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Rendezvous on an interval and a search game on a star

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“Rendezvous on an interval and a search game on a star”

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1 Introduction

Rendezvous problems and search games (with two players) are related problems. In a rendezvous problem the players like to meet as soon as possible, while in a search game one of them tries to avoid the meeting as long as possible. So in a rendezvous problem the players have a common interest and in a search game the players have conflicting interests. We consider in this thesis a rendezvous problem on a discrete labeled interval and a search game on a star with players having motion detection abilities.

This thesis consists of two parts. The first part (Chapter 2) is about the rendezvous problem on a discrete labeled interval. This problem was first introduced by Alpern in [1]. Most results for this problem are summarized in [4], Chapter 13. This thesis contains new results extending the results in [4]. The work was done in Delft in February 2008 until May 2008 and in July 2008. Moreover a part of the results in Chapter 2 were presented at the 13th International Symposium on Dynamics Games and Applications in Wrocław, Poland.

The second part of this thesis (Chapter 3) is about a search game on a star with players having motion detection abilities. This problem was proposed by Prof. S. Gal (University of Haifa, Israel) and until now no results were available. The work on this problem was done in June 2008 in Haifa resulting in (numerical) solutions for the game and insight in its asymptotic behavior.

2 Rendezvous search on a discrete labeled interval

2.1 Introduction to rendezvous search

In a *rendezvous search problem* two people (who we will call *players* or *searchers*) are both placed in a known search region Q according to known independent distributions supported on Q . They will move at unit speed and their common goal is to meet as soon as possible. The solution to the problem describes for both searchers a *search strategy* such that their expected meeting time is minimized. The rendezvous search problem in this form was first introduced by Alpern in [1].

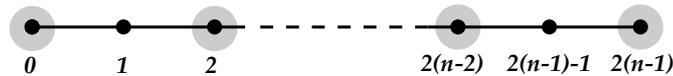


Figure 2.1: Discrete interval with $2n - 1$ nodes labeled as $0, 1, \dots, 2(n - 1)$.

We will consider a rendezvous search problem in which the search region is a discrete interval with $2n - 1$ nodes labeled as $0, 1, \dots, 2n - 2$. So in our problem the players have a common labeling of the nodes. Alpern describes in [4] also problems in which the searchers do not have a common labeling of the search region. The distance between two consecutive nodes is *one* and thus the searchers move each time step from one node to another. We assume the searchers can initially only be placed on the n even numbered nodes. In Figure 2.1 the discrete interval is showed and the marked nodes are even numbered nodes. Since the searchers move at unit speed they will be on even numbered nodes at even times and on odd numbered nodes at odd times, which ensures that they will always meet on an node and not between two nodes. This assumption was first made by Howard in [7] and he showed that in this way the discrete problem is a good discretization method for a continuous version of the rendezvous problem on a labeled interval.

Most work of Alpern and Howard concerning the rendezvous problem on the discrete labeled interval is summarized in [2] and [4], Chapter 13. Some additional results by Howard, Chester and Tütüncü can be found in [5] and [6]. Next we introduce the mathematical notation which will be used throughout this chapter. The notation is based on the notation used in [4].

2.1.1 Mathematical notation

Consider again a discrete interval with $2n - 1$ nodes such as defined above. Define \mathcal{N}_e as the set of even nodes such that

$$\mathcal{N}_e = \{0, 2, \dots, 2(n - 1)\}.$$

We will often use an index j with $j = 0, \dots, n - 1$, such that j corresponds to the even node $2j \in \mathcal{N}_e$. In this section we consider two versions of our rendezvous problem, namely an *asymmetric* version and a *symmetric* version. In the *asymmetric* rendezvous problem $\Gamma^a(p, q)$ two players are placed independently on \mathcal{N}_e according to respectively p and q , where p and q are both distributions supported on \mathcal{N}_e . As mentioned before, the players have the common

goal to meet as soon as possible by adopting some *search strategy*. In the *symmetric* problem $\Gamma^s(p)$ two players are placed independently on \mathcal{N}_e , but now according to the same distribution p supported on \mathcal{N}_e . Again the players have the common goal to meet as soon as possible, but now they must adopt the *same* search strategy. It is convenient to interpret p as a vector in \mathbb{R}^n with elements p_0, p_1, \dots, p_{n-1} . Then p_j for $j = 0, \dots, n-1$ is the probability that the corresponding player is initially placed on node $2j \in \mathcal{N}_e$. In the same way it is possible to interpret q as vector in \mathbb{R}^n .

A search strategy describes a path in time starting on node $2j$ at time $t = 0$ for each even node $2j \in \mathcal{N}_e$. Let \mathcal{S} denote the set of all search strategies which can be adopted by an individual player. Then a strategy $s \in \mathcal{S}$ describes for each $2j \in \mathcal{N}_e$ a path such that $s_j(0) = 2j$ and $s_j(t+1) = s_j(t) \pm 1$. In this way we model the assumption that each time step a player must move to an adjacent node. So consider for instance a rendezvous problem with $n = 4$. Then $\mathcal{N}_e = \{0, 2, 4, 6\}$ consists of four even points and a strategy $s \in \mathcal{S}$ could look like

$$\begin{cases} s_0(0) = 0, & s_0(1) = 1, & s_0(2) = 2, & s_0(3) = 3, & \dots \\ s_1(0) = 2, & s_1(1) = 3, & s_1(2) = 4, & s_1(3) = 3, & \dots \\ s_2(0) = 4, & s_2(1) = 5, & s_2(2) = 4, & s_2(3) = 3, & \dots \\ s_3(0) = 6, & s_3(1) = 5, & s_3(2) = 4, & s_3(3) = 3, & \dots \end{cases} \quad (2.1)$$

It is convenient to interpret the paths $s_0(\cdot), \dots, s_{n-1}(\cdot)$ as paths of a set of *agents* $\{s_0, \dots, s_{n-1}\}$ of a player, with agent j starting at even node $2j \in \mathcal{N}_e$. These paths can also be drawn in a picture, see Figure 2.2 for the picture corresponding to the example strategy. The lines

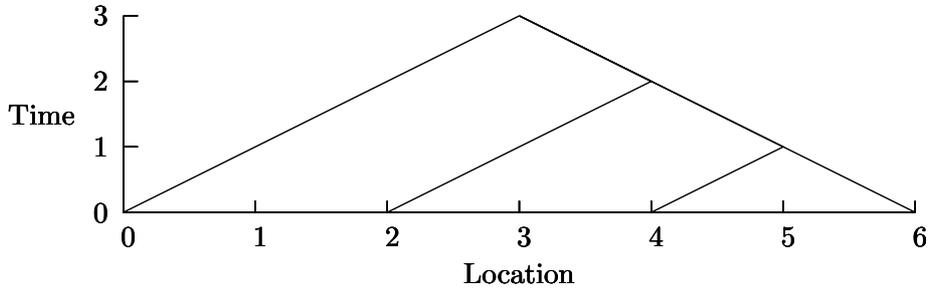


Figure 2.2: Search strategy represented in a picture

represent the paths of the different agents at the first three time steps. Here we used only solid lines, because in this case no group of agents split up after meeting. However in general to distinguish the paths it may be necessary to use different colors or line styles in such figures (for instance in Figure 2.3). The next step is to define the notion of *meeting time* and *meeting time matrix*. We will first do this for the more general asymmetric problem and then adapt the definitions to the symmetric problem.

Consider the asymmetric problem $\Gamma^a(p, q)$ and consider a pair of strategies $(f, g) \in \mathcal{S} \times \mathcal{S}$ such that player I adopts strategy $f \in \mathcal{S}$ and player II adopts strategy $g \in \mathcal{S}$. Then the *meeting time* $m_{f,g}(i, j)$ of agent f_i and agent g_j is defined as

$$m_{f,g}(i, j) = \min \{t \geq 0 \mid f_i(t) = g_j(t)\}. \quad (2.2)$$

So this defines the first time at which agents f_i and g_j are on the same node. Notice this is equivalent to the time at which player I and player II meet each other under the assumption that player I is initially placed on node i and player II on node j . Hence the expected meeting time \widehat{T} of player I and player II can be written as function of the chosen search strategies in the following way

$$\widehat{T}(f, g) = \sum_{0 \leq i, j \leq n-1} m_{f,g}(i, j) p_i q_j = p^T M(f, g) q,$$

where the *meeting time matrix* $M(f, g)$ is defined as a $n \times n$ -matrix with $m_{f,g}(i, j)$ being the element on the $(i + 1)^{th}$ row and $(j + 1)^{th}$ column. For the symmetric rendezvous problem $\Gamma^s(p)$ we can introduce the following simplified notation:

$$\widehat{T}(f) \equiv \widehat{T}(f, f), \quad M(f) \equiv M(f, f), \quad m_f(i, j) \equiv m_{f,f}(i, j).$$

The expected meeting time $\widehat{T}(f)$ is then given by

$$\widehat{T}(f) = \sum_{0 \leq i, j \leq n-1} m_f(i, j) p_i p_j = p^T M(f) p. \quad (2.3)$$

Notice that for any strategy $s' \in \mathcal{S}$ the meeting time matrix $M(s')$ is symmetric, because for all $i, j \in \{0, \dots, n-1\}$ we have $m_{s'}(i, j) = m_{s'}(j, i)$. Consider for instance again the strategy s such as defined in equation (2.1). For that strategy we have $m_s(0, 0) = 0$, $m_s(0, 1) = 3$, $m_s(1, 2) = 2$, etc. Computing all meeting times for this example results in the meeting time matrix

$$M(s) = \begin{bmatrix} 0 & 3 & 3 & 3 \\ 3 & 0 & 2 & 2 \\ 3 & 2 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{bmatrix}.$$

In both the asymmetric and the symmetric rendezvous problem each player can either adopt a *pure* search strategy such that he always plays that strategy or he can adopt a *mixed* search strategy in which he assigns a probability to each pure strategy. The next lemma shows that we can assume that the players will not adopt mixed strategies.

Lemma 2.1 *For the asymmetric rendezvous problem $\Gamma^a(p, q)$ and the symmetric problem $\Gamma^s(p)$, we can assume that the players will always adopt pure strategies.*

Proof: Consider the asymmetric problem $\Gamma^a(p, q)$. Suppose player I adopts a mixed strategy such that he plays the pure strategy $f^i \in \mathcal{S}$ with probability π_i and suppose player II adopts a mixed strategy such that he plays the pure strategy $g^j \in \mathcal{S}$ with probability ρ_j . The expected meeting time for a specific combination of pure strategies is described in (2.2) and consequently the expected meeting for this pair of mixed strategies is equal to

$$\sum_{i,j} \pi_i \rho_j \widehat{T}(f^i, g^j).$$

Now define i', j' such that $\widehat{T}(f_{i'}, g_{j'}) \leq \widehat{T}(f^i, g^j)$ for all i, j , then

$$\widehat{T}(f^{i'}, g^{j'}) \leq \sum_{i,j} \pi_i \rho_j \widehat{T}(f^i, g^j)$$

and hence the pure pair $(f^{i'}, g^{j'})$ results in an expected meeting time smaller than or equal to the expected meeting time for the mixed pair of strategies. Consequently we can assume that the players will always adopt pure strategies in $\Gamma^a(p, q)$. Using the same arguments with $p = q$, $\rho_j = \pi_j$ and $g^j = f^j$ proves the same result for $\Gamma^s(p)$. ■

We introduced notation in the section and moreover we just proved in Lemma 2.1.1 we can assume that the players will adopt pure strategies. Now we can define what is meant by an optimal solution for the rendezvous search problem. Remember in such a problem the players would like to meet as soon as possible, so the expected meeting time has to be minimized. In the asymmetric rendezvous problem $\Gamma^a(p, q)$ a pair of strategies $(f, g) \in \mathcal{S} \times \mathcal{S}$ minimizing $\widehat{T}(f, g)$ is called *optimal*. In the same way, in the symmetric rendezvous problem $\Gamma^s(p)$ a strategy $f \in \mathcal{S}$ which minimizes $\widehat{T}(f)$ is called *optimal*. Next we will outline the contents of the remaining part of this chapter.

2.1.2 Outline of the chapter

Both the symmetric and the asymmetric rendezvous problem as described above are still unsolved. Only for special distributions (like the uniform distribution) the optimal solution is known. All other known results describe ways to simplify the search for optimal solutions by restricting respectively \mathcal{S} and $\mathcal{S} \times \mathcal{S}$ to smaller sets. For information about the known results, see [2], [5] and [7].

In Section 2.2 we consider the symmetric rendezvous problem and relate search strategies in \mathcal{S} to proper binary trees. Next we use this relation to derive a recurrence relation describing the solution of the symmetric rendezvous problem for any distribution, such that this problem can be easily solved using the computer.

After that, we consider in Section 2.3 the more general asymmetric rendezvous problem $\Gamma^a(p, q)$. We introduce the term *sticky pairs* and describe an algorithm to approach the optimal pair of strategies for any p, q . Moreover we solve the asymmetric problem for a particular class of distribution in which p and q are defined such that the players are likely to start at opposite ends of the interval. Finally, we compare $\Gamma^a(p, p)$ with $\Gamma^s(p)$ and investigate whether the players can profit from being allowed to adopt distinct strategies.

In Section 2.4 the results from Section 2.2 and 2.3 are summarized and some ideas for future research are discussed.

2.2 Symmetric rendezvous problem

In the symmetric rendezvous problem $\Gamma^s(p)$ two players are placed independently on the set of even nodes \mathcal{N}_e according to a given distribution p . Their common goal is to adopt a strategy $f \in \mathcal{S}$ such that $\widehat{T}(f)$ is minimized. It is already known (see [2] and [7]) that the search for optimal strategies can be simplified by restricting \mathcal{S} to a smaller set containing at least one

optimal strategy. This restriction has nice properties, which we can use to derive a recurrence relation for solving the symmetric problem numerically. This algorithm is the main result of this section. Besides this result we will also prove some known results in a new way, which might eventually lead to new results in the future.

2.2.1 Restricting the set of strategies

The first restriction we can make, starts with an observation made in [7]. Consider the agents starting at opposite ends of the interval and call these agents respectively the *left sweeper* and the *right sweeper* just as in [7]. Since there are no agents starting left to the left sweeper or right to the right sweeper, it is optimal for the left sweeper to move always to the right and for the right sweeper to move always to the left. In this way the sweepers meet on node $n - 1$ at time $n - 1$. So the set of strategies \mathcal{S} can be restricted such that $s_0(t) = t$ and $s_{n-1}(t) = 2(n - 1) - t$ for all $s \in \mathcal{S}$. Moreover it seems reasonable that a strategy cannot be optimal if one of the agents passes a sweeper. This leads to the following result, which was already proven in [7].

Theorem 2.2 (see [7]) *The symmetric rendezvous problem $\Gamma^s(p)$ has an optimal strategy in which no agent passes one of the sweepers.*

Proof: Consider a strategy in which some agents pass a sweeper. Since there exist an agent who passes a sweeper at least one meeting takes place left to the left sweeper or right to right sweeper, namely the meeting with the other sweeper. Then modify this strategy such that every agent will follow a sweeper to the center when meeting him. Then any meeting which took place left to left sweeper or right to the right sweeper will now take place strictly earlier. Other meetings will take place at the same time as before. Therefore the modified strategy gives a lower expected meeting time than the original one. Hence $\Gamma^s(p)$ has an optimal strategy in which no agent passes a sweeper. ■

Consequently the set \mathcal{S} can be further restricted such that for all strategies $s \in \mathcal{S}$ and for all $i \in \{0, \dots, n - 1\}$ holds

$$\begin{cases} s_i(t) = s_0(t) & \implies \forall t' > t : s_i(t') = s_0(t'), \\ s_i(t) = s_{n-1}(t) & \implies \forall t' > t : s_i(t') = s_{n-1}(t'). \end{cases}$$

Assume from now on that \mathcal{S} consists of strategies corresponding to the restrictions above. Moreover notice these restrictions also imply that at time $n - 1$ all agents are in the center of the interval on node $n - 1$. Hence the meeting time of two agents is at most $n - 1$ and thus the minimal expected meeting time for $\Gamma^s(p)$ is less than or equal to $n - 1$ for any distribution p supported on \mathcal{N}_e . For more restrictions it is useful to introduce two other properties from [2]. In that article the properties are defined for general networks, but in this section the corresponding definitions for the symmetric problem on the interval will be given.

Definition 2.1 (see [2], Definition 2) *A strategy $s \in \mathcal{S}$ is **sticky** if for any time t , and any $i, j \in \{0, \dots, n - 1\}$ holds,*

$$s_i(t) = s_j(t) \implies \forall t' > t : s_i(t') = s_j(t').$$

So in a sticky strategy two agents stay together once they have met. For the symmetric rendezvous problem the following result is known.

Theorem 2.3 (see [2], Theorem 2) *Consider an instance of the symmetric rendezvous problem $\Gamma^s(p)$ with arbitrary distribution p . Then there exists an optimal strategy $s \in \mathcal{S}$, which is sticky.*

A formal proof of this theorem is given in [2], here we will only explain the intuitive idea behind the proof. Suppose two players have both adopted the strategy given in Figure 2.3. Notice then the agents starting at nodes 2 and 4 will meet on node 3 at time 1, but do not

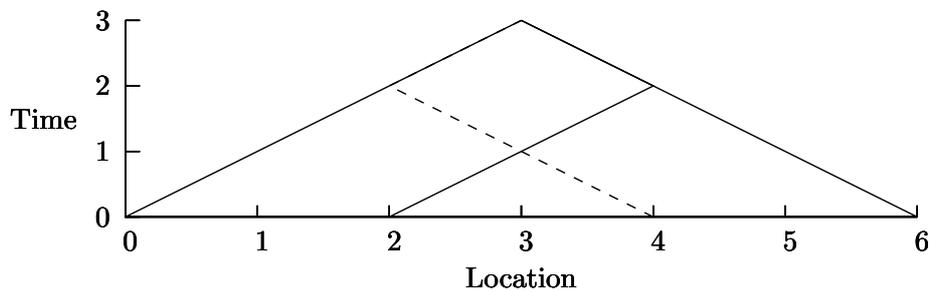


Figure 2.3: Non-sticky symmetric strategy

join each other. Moreover at time 1, they both have not met the other two agents yet. So it should be optimal for them to choose the same path between time 1 and 3.¹ Thus the players should play according to a *sticky* search strategy. In this example at least one of the two strategies in Figure 2.4 will be at least as good as the strategy in Figure 2.3. So Definition

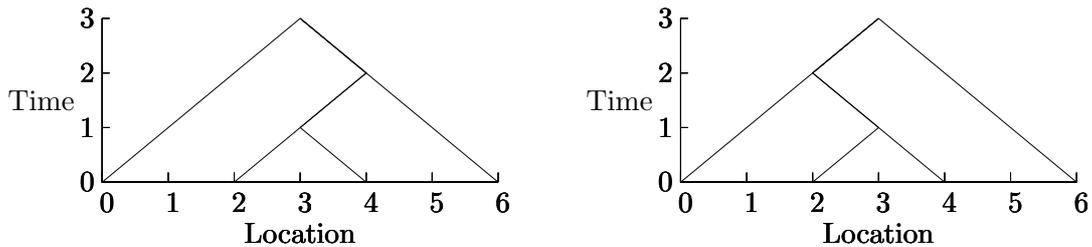


Figure 2.4: Possible improvements to the non-sticky strategy in Figure 2.3

2.1 gives a further restriction to the set of strategies \mathcal{S} . Next we define the property *geodesic* which gives a necessary condition for a strategy to be optimal. In [2] the property is defined for pairs of strategy such that it is also applicable to the asymmetric rendezvous problem. In this section we only give the definition for the symmetric rendezvous problem. Later on, in Section 2.3, we will give the full definition such as given in [2].

¹In some cases it may also be optimal for them to act according to Figure 2.3. However there certainly exists an optimal strategy such that they take the same path after time 1.

Definition 2.2 (see [2], Definition 1) *A strategy $s \in \mathcal{S}$ is **geodesic** if for any even node $2i \in \{0, \dots, n-1\}$ and any consecutive meetings at times t_0 and t_1 when agent s_i meets some distinct agents s_j (at time t_0) and $s_{j'}$ (at time t_1) of the another player (who has also adopted s), we have $|s_i(t_1) - s_i(t_0)| = t_1 - t_0$.*

In [4] the following result is mentioned for symmetric (and the asymmetric) rendezvous problem on the interval, however without really proving it.

Lemma 2.4 *Consider a geodesic strategy $s \in \mathcal{S}$, then an agent s_i can only change direction when meeting an agent of the other player.*

Proof: Consider a strategy $s \in \mathcal{S}$ and suppose agent s_i changes direction at time τ without meeting another agent. Every agent starts initially at some even node and will meet another agent at time 0, namely the agent of the other player which is initially placed at the same even node. At time $n-1$ the agent will meet one of the sweepers. Therefore at time τ agent s_i will always be between two consecutive meetings at times t_0 and t_1 . Since s_i changes direction between times t_0 and t_1 we must have $|s_i(t_1) - s_i(t_0)| < t_1 - t_0$ and thus s is not geodesic. Consequently if s is geodesic, then an agent can only change direction when meeting another agent. ■

Next a theorem which uses the notion of *geodesic* strategies and gives a necessary condition for a strategy to be optimal.

Theorem 2.5 (see [2], Theorem 1) *Consider a symmetric rendezvous problem $\Gamma^s(p)$ with $p_j > 0$ for $j = 0, \dots, n-1$. If a strategy $s \in \mathcal{S}$ is optimal for this problem, then s is a geodesic strategy.*

For a formal proof of this theorem see [2]. Again we will only give the intuitive idea behind the proof. Suppose in the symmetric rendezvous problem $\Gamma^s(p)$ with $p_j > 0$ for all $j \in \{0, \dots, n-1\}$ the searchers will play according to the strategy showed in Figure 2.5. The

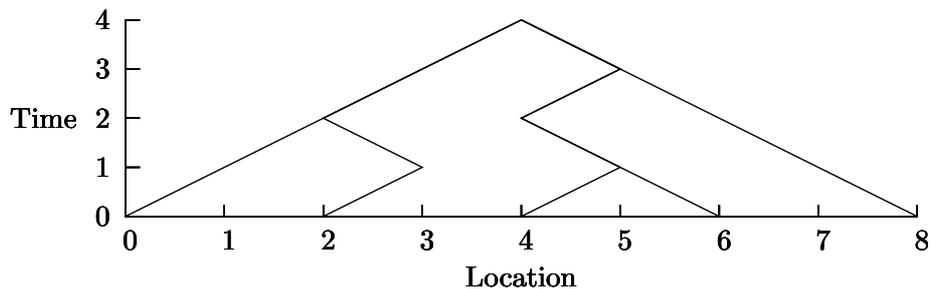


Figure 2.5: Non-geodesic symmetric strategy

agents starting at nodes 2, 4 and 6 all change direction without meeting another agent at some time. The agents starting at nodes 4 and 6 are together at this time, but they have already met in the previous time step. So from Lemma 2.4 it is obvious that the strategy in Figure 2.5 is not a geodesic strategy. In the figure it is not difficult to see how the strategy could be improved. The agent starting at node 2 should go to the left in the first time step and the

agents starting at nodes 4 and 6 should go to the right in the second time step. Applying these modifications results in the strategy showed in Figure 2.6. Notice this strategy is geodesic and all meetings between pairs of agents will take place earlier than or at the same time as the corresponding meetings in the original strategy. Moreover we assumed $p_j > 0$ for all $j = 0, \dots, n - 1$ and hence the modification improves the expected meeting time. If $p_j = 0$

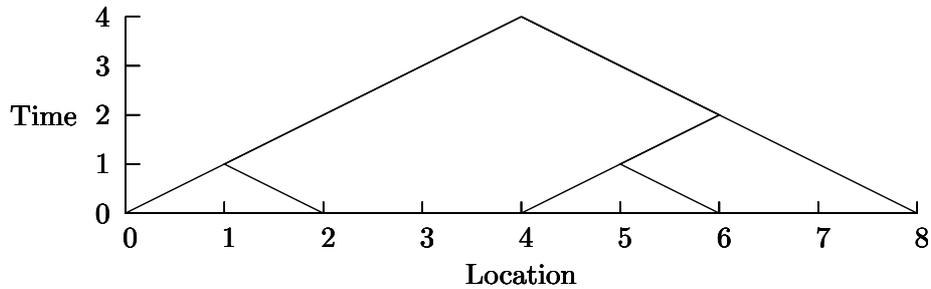


Figure 2.6: Improvement to the non-geodesic strategy in Figure 2.5

for some $j \in \{0, \dots, n - 1\}$, then the modification does not necessarily improve the expected meeting time (it can remain unchanged), but the expected meeting time will certainly not increase. Now the following corollary is a direct consequence of Theorems 2.3 and 2.5.

Corollary 2.6 *Define the subset $\mathcal{S}_{opt} \subset \mathcal{S}$ as the set of strategies in \mathcal{S} which are both sticky and geodesic. Then there exists an optimal strategy $s \in \mathcal{S}$ for $\Gamma^s(p)$ such that $s \in \mathcal{S}_{opt}$.*

Proof: Theorem 2.3 tells us that there exists an optimal strategy for $\Gamma^s(p)$ which is sticky. Moreover if $p_j > 0$ for all $j = 0, \dots, n - 1$, then such a strategy needs to be geodesic according to Theorem 2.5. If $p_j = 0$ for some $j \in \{0, \dots, n - 1\}$, then it is possible to modify every strategy such that it becomes geodesic without increasing the expected meeting time. Hence there exists an optimal strategy which is in \mathcal{S}_{opt} . ■

Consequently if we are looking for an optimal strategy for $\Gamma^s(p)$ we can restrict ourselves to strategies in \mathcal{S}_{opt} . Moreover for small n we can show that it is not possible to restrict ourselves to a smaller subset of \mathcal{S}_{opt} . This can be shown by constructing the optimal strategy for a big number of random distributions and observing that after some time each strategy in \mathcal{S}_{opt} was at least once the unique optimal strategy. For large n we can not perform this experiment, because the size of \mathcal{S}_{opt} grows exponentially, as we will show later on. However we may expect that for all n we cannot restrict ourselves to a smaller subset of \mathcal{S}_{opt} . This idea is summarized in the following conjecture.

Conjecture 2.7 *Consider a strategy $f \in \mathcal{S}_{opt}$ defined for a symmetric problem on \mathcal{N}_e . Then there exists a probability distribution p supported on \mathcal{N}_e such that f is the unique optimal strategy for the symmetric rendezvous problem $\Gamma^s(p)$.*

2.2.2 Relating strategies to proper binary trees

Figures 2.2-2.6 contain examples of strategies. These strategies can be either sticky, geodesic or both. The strategies which are both sticky and geodesic (Figures 2.2, 2.4, 2.6) and thus

contained in \mathcal{S}_{opt} , share an important feature which turns out to hold for all strategies in \mathcal{S}_{opt} . In the next subsection we will use this feature to derive a recurrence relation for solving the symmetric problem numerically.

Consider again Figures 2.2, 2.4 and 2.6. Removing all axes and labels and adding vertices to the meeting points and all possible starting locations (the even nodes on the horizontal axis) results in four trees, see Figure 2.7. These trees are all proper binary trees, which means

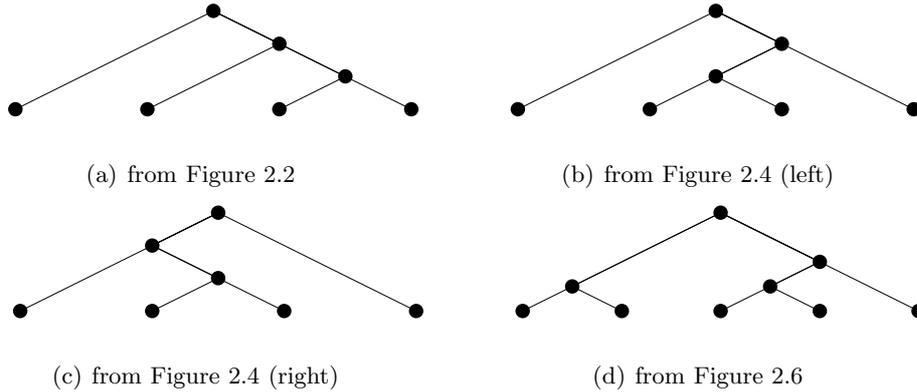


Figure 2.7: Trees constructed from strategies in Figures 2.2 and 2.4, 2.6.

that every vertex (except the leaves) has exactly two children. This is not surprising, because in a meeting point always exactly two groups of agents meet and in a sticky strategy we know groups of agents will not split up after meeting. So the described method for constructing a tree from a strategy results in a proper binary tree for all strategies in \mathcal{S}_{opt} . Notice that in this context the orientation of the trees is important, so the trees in Figures 2.7(b) and 2.7(c) are assumed to be different trees. By removing all vertices from the trees in Figure 2.7 and again

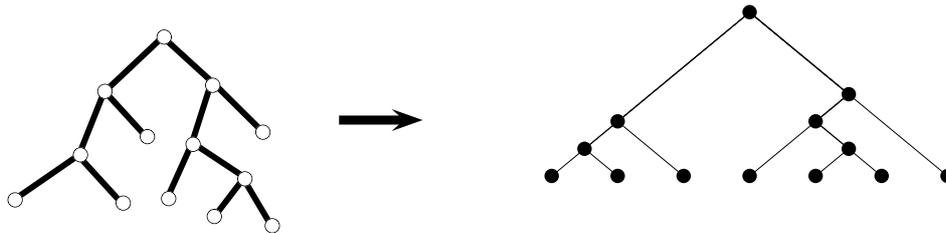


Figure 2.8: Drawing an arbitrary proper binary tree just as the ones in Figure 2.7

adding all axes and labels the original time-space plots for the strategies can be reconstructed. So the process of construction a proper binary tree from a sticky geodesic strategy is reversible. Moreover every proper binary tree can be drawn such that the corresponding sticky geodesic strategy is obvious: for an arbitrary proper binary tree draw the leaves on a horizontal line with the distance between two consecutive leaves equal to 2, next draw all edges with straight lines making an angle of 45 degrees with the horizontal line on which the leaves are placed. For an example of this procedure, see Figure 2.8. Consequently each proper binary tree corresponds to a strategy in \mathcal{S}_{opt} , which leads to the following lemma.

Lemma 2.8 *The method for constructing a proper binary tree from a sticky geodesic strategy describes a one-to-one correspondence between \mathcal{S}_{opt} and \mathcal{B} , where \mathcal{B} is the set of all proper binary trees.*

Proof: Take an arbitrary proper binary tree $B \in \mathcal{B}$ and construct a strategy $f \in \mathcal{S}_{opt}$ from it such as described above. Then constructing a tree from f again results in B . Since B was chosen arbitrarily, there exists a one-to-one correspondence between \mathcal{S}_{opt} and \mathcal{B} . ■

A proper binary tree representing a strategy can also be interpreted in a more intuitive way. Consider a strategy $f \in \mathcal{S}_{opt}$ with corresponding tree $B \in \mathcal{B}$ for a symmetric rendezvous problem played on n even nodes. Assume B is drawn in the same way as the trees in Figure 2.7. Number the leaves of B from left to right with $0, \dots, n-1$ and next place agent f_j in the leaf with number j for $0 \leq j \leq n-1$. Let the agents move at equal speed (without staying or returning) towards the root of the tree, then the agents will meet in exactly the same order as the agents in the rendezvous search on the interval where the players both adopt f . To obtain the same expected time we only have to give the agents an equal speed such that they arrive at the meeting points (branching points of B) at times equal to the times at which meetings in the interval problem take place. The exact value of the speed is not really important ($\sqrt{2}$ to be precise), we can just attach *time labels* with the correct meeting time to the branching points. Of course we need to know at which time the meetings occur and that information is given by Lemma 2.9.

Lemma 2.9 *Consider a strategy $s \in \mathcal{S}_{opt}$ and assume $t \leq n-1$. Next consider a meeting between two groups of agents, then this meeting takes place at time t if and only if exactly $t+1$ agents are involved in the meeting.*

Proof: Take an arbitrary strategy $s \in \mathcal{S}_{opt}$ and remember that such a strategy is both sticky and geodesic.

⇒ Suppose two groups of agents meet at time t . Since s is a sticky strategy agents will not split up once they have met. Therefore the agents involved in the considered meeting at time t were placed in consecutive even nodes at time 0. Let agent s_L be the left most of these agents (so the agent with the smallest index in the group). In the same way let agent s_R be the right most of these agents. Notice that agents s_L and s_R cannot be in the same group, because then using s is sticky we know all agents would have met before time t . So this also implies that agents s_L and s_R meet at time t .

The agents starting left to agent s_L are not involved in the meeting at time t . Moreover s is assumed to be a sticky strategy. So agent s_L will not meet any agent starting left to him before the considered meeting at time t takes place. Then it follows by Lemma 2.4 that once agent s_L moves to the left before time t he will certainly continue to move to the left until time t , because he will certainly not meet any agent starting left to him until time t . However if agent s_L moves to the left at time $t-1$ he cannot meet an agent right to him (and in particular agent s_R) at time t and hence agent s_L will certainly move to right in the first t time steps. In the same way it follows that agent s_R moves to the left in the first t time steps. So agents s_L and s_R move towards each

other as fast as possible and will meet at time t . Thus initially the distance between them must be equal to $2t$. The distance between two consecutive even nodes is equal to 2 and hence exactly $t + 1$ agents will be involved in the considered meeting at time t ; namely agents s_L , s_R and all agents starting in between them.

This holds for all meetings and thus it shows that if a meeting takes place at time t , then exactly $t + 1$ agents are involved in the meeting.

\Leftarrow Consider a meeting in which exactly $t + 1$ agents are involved. If this meeting takes place at time $t' \neq t$, then we just showed that exactly $t' + 1 \neq t + 1$ agents would be involved in the meeting. Hence the meeting must take place at time t .

These two results hold for all strategies $s \in \mathcal{S}_{opt}$. Hence combining the two results proves the lemma. \blacksquare

We can use this lemma to attach time labels to the nodes of the proper binary tree from Figure 2.8, which results in Figure 2.9. Notice the leaves of the tree are labeled with a zero, because they correspond to the placement of the players in the interval and that event takes place at time zero.

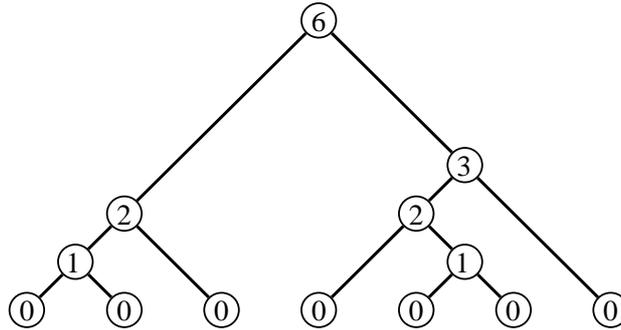


Figure 2.9: Time labels attached to the nodes of the tree from Figure 2.8

Lemma 2.8 shows the one-to-one correspondence between the set of sticky geodesic strategies and the set of proper binary trees. Now consider a symmetric rendezvous problem on the interval played with N even nodes, so $\mathcal{N}_e = \{0, 2, \dots, 2(N - 1)\}$. Then obviously there is an one-to-one correspondence between the sticky geodesic strategies for this particular problem and the set of proper binary trees with exactly N leaves.

Starting at the root of a proper binary tree the tree has to split exactly $N - 1$ times to end with N leaves. So a proper binary tree with N leaves has exactly $N - 1$ branching points and hence the number of such trees is given by the $(N - 1)^{th}$ Catalan number (see [10], Page 627, Fact 5). The Catalan numbers grow exponentially (see A.1) and hence solving the symmetric rendezvous problem by just analyzing all possible strategies (the brute force approach) takes too much time already for relatively small values of N . In the next subsection we will discuss a way to avoid this problem.

2.2.3 Recurrence relation for the symmetric problem

In the previous subsection we showed a one-to-one correspondence between sticky geodesic strategies and proper binary trees. We also showed how to interpret a proper binary tree as a strategy for the symmetric rendezvous problem on the interval. We can use this construction method to derive a recurrence relation describing the solution of the symmetric rendezvous problem.

Consider the symmetric rendezvous problem $\Gamma^s(p)$ with a fixed distribution p supported on \mathcal{N}_e and suppose $f \in \mathcal{S}_{opt}$ is an optimal strategy for this problem. Let $\mathcal{S}_{opt}^{(a,b)} \subset \mathcal{S}_{opt}$ denote the set of strategies such that there exists a meeting in which agents (and no other agents) starting at even nodes $2a, 2(a+1), \dots, 2b \in \mathcal{N}_e$ are involved. Next let $f^{(a,b)} \in \mathcal{S}_{opt}^{(a,b)}$ denote the best strategy for $\Gamma^s(p)$ contained in $\mathcal{S}_{opt}^{(a,b)}$. The strategy $f^{(a,b)} \in \mathcal{S}_{opt}^{(a,b)}$ contains a meeting in which agents (and no other agents) starting at even nodes $2a, 2(a+1), \dots, 2b \in \mathcal{N}_e$ are involved. Define $h(a, b)$ as the contribution of their mutual meetings to $\widehat{T}(f^{(a,b)})$.

By definition, the set $\mathcal{S}_{opt}^{(0,n-1)}$ is equal to \mathcal{S}_{opt} . Hence $f^{(0,n-1)}$ is an optimal strategy for $\Gamma^s(p)$. The contribution of the mutual meetings of all agents to the expected meeting time is just the expected meeting time and thus $\widehat{T}(f) = \widehat{T}(f^{(0,n-1)}) = h(0, n-1)$. The next theorem describes a recurrence relation which can be used to compute $h(0, n-1)$.

Theorem 2.10 *Consider the symmetric rendezvous problem $\Gamma^s(p)$ with a fixed distribution p supported on \mathcal{N}_e . Let $\mathcal{S}_{opt}^{(a,b)}$, $f^{(a,b)}$ and $h(a, b)$ be defined as above, then $h(a, a) = 0$ and for $a < b$ we have*

$$h(a, b) = \min_{a \leq k < b} \{h(a, k) + h(k+1, b) + 2(b-a)(p_a + \dots + p_k)(p_{k+1} + \dots + p_b)\}. \quad (2.4)$$

Proof: First consider the case where $a = b$. Notice $h(a, a)$ can be seen as the meeting between the agents of both players starting at the same even node $2a$. By definition these agents meet immediately and hence the contribution of this meeting to the optimal value of $\Gamma^s(p)$ is zero, which gives $h(a, a) = 0$ for $a = 0, \dots, n-1$. Next consider $a < b$. Let $B \in \mathcal{B}$ be the proper binary tree corresponding to $f^{(a,b)}$ and attach time labels to the nodes such as described in the previous subsection. Moreover number the leafs from left to right with numbers $j = 0, \dots, n-1$, such that agent $f_j^{(a,b)}$ will start in the leaf with number j in the corresponding search problem on B . Now we can use B to compute values for the rendezvous problem on the interval using the arguments from the previous subsection. So from now on we will consider in this proof the rendezvous search problem on B .

Suppose the meeting between the agents starting at leafs with numbers a, \dots, b takes place at branching point v of B . Next let w_1 and w_2 be the children of v such that the agents starting at leafs with numbers a, \dots, \bar{k} pass w_1 and agents starting at leafs with numbers $\bar{k} + 1, \dots, b$ pass w_2 (also see Figure 2.10). According to Lemma 2.9 the time label (the meeting time) attached to v is $b - a$. Next the probability that the actual meeting between the two players takes place at v is equal to

$$2(p_a + \dots + p_{\bar{k}})(p_{\bar{k}+1} + \dots + p_b).$$

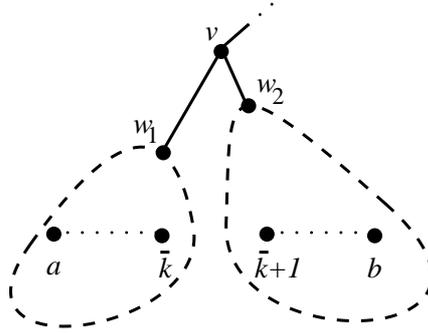


Figure 2.10: Schematic view of a part of the binary tree B

Hence $h(a, b)$ is the sum of $h(a, \bar{k})$, $h(\bar{k} + 1, b)$ and the contribution to $h(a, b)$ resulting from the meeting on v at time $b - a$, which gives

$$h(a, b) = h(a, \bar{k}) + h(\bar{k} + 1, b) + 2(b - a) (p_a + \dots + p_{\bar{k}}) (p_{\bar{k}+1} + \dots + p_b).$$

We defined \bar{f} as a minimizing strategy in $\mathcal{S}_{opt}^{(a,b)}$ and thus the previous expression is equal to

$$h(a, b) = \min_{a \leq k < b} \{h(a, k) + h(k + 1, b) + 2(b - a) (p_a + \dots + p_k) (p_{k+1} + \dots + p_b)\},$$

which proves the theorem. \blacksquare

Now we can apply Theorem 2.10 with $a = 0$ and $n - 1$ to obtain a recurrence relation describing the solution for a rendezvous search problem on \mathcal{N}_e .

Corollary 2.11 *Consider a rendezvous search problem $\Gamma^s(p)$ with p supported on \mathcal{N}_e . Then solving (2.4) with $a = 0$ and $b = n - 1$ gives the optimal value for $\Gamma^s(p)$.*

$$\min_{0 \leq k < n-1} \{h(0, k) + h(k + 1, n - 1) + 2(n - 1) (p_0 + \dots + p_k) (p_{k+1} + \dots + p_{n-1})\}. \quad (2.5)$$

Moreover an optimal strategy can be obtained by analyzing “the minimizing k ’s” each time (2.4) is evaluated for some a, b .

Proof: Take $a = 0$ and $b = n - 1$. All strategies in \mathcal{S}_{opt} contain a meeting in which agents starting at $0, 2, \dots, 2(n - 1)$ are involved and hence $\mathcal{S}_{opt}^{(a,b)} = \mathcal{S}_{opt}$. Of course this meeting is the one in which the two sweepers meet and thus after this meeting the players have certainly met. Hence computing $h(0, n - 1)$ using (2.4) gives the optimal value for $\Gamma^s(p)$. The minimizing k ’s describe the structure of the proper binary tree corresponding to a minimizing strategy. So an optimal strategy can be obtained by analyzing these k ’s. \blacksquare

In general the symmetric rendezvous problem can only be solved numerically using (2.4). However in some cases the solution can be obtained analytically. In the next subsection we will give such an example.

2.2.4 Analytical solutions to the recurrence relation

In [7] the symmetric rendezvous problem $\Gamma^s(p)$ with p being a distribution supported on \mathcal{N}_e such that $0 < p_0 < p_1 < \dots < p_{n-1}$ is solved.² In the unique optimal strategy for this particular problem all agents move to the right until meeting the right sweeper. Once an agent meets the right sweeper he follows him on his way to the center of the interval.

The proofs given in [7] are constructive proofs. Starting with an arbitrary strategy the proofs describe how a strategy can be improved each time step unless an optimal strategy is found. Using (2.4) the solution for this particular class of distribution can also be derived in a more analytical way. This is shown in the next example.

Example 2.1 Consider the symmetric rendezvous problem $\Gamma^s(p)$ with p being an distribution supported on \mathcal{N}_e such that $0 < p_0 < p_1 < \dots < p_{n-1}$. Then using (2.4) we can find

$$h(a, b) = 2 \sum_{j=a}^{b-1} p_j (b-j) (p_{j+1} + \dots + p_b) \quad (2.6)$$

and hence the minimal expected meeting time is equal to

$$h(0, n-1) = 2 \sum_{j=0}^{n-2} p_j (n-1-j) (p_{j+1} + \dots + p_{n-1}). \quad (2.7)$$

In the unique optimal strategy for this particular problem all agents move to the right until meeting the right sweeper. Once an agent meets the right sweeper he follows him on his way to the center of the interval.

Proof: We will prove (2.6) by induction. Evaluating (2.6) for $b = a$ gives an empty sum and hence $h(a, a) = 0$ which is indeed correct. Next assume (2.6) holds for $0 \leq a \leq m < n-1$. Then applying (2.4) to $h(a, m+1)$ gives $h(a, m+1) = \min\{X(k) \mid a \leq k < m+1\}$ with

$$X(k) = h(a, k) + h(k+1, m+1) + 2(m+1-a) \sum_{i=a}^k \sum_{j=k+1}^{m+1} p_i p_j.$$

Notice the assumption concerning (2.6) holds for any choice of a and is hence also applicable to $h(k+1, m+1)$. Since $X(a) < X(k)$ for $a < k < m+1$ (see A.2 for the proof) we have $h(a, m+1) = X(a)$ and hence

$$h(a, m+1) = \underbrace{h(a, a)}_{=0} + h(a+1, m+1) + 2(m+1-a) \sum_{i=a}^a \sum_{j=a+1}^{m+1} p_i p_j.$$

²Actually, the problem solved in [7] is the asymmetric rendezvous problem $\Gamma^a(p, q)$ with p, q supported on \mathcal{N}_e such that $0 < p_0 < p_1 < \dots < p_{n-1}$ and $0 < q_0 < q_1 < \dots < q_{n-1}$.

By assumption, $h(a+1, m+1)$ is given by (2.4) and thus

$$\begin{aligned} h(a, m+1) &= 2 \sum_{j=a+1}^m p_j(m+1-j)(p_{j+1} + \dots + p_{m+1}) + 2(m+1-a) \sum_{j=a+1}^{m+1} p_a p_j, \\ &= 2 \sum_{j=a}^m p_j(m+1-j)(p_{j+1} + \dots + p_{m+1}), \end{aligned}$$

which is equal to (2.6) with $b = m+1$. Then using induction we have now proved that (2.6) holds for all a, b where $0 \leq a \leq b \leq n-1$. It follows from the definition of h that the optimal value for $\Gamma^s(p)$ is equal to $h(0, n-1)$, which validates (2.7).

The computation in A.2 shows for all a, b that $k = a$ is the unique minimizing k in the evaluation of (2.6). Hence if the optimal strategy contains a meeting between agents (and no other agents) starting at $2a, 2(a+1), \dots, 2b$, then in this meeting two groups of agents meet. First a group consisting of a single agent starting at $2a$ and secondly a group consisting of the agents starting at nodes $2(a+1), \dots, 2b$. So starting this analysis with $a = 0$ and $b = n-1$ shows that every agent moves to the right until meeting the right sweeper (the agent starting at node $2(n-1)$). This proves the structure of the optimal strategy. ■

If the strict inequalities in Example 2.1 are relaxed such that $p_0 \leq p_1 \leq \dots \leq p_{n-1}$, then (2.7) is still correct. However now this value could be obtained by different strategies, because there might exist a, b such that the minimizing k in the evaluation of (2.4) is not unique. So the strategy described in Example 2.1 is still optimal in this relaxed problem, but there might exist other optimal strategies. If we have a symmetric rendezvous search problem in which $p_0 \geq p_1 \geq \dots \geq p_{n-1}$, then we can derive the same kind of formulas by using a symmetry argument.

Next we will give another example in which we assume an uniform distribution for the placement of the players. The optimal value can be computed using Example 2.1. As we mentioned above, in this case there might exist more than one optimal strategy. Alpern showed in [2] (Theorem 5) that every strategy in \mathcal{S}_{opt} is optimal for the symmetric rendezvous problem with uniform distribution. The next example in combination with Lemma 2.12 provides an alternative proof for that theorem.

Example 2.2 Consider the symmetric rendezvous problem $\Gamma^s(\bar{p})$ with \bar{p} being an uniform distribution supported on \mathcal{N}_e . So $\bar{p}_j = \frac{1}{n}$ for $j = 0, \dots, n-1$. Then the minimal expected meeting time (also see [7]) is equal to

$$\frac{2}{n^2} \sum_{m=1}^{n-1} m^2 = \frac{(2n-1)(n-1)}{3n}. \quad (2.8)$$

Moreover in this particular problem every sticky geodesic strategy has expected meeting time $\frac{(2n-1)(n-1)}{3n}$ and thus every sticky geodesic strategy is an optimal strategy for $\Gamma^s(\bar{p})$.

Proof: The uniform distribution satisfies $\bar{p}_0 \leq \bar{p}_1 \leq \dots \leq \bar{p}_{n-1}$. Hence we can apply (2.7)

with $\bar{p}_j = \frac{1}{n}$ for $j = 0, \dots, n-1$, which results in

$$2 \sum_{j=0}^{n-2} \bar{p}_j (n-1-j) (\bar{p}_{j+1} + \dots + \bar{p}_{n-1}) = 2 \sum_{j=0}^{n-2} \frac{(n-1-j)^2}{n^2} = \frac{2}{n^2} \sum_{m=1}^{n-1} m^2.$$

In Lemma 2.12 we will prove that for each $f \in \mathcal{S}_{opt}$ the sum of the elements in the meeting time matrix $M(f)$ is equal to $2 \sum_{m=1}^{n-1} m^2$. Hence we find

$$T(f) = \bar{p}^T M(f) \bar{p} = \frac{2}{n^2} \sum_{m=1}^{n-1} m^2, \quad \text{for all } f \in \mathcal{S}_{opt}.$$

So every sticky geodesic strategy is an optimal strategy for $\Gamma^s(\bar{p})$. ■

We still have to prove a lemma concerning the sum of the elements in a meeting time matrix. In this lemma we prove that the sum of the elements of all meeting time matrices corresponding to a strategy in \mathcal{S}_{opt} is the same.

Lemma 2.12 *Take an arbitrary strategy $f \in \mathcal{S}_{opt}$, then the sum of all elements in the meeting time matrix $M(f)$ is equal to $2 \sum_{k=1}^{n-1} k^2$.*

Proof: Take an arbitrary strategy $f \in \mathcal{S}_{opt}$ and let $B \in \mathcal{B}$ be the proper binary tree corresponding to f . Next let \mathcal{V} denote the set of all nodes of B and define $\mathcal{M}(v)$ for some $v \in \mathcal{V}$ as the set of agents being together in v . So if v is branching point of B , then $\mathcal{M}(v)$ is the set of all agents meeting in v . On the other hand if v is a leaf of B , then $\mathcal{M}(v)$ corresponds to the placement on the interval of a single agent and hence contains only one agent. Let $N(v)$ be the number of agents in $\mathcal{M}(v)$, so $N(v) = \#\mathcal{M}(v)$. Finally define $\sigma(v)$ for some $v \in \mathcal{V}$ as the sum of the mutual meeting times for the agents meeting in $\mathcal{M}(v)$, hence

$$\sigma(v) = \sum_{i,j \in \mathcal{M}(v), j \geq i} m_f(i,j).$$

Now we want to prove that for any $v \in \mathcal{V}$ holds

$$\sigma(v) = \sum_{k=1}^{N(v)-1} k^2. \tag{2.9}$$

Notice a leaf in the tree must corresponds to an event at time zero (the placement of an agent on the interval). So in any leaf $v \in \mathcal{V}$ holds $N(v) = 1$ and $\sigma(v) = 0$, which implies (2.9) is valid for all leaves of the tree. Now take an arbitrary branching point $w \in \mathcal{V}$ with children $w_1, w_2 \in \mathcal{V}$. Then $\mathcal{M}(w_1)$ and $\mathcal{M}(w_2)$ are disjoint sets and $\mathcal{M}(w) = \mathcal{M}(w_1) \cup \mathcal{M}(w_2)$. Moreover assume (2.9) is valid for w_1 and w_2 . The strategy f is sticky, so the agents in $\mathcal{M}(w_1)$ have not met the agents in $\mathcal{M}(w_2)$ yet and thus the number of meetings in w is $N(w_1)N(w_2)$. Moreover according to Lemma 2.9 the meeting between $\mathcal{M}(w_1)$ and $\mathcal{M}(w_2)$ takes place at time $N(w) - 1$. Now we know that the sum of the mutual meeting times for

the agents in $\mathcal{M}(w)$ is equal to

$$\begin{aligned}
\sigma(w) &= N(w_1)N(w_2)(N(w) - 1) + \sigma(w_1) + \sigma(w_2), \\
&= N(w_1)N(w_2)(N(w) - 1) + \sum_{k=1}^{N(w_1)-1} k^2 + \sum_{k=1}^{N(w_2)-1} k^2, \\
&= N(w_1)N(w_2)(N(w) - 1) + \sum_{k=1}^{N(w_1)-1} k^2 - \sum_{k=N(w_2)}^{N(w)-1} k^2 + \sum_{k=1}^{N(w)-1} k^2.
\end{aligned}$$

Then use $N(w) = N(w_1) + N(w_2)$ and thus $N(w_1) = N(w) - N(w_2)$ to obtain

$$\sigma(w) = N(w_2)(N(w) - N(w_2))(N(w) - 1) + \sum_{k=1}^{N(w)-N(w_2)-1} k^2 - \sum_{k=N(w_2)}^{N(w)-1} k^2 + \sum_{k=1}^{N(w)-1} k^2. \quad (2.10)$$

For positive integers a, b with $a < b$ holds (see A.3)

$$\sum_{k=a}^{b-1} k^2 - \sum_{k=1}^{b-a-1} k^2 = a(b-a)(b-1).$$

Since $N(w_2) < N(w)$ we can apply this relation with $a = N(w_2)$ and $b = N(w)$ to conclude that the sum of the terms on the right hand side of (2.10) except the last one, is equal to zero and consequently

$$\sigma(w) = \sum_{k=1}^{N(w)-1} k^2.$$

Then applying induction to the branching points of the tree proves that (2.9) holds for all $v \in \mathcal{V}$. Let v_r be the root of the tree, then $\mathcal{M}(v_r)$ is the set of all agents and $h(v_r) = n$. Consequently, applying (2.9) to v_r proves the theorem, because

$$\sum_{0 \leq i, j \leq n-1} m_f(i, j) = \sum_{0 \leq i < j \leq n-1} m_f(i, j) + \sum_{0 \leq j < i \leq n-1} m_f(i, j) + \sum_{i=0}^{n-1} m_f(i, i), \quad (2.11)$$

$$= \sigma(v_r) + \sigma(v_r) + \sum_{i=0}^{n-1} 0, \quad (2.12)$$

$$= 2 \sum_{k=1}^{n-1} k^2. \quad (2.13)$$

■

For an arbitrary distribution we have no way to solve (2.4) analytically. However we can always solve the symmetric rendezvous problem numerically for an arbitrary distribution by using (2.4). In the next subsection we will discuss some numerical results.

2.2.5 Numerically solving the recurrence relation

In this subsection we will first describe a way to solve the symmetric rendezvous problem (and thus $h(0, n-1)$) numerically using (2.4). After that we will numerically verify a result of Chester and Tütüncü (see [5]). Finally we will give some other numerical examples from which we may be able to make useful observations.

The way we used to compute $h(0, n-1)$ numerically is as follows. By definition $h(a, a) = 0$ for $a = 0, \dots, n-1$. This knowledge can be used to compute $h(a, a+1)$ for $a = 0, \dots, n-2$ and these values can be stored. Next it is possible to compute $h(a, a+2)$ for $a = 0, \dots, n-3$ and again these values can be stored. Repeatedly applying such a step will finally result in the value of $h(0, n-1)$ and hence the solution for the considered rendezvous problem. It is possible to construct an optimal strategy from all stored subvalues. In the next small example we take $n = 4$ and the probability distribution consists of simple fractions. We will compute $h(0, 3)$ by hand to visualize the method described above. In general n can be large and the probability distribution will not always be a nice one. Thus in general it will be very hard (or impossible) to do the computations by hand, but relatively easy using the computer.

Example 2.3 Consider a symmetric rendezvous problem $\Gamma^s(q)$ in which $n = 3$ and q is given by

$$q_0 = \frac{1}{6}, \quad q_1 = \frac{1}{4}, \quad q_2 = \frac{1}{12}, \quad q_3 = \frac{1}{2}.$$

Initially we have $h(0, 0) = h(1, 1) = h(2, 2) = h(3, 3) = 0$. Then we find

$$h(0, 1) = 2 \cdot \frac{1}{6} \cdot \frac{1}{4} = \frac{1}{12}, \quad h(1, 2) = 2 \cdot \frac{1}{4} \cdot \frac{1}{12} = \frac{1}{24}, \quad h(2, 3) = 2 \cdot \frac{1}{12} \cdot \frac{1}{2} = \frac{1}{12}.$$

Next we compute

$$\begin{aligned} h(0, 2) &= \min\{h(0, 0) + h(1, 2) + 4q_0(q_1 + q_2), h(0, 1) + h(2, 2) + 4q_2(q_0 + q_1)\}, \\ &= \min\left\{0 + \frac{1}{24} + 4 \cdot \frac{1}{6} \cdot \frac{1}{3}, \frac{1}{12} + 0 + 4 \cdot \frac{1}{12} \cdot \frac{5}{12}\right\}, \\ &= \min\left\{\frac{19}{72}, \frac{2}{9}\right\} = \frac{2}{9} \end{aligned}$$

and in the same way

$$\begin{aligned} h(1, 3) &= \min\{h(1, 1) + h(2, 3) + 4q_1(q_2 + q_3), h(1, 2) + h(3, 3) + 4q_3(q_1 + q_2)\}, \\ &= \min\left\{0 + \frac{1}{12} + 4 \cdot \frac{1}{4} \cdot \frac{7}{12}, \frac{1}{24} + 0 + 4 \cdot \frac{1}{2} \cdot \frac{1}{3}\right\}, \\ &= \min\left\{\frac{2}{3}, \frac{17}{24}\right\} = \frac{2}{3}. \end{aligned}$$

Then finally we can find

$$h(0, 3) = \min\left\{\frac{3}{2}, \frac{13}{8}, \frac{31}{18}\right\} = \frac{3}{2}.$$

All computed values are given in the table below. So for instance $h(1, 2) = \frac{1}{24}$ as we computed

above.

h	0	1	2	3
0	0	$\frac{1}{12}$	$\frac{2}{9}$	$\frac{3}{2}$
1		0	$\frac{1}{24}$	$\frac{3}{2}$
2			0	$\frac{1}{12}$
3				0

By analyzing which term is the minimizing one and starting at $h(0, 3)$ we can construct a binary tree representing an optimal strategy. For instance for $h(0, 3)$ the minimizing term is the first one. Since the first term is the sum of $h(0, 0)$, $h(1, 3)$ and some other value we can mark $h(0, 0)$ and $h(1, 3)$. Moreover we could draw a line between the value of $h(0, 3)$ and the values corresponding to respectively $h(0, 0)$ and $h(1, 3)$. Next we can analyze $h(1, 3)$, etc. In this example this will eventually result in the binary tree given in Figure 2.7(a). Hence (2.1) is an optimal strategy for $\Gamma^s(q)$.

In this last example the value of $h(a, b)$ for all a, b with $0 \leq a < b \leq n - 1$ has to be computed. This is true in general and hence (2.4) has to be evaluated $\mathcal{O}(n^2)$ times to obtain the value of $h(0, n - 1)$. Moreover all computed values have to be stored, because they are needed later on to construct an optimal strategy. So we also need to store $\mathcal{O}(n^2)$ values. Both the computation time and the amount of memory grows quadratically. Remember that the size of strategies in \mathcal{S}_{opt} for a problem on n even nodes is given by the $(n - 1)^{th}$ Catalan numbers. This means that if we have to evaluate all possible strategies to find an optimal strategy we have to evaluate more than 10^{10} strategies for $n = 21$. However for the numerical method based on the recurrence relation (2.4) we can find the answer for $n = 2048$ in about five minutes.

In the previous subsection Examples 2.1 and 2.2 show two analytical examples for a particular sets of distributions. The analytical results in [5] are also about such a particular set of probability distributions. In [5] these distributions are called *symmetric and strictly monotone* distributions. Such distributions are symmetric around the center of the considered discrete interval. Moreover the probability of starting at some node decreases when moving away from the center. Translating Definition 2.1 in [5] to our notation results in the following definition.³

Definition 2.3 Consider a rendezvous problem defined on the set of even nodes \mathcal{N}_e . Let p be a distribution supported on these n even nodes. Then p is called *symmetric and strictly monotone* if

- ▶ $p_i = p_{n-1-i}$ for $i = 0, 1, \dots, n - 1$,
- ▶ $\left| \frac{n-1}{2} - i \right| < \left| \frac{n-1}{2} - j \right| \implies p_i > p_j > 0$.

Now consider a symmetric rendezvous problem $\Gamma^s(\tilde{p})$ with \tilde{p} being a symmetric and strictly monotone distribution. Then it is proven in [5] that in the optimal solution of $\Gamma^s(\tilde{p})$ all agents always move to center of the interval (except when an agent is exactly at the center). Suppose from this point that \tilde{p} is defined by the circles in Figure 2.12. Notice \tilde{p} fulfils the two criteria

³In [5] the interval is labeled as $-n, -n + 2, \dots, -2, 0, 2, \dots, n - 2, n$. Such a labeling is more practical for this particular problem. However we will stick to the the labeling we used throughout this chapter.

in Definition 2.3. Moreover $\tilde{p}_j \geq 0$ for $j = 0, \dots, n-1$ and $\sum_{j=0}^{n-1} \tilde{p}_j = 1$, which shows that \tilde{p} is indeed a probability distribution. Solving $\Gamma^s(\tilde{p})$ numerically for $n = 51$ results in the

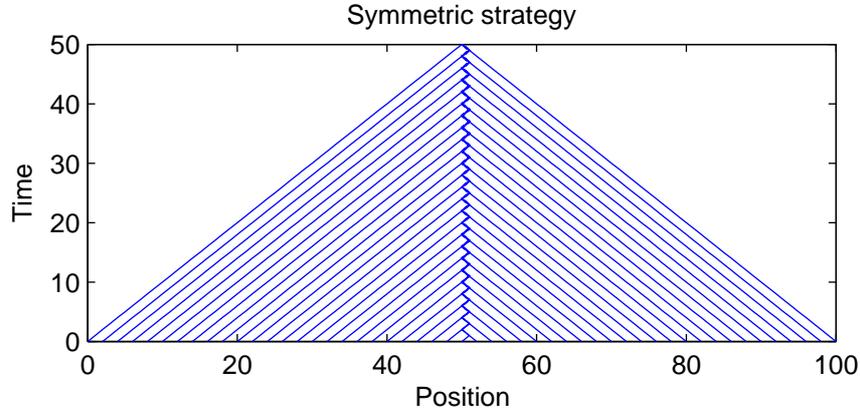


Figure 2.11: An optimal strategy for $\Gamma^s(\tilde{p})$ with $n = 51$

strategy given in Figure 2.11. So Figure 2.11 shows an optimal strategy for $\Gamma^s(\tilde{p})$ with $n = 51$. Indeed all agents move to the center of the interval (except when one of them is exactly at the center). This was already proved analytically in [5] as we mentioned before.

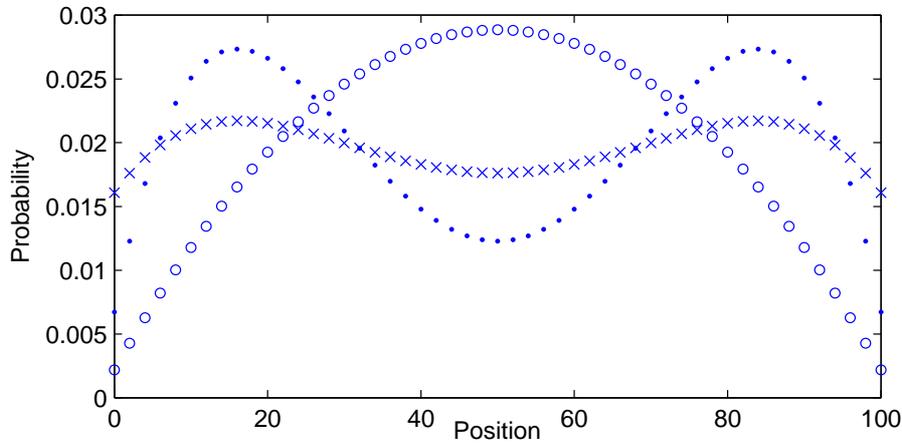


Figure 2.12: Some distributions supported on the set of even nodes for $n = 51$.

The other two distributions in Figure 2.12 (crosses and dots) are also distributions for the symmetric rendezvous problem with $n = 51$. Let q denote the distribution defined by the dots and r the distribution defined by the crosses. These two distributions are also symmetric around the center of the interval. However both do not fulfill the second property in Definition 2.3. Notice distributions q, r are related by

$$r_i = \frac{\log_{10} \left(q_i + \frac{21}{20} \right)}{\sum_{j=0}^{n-1} q_j}, \quad \text{for } j = 0, \dots, n-1.$$

The logarithm is an increasing function on the positive numbers and thus the peaks of those two distributions are at the same places. Moreover the sign of $q_j - q_{j-1}$ is equal to the sign of $r_j - r_{j-1}$ for $i = 1, \dots, n - 1$. An optimal strategy for both $\Gamma^s(q)$ and $\Gamma^s(r)$ is depicted in

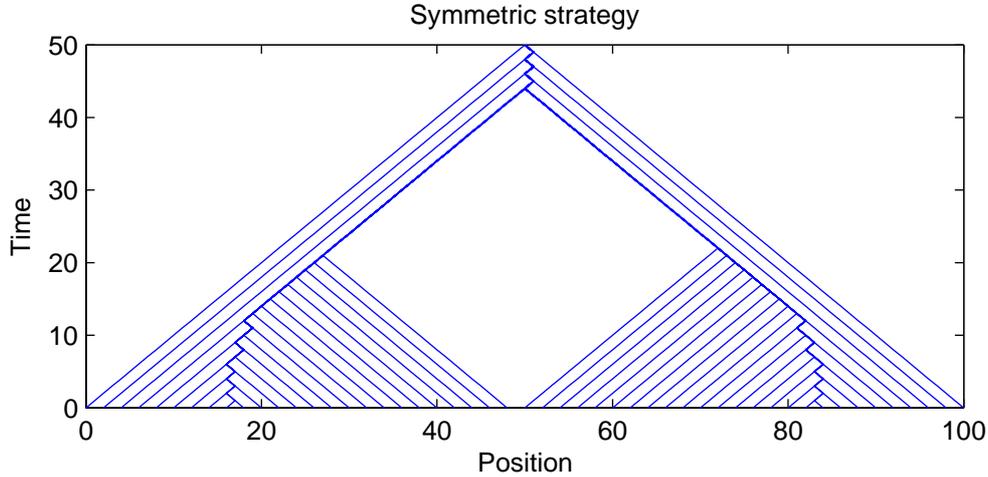


Figure 2.13: Optimal strategy corresponding both the crosses and the dots from Figure 2.12

Figure 2.13. This example (and a lot of other such examples) lead to the following still open questions.

- ▶ Consider a symmetric rendezvous problem $\Gamma^s(p)$ with p supported on \mathcal{N}_e such that p is symmetric around the center of the interval. Does $\Gamma^s(p)$ always have an (almost) symmetric optimal strategy? ‘Almost’ means that for an odd number of the even nodes the agent at the central node has to move and hence ‘disturbs’ the symmetry. Notice this occurs for instance in Figure 2.13 where $n = 51$.
- ▶ Consider two symmetric rendezvous problems $\Gamma^s(p)$ and $\Gamma^s(q)$ with p, q supported on \mathcal{N}_e such that p and q are both symmetric around the center of the interval. Moreover assume the sign of $p_j - p_{j-1}$ to be equal to the sign of $q_j - q_{j-1}$ for $j = 1, \dots, n - 1$. Then do $\Gamma^s(p)$ and $\Gamma^s(q)$ have the same optimal strategies? Or do they have at least one optimal strategy in common?

This example shows we can get new ideas for rules describing optimal strategies from the numerical solutions. We cannot prove these rules as of yet.

2.3 Asymmetric rendezvous problem

In this section we will consider the asymmetric rendezvous problem $\Gamma^a(p, q)$. In this problem two players are placed independently on the set of even nodes \mathcal{N}_e according to a given distribution p for player I and q for player II. Their common goal is to adopt respectively strategies $f, g \in \mathcal{S}$ such that $\hat{T}(f, g)$ is minimized. Again the search for optimal strategies can be simplified by restricting $\mathcal{S}^2 := \mathcal{S} \times \mathcal{S}$ to a smaller set using *sweepers* and the notion of *geodesic strategies* (see [2] and [7]). In the next subsection we will discuss those restrictions and we will also show that it is also possible to generalize the idea of *sticky strategies* to the

asymmetric problem. After that we will consider an algorithm for the asymmetric problem and two restricted versions of this problem.

2.3.1 Restricting the set of strategies

After the introduction of sweepers in Section 2.2 on the symmetric rendezvous problem, the set \mathcal{S} was redefined as the set all strategies in which agents cannot pass both the left and the right sweeper. The same arguments can be applied to the asymmetric problem and thus there exists no pair of strategies which is possibly optimal, but not in \mathcal{S}^2 . For the symmetric problem it was possible to restrict the set \mathcal{S} using the properties *sticky* and *geodesic*. The original definition for *geodesic* such as given in [2] is defined for pairs of strategies and gives a restriction for the asymmetric problem. On the other hand the original definition for *sticky* is only defined for strategies and not for pairs of strategies. However it is possible to generalize the idea of sticky strategies to *sticky pairs* of strategies as we will show in this section.

Definition 2.4 (see [2], Definition 1) *A pair $(f, g) \in \mathcal{S}^2$ is **geodesic** if for any even node $2i \in \{0, \dots, n-1\}$ and any consecutive meetings at times t_0 and t_1 when agent f_i meets some distinct agents g_j (at time t_0) and $g_{j'}$ (at time t_1) of the other player, we have $|f_i(t_1) - f_i(t_0)| = t_1 - t_0$. The corresponding condition must also hold for g .*

In [4] the following result is mentioned. This corollary gives the same result as Lemma 2.4, but now for the asymmetric problem.

Lemma 2.13 *An agent in a geodesic pair (f, g) can only change direction when meeting an agent of the other player.*

Proof: Consider a strategy pair $(f, g) \in \mathcal{S}^2$ and suppose agent f_i changes direction at time τ without meeting an agent of the other agent. Every agent starts initially at some even node and meets another agent at time 0, namely the agent of the other player starting at the same even node. At time $n-1$ the agent meets one of the sweepers. Therefore at time τ agent f_i will always be between two consecutive meetings with agents of the other player at times t_0 and t_1 . Since f_i changes direction between times t_0 and t_1 we must have $|f_i(t_1) - f_i(t_0)| < t_1 - t_0$ and thus (f, g) is not geodesic. Consequently if (f, g) is geodesic, then an agent of the player adopting f can only change direction when meeting an agent of the other player. In the same way we can show that this result also holds for agents of the player who adopts strategy g . Hence any agent in a geodesic pair (f, g) can only change direction when meeting an agent of the other player. ■

Next a theorem which gives a necessary condition for a pair of strategies to be optimal.

Theorem 2.14 (see [2], Theorem 1) *If a pair $(f, g) \in \mathcal{S}^2$ is optimal for the asymmetric rendezvous problem $\Gamma^a(p, q)$ with $p_i > 0$ and $q_j > 0$ for $i, j = 0, \dots, n-1$, then (f, g) is a geodesic pair of strategies.*

Again we will not give the proof, but we will give an example just as in Section 2.2. In Figure 2.14 a non-geodesic pair of strategies is given. This pair is not geodesic, because the agent of player I starting at node 4 changes direction after one time step without meeting an

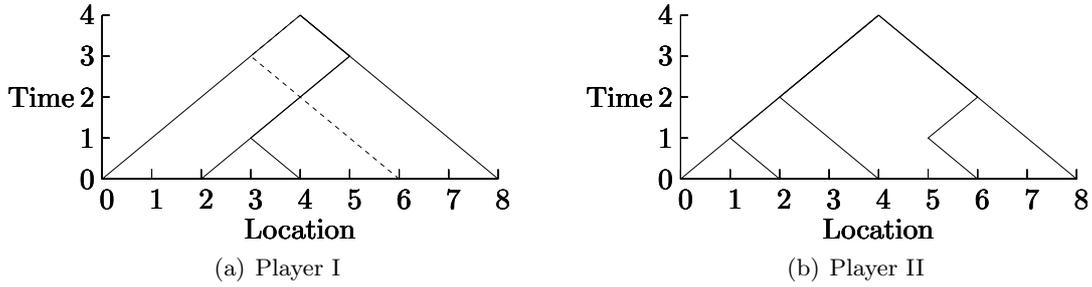


Figure 2.14: Non-geodesic pair of strategies

agent of player II. Also the agent of player II starting at node 6 changes direction without meeting an agent of player II at time 1. The first meeting for the agent of player I starting at node 4 after time 0 is the meeting with the right sweeper. So it is certainly an improvement for this agent to move directly towards the right sweeper. Such an observation can also be made for the agent of player II starting at node 6. Applying these improvements leads to the geodesic pair represented in Figure 2.15. In this improved pair of strategies all meetings take place earlier than or at the same time as in the original one.

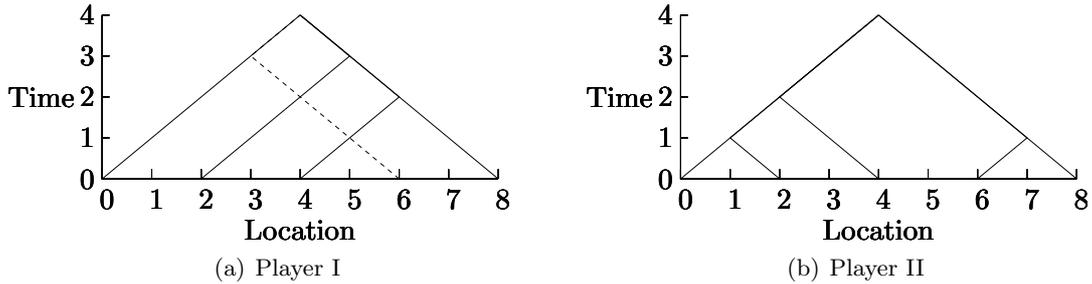


Figure 2.15: Geodesic pair of strategies improving the non-geodesic pair from Figure 2.14

Now consider the asymmetric problem with two agents of player I who meet at some time t and who have met the same set of agents of player II up to time t . Then it seems reasonable that they will move on together, because at time t they face the same optimization problem for the remaining part of their path. This idea leads to a generalization of sticky strategies to sticky pairs of strategies. First define the set of g -agents that a particular f -agent $i \in \{0, \dots, n-1\}$ has met up to time t by

$$\mathcal{M}_{f,g}(i,t) = \{j \mid m_{f,g}(i,j) \leq t\}.$$

Using these sets the term ‘sticky’ as defined in [2] for individual strategies can be generalized to pairs of strategies as follows.

Definition 2.5 A pair $(f,g) \in \mathcal{S} \times \mathcal{S}$ is called a **sticky pair** if for any time t , and for all $i, j \in \{0, \dots, n-1\}$ holds

$$f_i(t) = f_j(t) \wedge \mathcal{M}_{f,g}(i,t) = \mathcal{M}_{f,g}(j,t) \implies \forall t' > t : f_i(t') = f_j(t') \quad (2.14)$$

and

$$g_i(t) = g_j(t) \wedge \mathcal{M}_{g,f}(i,t) = \mathcal{M}_{g,f}(j,t) \implies \forall t' > t : g_i(t') = g_j(t'). \quad (2.15)$$

So in a sticky pair two agents of a player who are on the same node at a certain time τ and have met the same agents of the other players up to that time, will move on together. In such a case they will face the same optimization problem at time τ for determining the remaining part of their path. So it seems reasonable that for asymmetric rendezvous problem there exists an optimal sticky pair of strategies. In the next theorem this is proven.

Theorem 2.15 *For any asymmetric rendezvous problem $\Gamma^a(p,q)$ there exists an optimal sticky pair of strategies.*

Proof: Take an arbitrary pair (f,g) . First notice $f_i(0) \neq f_j(0)$ and $g_i(0) \neq g_j(0)$ for all $i,j \in \{0, \dots, n-1\}$ with $i \neq j$. So it is obvious that (2.14) and (2.15) hold for (f,g) at $t=0$ for all $i,j \in \{0, \dots, n-1\}$. Now suppose (2.14) and (2.15) hold for $t \leq \tau-1$ (with $\tau \geq 1$) and for all $i,j \in \{0, \dots, n-1\}$. Then we will show that there exists a pair (\tilde{f}, \tilde{g}) fulfilling (2.14) and (2.15) for $t \leq \tau$ and for all $i,j \in \{0, \dots, n-1\}$, such that $\widehat{T}(\tilde{f}, \tilde{g}) \leq \widehat{T}(f, g)$. Consider the following cases:

- If (f,g) satisfies (2.14) and (2.15) for $t \leq \tau$ and for all $i,j \in \{0, \dots, n-1\}$, then we can take $(\tilde{f}, \tilde{g}) = (f, g)$.
- Next investigate whether or not there exist $x, y \in \{0, \dots, n-1\}$, such that

$$f_x(\tau) = f_y(\tau), \quad \mathcal{M}_{f,g}(x, \tau) = \mathcal{M}_{f,g}(y, \tau), \quad \exists \tau' > \tau : f_x(\tau') \neq f_y(\tau').$$

If not take $\tilde{f} = f$ and proceed to take next point. If there exist such x, y , then notice that the contribution of agent x after time τ to $\widehat{T}(f, g)$ is

$$C_1 = \sum_{k \in \{0, \dots, n-1\} \setminus \mathcal{M}_{f,g}(x, \tau)} m_{f,g}(x, k) p_x q_k$$

and the contribution of agent y after time τ to $\widehat{T}(f, g)$ is

$$C_2 = \sum_{k \in \{0, \dots, n-1\} \setminus \mathcal{M}_{f,g}(y, \tau)} m_{f,g}(y, k) p_y q_k.$$

If $C_1 \geq C_2$, then define $\tilde{f} = f$ and set $\tilde{f}_x(t) = f_y(t)$ for $t > \tau$. If $C_1 < C_2$, then define $\tilde{f} = f$ and set $\tilde{f}_y(t) = f_x(t)$ for $t > \tau$. Now for the pair (\tilde{f}, g) the definition of sticky pairs holds for x and y . Moreover $\widehat{T}(\tilde{f}, g) \leq \widehat{T}(f, g)$. Repeat this step until (\tilde{f}, g) fulfils (2.14) for $t \leq \tau$ and for all $i, j \in \{0, \dots, n-1\}$.

- Finally we have constructed a pair (\tilde{f}, g) such that (2.14) holds for $t = \tau$ and for all $i, j \in \{0, \dots, n-1\}$. Now investigate whether or not there exist $x, y \in \{0, \dots, n-1\}$, such that

$$g_x(\tau) = g_y(\tau), \quad \mathcal{M}_{g,f}(x, \tau) = \mathcal{M}_{g,f}(y, \tau), \quad \exists \tau' > \tau : g_x(\tau') \neq g_y(\tau').$$

If not take $\tilde{g} = g$. On the other hand if there exist such x, y , then with the same kind of arguments as above we can construct a new pair (\tilde{f}, \tilde{g}) such that $\widehat{T}(\tilde{f}, \tilde{g}) \leq \widehat{T}(\tilde{f}, g)$ and (2.15) holds for x and y . Notice \tilde{g} is equal to g for $t \leq \tau$, thus $\mathcal{M}_{\tilde{f}, \tilde{g}}(i, \tau) = \mathcal{M}_{\tilde{f}, g}(i, \tau)$ for all $i \in \{0, \dots, n-1\}$. This implies that (2.14) also holds for (\tilde{f}, \tilde{g}) and for all $i, j \in \{0, \dots, n-1\}$. Repeat this step until (\tilde{f}, \tilde{g}) fulfils (2.15) for $t \leq \tau$ and for all $i, j \in \{0, \dots, n-1\}$.

Since each case can only occur a finite number of times, this gives a way to construct a pair (\tilde{f}, \tilde{g}) which fulfils (2.14)-(2.15) for $t \leq \tau$ and for all $i, j \in \{0, \dots, n-1\}$, such that $\widehat{T}(\tilde{f}, \tilde{g}) \leq \widehat{T}(\tilde{f}, g)$. Now apply induction to t , then we have proven that we can construct a pair (\tilde{f}, \tilde{g}) fulfilling (2.14)-(2.15) for all t and for all $i, j \in \{0, \dots, n-1\}$, such that $\widehat{T}(\tilde{f}, \tilde{g}) \leq \widehat{T}(\tilde{f}, g)$. Hence for any asymmetric rendezvous problem $\Gamma^a(p, q)$ there exists an optimal *sticky pair* of strategies. ■

Notice this result also holds on general networks, because nowhere in the proof details concerning the discrete search region are used. Now consider the pair of strategies in Figure 2.16. At time 1 the agents of player II starting at nodes 4 and 6 have met the same set of

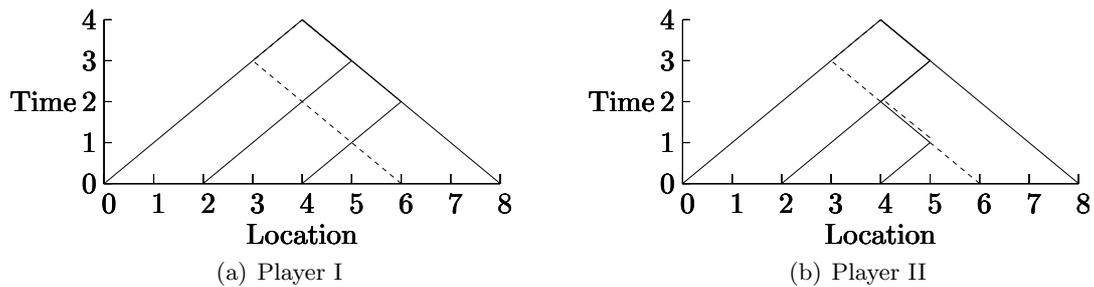


Figure 2.16: Non-sticky pair of strategies

agents of player I and hence they should stay together. However they split up at time 2 and so the pair of strategies in Figure 2.16 is not a sticky pair. If those two agents do not split up at time 2, then the pair of strategies will be a sticky pair. So the pair in Figure 2.16 can be modified in at least two ways (the agents of player II starting at nodes 4 and 6 can both move to the left or to the right at time 2) such that the resulting pair is a sticky pair. It is

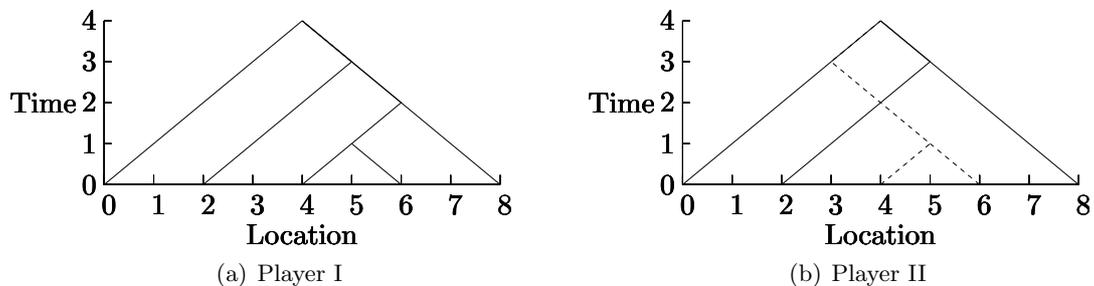


Figure 2.17: Pair of strategies which possibly improves the pair from Figure 2.16

obvious that at least one of those modifications will result in a pair which is at least as good

as the original pair in Figure 2.16 (this is exactly what is proven in Theorem 2.15). The same kind of arguments also hold for the agents of player I starting at nodes 4 and 6 and meeting at time 1. Then finally we will obtain four possible modifications from which at least one is as good as the original pair from Figure 2.16. One of these possibilities is depicted in Figure 2.17.

Corollary 2.6 showed that in the symmetric problem we can restrict ourselves to \mathcal{S}_{opt} . Such a corollary also exists for the asymmetric problem and is a direct consequence of Theorems 2.14 and 2.15.

Corollary 2.16 *Define the subset $\mathcal{S}_{opt}^2 \subset \mathcal{S}^2$ as the set of all pairs of strategies in \mathcal{S}^2 which are both sticky and geodesic. Then there exists a optimal pair of strategies $(f, g) \in \mathcal{S}^2$ for $\Gamma^a(p, q)$ such that $(f, g) \in \mathcal{S}_{opt}^2$.*

Proof: Theorem 2.15 tells us that there exists an optimal pair of strategies for $\Gamma^a(p, q)$ which is sticky. Moreover if $p_i > 0$ for all $i = 0, \dots, n-1$ and $q_j > 0$ for all $j = 0, \dots, n-1$, then such a strategy needs to be geodesic according to Theorem 2.14. If $p_i = 0$ for some $i = 0, \dots, n-1$ or $q_j = 0$ for some $j = 0, \dots, n-1$, then it is possible to modify every pair of strategies such that it becomes geodesic without increasing the expected meeting time. Hence there exists an optimal strategy which is an element of the set $\mathcal{S}_{opt}^2 \subset \mathcal{S}^2$. ■

Consequently if we are looking for an optimal strategy for $\Gamma^a(p, q)$ we can restrict ourselves to strategies in \mathcal{S}_{opt}^2 . It would be nice if it is possible to characterize \mathcal{S}_{opt}^2 by some set of trees or graphs similar to the relation between \mathcal{S}_{opt} and proper binary trees in the symmetric problem from section 2.2, but we were not able to find such a characterization for \mathcal{S}_{opt}^2 .

For small n we counted the number of sticky geodesic pairs of strategies in the hope to find a known integer sequence (like the Catalan numbers for the symmetric problem), which would maybe give a hint for an alternative representation of the strategies in \mathcal{S}_{opt}^2 . However Sloane's online database of integer sequences⁴ does not contain a sequence containing 2, 4, 37, 908, 66314, Notice for $n = 6$ we had to test for 6,250,000 pairs of strategies

n	symmetric problem	asymmetric problem
2	1	2
3	2	4
4	5	37
5	14	908
6	42	66314

Table 2.1: Number of (pairs of) strategies in respectively \mathcal{S}_{opt} and \mathcal{S}_{opt}^2 for small n

whether or not the pair was an element of \mathcal{S}_{opt}^2 . For $n = 7$ this number already grows to 26,244,000,000 which would take too much time. In general the number of pairs we should test is

$$\left(\prod_{i=0}^{n-1} \binom{n-1}{i} \right)^2.$$

⁴<http://www.research.att.com/~njas/sequences/>

Just as for the symmetric problem we can determine for small n the optimal pair of strategies for a big number of random distributions by trying all possible pair, and observe that \mathcal{S}_{opt}^2 can probably not be restricted to smaller set. Again (compare to Conjecture 2.7) we state a conjecture based on this experiment.

Conjecture 2.17 *Consider a pair of strategies $(f, g) \in \mathcal{S}_{opt}^2$ defined for an asymmetric rendezvous problem on \mathcal{N}_e . Then there exist probability distributions p, q supported on \mathcal{N}_e such that (f, g) is the unique optimal pair of strategies for the asymmetric rendezvous problem $\Gamma^a(p, q)$.*

The experiment leading to this observation is in some sense more difficult than the experiment performed in Section 2.2, because for some pairs of strategies it takes a lot of effort to find distributions such that the concerning pair is the unique optimal one. For instance take $n = 5$ and consider a pair of strategies $(f, g) \in \mathcal{S}_{opt}^2$ which is not optimal for the asymmetric problem $\Gamma^a(\bar{p}, \bar{p})$ with \bar{p} being the uniform distribution on five even points. Then the experiment leads to the observation that one can expect to take between 10^5 and 10^6 random choices for the distributions before finding p, q such that (f, g) is optimal for $\Gamma^a(p, q)$. To explain why a pair can be in \mathcal{S}_{opt}^2 but is not optimal for the asymmetric problem with

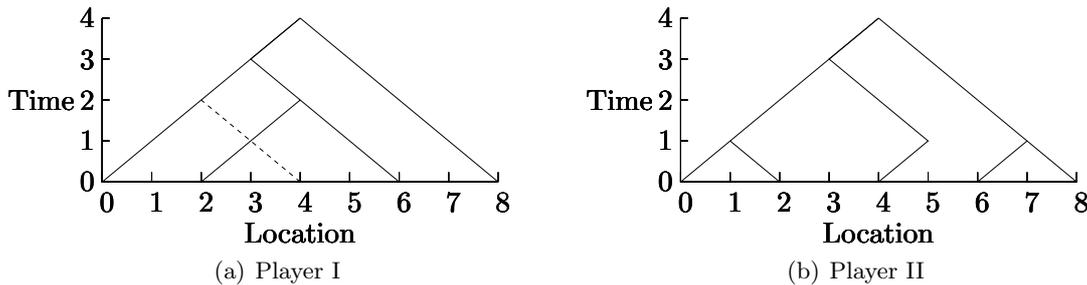


Figure 2.18: In \mathcal{S}_{opt}^2 , but not optimal for the asymmetric problem with uniform distributions

uniform distribution we will now introduce the notion of *strictly geodesic* pairs of strategies (the term was first used by Howard in [6]).

Definition 2.6 *A pair of strategies $(f, g) \in \mathcal{S}_{opt}^2$ is called **strictly geodesic** if an agent who is not moving together with a sweeper, will only change direction at some time when he has met all the agents he could have possibly met at that time.*

Howard proved in [6] that a pair of strategies in \mathcal{S}_{opt}^2 is optimal for the asymmetric problem with uniform distributions if and only if that pair is strictly geodesic. For instance the pair of strategies in Figure 2.18 is not strictly geodesic, because the agent of player I starting at node 2 changes direction at time 2 while he has not met all agents he could possibly have met (the agents of player II starting at nodes 4 and 6). So Figure 2.18 shows an example of a pair in \mathcal{S}_{opt}^2 which is not optimal for the asymmetric problem with uniform distributions. However for the following distributions the pair in Figure 2.18 is the unique optimal pair of

strategies.

$$p = \left[\frac{3619}{10000}, \frac{740}{10000}, \frac{123}{10000}, \frac{3530}{10000}, \frac{1988}{10000} \right]^T,$$

$$q = \left[\frac{835}{10000}, \frac{5053}{10000}, \frac{3451}{10000}, \frac{651}{10000}, \frac{46}{10000} \right]^T.$$

This particular example also shows that \mathcal{S}_{opt}^2 cannot be restricted to the smaller set of sticky and *strictly geodesic* pairs of strategies, because the pair in Figure 2.18 is not a strictly geodesic pair of strategies. We had to take a problem on five even nodes to give such an example, because every strategy pair in \mathcal{S}_{opt}^2 for a problem with at most four even nodes is always strictly geodesic. For $n \leq 3$ this is trivial. For $n = 4$ we can verify that all 37 strategy pairs (see Table 2.1) in \mathcal{S}_{opt}^2 are optimal pairs for the asymmetric problem with uniform distribution and so by Howard's result in [6] it follows that each of these pairs is strictly geodesic.

At this point we know that if we are looking for an optimal pair of strategies for an instance of the asymmetric rendezvous problem, we can restrict ourselves to the set \mathcal{S}_{opt}^2 of sticky and geodesic pairs of strategies. Although we have this restriction, solving the asymmetric rendezvous problem for arbitrary distributions remains a difficult problem. In general even the restricted set of pairs of strategies is too large to compute the expected meeting time for all pairs and we are also not able to characterize \mathcal{S}_{opt}^2 by some set of trees or graphs like for the symmetric problem. However we are still interested in the optimal pair of strategies for such problems. In the next subsection we will give algorithm which tries to find an optimal pair of strategies.

2.3.2 Algorithm for constructing equilibrium pairs of strategies

Consider an optimal pair of strategies $(f, g) \in \mathcal{S}_{opt}^2$ for the asymmetric rendezvous problem $\Gamma^a(p, q)$, where both p and q can be any distribution supported on \mathcal{N}_e . In other words, there exists no pair of strategies which is better than (f, g) and thus for all $s \in \mathcal{S}$ we have $\widehat{T}(f, g) \leq \widehat{T}(f, s)$ and $\widehat{T}(f, g) \leq \widehat{T}(s, g)$.

On the other hand suppose we have a pair $(f', g') \in \mathcal{S}_{opt}^2$ for $\Gamma^a(p, q)$ such that for all $s \in \mathcal{S}$ holds $\widehat{T}(f, g) \leq \widehat{T}(f, s)$ and $\widehat{T}(f, g) \leq \widehat{T}(s, g)$. We will call such a pair of strategies an *equilibrium pair*.⁵

Definition 2.7 Consider a asymmetric rendezvous problem $\Gamma^a(p, q)$ with p, q being arbitrary distributions supported on \mathcal{N}_e . Then a pair of strategies $(f', g') \in \mathcal{S}_{opt}^2$ is called an *equilibrium pair* if and only if for all $s \in \mathcal{S}$ holds

$$\widehat{T}(f', g') \leq \widehat{T}(f', s), \quad \text{and} \quad \widehat{T}(f', g') \leq \widehat{T}(s, g'). \quad (2.16)$$

So if the players have adopted an equilibrium pair of strategies, then they cannot improve the expected meeting time unilaterally. We can ask ourselves whether or not the expected meeting time for all equilibrium pairs is equal? This question is important, because we will

⁵Notice the similarity to the definition of Nash equilibria for non-cooperative games, see [9] for more details. Also compare to (3.1) in Section 3.1.

present an algorithm to construct such equilibrium pairs. Since an optimal pair of strategies is an equilibrium pair itself, a positive answer to the question would imply that the algorithm will construct an optimal pair of strategies. The algorithm tries to find unilateral improvements to a given strategy pair, such that after a finite number of steps we find a strategy pair satisfying (2.16). We will now present the algorithm.

1. Choose an initial strategy $f \in \mathcal{S}$ for player I. (Of course we can also start with choosing an initial strategy for player II. Then we have to replace player I by player II and player II by player I everywhere in the description of the algorithm.)
2. Construct the set of strategies $\tilde{\mathcal{S}}_g = \{s \in \mathcal{S} \mid (f, s) \in \mathcal{S}_{opt}^2\}$. If $\tilde{\mathcal{S}}_g = \emptyset$, then repeat this step with another choice for f .
3. Compute for all strategies in $s \in \tilde{\mathcal{S}}_g$ the expected meeting time $\hat{T}(f, s)$ and let player II adopt a strategy g such that

$$g \in \arg \min \left\{ \hat{T}(f, s) \mid s \in \tilde{\mathcal{S}}_g \right\}.$$

So the best strategy found so far is $T(f, g)$.

4. Construct the set $\tilde{\mathcal{S}}_f = \{s \in \mathcal{S} \mid (s, g) \in \mathcal{S}_{opt}^2\}$. This set is nonempty, because it contains at least (f, g) . Now take a strategy $f' \in \tilde{\mathcal{S}}_f$ such that

$$f' \in \arg \min \left\{ \hat{T}(s, g) \mid s \in \tilde{\mathcal{S}}_f \right\}.$$

If $\hat{T}(f', g) = \hat{T}(f, g)$, then (f, g) is an equilibrium pair and the algorithm stops. If not, then set f equal to f' and continue with the next step.

5. Construct the set $\tilde{\mathcal{S}}_g = \{s \in \mathcal{S} \mid (f, s) \in \mathcal{S}_{opt}^2\}$. This set is nonempty, because it contains at least (f, g) . Now take a strategy $g' \in \mathcal{S}$ such that

$$g' \in \arg \min \left\{ \hat{T}(f, s) \mid s \in \tilde{\mathcal{S}}_g \right\}.$$

If $\hat{T}(f, g') = \hat{T}(f, g)$, then (f, g) is an equilibrium pair and the algorithm stops. If not, then set g equal to g' and return to step 4.

Notice in step 2 we do not have to consider all strategies in \mathcal{S}_{opt}^2 to construct $\tilde{\mathcal{S}}_g$, because we can construct all strategies g such that $(f, g) \in \mathcal{S}_{opt}^2$ by analyzing the paths of the agents of player I. Such analysis gives information about which meetings must take place at certain time and certain location (because an agent of player I changes direction), and also about where and when agents of player II can change direction (because an agent can only change direction when meeting an agent of the other player). The same argument holds for constructing the sets in steps 4 and 5. The algorithm will certainly find an equilibrium pair after a finite number of steps. This follows from the observation that if the algorithm stops the constructed pair is always an equilibrium pair. Moreover we have a finite number of strategy pairs and each time a strategy is replaced by a new one the expected meeting time decreases. Thus the algorithm will certainly stop in finite time.

Now return to the question we asked ourselves before. Is an equilibrium pair always an optimal pair? Unfortunately the answer is *no*. Hence the algorithm does not solve the asymmetric rendezvous problem, because we do not know whether the algorithm will converge to an optimal pair of strategies or to another equilibrium pair.

Example 2.4 Consider an asymmetric rendezvous problem $\Gamma^a(p, q)$ with

$$p = \left[\frac{2143}{10000}, \frac{2994}{10000}, \frac{1405}{10000}, \frac{2673}{10000}, \frac{785}{10000} \right]^T,$$

$$q = \left[\frac{2799}{10000}, \frac{1778}{10000}, \frac{2657}{10000}, \frac{2042}{10000}, \frac{724}{10000} \right]^T.$$

Applying the algorithm with one of the two in strategies in Figure 2.19 as initial strategy for Player I and also for Player II after changing roles, results in all four cases in the equilibrium pair depicted in Figure 2.20 with expected meeting time 2.05693553. However the minimal expected meeting time for the players is 2.05265804 which they can obtain by adopting to strategy pair in Figure 2.21.

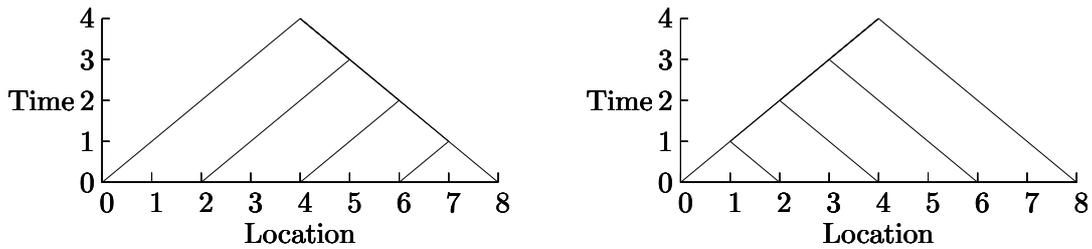


Figure 2.19: Initial strategies chosen for both player I and II in Example 2.4

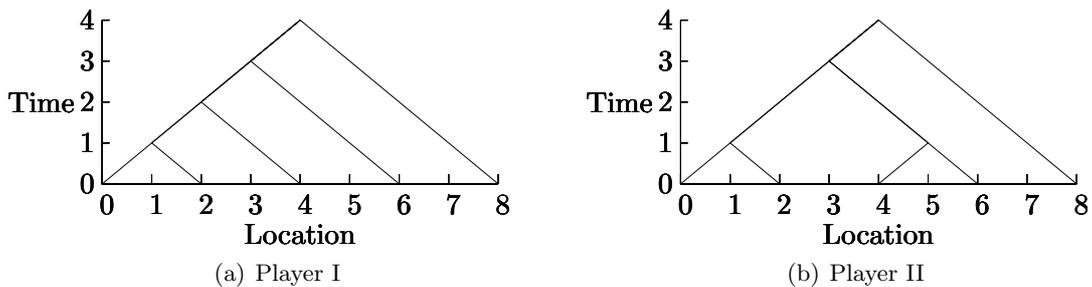


Figure 2.20: Equilibrium point resulting from all four runs of the algorithm

In the above examples the algorithm does not converge to an optimal pair of strategies but to another equilibrium pair. However it is good to realize that the equilibrium point to which the algorithm converges depends strongly on the chosen initial strategy. For instance if we had chosen either the left strategy or the right strategy in Figure 2.21 for respectively player I or player II as initial strategy, then the algorithm would have converged to the optimal pair. Moreover in this example the algorithm converged to the same equilibrium pair for all four considered initializations. However this is certainly not a general rule. On the contrary,

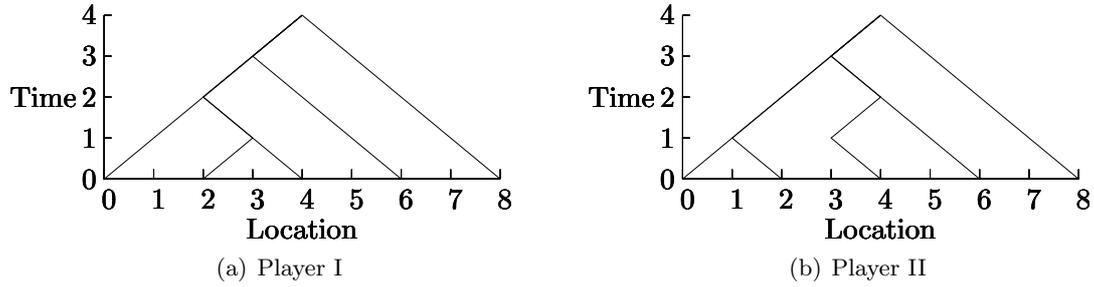


Figure 2.21: Optimal pair of strategies for Example 2.4

most of the time the equilibrium pairs will differ. The relative gap in Example 2.4 between the expected meeting time for the constructed equilibrium pair and the optimal pair is small (only 0.2%). We will do a number of simulations for randomly chosen distributions p and q to see whether this is general behavior. In the same experiment we will also count how often the constructed equilibrium pair differs from the optimal pair using the same four initialization as in Example 2.4. The results are listed in Table 2.2. For each value of n we have run the

n	constructed pair is not optimal	average relative error	maximal relative error
4	0	-	-
5	165	0.53%	2.3%
6	600	0.48%	2.36%

Table 2.2: Results for 10000 simulations with randomly chosen distributions for $n = 4, 5, 6$.

algorithm 10000 times. The second column gives the number of runs in which we found an equilibrium pair not equal to an optimal pair. The third and fourth column give for these pairs respectively the average and the maximal relative error. For $n > 6$ we cannot run this experiment, because it takes too much time to find the optimal pair. Nevertheless this experiment gives rise to some interesting questions.

- ▶ For $n = 4$ we found an optimal strategy in all runs. Is it possible to prove that the algorithm in combination with the four chosen initializations always gives the optimal strategy?
- ▶ For $n = 5$ and $n = 6$ the relative errors remains more or less constant. Does this observation also hold in general for $n \geq 5$? So in other words, is the error independent of n ?
- ▶ Can we construct a set of initializations (for instance based on p and q) such that the average and/or maximal relative error are smaller?

The first question is answered. After another 100000 runs for $n = 4$ we found one particular example in which the constructed optimal pair was not an optimal one. In that case the relative error was about 0.02%. The other two question are unanswered, but still interesting. For $n = 50$ the algorithm still can be executed in a reasonable time. So a positive answer to the second question will give us a way to approximate the minimal expected meeting time

for $\Gamma^a(p, q)$ (with p, q arbitrary distributions) at least up to $n = 50$. Until now we assumed an asymmetric rendezvous problem with arbitrary distributions and tried to derive general properties. In the next subsections we will approach the problem from another point of view. We will first restrict the problem such that the problem may become easier and then we will analyze the restricted (and hopefully simplified) problem.

2.3.3 Asymmetric rendezvous problem with players likely to start at opposite ends of the interval

In the subsection we will analyze the asymmetric rendezvous problem $\Gamma^a(p, q)$ for a particular class of distributions p, q . Assume in the remaining part of this subsection that p, q are distributions supported on \mathcal{N}_e such that

$$\min\{p_i q_j \mid 0 \leq j < i \leq n - 1\} > \max\{p_i q_j \mid 0 \leq i < j \leq n - 1\} \quad (2.17)$$

At first sight it may not be immediately clear what this restriction for p and q means. So we will first explain (2.17). After that, we will show what pair of strategies is optimal for $\Gamma^a(p, q)$. For this problem no solution can be found in the known literature and thus this is a new result.

First notice computing pq^T (assume p, q are column vectors) gives a $n \times n$ -matrix, which can be interpreted as a joint probability table for the distributions p and q . And thus if p and q fulfil (2.17), then the minimum of all elements below the diagonal of this joint probability table is larger than the maximum of all elements above its diagonal.

Example 2.5 Take $n = 4$ and

$$p = [0.02, 0.03, 0.05, 0.9], \quad q = [0.8, 0.1, 0.09, 0.01],$$

then

$$pq^T = \begin{bmatrix} 0.0160 & 0.0020 & 0.0018 & 0.0002 \\ 0.0240 & 0.0030 & \mathbf{0.0027} & 0.0003 \\ 0.0400 & \mathbf{0.0050} & 0.0045 & 0.0005 \\ 0.7200 & 0.0900 & 0.0810 & 0.0090 \end{bmatrix}.$$

The minimal value of the elements below the diagonal is 0.0050 and the maximal of the elements above the diagonal 0.0027. So this particular choice of p, q fulfils (2.17). Here p and q were chosen such that player I and II are likely to start at respectively the right end and the left end of the interval.

We can consider another case in which the probability for the players of starting at respectively the right and the left end is less extreme. Take again $n = 4$ and now we take

$$p = [0.12, 0.19, 0.33, 0.36]^T, \quad q = [0.42, 0.3, 0.2, 0.01]^T,$$

then

$$pq^T = \begin{bmatrix} 0.0504 & 0.0360 & 0.0240 & 0.0096 \\ 0.0798 & 0.0570 & \mathbf{0.0380} & 0.0152 \\ 0.1386 & 0.0990 & 0.0660 & 0.0264 \\ 0.1512 & 0.1080 & \mathbf{0.0720} & 0.0288 \end{bmatrix}$$

Observe that also this choice of p, q fulfils (2.17). In both choices for p and q we considered so far, we have $p_0 < p_1 < p_2 < p_3$ and $q_0 > q_1 > q_2 > q_3$. So we could think that all such distribution fulfil (2.17), however that is not the true, as will be showed in the next case. Also this time we take $n = 4$ and for p, q we take

$$p = [0.01, 0.02, 0.48, 0.49]^T, \quad q = [0.45, 0.44, 0.06, 0.05]^T.$$

Then

$$pq^T = \begin{bmatrix} 0.0045 & 0.0044 & 0.0006 & 0.0005 \\ \mathbf{0.0090} & 0.0088 & 0.0012 & 0.0010 \\ 0.2160 & 0.2112 & 0.0288 & \mathbf{0.0240} \\ 0.2205 & 0.2156 & 0.0294 & 0.0245 \end{bmatrix}$$

and hence this choice of p, q does not fulfil (2.17).

A lot of distribution pairs of the form $p_0 < p_1 < \dots < p_{n-1}$ and $q_0 > q_1 > \dots > q_{n-1}$ fulfil (2.17). However certainly not all distribution pairs of this form will fulfil (2.17). For each $n > 4$ for instance we can construct such an example from the third case in Example 2.5.⁶

After explaining (2.17), we will now show that in $\Gamma^a(p, q)$ it is optimal for the first player to move to the left until meeting the left sweeper and for the second player to move to the right until meeting the right sweeper. Let $(s_L, s_R) \in \mathcal{S}_{opt}^2$ denote this pair of strategies. It is easy to see that this pair is indeed an element of \mathcal{S}_{opt}^2 , because all agents only change direction when meeting a sweeper and thus (s_L, s_R) is both a sticky pair and a geodesic pair. To show

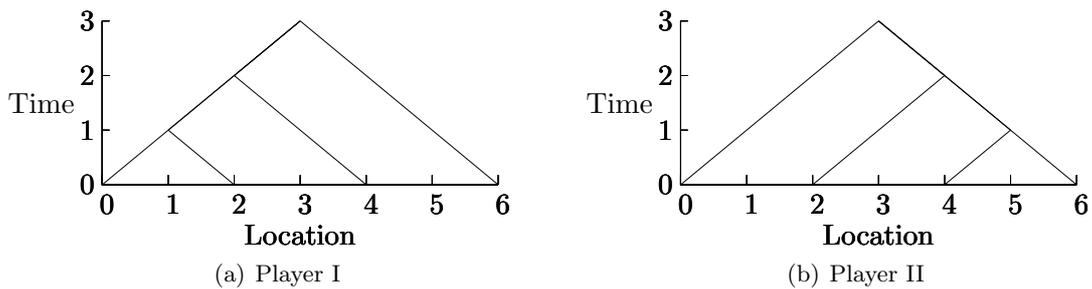


Figure 2.22: The pair $(s_L, s_R) \in \mathcal{S}_{opt}^2$ for $n = 4$

that (s_L, s_R) is an optimal pair of strategies for $\Gamma^a(p, q)$ we first need the meeting time matrix $M(s_L, s_R)$. For $n = 4$ the pair (s_L, s_R) is depicted in Figure 2.22. It is not hard to see that

⁶First, the idea was that (2.17) holds for all distribution pairs of the form $0 < p_0 < p_1 < \dots < p_{n-1}$ and $q_0 > q_1 > \dots > q_{n-1} > 0$. However the third case in Example 2.5 shows this idea was not correct.

the corresponding meeting is equal to

$$\begin{bmatrix} 0 & 3 & 3 & 3 \\ 1 & 0 & 3 & 3 \\ 2 & 1 & 0 & 3 \\ 3 & 2 & 1 & 0 \end{bmatrix}.$$

In general the following lemma describes the structure of the meeting time matrix and gives also information about the sum of the elements in the matrix.

Lemma 2.18 *The strategies s_L and s_R describe for respectively player I and player II a path in time for each even node on the discrete interval $0, 1, \dots, 2(n-1)$. There are n even nodes and hence the meeting time matrix $M(s_L, s_R)$ is an $n \times n$ matrix, then for $0 \leq i, j \leq n-1$:*

$$m_{s_L, s_R}(i, j) = \begin{cases} 0, & \text{for } i = j, \\ n-1, & \text{for } i < j, \\ i-j, & \text{for } i > j. \end{cases} \quad (2.18)$$

Moreover the sum of the elements of $M(s_L, s_R)$ is equal to

$$\sum_{0 \leq i, j \leq n-1} m_{s_L, s_R}(i, j) = \frac{n(2n-1)(n-1)}{3}. \quad (2.19)$$

(See equation (2.2) for the definition of $m_{s_L, s_R}(i, j)$.)

Proof: First of all, $m_{s_L, s_R}(i, i)$ corresponds for $i = 0, \dots, n-1$ to a meeting between the agents of player I and player II starting at the same even node. It is obvious that this meeting occurs at time 0 and hence $m_{s_L, s_R}(i, i) = 0$ for $i = 0, \dots, n-1$.

Secondly consider $m_{s_L, s_R}(i, j)$ for the case in which $i < j$. In this case the agent of player I starts at node $2i$ and the agent of player II starts at node $2j$. Notice the agent of player I starts left to the agent of player II. The agent of player I moves to the left until meeting the left sweeper and hence he will meet the left sweeper before meeting the agent of player II. In the same way, the agent of player II moves to the right until meeting the right sweeper and hence he will meet the right sweeper before meeting the agent of player I. So the agents of player I and player II meet in this case when the sweepers meet, which implies they will meet at time $n-1$. Thus for $i < j$ we find $m_{s_L, s_R}(i, j) = n-1$.

In the remaining case we consider $m_{s_L, s_R}(i, j)$ for $i > j$. In this case the agent of player I starts right to the agent of player II. We will again use that the agent of player I moves to the left until meeting the left sweeper and the agent of player II moves to the right until meeting the right sweeper. This implies that the two agents move initially towards each other until they meet (after that they immediately split up and continue their way to meet respectively the left and right sweeper). The initial distance between the two agents is $2(i-j)$ and thus they will at time $i-j$, which implies $m_{s_L, s_R}(i, j) = i-j$ for $i > j$.

So now we have proved (2.18). Next we can compute the sum of the elements of $M(s_L, s_R)$.

$$\begin{aligned}
\sum_{0 \leq i, j \leq n-1} m_{s_L, s_R}(i, j) &= \sum_{i=0}^{n-1} \sum_{j=0}^{i-1} m_{s_L, s_R}(i, j) + \sum_{i=0}^{n-1} \sum_{j=i+1}^{n-1} m_{s_L, s_R}(i, j) + \sum_{i=0}^{n-1} m_{s_L, s_R}(i, i), \\
&= \sum_{i=0}^{n-1} \sum_{j=0}^{i-1} (i - j) + \sum_{i=0}^{n-1} \sum_{j=i+1}^{n-1} (n - 1), \\
&= \sum_{i=0}^{n-1} \left\{ \frac{i(i+1)}{2} + (n - i - 1)(n - 1) \right\}.
\end{aligned}$$

It can be showed that this last expression is equal to $\frac{n(2n-1)(n-1)}{3}$ and consequently the sum of the elements of $M(s_L, s_R)$ is indeed given by (2.19), which completes the proof. \blacksquare

Now we know more details about the meeting time matrix $M(s_L, s_R)$ we can prove that (s_L, s_R) is an optimal pair of strategies for $\Gamma^a(p, q)$. Before we proof this result in Theorem 2.20 we need some other useful observations summarized in the following lemma.

Lemma 2.19 *Consider a pair of strategies $(f, g) \in \mathcal{S}_{opt}^2$ for an asymmetric rendezvous problem on the set \mathcal{N}_e . Consider the meeting time matrix $M(f, g)$, then*

- (i) *all diagonal elements of $M(f, g)$ are equal to zero,*
- (ii) *$|i - j| \leq m_{f, g}(i, j) \leq n - 1$, for $i \neq j$,*
- (iii) *the sum of the elements of $M(f, g)$ is greater than or equal to $\frac{n(2n-1)(n-1)}{3}$.*

Proof:

- (i) The diagonal elements of $M(f, g)$ correspond to a meeting of agents starting at the same location. These agents meet immediately and thus all diagonal elements of $M(f, g)$ are equal to zero.
- (ii) The element $m_{f, g}(i, j)$ with $i \neq j$ corresponds to a meeting between agent f_i of player I and agent g_j of player II (starting at distinct locations). The initial distance between these two agents is $2|i - j|$ and hence they cannot meet before time $|i - j|$. Moreover the left and the right sweeper meet at time $n - 1$ and hence agents f_i and g_j cannot meet after time $n - 1$. Combining these two observations yields $|i - j| \leq m_{f, g}(i, j) \leq n - 1$.
- (iii) Consider the problem $\Gamma^a(\bar{p}, \bar{p})$ with \bar{p} being a uniform distribution supported on \mathcal{N}_e (so $\bar{p}_j = \frac{1}{n}$ for $j = 0, \dots, n - 1$). Then Howard proved (see [7], Page 553) that (s_R, s_R) is an optimal pair of strategies for $\Gamma^a(\bar{p}, \bar{p})$. Using (2.8) we know $\widehat{T}(s_R, s_R) = \frac{(2n-1)(n-1)}{3n}$. So the expected meeting time $\widehat{T}(f, g)$ is larger than or equal to this value. Then

$$\frac{(2n-1)(n-1)}{3n} \leq \widehat{T}(f, g) = \bar{p}^T M(f, g) \bar{p} = \frac{1}{n^2} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} m_{f, g}(i, j)$$

and thus the sum of the elements of $M(f, g)$ is greater than or equal to $\frac{n(2n-1)(n-1)}{3}$. \blacksquare

Finally we are now able to prove the theorem, where will make use of Lemmas 2.18 and 2.19.

Theorem 2.20 Consider the asymmetric rendezvous problem $\Gamma^a(p, q)$ with p, q supported on \mathcal{N}_e such that $p_i > 0, q_j > 0$ for $i, j = 0, \dots, n-1$, and

$$\min\{p_i q_j \mid 0 \leq j < i \leq n-1\} > \max\{p_i q_j \mid 0 \leq i < j \leq n-1\}. \quad (2.20)$$

Then (s_L, s_R) is the unique optimal strategy for this problem.

Proof: Consider an arbitrary pair of strategies $(f, g) \in \mathcal{S}_{opt}^2$ such that $(f, g) \neq (s_L, s_R)$. Notice for $M(s_L, s_R)$ we know that all elements above the diagonal are equal to $n-1$ (see Lemma 2.18). Moreover Lemma 2.19 tells us that in a meeting matrix the maximum value of an element above the diagonal is $n-1$. Consequently, we find

$$m_{f,g}(i, j) - m_{s_L, s_R}(i, j) \leq 0, \text{ for all } 0 \leq i < j \leq n-1. \quad (2.21)$$

We also know for $M(s_L, s_R)$ that all elements below the diagonal are equal to $i-j$ (again see Lemma 2.18). This time Lemma 2.19 tells us that in a meeting matrix the minimal value of an element below the diagonal is given by the row number minus the column number. Thus this results in the following relation

$$m_{f,g}(i, j) - m_{s_L, s_R}(i, j) \geq 0, \text{ for all } 0 \leq j < i \leq n-1. \quad (2.22)$$

Next we can conclude from Lemma 2.18 and Lemma 2.19 that the sum of the elements in $M(f, g)$ is greater than or equal to the sum of the elements in $M(s_L, s_R)$ and thus

$$\sum_{i=0}^{n-1} \sum_{j=0}^{n-1} (m_{f,g}(i, j) - m_{s_L, s_R}(i, j)) \geq 0 \quad (2.23)$$

Combining (2.21), (2.22) and (2.23) shows

$$\sum_{i=0}^{n-1} \sum_{j=0}^{i-1} (m_{f,g}(i, j) - m_{s_L, s_R}(i, j)) \begin{cases} = 0, & \text{for } (f, g) = (s_L, s_R) \\ > 0, & \text{for } (f, g) \neq (s_L, s_R) \end{cases}. \quad (2.24)$$

Remember we chose $(f, g) \neq (s_L, s_R)$. Now to complete the proof of this lemma we have to compute $\widehat{T}(f, g) - \widehat{T}(s_L, s_R)$, which can also be written as

$$\begin{aligned} \widehat{T}(f, g) - \widehat{T}(s_L, s_R) &= p^T M(f, g)q - p^T M(s_L, s_R)q, \\ &= \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} m_{f,g}(i, j) p_i q_j - \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} m_{s_L, s_R}(i, j) p_i q_j, \\ &= \sum_{i=0}^{n-1} \sum_{j=i+1}^{n-1} \underbrace{(m_{f,g}(i, j) - m_{s_L, s_R}(i, j))}_{\leq 0, \text{ by (2.21)}} p_i q_j + \sum_{i=0}^{n-1} \sum_{j=0}^{i-1} \underbrace{(m_{f,g}(i, j) - m_{s_L, s_R}(i, j))}_{\geq 0, \text{ by (2.22)}} p_i q_j. \end{aligned}$$

In the last step we used that the diagonal elements of a meeting matrix are always zero.

Next using the information provided by (2.21) and (2.22), we find

$$\begin{aligned} \widehat{T}(f, g) - \widehat{T}(s_L, s_R) &\geq \max\{p_i q_j \mid 0 \leq i < j \leq n-1\} \cdot \sum_{i=0}^{n-1} \sum_{j=i+1}^{n-1} (m_{f,g}(i, j) - m_{s_L, s_R}(i, j)) \\ &\quad + \min\{p_i q_j \mid 0 \leq j < i \leq n-1\} \cdot \underbrace{\sum_{i=0}^{n-1} \sum_{j=0}^{i-1} (m_{f,g}(i, j) - m_{s_L, s_R}(i, j))}_{>0, \text{ by (2.24)}}. \end{aligned}$$

After that by using the assumption on p and q (see (2.20)), we can conclude that

$$\begin{aligned} \widehat{T}(f, g) - \widehat{T}(s_L, s_R) &> \max\{p_i q_j \mid 0 \leq i < j \leq n-1\} \cdot \sum_{i=0}^{n-1} \sum_{j=i+1}^{n-1} (m_{f,g}(i, j) - m_{s_L, s_R}(i, j)) \\ &\quad + \max\{p_i q_j \mid 0 \leq i < j \leq n-1\} \cdot \sum_{i=0}^{n-1} \sum_{j=0}^{i-1} (m_{f,g}(i, j) - m_{s_L, s_R}(i, j)), \\ &= \max\{p_i q_j \mid 0 \leq i < j \leq n-1\} \cdot \underbrace{\sum_{i=0}^{n-1} \sum_{j=0}^{n-1} (m_{f,g}(i, j) - m_{s_L, s_R}(i, j))}_{\geq 0, \text{ by (2.23)}} \\ &> 0. \end{aligned}$$

Thus for $(f, g) \neq (s_L, s_R)$ we find $\widehat{T}(f, g) - \widehat{T}(s_L, s_R) > 0$ and hence (s_L, s_R) is the unique optimal pair of strategies for $\Gamma^a(p, q)$. \blacksquare

Using the structure of the meeting time matrix $M(s_L, s_R)$ such as described in 2.18 we find that

$$\widehat{T}(s_L, s_R) = p^t M(s_L, s_R) q = \sum_{i=0}^{n-1} \sum_{j=0}^{i-1} (i-j) p_i q_j + (n-1) \sum_{i=0}^{n-1} \sum_{j=i+1}^{n-1} p_i q_j$$

is the minimal expected meeting time for $\Gamma^a(p, q)$. If (2.20) is not a strict inequality, then we can prove using the same reasoning as above that $\widehat{T}(f, g) - \widehat{T}(s_L, s_R) \geq 0$ for an arbitrary pair (f, g) not equal to (s_L, s_R) . Hence in that case (s_L, s_R) is still an optimal pair of strategies for $\Gamma^a(p, q)$, however not necessarily the unique one.

Intuitively the result proven in this chapter is also correct, because distribution pairs fulfilling (2.20) are such that it is likely for player I to start at the right end of the interval and for player II to start at the left of the interval. So it seems reasonable that player I prefers to move to the left as long as possible and player II prefers to move to the right as long as possible.

In this subsection we solved the asymmetric rendezvous problem for a particular class of distributions. In the next subsection we will consider another restricted version of the asymmetric problem, which can also be seen as an extension of the symmetric problem from Section 2.2.

2.3.4 Asymmetric rendezvous problem with equal distributions

In the asymmetric rendezvous problem $\Gamma^a(p, q)$ the players are in general placed on the interval according to different distributions. However we could think of situations in which players may adopt different strategies, but where it is not likely that the distributions for the starting position will differ. For instance if the circumstances prior to the rendezvous problem are equal for both players. Therefore we will restrict ourselves in this subsection to problems of the form $\Gamma^a(p, p)$ where p can be any distribution supported on \mathcal{N}_e . In this restricted asymmetric rendezvous problem the players can still adopt different strategies, but now they are placed independently according to the *same* distribution like in the symmetric rendezvous problem from Section 2.2. Another way of looking to this problem is to interpret it as an extension of the symmetric problem in which the players are still placed on interval according to the same distribution, but now they are allowed to adopt different strategies. If we compare the problems $\Gamma^a(p, p)$ and $\Gamma^s(p)$ for some p , then it is interesting to know how much the players can profit from being allowed to adopt different strategies. We have done an numerical experiment and answered this question for many arbitrarily chosen distributions. The results⁷ are listed

n	# of simulations	# of times profitable to adopt different strategies	average relative difference	maximal relative difference
4	$> 10^8$	0 (0%)	-	-
5	$5 \cdot 10^7$	580 (0.0012%)	0.0196%	0.0902%
6	$\approx 2.6 \cdot 10^6$	119 (0.0046%)	0.0231%	0.0808%

Table 2.3: Results for experiment in which we compare $\Gamma^a(p, p)$ and $\Gamma^s(p)$ for a large number of distributions p and for $n = 4, 5, 6$.

in Table 2.3 and are rather surprising. It turns out that in most cases it is not profitable for the players to adopt distinct strategies. Moreover in the very few cases that it is profitable, the difference with the optimal symmetric strategy is very small. For $n = 4$ it seems that it is never beneficial to adopt different strategies. Indeed, we will conclude this observation later on from a more general result (see Corollary 2.22). Of course it would be very nice to know whether or not the observed behavior for $n = 5, 6$ also holds for large n , because then for any distribution p we could approach the optimal solution of $\Gamma^a(p, p)$ by solving $\Gamma^s(p)$. This question is still unanswered, but in Theorem 2.21 we will prove an equality which relates pairs of strategies for $\Gamma^a(p, p)$ to strategies for $\Gamma^s(p)$ and hence might be useful for answering the question.

First consider Figure 2.23. The plot left of the equality sign shows a time-space plot representing a strategy pair for an asymmetric problem in which player I plays according to the solid lines and player II according to the dotted lines. The grey circles indicate where and when meetings between agents occur. Both plots right to the equality sign are time-space plots representing a strategy for a symmetric problem. Again the grey circles indicate meetings between agents. The ‘2x’ close to some circle means that at that particular circle two meetings between agents take place. Here the occurrence of the ‘2x’ is obvious, because

⁷Notice the *average relative difference* is the only the average of the differences larger than zero. In other words, only for the cases in which it is beneficial for the players to adopt different strategies.

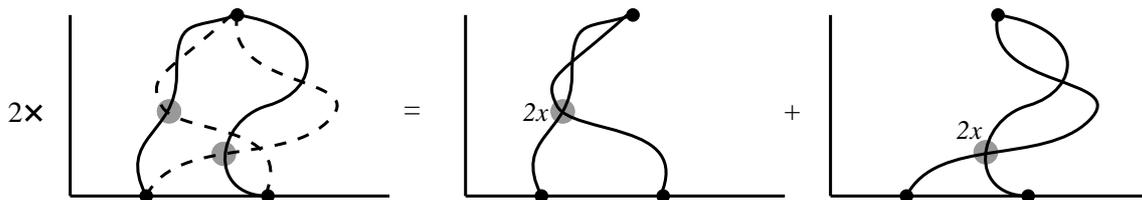


Figure 2.23: Relation between asymmetric strategy pair and symmetric strategy for two locations

the plots represent a strategy for a symmetric problem. If in such a problem agent i of player I meets agent j of player II, then agent i of player II also meets agent j of player I which results for $i \neq j$ in two meetings at the same place at the same time. Now observe that the strategies on the right hand side are combinations of paths in the strategy pair on the left hand side. Next notice if we take twice the strategy pair on the left hand side and once the two strategies on the right hand side, then on both sides of the equality sign exactly the same number (four) of meetings occur. Moreover the places and times at which the meetings occur are exactly the same on both sides. In Figure 2.24 a more complicated example is given.

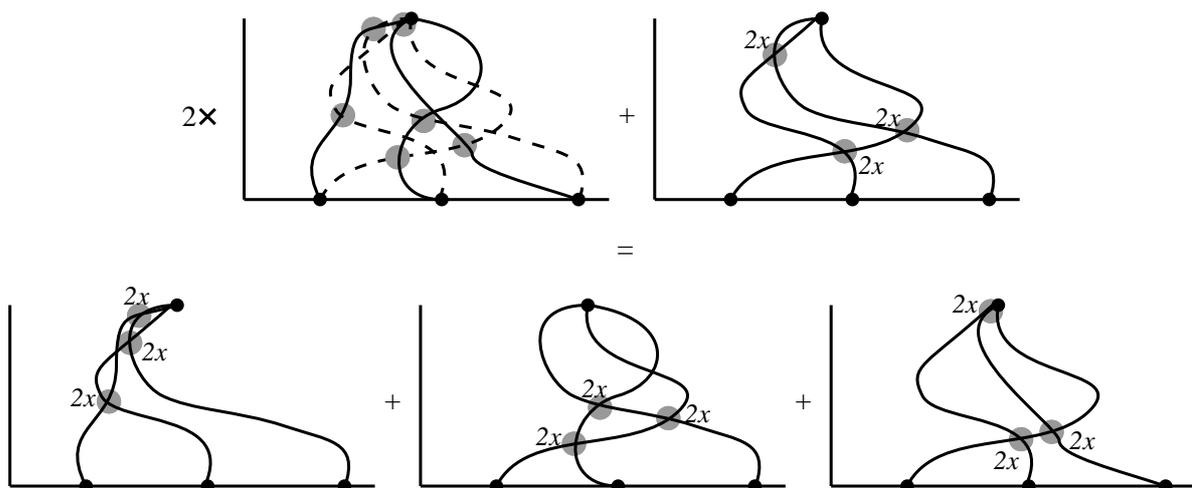


Figure 2.24: Relation between asymmetric strategy pair and symmetric strategy for three locations

Again the plot with both solid and dotted lines is a pair of strategies and the plots with only solid lines represent a strategy adopted by both players. Also in this figure the strategies are all composed from paths in the strategy pair. In this example we had to add a symmetric strategy to the left hand side to compensate for meetings that occurred too much on the right hand side.

These two examples give an intuitive idea how we can add and subtract strategies and strategy pairs, which has formalized by the following theorem.⁸

⁸In the next theorem and also the corollaries after that we assume $n \geq 4$, because for $n \leq 3$ the results are trivial.

Theorem 2.21 Consider the asymmetric rendezvous problem $\Gamma^a(p, p)$ where p can be any distribution supported on \mathcal{N}_e . Assume $n \geq 4$ and suppose the players adopt strategy pair $(f, g) \in \mathcal{S}^2$. Next consider the symmetric rendezvous problem $\Gamma^s(p)$ and suppose the players adopt a strategy composed from paths in (f, g) . Let \mathbf{x} for $i = 0, \dots, n-1$ be a ‘zero and one vector’ in \mathbb{R}^n with the coordinates numbered as $0, \dots, n-1$ and representing a composed strategy such that

$$x_j = \begin{cases} 1, & \text{if the agent starting at } 2j \in \mathcal{N}_e \text{ takes the path of agent } f_j, \\ 0, & \text{if the agent starting at } 2j \in \mathcal{N}_e \text{ takes the path of agent } g_j. \end{cases}$$

Define the expected meeting time $\widehat{T}(\mathbf{x})$ as if \mathbf{x} is the strategy in \mathcal{S} represented by \mathbf{x} . Let \mathbf{e}^i for $i = 0, \dots, n-1$ denote a unit vector in \mathbb{R}^n with elements numbered as $0, \dots, n-1$ such that $\mathbf{e}_j^i = \delta_{ij}$ for $j = 0, \dots, n-1$. Moreover let $\mathbf{1}$ denote a vector in \mathbb{R}^n containing only ones. Then we have the following relation between $\widehat{T}(f, g)$ and a number of composed strategies.

$$2\widehat{T}(f, g) + (n-4)\widehat{T}(g) = \sum_{i=1}^{n-2} \widehat{T}(\mathbf{e}^i)$$

Proof: Let $M(\mathbf{x})$ be the meeting time matrix corresponding to \mathbf{x} . Notice \mathbf{x} is composed from paths in strategies f and g . Hence we can express $M(\mathbf{x})$ in terms of the meeting time matrices $M(f)$, $M(g)$ and $M(f, g)$ by selecting the right values from the matrices. This can be done as follows:

$$M(\mathbf{x}) = D(\mathbf{x})M(f)D(\mathbf{x}) + D(\mathbf{1} - \mathbf{x})M(g)D(\mathbf{1} - \mathbf{x}) + D(\mathbf{x})\{M(f, g) + M(g, f)^T\}D(\mathbf{1} - \mathbf{x}),$$

where $D(\mathbf{y})$ is defined as a diagonal matrix with \mathbf{y} on the diagonal. Note $D(\mathbf{1} - \mathbf{e}^i) = I - D(\mathbf{e}^i)$ where I is the $n \times n$ identity matrix. If we choose $\mathbf{x} = \mathbf{e}^i$ in the above expression for $M(\mathbf{x})$ with $i = 0, \dots, n-1$ and use

- a meeting time matrix always has zeroes on the diagonal (see Lemma 2.19),
- $M(g)$ is a symmetric matrix,
- $M(f, g) = M(g, f)^T$,

then we find

$$\begin{aligned} M(\mathbf{e}^i) &= \underbrace{D(\mathbf{e}^i)M(f)D(\mathbf{e}^i)}_{=0} + D(\mathbf{1} - \mathbf{e}^i)M(g)D(\mathbf{1} - \mathbf{e}^i) + D(\mathbf{e}^i)\{M(f, g) + M(g, f)^T\}D(\mathbf{1} - \mathbf{e}^i), \\ &= M(g) + D(\mathbf{e}^i)\{M(f, g) + M(g, f)^T\} - \underbrace{D(\mathbf{e}^i)\{M(f, g) + M(g, f)^T\}D(\mathbf{e}^i)}_{=0} \\ &\quad - D(\mathbf{e}^i)M(g) - M(g)D(\mathbf{e}^i) + \underbrace{D(\mathbf{e}^i)M(g)D(\mathbf{e}^i)}_{=0}, \\ &= M(g) + 2D(\mathbf{e}^i)M(f, g) - 2D(\mathbf{e}^i)M(g). \end{aligned}$$

Next summing over all $i = 0, \dots, n-1$ results in

$$\begin{aligned} \sum_{i=0}^{n-1} M(\mathbf{e}^i) &= nM(g) + 2 \left(\sum_{i=0}^{n-1} D(\mathbf{e}^i) \right) M(f, g) - 2 \left(\sum_{i=0}^{n-1} D(\mathbf{e}^i) \right) M(g), \\ &= 2M(f, g) + (n-2)M(g). \end{aligned} \quad (2.25)$$

Now remember that \mathbf{e}^0 corresponds to a strategy in which the agent starting at node 0 takes the path of the corresponding f -agent and the other agents take the path of the corresponding g -agent. However it makes no difference whether the agent starting at zero takes the path from f or the path from g , because in both cases he takes the path of the left sweeper. Consequently we have $M(\mathbf{e}^0) = M(g)$. Using the same arguments but now for the right sweeper, we also find $M(\mathbf{e}^{n-1}) = M(g)$. Hence (2.25) is equivalent to

$$\sum_{i=1}^{n-2} M(\mathbf{e}^i) = 2M(f, g) + (n-4)M(g).$$

Next notice

$$\sum_{i=1}^{n-2} \widehat{T}(\mathbf{e}^i) = \sum_{i=1}^{n-2} p^T M(\mathbf{e}^i) p = p^T \left(\sum_{i=1}^{n-2} M(\mathbf{e}^i) \right) p$$

and consequently

$$\begin{aligned} \sum_{i=1}^{n-2} \widehat{T}(\mathbf{e}^i) &= p^T (2M(f, g) + (n-4)M(g)) p, \\ &= 2\widehat{T}(f, g) + (n-4)\widehat{T}(g) \end{aligned}$$

which completes the proof. ■

Using this theorem it is now not difficult to explain why it is not beneficial for two players to adopt different strategies in the restricted asymmetric problem $\Gamma^a(p, p)$ with $n = 4$ (also see Table 2.3).

Corollary 2.22 *Consider an asymmetric rendezvous problem $\Gamma^a(p, p)$ with p an arbitrary distribution supported on \mathcal{N}_e and assume $n = 4$. Take an arbitrary $(f, g) \in \mathcal{S}_{opt}^2$, then there exists a strategy in $s \in \mathcal{S}$ such that $\widehat{T}(s) \leq \widehat{T}(f, g)$. Moreover s is a combination of the paths in f and g .*

Proof: Applying Theorem 2.21 with $n = 4$ gives

$$2\widehat{T}(f, g) = \widehat{T}(s') + \widehat{T}(s''),$$

with $s', s'' \in \mathcal{S}_{opt}$ both combinations of paths of f and g . Because of the equality we have either $\widehat{T}(s') < \widehat{T}(f, g)$, $\widehat{T}(s'') < \widehat{T}(f, g)$ or $\widehat{T}(s') = \widehat{T}(s'') = \widehat{T}(f, g)$. Thus there exists always a strategy $s \in \mathcal{S}$, such that s is a combination of paths in f, g and such that $\widehat{T}(s) \leq \widehat{T}(f, g)$. ■

The last proof in this chapter will be another consequence of Theorem 2.21. The results in Table 2.3 show that for $n = 5$ and $n = 6$ in most cases an optimal solution for $\Gamma^s(p)$ is also an optimal solution for $\Gamma^a(p, p)$. Moreover for $n = 4$ we just proved in Corollary 2.22 that an optimal solution for $\Gamma^s(p)$ is always an optimal solution for $\Gamma^a(p, p)$. However we have no general result which tells us whether or not an optimal solution for $\Gamma^s(p)$ is optimal for $\Gamma^a(p, p)$, but we can prove that it is always an equilibrium pair.

Corollary 2.23 *Assume $n \geq 4$ and consider the asymmetric problem $\Gamma^a(p, p)$ with p an arbitrary distribution supported on \mathcal{N}_e . Let $s \in \mathcal{S}_{opt}$ be an optimal strategy for the symmetric problem $\Gamma^s(p)$, then $(s, s) \in \mathcal{S}_{opt}^2$ is an equilibrium pair for $\Gamma^a(p, p)$.*

Proof: Suppose s is an optimal strategy for $\Gamma^s(p)$, but (s, s) is *not* an equilibrium pair for $\Gamma^a(p, p)$. Then there exists a strategy $s' \in \mathcal{S}$ such that $\widehat{T}(s', s) < \widehat{T}(s)$. Using Theorem 2.21 we find that there exists strategies $s^1, \dots, s^{n-2} \in \mathcal{S}$ such that

$$2\widehat{T}(s', s) + (n-2)\widehat{T}(s) = \sum_{i=1}^{n-2} \widehat{T}(s^i).$$

Since s is an optimal strategy for $\Gamma^s(p)$ we have $\widehat{T}(s) \leq \widehat{T}(s^i)$ and thus

$$2\widehat{T}(s', s) = \sum_{i=1}^{n-2} \widehat{T}(s^i) - (n-2)\widehat{T}(s) \geq \widehat{T}(s^1) + \widehat{T}(s^2).$$

Notice we have chosen s^1 and s^2 on the right hand side, but could have chosen s^i and s^j for any $i, j \in \{1, \dots, n-2\}$ with $i \neq j$. Moreover we can conclude that either $\widehat{T}(s^1) \leq \widehat{T}(s', s)$, $\widehat{T}(s^2) \leq \widehat{T}(s', s)$ or both. Hence s cannot be the optimal strategy for $\Gamma^s(p)$, because we assumed $\widehat{T}(s', s) < \widehat{T}(s)$. Notice we now have a contradiction and thus if s is an optimal strategy for $\Gamma^s(p)$, then (s, s) is an equilibrium pair for $\Gamma^a(p, p)$. ■

2.4 Conclusion

This chapter contains an overview of all known results for the rendezvous problem on the labeled interval. However more importantly, in this chapter also a number of new results (both analytical and numerical) for rendezvous problems on the discrete interval are derived.

In Section 2.2 we proved an one-to-one correspondence between sticky geodesic strategies and proper binary trees. We used this result to derive a recurrence relation describing the minimal expected meeting time of the symmetric rendezvous problem $\Gamma^s(p)$ for any p . This recurrence relation provides us with an relatively easy way for numerically solving the symmetric rendezvous problem.

After that in Section 2.3, we considered the more general asymmetric rendezvous problem. We generalized the definition of sticky strategies to *sticky pairs* of strategies. We used these sticky pairs in an algorithm for approaching the optimal strategy pair of the asymmetric rendezvous problem $\Gamma^a(p, q)$ where both p and q can be any distribution. In this section we also considered two restricted problems. In the first one we solved the asymmetric problem

with players likely to start at opposite ends of the interval. Next we compared the restricted asymmetric problem $\Gamma^a(p, p)$ to the symmetric problem $\Gamma^s(p)$ where we found some surprising numerical results. We also showed how we can express a strategy pair for $\Gamma^a(p, p)$ as a combination of strategies for $\Gamma^s(p)$.

We can conclude that rendezvous search problems on the discrete labeled interval (and also on other search spaces) remain difficult problems. The results in this chapter are a nice contribution to the research in this area. However there are still many open questions to answer. For instance, it would be nice to know whether there exists a one-to-one correspondence between \mathcal{S}_{opt}^2 and some set of trees or graphs. Such a relation might lead to an algorithm for solving the asymmetric problem on the interval. Also the conjectures stated in this chapter are open questions as of yet. Another idea for future research is to generalize the results in this chapter to rendezvous problems on trees, graphs, a continuous interval, rendezvous problem in which the players are not placed independently, etc.

3 Search game on a star with players having motion detection abilities

3.1 Introduction to the search game

In the previous chapter we considered a rendezvous search problem on a known search region (a finite interval in our case). In this problem two players located in the search region have the common goal to meet as soon as possible. One could also think of a related problem in which one of them likes to avoid the meeting as long as possible. In fact such problems (‘Princess and Monster’ games) were introduced by Isaacs in [8] before rendezvous search problems were considered.⁹ In a ‘Princess and Monster’ game a monster and a princess are located on a known region and the monster’s goal is to catch the princess as soon as possible. Of course the princess does not like to be caught by the monster and hence her objective is to maximize the expected capture time. It is important to notice that the monster and the princess will not have any visual contact and thus the monster can only catch the princess when he is ‘close enough’. The ‘Princess and Monster’ game is an example of a *search game*.

In general, a search game is a problem in which a searcher (the monster) and a hider (the princess) are located on a known region. The searcher is allowed to move at some bounded speed and his objective is to catch the hider as soon as possible by adopting a search strategy. The hider can be either mobile or immobile and his objective is to maximize the expected time to be captured by the searcher by adopting some hiding strategy. Both a search strategy and a hiding strategy can be a pure strategy or a mixed strategy. In a pure strategy the searcher always plays the same strategy and in a mixed strategy he assigns a probability to each pure strategy.

Next we have to describe what we will call a solution to our problem. Therefore we will informally introduce the concept of *saddle-point equilibria* in the context of search games.¹⁰ Let p and q be vectors representing mixed strategies with probabilities assigned to each pure strategy of respectively the searcher and the hider. Next suppose $f(p, q)$ is a function describing the expected time at which the searcher will capture the hider as a function of p and q . Now a pair (p, q) is called a saddle-point equilibrium if and only if for all other mixed strategies p' and q' holds

$$f(p, q') \leq f(p, q) \leq f(p', q). \quad (3.1)$$

We will call such a pair (p, q) with value $f(p, q)$ a solution for our search game. The strategies p and q are *optimal strategies* for respectively the searcher and the hider.

After some remarks we can now introduce our problem in which the search region is a star with n arcs of length $\frac{1}{2}$. We will assume $n \geq 3$, because else we just have an interval with the searcher starting at one the ends (for $n = 1$) or in the center (for $n = 2$). Initially the searcher is located in the origin O of the star and the hider at one of the n ends of the arcs according to a uniform distribution. Assume the searcher will catch the hider if and only if they are at exactly the same position. Such a meeting does not necessarily have to take place in the origin or at the end of an arc, but can also take place halfway an arc. Our problem

⁹Also see [3] for the ‘Princess and Monster’ game on an interval.

¹⁰Such an equilibrium can interpreted as a Nash equilibrium for a zero-sum game (also see [9] and [11]).

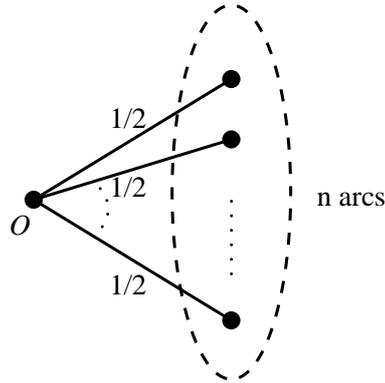


Figure 3.1: Star with origin O and n arcs of length $\frac{1}{2}$.

will be in discrete time and each time step the searcher can either

- ▶ ambush the hider by staying in the origin of the star,
- ▶ or move at unit speed to the end of one of the n arcs and then back to the origin.

We will call the first strategy *ambush* and the second strategy *search*. The hider also has two options each time step. He can either

- ▶ stay at his current position,
- ▶ or move at unit speed to the origin and then to the end of one of the $n - 1$ remaining arcs.

We will refer to these strategies as respectively *stay* and *move*. Notice the searcher and the hider make their moves simultaneously. In this chapter we will consider two models for this search game on the star. First in Section 3.2 we will consider a simple model in which both the searcher and the hider have no memory. In this model the searcher does not take the number of arcs he has searched already into account, when he decides his strategy for the next time step. We will conclude that this model is too simple. After that in Section 3.3 we will analyze an extended model in which we add information such that the searcher and the hider *both* know in how many arcs the hider can possibly stay (from the searcher's point of view). For this extended model we derived a number of results, both analytical and numerical.

3.2 Simple model without memory

We will discuss a simple model without memory, such that each time step the searcher and the hider will play the same game until the hider is caught. In general we cannot assume that the searcher and the hider will adopt pure strategies. So we introduce variables p and q with $0 \leq p, q \leq 1$, such that in each time step

- ▶ the searcher searches with probability p and ambushes with probability $1 - p$,
- ▶ the hider stays with probability q and moves with probability $1 - q$.

Now the question is which p and q are optimal? As mentioned in the introduction of this chapter we are looking for saddle-point equilibria. So p and q are optimal if and only if they are a saddle-point equilibrium.

Let $T(p, q)$ be a stochastic variable denoting the time at which the searcher will catch the hider as function of p and q . To compute the expected capture time $\mathbb{E}[T(p, q)]$ at which the searcher captures the hider we can condition on the strategies chosen in the first time step. If the hider is not caught after the first time step, then in the next time step the searcher and the hider will play the same game as in the first time step. Hence in that case the expected capture time after one time step is $\mathbb{E}[T(p, q)] + 1$. Conditioning on the strategies played in the first time step results in the following four conditional expectations.

- ▶ In the first place suppose in the first time step the searcher *ambushes* and the hider *stays*. Then obviously the searcher does not catch the hider and hence in this case the expected capture time is $\mathbb{E}[T(p, q)] + 1$.
- ▶ Secondly, suppose in the first time step the searcher *ambushes* and the hider *moves*. Then the hider runs into the searcher's trap at the origin at time $\frac{1}{2}$. Hence the expected capture time in this case is $\frac{1}{2}$.
- ▶ Next suppose in the first time step the searcher *searches* and the hider *stays*. If the searcher takes the arc in which the hider stays, then he catches the hider at the end of the arc at time $\frac{1}{2}$. Else he returns to origin without catching the hider. Hence the expected capture time in this case is

$$\frac{1}{n} \cdot \frac{1}{2} + \frac{n-1}{n} (\mathbb{E}[T(p, q)] + 1) = \frac{2n-1}{2n} + \frac{n-1}{n} \mathbb{E}[T(p, q)].$$

- ▶ Finally, suppose in the first time step the searcher *searches* and the hider *moves*. If the searcher takes the arc from which the hider starts, he catches him halfway that arc after $\frac{1}{4}$ time steps. If not, then the hider is not safe yet. After $\frac{3}{4}$ time steps he can still run into the searcher, who is returning to the origin from the end of an arc (this cannot be the arc just left by the hider) at that moment. Hence in this case the expected capture time is

$$\frac{1}{n} \cdot \frac{1}{4} + \frac{n-1}{n} \left(\frac{1}{n-1} \cdot \frac{3}{4} + \frac{n-2}{n-1} (\mathbb{E}[T(p, q)] + 1) \right) = \frac{n-1}{n} + \frac{n-2}{n} \mathbb{E}[T(p, q)].$$

The expected capture times for the four possible combinations of pure strategies which can be played in the first time step, are summarized in Table 3.1. Combining all conditional

Searcher	Hider	Probability	Conditional expectation
ambush	stay	$(1-p)q$	$\mathbb{E}[T(p, q)] + 1$
ambush	move	$(1-p)(1-q)$	$\frac{1}{2}$
search	stay	pq	$\frac{2n-1}{2n} + \frac{n-1}{n} \mathbb{E}[T(p, q)]$
search	move	$p(1-q)$	$\frac{n-1}{n} + \frac{n-2}{n} \mathbb{E}[T(p, q)]$

Table 3.1: Conditional expectations for the four possible combinations of pure strategies

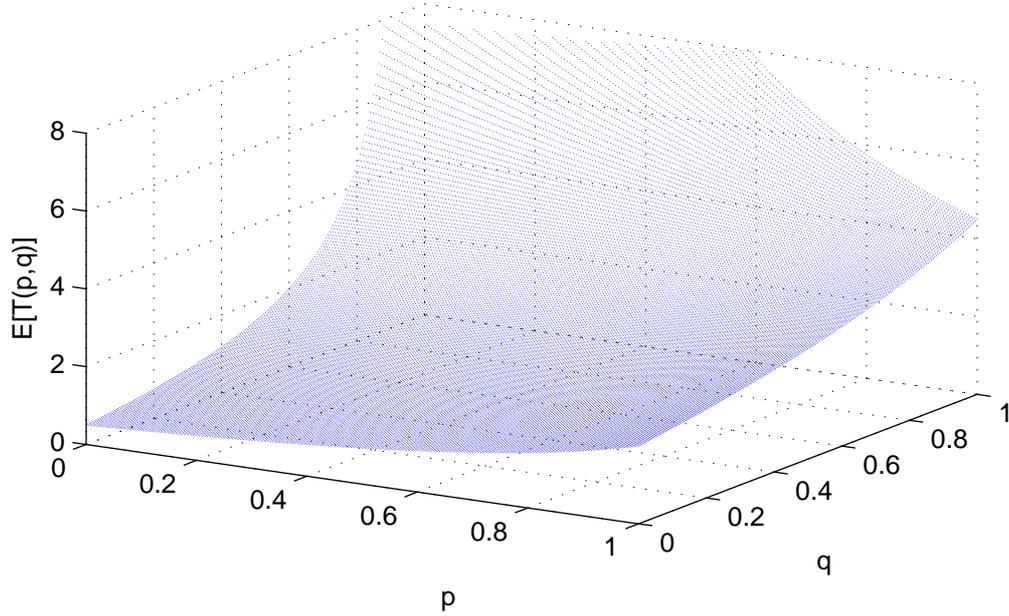


Figure 3.2: $\mathbb{E}[T(p, q)]$ plotted for $n = 5$ as function of p and q

expectations results in the following implicit formula for $\mathbb{E}[T(p, q)]$,

$$\begin{aligned} \mathbb{E}[T(p, q)] &= (1-p)q(\mathbb{E}[T(p, q)] + 1) + \frac{(1-p)(1-q)}{2} + pq \left(\frac{2n-1}{2n} + \frac{n-1}{n} \mathbb{E}[T(p, q)] \right) \\ &\quad + p(1-q) \left(\frac{n-1}{n} + \frac{n-2}{n} \mathbb{E}[T(p, q)] \right). \end{aligned}$$

Solving for $\mathbb{E}[T(p, q)]$ results in

$$\mathbb{E}[T(p, q)] = \frac{n}{n(1-p)(1-q) - p(q-2)} - \frac{1}{2}. \quad (3.2)$$

In Figure 3.2 this expression for $\mathbb{E}[T(p, q)]$ is plotted for $n = 5$ as function of p and q . For $p = 0$ and $q = 1$ the value tends to infinity, which is correct because it corresponds to the situation where the searcher will always ambush and where the hider will always stay. Hence in that case the hider is never caught. However we are more interested in values of p and q such that both the searcher and the hider cannot unilaterally change the expected capture time. So we are looking for a pair p, q such that for all p', q' with $0 \leq p', q' \leq 1$ holds

$$\mathbb{E}[T(p, q')] \leq \mathbb{E}[T(p, q)] \leq \mathbb{E}[T(p', q)]$$

Such a pair (p, q) with expected capture time $\mathbb{E}[T(p, q)]$ is a saddle-point equilibrium and hence a solution for our problem as we already mentioned before in Section 3.1. Next we will prove that in the unique optimal solution for the search game considered in this section the searcher will always search and the hider will always stay.

Theorem 3.1 *Consider the search game such as defined in this section. Then in the unique solution for this problem the searcher will always search and the hider will always stay. The expected time at which the hider is caught by the searcher is $n - \frac{1}{2}$.*

Proof: Differentiating (3.2) with respect to q gives

$$\frac{\partial}{\partial q} \mathbb{E}[T(p, q)] = \frac{n^2(1-p) + np}{(n(1-p)(1-q) - p(q-2))^2} > 0, \quad \text{for all } p, q \in [0, 1].$$

This derivative shows that for $q < 1$ the hider can always increase the expected capture time by increasing the probability of staying, no matter what strategy the searcher plays. Thus in a solution for the problem q will equal to be 1. (Notice this result could also be observed from Figure 3.2.) Next we have to find the best response of the searcher to this hider's strategy. Therefore consider

$$\mathbb{E}[T(p, 1)] = \frac{n}{p} - \frac{1}{2}.$$

So if $q = 1$, the searcher can either always stay ($p = 0$) and never catching the hider or always search ($p = 1$) and expect to catch the hider after $\mathbb{E}[T(1, 1)] = n - \frac{1}{2}$ time steps. Hence in the unique solution to problem the searcher will always search and the hider will always stay. Then the expected time at which the hider is caught by the searcher is $n - \frac{1}{2}$. ■

This result can also immediately be concluded from Table 3.1, because for both pure strategies the searcher can play it is beneficial for the hider to stay. In real life it seems reasonable that the searcher will alternate between searching and ambushing instead of searching all the time. So probably the simple search game considered in this section is too simple to model realistic behavior. In reality the searcher can probably distinguish the arcs (as if they are labeled for him) and count how many times he has visited an arc without catching the hider. Moreover if the hider will not move for too long, then at some moment there is only one arc left where he can possibly stay from the searcher's point of view. So intuitively it would be optimal for the hider to move to another arc every now and then. In the next section we discuss an extended model in which we try to include some of these intuitive ideas.

3.3 Extended model with motion detection

3.3.1 Introduction to the extended model

Consider again the star with n arcs of length $\frac{1}{2}$. In the previous section we considered a model in which both the searcher and the hider have no memory. In that model they played the same game at each time step until the hider was caught by the searcher. In this section we will assume that both the searcher and the hider have some motion detecting abilities such that after each time step they know whether the opponent has moved or not. Example 3.1 shows that such an assumption is not unrealistic.

Example 3.1 *Consider a fox trying to catch a rabbit which can hide in a number of holes. The fox can either wait outside the hole and try to lure the rabbit, or the fox can search in*

one of the holes. The rabbit can hide in one of his holes or he can try to run to a safer place. This small example is a game with a searcher (the fox) and a hider (the rabbit), which can be modeled as a search game on a star. The place outside where the fox tries to lure the rabbit, is the origin of the star and the end of each arc represents a hole. If the rabbit stays in his hole, then it can notice the fox searching through vibrations in the ground, some sound, etc. On the other hand if the fox is searching a hole and meanwhile the rabbit moves to another hole, then it is reasonable to assume that the fox can hear or smell this. So if we model this problem in discrete time, then both the fox and the rabbit have the ability to know after each time step whether or not the other has moved.

This example shows that our extended model could be realistic and hence it supports the decision to put effort in analyzing this model. In the extended model the searcher and the hider will play a *stochastic game* (see [11]) with n states numbered as $1, \dots, n$. In each state they will play a subgame denoted as $\Gamma_n(k)$, for $k = 1, \dots, n$ where n is the number of arcs and k is the number of arcs where the hider can possibly stay from the searcher's point of view. So $\Gamma_n(k)$ is the subgame in which the searcher has restricted the set of arcs where the hider can stay from n arcs to only k arcs. Initially the searcher has no information about the position of the hider and thus $k = n$ at time zero. In other words, the game starts in subgame $\Gamma_n(n)$. The stochastic game played by the searcher and the hider is the collection $\Gamma_n = \{\Gamma_n(k) \mid k = 1, \dots, n\}$ where initially subgame $\Gamma_n(n)$ is played.

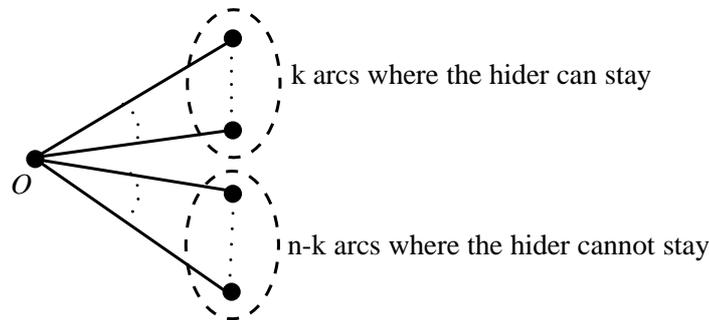


Figure 3.3: Subgame $\Gamma_n(k)$ where k arcs (from n in total) are left in which the hider can stay.

Just as in the simple model from Section 3.2 both the searcher and the hider have two options. Again the searcher can either *ambush* or *search*, and the hider can either *stay* or *move*. In Section 3.2 we introduced variables p and q to model their choice. Now their choice can be based on the subgame that is played. So we introduce vectors $\mathbf{p}, \mathbf{q} \in \mathbb{R}^n$ with $0 \leq p_k, q_k \leq 1$ for $k = 1, \dots, n$, such that in subgame $\Gamma_n(k)$

- ▶ the searcher searches with probability p_k one of the k arcs where the hider can stay and ambushes with probability $1 - p_k$,
- ▶ the hider stays with probability q_k and moves to the end of another arc with probability $1 - q_k$.

If the pair (\mathbf{p}, \mathbf{q}) is chosen such that it is a saddle-point equilibrium, then \mathbf{p} and \mathbf{q} are optimal strategies for Γ_n . Define $v_n(k)$ as the expected capture time at which the hider is caught in case the game would start in subgame $\Gamma_n(k)$ and the players respectively adopt strategies \mathbf{p} and \mathbf{q} . We call $v_n(k)$ the *value* of $\Gamma_n(k)$. Moreover $v_n \equiv v_n(n)$ is called the *value* of the game Γ_n . In Lemma 3.5 the uniqueness of the values and the existence of optimal strategies is proved using Shapley's theorem (see [11]).

Next suppose a subgame $\Gamma_n(k)$ is played and the hider is not caught. Then the stochastic game proceeds to another subgame based on the actions of the searcher and the hider.¹¹ On the other hand, if the searcher catches the hider, then the game ends. For $k = 1$ it is not hard to determine the value $v_n(1)$, see the next lemma. After this lemma we will analyze $v_n(k)$ for $2 \leq k \leq n$.

Lemma 3.2 *In subgame $\Gamma_n(1)$ the searcher will always search ($p_1 = 1$) and the hider will always stay ($q_1 = 0$) for any n . The value $v_n(1)$ of this subgame is equal to $\frac{1}{2}$.*

Proof: In subgame $\Gamma_n(1)$ there is only one arc left in which the hider can stay. So the searcher will certainly search that arc and thus $p_1 = 1$. Then the hider can choose either to move and be captured after $\frac{1}{4}$ time steps or to stay and be captured after $\frac{1}{2}$ time steps. The hider wants to maximize the expected capture time and thus he will certainly stay. Hence $q_1 = 1$ and $v_n(1) = \frac{1}{2}$. ■

For $2 \leq k \leq n$ we restrict ourselves to *stationary strategies*, which prescribe the same behavior for both players every time the same subgame is played. In that way we can express $v_n(k)$ in terms of the values of other subgames by assuming $\Gamma_n(k)$ is played at the first time step and then conditioning on the strategies played in $\Gamma_n(k)$. Remember we assumed both the searcher and hider to have some motion detecting abilities, such that after each time step they do know the opponents' action in the last turn.

- ▶ In the first place suppose the searcher ambushes in $\Gamma_n(k)$ and the hider stays in $\Gamma_n(k)$. Then the searcher does not catch the hider in this subgame. Moreover both the searcher and the hider do not move (and by assumption they know it from each other!) and thus the state does not change. So in the next time step $\Gamma_n(k)$ is played another time and thus in this case we find $v_n(k) = v_n(k) + 1$.
- ▶ Secondly, suppose the searcher ambushes in $\Gamma_n(k)$ and the hider moves in $\Gamma_n(k)$. Then the searcher certainly catches the hider after $\frac{1}{2}$ time steps and hence in this case we find $v_n(k) = \frac{1}{2}$.
- ▶ Next suppose the searcher searches in $\Gamma_n(k)$ and the hider stays in $\Gamma_n(k)$. Then the searcher chooses one of the k arcs in which the hider can possibly stay and hence he catches the hider with probability $\frac{1}{k}$ after $\frac{1}{2}$ time steps. With probability $\frac{k-1}{k}$ he does not catch the hider, but then he knows there are only $k - 1$ arcs left where the hider can possibly stay and thus the game will proceed to $\Gamma_n(k - 1)$. So in this case we find

¹¹In Shapley's definition in [11] the game proceeds from subgame to subgame according to transition probabilities. Moreover he introduces for each subgame a stopping probability. We define our stochastic game in a slightly different form, but we could reformulate it in Shapley's notation.

for $v_n(k)$,

$$v_n(k) = \frac{1}{k} \cdot \frac{1}{2} + \frac{k-1}{k} \cdot (v_n(k-1) + 1) = \frac{2k-1}{2k} + \frac{k-1}{k} v_n(k-1).$$

- Finally suppose the searcher searches in $\Gamma_n(k)$ and the hider moves in $\Gamma_n(k)$. Then the searcher again chooses one of the k arcs in which the hider can possibly and hence he catches the hider with probability $\frac{1}{k}$ after $\frac{1}{4}$ time steps. With probability $\frac{k-1}{k}$ he chooses the wrong arc, but then the hider is not safe yet. At the moment the hider reaches the origin, with probability $\frac{1}{n-1}$ he will choose the arc from which the searcher is returning and hence run into the searcher after $\frac{3}{4}$ time steps. If the hider is not caught in this step, then the game proceeds to $\Gamma_n(n-1)$ because the searcher has detected the movement of the hider and hence at this point the searcher only knows that the hider cannot stay in the arc from which he just returned. Combining all these observation leads in this case to

$$\begin{aligned} v_n(k) &= \frac{1}{k} \cdot \frac{1}{4} + \frac{k-1}{k} \left(\frac{1}{n-1} \cdot \frac{3}{4} + \frac{n-2}{n-1} (v_n(n-1) + 1) \right), \\ &= \frac{4kn - 3n - 5k + 4}{4k(n-1)} + \frac{(k-1)(n-2)}{k(n-1)} v_n(n-1). \end{aligned}$$

In Table 3.2 these four results are summarized in a *payoff*¹² table for $\Gamma_n(k)$ with $2 \leq k \leq n$. In such a table the expected capture time for each pair of pure strategies is given. To find the

$\Gamma_n(k)$		Hider	
		<i>stay</i>	<i>move</i>
Searcher	<i>search</i>	$\frac{2k-1}{2k} + \frac{k-1}{k} v_n(k-1)$	$\frac{4kn-3n-5k+4}{4k(n-1)} + \frac{(k-1)(n-2)}{k(n-1)} v_n(n-1)$
	<i>ambush</i>	$v_n(k) + 1$	$\frac{1}{2}$

Table 3.2: Payoff table for subgame $\Gamma_n(k)$ with $2 \leq k \leq n$.

expected capture time for mixed strategies, we only have to take a combination of the values in Table 3.2 corresponding to the probabilities in the mixed strategy. Here we added labels in Table 3.2 indicating the searcher is the row player and the hider is the column player. However normally we will omit these labels.

Of course we want to know the solution of the problem. Before we try to find the solution in subsection 3.3.3 for $n = 3, 4, 5$ and in subsection 3.3.4 for general n , it is useful first to prove some properties of the extended model in subsection 3.3.2.

¹²In this section we will often use the more general term *payoff* to denote the capture time corresponding to some search and hiding strategies.

3.3.2 Properties of the extended model

All four lemmas in this subsection will be used in subsections 3.3.3 and 3.3.4. Below, a brief description of each lemma is given. After that, the lemmas are presented and proved one by one. Between the proofs of Lemma 3.5 and Lemma 3.6 we will also give a small example how to compute optimal mixed strategies from a payoff table.

1. First we show which pure strategies can possibly be optimal in a certain subgame. (Lemma 3.3)
2. Secondly, we will derive simple but useful bounds for v_n (Lemma 3.4).
3. The next lemma concerns the existence of optimal strategies and the uniqueness of the value of each subgame and hence also for the search game. (Lemma 3.5).
4. Finally we will prove which strategies are optimal for the searcher and the hider in subgames $\Gamma_n(n)$ and $\Gamma_n(n-1)$ (Lemma 3.6).

Lemma 3.3 *Consider a subgame $\Gamma_n(k)$ with $1 \leq k \leq n$ and suppose it is optimal for the players to adopt pure strategies. Then the searcher will search and the hider will stay in this subgame. Moreover another combination of pure strategies cannot be optimal.*

Proof: Suppose it is optimal for the players to adopt pure strategies in subgame $\Gamma_n(k)$. The hider's objective is to maximize the expected capture time. So if the searcher would choose to ambush, then the hider would certainly stay. On the other hand, the searcher's objective is to minimize the expected capture time. So if the hider would move, then the searcher would certainly ambush. Thus if the players will play pure strategies such that either the hider will move, the searcher will ambush or both, then at least one player can unilaterally change his strategy in his own advantage. Consequently such a strategy cannot be optimal and hence if it is optimal to adopt pure strategies in $\Gamma_n(k)$, then certainly the searcher will search and the hider will stay. ■

Lemma 3.4 *Consider the stochastic game Γ_n with value v_n for $n \geq 3$. Then $\frac{n}{2} \leq v_n \leq n$.*

Proof: This proof consists of two parts. First we will prove the lower bound and secondly we will prove the upper bound.

- Suppose the hider will always stay. Then we are interested in the searcher's best response to this strategy of the hider. If the searcher will ambush in some subgame, then he is wasting his time because the hider never moves. Hence the searcher's best response is to search in every subgame. Now let $u_n(k)$ be the expected time at which the hider is caught while playing these strategies and for a game starting in $\Gamma_n(k)$, then we find $u_n(k) = \frac{2k-1}{2k} + \frac{k-1}{k}u_n(k-1)$ for $k = 2, \dots, n-1$. Moreover from Lemma 3.2 we know $u_n(1) = \frac{1}{2}$. Solving this recurrence relation gives $u_n(k) = \frac{k}{2}$ and thus in particular $u_n(n) = \frac{n}{2}$. So if the hider will always stay in every subgame, then he knows for sure that the expected capture time is at least $\frac{n}{2}$ and consequently $\frac{n}{2} \leq v_n$.

- ▶ Next assume that the searcher will randomly search an arc in every subgame $\Gamma_n(k)$ for $k = 2, \dots, n$. Then each time step he will catch the hider with a probability of at least $\frac{1}{n}$. So v_n is smaller than or equal to the expected value of a geometric distribution with parameter $\frac{1}{n}$ and hence $v_n \leq n$.

Combining these two parts completes the proof of this lemma. ■

In the proof of the next lemma we will use Shapley's Theorem (see [11]). This theorem states that every stochastic game with

- ▶ a finite number of subgames,
- ▶ a finite number of possible pure strategies for both players in each subgame,
- ▶ a positive *stopping probability* for each combination of pure strategies in each subgame,

has a *unique* value for each subgame. Moreover this theorem ensures the existence of optimal (mixed) strategies. Note Shapley used the last condition to ensure that the game ends in finite time. In our search game a positive stopping probability means that the probability that the searcher captures the hider is positive.

Lemma 3.5 *There exist optimal stationary strategies for Γ_n . Moreover the value $v_n(k)$ for subgame $\Gamma_n(k)$ is uniquely determined by these optimal stationary strategies.*

Proof: Our game has n subgames and both players have two options in each subgame. So it is obvious that our game fulfils the first two criteria of Shapley's Theorem. Thus we only need to prove that our game also fulfils the last criterion. In each subgame there are four possible combinations of pure strategies. In three of them, the hider can be caught and hence those combinations have a positive stopping probability. However there is one combination (the search will ambush and the hider will stay) where the hider is certainly not caught and thus the stopping probability for this combination is zero. Consequently, in the current form our search game does not fulfil the third criterion of Shapley's Theorem.

However we can use a trick to avoid this problem. Remember that if the searcher adopts a pure strategy in subgame $\Gamma_n(k)$, then by Lemma (3.3) he will search. In other words, the searcher will never ambush with probability one in $\Gamma_n(k)$. Since the searcher ambushes in $\Gamma_n(k)$ with probability $1 - p_k$, this implies $p_k > 0$ and hence there exists an $\epsilon_k > 0$ such that $p_k \geq \epsilon$. So if we can replace the searcher's current pure strategies by two new pure strategies for the searcher such that he can search with any probability $p_k \in [\epsilon_k, 1]$ by mixing those pure strategies, then we do not restrict the search game.

Now take $\epsilon = \min\{\epsilon_k \mid 1 \leq k \leq n\}$ and replace the searcher's current pure strategies in $\Gamma_n(k)$ by the following two pure strategies.

- M1: Search one of the k arcs where the hider can stay (in fact this is the one of the two original pure strategies).
- M2: Search one of the k arcs where the hider can stay with probability ϵ and ambush with probability $1 - \epsilon$.

If the searcher would originally *search* in $\Gamma_n(k)$ with probability $p_k \in [\epsilon, 1]$ and *ambush* with probability $1 - p_k$, then this equal to playing strategy M1 with probability $\rho_k := \frac{p_k - \epsilon}{1 - \epsilon}$ and M2 with probability $1 - \rho_k$, because

$$\rho_k + (1 - \rho_k)(1 - \epsilon) = \frac{p_k - \epsilon}{1 - \epsilon} + \left(1 - \frac{p_k - \epsilon}{1 - \epsilon}\right) \epsilon = \underbrace{\frac{(1 - \epsilon)(p_k - \epsilon)}{1 - \epsilon}}_{\geq \epsilon} + \epsilon = p_k$$

and

$$(1 - \rho_k)(1 - \epsilon) = \left(1 - \frac{p_k - \epsilon}{1 - \epsilon}\right) (1 - \epsilon) = 1 - \epsilon - p_k + \epsilon = 1 - p_k.$$

Moreover $\rho_k \in [0, 1]$ for $p_k \in [\epsilon, 1]$ and thus ρ_k denotes indeed a probability.

Since $\epsilon \leq \epsilon_k$, it follows that we can assume that the searcher plays in $\Gamma_n(k)$ according to a mixture of M1 and M2 instead of a mixture of *search* and *ambush*. Notice the searcher has in both M1 and M2 a positive probability for catching the hider and thus a positive stopping probability. Hence our search game fulfils the third condition in Shapley's Theorem, which completes the proof of this lemma. \blacksquare

Example 3.2 Consider a simple search game in which the searcher and the hider can both choose from two strategies in each time step with payoffs respectively a, b, c and d . We will solve this simple problem in a detailed way, because later on we will use the same approach to derive formulas describing our stochastic game.

	hider's strategy H_1	hider's strategy H_2
searcher's strategy S_1	a	b
searcher's strategy S_2	c	d

Assume in this example that it is optimal to adopt mixed strategies and let V denote the value of this game. Next suppose it is optimal for the searcher to play strategy S_1 with probability $p \in (0, 1)$ and for the hider to play strategy H_1 with probability $q \in (0, 1)$. Then V is equal to

$$\begin{aligned} V &= pqa + p(1 - q)b + (1 - p)qc + (1 - p)(1 - q)d, \\ &= p(qa + (1 - q)b) + (1 - p)(qc + (1 - q)d), \\ &= q(pa + (1 - p)c) + (1 - q)(pb + (1 - p)d). \end{aligned}$$

We assumed p, q to be optimal and thus there exists no $p' \in [0, 1]$ such that

$$p'(qa + (1 - q)b) + (1 - p')(qc + (1 - q)d) < p(qa + (1 - q)b) + (1 - p)(qc + (1 - q)d) \quad (3.3)$$

and no $q' \in [0, 1]$ such that

$$q'(pa + (1 - p)c) + (1 - q')(pb + (1 - p)d) > q(pa + (1 - p)c) + (1 - q)(pb + (1 - p)d). \quad (3.4)$$

Consequently,

$$pa + (1 - p)c = pb + (1 - p)d \quad \text{and} \quad qa + (1 - q)b = qc + (1 - q)d. \quad (3.5)$$

Notice if (3.3) or (3.4) does not hold, then we could respectively construct such a p' or q' . Next solving (3.5) for respectively p and q gives

$$p = \frac{c - d}{b + c - a - d} \quad \text{and} \quad q = \frac{b - d}{b + c - a - d}$$

and hence for V ,

$$V = pa + (1 - p)c = pb + (1 - p)d = qa + (1 - q)b = qc + (1 - q)d = \frac{cb - ad}{b + c - a - d}. \quad (3.6)$$

This example shows how we can compute p , q and V under the assumption that it is optimal to adopt mixed strategies. This approach can also be used in stochastic games, and in particular in the one that we consider in this section.

Lemma 3.6 Consider the stochastic game Γ^n with $n \geq 3$. Then in subgames $\Gamma_n(n)$ and $\Gamma_n(n - 1)$ the searcher will always search and the hider will always stay. Consequently

$$\left. \begin{aligned} v_n(n) &= \frac{2n-1}{2n} + \frac{n-1}{n}v_n(n-1) \\ v_n(n-1) &= \frac{2n-3}{2n-2} + \frac{n-2}{n-1}v_n(n-2) \end{aligned} \right\} \implies v_n(n) = \frac{2(n-1)}{n} + \frac{n-2}{n}v_n(n-2).$$

Proof: First consider the payoff table for $\Gamma_n(n)$.

$\Gamma_n(n)$	stay	move
search	$\frac{2n-1}{2n} + \frac{n-1}{n}v_n(n-1)$	$\frac{n-1}{n} + \frac{n-2}{n}v_n(n-1)$
ambush	$v_n(n) + 1$	$\frac{1}{2}$

Table 3.3: Payoff table for subgame $\Gamma_n(n)$.

Notice no matter what the searcher does in this subgame, it is always beneficial for the hider to stay, because the left column *dominates* the right column. This means that both payoffs in the left column are more beneficial for the hider than the payoffs in the right column.¹³ Here this is indeed the case, because

$$\frac{2n-1}{2n} + \frac{n-1}{n}v_n(n-1) > \frac{n-1}{n} + \frac{n-2}{n}v_n(n-1) \quad \text{and} \quad v_n(n) + 1 > \frac{1}{2}.$$

If the hider will always stay in a subgame, then it is obviously that the searcher responds by always searching in that particular subgame. Hence in $\Gamma_n(n)$ the searcher will search and the hider will stay and thus $v_n(n) = \frac{2n-1}{2n} + \frac{n-1}{n}v_n(n-1)$.

¹³If the column players' objective was to minimize the value of the game, then the right column would have dominated the left column in this case. Moreover in payoff table it also possible that a row is dominated by another row.

Next consider subgame $\Gamma_n(n-1)$ with corresponding payoff table. Suppose it is optimal

$\Gamma_n(\mathbf{n}-1)$	<i>stay</i>	<i>move</i>
<i>search</i>	$\frac{2n-3}{2n-2} + \frac{n-2}{n-1}v_n(n-2)$	$\frac{(2n-3)^2}{4(n-1)^2} + \frac{(n-2)^2}{(n-1)^2}v_n(n-1)$
<i>ambush</i>	$v_n(n-1) + 1$	$\frac{1}{2}$

Table 3.4: Payoff table for subgame $\Gamma_n(n-1)$.

for the players to adopt mixed strategies in $\Gamma_n(n-1)$ such that the searcher will search with probability $p_{n-1} \in (0, 1)$ in $\Gamma_n(n-1)$ and the hider will stay in $\Gamma_n(n-1)$ with probability $q_{n-1} \in (0, 1)$. Both the searcher and the hider cannot profit from unilaterally changing their strategy. Hence (compare to (3.6) in Example 3.2)

$$v_n(n-1) = p_{n-1} \left\{ \frac{(2n-3)^2}{4(n-1)^2} + \frac{(n-2)^2}{(n-1)^2}v_n(n-1) \right\} + \frac{1-p_{n-1}}{2},$$

and next solving for p_{n-1} results in

$$p_{n-1} = \frac{2(n-1)^2(2v_n(n-1)-1)}{2n^2-8n+7+4(n-2)^2v_n(n-1)}$$

Differentiating this expression with respect to $v_n(n-1)$ gives

$$\frac{\partial}{\partial v_n(n-1)} p_{n-1} = \frac{4(n-1)^2(2n-3)(2n-5)}{(2n^2-8n+7+4(n-2)^2v_n(n-1))^2} > 0, \quad \text{for } n \geq 3.$$

So p_{n-1} can be interpreted as an increasing function of $v_n(n-1)$. From Lemma 3.4 we know $v_n(n) \geq \frac{n}{2}$. Moreover we showed in this proof that $v_n(n) = \frac{2n-1}{2n} + \frac{n-1}{n}v_n(n-1)$ or equivalently $v_n(n-1) = \frac{n}{n-1}v_n(n) - \frac{2n-1}{2n-2}$. Consequently we know $v_n(n-1) \geq \frac{n}{n-1} \cdot \frac{n}{2} - \frac{2n-1}{2n-2} = \frac{n-1}{2}$. Then we find for p_{n-1} ,

$$\begin{aligned} p_{n-1} &\geq \frac{2(n-1)^2(2v_n(n-1)-1)}{2n^2-8n+7+4(n-2)^2v_n(n-1)} \Big|_{v_n(n-1)=\frac{n-1}{2}}, \\ &= \frac{2(n-1)^2(n-2)}{2n^3-8n^2+8n-1}. \end{aligned}$$

Both the numerator and the denominator of this fraction are positive for $n \geq 3$. Moreover

$$2(n-1)^2(n-2) - (2n^3-8n^2+8n-1) = 2n-3 > 0, \quad \text{for } n \geq 3$$

and thus we know $p_{n-1} > 1$ for $n \geq 3$. However p_{n-1} cannot be larger than one, because it denotes a probability. Hence we have a contradiction to the assumption that it was optimal for the players to adopt mixed strategies in $\Gamma_n(n-1)$. So the players both adopt a pure strategy in $\Gamma_n(n-1)$. Then according to Lemma 3.3 the searcher will search and the hider

will stay in $\Gamma_n(n-1)$. Consequently $v_{n-1} = \frac{2n-3}{2n-2} + \frac{n-2}{n-1}v_n(n-2)$ and next substituting this expression into the the relation between $v_n(n)$ and $v_n(n-1)$ obtained in the first part of this proof, results in the relation between $v_n(n)$ and $v_n(n-2)$ stated in the lemma. ■

3.3.3 Solving the extended model for $n = 3, 4, 5$

In this subsection we will solve the stochastic games Γ_3 , Γ_4 and Γ_5 . We will see that Γ_3 is easy to solve, Γ_4 is a bit more difficult and analytically solving Γ_5 becomes too complicated.

- The solution for Γ_3 is an immediately consequence of Lemma 3.2 and Lemma 3.6. In the first lemma we proved $v_3^1 = \frac{1}{2}$ and in the second lemma we proved $v_3 = v_3^3 = \frac{4}{3} + \frac{1}{3}v_3^1$. So combining these two lemmas shows $v_3 = \frac{3}{2}$ and also that the searcher will always search and that the hider will always stay in each subgame of Γ_3 .
- Next consider Γ_4 . We know using Lemma 3.2 and Lemma 3.6 that in Γ_4^1 , Γ_4^3 and Γ_4^4 the searcher will always search and the hider will always stay. So we only need to consider the payoff table for Γ_4^2 . We know $v_4(1) = \frac{1}{2}$ (see Lemma 3.2) and $v_4(3) = \frac{5}{6} + \frac{2}{3}v_4(2)$

$\Gamma_4(2)$	<i>stay</i>	<i>move</i>	\implies	$\Gamma_4(2)$	<i>stay</i>	<i>move</i>
<i>search</i>	$\frac{3}{4} + \frac{1}{2}v_4(1)$	$\frac{7}{12} + \frac{1}{3}v_4(3)$		<i>search</i>	1	$\frac{31}{36} + \frac{2}{9}v_4(2)$
<i>ambush</i>	$v_4(2) + 1$	$\frac{1}{2}$		<i>ambush</i>	$v_4(2) + 1$	$\frac{1}{2}$

Table 3.5: Payoff table for Γ_4^2

(see Lemma 3.6). Substituting these equalities in the payoff table for Γ_4^2 results in a table only containing $v_4(2)$ and thus $v_4(2)$ can be expressed in terms of itself.

If it would be optimal for the players to play pure strategies in $\Gamma_4(2)$, then the only possibility is that the searcher would search and the hider would stay (see Lemma 3.3). So in that case $v_4(2)$ would be equal to 1, but then it is beneficial for the hider to move because $\frac{31}{36} + \frac{2}{9} > 1$. Hence in an optimal solution the players do not adopt pure strategies in $\Gamma_4(2)$, because then the solution cannot be a saddle-point equilibrium and thus cannot be optimal.

So we know the searcher will search with probability $p_2 \in (0, 1)$ and the hider will stay with probability $q_2 \in (0, 1)$. Then we find

$$v_4(2) = p_2q_2 + p_2(1 - q_2) \left(\frac{31}{36} + \frac{2}{9}v_4(2) \right) + (1 - p_2)q_2 (v_4(2) + 1) + \frac{(1 - p_2)(1 - q_2)}{2}$$

and after solving for $v_4(2)$,

$$v_4(2) = \frac{18 + 13p_2 + 18q_2 - 13p_2q_2}{4(11p_2q_2 - 2p_2 - 9q_2 + 9)}.$$

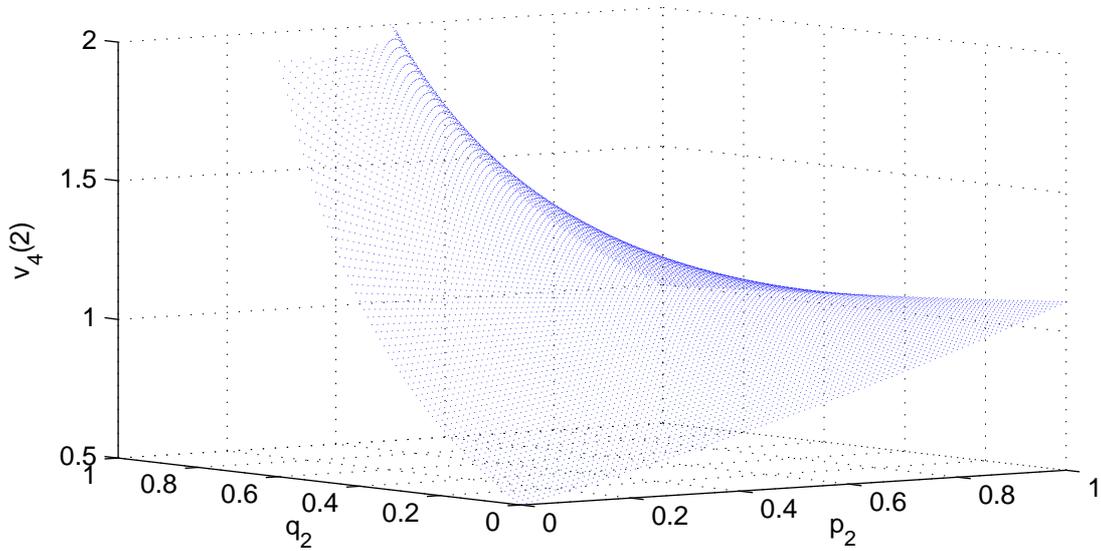


Figure 3.4: Saddle point indicating value of $v_4(2)$.

To solve the problem we have find the saddle point visible in Figure 3.4. So we have to solve

$$\begin{cases} \frac{\partial}{\partial p_2} v_4(2) = \frac{9(17-44q_2-9q_2^2)}{4(11p_2q_2-2p_2-9q_2+9)^2} = 0, \\ \frac{\partial}{\partial q_2} v_4(2) = \frac{9(36-26p_2-13p_2^2)}{4(11p_2q_2-2p_2-9q_2+9)^2} = 0. \end{cases} \implies p_2 = \frac{7}{13}\sqrt{13} \pm 1, \quad q_2 = \frac{7\sqrt{13} \pm 22}{9}.$$

Since p_2 and q_2 are defined as probabilities we find $p_2 = \frac{7}{13}\sqrt{13} - 1 \approx 0.9415$ and $q_2 = \frac{7\sqrt{13}-22}{9} \approx 0.3599$.¹⁴ We can substitute these values in the expression for $v_4(2)$ and then using $v_4 = v_4(4) = \frac{3}{2} + \frac{1}{2}v_4(2)$ (see Lemma 3.6), we can conclude

$$v_4 = \frac{209\sqrt{13} - 833}{160\sqrt{13} - 616} \approx 2.0311.$$

- Finally consider Γ_5 for which we need the payoff tables for Γ_5^2 and Γ_5^3 . Just as for Γ_4^2 , also in these tables the relations provided by Lemma 3.2 and Lemma 3.6 can be substituted. We expect mixed strategies in $\Gamma_5(2)$ and $\Gamma_5(3)$ and thus we can try to find saddle points for those subgames. We cannot represent the saddle points in a plot anymore (assuming there exists at least one saddle point), because we now have four variables. However the idea remains the same and we can still compute the saddle points. Notice if it is not optimal to play mixed strategies in either $\Gamma_5(2)$ or $\Gamma_5(3)$, then we do not find saddle points such that p_2, p_3, q_2 and q_3 all take values in $(0, 1)$. On the other hand if we find a saddle point, then we know the corresponding strategies are optimal because each

¹⁴Since $\frac{7}{13}\sqrt{13} + 1 > 1$ and $\frac{7\sqrt{13}+22}{9} > 1$, the remaining solutions do not satisfy the condition that both p_2 and q_2 are defined to be probabilities.

$\Gamma_5(2)$	<i>stay</i>	<i>move</i>		$\Gamma_5(2)$	<i>stay</i>	<i>move</i>
<i>search</i>	$\frac{3}{4} + \frac{1}{2}v_5(1)$	$\frac{19}{32} + \frac{3}{8}v_5(4)$	\implies	<i>search</i>	1	$\frac{59}{64} + \frac{9}{32}v_5(3)$
<i>ambush</i>	$v_5(2) + 1$	$\frac{1}{2}$		<i>ambush</i>	$v_5(2) + 1$	$\frac{1}{2}$
$\Gamma_5(3)$	<i>stay</i>	<i>move</i>		$\Gamma_5(3)$	<i>stay</i>	<i>move</i>
<i>search</i>	$\frac{5}{6} + \frac{2}{3}v_5(2)$	$\frac{17}{24} + \frac{1}{2}v_5(4)$	\implies	<i>search</i>	$\frac{5}{6} + \frac{2}{3}v_5(2)$	$\frac{55}{48} + \frac{3}{8}v_5(3)$
<i>ambush</i>	$v_5(3) + 1$	$\frac{1}{2}$		<i>ambush</i>	$v_5(3) + 1$	$\frac{1}{2}$

Table 3.6: Payoff table for Γ_5^2 and Γ_5^3 .

subgame has a unique value according to Lemma 3.5.

To find a saddle point notice that if the searcher will search with probability p_2 in $\Gamma_5(2)$ in such a point, then the hider must always have the same payoff for each choice of q_2 . Therefore p_2 fulfils the following relation (also see Example 3.2).

$$p_2 + (1-p_2)(v_5(2)+1) = p_2 \left(\frac{59}{64} + \frac{9}{32}v_5(3) \right) + \frac{1-p_2}{2} \implies p_2 = \frac{32(2v_5(2) + 1)}{64v_5(2) + 18v_5(3) + 27}.$$

On the other hand, the hider will stay with probability q_2 such that the search will always have the same payoff for each choice of p_2 . Then we find

$$q_2 + (1-q_2) \left(\frac{59}{64} + \frac{9}{32}v_5(3) \right) = \underbrace{q_2(v_5(2) + 1) + \frac{1-q_2}{2}}_{= v_5(2)} \implies q_2 = \frac{9(2v_5(3) + 3)}{64v_5(2) + 18v_5(3) + 27}.$$

Moreover both terms on the left hand side of the arrow (we need the term on the right hand side of the equality sign) must be equal to $v_5(2)$ (see Example 3.2) and thus

$$v_5(2) = q_2(v_5(2) + 1) + \frac{1-q_2}{2} \implies v_5(2) = \frac{1+q_2}{2(1-q_2)} = \frac{18v_5(3) + 32v_5(2) + 27}{64v_5(2)}. \quad (3.7)$$

In the same way we can find for $\Gamma_5(3)$,

$$p_3 = \frac{24(2v_5(3) + 1)}{66v_5(3) - 32v_5(2) + 39},$$

$$q_3 = \frac{18v_5(3) + 31}{66v_5(3) - 32v_5(2) + 39},$$

and thus

$$v_5(3) = \frac{1+q_3}{2(1-q_3)} = \frac{42v_5(3) - 16v_5(2) + 35}{8(1+6v_5(3) - 4v_5(2))}. \quad (3.8)$$

Combining (3.7) and (3.8) shows we have to solve the following system with two

equations and two unknowns,

$$\begin{cases} 64v_5(2)^2 &= 18v_5(3) + 32v_5(2) + 27, \\ 8v_5(3)(1 + 6v_5(3) - 4v_5(2)) &= 42v_5(3) - 16v_5(2) + 35 \end{cases} \quad (3.9)$$

It is possible to eliminate $v_5(2)$ (we will not show the details) and obtain a fourth order polynomial describing $v_5(3)$ such that

$$2304v_5(3)^4 - 4320v_5(3)^3 - 1420v_5(3)^2 + 3028v_5(3) + 837 = 0. \quad (3.10)$$

Theoretically the roots of this polynomial can be found analytically, however that is not practical. Numerically we find that the roots are approximately -0.6869 , -0.2750 , 1.1209 and 1.7159 . The first two roots are not useful, because $v_5(3)$ cannot be negative. The third root is not useful, because the corresponding value for $v_5(2)$ is negative (approximately -0.6442 using (3.8)). The fourth root is useful, because for this root the corresponding value for $v_5(2)$ is approximately 1.2334 . We expressed p_2 , q_2 , p_3 and q_3 in terms of $v_5(2)$ and $v_5(3)$ and now substituting $v_5(2)$ and $v_5(3)$ in these expressions results in

$$p_2 \approx 0.8108, \quad q_2 \approx 0.4231, \quad p_3 \approx 0.9431, \quad q_3 \approx 0.5487. \quad (3.11)$$

These values are all in the open interval $(0, 1)$ and hence our assumption concerning mixed strategies in $\Gamma_5(2)$ and $\Gamma_5(3)$ was correct. So we can conclude that is optimal for the searcher and the hider to behave in subgames $\Gamma_5(2)$ and $\Gamma_5(3)$ according to (3.11). We already knew that in subgames $\Gamma_5(1)$, $\Gamma_5(4)$ and $\Gamma_5(5)$, it is optimal for the searcher always to search and for the hider always to stay. The value of Γ_5 we find using Lemma 3.6,

$$v_5 = v_5(5) = \frac{8}{5} + \frac{3}{5}v_5(3) \approx 2.6296.$$

The computations in this subsection show that the value of Γ_3 can be found easily and is given by a nice fraction, namely $\frac{3}{2}$. For Γ_4 it is still not very hard to compute the answer analytically, however we do not longer find a nice answer. After that, we performed a lot of computations finally resulting in approximation for the value of Γ_5 . We showed these computations for Γ_5 , because in the next subsection we will use almost the same approach to approximate the value of Γ_n for $n > 5$. The only difference is that we will not approximate the roots of a polynomial, but numerically solve a fixed point problem. For $n = 5$ we would be interested in the fixed points of the vector function f defined by (compare to (3.7) and (3.8))

$$f(v_5(2), v_5(3)) = \begin{bmatrix} \frac{18v_5(3) + 32v_5(2) + 27}{64v_5(2)} \\ \frac{42v_5(3) - 16v_5(2) + 35}{8(1 + 6v_5(3) - 4v_5(2))} \end{bmatrix}.$$

This vector function has four fixed points corresponding to the four roots of the polynomial in (3.10) and the interesting fixed point is of course the one corresponding to $v_5(3) \approx 1.7159$.

In general, we are interested in fixed points such that each subgame has a positive value and such that the variables p_k, q_k for $k = 2, \dots, n-2$ take values in the open interval $(0, 1)$. If such a fixed points would not exist, then we know our assumption concerning mixed strategies

in $\Gamma_n(2), \dots, \Gamma_n(n-2)$ was not correct. Hence in that case in at least one of those subgames it is optimal for the players to adopt a pure strategy instead of a mixed strategy. In that case we should find such a subgame and formulate a new fixed point problem incorporating this new knowledge.

3.3.4 Numerical analysis of the extended model

In this subsection we numerically solve the extended model. After that, based on the numerical solutions we make a guess for the optimal strategies and the values of the subgames for n tending to infinity.

To obtain the numerical solution for Γ_n we will start using the same approach as for Γ_5 . So first assume that it is optimal for the players to adopt mixed strategies in subgames $\Gamma_n(2), \dots, \Gamma_n(n-2)$. Then using Table 3.2 we know

$$\begin{aligned} p_k & \left(\frac{2k-1}{2k} + \frac{k-1}{k} v_n(k-1) \right) + (1-p_k)(v_n(k)+1) \\ & = p_k \left(\frac{4kn-3n-5k+4}{4k(n-1)} + \frac{(k-1)(n-2)}{k(n-1)} v_{n(n-1)} \right) + \frac{1-p_k}{2}. \end{aligned}$$

and

$$\begin{aligned} q_k & \left(\frac{2k-1}{2k} + \frac{k-1}{k} v_n(k-1) \right) + (1-q_k) \left(\frac{4kn-3n-5k+4}{4k(n-1)} + \frac{(k-1)(n-2)}{k(n-1)} v_{n(n-1)} \right) \\ & = q_k(v_n(k)+1) + \frac{1-q_k}{2}. \end{aligned}$$

Solving these two equalities for respectively p_k and q_k results in

$$p_k = \frac{2k(n-1)(2v_n(k)+1)}{2+2kn-3k-n-4(k-1)(n-1)v_n(k-1)+4k(n-1)v_n(k)+4(k-1)(n-2)v_n(n-1)}$$

and

$$q_k = \frac{4-3n-3k+2kn+4(k-1)(n-2)v_n(n-1)}{2+2kn-3k-n-4(k-1)(n-1)v_n(k-1)+4k(n-1)v_n(k)+4(k-1)(n-2)v_n(n-1)}.$$

In the analysis of Γ_5 we found $v_5(2) = \frac{1+q_2}{2(1-q_2)}$ and $v_5(3) = \frac{1+q_3}{2(1-q_3)}$. Such a relation also holds in general for all n and $k = 2, \dots, n-2$ such that $v_n(k) = \frac{1+q_k}{2(1-q_k)}$. Next we find

$$v_n(k) = \frac{(2n-3)(k-1) - 2(k-1)(n-1)v_n(k-1) + 2k(n-1)v_n(k) + 4(k-1)(n-2)v_n(n-1)}{2n-2-4(k-1)(n-1)v_n(k-1)+4k(n-1)v_n(k)}.$$

So for each $v_n(k)$ with $k = 2, \dots, n-2$ ($n-3$ in total), we can express $v_n(k)$ in terms of $v_n(2), \dots, v_n(n-2)$ (where we also use $v_n(1) = \frac{1}{2}$ by Lemma 3.2 and $v_n(n-1) = \frac{2n-3}{2n-2} + \frac{n-2}{n-1}v_n(n-2)$ by Lemma 3.6.) Now we have a fixed point problem in \mathbb{R}^{n-3} for which we can try to find a fixed point resulting in a solution for Γ_n , such that all values for the subgames are positive and p_k, q_k are between 0 and 1 for $k = 2, \dots, n-2$.

In Figures 3.5, 3.6 and 3.7 some numerical results are given. We derived the numerical

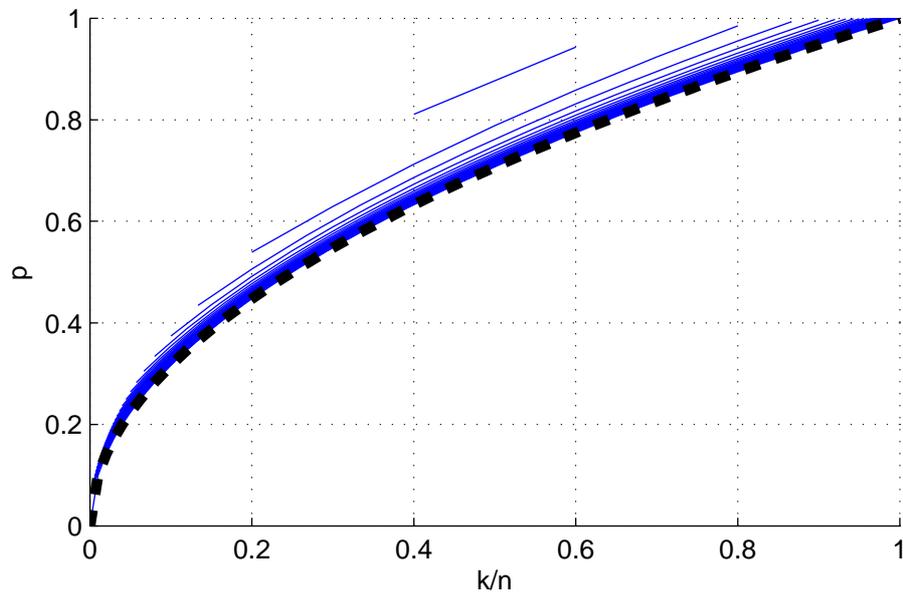


Figure 3.5: Searcher's optimal strategy for different values of n .

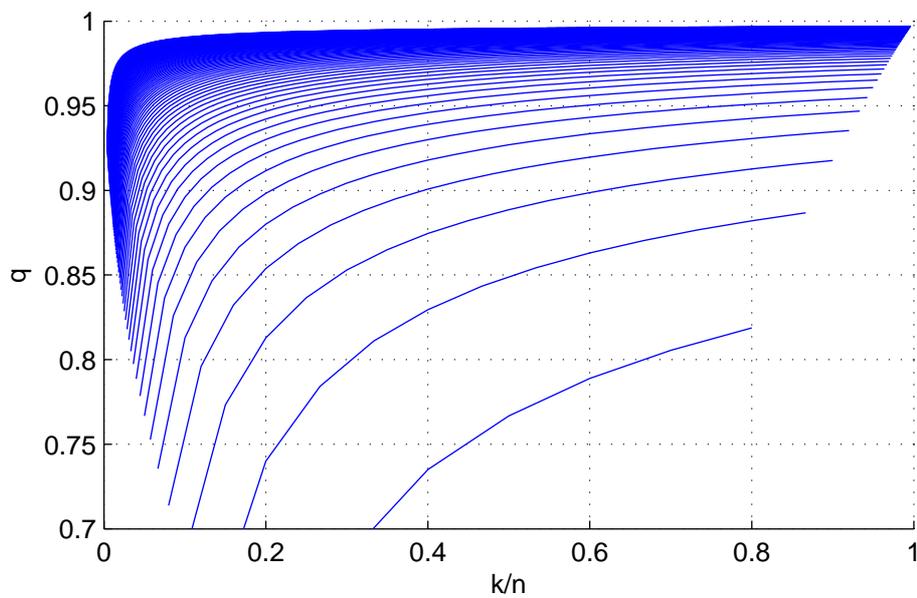


Figure 3.6: Hider's optimal strategy for different values of n .

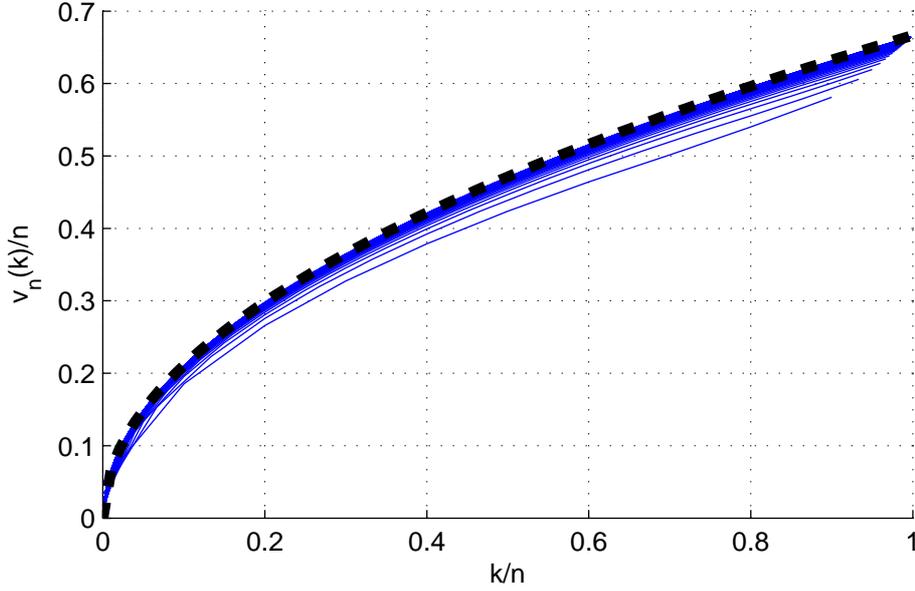


Figure 3.7: Ratio between values and n for subgames $\Gamma_n(k)$ with $k = 1, \dots, n$ for different values of n .

solutions for a large number of different values for n . To show the searcher's optimal strategies, the hider's optimal strategies and the values of the subgames in a single plot we rescaled the interval $[0, n]$ to $[0, 1]$ for each n . In Figures 3.5 and 3.7 the numerical results on the normalized scale converge to the dotted curves for large n . The dotted curve in Figure 3.5 corresponds to the function $\sqrt{\frac{k}{n}}$ and hence it appears $p_k \approx \sqrt{\frac{k}{n}}$ for n tending to infinity. In the same way, the dotted curve in Figure 3.7 corresponds to $\frac{2}{3}\sqrt{\frac{k}{n}}$ and hence from the values of the subgames it appears

$$\frac{v_n(k)}{n} \approx \frac{2}{3}\sqrt{\frac{k}{n}} \iff v_n(k) \approx \frac{2}{3}\sqrt{nk}$$

for n tending to infinity. From Figure 3.6 such an observation for the hider's optimal strategy is not immediately clear, however further analysis suggests

$$q_k \approx 1 - \frac{1}{\frac{2}{3}\sqrt{nk}}.$$

We summarize these observations in Conjecture 3.7.

Conjecture 3.7 Consider the stochastic game Γ_n , then $\lim_{n \rightarrow \infty} \frac{v_n}{n} = \frac{2}{3}$ and for $n \rightarrow \infty$,

$$p_k \approx \sqrt{\frac{k}{n}}, \quad q_k \approx 1 - \frac{1}{\frac{2}{3}\sqrt{nk}}, \quad v_n(k) \approx \frac{2}{3}\sqrt{nk}.$$

3.3.5 Asymptotic behavior of the expected capture time

We aim in this subsection at proving the statement concerning the asymptotic behavior of the expected capture time presented in the last statement of Conjecture 3.7. Therefore we will first try to improve the bounds for v_n derived in Lemma 3.4. The next theorem provides us with improved bounds, under the assumption that we know in which subgames it is optimal to play pure strategies.

Theorem 3.8 *Suppose it is optimal for the players to adopt pure strategies in each subgame $\Gamma_n(n - m + 1)$ for $m = 1, \dots, j$ with $2 \leq j \leq n - 2$ and not optimal to adopt a pure strategy in $\Gamma_n(n - j)$, then*

$$\alpha(n, j) \leq v_n(n - 1) \leq \alpha(n, j + 1)$$

with

$$\alpha(n, x) = \frac{(2nx - 2n - 2x + 1)(2n - x - 1)}{4(nx - 2x + 1)}.$$

Proof: We will first prove the upper bound and then the lower bound.

- It is optimal to play pure strategies in $\Gamma_n(n - 1), \dots, \Gamma_n(n - j + 1)$. So using Lemma 3.3 we know the searcher will search and the hider will stay in those subgames. Then from the payoff table for $\Gamma_n(n - m + 1)$ (see Table 3.2 with $k = n - m + 1$) we find

$$v_n(n - m + 1) = \frac{2n - 2m + 1}{2n - 2m + 2} + \frac{n - m}{n - m + 1}v_n(n - m), \quad \text{for } m = 2, \dots, j.$$

Notice this relation gives us a way to express $v_n(n - j)$ in terms of $v_n(n - 1)$. It turns out (see Appendix B.1) that

$$v_n(n - j) = \frac{n - 1}{n - j}v_n(n - 1) - \frac{(2n - j - 1)(j - 1)}{2(n - j)}. \quad (3.12)$$

Next suppose the searcher chooses always to search in $\Gamma_n(n - j)$, then the hider's best response in $\Gamma_n(n - j)$ is certainly not staying. If the hider's best response would be staying, then it would be optimal to play a pure strategy in $\Gamma_n(n - j)$ which we assumed not to be true. Consequently his best response must be moving and thus the searcher

$\Gamma_n(n - j)$	<i>stay</i>	<i>move</i>
<i>search</i>	$\frac{2n-2j-1}{2(n-j)} + \frac{n-j-1}{n-j}v_n(n - j - 1)$	$\frac{4n^2-4nj-8n+5j+4}{4(n-j)(n-1)} + \frac{(n-j-1)(n-2)}{(n-j)(n-1)}v_n(n - 1)$
<i>ambush</i>	$v_n(n - j) + 1$	$\frac{1}{2}$

Table 3.7: Payoff table for subgame $\Gamma_n(n - j)$.

can in this way ensure (see Table 3.7)

$$v_n(n - j) \leq \frac{4n^2 - 4nj - 8n + 5j + 4}{4(n - j)(n - 1)} + \frac{(n - j - 1)(n - 2)}{(n - j)(n - 1)}v_n(n - 1)$$

or equivalently

$$v_n(n-1) \geq \frac{(n-j)(n-1)}{(n-j-1)(n-2)}v_n(n-j) - \frac{4n^2 - 4nj - 8n + 5j + 4}{4(n-j-1)(n-2)}.$$

Then substituting (3.12) gives

$$\begin{aligned} v_n(n-1) &\geq \frac{(n-j)(n-1)}{(n-j-1)(n-2)} \left\{ \frac{n-1}{n-j}v_n(n-1) - \frac{(2n-j-1)(j-1)}{2(n-j)} \right\} \\ &\quad - \frac{4n^2 - 4nj - 8n + 5j + 4}{4(n-j-1)(n-2)}, \\ &= \underbrace{\frac{(n-1)^2}{(n-j-1)(n-2)}}_{> 1} v_n(n-1) - \frac{(2nj - 2j - 1)(2n - j - 2)}{4(n-j-1)(n-2)}, \end{aligned}$$

and next

$$\begin{aligned} \left(\frac{(n-1)^2}{(n-j-1)(n-2)} - 1 \right) v_n(n-1) &= \frac{n + jn - 2j - 1}{(n-j-1)(n-2)}v_n(n-1), \\ &\leq \frac{(2nj - 2j - 1)(2n - j - 2)}{4(n-j-1)(n-2)}. \end{aligned}$$

Thus finally we find

$$v_n(n-1) \leq \frac{(2nj - 2j - 1)(2n - j - 2)}{4(n + jn - 2j - 1)} = \alpha(n, j + 1),$$

which completes the first part of the proof.

- The players will adopt pure strategies in $\Gamma_n(n-j+1)$. So the searcher will search in this subgame and the hider will stay (see Lemma 3.3). So we know from the payoff table for $\Gamma_n(n-j+1)$ that

$$\begin{aligned} v_n(n-j+1) &= \frac{2n-2j+1}{2n-2j+2} + \frac{n-j}{n-j+1}v_n(n-j), \\ &= \frac{4n^2 - 4nj - 2n + 4j - 2}{4(n-j+1)(n-1)} + \frac{n-j}{n-j+1}v_n(n-j). \end{aligned}$$

If the hider would move in $\Gamma_n(n-j+1)$, then $v_n(n-j+1)$ would be equal to

$$v_n(n-j+1) = \frac{4n^2 - 4nj - 4n + 5j - 1}{4(n-j+1)(n-1)} + \frac{(n-2)(n-j)}{(n-j+1)(n-1)}v_n(n-1).$$

However it is not optimal for the hider to stay and thus we find

$$\begin{aligned} &\frac{4n^2 - 4nj - 2n + 4j - 2}{4(n-j+1)(n-1)} + \frac{n-j}{n-j+1}v_n(n-j) \\ &\geq \frac{4n^2 - 4nj - 4n + 5j - 1}{4(n-j+1)(n-1)} + \frac{(n-2)(n-j)}{(n-j+1)(n-1)}v_n(n-1). \end{aligned}$$

Next simplifying gives

$$v_n(n-j) \geq \frac{j+1-2n}{4(n-1)(n-j)} + \frac{(n-2)}{(n-1)}v_n(n-1).$$

Now we again use (3.12) to find

$$\frac{n-1}{n-j}v_n(n-1) - \frac{(2n-j-1)(j-1)}{2(n-j)} \geq \frac{j+1-2n}{4(n-1)(n-j)} + \frac{(n-2)}{(n-1)}v_n(n-1).$$

Finally we can rewrite this inequality such that

$$v_n(n-j) \geq \frac{(2nj-2n-2j+1)(2n-j-1)}{4(nj-2j+1)} = \alpha(n, j).$$

So now we can combine the two parts of this proof to conclude $\alpha(n, j) \leq v_n(n-1) \leq \alpha(n, j+1)$. \blacksquare

So as we mentioned before, if we know in which subgames it is optimal to play pure strategies then this theorem will provide us with improved bounds for $v_n(n)$. The numerical results from subsection 3.3.4 suggest it is only optimal to play pure strategies in subgames $\Gamma_n(n)$, $\Gamma_n(n-1)$ and $\Gamma_n(1)$. Therefore these numerical results suggest we can apply Theorem 3.8 with $j = 2$ to obtain

$$\frac{(2n-3)}{4} \leq v_n(n-1) \leq \frac{(4n-5)(n-2)}{2(3n-5)}.$$

and hence using $v_n = \frac{2n-1}{2n} + \frac{n-1}{n}v_n(n-1)$ (see Lemma 3.6),

$$\frac{2n^2-n-1}{4n} \leq v_n \leq \frac{4n^3-11n^2+10n-5}{2n(3n-5)} = \frac{2n}{3} - \frac{\overbrace{13n^2-30n+15}^{> 0, \text{ for } n \geq 4}}{6n(3n-5)} < \frac{2n}{3}$$

such that

$$v_n < \frac{2n}{3}, \text{ for all } n \geq 4 \quad \text{and} \quad \frac{1}{2} \leq \lim_{n \rightarrow \infty} \frac{v_n}{n} \leq \frac{2}{3}.$$

We were not able to prove this, but we expect it to be true.

Figure 3.8 shows there exists an upper bound for the value of j , because for each fixed n there exists a smallest $y_n > 0$ such that $\alpha(n, x)$ is decreasing for $y_n < x \leq n-2$. Since $\alpha(n, j) \geq \alpha(n, j+1)$ must hold (see Theorem 3.8), this implies $j < y_n$. However this upper bound for j does not lead to an improved upper bound for $v_n(n-1)$, because it turns out that $\lim_{n \rightarrow \infty} \frac{\alpha(n, y_n)}{n} = 1$.

3.4 Conclusion

This chapter can be seen as a small introduction to search games on a star with arcs of equal length. We assumed the searcher starts in the origin of star and the hider starts at the end of

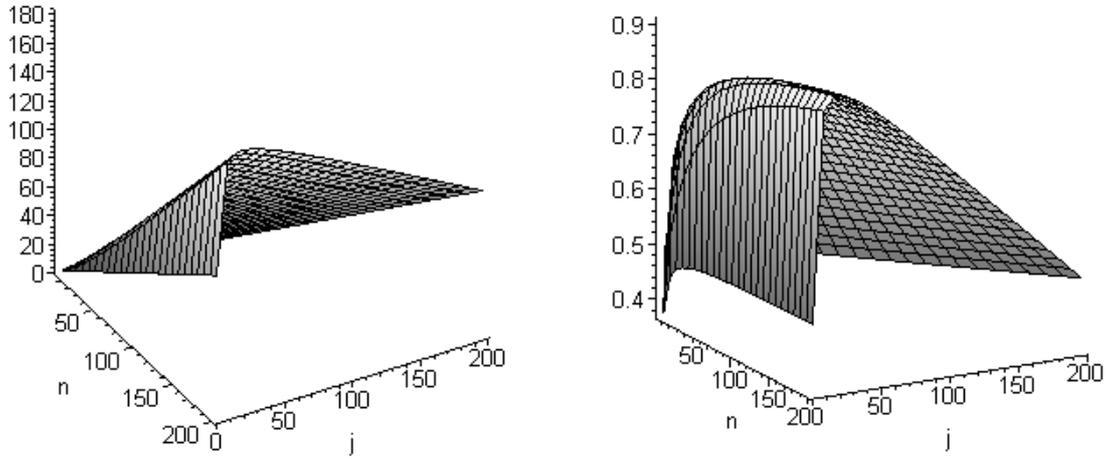


Figure 3.8: Left picture shows $\alpha(n, x)$ and right picture shows $\frac{\alpha(n, x)}{n}$

one of the arcs. Each time step both the searcher and the hider could choose either to adopt a passive strategy (respectively *ambush* and *stay*) or to adopt an active strategy (respectively *search* and *move*).

First we analyzed in Section 3.2 a simple model in which both players did not use knowledge from the past. So in that model the players are playing each time step the same game until the hider is caught by the searcher. It was not hard to solve this problem and it turned out the searcher will always search and the hider will always stay. It appeared this model was too simple and therefore was considered an extended version of the problem in Section 3.3. In this extended model we assumed both players possess some motion detection abilities such that after each time step they know whether or not the opponent has moved in that particular time step. We solved this game and the obtained numerical solutions suggest that the value of the game tends to $\frac{2n}{3}$ for $n \rightarrow \infty$. The numerical results also suggest what will be optimal strategies (see Conjecture 3.7) for both the searcher and the hider. The formulas in this conjecture contain *square roots* and it is interesting to ask ourselves why and how those square roots appear in the extended search game? Another interesting open question for the extended problem is whether or not we can apply Theorem 3.8 with $j = 2$ to conclude $v_n \leq \frac{2n}{3}$?

In the future we can also consider other search games on a star which might lead to new interesting results. We can for instance consider search games on the star in which the players have more advanced motion detection abilities, the initial location of the players is different, the game is played in continuous time, etc. It might also be interesting to introduce a probability that the hider can sneak past the searcher and see how this modification influences the optimal strategies and the expected capture time.

A Computations for Chapter 2

A.1 Catalan numbers grow exponentially

Let C_m be the m^{th} Catalan number, then (see [10])

$$C_m = \frac{1}{m+1} \binom{2m}{m}.$$

We can use Stirling's formula

$$m! \approx \sqrt{2\pi m} \left(\frac{m}{e}\right)^m$$

to approximate C_m for large m . Then we find

$$\begin{aligned} C_m &= \frac{1}{m+1} \binom{2m}{m}, \\ &= \frac{1}{m+1} \frac{(2m)!}{(m!)^2}, \\ &\approx \frac{1}{m+1} \frac{\sqrt{4\pi m}}{2\pi m} \left(\frac{2m}{e}\right)^{2m} \left(\frac{m}{e}\right)^{-2m}, \\ &= \frac{2^{2m}}{(m+1)\sqrt{\pi m}}. \end{aligned}$$

Hence the Catalan numbers grow exponentially.

A.2 Computation for Example 2.1

We have $p_0 < p_1 < \dots < p_{n-1}$ and

$$X(k) = h(a, k) + h(k+1, m+1) + 2(m+1-a) \sum_{i=a}^k \sum_{j=k+1}^{m+1} p_i p_j$$

with

$$h(a, k) = 2 \sum_{j=a}^{k-1} p_j (k-j) (p_{j+1} + \dots + p_k)$$

and

$$h(k+1, m+1) = 2 \sum_{j=k+1}^m p_j (m+1-j) (p_{j+1} + \dots + p_{m+1}).$$

And thus in particular we have

$$X(a) = 2 \sum_{j=a+1}^m p_j (m+1-j) (p_{j+1} + \dots + p_{m+1}) + 2(m+1-a) \sum_{j=a+1}^{m+1} p_a p_j$$

For $a < k < m + 1$ we find

$$\begin{aligned}
X(k) - X(a) &= 2 \sum_{j=a}^{k-1} p_j(k-j)(p_{j+1} + \dots + p_k) + 2 \sum_{j=k+1}^m p_j(m+1-j)(p_{j+1} + \dots + p_{m+1}) \\
&\quad + 2(m+1-a) \sum_{i=a}^k \sum_{j=k+1}^{m+1} p_i p_j \\
&\quad - 2 \sum_{j=a+1}^m p_j(m+1-j)(p_{j+1} + \dots + p_{m+1}) - 2(m+1-a) \sum_{j=a+1}^{m+1} p_a p_j, \\
&= 2 \sum_{j=a}^{k-1} \sum_{l=j+1}^k p_j p_l(k-j) + 2 \sum_{j=k+1}^m \sum_{l=j+1}^{m+1} p_j p_l(m+1-j) + 2(m+1-a) \sum_{i=a}^k \sum_{j=k+1}^{m+1} p_i p_j \\
&\quad - 2 \sum_{j=a+1}^m \sum_{l=j+1}^{m+1} p_j p_l(m+1-j) - 2(m+1-a) \sum_{j=a+1}^{m+1} p_a p_j, \\
&= 2 \sum_{j=a}^{k-1} \sum_{l=j+1}^k p_j p_l(k-j) - 2 \sum_{j=a+1}^k \sum_{l=j+1}^{m+1} p_j p_l(m+1-j) \\
&\quad + 2(m+1-a) \left\{ \sum_{i=a}^k \sum_{j=k+1}^{m+1} p_i p_j - \sum_{j=a+1}^{m+1} p_a p_j \right\}, \\
&= 2 \sum_{l=a+1}^k p_a p_l(k-a) - 2 \sum_{j=a+1}^k \sum_{l=k+1}^{m+1} p_j p_l(m+1-j) - 2 \sum_{j=a+1}^{k-1} \sum_{l=j+1}^k p_j p_l(m+1-k) \\
&\quad + 2(m+1-a) \left\{ \sum_{i=a+1}^k \sum_{j=k+1}^{m+1} p_i p_j - \sum_{j=a+1}^k p_a p_j \right\}.
\end{aligned}$$

Analyzing this last expression reveals that some terms can be combined (sometimes after relabeling the summation), which results in

$$X(k) - X(a) = 2 \sum_{j=a+1}^k \sum_{l=k+1}^{m+1} p_j p_l(j-a) - 2 \sum_{j=a}^{k-1} \sum_{l=j+1}^k p_j p_l(m+1-k).$$

Next we can interchange the order of summation in the second term. Then we find

$$X(k) - X(a) = 2 \sum_{j=a+1}^k \sum_{l=k+1}^{m+1} p_j p_l(j-a) - 2 \sum_{l=a+1}^k \sum_{j=a}^{l-1} p_j p_l(m+1-k).$$

Next relabeling the second term such that j is replaced by l and l is replaced by j gives

$$\begin{aligned}
X(k) - X(a) &= 2 \sum_{j=a+1}^k \sum_{l=k+1}^{m+1} p_j p_l(j-a) - 2 \sum_{j=a+1}^k \sum_{l=a}^{j-1} p_j p_l(m+1-k), \\
&= 2 \sum_{j=a+1}^k p_j \left\{ \sum_{l=k+1}^{m+1} p_l(j-a) - \sum_{l=a}^{j-1} p_l(m+1-k) \right\}.
\end{aligned}$$

Now we can finally use $p_0 < p_1 < \dots < p_{n-1}$ to find

$$\begin{aligned}
X(k) - X(a) &\geq 2 \sum_{j=a+1}^k p_j \left\{ \sum_{l=k+1}^{m+1} p_{k+1}(j-a) - \sum_{l=a}^{j-1} p_{k-1}(m+1-k) \right\}, \\
&= 2 \sum_{j=a+1}^k p_j \{ p_{k+1}(m+1-k)(j-a) - p_{k-1}(m+1-k)(j-a) \}, \\
&= 2 \sum_{j=a+1}^k \underbrace{p_j}_{>0} \underbrace{(m+1-k)}_{>0} \underbrace{(j-a)}_{>0} \underbrace{\{p_{k+1} - p_{k-1}\}}_{>0}, \\
&> 0.
\end{aligned}$$

Consequently $X(a) < X(k)$ for $a < k < m+1$, which completes the proof.

A.3 Computation for Lemma 2.12

Let a, b be positive integers such that $a < b$, then

$$\sum_{k=a}^{b-1} k^2 - \sum_{k=1}^{b-a-1} k^2 = a(b-a)(b-1). \quad (\text{A.1})$$

Proof: Take arbitrary positive integers a, b such that $b = a+1$, then

$$\sum_{k=a}^{b-1} k^2 - \sum_{k=1}^{b-a-1} k^2 = \sum_{k=b-1}^{b-1} k^2 = (b-1)^2, \quad a(b-a)(b-1) = (b-1)^2.$$

Thus (A.1) holds if $b = a+1$. Now suppose (A.1) holds for positive integers a, b such that $b = a+m$ with $m \geq 1$. For $b = a+m+1$ we can write

$$\sum_{k=a}^{b-1} k^2 - \sum_{k=1}^{b-a-1} k^2 = \sum_{k=a}^{a+m} k^2 - \sum_{k=1}^m k^2 = \left(\sum_{k=a}^{a+m-1} k^2 - \sum_{k=1}^{m-1} k^2 \right) + (a+m)^2 - m^2,$$

then using our assumption for $b = a+m$ gives for $b = a+m+1$

$$\sum_{k=a}^{a+m} k^2 - \sum_{k=1}^m k^2 = am(a+m-1) + (a+m)^2 - m^2 = a(m+1)(a+m).$$

The last term in this equation is exactly (A.1) for $b = a+m+1$. Hence by induction to m we can conclude that (A.1) holds for all positive integers a, b with $b > a$. ■

B Computations for Chapter 3

B.1 Computation for Theorem 3.8

We know

$$v_n(n-m+1) = \frac{2n-2m+1}{2n-2m+2} + \frac{n-m}{n-m+1}v_n(n-m), \quad \text{for } m = 2, \dots, j, \quad (\text{B.1})$$

where $2 \leq j \leq n-2$. We would like to prove

$$v_n(n-1) = \frac{(2n-j-1)(j-1)}{2(n-1)} + \frac{n-j}{n-1}v_n(n-j). \quad (\text{B.2})$$

For $j = 2$ it is obvious that this expression is correct, because applying (B.1) with $m = 2$ immediately gives the desired result. So now assume (B.2) holds for $j = k-1 \leq n-3$, then for $j = k$ we find

$$\begin{aligned} v_n(n-1) &= \frac{(2n-k)(k-2)}{2(n-1)} + \frac{n-k+1}{n-1}v_n(n-k+1), \\ &= \frac{(2n-k)(k-2)}{2(n-1)} + \frac{n-k+1}{n-1} \left(\frac{2n-2k+1}{2n-2k+2} + \frac{n-k}{n-k+1}v_n(n-k) \right), \\ &= \frac{(2n-k)(k-2)}{2(n-1)} + \frac{2n-2k+1}{2(n-1)} + \frac{n-k}{n-1}v_n(n-k), \\ &= \frac{(2n-k-1)(k-1)}{2(n-1)} + \frac{n-k}{n-1}v_n(n-k), \end{aligned}$$

which is exactly (B.2) for $j = k$. Hence by using induction, we can now conclude (B.2) is correct for $j = 2, \dots, n-2$. Moreover we can rewrite (B.2) as

$$v_n(n-j) = \frac{n-1}{n-j}v_n(n-1) - \frac{(2n-j-1)(j-1)}{2(n-j)},$$

which is the version used in Theorem 3.8.

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