In order to get more precise results instead of a rough indication, the calculation can be done partially with the derived equation for this method. The result of this is the correct thrust line which can then be drawn in a figure together with the structure. Projecting the forces from the thrust line onto the structure will give the designer insight in why bending moments for instance are bigger in some places and smaller in other places. It can also give an indication of which parts of the structure can be made smaller or bigger.

A guideline during design

Not only does this method give an indication of the forces in an arch as mentioned in the previous section, it can also be used as a guideline during the design of an arch. Consider for example a certain space that needs to be covered by an arch shown in Figure 52 (a) which needs to carry a certain load. For this load, an infinite amount of thrust lines can be drawn (b). A designer can start designing with the idea that (i) an arch is the most efficient if the eV is kept small, so the vertical distance between structure and a thrust line should be kept as small as possible. Which thrust line this is does not matter, the designer can choose a thrust line depending on the concept or the program of the building. In this case, it could be the one in (c) for example. The designer can design an arch with some deviations of the thrust line as desired, knowing that a bigger deviation will result in a bigger bending moment, resulting in either a simpler arch (d) or an arch which is reinforced or made thicker in some places (e).

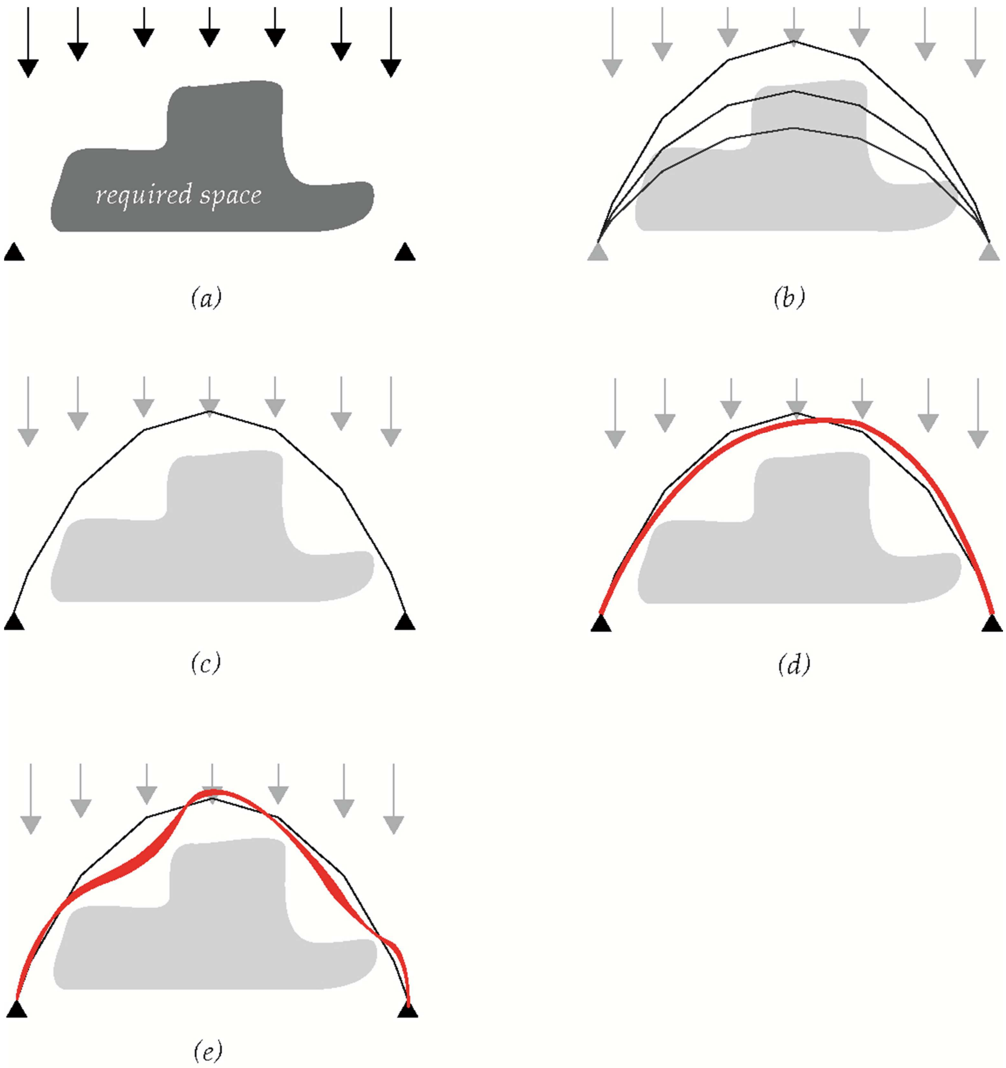


Figure 52 The calculation method as a tool for design

3.4.8 Conclusions

The newly developed method, minimizing only the energy due to bending moments, provides a simple and direct method to calculate the correct thrust line. It is extended for a three-bar structure now, which has all the variables that any structure has, and can thus be expanded to structures with more than three bars. It is also a very accurate method as shown in section 3.4.6, even with structures with a thickness as will probably never be applied in practice, the differences are no more than 1,5%.

CHAPTER 4 –RESULTS AND CONCLUSIONS

4.1 RESULTS AND CONCLUSIONS

This section will discuss the results and conclusions of the research. The first three subsections discuss the three sub questions. The final subsection discusses the main research question and gives the final conclusions of the research.

4.1.1 PROVING THE EQUAL AREA METHOD

The first research question was: *Can the method of equal areas be proven for arches?* This question was chosen because the equal area method seemed a promising method at the beginning of the research.

Results

- Section 3.1 shows a description of the complementary energy which was derived in order to connect it to the equal area method and prove it. This results in two equations which together can express the complementary energy due to normal forces:

$$E_c = \sum \frac{cos^2\theta_{n+1} \cdot {F_{zn}}^2}{sin^2(\theta_n - \theta_{n+1})} \cdot \frac{\delta x_n}{\cos\theta_n}$$

$$\tan\theta_n = \frac{F_{Vn}}{F_H} = F_{Vn} \cdot \frac{\tan\theta_1 + \tan\theta_c}{F_{v1}}$$

  This path of research was not continued because combining these two equations and writing them out for every separate bar of a structure would result in an equation which would be too extensive.

- The iterative equal area method was translated into a direct method by describing it in a mathematical way. The resulting equation (see section 3.2) was translated into a Grasshopper tool as described in section 3.2. Connections to the complementary energy method were not found. This tool provided a simple way to compare the equal area method to other methods.
- The equal area method is compared to the Finite Element Method and the complementary energy method in section 3.3. The complementary energy method and the FEM both give the same results for all the cases. The equal area method does not give the same result for most of the cases. For structures which are (almost) symmetrical the equal area method often gives small deviations. For structures which are not symmetrical at all the method gives bigger deviations of more than 25%.
- The equal area method is proven in a mathematical way for a very specific situation as described in section 3.4.3. This is a situation consisting of two supports, two bars and one load. In this situation the thrust line coincides with the structure.

Conclusions

From these results, it can be concluded that the equal area method is not applicable as a general way to calculate arches. This explains why different attempts to prove this theory gave no result, it can not be proven because it is not a valid method. The method gets more inaccurate when the arch is more irregular shaped and has bigger bending moments.

The method is applicable to some cases however. It is proven for a particular situation consisting of only two bars . Because the thrust line coincides with the structure there are no bending moments and the areas of both are equal to each other.

This proof, together with the fact that the equal area method gives more accurate results in case of smaller bending moments can explain why the reason seemed to be valid in the first calculations. As long as situations are chosen with only small irregularities, the bending moments will remain small and the deviations that this method gives will remain small as well.

This means that the equal area method is applicable to situations for which there are small or no bending moments. Whether or not a structure has bending moments is what the method aims to calculate but is also needed to know to determine the accuracy of the method. This renders the method not useable for this purpose.

4.1.2 Applying calculation methods to shells

The second research question was: *Which methods can be made applicable to shell structures?* The expectation was that once the equal area method was proven, this could be translated into shell structures.

Results

- Since the equal area method turned out not to be generally valid for arches, no attempt was made to translate this method to shell structures
- Developing a method for arches turned out to be a bigger challenge than expected. For this reason the research has been limited to developing the method using complementary energy due to bending for arches only.

Conclusions

The equal area method does not provide a general method to calculate shell structures. It can be applied however to shells without bending moments in a same way as it can be applied to arches without bending moments, simply because in this case the thrust surface coincides with the shell structure. This makes it plausible that this method can be applied to shells with only small bending moments, just like arches with small bending moments can be calculated using this method. To be sure about this, calculations should be made for a number of cases, comparing FEM to the equal area method.

However, applying this method to shells will hold the same problem as mentioned for arches: the aim of the method is to calculate (amongst others) the bending moments in the structure, but the accuracy of the method is dependent on the magnitude of the bending moments. Because of this, the accuracy of the method can not be known for sure unless another calculation method is used as well.

It is not directly clear whether or not the method using minimum bending energy can be made applicable to shells. It seems plausible because the deformations due to bending are often accountable for the most complementary energy. A phenomenon that occurs in shells but does not occur in arches are hoop forces. Since hoop forces consist of normal forces, these can influence the ratio between bending moments and normal forces, which influences the accuracy of this method. To be sure whether this method can be applied, more research is needed.

4.1.3 Modeling methods into Grasshopper

This section reflects on the subquestion: *How can this calculation method be translated into a computational algorithm and modelled in a 3D visualization program?* This was a follow-up question presuming that a method for calculating shells would be found. Even though this is not the case, there are some results related to this question which can be answered.

Results

- After the equal area method was described in a direct way, the resulting equation was translated into a Grasshopper algorithm as described in section 3.2.2. This algorithm required as input the arch to be calculated and the loads acting on this arch. The output was the corresponding thrust line according to the equal area method. This tool was intended to be further developed and to be used as a design tool. It is only used as a research tool however, since it allowed for testing the equal area method. Because the equal area method turned out not to be valid, developing this tool was not continued.
- The method using minimum complementary energy due to bending is described for a 3-bar structure. Theoretically this can be expanded to a general method applicable to all arches. But because this method is not described in a general way yet, it was not relevant to turn this into a Grasshopper algorithm.
- The start of a graphical method is described in section 3.4.7. The exact calculation of the correct thrust line is only partially a graphical method, as a start it still requires the equations derived in section 3.4.4. This is an opportunity where a Grasshopper tool could be used.

Conclusions

When the tool using the equal area method was developed thus far that it could directly generate a thrust line, it was used as a research tool. The thrust lines from this tool were compared to thrust lines calculated by FEM and by the complementary energy method (section 3.3 and appendix II). The tool can be used if more research were to be done into the equal area method.

Since the $E_{c;M}$-method still uses an equation to exactly determine the thrust line, the non-graphical part of this method can be modeled in a computational tool. This would result in a tool which allows the user to focus on the graphical part only. This tool would require the shape of an arch and the loads acting on it as input. The tool then calculates the correct thrust line and gives that as output. This will give the designer insight in how the loads will flow through the structure.

Additionally this tool can offer the functions to output the diagrams and values for bending moments, normal forces and shear forces. This can be calculated in the way described in section 2.2, which can easily be modeled in such a tool.

This will provide a simple calculation tool which lets the user change the structure, giving instant feedback on what this does to the thrust line and thus to the normal forces and bending moments of the structure.

4.1.4 Finding a graphical calculation method

The main research question was: *How can the structural performance of a shell structure be calculated in such a way that the relation between the geometry and the structural performance is shown?* Though the step to shell structures could not be made within this research, the question will be discussed for arches and for shells.

Results

Most of the results are discussed when answering the subquestions in the previous sections. For this reason they are only summarized here.

- The equal area method seemed a good method which could show the relation between structure and performance (section 3.1). It turned out not to be valid however.
- The start of a method minimizing the total complementary energy is made in section 3.2. The equation resulting from this method turned out to be too extensive to continue.
- The method using minimum complementary energy due to bending ($E_{c;M}$-method) is developed as described in section 3.4. This method is based on the idea that the normal forces attribute to such a small fraction of the total complementary energy that they can be ignored when determining the thrust line.
- The results of the $E_{c;M}$-method are compared to results of the Finite Element Method. This shows that the deviations are negligible (section 3.3 and appendix II).
- The step from an iterative to a direct calculation is made for the $E_{c;M}$-method for all three-bar structures as described in section 3.4.4.
- The boundaries of the $E_{c;M}$-method are explored showing that even for very thick structures which are not built in practice, the deviations are smaller than 2%. (section 3.4.6)
- The ways the $E_{c;M}$-method can be used in practice are explored in section 3.4.7, showing how the relation between structure and performance is preserved.

Conclusions

The equal area method can not be used as a method for calculating the correct thrust line. Even though it gives correct results for some cases, it is not clear for which cases this is, making this method not usable.

Finding the correct thrust line by minimizing the total complementary energy gives the correct results when done in an iterative way. Since the rewriting of this method into a direct one resulted in a too extensive equation, even without the bending moments added, it is not expected to result in a simple graphical method. However, since this path of research is not continued, it can not be ruled out that this can lead to a simple graphical method.

The $E_{c;M}$-method is rewritten from an indirect to a direct method which is shown by the mathematical proof in section 3.4.4. There is no doubt about the validity of this part of the method.

The assumption that normal forces can be ignored is very likely according to the results. A broad range of structures is calculated, resulting in differences of less than 2%, even for extreme cases.

The method is not an exclusively graphical method, part of it still uses the equation. But the use of this equation in combination with graphic statics can give more insight in a structure in an early stage of the design. Section 3.4.7. gives some examples of graphical applications of this method. Designing an arch using this method gives the designer an idea of how the shape of an arch determines the magnitude of the forces and moments in the arch.

As described in section 4.1.3, this provides an excellent opportunity to design a computational tool which allows both the mathematical part and the graphical part of the method to be used together. The mathematical part will be modeled in a computational tool and the graphical part can be either interpreted by the user, but can be output by the tool as well.

The $E_{c;M}$-method can probably be applied to shells as mentioned in section 4.1.2, since the normal forces in shells are quite small as well in relation to bending moments.

4.2 Relevance

The two most relevant results of this research are (i) the conclusion that the equal area method is not valid and (ii) the developed method to calculate a thrust line through minimizing the complementary energy due to bending. The relevance of both the results will be discussed in this section.

Disproving the equal area method

The proof that the equal area method is not valid as a general calculation method is relevant because it first seemed to be a promising and simple method. It seemed a good starting point for a method to calculate shells, so it needed to be either proven or disproven.

Next to that, trying to prove this method led to the insight that the bending moment is the most important, or even the only factor to determine which thrust line contains the lowest energy. The newly developed $E_{c;M}$-method is based on this idea.

Minimizing bending energy as a step towards calculating shells

The idea of omitting the energy due to normal forces may seem logical and maybe even obvious in hindsight, however, it is not used before and thus a completely new method. The big advantage of this method is that the calculations get a lot simpler now half of the equation can be omitted. This can be beneficial to the research towards a direct graphical calculation method for shells. As mentioned before, because in shells the deformation due to bending moments are a lot bigger than deformations due to normal forces as well, the expectation is that this method can be applied to shells as well. When this method is expanded to shells, which are more complex than arches, this can simplify the calculations a lot and may lead to a direct graphical way to calculate shells as well. So the result of this research provides an interesting hypothesis on how to find a direct graphical method for calculating shells.

Insight in arches as a structural element

Not only does this pave the way to a direct method for calculating shells, it also gives insight in the way arches work. The resulting equation shows that the way arches carry loads is dependent on two factors. First it is dependent on the way the loads are distributed and thus the possible thrust lines. And second, it is dependent on which thrust line is the correct one, which is in turn determined by the total vertical eccentricity of that line, the horizontal component of the forces and the total length of the structure.

Minimizing bending energy during the design process

Section 3.4.7 describes how this method can be used as a graphic calculation method during the design process. First of all it can be used as a guideline during the process. Drawing a possible thrust line and designing the structure from the thrust line can give the designer an idea of why a structure should have a certain shape. Next to that it can be used as a tool to calculate structures whilst the details of the structure are still unknown. This can give the designer early in the design process an indication of what the forces in the structure are and from that a rough dimension can be determined.

4.3 Recommendations

There is still a lot which needs to be researched in this particular area of structural mechanics. Following are a few recommendations on whether and how this research can be continued:

- Research needs to be done into the method of minimizing bending moments. A study could be done to how accurate this method is. What does the accuracy exactly depend on? It is shown that it will be applicable to most of the situations in today's building practice, since the thickness is always quite small in relation to the span, but it is not known exactly how the accuracy is determined.
- The $E_{c;M}$-method is written out for a three-bar structure. If the method is written out for a *n*-bar structure in a similar way, the result will be a general theory, applicable to all arches.
- Once the $E_{c;M}$-method is written out for all arches, research can be done into applying this method to shell structures.
- Since the equal area method does not hold for situations with irregular shapes, it is recommended not to continue this research. However, to be completely sure, a research could be done to determine for which situations this method does give an accurate result and for which it does not.
- The research from section 3.1 on minimizing the total complementary energy could be continued, however, if the $E_{c;M}$-method is made more general and expanded to shells, it is a less interesting path to follow, since it will be of less use.

4.4 Process

During the process a lot of time went into trying to prove the equal area method. The process could have been more efficient if the research would have started with calculating different types of arches through the equal area method to compare them with the Finite Element Method. This might have resulted in the conclusion that the equal area method is not valid for all arches. This way less time would have been invested into trying to prove this theory.

The research started off with the aim to prove the equal area method directly at the beginning and then extend this method to shells. When this method turned out not to be correct, the focus had to be adjusted to finding a new method of determining the correct thrust lines. This is inherent to doing this kind of research, when trying to find a new method it can not be known in advance whether the hypotheses turn out to be correct. Because the focus had to be adjusted to developing a new calculation method, the research into expanding the method to shells could not be conducted.

List of symbols

δh or δx	vertical component of the length of a part of a structure or thrust line
δx	horizontal component of the length of a part of a structure or thrust line
θ_c	angle between the closing line of a structure and the horizontal axis
θ_n	angle between the *n*th bar of a structure and the horizontal axis
A	area
A_{str}	the area between a structure and its closing line
A_{th}	the area between a thrust line and its closing line
E	Young's modulus
e_V	the vertical eccentricity of a force in a thrust line
FD	force density
F_H	horizontal component of a force
F_n	force in the *n*th bar of a structure
F_V	vertical component of a force
F_z	gravitational force
I	area moment of inertia
l	length
M	bending moment
t	thickness

List of figures

Note: all images in this report are created by the author, unless stated otherwise

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SOFTWARE USED

Rhinoceros 3D, NURBS-based 3D-modeling software, Robert McNeel & Assoc., Seattle, WA, USA

Grasshopper, extension for Rhinoceros 3D, Robert McNeel & Assoc., Seattle, WA, USA

GSA Suite, structural design and analysis software, Oasys software, Newcastle-Upon-Tyne, UK

Appendix I Excel and Grasshopper calculations

The CD included with this report contains a folder with all the files referenced to in this report. Next to that, it also contains the digital version of this report and more files created during the process.

The files referenced in this report are:

Name	Type
3.1.3_minEcN	Microsoft Excel Spreadsheet
3.2.2_eqA	Grasshopper Document
3.2.3_eqA_minEc	Microsoft Excel Spreadsheet
3.3.1_eqA_minEc_1	Microsoft Excel Spreadsheet
3.3.1_eqA_minEc_2	Microsoft Excel Spreadsheet
3.3.1_minE	Grasshopper Document
3.4.5_FEM (1)	GSA Document
3.4.5_FEM (2)	GSA Document
3.4.5_FEM (3)	GSA Document
3.4.5_minEcM	Microsoft Excel Spreadsheet
3.4.6_FEM_thickness	GSA Document
3.4.6_minEcM	Microsoft Excel Spreadsheet
Appendix II_comparison	Rhino 3-D Model
Appendix II_comparison	Grasshopper Document

Appendix II Comparing minimum energy methods to equal area method

To determine the accuracy of different methods and to discuss why the equal area method seemed to work at first, several arches were calculated using three different methods. Below, the arches with their thrust lines are plotted and the F_H of all the thrust lines is compared in the tables. For each arch, both the projected load situation and the own weight situation are calculated. How big the loads are is not relevant because that does not affect the correct thrust line.

The first four arches are the same arches as tested in the graduation report of van Dijk (2014). This gives an indication of why the equal area method seemed so promising: these arches are symmetrical enough to give an accurate result. Only in asymmetrical arches like arch 5 and 7, the differences get bigger, which shows why the equal area method can not be used. For each arch the minimum E_c method is also compared to the newly developed minimum $E_{c;M}$ method, showing the accuracy for this method.

In these comparisons, the thickness of each arch is considered to be 0,2 m and the span 10 m.

The red line is the line calculated by the equal area method. Both thrust lines calculated by the minimum E_c and minimum $E_{c;M}$ coincide for all the situations and are represented by the green line.

The files used for calculations can be found on the CD.

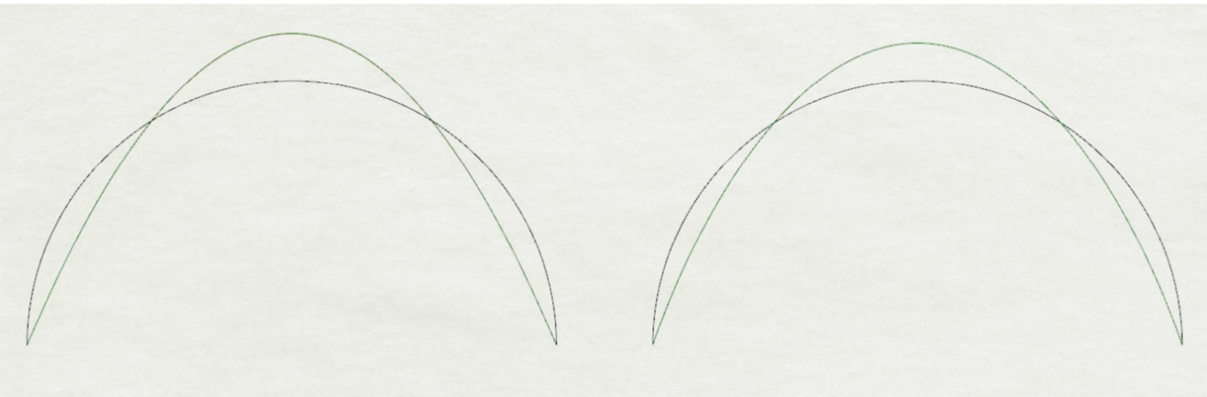

Figure 53 Arch 1 from van Dijk (2014)

own weight	F_H	
min. E_c	5,50	
min. $E_{c;M}$	5,50	0,00 %
equal A	5,50	0,00 %

projected load	F_H	
min. E_c	4,67	
min. $E_{c;M}$	4,67	0,00 %
equal A	4,67	0,00 %

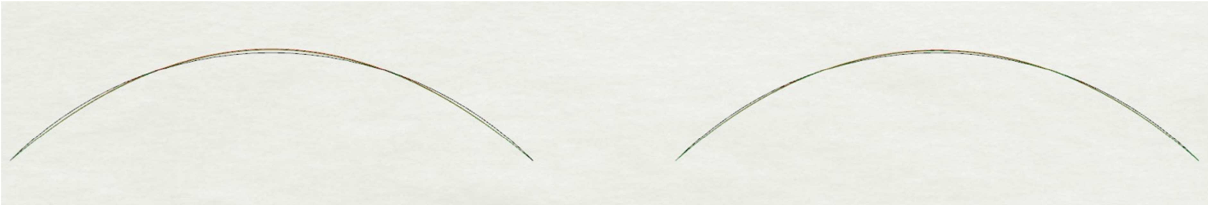

Figure 54 Arch 2 from van Dijk (2014)

own weight	F_H	
min. E_c	13,70	
min. $E_{c;M}$	13,70	0,00 %
equal A	13,62	0,58 %

projected load	F_H	
min. E_c	12,94	
min. $E_{c;M}$	12,95	0,07 %
equal A	12,85	0,70 %

Figure 55 Arch 3 from van Dijk (2014)

own weight	F_H	
min. E_c	8,54	
min. $E_{c;M}$	8,55	0,12 %
equal A	8,71	1,99 %

projected load	F_H	
min. E_c	8,03	
min. $E_{c;M}$	8,04	0,12 %
equal A	8,17	1,74%

Figure 56 Arch 4 from van Dijk (2014)

own weight	F_H	
min. E_c	26,71	
min. $E_{c;M}$	26,72	0,04 %
equal A	26,72	0,04 %

projected load	F_H	
min. E_c	25,60	
min. $E_{c;M}$	25,60	0,00 %
equal A	25,63	0,12 %

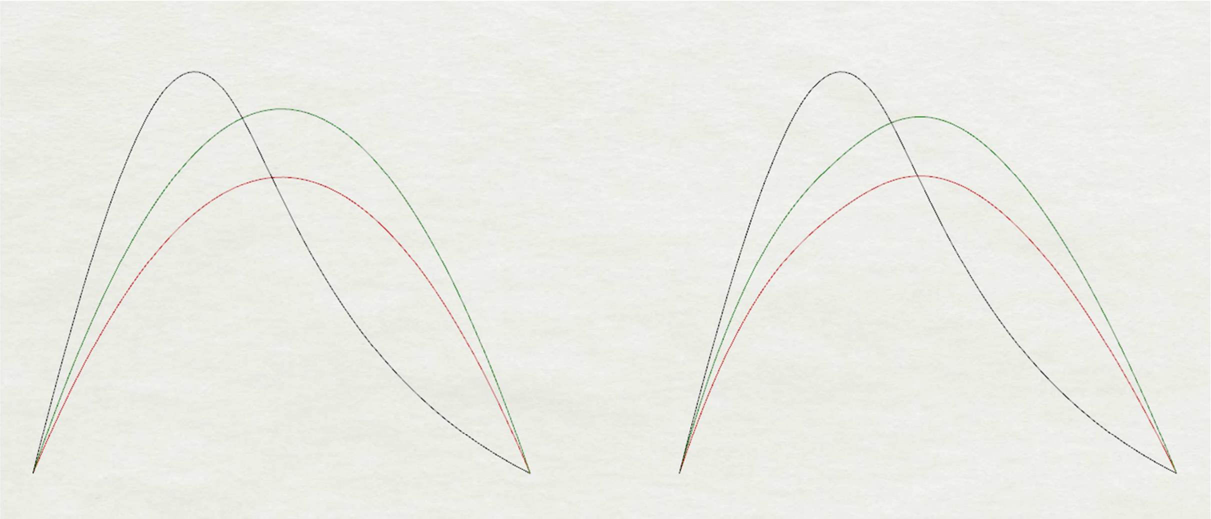

Figure 57 Arch 5

own weight	F_H	
min. E_c	6,48	
min. $E_{c;M}$	6,48	0,00 %
equal A	7,77	19,91 %

projected load	F_H	
min. E_c	3,41	
min. $E_{c;M}$	3,41	0,00 %
equal A	4,35	27,57 %

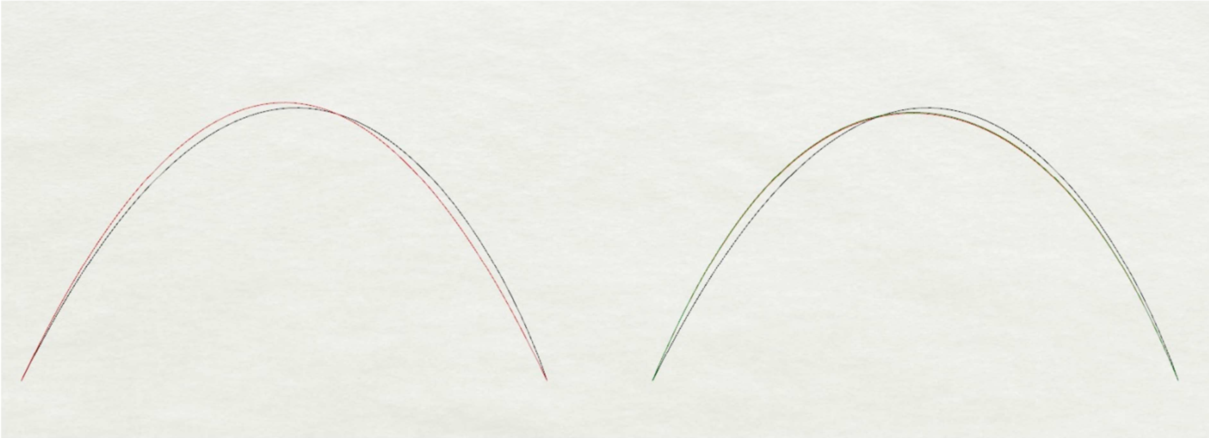

Figure 58 Arch 6

own weight	F_H	
min. E_c	6,19	
min. $E_{c;M}$	6,20	0,16 %
equal A	6,21	0,32 %

projected load	F_H	
min. E_c	4,73	
min. $E_{c;M}$	4,73	0,00 %
equal A	4,73	0,00 %

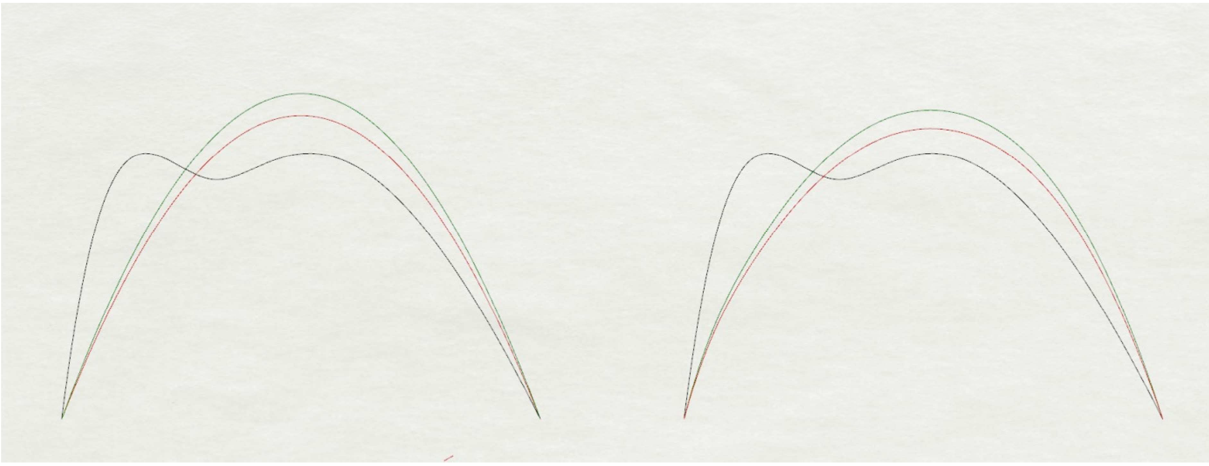

Figure 59 Arch 7

own weight	F_H	
min. E_c	13,23	
min. $E_{c;M}$	13,23	0,00 %
equal A	13,40	1,28 %

projected load	F_H	
min. E_c	3,68	
min. $E_{c;M}$	3,68	0,00 %
equal A	3,94	7,07 %

Appendix III Reflection

Research proposal

In the research proposal the methods to be used are described. These methods will be described here as well.

Approach and methodology

Literature study

The whole research will be done within the field of structural mechanics. For this reason, the literature to be studied is mainly in the field of structural mechanics. To get the research started, a literature study on several subjects needs to be done. Part of these subjects are studied already. The following subjects will be studied:

- Complementary energy method
- Arches and thrust lines
- Graphic statics in arches
- Hoop forces
- Split in surfaces
- Curvature

Method development

From the literature study hypotheses will emerge. From these hypotheses a method to calculate shell structures will be developed.

Design computational algorithm

The found method will be translated into a computational algorithm. For this algorithm, the 3D program Rhino will be used, with the Grasshopper-plugin.

Validate method

The computational algorithm will be compared to FEM calculations for several case studies. Differences in results from these calculations will show whether or not the method is valid.

Relevance

Societal relevance

This research aims to provide in a tool for designers which gives them earlier in the design process insight in the structural performance of a shell structure. This will lead to a less time-consuming design process, but also to a more direct feedback on the design changes. It will probably lead to more efficient structural design, in which less material can be used for a similar performance.

Scientific relevance

Currently it is still unknown what the mechanics are behind shell structures. This research aims to give more insight in these mechanics.

Methods during the research

During the research, some of the research was conducted as planned but not all of it. This section will reflect on how these methods worked out.

Literature study

All the subjects in the literature study were researched extensively. However, alongside this study, the first hypotheses emerged. Some of the topics studied turned out to be less relevant to the hypotheses which were to be tested.

Method development

The method development turned out to be a bit less structured then imagined in the first instance. Even though from the literature study some ideas emerged, only when the first calculations are done, you really understand how the theories work. This results in constantly changing of ideas of what might work and how to test it. Some hypotheses could be tested quite quickly (in a day or two) and if they didn't seem to work out, they were not included in any report.

Design computational algorithm

The design of the algorithm was done before the theory was proven, making it more of a research tool than a final product.

Validate method

The theories that seemed to be promising were always compared to a FEM calculation to see whether it was accurate or not.

Results

These methods resulted in several products:

- A summary of some of the methods to calculate the bending energy in arches (chapter 2 and section 3.4.1)
- An equation for calculating the energy due to normal forces
- An equation for directly calculating the thrust line using the equal area method
- An algorithm using the equal area equation
- FEM calculations compared to the algorithm, which prove that the equal area method is not valid for a lot of situations
- A proof of the equal area method for one situation in which it is valid, the situation in which the thrust line coincides with the structure
- A hypothesis on how to calculate the correct thrust line, by only minimizing the bending energy
- An equation using the minimizing of bending energy to find the correct thrust line for three-bar structures
- FEM calculations showing how accurate this method is

Conclusions

As can be seen from the results, the methods were adequate for this type of research. The aim of the research however is not fully achieved, finding a method to calculate shells. It turned out that for arches, there was still so much to be discovered that the step to shells could not be made in this time frame. This results for instance in some of the subjects (*curvature* and *hoop forces*) which are included in the literature study but not in the final report, since they mainly deal with shell structures. The process might have been more efficiently if the literature study was more fragmented, by studying arch related literature first, conducting that part of the research after that and wait with the second part of the literature study until the problem is solved for arches.

Since the equal area method is proven to be invalid, the second and third product in the list are less relevant than they would have been if it turned out to be valid. This inefficiency could have been prevented if some more extensive calculations were done on this subject. This way the theory would probably have been proven wrong earlier in the process, making sure that less time was spent on trying to prove this subject.

Apart from these two inefficiencies, the methods turned out to fit the problem quite well. Even though the scope of the research was during the process limited to arches, the methods could be applied to this part of the subject as well.