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Existence of solutions to the generalized periodic fractional boundary value problem

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We study a boundary value problem for a Caputo-type fractional differential equation subjected to periodic boundary conditions. For an auxiliary problem with the simplified right-hand side, we explicitly construct its unique solution. In addition, based on the theory of the topological index, we prove existence of at least one solution to the original problem.

KEYWORDS
existence of solutions, fractional boundary value problem, periodic boundary conditions, topological index

MSC CLASSIFICATION
34A08

1 | INTRODUCTION

In the theory of applied fractional calculus, boundary value problems for periodic[1–6] and anti-periodic[7–10] boundary conditions play an important role. They often occur in mathematical models of the real-world problems, for example, in epidemiology, ecology, physics, and material sciences (see Kilbas et al[11] and Podbulny[12]). This motivates the in-depth study of these types of problems aiming to prove existence and/or uniqueness of their solutions.

Among the already known results in study of the periodic FBVPs including the construction of an explicit solution are those obtained in previous studies.[3–5] For instance, Fečkan and Marynets[4] study the BVP

$$ {}_0^C D_t^p x(t) = f(t, x(t)), \; p \in (0,1), \tag{1} $$

$$ x(0) = x(T), \tag{2} $$

where ${}_0^C D_t^p$ is the generalized Caputo fractional derivative with lower limit at 0 (see Podbulny[12] and Zhou[12]), $t \in [0,T]$, $x : [0,T] \to D$, $f : G \to \mathbb{R}^n$ are continuous functions, $G := [0,T] \times D$ and $D \subset \mathbb{R}^n$ is a closed and bounded domain. Under assumptions that function f in the system (1) is bounded by a constant vector $M = (M_1, M_2, \dots, M_n)^T \in \mathbb{R}^n$ and it satisfies the Lipschitz condition with a non-negative real matrix $K = (k_{ij})_{i,j=1}^n$, they construct a sequence of functions $\{x_m(t, x_0)\}$ given by

$$ x_m(t,x_0) := x_0 + \frac{1}{\Gamma(p)} \left[\int_0^t (t-s)^{p-1} f(s, x_{m-1}(s,x_0)) ds - \left(\frac{t}{T}\right)^p \int_0^T (T-s)^{p-1} f(s, x_{m-1}(s,x_0)) ds \right], \tag{3} $$

that satisfies both the differential equation (1) and the boundary conditions (2). Moreover, they prove that the sequence (3) converges uniformly to the exact solution of (1), (2).

This approach was extended in Fečkan and Marynets[5] to the study of a mixed order FDS

$$ \begin{array}{l} {}_0^C D_t^p x = f(t, x(t), y(t)), \\ {}_0^C D_t^q y = g(t, x(t), y(t)), \end{array} \quad p, q \in (0,1], \tag{4} $$

subjected to periodic boundary conditions

$$ x(0) = x(T), \;\; y(0) = y(T), \tag{5} $$

with its further application to the fractional-order Duffing equation.[14]

The most general case of the periodic FBVP was studied in Fečkan et al,[3] where the authors looked at the differential system

$$ {}_0^c D_t^p x(t) = f(t, x(t)), \, p \in (m, m+1), \, m \in \mathbb{N}, \tag{6} $$

with periodic boundary conditions

$$ \begin{array}{c} x(0) = x(T), \\ x'(0) = x'(T), \\ \dots \\ x^{(m)}(0) = x^{(m)}(T), \end{array} \tag{7} $$

where $t \in [0,T]$, $T > 0$, $x \in C^m([0,T], D)$, $D \subset \mathbb{R}^n$ is open, $f \in C(G, \mathbb{R}^n)$, $G := [0,T] \times D$. They did not only construct an iterative scheme for approximation of solutions to (6), (7) in the form

$$ \begin{aligned} x_{k+1}(t) = \xi_0 + \sum_{j=1}^m \frac{T^{j-1}}{j!} \left[B_j\left(\frac{t}{T}\right) - B_j \right] & \left[-\frac{1}{\Gamma(p-j+1)} \int_0^T (T-s)^{p-j} f(s, x_k(s)) ds \right. \\ & \left. + \frac{(p-m)T^{m-j+1}}{\Gamma(p-j+2)} \int_0^T (T-s)^{p-m-1} f(s, x_k(s)) ds \right] \\ & + \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} f(s, x_k(s)) ds - \frac{(p-m)t^p}{T^{p-m}\Gamma(p+1)} \int_0^T (T-s)^{p-m-1} f(s, x_k(s)) ds, \\ k = 0, 1, \dots, & \end{aligned} \tag{8} $$

for B_j being Bernoulli numbers (see Sebah and Gourdon[15]) but also proved some asymptotic results, based on the Landesman–Lazer-type conditions.

Even though these results provide us with solutions to the periodic FBVPs, but the conditions we put on the right-hand side of the studied FDEs and on the domain itself are very restrictive.

Motivated by Wang et al,[7] in this paper, we prove existence and uniqueness of solutions to a higher order FDE in all $\mathbb{R}$, using techniques of the topological degree (for details, see Gaines and Mawhin[16]). The outline of the paper is the following. In Section 2, we formulate the problem setting and prove some auxiliary existence and uniqueness result for a simplified FDE, where the right-hand side does not depend on the unknown function. Section 3 contains the main result that shows existence and uniqueness of solutions of the studied problem.

2 | HIGHER ORDER PERIODIC FBVP

2.1 | Problem setting

Consider the Caputo-type fractional differential equation

$$ {}_0^C D_t^q x(t) = f(t, x(t)),\ q \in (m-1, m),\ m \in \mathbb{N}, \tag{9} $$

subjected to periodic boundary conditions of the form

$$ x^{(k)}(0) = x^{(k)}(T),\ k \in \overline{0, m-1}, \tag{10} $$

where $t \in [0, T]$, with $T > 1$, $x \in C^{(m-1)}([0, T], \mathbb{R})$, $f \in C([0, T], \mathbb{R})$ and ${}_0^C D_t^p$ being the generalized Caputo fractional derivative with lower limit at 0 (see Podbulny[12]).

Assume that $X = C([0, T])$ is a Banach space with a maximum norm

$$ ||x|| = \max_{t\in[0,T]} \{|x(t)|, |x'(t)|, \dots, |x^{(m-1)}(t)|\}, $$

and let us define operators

$$ \mathcal{L} : \mathrm{dom}\mathcal{L} \to X \text{ and } \mathcal{N} : X \to X, $$

with

$$ \mathrm{dom}\mathcal{L} = \left\{ x \in C^{(m-1)}([0, T]) : {}_0^C D_t^q x(t) \in X, x^{(k)}(0) = x^{(k)}(T), k \in \overline{0, m-1} \right\}, $$

as follows:

$$ \mathcal{L}x := {}_0^c D_t^q x \text{ and } \mathcal{N}(x)(t) := f(t, x(t)). $$

Then, periodic FBVP (9), (10) can be rewritten in an operator form

$$ \mathcal{L}x = \mathcal{N}(x),\ x \in \mathrm{dom}\mathcal{L}. $$

2.2 | Auxiliary results and explicit solution

Consider now an auxiliary periodic FBVP:

$$ {}_0^C D_t^q x(t) = z(t),\ q \in (m-1, m),\ m \in \mathbb{N}, \tag{11} $$

$$ x^{(k)}(0) = x^{(k)}(T),\ k \in \overline{0, m-1}, \tag{10} $$

with the right-hand side being independent of $x(t)$.

For the problem (11), (10), the following result holds.

Theorem 1. *The mapping $\mathcal{L} : dom\mathcal{L} \subset X$ is a Fredholm operator of index zero. Furthermore,*

$$Im\mathcal{L} = \left\{ z \in X : \int_0^T (T-s)^{q-m} z(s)ds = 0 \right\}, \tag{12}$$
$$\ker \mathcal{L} = \{constant\ functions\}.$$

So if a function $z \in Im\mathcal{L}$, that is, it satisfies a relation:

$$\int_0^T (T-s)^{q-m} z(s)ds = 0, \tag{13}$$

then a unique solution $x : [0, T] \to X$ of the FBVP (11), (10) with (13) is given by

$$x(t) = I^q z(t) - \sum_{k=0}^{m-1} b_k t^k, \tag{14}$$

where

$$b_0 = \frac{\Gamma(q-m+2)}{T^{q-m+1}} I^{2q-m+1} z(T) - (q-m+1) \sum_{k=1}^{m-1} B(q-m+1, k+1) b_k T^k, \tag{15}$$

$$b_{j+1} = \frac{1}{(j+1)!T} \left[I^{q-j} z(T) - \sum_{k=j+2}^{m-1} \frac{k!}{(k-j)!} b_k T^{k-j} \right], j = \overline{0, m-3}, \tag{16}$$

$$b_{m-1} = \frac{1}{(m-1)!T} I^{q-(m-2)} z(T), \tag{17}$$

with $B(q-m+1, k+1) = \frac{\Gamma(q-m+1)\Gamma(k+1)}{\Gamma(q-m+k+2)}$ being the Beta function[12] and $I^p y(t)$ is a Riemann–Liouville fractional integral operator of order p, defined by[12]

$$I^p y(t) := \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} y(s)ds.$$

Proof. By Lemma 2.2 in Zhang[9] for $q > 0$, the general solution of the homogeneous FDE

$${}_0^cD_t^q u(t) = 0$$

is given by

$$u(t) = b_0 + b_1 t + \ldots + b_{m-1} t^{m-1}, \; m = [q] + 1,$$

where $b_i, \overline{0, m-1}$ are real constants.

Moreover, using the result of Lemma 2.3 from Zhang,[9] we deduce that the general solution of the perturbed equation (11) has the form (14), where coefficients $b_k, k = \overline{0, m-1}$ to be defined.

From the $(m-1)$th derivative of solution (14), we obtain

$$x^{(m-1)}(t) = I^{q-m+1} z(t) - (m-1)! b_{m-1}.$$

Thus, in order for the boundary condition

$$x^{(m-1)}(0) - x^{(m-1)}(T) = 0$$

to hold, function $z(t)$ in the right-hand side of the FDS (11) should satisfy the relation:

$$\int_0^T (T-s)^{q-m} z(s)ds = 0. \tag{18}$$

On the other hand, assume $z \in X$ satisfying (18), and let

$$x(t) = I^q z(t) - \sum_{k=0}^{m-1} b_k t^k.$$

It is easy to prove that $x(t)$, defined by (14), satisfies the rest of the periodic boundary conditions (10), or that $x \in \mathrm{dom}\mathcal{L}$.

Indeed, substitution of (14) into the first boundary condition in (10) (for $k = 0$) gives:

$$\begin{aligned} x(0) - x(T) &= -b_0 - I^q z(T) + b_0 + b_1 T + \sum_{k=2}^{m-1} b_k T^k = 0, \\ b_1 &= \frac{1}{T}\left[I^q z(T) - \sum_{k=2}^{m-1} b_k T^k \right]. \end{aligned} \tag{19}$$

For $k = 1$, we get

$$\begin{aligned} x'(0) - x'(T) &= -b_1 - I^{q-1} z(T) + b_1 + 2b_2 T + \sum_{k=3}^{m-1} k b_k T^{k-1} = 0, \\ b_2 &= \frac{1}{2!T}\left[I^{q-1} z(T) - \sum_{k=3}^{m-1} k b_k T^{k-1} \right]. \end{aligned} \tag{20}$$

For $2 \le j \le m-3$, we have

$$\begin{aligned} x^{(j)}(0) - x^{(j)}(T) &= -I^{q-j} z(T) + (j+1)! T b_{j+1} + \sum_{k=j+2}^{m-1} \frac{k!}{(k-j)!} b_k T^{k-j} = 0, \\ b_{j+1} &= \frac{1}{(j+1)!T}\left[I^{q-j} z(T) - \sum_{k=j+2}^{m-1} \frac{k!}{(k-j)!} b_k T^{k-j} \right]. \end{aligned} \tag{21}$$

For $k = m-2$,

$$\begin{aligned} x^{(m-2)}(0) - x^{(m-2)}(T) &= -I^{q-(m-2)} z(T) + (m-1)! b_{m-1} T = 0, \\ b_{m-1} &= \frac{1}{(m-1)!T} I^{q-(m-2)} z(T). \end{aligned} \tag{22}$$

Thus, we conclude that function $x(t)$, defined by the relation (14), satisfies periodic boundary conditions (10), where parameters b_k, $k = \overline{1, m-1}$ are in the form (16), (17). In addition, it follows that

$$\mathrm{Im}\mathcal{L} = \left\{ z \in X : \int_0^T (T-s)^{q-m} z(s)ds = 0 \right\}. \tag{23}$$

Consider now two linear operators $\mathcal{P} : X \to X$ and $\mathcal{Q} : X \to X$ defined by

$$\mathcal{P}x(t) = (q-m+1)\int_0^T (T-s)^{q-m} x(s)ds,\ t \in [0, T], \tag{24}$$

and

$$\mathcal{Q}z(t) = (q-m+1)\int_0^T (T-s)^{q-m}z(s)ds,\ t \in [0,T]. \tag{25}$$

For $x \in X$, we get

$$\mathcal{P}(\mathcal{P}x) = \mathcal{P}\left[(q-m+1)\int_0^T (T-s)^{q-m}x(s)ds\right] = (q-m+1)\int_0^T (T-s)^{q-m}x(s)ds = \mathcal{P}x.$$

Thus, $\mathcal{P}^2 = \mathcal{P}$. Similarly, we obtain that $\mathcal{Q}^2 = \mathcal{Q}$. Note that $\mathrm{Im}\mathcal{P} = \ker\mathcal{L}$ and $\ker\mathcal{Q} = \mathrm{Im}\mathcal{L}$.

From the relation

$$\mathrm{ind}\mathcal{L} = \dim\ker\mathcal{L} - \mathrm{codim}\ker\mathcal{L} = 0,$$

it follows that $\mathcal{L}$ is the Fredhold operator of index zero.

Since $x \in \ker\mathcal{P}$, that is,

$$(q-m+1)\int_0^T (T-s)^{q-m}x(s)ds = 0, \tag{26}$$

we deduce that

$$\int_0^T (T-s)^{q-m}\left[I^q z(s) - b_0 - \sum_{k=1}^{m-1} b_k s^k\right] ds = 0, \tag{27}$$

and thus,

$$b_0 = \frac{\Gamma(q-m+2)}{T^{q-m+1}} I^{2q-m+1}z(T) - (q-m+1)\sum_{k=1}^{m-1} B(q-m+1,k+1)b_k T^k, \tag{28}$$

where b_k, $k = \overline{1, m-1}$ are defined by (16), (17).

Substituting b_0 into (14), we obtain the unique solution to the FBVP (11), (10). □

Example. To verify the result of Theorem 1, consider a periodic FBVP of the form:

$${}_0^C D_t^{2.5} x(t) = -3t + 2\ \left(:= z(t)\right),\ t \in [0,1], \tag{29}$$

$$x(0) = x(1),\ x'(0) = x'(1),\ x''(0) = x''(1), \tag{30}$$

where $m = 3$.

It is easy to check that function $z(t)$ satisfies condition (13), and thus, solution $x(t)$ of (29), (30) can be written in the form:

$$x(t) = I^{2.5}z(t) - b_0 - b_1 t - b_2 t^2,$$

where parameters b_i, $i = \overline{0, 2}$ to be calculated following the process presented in the proof of Theorem 1.

Using mathematical software Maple 2022, we find that

$$b_0 = \frac{\pi}{2} + \frac{352}{105},\ b_1 = \frac{8}{105\sqrt{\pi}},\ b_2 = \frac{8}{15\sqrt{\pi}},$$

and thus, solution $x(t)$ of the periodic FBVP (29), (30) can be written as

$$x(t) = \frac{-96t^{3.5} + 224t^{2.5} - 112t^2 - 16t - 704\pi^{0.5} - 105\pi^{1.5}}{210\sqrt{\pi}}. \tag{31}$$

Direct substitution of (31) into (29), (30) shows that $x(t)$ indeed satisfies both. As an additional verification, we use the fact that integration of the FDE (29) leads to a solution of the form:

$$x(t) = x(0) + tx'(0) + \frac{t^2}{2}x''(0) + I^{2.5}z(t). \tag{32}$$

Substituting (31) into (32) and comparing its left- and right-hand sides, we conclude that they coincide.

Thus, we have demonstrated on a concrete example of a periodic FBVP that under conditions of Theorem 1, solution to the problem indeed has the form (14) with coefficients b_i, $i = \overline{0, m-1}$ being calculated according to formulas (15)–(17).

3 | EXISTENCE RESULT

Consider now a projection $\mathcal{P} : X \to X$ defined by

$$\mathcal{P}x(t) = (q - m + 1)\int_0^T (T-s)^{q-m}x(s)ds, \; t \in [0, T]. \tag{33}$$

Let us set $\mathcal{Q} = \mathcal{I} - \mathcal{P}$ and note that $\ker\mathcal{Q} = \ker\mathcal{L}$ and $\ker\mathcal{P} = \mathrm{Im}\mathcal{L}$. Additionally, we denote by $\mathcal{L}^{-1} : \ker\mathcal{P} \to \ker\mathcal{P} \cap \mathrm{dom}\mathcal{L} \subset X$ an inverse operator given by (14).

The following existence result holds.

Theorem 2. *Assume that there exist positive constants M_1 and k_0 such that $|f(t,x)| \le M_1$, for $t \in [0,T]$, $x \in \mathbb{R}$, and*

$$\begin{gathered} \textit{either} \\ \int_0^T (T-s)^{q-m}f(s,x)ds > 0, \; \forall x \ge k_0, \quad \int_0^T (T-s)^{q-m}f(s,x)ds < 0, \; \forall x \le -k_0, \\ \textit{or} \\ \int_0^T (T-s)^{q-m}f(s,x)ds < 0, \; \forall x \ge k_0, \quad \int_0^T (T-s)^{q-m}f(s,x)ds > 0, \; \forall x \le -k_0. \end{gathered} \tag{34}$$

Then, the problem (9), (10) has at least one solution.

Proof. From Theorem 1, it follows that function $x(t)$ is a solution of the periodic FBVP (9), (10) if and only if $\mathcal{P}\mathcal{N}(x) = 0$ and then $\mathcal{Q}x = \mathcal{L}^{-1}\mathcal{Q}\mathcal{N}(x)$, which is equivalent to an equation

$$\mathcal{T}(x) = \mathcal{P}\mathcal{N}(x) + \mathcal{Q}x - \mathcal{L}^{-1}\mathcal{Q}\mathcal{N}(x) = 0. \tag{35}$$

Note that $\mathcal{Q}x - \mathcal{L}^{-1}\mathcal{Q}\mathcal{N}(x) \in \ker\mathcal{P}$.

Next, we will show that the operator $\mathcal{T} : X \to X$ is completely continuous.

Let $B \subset X$ be a bounded set. By the assumption that $|f(t, x(t))| \le M_1$, for $x \in B$, we deduce

$$\begin{aligned} |\mathcal{T}x(t)| &\le \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}|f(s,x(s))|ds + |b_0| + \sum_{k=1}^{m-1}|b_k|T^k \\ &\le M_1T^q\left[\frac{1}{\Gamma(q+1)} + \frac{\Gamma(q-m+2)}{\Gamma(2q-m+2)} + \sum_{k=1}^{m-1}\left\{\frac{\Gamma(q-m+2)}{\Gamma(q-m+k+2)\Gamma(q-k)} + \frac{1}{k!\Gamma(q-k)}\right\}\right] := M_2. \end{aligned} \tag{36}$$

Indeed, calculations show that the following estimates hold:

$$\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}|f(s,x(s))|ds \le \frac{M_1}{\Gamma(q)}\int_0^t (t-s)^{q-1}ds = \frac{M_1 t^q}{\Gamma(q+1)} \le \frac{M_1 T^q}{\Gamma(q+1)}$$

$$\begin{aligned}|b_{m-1}|T^{m-1} &\le \frac{T^{m-2}}{(m-1)!\Gamma(q-m+2)}\int_0^T (T-s)^{q-m+1}|f(s,x(s))|ds\\ &\le \frac{M_1 T^{m-2}}{(m-1)!\Gamma(q-m+2)}\int_0^T (T-s)^{q-m+1}ds = \frac{M_1 T^q}{(m-1)!\Gamma(q-m+3)},\end{aligned}$$

and thus,

$$\begin{aligned}|b_{m-1}| &\le \frac{M_1 T^{q-m+1}}{(m-1)!\Gamma(q-m+1)}\cdot\frac{1}{(q-m+2)(q-m+1)} < \frac{M_1 T^{q-m+1}}{(m-1)!\Gamma(q-m+1)},\\ |b_{m-2}| &\le \frac{M_1 T^{q-m+2}}{(m-2)!\Gamma(q-m+2)}\left[\frac{1}{(q-m+3)(q-m+2)} + \frac{1}{2!(q-m+2)}\right] < \frac{M_1 T^{q-m+2}}{(m-2)!\Gamma(q-m+2)},\\ |b_{m-3}| &\le \frac{M_1 T^{q-m+3}}{(m-3)!\Gamma(q-m+3)}\left[\frac{1}{(q-m+3)(q-m+4)} + \frac{1}{2!}\left\{\frac{1}{(q-m+3)} + \frac{1}{2!}\right\} + \frac{1}{3!}\right] < \frac{M_1 T^{q-m+3}}{(m-3)!\Gamma(q-m+3)}.\end{aligned}$$

Continuing computations further, we obtain that

$$\begin{aligned}|b_j| &\le \frac{M_1 T^{q-j}}{j!\Gamma(q-j)},\ j=\overline{1,m-1},\\ |b_0| &\le \frac{\Gamma(q-m+2)M_1 T^q}{\Gamma(2q-m+2)} + \sum_{k=1}^{m-1}\frac{\Gamma(q-m+2)M_1 T^q}{\Gamma(q-m+k+2)\Gamma(q-k)}.\end{aligned}$$

Furthermore, calculations show that the inequality holds:

$$\begin{aligned}|(\mathcal{T}x)'(t)| &\le \frac{1}{\Gamma(q-1)}\int_0^t (t-s)^{q-2}|f(s,x(s))|ds + \sum_{k=1}^{m-1}|kb_k|T^{k-1}\\ &\le M_1 T^{q-1}\left[\frac{1}{\Gamma(q)} + \sum_{k=1}^{m-1}\frac{1}{(k-1)!\Gamma(q-k)}\right] := M_3.\end{aligned} \tag{37}$$

Hence, for $t_1, t_2 \in [0,T]$, we conclude that

$$|(\mathcal{T}x)(t_2) - (\mathcal{T}x)(t_1)| \le \int_{t_1}^{t_2}|(\mathcal{T}x)'(s)|ds \le M_3(t_2 - t_1). \tag{38}$$

This proves that $\mathcal{T}$ is equicontinuous on $[0,T]$. Hence, by Arzela–Ascoli theorem, the operator $\mathcal{T} : X \to X$ is completely continuous.

Now, we need to show that there exists at least one solution $x \in C[0,T]$ satisfying (35). For this purpose, we take a homotopy

$$\mathcal{T}_\lambda(x) = \mathcal{P}\mathcal{N}(x) + \mathcal{Q}x - \lambda\mathcal{L}^{-1}\mathcal{Q}\mathcal{N}(x),\ \lambda \in [0,1], \tag{39}$$

and consider the set

$$\Omega = \{x \in X : \|\mathcal{P}x\| \le k_1,\ \|\mathcal{Q}x\| \le k_2\},$$

for

$$k_1 = k_0 + k_2,\ k_2 = \|\mathcal{L}^{-1}\|\|\mathcal{Q}\|M_1 + 1.$$

Then, we claim

$$\mathcal{T}_\lambda(x) \neq 0, \forall x \in \partial\Omega. \tag{40}$$

Indeed, if $\mathcal{T}_\lambda(x) = 0$ for $\|\mathcal{Q}x\| = k_2$, then

$$\|\mathcal{L}^{-1}\|\|\mathcal{Q}\|M_1 + 1 = k_2 = \|\mathcal{Q}x\| = \|\lambda\mathcal{L}^{-1}\mathcal{Q}\mathcal{N}(x)\| \leq \|\mathcal{L}^{-1}\|\|\mathcal{Q}\|M_1,$$

which is a contradiction. If $\mathcal{T}_\lambda(x) = 0$ for $\|\mathcal{P}x\| = k_1$, then

$$|x(t)| \geq |\mathcal{P}x| - \|\mathcal{Q}x\| \geq k_1 - k_2 = k_0,$$

and

$$0 = \int_0^T (T-s)^{q-m} f(s, x(s))ds \neq 0,$$

by (34), which is again a contradiction.

Thus, we have

$$\begin{aligned} \deg(\mathcal{T}, \Omega, 0) &= \deg(\mathcal{T}_0, \Omega, 0) = \deg(\mathcal{P}\mathcal{N} + \mathcal{Q}, \Omega, 0) \\ &= \deg(\mathcal{P}\mathcal{N}, \Omega \cap \ker \mathcal{N}, 0) = \deg\left(\int_0^T (T-s)^{q-m} f(s,x)ds, (-k_0, k_0), 0\right) = \pm 1 \neq 0. \end{aligned}$$

This means that the operator $\mathcal{T}$ has at least one zero point in Ω, which implies that (9), (10) has at least one solution. □

We present an example. Consider the Caputo-type fractional differential equation

$$ {}_0^C D_t^q x(t) = \mu \tanh x(t) + \nu \cos t,\ q \in (m-1, m),\ m \in \mathbb{N}, \tag{41}$$

for $\mu, \nu \in \mathbb{R}\backslash\{0\}$ and subjected to periodic boundary conditions

$$x^{(k)}(0) = x^{(k)}(2\pi), k \in \overline{0, m-1}. \tag{42}$$

We set $\alpha = q - m \in (0, 1)$. Then, $T = 2\pi$, $f(x, t) = \mu \tanh x + \nu \cos t$, and hence,

$$\begin{aligned} \int_0^{2\pi} (2\pi - s)^\alpha f(s, x)ds &= \int_0^{2\pi} (2\pi - s)^\alpha(\mu \tanh x + \nu \cos s)ds = \mu \tanh x \int_0^{2\pi} (2\pi - s)^\alpha ds + \nu \int_0^{2\pi} (2\pi - s)^\alpha \cos s)ds \\ &= \mu \frac{(2\pi)^{\alpha+1}}{\alpha+1} \tanh x + \nu \int_0^{2\pi} s^\alpha \cos s ds \to_{x\to\pm\infty} \pm\mu\frac{(2\pi)^{\alpha+1}}{\alpha+1} + \nu \int_0^{2\pi} s^\alpha \cos s ds. \end{aligned}$$

Thus, condition (34) is satisfied if it holds

$$-\mu^2 \frac{(2\pi)^{2(\alpha+1)}}{(\alpha+1)^2} + \nu^2 \left(\int_0^{2\pi} s^\alpha \cos s ds\right)^2 < 0,$$

which is equivalent to

$$\frac{\alpha+1}{(2\pi)^{\alpha+1}} \left|\int_0^{2\pi} s^\alpha \cos s ds\right| = \left|{}_1F_2\left(1; \frac{a}{2}+1, \frac{a}{2}+\frac{3}{2}; -\pi^2\right)\right| < \frac{|\mu|}{|\nu|}, \tag{43}$$

where ${}_1F_2$ is a generalized hypergeometric function. By applying Theorem 2, we arrive at the following result.

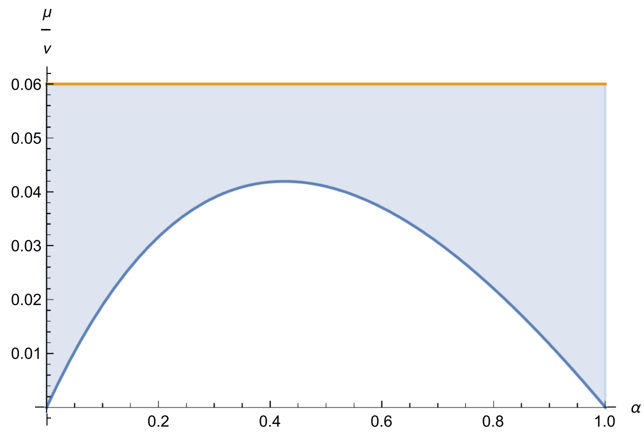

FIGURE 1 Graph of (43). [Colour figure can be viewed at wileyonlinelibrary.com]

Theorem 3. *If condition (43) holds, then (41) with (42) has a solution.*

Using the expansion

$$\int_0^{2\pi} s^\alpha \cos s ds = \sum_{k=0}^{\infty} (-1)^k \frac{(2\pi)^{2k+\alpha+1}}{(2k)!(2k+\alpha+1)},$$

we can approximately compute (43) with the graph on Figure 1.

4 | SYSTEMS OF FDE

The result of Theorem 2 can be generalized to the case of systems of FDEs. So we consider the case $f \in C([0,T],\mathbb{R}^n)$ for $n \geq 2$. Then, conditions (34) need to be modified by evaluating the quantity $|x|$ instead of x itself. To be more concrete, we have the following results.

Theorem 4. *Assume that there exist positive constants M_1 and k_0 such that $|f(t,x)| \leq M_1$, for $t \in [0,T]$, $x \in \mathbb{R}^n$ such that*

$$\int_0^T (T-s)^{q-m} f(s,x)ds \neq 0, \ \forall |x| \geq k_0, \tag{44}$$

and

$$\deg\left(\int_0^T (T-s)^{q-m} f(s,x)ds, B(k_0), 0\right) \neq 0, \tag{45}$$

for the ball $B(k_0) = \{x \in \mathbb{R}^n : |x| \leq k_0\}$. Then, the problem (9), (10) has at least one solution.

Proof. The result follows directly from the proof of Theorem 2. □

Theorem 5. *Assume there is a scalar product $(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ and a positive constant k_0 such that*

$$\int_0^T (T-s)^{q-m}(f(s,x)ds, x) \neq 0, \ \forall |x| \geq k_0. \tag{46}$$

Then, the problem (9), (10) has at least one solution.

Proof. We apply Theorem 4. Clearly, (46) implies (44). Since the set $\{x \in \mathbb{R}^n : |x| \geq k_0\}$ is connecting, we have the following two possibilities:

1. $\int_0^T (T-s)^{q-m}(f(s,x)ds, x) > 0$, $\forall |x| \geq k_0$. Then, we derive

$$\left(\lambda \int_0^T (T-s)^{q-m} f(s,x)ds + (1-\lambda)x\right) = \lambda \int_0^T (T-s)^{q-m}(f(s,x)ds, x) + (1-\lambda)(x,x) > 0,$$

for $\lambda \in [0,1]$ and $|x| \geq k_0$. This implies

$$\deg\left(\int_0^T (T-s)^{q-m} f(s,x)ds, B(k_0), 0\right) = \deg(I, B(k_0), 0) = 1 \neq 0.$$

Hence, (45) holds.

2. $\int_0^T (T-s)^{q-m}(f(s,x)ds, x) < 0$, $\forall |x| \geq k_0$. Then, we derive

$$\left(\lambda \int_0^T (T-s)^{q-m} f(s,x)ds - (1-\lambda)x\right) = \lambda \int_0^T (T-s)^{q-m}(f(s,x)ds, x) - (1-\lambda)(x,x) < 0,$$

for $\lambda \in [0,1]$ and $|x| \geq k_0$. This implies

$$\deg\left(\int_0^T (T-s)^{q-m} f(s,x)ds, B(k_0), 0\right) = \deg(-I, B(k_0), 0) = (-1)^n \neq 0.$$

Hence, (45) holds. The proof is finished.

□

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CONFLICT OF INTEREST STATEMENT

This work does not have any conflicts of interest.

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