Master of Science Thesis

Energy, vorticity and enstrophy conserving mimetic spectral method for the Euler equation

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5 September 2013

Faculty of Aerospace Engineering · Delft University of Technology

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Summary

The behaviour of an inviscid, constant density fluid on which no body forces act, may be modelled by the two-dimensional incompressible Euler equations, a non-linear system of partial differential equations. If a fluid whose behaviour is described by these equations, is confined to a space where no fluid flows in or out, the kinetic energy, vorticity integral and enstrophy integral within that space remain constant in time. Solving the Euler equations accompanied by appropriate boundary and initial conditions may be done analytically, but more often than not, no analytical solution is available. The solution, however, can always be approximated and usually a computer is used to provide a numerical approximation.

However, so far no numerical technique exists that yields a solution which actually keeps both the kinetic energy, the vorticity integral and the enstrophy integral constant. This means that for long-time simulations, the numerical approximation of the velocity or vorticity field will not at all accurately represent the solution to the problem at hand.

In this thesis we develop a numerical method that approximates the two-dimensional incompressible Euler equations such that all of those invariants are conserved within a fixed space where no fluid flows in or out. We first rewrite the Euler equations and the corresponding vorticity equation in terms of differential geometry and then discretize the resulting equations using a structure preserving spectral method on a square domain. The numerical tests performed show conservation of kinetic energy, vorticity integral and enstrophy integral. Deviations from their initial values, which are of the order one, were always less than $1.2 \cdot 10^{-10}$ and may be attributed to the accuracy of the implementation of the method instead of the method itself. Future work may consist of applying the method to a benchmark flow problem like the Taylor-Green vortex, creating a spectral element method based on the described spectral method and adding the viscous terms from the Navier-Stokes equations to our method.

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Chapter 1

Introduction

In many branches of physics, partial differential equations are used to model nature's behaviour, and the branch of aerodynamics is no exception. Solving a set of partial differential equations accompanied by appropriate boundary or initial conditions may be done analytically, but more often than not, no analytical solution is available. The solution, however, can always be approximated and usually a computer is used to provide a numerical approximation. Many numerical techniques exist, such as the finite difference method, the finite volume method and the finite element method.

The behaviour of an inviscid, constant density fluid on which no body forces act, may be modelled by the two-dimensional incompressible Euler equations. If a fluid whose behaviour is described by these equations, is confined to a space where no fluid flows in or out, the kinetic energy, vorticity integral and enstrophy integral within that space remain constant in time. However, so far no numerical technique exists that yields a solution which actually keeps all of those invariants constant. This means that for long-time simulations, the numerical approximation of the velocity or vorticity field will not at all accurately represent the solution to the problem at hand. Furthermore, Arakawa [1] shows that only if both kinetic energy and vorticity integral are conserved, there is a constraint on the change in spectral energy, namely that there is an average wave number that should remain constant.

Perot [11] and Zhang et al. [12] have shown conservation of both kinetic energy and vorticity integral for unstructured staggered mesh methods using the rotational form of the Navier-Stokes equation, but the enstrophy integral is not conserved in their method. Elcott et al. [3] consider vorticity to be a 2-form and, using concepts from algebraic topology and differential geometry, develop a structure-preserving method that conserves the vorticity integral along any discrete loop in the mesh. Employing ideas from those branches of mathematics as well, Mullen et al. [9], with the purpose of fluid animation in computer graphics in mind, have developed a kinetic energy conserving method for the Euler equations. Pavlov et al. [10] have combined concepts from differential geometry and variational mechanics into a structure-preserving discretization method for the Euler equations. Their result shows good, but not exact, long-term energy conservation.

Based on this work, Gawlik et al. [5] have extended the application of their method to magnetohydrodynamics and complex fluids.

The purpose of our research is to develop a numerical method that approximates the two-dimensional incompressible Euler equations such that kinetic energy, vorticity integral and enstrophy integral are conserved within a fixed space where no fluid flows in or out. We will do so by first rewriting the Euler equations and the corresponding vorticity equation in terms of differential geometry and then discretizing the resulting equations using a structure preserving spectral method on a square domain.

Chapter 2 will introduce the Euler equations and the corresponding vorticity equation. We will at the continuous level derive the conservation laws that we want to satisfy discretely and we will mention how differential geometry and algebraic topology relate to our problem. A short introduction to differential geometry is then given in chapter 3, which concludes with a restatement of the Euler equations and the vorticity equation in terms of the newly developed language. They are subsequently discretized in chapter 4, where we show how the method is constructed such that kinetic energy and enstrophy integral are conserved and how this also implies conservation of vorticity integral. Chapter 5 shows the results obtained from an implementation of the method developed and in chapter 6 we compare the results to the goal of our work and we do some suggestions concerning future work.

Chapter 2

The Euler equations

In this chapter we derive from the two-dimensional incompressible Euler equations the three conservation laws, namely conservation of kinetic energy, vorticity integral and enstrophy integral, which we strive to satisfy in the numerical method that we will develop. After that, we will touch upon how concepts from vector calculus may be represented in terms of differential geometry.

In this thesis we consider two-dimensional incompressible fluid flows governed by the Euler equations. In familiar terms, they are written as

$$\begin{aligned} \nabla \cdot \mathbf{v} &= 0, \\ \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} + \nabla p &= 0. \end{aligned} \tag{2.1}$$

These are the continuity equation and the momentum equation respectively. Taking the curl of the latter yields the related vorticity equation,

$$\frac{\partial \omega}{\partial t} + (\mathbf{v} \cdot \nabla)\omega = 0. \tag{2.2}$$

If the behaviour of a fluid in a domain Ω is described by these equations and the normal velocity at the boundary of the domain $\partial\Omega$ is zero, then some global conservation laws hold:

$$\begin{aligned} \frac{d}{dt}\int_{\Omega} \frac{1}{2}||\mathbf{v}||^2 \, d\Omega &= 0, \\ \frac{d}{dt}\int_{\Omega} \omega \, d\Omega &= 0, \\ \frac{d}{dt}\int_{\Omega} \frac{1}{2}\omega^2 \, d\Omega &= 0, \end{aligned}$$

representing conservation of kinetic energy K, vorticity integral V and enstrophy integral E respectively, where

$$K = \int_\Omega \frac{1}{2}||\mathbf{v}||^2 \, d\Omega,$$
$$V = \int_\Omega \omega \, d\Omega,$$
$$E = \int_\Omega \frac{1}{2}\omega^2 \, d\Omega.$$

Conservation of kinetic energy can be shown by taking the inner product of equation 2.1 with $\mathbf{v}$ and integrating over the domain, yielding

$$\begin{aligned} 0 &= \int_\Omega \mathbf{v} \cdot \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \{(\mathbf{v} \cdot \nabla)\mathbf{v}\} + \mathbf{v} \cdot \nabla p \, d\Omega \\ &= \int_\Omega \mathbf{v} \cdot \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \{\nabla \cdot [\mathbf{v} \otimes \mathbf{v}]\} + \mathbf{v} \cdot \nabla p \, d\Omega. \end{aligned} \tag{2.3}$$

We may write $\mathbf{v} \cdot \{\nabla \cdot [\mathbf{v} \otimes \mathbf{v}]\} = \nabla \cdot \left(\mathbf{v}\frac{1}{2}||\mathbf{v}||^2\right) + \frac{1}{2}||\mathbf{v}||^2 \nabla \cdot \mathbf{v}$, for which a hint of a proof in Cartesian coordinates is that both sides of the equality sign reduce to $(2u^2 + v^2)\frac{\partial u}{\partial x} + uv\frac{\partial u}{\partial y} + uv\frac{\partial v}{\partial x} + (u^2 + 2v^2)\frac{\partial v}{\partial y}$ if $\mathbf{v}$ has components u and v, where we used that $\nabla \cdot \mathbf{v} = 0$. Equation 2.3 may now be stated as

$$\begin{aligned} 0 &= \int_\Omega \mathbf{v} \cdot \frac{\partial \mathbf{v}}{\partial t} + \nabla \cdot \left(\mathbf{v}\frac{1}{2}||\mathbf{v}||^2\right) + \frac{1}{2}||\mathbf{v}||^2 \nabla \cdot \mathbf{v} + \mathbf{v} \cdot \nabla p \, d\Omega \\ &= \int_\Omega \mathbf{v} \cdot \frac{\partial \mathbf{v}}{\partial t} + \frac{1}{2}||\mathbf{v}||^2 \nabla \cdot \mathbf{v} + \mathbf{v} \cdot \nabla \left(\frac{1}{2}||\mathbf{v}||^2\right) + \frac{1}{2}||\mathbf{v}||^2 \nabla \cdot \mathbf{v} + \mathbf{v} \cdot \nabla p \, d\Omega \\ &= \int_\Omega \mathbf{v} \cdot \frac{\partial \mathbf{v}}{\partial t} + ||\mathbf{v}||^2 \nabla \cdot \mathbf{v} + \mathbf{v} \cdot \nabla \left(p + \frac{1}{2}||\mathbf{v}||^2\right) \, d\Omega. \end{aligned}$$

Again using that $\nabla \cdot \mathbf{v} = 0$ and applying higher-dimensional integration by parts results in

$$\begin{aligned} 0 &= \int_\Omega \mathbf{v} \cdot \frac{\partial \mathbf{v}}{\partial t} \, d\Omega + \int_{\partial\Omega} \left(p + \frac{1}{2}||\mathbf{v}||^2\right) \mathbf{v} \cdot \mathbf{n} \, d\Gamma - \int_\Omega \left(p + \frac{1}{2}||\mathbf{v}||^2\right) \nabla \cdot \mathbf{v} \, d\Omega \\ &= \int_\Omega \mathbf{v} \cdot \frac{\partial \mathbf{v}}{\partial t} \, d\Omega \end{aligned}$$

since $\mathbf{v} \cdot \mathbf{n} = 0$ on the boundary $\partial\Omega$ and $\nabla \cdot \mathbf{v} = 0$. If Ω does not change in time, we find that

$$\int_{\Omega} \mathbf{v} \cdot \frac{\partial \mathbf{v}}{\partial t} \, d\Omega = \int_{\Omega} \frac{\partial}{\partial t} \frac{1}{2} ||\mathbf{v}||^2 \, d\Omega = \frac{d}{dt} \int_{\Omega} \frac{1}{2} ||\mathbf{v}||^2 \, d\Omega = 0.$$

The global kinetic energy does not change over time.

In order to show conservation of the vorticity integral, we integrate the vorticity equation over the domain. Remembering that $\nabla \cdot \mathbf{v} = 0$ we may write

$$(\mathbf{v} \cdot \nabla)\omega = (\mathbf{v} \cdot \nabla)\omega + \omega(\nabla \cdot \mathbf{v}) = \nabla \cdot (\omega \mathbf{v}),$$

so using that $\mathbf{v} \cdot \mathbf{n} = 0$ on the boundary, we have, if Ω does not change in time,

$$\begin{aligned} 0 &= \int_{\Omega} \frac{\partial \omega}{\partial t} + (\mathbf{v} \cdot \nabla)\omega \, d\Omega \\ &= \int_{\Omega} \frac{\partial \omega}{\partial t} + \nabla \cdot (\omega \mathbf{v}) \, d\Omega \\ &= \int_{\Omega} \frac{\partial \omega}{\partial t} \, d\Omega + \int_{\partial\Omega} \omega \mathbf{v} \cdot \mathbf{n} \, d\Gamma \\ &= \frac{d}{dt} \int_{\Omega} \omega \, d\Omega. \end{aligned} \tag{2.4}$$

This shows that the integral of the vorticity is conserved.

We demonstrate conservation of the integral of the enstrophy by multiplying the vorticity equation by ω and integrating the resulting equation over the domain. Using the identity

$$\begin{aligned} \omega(\mathbf{v} \cdot \nabla)\omega &= \mathbf{v}\omega \cdot \nabla\omega \\ &= \mathbf{v} \cdot \nabla \left(\frac{1}{2}\omega^2 \right) \\ &= \nabla \cdot \left(\frac{1}{2}\omega^2 \mathbf{v} \right) - \frac{1}{2}\omega^2 (\nabla \cdot \mathbf{v}) \\ &= \nabla \cdot \left(\frac{1}{2}\omega^2 \mathbf{v} \right), \end{aligned}$$

where again we took advantage of $\nabla \cdot \mathbf{v} = 0$, yields

$$\begin{aligned}
0 &= \int_\Omega \omega \frac{\partial \omega}{\partial t} + \omega(\mathbf{v}\cdot\nabla)\omega \, d\Omega \\
&= \int_\Omega \frac{\partial}{\partial t}\left(\frac{1}{2}\omega^2\right) + \nabla\cdot\left(\frac{1}{2}\omega^2\mathbf{v}\right) \, d\Omega \\
&= \frac{d}{dt}\int_\Omega \frac{1}{2}\omega^2 \, d\Omega + \int_{\partial\Omega} \frac{1}{2}\omega^2 \mathbf{v}\cdot\mathbf{n} \, d\Gamma \\
&= \frac{d}{dt}\int_\Omega \frac{1}{2}\omega^2 \, d\Omega
\end{aligned}$$

for domains that do not change in time and again since $\mathbf{v}\cdot\mathbf{n} = 0$ at the boundary. This demonstrates conservation of the enstrophy integral.

In the derivations of each conservation law we used the identity $\nabla\cdot\mathbf{v} = 0$ many times. This is the expression stating that the velocity field is divergence free. The concept of divergence is elegantly generalised in the mathematical branch of differential geometry: given an n-dimensional vector field $\mathbf{v}$ on a manifold M^n, the divergence of $\mathbf{v}$, $\operatorname{div}\mathbf{v}$, is such that

$$\mathcal{L}_\mathbf{v} \operatorname{vol}^n = (\operatorname{div}\mathbf{v}) \operatorname{vol}^n \tag{2.5}$$

where $\mathcal{L}_\mathbf{v}$ is the Lie derivative with respect to $\mathbf{v}$ and vol^n is the n-dimensional volume form. The Lie derivative with respect to $\mathbf{v}$ on volume forms is given by $d\imath_\mathbf{v}$, where d is the exterior derivative and $\imath_\mathbf{v}$ is the interior product with respect to $\mathbf{v}$.

If a vector field $\mathbf{v}$ is divergence free, then application of $d\imath_\mathbf{v}$ to the volume form yields zero, and since for any differential form $\alpha^{(k)}$, $dd\alpha^{(k)}$, there is an $(n-2)$-form $\psi^{(n-2)}$ such that $d\psi^{(n-2)} = q^{(n-1)} = \imath_\mathbf{v} \operatorname{vol}^n$. The latter is a closed form since taking its exterior derivative returns zero. $q^{(n-1)}$ represents the divergence free velocity field and describes fluxes through the boundary of volumes.

In vector calculus the relation between vorticity and velocity is written as $\omega = \nabla\times\mathbf{v}$ whereas in differential geometry, it reads $\omega^{(2)} = dv^{(1)}$. $v^{(1)}$ is also a representation of the velocity field, but it is not equal to $q^{(n-1)}$. The two representations are related by the Hodge star operator $*$ such that $q^{(n-1)} = *v^{(1)}$. $v^{(1)}$ describes the velocity along curves: the integral of the vorticity over an area equals the integral of the tangential velocity along the boundary of that area.

A natural way to deal with the discretization of the different velocity representations, those along lines and through volume boundaries, is to use two cell complexes, a concept provided by algebraic topology. An n-dimensional cell complex is a description of how points and other k-dimensional objects, such as lines, surfaces and volumes, are connected, where $1 \le k \le n$ and where the boundary of a k-dimensional object consists of $(k-1)$-dimensional objects. $v^{(1)}$ is represented on an inner oriented cell complex and $q^{(n-1)}$ on an outer oriented cell complex. One cell complex, the primal cell complex, is dual to the other cell complex, the dual cell complex, in the sense that a point in one cell complex corresponds to a volume in the other cell complex, a line in one cell complex corresponds to an $(n-1)$-dimensional hypersurface in the other cell complex, etc. There is no general choice for whether the primal or dual cell complex is the inner oriented one and which is the outer oriented one.

The paragraphs above the previous one have mentioned many concepts from differential geometry. Furthermore, a multitude of operators in differential geometry correspond to equivalent discrete operators in algebraic topology. The exterior derivative, for example, finds its discrete equivalent in the coboundary operator. The reader interested in the link between differential geometry and algebraic topology is refered to [8]. This motivates us to rewrite the Euler equations and the vorticity equation in terms of differential geometry. The next chapter will very shortly introduce some of the concepts belonging to that branch of mathematics, leaving out many bits that are less relevant for this thesis.

Chapter 3

Differential geometry

This chapter introduces some concepts from the mathematical branch of differential geometry. First we see how vectors operate on functions, motivating us to use differential operators as basis vectors. After that we introduce the concepts of differential forms, exterior derivative, interior product, Lie derivative, Lie bracket and Hodge star operator. Stokes' theorem then generalises theorems like the fundamental theorem of calculus, Green's theorem, the curl theorem and the divergence theorem. Next we define pointwise and global scalar products and finally, we rewrite the Euler equations and the corresponding vorticity equation. This chapter is based on [2] and [4].

3.1 Vectors

Let p be a point on an n-dimensional manifold M^n and let f be a real-valued function near p.

Definition 1. *[4] If* $\mathbf{X}$ *is a vector at* p *we define the derivative of* f *with respect to the vector* $\mathbf{X}$ *by*

$$\mathbf{X}_p(f) \doteq D_{\mathbf{X}}(f) \doteq \sum_j X^j \left[\frac{\partial f}{\partial x^j}\right](p). \tag{3.1}$$

X^j are the components of $\mathbf{X}$ in the coordinate system x. A vector at p may act as a differential operator on a function defined near p and as such is of the form

$$\mathbf{X}_p = \sum_j X^j \left.\frac{\partial}{\partial x^j}\right|_p .$$

At p, there are vectors $\boldsymbol{\partial/\partial x^j}$ corresponding to the operators $\partial/\partial x^j$. These vectors form a basis of the tangent space at p. We will also write $\partial/\partial x^j$ as ∂_j. In a two-dimensional Euclidean vector space, a vector $\mathbf{X}$ may be written as

$$\mathbf{X} = X^x \frac{\boldsymbol{\partial}}{\boldsymbol{\partial x}} + X^y \frac{\boldsymbol{\partial}}{\boldsymbol{\partial y}}.$$

3.2 Differential forms

Definition 2. *A function is multilinear if it is linear in each argument when all other arguments are kept constant.*

Definition 3. *[4] A covariant tensor of rank r is a multilinear real-valued function of r-tuples of vectors.*

Definition 4. *[4] An (exterior) p-form is a covariant p-tensor that is antisymmetric in each pair of entries.*

A p-form is also called an exterior differential p-form or differential p-form. Let for any basis $\boldsymbol{\partial}/\boldsymbol{\partial x}^j$, $\mathbf{X}_p = \sum_j X^j \boldsymbol{\partial}/\boldsymbol{\partial x}^j$ be a vector and $\alpha^{(1)}$ be a 1-form. Then

$$\alpha^{(1)}(\mathbf{X}) = \alpha^{(1)}\left(\sum_j X^j \frac{\boldsymbol{\partial}}{\boldsymbol{\partial x}^j}\right) = \sum_j \alpha^{(1)}\left(\frac{\boldsymbol{\partial}}{\boldsymbol{\partial x}^j}\right) X^j = \sum_j a_j X^j$$

with $a_j = \alpha^{(1)}(\boldsymbol{\partial}/\boldsymbol{\partial x}^j)$. Note that this is *not* an inner product: inner products depend on a metric tensor whereas the evaluation of a 1-form acting on a vector does not.

Definition 5. *The dual vector space E^* to a (primal) vector space E consists of all 1-tensors acting on E.*

Let $\alpha, \beta \in E^*$, $\mathbf{X} \in E$ and $c \in \mathbb{R}$, then $(\alpha + \beta)(\mathbf{X}) = \alpha(\mathbf{X}) + \beta(\mathbf{X})$ and $(c\alpha)(\mathbf{X}) = c\alpha(\mathbf{X})$.

Let p again be a point on an n-dimensional manifold M^n and let f be a real-valued function near p.

Definition 6. *[4] The differential of f at p, written df, is the linear functional defined by $df(\mathbf{X}) = \mathbf{X}_p(f)$.*

From equation 3.1, it follows that $df(\mathbf{X}) = D_{\mathbf{X}}(f)$. $df(\boldsymbol{\partial}/\boldsymbol{\partial x}^j) = \partial f/\partial x^j$, so the differential of a coordinate function x^i is $dx^i(\boldsymbol{\partial}/\boldsymbol{\partial x}^j) = \partial x^i/\partial x^j = \delta^i_j$. By linearity, applying dx^i to $\mathbf{X}$ yields component X^i. dx^j is the basis of E^* dual to $\boldsymbol{\partial}/\boldsymbol{\partial x}^j$.

A 1-form $\alpha^{(1)}$ can be written as $\sum_j a_j dx^j$ and applying it to a vector $\mathbf{X}$ yields

$$\alpha^{(1)}(\mathbf{X}) = \sum_i a_i dx^i \left(\sum_j X^j \frac{\boldsymbol{\partial}}{\boldsymbol{\partial x}^j}\right) = \sum_i \sum_j a_i X^j dx^i \left(\frac{\boldsymbol{\partial}}{\boldsymbol{\partial x}^j}\right) = \sum_i \sum_j a_i X^j \delta^i_j = \sum_i a_i X^i.$$

In a two-dimensional Euclidean dual vector space, a 1-form $\alpha^{(1)}$ may be written as $\alpha^{(1)} = a_1(x,y)dx + a_2(x,y)dy$. A 0-form has no vectors as arguments and is merely a function: in a two-dimensional setting, $\beta^{(0)} = b(x,y)$.

In the succeding we will use multi-indices. $(i_1, i_2, \ldots, i_p)$ will be abbreviated to I, $(\mathbf{X}_{i_1}, \mathbf{X}_{i_2}, \ldots, \mathbf{X}_{i_p})$ to $\mathbf{X}_I$, and so on. When $i_1 < i_2 < \ldots < i_p$, we will write $\underset{\rightarrow}{I}$.

A p-form $\alpha^{(p)}$ on vectors in n-dimensional space has n^p components $a_I = \alpha(\boldsymbol{\partial}_I)$. However, due to the antisymmetry of differential forms, only $\binom{n}{p}$ components are independent. The values $a_{\underset{\rightarrow}{I}} = \alpha(\boldsymbol{\partial}_{\underset{\rightarrow}{I}})$ determine all components of $\alpha^{(p)}$. Using Einstein's summation convention, we may write $\alpha^{(p)} = a_{\underset{\rightarrow}{I}} dx^I$.

Definition 7. *Suppose that all entries in I are distinct and so are all entries of J. The generalized Kronecker delta is then defined as*

$$\delta^I_J \doteq \begin{cases} 1 & \text{if } J \text{ is an even permutation of } I, \\ -1 & \text{if } J \text{ is an odd permutation of } I, \\ 0 & \text{if } J \text{ is not a permutation of } I. \end{cases}$$

A permutation is even (odd) if an even (odd) number of transpositions of two adjacent indices is required to obtain the one permutation from the other. Furthermore, the permutation symbol ϵ_I is defined as

$$\epsilon_I = \epsilon_{i_1, i_2, \ldots, i_n} \doteq \delta^I_{12\ldots n}.$$

Definition 8. *Let $\alpha^{(p)}$ and $\beta^{(q)}$ be differential forms. Then the exterior or wedge product $\alpha^{(p)} \wedge \beta^{(q)}$ is such that*

$$\left(\alpha^{(p)} \wedge \beta^{(q)}\right)(\mathbf{X}_I) = \sum_{\underset{\rightarrow}{K}} \sum_{\underset{\rightarrow}{J}} \delta^{JK}_I \alpha(\mathbf{X}_J) \beta(\mathbf{X}_K),$$

the components of which are

$$\left(\alpha^{(p)} \wedge \beta^{(q)}\right)_I = \sum_{\underset{\rightarrow}{K}} \sum_{\underset{\rightarrow}{J}} \delta^{JK}_I a_J b_K.$$

Since forms are multilinear, the wedge products of two forms is also multilinear. The wedge product is also antisymmetric:

$$\begin{aligned} \left(\alpha^{(p)} \wedge \beta^{(q)}\right)_{\ldots i \ldots j \ldots} &= \sum_{\underset{\rightarrow}{K}} \sum_{\underset{\rightarrow}{J}} \delta^{JK}_{\ldots i \ldots j \ldots} a_J b_K \\ &= -\sum_{\underset{\rightarrow}{K}} \sum_{\underset{\rightarrow}{J}} \delta^{JK}_{\ldots j \ldots i \ldots} a_J b_K \\ &= -\left(\alpha^{(p)} \wedge \beta^{(q)}\right)_{\ldots j \ldots i \ldots} \end{aligned}$$

as

$$\delta^R_{\dots i \dots j \dots} = \delta^R_{\dots\dots ij\dots}(-1)^r = \delta^R_{\dots\dots ji\dots}(-1)^{r+1} = \delta^R_{\dots j\dots i\dots}(-1)^{2r+1} = -\delta^R_{\dots j\dots i\dots}.$$

The wedge product is also distributive, so $(\alpha+\beta)\wedge(\eta+\theta) = \alpha\wedge\eta+\alpha\wedge\theta+\beta\wedge\eta+\beta\wedge\theta$. It is not always commutative:

$$\left(\beta^{(q)}\wedge\alpha^{(p)}\right)_I = \sum_{\underset{\rightarrow}{J}}\sum_{\underset{\rightarrow}{K}}\delta_I^{KJ} b_K a_J = (-1)^{pq}\sum_{\underset{\rightarrow}{K}}\sum_{\underset{\rightarrow}{J}}\delta_I^{JK} a_J b_K = (-1)^{pq}\left(\alpha^{(p)}\wedge\beta^{(q)}\right)_I$$

since pq transpositions are required to turn KJ into JK. Note that $dx\wedge dy = -dy\wedge dx$ and $dx\wedge dx = -dx\wedge dx = 0$. In two dimensions, the only differential forms that are not identically zero are 0-forms $f(x,y)$, 1-forms $a(x,y)\,dx + b(x,y)\,dy$ and 2-forms $c(x,y)\,dx\wedge dy$, and we have the following identities:

$$\begin{aligned}
&1\wedge 1 = 1,\\
&1\wedge dx = dx\wedge 1 = dx,\\
&1\wedge dy = dy\wedge 1 = dy,\\
&dx\wedge dy = -dy\wedge dx,\\
&1\wedge(dx\wedge dy) = (dx\wedge dy)\wedge 1 = dx\wedge dy.
\end{aligned}$$

Definition 9. *A p-form $\alpha^{(p)} = a(x)dx^I$ can be integrated over a p-dimensional region Ω. The integral is defined as*

$$\int_\Omega \alpha^{(p)} \doteq \int_\Omega a(x)dx^1\dots dx^p$$

where the right-hand side is the familiar multiple integral of $a(x)$.

3.3 Exterior derivative

Definition 10. *There is an operator that transforms p-forms into $(p+1)$-forms, called the exterior derivative d. Acting on a 0-form f, it is the familiar differential operator $df = \sum\limits_i(\partial_i f)dx^i$. Let dx^I denote $dx^{i_1}\wedge\ldots\wedge dx^{i_p}$ and $f(x)$ a function in coordinates x. A p-form may then be written as $\sum\limits_{\underset{\rightarrow}{I}} f(x)dx^I$. We then define the exterior derivative of $f(x)dx^I$ to be*

$$d\left(f(x)dx^I\right) = df(x)\wedge dx^I.$$

It follows that

$$d\left[\sum_{\underset{\rightarrow}{I}} f_I(x)dx^I\right] = \sum_{\underset{\rightarrow}{I}} df_I(x) \wedge dx^I.$$

Defined in this way, one can show (see [4]) that

$$d\left(\alpha^{(p)} \wedge \beta^{(q)}\right) = d\alpha^{(p)} \wedge \beta^{(q)} + (-1)^p \alpha^{(p)} \wedge d\beta^{(q)} \tag{3.2}$$

and

$$d\left(d\alpha^{(p)}\right) = 0.$$

In two dimensions,

$$\begin{aligned} df(x,y) &= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy, \\ d[a(x,y)dx + b(x,y)dy] &= \left(\frac{\partial b}{\partial x} - \frac{\partial a}{\partial y}\right) dx \wedge dy, \end{aligned}$$

and in particular,

$$d(udy - vdx) = \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) dx \wedge dy.$$

3.4 Interior product

Definition 11. *The interior product is an operator $\imath$ depending on a vector $\mathbf{X}$, acting on p-forms $\alpha^{(p)}$ and yielding $(p-1)$-forms such that*

$$\begin{aligned} &\imath_{\mathbf{X}} \alpha^{(0)} = 0, \\ &\imath_{\mathbf{X}} \alpha^{(1)} = \alpha^{(1)}(\mathbf{X}), \\ &\imath_{\mathbf{X}} \alpha^{(p)}(\mathbf{X}_2, \ldots, \mathbf{X}_p) = \alpha^{(p)}(\mathbf{X}, \mathbf{X}_2, \ldots, \mathbf{X}_p). \end{aligned}$$

Written out in components,

$$\imath_{\mathbf{X}} \alpha^{(p)} = \sum_{i_2 < \ldots < i_p} \left(\sum_j X^j a_{j i_2 \ldots i_p}\right) dx^{i_2} \wedge \ldots \wedge dx^{i_p}.$$

One can show that (see [4])

$$\imath_{\mathbf{v}}\left(\alpha^{(p)}\wedge\beta^{(q)}\right)=\imath_{\mathbf{v}}\alpha^{(p)}\wedge\beta^{(q)}+(-1)^p\alpha^{(p)}\wedge\imath_{\mathbf{v}}\beta^{(q)}. \tag{3.3}$$

In two dimensions,

$$\begin{aligned}\imath_{\mathbf{X}}[a(x,y)dx+b(x,y)dy]&=X^x a(x,y)+X^y b(x,y),\\ \imath_{\mathbf{X}}[c(x,y)dx\wedge dy]&=c(x,y)\left(X^x dy-X^y dx\right).\end{aligned}$$

3.5 Lie derivative

Let $\mathbf{X}$ be a vector field, generating a local flow. The Lie derivative of a vector field or tensor field with respect to $\mathbf{X}$, $\mathcal{L}_{\mathbf{X}}$, is a vector's or tensor's rate of change as observed by an observer that moves and rotates with mentioned local flow. In this thesis we will only take the Lie derivative of p-forms explicitly. As such, it suffices to use the following as a definition.

Definition 12. *The Lie derivative $\mathcal{L}_{\mathbf{X}}$ of a p-form $\alpha^{(p)}$ is defined as*

$$\mathcal{L}_{\mathbf{X}}\alpha^{(p)}\doteq\imath_{\mathbf{X}}\circ d\alpha^{(p)}+d\circ\imath_{\mathbf{X}}\alpha^{(p)}.$$

The Lie derivative of a 0-form $\alpha^{(0)}=\alpha$ is

$$\mathcal{L}_{\mathbf{X}}\alpha^{(0)}=\imath_{\mathbf{X}}\circ d\alpha^{(0)}+d\circ\imath_{\mathbf{X}}\alpha^{(0)}=\imath_{\mathbf{X}}\circ d\alpha=d\alpha(\mathbf{X}).$$

In two dimensions,

$$\begin{aligned}\mathcal{L}_{\mathbf{X}}[f(x,y)]&=X^x\frac{\partial f}{\partial x}+X^y\frac{\partial f}{\partial y},\\ \mathcal{L}_{\mathbf{X}}[a(x,y)dx+b(x,y)dy]&=\left(X^x\frac{\partial a}{\partial x}+a\frac{\partial X^x}{\partial x}+X^y\frac{\partial a}{\partial y}+b\frac{\partial X^y}{\partial x}\right)dx\\ &\quad+\left(X^x\frac{\partial b}{\partial x}+a\frac{\partial X^x}{\partial y}+X^y\frac{\partial b}{\partial y}+b\frac{\partial X^y}{\partial y}\right)dy,\\ \mathcal{L}_{\mathbf{X}}[c(x,y)dx\wedge dy]&=\left(\frac{\partial}{\partial x}\left(cX^x\right)+\frac{\partial}{\partial y}\left(cX^y\right)\right)dx\wedge dy.\end{aligned}$$

On a manifold, a Riemannian metric defines in each tangent space a symmetric, positive definite inner product $\langle\cdot,\cdot\rangle$ in a differentiable fashion. The metric tensor can be written as a matrix G with entries $g_{ij}(x)=\left\langle\frac{\boldsymbol{\partial}}{\boldsymbol{\partial x^i}},\frac{\boldsymbol{\partial}}{\boldsymbol{\partial x^j}}\right\rangle$. The inverse of G, G^{-1}, has components g^{ij}. For orthonormal bases, $g_{ij}(x)=\delta_{ij}$ and $g^{ij}(x)=\delta^{ij}$, where δ_{ij} and δ^{ij} both represent the number δ_i^j but where the lowered indices represent a reference to the primal vector space and the raised indices refer to the dual vector space. The determinant of G at point x is $g(x)$, which for orthonormal bases equals one.

Definition 13. *We define the volume form of an n-dimensional manifold M^n, expressed in local coordinates, as*

$$\mathrm{vol}^n \doteq \sqrt{|g|}dx^1 \wedge \ldots \wedge dx^n.$$

Applying the volume form to an orthonormal basis of a vector space yields 1. In a two-dimensional Euclidean vector space, $\mathrm{vol}^2 = dx \wedge dy$. If furthermore, $\mathbf{v} = u\frac{\boldsymbol{\partial}}{\boldsymbol{\partial x}} + v\frac{\boldsymbol{\partial}}{\boldsymbol{\partial y}}$, we may write equation 2.5 as

$$(\mathrm{div}\,\mathbf{v})\,\mathrm{vol}^n = \mathcal{L}_{\mathbf{v}}\,\mathrm{vol}^n = \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) dx \wedge dy.$$

3.6 Lie bracket

Definition 14. *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be two left invariant vector fields. We may then form a third vector field* $[\mathbf{X}, \mathbf{Y}]$*, the Lie bracket for left invariant vector fields, such that for any function* f*,* $[\mathbf{X}, \mathbf{Y}](f) = \mathbf{X}(\mathbf{Y}(f)) - \mathbf{Y}(\mathbf{X}(f))$*.*

However, in this thesis we deal with right invariant vector fields. In that case (see [2]) the sign should be changed:

Definition 15. *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be two right invariant vector fields. We may then form a third vector field* $[\mathbf{X}, \mathbf{Y}]$*, the Lie bracket for right invariant vector fields, such that for any function* f*,* $[\mathbf{X}, \mathbf{Y}](f) = \mathbf{Y}(\mathbf{X}(f)) - \mathbf{X}(\mathbf{Y}(f))$*.*

The Lie bracket for right invariant vectors is right invariant itself as well. Component i of $[\mathbf{X}, \mathbf{Y}]$ is given by

$$[\mathbf{X}, \mathbf{Y}]^i = \sum_j \left\{ Y^j \left(\frac{\partial X^i}{\partial x^j}\right) - X^j \left(\frac{\partial Y^i}{\partial x^j}\right) \right\}.$$

From now on, the reader may assume that when a Lie bracket is mentioned, we mean one for right invariant vectors. Note that both Lie brackets are antisymmetrical. In two dimensions,

$$\begin{aligned}[\mathbf{X}, \mathbf{Y}] = &\left\{ Y^x \left(\frac{\partial X^x}{\partial x}\right) - X^x \left(\frac{\partial Y^x}{\partial x}\right) + Y^y \left(\frac{\partial X^x}{\partial y}\right) - X^y \left(\frac{\partial Y^x}{\partial y}\right) \right\} \frac{\partial}{\partial x} \\ &+ \left\{ Y^x \left(\frac{\partial X^y}{\partial x}\right) - X^x \left(\frac{\partial Y^y}{\partial x}\right) + Y^y \left(\frac{\partial X^y}{\partial y}\right) - X^y \left(\frac{\partial Y^y}{\partial y}\right) \right\} \frac{\partial}{\partial y}.\end{aligned}$$

3.7 Hodge star operator

The Hodge star operator associates to every p-form $\alpha^{(p)}$ the $(n-p)$-form $*\alpha^{(p)}$ dual to $\alpha^{(p)}$.

Definition 16. *[4] If* $\alpha^{(p)} = a_{\underrightarrow{I}}\, dx^I$ *then* $*\alpha^{(p)} = a^*_{\underrightarrow{J}}\, dx^J$ *where*

$$a^*_{\underrightarrow{J}} \doteq \sqrt{|g|}\alpha^K \epsilon_{\underrightarrow{K}\,\underrightarrow{J}}$$

where $\alpha^{a\ldots b} = g^{ac} \ldots g^{bd} \alpha_{c\ldots d}$.

Note that $*(f\alpha^{(p)}) = f *\alpha^{(p)}$. In a two-dimensional Euclidean vector space,

$$*1 = dx \wedge dy, \quad *dx = dy, \quad *dy = -dx, \quad *(dx \wedge dy) = 1.$$

3.8 Stokes' theorem

The most general space over which differential forms may be integrated are manifolds, one characteristic of which is that they are open sets. An n-dimensional manifold with boundary consists of an interior that is an n-dimensional manifold M^n and a boundary ∂M. The boundary is an $(n-1)$-dimensional, boundaryless manifold: the $(n-2)$-dimensional boundary of the boundary, $\partial\partial M$, is empty.

Theorem 1 (Stokes' theorem)**.** *Let $M^p \subset N^n$ be a p-dimensional submanifold of an n-dimensional manifold N^n together with its (possibly empty) boundary, $\partial M \subset N^n$, and let $\alpha^{(p-1)}$ be a continuously differentiable $(p-1)$-form defined on N^n. Then*

$$\int_M d\alpha^{(p-1)} = \int_{\partial M} \alpha^{(p-1)}.$$

3.9 Global scalar products

Definition 17. *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be two vector fields on a manifold M^n. We then define their pointwise scalar product to be*

$$\left\langle \sum_i X^i \frac{\boldsymbol{\partial}}{\boldsymbol{\partial x}^i}, \sum_j Y^j \frac{\boldsymbol{\partial}}{\boldsymbol{\partial x}^j} \right\rangle \doteq \sum_i \sum_j X^i Y^j \left\langle \frac{\boldsymbol{\partial}}{\boldsymbol{\partial x}^i}, \frac{\boldsymbol{\partial}}{\boldsymbol{\partial x}^j} \right\rangle.$$

For a two-dimensional orthonormal coordinate basis,

$$\langle \mathbf{X}, \mathbf{Y} \rangle = \sum_i \sum_j X^i Y^j \delta_{ij} = \sum_i X^i Y^i = X^x Y^x + X^y Y^y.$$

Definition 18. *Similarly, we define a pointwise scalar product for two p-forms* $\alpha^{(p)} = a_{\underset{\to}{I}} dx^I$ *and* $\beta^{(p)} = b_{\underset{\to}{I}} dx^I$*:*

$$\left\langle \alpha^{(p)}, \beta^{(p)} \right\rangle \doteq a_{\underset{\to}{I}} b^I .$$

It can be shown (see [4]) that

$$\alpha^{(p)} \wedge *\beta^{(p)} = \left\langle \alpha^{(p)}, \beta^{(p)} \right\rangle \mathrm{vol}^n .$$

Definition 19. *The global scalar product of two vectors* $\mathbf{X}$ *and* $\mathbf{Y}$ *is*

$$(\mathbf{X}, \mathbf{Y}) \doteq \int\limits_{M^n} \langle \mathbf{X}, \mathbf{Y} \rangle \, \mathrm{vol}^n .$$

Definition 20. *The global scalar product of two p-forms* $\alpha^{(p)}$ *and* $\beta^{(p)}$ *is*

$$\left(\alpha^{(p)}, \beta^{(p)}\right) \doteq \int\limits_{M^n} \left\langle \alpha^{(p)}, \beta^{(p)} \right\rangle \mathrm{vol}^n = \int\limits_{M^n} \alpha^{(p)} \wedge *\beta^{(p)}.$$

When we talk about inner products in this thesis, we mean one of these global scalar products.

3.10 Rewriting the Euler equations

Definition 21. *Let* $\mathbf{X}$*,* $\mathbf{Y}$ *and* $\mathbf{Z}$ *be three vectors from the same Lie algebra.* $B(\mathbf{X}, \mathbf{Y})$ *is then the unique vector in that Lie algebra for which*

$$\langle B(\mathbf{X}, \mathbf{Y}), \mathbf{Z} \rangle = \langle [\mathbf{Y}, \mathbf{Z}], \mathbf{X} \rangle .$$

Arnold [2] shows that in order to obtain the velocity field $\mathbf{v}$ from the incompressible Euler equations 2.1 and the condition that $\mathbf{v} \cdot \mathbf{n}$ on the boundary, we may as well solve the system of equations

$$\begin{aligned} \nabla \cdot \mathbf{v} &= 0, \\ \frac{\partial \mathbf{v}}{\partial t} + B(\mathbf{v}, \mathbf{v}) &= 0 \end{aligned} \tag{3.4}$$

together with the boundary condition. The pressure term has disappeared from the equation since it is L^2-orthogonal to any divergence free velocity field $\mathbf{v}$ satisfying the boundary condition: in terms of vector calculus,

$$
\begin{aligned}
(\nabla p, \mathbf{v}) &= \int_\Omega \nabla p \cdot \mathbf{v} \, d\Omega \\
&= \int_\Omega \nabla \cdot (p\mathbf{v}) - p(\nabla \cdot \mathbf{v}) \, d\Omega \\
&= \int_{\partial\Omega} p\mathbf{v} \cdot \mathbf{n} - \int_\Omega p(\nabla \cdot \mathbf{v}) \, d\Omega \\
&= 0
\end{aligned}
$$

for a domain Ω with boundary $\partial\Omega$, since $\nabla \cdot \mathbf{v} = 0$ in the domain and $\mathbf{v} \cdot \mathbf{n} = 0$ on the boundary.

To the vector $\mathbf{v}$ we may associate a set of 1-forms $v^{(1)}$ which all satisfy the duality pairing

$$\left\langle v^{(1)}, \mathbf{v} \right\rangle \doteq \int_{M^n} \imath_{\mathbf{v}} v^{(1)} \, \mathrm{vol}^n$$

(see [7]). $v^{(1)}$ is determined up to an exact 1-form, i.e. a 1-form which can be written as the exterior derivative of a 0-form: for an exact 1-form $d\phi^{(0)}$, it can be shown that $\left\langle d\phi^{(0)}, \mathbf{v} \right\rangle = 0$ (see appendix D, equation D.1).

The vorticity may be considered to be a 2-form: the exterior derivative generalises concepts such as the gradient, the curl and the divergence, so instead of $\boldsymbol{\omega} = \nabla \times \mathbf{v}$, we may write $\omega^{(2)} = dv^{(1)}$. In terms of differential geometry, the vorticity equation 2.2 may be written as

$$\frac{\partial \omega^{(2)}}{\partial t} + \mathcal{L}_{\mathbf{v}} \omega^{(2)} = 0. \tag{3.5}$$

Adding any exact 1-form $d\phi^{(0)}$ to $v^{(1)}$ does not change the vorticity $\omega^{(2)}$ since $d\left(v^{(1)} + d\phi^{(0)}\right) = dv^{(1)} + dd\phi^{(0)} = dv^{(1)} = \omega^{(2)}$. Equations 3.4 and 3.5 are the equations we will consider in the remainder of this thesis.

Chapter 4

Numerical modelling

In this chapter we develop the numerical method that approximates the evolution of the velocity and vorticity fields of a fluid whose behaviour is described by the two-dimensional incompressible Euler equations, and which is confined to a domain without in- or outflow at the boundary, for a given intitial velocity field. We will first discuss how the equations simplify in a two-dimensional setting, and introduce the domain on which we will solve the flow problem. We then introduce several basis functions and vector fields, after which some commutators are introduced. Next, we find a semi-discrete momentum equation, which we subsequently fully discretize in such a way that the kinetic energy is conserved. Finally we discuss a way to discretize the vorticity equation such that under certain conditions, the vorticity and enstrophy integrals are conserved as the flow develops.

In this thesis we will approximate the development of the velocity field in time based on the two-dimensional version of the incompressible Euler equations 3.4 with a mimetic spectral method. As we consider an incompressible flow, the velocity field is divergence free at any time instance. In our two-dimensional Cartesian setting, the velocity field may therefore be derived from a stream function. We describe the velocity field as

$$\mathbf{v}(x,y) = u(x,y)\frac{\partial}{\partial x} + v(x,y)\frac{\partial}{\partial y}$$

where the components $u(x,y)$ and $v(x,y)$ are given by (plus or minus) the partial derivatives of a stream function $\psi(x,y)$:

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x}.$$

Note that the difference between the values of the stream function at two points is a measure for the nett flux through *any* line connecting the two points. After having approximated the development of the velocity field, we will approximate the development of the related vorticity field. The starting point of our discussion is the vorticity equation, equation 3.5,

$$\frac{\partial \omega^{(2)}}{\partial t} + \mathcal{L}_{\mathbf{v}}\omega^{(2)} = 0.$$

On a two-dimensional manifold, the Lie derivative satisfies $\mathcal{L}_{\mathbf{v}}\omega^{(2)} = d\imath_{\mathbf{v}}\omega^{(2)}$ as $\imath_{\mathbf{v}} d\omega^{(2)} = \imath_{\mathbf{v}} 0 = 0$, so we may rewrite the vorticity equation as

$$\frac{\partial \omega^{(2)}}{\partial t} + d\imath_{\mathbf{v}}\omega^{(2)} = 0.$$

We may write the vorticity field as $\omega^{(2)} = \omega(x, y, t)dx \wedge dy$, so, dropping the arguments, we express $d\imath_{\mathbf{v}}\omega^{(2)}$ as

$$\begin{aligned} d\imath_{\mathbf{v}}\omega^{(2)} = d(\omega u dy - \omega v dx) &= \left(\frac{\partial}{\partial x}(\omega u) + \frac{\partial}{\partial y}(\omega v)\right) dx \wedge dy \\ &= \left(\omega \underbrace{\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)}_{0} + u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y}\right) dx \wedge dy \\ &= \left(u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y}\right) dx \wedge dy. \end{aligned}$$

The vorticity equation may now be written as

$$\left(\frac{\partial \omega}{\partial t} + u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y}\right) dx \wedge dy = 0. \tag{4.1}$$

The manifold M^2 on which we approximate the velocity and vorticity field developments is a square having sides of length 2. We attach a coordinate system to it, the origin of which is at the center of the square and the axes of which are parallel to the square's sides. The square has nodes at (x_i, y_j), $i \in \{0, \ldots, M\}$, $j \in \{0, \ldots, N\}$ with $-1 = x_0 < x_1 < \ldots < x_M = 1$ and $-1 = y_0 < y_1 < \ldots < y_N = 1$, and at $(\bar{x}_i, \bar{y}_j)$, $i \in \{0, \ldots, C\}$, $j \in \{0, \ldots, D\}$ with $-1 = \bar{x}_0 < \bar{x}_1 < \ldots < \bar{x}_C = 1$ and $-1 = \bar{y}_0 < \bar{y}_1 < \ldots < \bar{y}_D = 1$. From the nodes (x_i, y_j) we construct an outer oriented cell complex in order to approximate the velocity field development. We will call this the primal mesh. The nodes $(\bar{x}_i, \bar{y}_j)$ are used to generate an inner oriented cell complex which is dual to the primal mesh and which we will call the dual mesh. We will use it to approximate the vorticity field development. Note that the nodes $(\bar{x}_i, \bar{y}_j)$ that lie on the boundary of the square are no elements of the cell complex itself. We merely use them to perform calculations on the surfaces dual to the nodes of the primal mesh. Since the cell complexes are dual to each other, we have that $-1 = x_0 = \bar{x}_0 < \bar{x}_1 < x_1 < \ldots < x_{M-1} < \bar{x}_M < x_M = \bar{x}_{M+1} = 1$ and $-1 = y_0 = \bar{y}_0 < \bar{y}_1 < y_1 < \ldots < y_{N-1} < \bar{y}_N < y_N = \bar{y}_{N+1} = 1$: we see that $C = M + 1$ and $D = N + 1$. In order to keep the equations relatively small, we will choose $M = N$, $x_i = y_i$ and $\bar{x}_i = \bar{y}_i$. Figure 4.1 gives an overview of the discretization of the square.

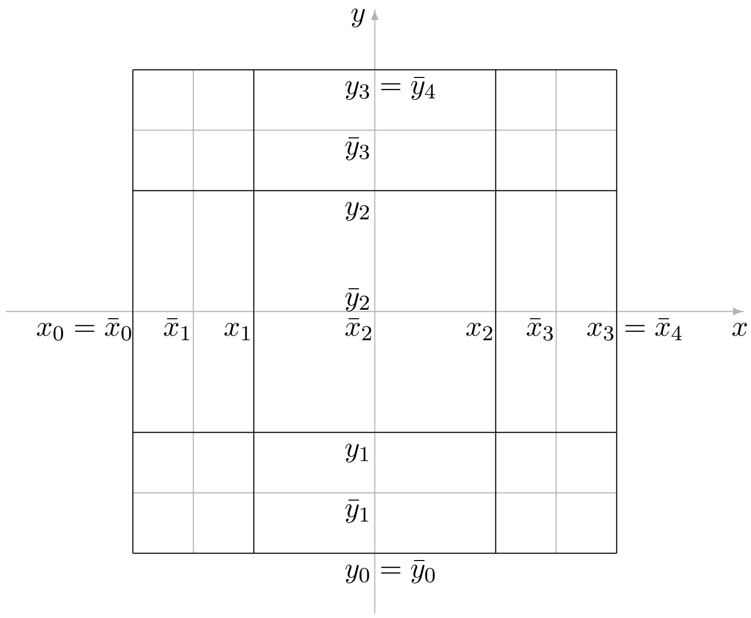

Figure 4.1: Schematic overview of the mesh for $N = 3$.

4.1 Basis functions and basis vector fields

Suppose we have nodes on the real line at x_i, $i \in \{0, \ldots, N\}$. The Lagrange polynomials, each of which takes the value 1 at the corresponding node and 0 on all other nodes and are of order N, are given by

$$h_i(x) = \prod_{\substack{k=0 \\ k \neq i}}^{N} \left(\frac{x - x_k}{x_i - x_k} \right),$$

the derivatives of which are

$$h_i'(x) = \frac{\sum\limits_{\substack{k=0 \\ k \neq i}}^{N} \prod\limits_{\substack{l=0 \\ l \neq i \\ l \neq k}}^{N} (x - x_l)}{\prod\limits_{\substack{k=0 \\ k \neq i}}^{N} (x_i - x_k)}$$

and the second derivatives of which are

$$h_i''(x) = \frac{\displaystyle\sum_{\substack{k=0\\k\neq i}}^{N} \sum_{\substack{l=0\\l\neq i\\l\neq k}}^{N} \prod_{\substack{m=0\\m\neq i\\m\neq k\\m\neq l}}^{N} (x-x_m)}{\displaystyle\prod_{\substack{k=0\\k\neq i}}^{N}(x_i-x_k)} = \frac{\displaystyle 2\sum_{\substack{k=0\\k\neq i}}^{N} \sum_{\substack{l=k+1\\l\neq i}}^{N} \prod_{\substack{m=0\\m\neq i\\m\neq k\\m\neq l}}^{N} (x-x_m)}{\displaystyle\prod_{\substack{k=0\\k\neq i}}^{N}(x_i-x_k)}.$$

The stream function ψ at (x_i, y_j) has the value $\psi(x_i, y_j) = \psi^{ij}$. We may then approximate the stream function as

$$\psi(x,y) \approx \psi_h(x,y) = \sum_{i=0}^{N}\sum_{j=0}^{N} \psi^{ij} h_i(x) h_j(y) = \sum_{i=0}^{N}\sum_{j=0}^{N} \psi^{ij} \tilde{\psi}_{i+(N+1)j}(x,y)$$

where $\tilde{\psi}_{i+(N+1)j}(x,y) = h_i(x)h_j(y)$ are the basis functions of the stream function approximation. They have the value 1 in node (x_i, y_j) and 0 in all other nodes of the primal mesh. The partial derivatives of the stream function are approximated by

$$\begin{aligned} u = \frac{\partial\psi}{\partial y} \approx u_h = \frac{\partial\psi_h}{\partial y} &= \sum_{i=0}^{N}\sum_{j=0}^{N} \psi^{ij} h_i(x) h_j'(y) = \sum_{i=0}^{N}\sum_{j=0}^{N} \psi^{ij} \tilde{u}_{i+(N+1)j}(x,y), \\ -v = \frac{\partial\psi}{\partial x} \approx -v_h = \frac{\partial\psi_h}{\partial x} &= \sum_{i=0}^{N}\sum_{j=0}^{N} \psi^{ij} h_i'(x) h_j(y) = \sum_{i=0}^{N}\sum_{j=0}^{N} \psi^{ij} (-\tilde{v}_{i+(N+1)j}(x,y)) \end{aligned}$$

where $\tilde{u}_{i+(N+1)j}(x,y) = h_i(x)h_j'(y)$ and $\tilde{v}_{i+(N+1)j}(x,y) = -h_i'(x)h_j(y)$ are the basis functions of the velocity component approximations. The velocity vector field $\mathbf{v}$ can now be approximated as

$$\begin{aligned} \mathbf{v} = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} \approx \mathbf{v}_h &= u_h\frac{\partial}{\partial x} + v_h\frac{\partial}{\partial y} \\ &= \left(\sum_{i=0}^{N}\sum_{j=0}^{N} \psi^{ij}\tilde{u}_{i+(N+1)j}(x,y)\right)\frac{\partial}{\partial x} + \left(\sum_{i=0}^{N}\sum_{j=0}^{N} \psi^{ij}\tilde{v}_{i+(N+1)j}(x,y)\right)\frac{\partial}{\partial y} \\ &= \sum_{i=0}^{N}\sum_{j=0}^{N} \psi^{ij}\left(\tilde{u}_{i+(N+1)j}(x,y)\frac{\partial}{\partial x} + \tilde{v}_{i+(N+1)j}(x,y)\frac{\partial}{\partial y}\right) \\ &= \sum_{k=0}^{(N+1)^2-1} \alpha^k \tilde{\mathbf{v}}_k(x,y) \end{aligned}$$

where $\alpha^{i+(N+1)j} = \psi_{ij}$, and $\tilde{\mathbf{v}}_k(x,y) = \tilde{u}_k(x,y)\frac{\partial}{\partial x} + \tilde{v}_k(x,y)\frac{\partial}{\partial y}$ are the basis vector fields of the velocity vector field approximation. Each basis function of the stream function approximation yields a basis vector field of the velocity vector field approximation: to each $\tilde{\psi}_k(x,y)$ corresponds a $\tilde{\mathbf{v}}_k(x,y)$. Note that $\tilde{u}_{i+(N+1)j} = \frac{\partial}{\partial y}\tilde{\psi}_{i+(N+1)j}$ and $\tilde{v}_{i+(N+1)j} = -\frac{\partial}{\partial x}\tilde{\psi}_{i+(N+1)j}$: the basis vector fields are divergence free.

4.2 Commutator of two basis vector fields

Taking the commutator of two basis vector fields of the velocity field approximation yields another vector field

$$\begin{aligned}
[\tilde{\mathbf{v}}_k, \tilde{\mathbf{v}}_l] &= \frac{\partial}{\partial y}\left(\frac{\partial\tilde{\psi}_k}{\partial x}\frac{\partial\tilde{\psi}_l}{\partial y} - \frac{\partial\tilde{\psi}_k}{\partial y}\frac{\tilde{\psi}_l}{\partial x}\right)\frac{\partial}{\partial x} - \frac{\partial}{\partial x}\left(\frac{\partial\tilde{\psi}_k}{\partial x}\frac{\partial\tilde{\psi}_l}{\partial y} - \frac{\partial\tilde{\psi}_k}{\partial y}\frac{\tilde{\psi}_l}{\partial x}\right)\frac{\partial}{\partial y} \\
&= \frac{\partial\tilde{\psi}_{[\tilde{\mathbf{v}}_k,\tilde{\mathbf{v}}_l]}}{\partial y}\frac{\partial}{\partial x} - \frac{\partial\tilde{\psi}_{[\tilde{\mathbf{v}}_k,\tilde{\mathbf{v}}_l]}}{\partial x}\frac{\partial}{\partial y} \\
&= \tilde{u}_{[\tilde{\mathbf{v}}_k,\tilde{\mathbf{v}}_l]}\frac{\partial}{\partial x} + \tilde{v}_{[\tilde{\mathbf{v}}_k,\tilde{\mathbf{v}}_l]}\frac{\partial}{\partial y} \\
&= \tilde{\mathbf{v}}_{[\tilde{\mathbf{v}}_k,\tilde{\mathbf{v}}_l]}
\end{aligned}$$

(see appendix D, equation D.2) where

$$\begin{aligned}
\tilde{\psi}_{[\tilde{\mathbf{v}}_k,\tilde{\mathbf{v}}_l]} &\doteq \frac{\partial\tilde{\psi}_k}{\partial x}\frac{\partial\tilde{\psi}_l}{\partial y} - \frac{\partial\tilde{\psi}_k}{\partial y}\frac{\tilde{\psi}_l}{\partial x}, \\
\tilde{u}_{[\tilde{\mathbf{v}}_k,\tilde{\mathbf{v}}_l]} &\doteq \frac{\partial}{\partial y}\tilde{\psi}_{[\tilde{\mathbf{v}}_k,\tilde{\mathbf{v}}_l]}, \\
\tilde{v}_{[\tilde{\mathbf{v}}_k,\tilde{\mathbf{v}}_l]} &\doteq -\frac{\partial}{\partial x}\tilde{\psi}_{[\tilde{\mathbf{v}}_k,\tilde{\mathbf{v}}_l]}, \\
\tilde{\mathbf{v}}_{[\tilde{\mathbf{v}}_k,\tilde{\mathbf{v}}_l]} &\doteq \tilde{u}_{[\tilde{\mathbf{v}}_k,\tilde{\mathbf{v}}_l]}\frac{\partial}{\partial x} + \tilde{v}_{[\tilde{\mathbf{v}}_k,\tilde{\mathbf{v}}_l]}\frac{\partial}{\partial y}.
\end{aligned}$$

If we define the commutator of two functions $f(x,y)$ and $g(x,y)$ as

$$[f,g] \doteq \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x},$$

we can write

$$\tilde{\psi}_{[\tilde{\mathbf{v}}_k,\tilde{\mathbf{v}}_l]} = \left[\tilde{\psi}_k, \tilde{\psi}_l\right].$$

This operator is antisymmetric. By applying the commutator to two basis functions of the stream function approximation, we find a new basis function:

$$\begin{aligned}\left[\tilde{\psi}_{i_1+(N+1)j_1}, \tilde{\psi}_{i_2+(N+1)j_2}\right] &= \frac{\partial \tilde{\psi}_{i_1+(N+1)j_1}}{\partial x} \frac{\partial \tilde{\psi}_{i_2+(N+1)j_2}}{\partial y} - \frac{\partial \tilde{\psi}_{i_1+(N+1)j_1}}{\partial y} \frac{\tilde{\psi}_{i_2+(N+1)j_2}}{\partial x} \\ &= -\tilde{v}_{i_1+(N+1)j_1} \tilde{u}_{i_2+(N+1)j_2} + \tilde{u}_{i_1+(N+1)j_1} \tilde{v}_{i_2+(N+1)j_2} \\ &= h'_{i_1}(x) h_{j_1}(y) h_{i_2}(x) h'_{j_2}(y) - h_{i_1}(x) h'_{j_1}(y) h'_{i_2}(x) h_{j_2}(y) \\ &= \tilde{\psi}_{\left[\tilde{\mathbf{v}}_{i_1+(N+1)j_1}, \tilde{\mathbf{v}}_{i_2+(N+1)j_2}\right]}.\end{aligned}$$

From these basis functions new basis functions of the velocity component approximations may be obtained:

$$\begin{aligned}\tilde{u}_{\left[\tilde{\mathbf{v}}_{i_1+(N+1)j_1}, \tilde{\mathbf{v}}_{i_2+(N+1)j_2}\right]} &= \frac{\partial}{\partial y} \tilde{\psi}_{\left[\tilde{\mathbf{v}}_{i_1+(N+1)j_1}, \tilde{\mathbf{v}}_{i_2+(N+1)j_2}\right]} \\ &= h'_{i_1}(x) h_{i_2}(x) \left\{ h'_{j_1}(y) h'_{j_2}(y) + h_{j_1}(y) h''_{j_2}(y) \right\} \\ &\quad - h_{i_1}(x) h'_{i_2}(x) \left\{ h'_{j_1}(y) h'_{j_2}(y) + h''_{j_1}(y) h^{y}_{j_2}(y) \right\},\end{aligned}$$

$$\begin{aligned}\tilde{v}_{\left[\tilde{\mathbf{v}}_{i_1+(N+1)j_1}, \tilde{\mathbf{v}}_{i_2+(N+1)j_2}\right]} &= -\frac{\partial}{\partial x} \tilde{\psi}_{\left[\tilde{\mathbf{v}}_{i_1+(N+1)j_1}, \tilde{\mathbf{v}}_{i_2+(N+1)j_2}\right]} \\ &= -h_{j_1}(y) h'_{j_2}(y) \left\{ h'_{i_1}(x) h'_{i_2}(x) h''_{i_1}(x) h_{i_2}(x) \right\} \\ &\quad + h'_{j_1}(y) h_{j_2}(y) \left\{ h'_{i_1}(x) h'_{i_2}(x) + h^{x}_{i_1}(x) h''_{i_2}(x) \right\}.\end{aligned}$$

4.3 Approximating the momentum equation

The momentum equation that we would like to solve is

$$\frac{\partial \mathbf{v}}{\partial t} + B(\mathbf{v}, \mathbf{v}) = 0,$$

supplemented with the continuity equation $\nabla \cdot \mathbf{v} = 0$ and the condition that $\mathbf{v} \cdot \mathbf{n} = 0$ on the boundary. Remembering that

$$\mathbf{v}_h = \sum_{k=0}^{(N+1)^2-1} \alpha^k \tilde{\mathbf{v}}_k(x, y)$$

and recognising that α^k may be a function of time, we may approximate the momentum equation by

$$\frac{\partial}{\partial t} \sum_{k=0}^{(N+1)^2-1} \alpha^k(t) \tilde{\mathbf{v}}_k(x, y) \approx -B\left(\left[\sum_{k=0}^{(N+1)^2-1} \alpha^k(t) \tilde{\mathbf{v}}_k(x, y) \right], \left[\sum_{l=0}^{(N+1)^2-1} \alpha^l(t) \tilde{\mathbf{v}}_l(x, y) \right] \right) \iff$$

$$\sum_{k=0}^{(N+1)^2-1} \tilde{\mathbf{v}}_k(x, y) \frac{d\alpha^k(t)}{dt} \approx -\sum_{k=0}^{(N+1)^2-1} \sum_{l=0}^{(N+1)^2-1} \alpha^k(t) \alpha^l(t) B\left(\tilde{\mathbf{v}}_k(x, y), \tilde{\mathbf{v}}_l(x, y)\right).$$

Taking the inner product of each side of this equation with $\tilde{\mathbf{v}}_m(x,y)$, $m \in \{0,\ldots,(N+1)^2-1\}$ yields (dropping the arguments of the functions)

$$\left(\left[\sum_{k=0}^{(N+1)^2-1} \tilde{\mathbf{v}}_k \frac{d\alpha^k}{dt}\right], \tilde{\mathbf{v}}_m\right) = \left(\left[-\sum_{k=0}^{(N+1)^2-1}\sum_{l=0}^{(N+1)^2-1} \alpha^k\alpha^l B\left(\tilde{\mathbf{v}}_k, \tilde{\mathbf{v}}_l\right)\right], \tilde{\mathbf{v}}_m\right) \iff$$

$$\sum_{k=0}^{(N+1)^2-1} \left(\tilde{\mathbf{v}}_k, \tilde{\mathbf{v}}_m\right) \frac{d\alpha^k}{dt} = -\sum_{k=0}^{(N+1)^2-1}\sum_{l=0}^{(N+1)^2-1} \left(B\left(\tilde{\mathbf{v}}_k, \tilde{\mathbf{v}}_l\right), \tilde{\mathbf{v}}_m\right)\alpha^k\alpha^l$$

$$= -\sum_{k=0}^{(N+1)^2-1}\sum_{l=0}^{(N+1)^2-1} \left(\left[\tilde{\mathbf{v}}_l, \tilde{\mathbf{v}}_m\right], \tilde{\mathbf{v}}_k\right)\alpha^k\alpha^l.$$

This is a system of $(N+1)^2$ nonlinear ordinary differential equations (one for each $\tilde{\mathbf{v}}_m$) having the $(N+1)^2$ functions $\alpha^k(t)$ to be solved for. $(\tilde{\mathbf{v}}_k, \tilde{\mathbf{v}}_m)$ and $([\tilde{\mathbf{v}}_l, \tilde{\mathbf{v}}_m], \tilde{\mathbf{v}}_k)$ are numbers that are independent of the solution of the system of equations. In terms of the Lagrange polynomials and their derivatives, these numbers are

$$\left(\tilde{\mathbf{v}}_{i_1+(N+1)j_1}, \tilde{\mathbf{v}}_{i_2+(N+1)j_2}\right) = \int_{-1}^{1} h_{i_1}(x)h_{i_2}(x)\,dx \cdot \int_{-1}^{1} h'_{j_1}(y)h'_{j_2}(y)\,dy + \int_{-1}^{1} h'_{i_1}(x)h'_{i_2}(x)\,dx \cdot \int_{-1}^{1} h_{j_1}(y)h_{j_2}(y)\,dy$$

and

$$\begin{aligned}
&\left(\left[\tilde{\mathbf{v}}_{i_1+(N+1)j_1}, \tilde{\mathbf{v}}_{i_2+(N+1)j_2}\right], \tilde{\mathbf{v}}_{i_3+(N+1)j_3}\right) \\
&= \int_{-1}^{1} h'_{i_1}(x)h_{i_2}(x)h_{i_3}(x)\,dx \cdot \int_{-1}^{1} \left\{h'_{j_1}(y)h'_{j_2}(y) + h_{j_1}(y)h''_{j_2}(y)\right\} h'_{j_3}(y)\,dy \\
&\quad - \int_{-1}^{1} h_{i_1}(x)h'_{i_2}(x)h_{i_3}(x)\,dx \cdot \int_{-1}^{1} \left\{h'_{j_1}(y)h'_{j_2}(y) + h''_{j_1}(y)h_{j_2}(y)\right\} h'_{j_3}(y)\,dy \\
&\quad + \int_{-1}^{1} \left\{h'_{i_1}(x)h'_{i_2}(x) + h''_{i_1}(x)h_{i_2}(x)\right\} h'_{i_3}(x)\,dx \cdot \int_{-1}^{1} h_{j_1}(y)h'_{j_2}(y)h_{j_3}(y)\,dy \\
&\quad - \int_{-1}^{1} \left\{h'_{i_1}(x)h'_{i_2}(x) + h_{i_1}(x)h''_{i_2}(x)\right\} h'_{i_3}(x)\,dx \cdot \int_{-1}^{1} h'_{j_1}(y)h_{j_2}(y)h_{j_3}(y)\,dy
\end{aligned}$$

(see appendix D, equations D.3 and D.4). Since the integrands are all polynomials, we may calculate these integrals using the Gauss-Legendre quadrature rule (see appendix A). We may write

$$\sum_{k=0}^{(N+1)^2-1} (\tilde{\mathbf{v}}_k, \tilde{\mathbf{v}}_m) \frac{d\alpha^k}{dt} = - \sum_{k=0}^{(N+1)^2-1} \sum_{l=0}^{(N+1)^2-1} ([\tilde{\mathbf{v}}_l, \tilde{\mathbf{v}}_m], \tilde{\mathbf{v}}_k) \alpha^k \alpha^l$$

as

$$A\dot{\boldsymbol{\alpha}} = -B(\boldsymbol{\alpha})\boldsymbol{\alpha}$$

where component

$$A_{mk} = (\tilde{\mathbf{v}}_k, \tilde{\mathbf{v}}_m)$$

and where

$$B(\boldsymbol{\alpha}) = \sum_{k=0}^{(N+1)^2-1} \alpha^k C_k$$

with component ml of matrix C_k

$$(C_k)_{ml} = ([\tilde{\mathbf{v}}_l, \tilde{\mathbf{v}}_m], \tilde{\mathbf{v}}_k).$$

Note that A is symmetric as $(\tilde{\mathbf{v}}_k, \tilde{\mathbf{v}}_m) = (\tilde{\mathbf{v}}_m, \tilde{\mathbf{v}}_k)$, and that C_k is antisymmetric for all k since $([\tilde{\mathbf{v}}_l, \tilde{\mathbf{v}}_m], \tilde{\mathbf{v}}_k) = -([\tilde{\mathbf{v}}_m, \tilde{\mathbf{v}}_l], \tilde{\mathbf{v}}_k)$, from which follows that $B(\boldsymbol{\alpha})$ is antisymmetric as it is a linear combination of antisymmetric matrices.

4.4 Kinetic energy conserving numerical time integration

The kinetic energy K within a volume Ω at a certain moment is

$$K = \int_\Omega \frac{1}{2}\|\mathbf{v}\|^2 \, d\Omega = \frac{1}{2}\int_\Omega \langle \mathbf{v}, \mathbf{v}\rangle \, d\Omega = \frac{1}{2}(\mathbf{v}, \mathbf{v}).$$

Remembering that

$$\mathbf{v}_h = \sum_{k=0}^{(N+1)^2-1} \alpha^k \tilde{\mathbf{v}}_k(x, y),$$

the kinetic energy is approximated by

$$
\begin{aligned}
K \approx K_h &= \frac{1}{2}\left(\sum_{k=0}^{(N+1)^2-1} \alpha^k \tilde{\mathbf{v}}_k, \sum_{l=0}^{(N+1)^2-1} \alpha^l \tilde{\mathbf{v}}_l\right) \\
&= \sum_{k=0}^{(N+1)^2-1} \sum_{l=0}^{(N+1)^2-1} \frac{1}{2}\alpha^k \alpha^l (\tilde{\mathbf{v}}_k, \tilde{\mathbf{v}}_l) \\
&= \frac{1}{2}\boldsymbol{\alpha}^T A \boldsymbol{\alpha}.
\end{aligned}
$$

At the continuous level, the Euler equations conserve kinetic energy in domains not subject to in- or outflow, a property which we strive to maintain in the numerical time integration. Let $\boldsymbol{\alpha}^n$ be the value of $\boldsymbol{\alpha}$ at time step $n \in \mathbb{Z}$. Then conservation of kinetic energy in the numerical scheme means that for all n,

$$\frac{1}{2}\left(\boldsymbol{\alpha}^{n+1}\right)^T A \boldsymbol{\alpha}^{n+1} = \frac{1}{2}\left(\boldsymbol{\alpha}^n\right)^T A \boldsymbol{\alpha}^n$$

or

$$
\begin{aligned}
0 &= \frac{\left(\boldsymbol{\alpha}^{n+1}\right)^T A \boldsymbol{\alpha}^{n+1} - \left(\boldsymbol{\alpha}^n\right)^T A \boldsymbol{\alpha}^n}{2\,\Delta t} \\
&= \frac{\left(\boldsymbol{\alpha}^{n+1}\right)^T A \boldsymbol{\alpha}^{n+1} + \left(\boldsymbol{\alpha}^{n+1}\right)^T A \boldsymbol{\alpha}^n - \left(\boldsymbol{\alpha}^{n+1}\right)^T A \boldsymbol{\alpha}^n - \left(\boldsymbol{\alpha}^n\right)^T A \boldsymbol{\alpha}^n}{2\,\Delta t} \\
&= \frac{\left(\boldsymbol{\alpha}^{n+1}\right)^T A \boldsymbol{\alpha}^{n+1} + \left(\left(\boldsymbol{\alpha}^{n+1}\right)^T A \boldsymbol{\alpha}^n\right)^T - \left(\boldsymbol{\alpha}^{n+1}\right)^T A \boldsymbol{\alpha}^n - \left(\boldsymbol{\alpha}^n\right)^T A \boldsymbol{\alpha}^n}{2\,\Delta t} \\
&= \frac{\left(\boldsymbol{\alpha}^{n+1}\right)^T A \boldsymbol{\alpha}^{n+1} + \left(\boldsymbol{\alpha}^n\right)^T A \boldsymbol{\alpha}^{n+1} - \left(\boldsymbol{\alpha}^{n+1}\right)^T A \boldsymbol{\alpha}^n - \left(\boldsymbol{\alpha}^n\right)^T A \boldsymbol{\alpha}^n}{2\,\Delta t} \\
&= \frac{\left(\boldsymbol{\alpha}^{n+1}\right)^T + \left(\boldsymbol{\alpha}^n\right)^T}{2} A \frac{\boldsymbol{\alpha}^{n+1} - \boldsymbol{\alpha}^n}{\Delta t} \\
&= \left(\frac{\boldsymbol{\alpha}^{n+1} + \boldsymbol{\alpha}^n}{2}\right)^T A \frac{\boldsymbol{\alpha}^{n+1} - \boldsymbol{\alpha}^n}{\Delta t}
\end{aligned}
$$

where $\Delta t \in \mathbb{R}_{\neq 0}$. For any vector of the right dimension $\mathbf{b}$, $\mathbf{b}^T B \mathbf{b}$ equals a number b, but also $\left(\mathbf{b}^T B \mathbf{b}\right)^T = b$ or $\mathbf{b}^T B^T \mathbf{b} = b$. However, since B is skew-symmetric, $\mathbf{b}^T B^T \mathbf{b} = -\mathbf{b}^T B \mathbf{b} = b$. We see that $\mathbf{b}^T B \mathbf{b} = -\mathbf{b}^T B \mathbf{b}$, or $\mathbf{b}^T B \mathbf{b} = 0$. One particular choice for which this holds is

$$\mathbf{b} = \frac{\boldsymbol{\alpha}^{n+1} + \boldsymbol{\alpha}^n}{2}.$$

We may now write the equality $0 = 0$ as

$$\left(\frac{\boldsymbol{\alpha}^{n+1} + \boldsymbol{\alpha}^n}{2}\right)^T A \frac{\boldsymbol{\alpha}^{n+1} - \boldsymbol{\alpha}^n}{\Delta t} = -\left(\frac{\boldsymbol{\alpha}^{n+1} + \boldsymbol{\alpha}^n}{2}\right)^T B \frac{\boldsymbol{\alpha}^{n+1} + \boldsymbol{\alpha}^n}{2}. \tag{4.2}$$

If we approximate

$$\dot{\boldsymbol{\alpha}} \approx \frac{\boldsymbol{\alpha}^{n+1} - \boldsymbol{\alpha}^n}{\Delta t},$$
$$\boldsymbol{\alpha} \approx \frac{\boldsymbol{\alpha}^{n+1} + \boldsymbol{\alpha}^n}{2},$$

then the semi-discrete momentum equation $A\dot{\boldsymbol{\alpha}} = -B(\boldsymbol{\alpha})\boldsymbol{\alpha}$ yields the fully discrete momentum equation

$$A\frac{\boldsymbol{\alpha}^{n+1} - \boldsymbol{\alpha}^n}{\Delta t} = -B\left(\frac{\boldsymbol{\alpha}^{n+1} + \boldsymbol{\alpha}^n}{2}\right)\frac{\boldsymbol{\alpha}^{n+1} + \boldsymbol{\alpha}^n}{2}.$$

When Δt and $\boldsymbol{\alpha}^{m_1}$, $m_1 \in \mathbb{Z}$ are given, we may solve this equation $|m_2 - m_1|$ times iteratively to yield a solution $\boldsymbol{\alpha}^{m_2}$, $m_2 \in \mathbb{Z}$, either by solving it for $\boldsymbol{\alpha}^{n+1}$ for given $\boldsymbol{\alpha}^n$ or the other way around. Each time we solve the fully discrete momentum equation, the kinetic energy is conserved: pre-multiplying by the transpose of the approxmation of $\boldsymbol{\alpha}$ yields equation 4.2.

We may rewrite the fully discrete momentum equation, dropping the argument of B, as

$$\begin{aligned} A\frac{\boldsymbol{\alpha}^{n+1} - \boldsymbol{\alpha}^n}{\Delta t} &= -B\frac{\boldsymbol{\alpha}^{n+1} + \boldsymbol{\alpha}^n}{2} \iff \\ 2A\left(\boldsymbol{\alpha}^{n+1} - \boldsymbol{\alpha}^n\right) &= -\Delta t B\left(\boldsymbol{\alpha}^{n+1} + \boldsymbol{\alpha}^n\right) \iff \\ (2A + \Delta t B)\boldsymbol{\alpha}^{n+1} &= (2A - \Delta t B)\boldsymbol{\alpha}^n. \end{aligned} \tag{4.3}$$

Solving it for $\boldsymbol{\alpha}^{n+1}$ for given Δt and $\boldsymbol{\alpha}^n$ is not possible as the matrix $(2A + \Delta t B)$ is singular but prescribing one component of $\boldsymbol{\alpha}^{n+1}$ determines all other components (see appendix B). In this thesis, we restrict the problem even further: we require that the flow is tangential to the boundary of the considered square. Setting the normal velocity to zero at the boundary is done by prescribing that all components of $\boldsymbol{\alpha}^n$ for any n that correspond to a boundary node are set to the same constant Q.

Solving the fully discrete momentum equation for $\boldsymbol{\alpha}^{n+1}$ for given Δt, $\boldsymbol{\alpha}^n$ and mentioned boundary conditions is done as follows. We first choose an initial estimate of $\boldsymbol{\alpha}^{n+1}$, which we call $\left(\boldsymbol{\alpha}^{n+1}_{\text{temp}}\right)_0$. Then we calculate $B\left(\left[\left(\boldsymbol{\alpha}^{n+1}_{\text{temp}}\right)_0 + \boldsymbol{\alpha}^n\right]/2\right)$. The next step is to calculate from the fully discrete momentum equation a new estimate of $\boldsymbol{\alpha}^{n+1}$, which we will call $\left(\boldsymbol{\alpha}^{n+1}_{\text{temp}}\right)_1$. We calculate it from

$$\left(2A + \Delta t B\left(\frac{\left(\boldsymbol{\alpha}^{n+1}_{\text{temp}}\right)_0 + \boldsymbol{\alpha}^n}{2}\right)\right)\left(\boldsymbol{\alpha}^{n+1}_{\text{temp}}\right)_1 = \left(2A - \Delta t B\left(\frac{\left(\boldsymbol{\alpha}^{n+1}_{\text{temp}}\right)_0 + \boldsymbol{\alpha}^n}{2}\right)\right)\boldsymbol{\alpha}^n.$$

In the same way we calculated $\left(\boldsymbol{\alpha}^{n+1}_{\text{temp}}\right)_1$ from $\left(\boldsymbol{\alpha}^{n+1}_{\text{temp}}\right)_0$, we may calculate $\left(\boldsymbol{\alpha}^{n+1}_{\text{temp}}\right)_{k+1}$ from $\left(\boldsymbol{\alpha}^{n+1}_{\text{temp}}\right)_k$ for any $k \in \mathbb{N}$. We do this until $\left(\boldsymbol{\alpha}^{n+1}_{\text{temp}}\right)_{k+1}$ is sufficiently close to $\left(\boldsymbol{\alpha}^{n+1}_{\text{temp}}\right)_k$, by which we mean that the magnitude of each component of the vector $\left(\left(\boldsymbol{\alpha}^{n+1}_{\text{temp}}\right)_{k+1} - \left(\boldsymbol{\alpha}^{n+1}_{\text{temp}}\right)_{k}\right)$ is smaller than a certain small positive number ϵ, the order of magnitude of which is the resolution with which numbers are represented on a computer. In the numerical tests we performed, we chose $\epsilon = 10^{-13}$. When $\left(\boldsymbol{\alpha}^{n+1}_{\text{temp}}\right)_{k+1}$ is sufficiently close to $\left(\boldsymbol{\alpha}^{n+1}_{\text{temp}}\right)_{k}$, we say that $\boldsymbol{\alpha}^{n+1} = \left(\boldsymbol{\alpha}^{n+1}_{\text{temp}}\right)_{k+1}$.

Solving the fully discrete momentum equation for $\boldsymbol{\alpha}^n$ for given Δt, $\boldsymbol{\alpha}^{n+1}$ and mentioned boundary conditions may be done in a similar fashion.

Finally, the velocity field approximation $\mathbf{v}_h^n$ at time step n is

$$\mathbf{v}_h^n = \sum_{k=0}^{(N+1)^2-1} (\alpha^n)^k \, \tilde{\mathbf{v}}_k(x,y)$$

and at each time step n, the kinetic energy is the same.

4.5 Conservation of vorticity and enstrophy

For the approximation of $\omega^{(2)}$, we use the nodes $(\bar{x}_i, \bar{y}_j)$, to which the dual mesh is associated, and we use edge functions, introduced by Gerritsma [6].

Suppose we have nodes on the real line at z_i, $i \in \{0, \ldots, P\}$ and the corresponding Lagrange polynomials $h_i(z)$. The edge function $e_i(z)$, $i \in \{1, \ldots, P\}$ is the 1-form $\epsilon_i(z)dz$ having $P-1$ as the order of $\epsilon_i(z)$ and being such that integration from z_{j-1} to z_j yields the number δ^i_j. It satisfies

$$e_i(z) = -\sum_{j=0}^{i-1} dh_j(z) = -\sum_{j=0}^{i-1} h'_j(z)dz = \epsilon_i(z)\, dz$$

(see Appendix C) where

$$\epsilon_i(z) = -\sum_{j=0}^{i-1} h'_j(z).$$

We now approximate $\omega^{(2)}$ as

$$\begin{aligned} \omega^{(2)} \approx \omega_h^{(2)} &= \sum_{i=1}^{N+1}\sum_{j=1}^{N+1} \omega^{i+(N+1)(j-1)} e_i^x(x) \wedge e_j^y(y) \\ &= \sum_{i=1}^{N+1}\sum_{j=1}^{N+1} \omega^{i+(N+1)(j-1)} \tilde{\omega}^{(2)}_{i+(N+1)(j-1)} \end{aligned} \tag{4.4}$$

where

$$
\begin{aligned}
\tilde{\omega}^{(2)}_{i+(N+1)(j-1)} &= e^x_i(x) \wedge e^y_j(y) \\
&= \epsilon^x_i(x)\epsilon^y_j(y) dx \wedge dy \\
&= \tilde{\omega}_{i+(N+1)(j-1)}(x,y) dx \wedge dy
\end{aligned}
$$

with $\tilde{\omega}_{i+(N+1)(j-1)}(x,y) = \epsilon^x_i(x)\epsilon^y_j(y)$, and where $e^x_i(x)$ and $e^y_j(y)$ are the edge functions in x- and y-directions respectively. Integration of $\tilde{\omega}^{(2)}_{i+(N+1)(j-1)}$ over a rectangular part of the square $[\bar{x}_{k-1}, \bar{x}_k] \times [\bar{y}_{l-1}, \bar{y}_l]$, $k, l \in \{1, \dots, N+1\}$ yields

$$
\int_{\bar{x}_{k-1}}^{\bar{x}_k} \int_{\bar{y}_{l-1}}^{\bar{y}_l} e_i(x) \wedge e_j(y) = \int_{\bar{x}_{k-1}}^{\bar{x}_k} e_i(x) \cdot \int_{\bar{y}_{l-1}}^{\bar{y}_l} e_j(y) = \delta^k_i \delta^l_j
$$

so we may interpret the coefficient $\omega^{i+(N+1)(j-1)}$ to be the integral of the vorticity approximation over the rectangle $[\bar{x}_{i-1}, \bar{x}_i] \times [\bar{y}_{j-1}, \bar{y}_j]$. Note that $\omega^{i+(N+1)(j-1)}$ are in general a function of time and that

$$
\begin{aligned}
*\tilde{\omega}^{(2)}_{i+(N+1)(j-1)} &= *\Big(\epsilon^x_i(x)\epsilon^y_j(y) dx \wedge dy\Big) \\
&= \epsilon^x_i(x)\epsilon^y_j(y) *(dx \wedge dy) \\
&= \epsilon^x_i(x)\epsilon^y_j(y).
\end{aligned} \tag{4.5}
$$

Filling in the approximation of $\omega^{(2)}$ and those of the velocity components,

$$
u \approx u_h = \sum_{l=0}^{(N+1)^2-1} \alpha^l \tilde{u}_l, \qquad v \approx v_h = \sum_{l=0}^{(N+1)^2-1} \alpha^l \tilde{v}_l, \tag{4.6}
$$

in the vorticity equation, equation 4.1, gives

$$
\sum_{k=1}^{(N+1)^2} \left\{ \tilde{\omega}^{(2)}_k \frac{d\omega^k}{dt} + \sum_{l=0}^{(N+1)^2-1} \alpha^l \omega^k \left(\tilde{u}_l \frac{\partial}{\partial x} \left(\tilde{\omega}^{(2)}_k \right) + \tilde{v}_l \frac{\partial}{\partial y} \left(\tilde{\omega}^{(2)}_k \right) \right) \right\} \approx 0 \tag{4.7}
$$

and then taking the inner product of each side with $\tilde{\omega}^{(2)}_m$, $m \in \{1, \dots, (N+1)^2\}$ yields

$$
\begin{aligned}
&\left(\sum_{k=1}^{(N+1)^2} \left\{ \tilde{\omega}^{(2)}_k \frac{d\omega^k}{dt} + \sum_{l=0}^{(N+1)^2-1} \alpha^l \omega^k \left(\tilde{u}_l \frac{\partial}{\partial x} \left(\tilde{\omega}^{(2)}_k \right) + \tilde{v}_l \frac{\partial}{\partial y} \left(\tilde{\omega}^{(2)}_k \right) \right) \right\}, \tilde{\omega}^{(2)}_m \right) = 0 \iff \\
&\sum_{k=1}^{(N+1)^2} \left(\tilde{\omega}^{(2)}_k, \tilde{\omega}^{(2)}_m \right) \frac{d\omega^k}{dt} = - \sum_{k=1}^{(N+1)^2} \sum_{l=0}^{(N+1)^2-1} \left(\left(\tilde{u}_l \frac{\partial}{\partial x} \left(\tilde{\omega}^{(2)}_k \right) + \tilde{v}_l \frac{\partial}{\partial y} \left(\tilde{\omega}^{(2)}_k \right) \right), \tilde{\omega}^{(2)}_m \right) \alpha^l \omega^k.
\end{aligned} \tag{4.8}
$$

This is a system of $(N+1)^2$ linear ordinary differential equations (one for each $\tilde{\omega}_m^{(2)}$) having the $(N+1)^2$ functions $\omega^k(t)$ to be solved for. $\left(\tilde{\omega}_k^{(2)}, \tilde{\omega}_m^{(2)}\right)$ and $\left(\left(\tilde{u}_l \frac{\partial}{\partial x}\left(\tilde{\omega}_k^{(2)}\right) + \tilde{v}_l \frac{\partial}{\partial y}\left(\tilde{\omega}_k^{(2)}\right)\right), \tilde{\omega}_m^{(2)}\right)$ are numbers that are independent of the solution of the system of equations. Let us now calculate these numbers explicitly, where we will refer to $\epsilon_i^x(x) = \epsilon_i^y(x)$ as $\epsilon_i(x)$ since $\bar{x}_i = \bar{y}_i$ for all i:

$$\left(\tilde{\omega}_{i_1+(N+1)(j_1-1)}^{(2)}, \tilde{\omega}_{i_2+(N+1)(j_2-1)}^{(2)}\right) = \int_{-1}^{1} \epsilon_{i_1}(x)\epsilon_{i_2}(x)dx \cdot \int_{-1}^{1} \epsilon_{j_1}(y)\epsilon_{j_2}(y)dy$$

and

$$\left(\left(\tilde{u}_{i_3+(N+1)j_3} \frac{\partial}{\partial x}\left(\tilde{\omega}_{i_1+(N+1)(j_1-1)}^{(2)}\right) + \tilde{v}_{i_3+(N+1)j_3} \frac{\partial}{\partial y}\left(\tilde{\omega}_{i_1+(N+1)(j_1-1)}^{(2)}\right)\right), \tilde{\omega}_{i_2+(N+1)(j_2-1)}^{(2)}\right)$$
$$= \int_{-1}^{1} \epsilon'_{i_1}(x)\epsilon_{i_2}(x)h_{i_3}(x)dx \cdot \int_{-1}^{1} \epsilon_{j_1}(y)\epsilon_{j_2}(y)h'_{j_3}(y)dy - \int_{-1}^{1} \epsilon_{i_1}(x)\epsilon_{i_2}(x)h'_{i_3}(x)dx \cdot \int_{-1}^{1} \epsilon'_{j_1}(y)\epsilon_{j_2}(y)h_{j_3}(y)dy$$

(see appendix D, equations D.5 and D.6). The integrands are all polynomials so we may calculate these integrals using the Gauss-Legendre quadrature rule (see appendix A). We may write equation 4.8,

$$\sum_{k=1}^{(N+1)^2} \left(\tilde{\omega}_k^{(2)}, \tilde{\omega}_m^{(2)}\right) \frac{d\omega^k}{dt} = -\sum_{k=1}^{(N+1)^2} \sum_{l=0}^{(N+1)^2-1} \left(\left(\tilde{u}_l \frac{\partial}{\partial x}\left(\tilde{\omega}_k^{(2)}\right) + \tilde{v}_l \frac{\partial}{\partial y}\left(\tilde{\omega}_k^{(2)}\right)\right), \tilde{\omega}_m^{(2)}\right) \alpha^l \omega^k,$$

as

$$F\dot{\boldsymbol{\omega}} = -G(\boldsymbol{\alpha})\boldsymbol{\omega} \tag{4.9}$$

where component

$$F_{mk} = \left(\tilde{\omega}_k^{(2)}, \tilde{\omega}_m^{(2)}\right)$$

and where

$$G(\boldsymbol{\alpha}) = \sum_{l=0}^{(N+1)^2-1} \alpha^l H_l \tag{4.10}$$

with component mk of matrix H_l

$$(H_l)_{mk} = \left(\left(\tilde{u}_l \frac{\partial}{\partial x}\left(\tilde{\omega}_k^{(2)}\right) + \tilde{v}_l \frac{\partial}{\partial y}\left(\tilde{\omega}_k^{(2)}\right)\right), \tilde{\omega}_m^{(2)}\right).$$

Note that F is symmetric as $\left(\tilde{\omega}_k^{(2)}, \tilde{\omega}_m^{(2)}\right) = \left(\tilde{\omega}_m^{(2)}, \tilde{\omega}_k^{(2)}\right)$.

4.5.1 Conservation of enstrophy

The enstrophy integral E within the square considered at a certain moment is

$$E = \frac{1}{2} \int_{-1}^{1} \int_{-1}^{1} (\omega(x,y))^2 dx \wedge dy$$

where $\omega(x,y)$ is the function in $\omega^{(2)} = \omega(x,y) dx \wedge dy$. Remembering that for a given time,

$$\omega(x,y) \approx \omega_h(x,y) = \sum_{k=1}^{(N+1)^2} \omega^k \tilde{\omega}_k(x,y),$$

where $\tilde{\omega}_k(x,y)$ is the function in $\tilde{\omega}^{(2)} = \tilde{\omega}_k(x,y) dx \wedge dy$, the enstrophy integral is approximated by

$$\begin{aligned} E \approx E_h &= \frac{1}{2} \sum_{k=1}^{(N+1)^2} \sum_{l=1}^{(N+1)^2} \omega^k \omega^l \left(\tilde{\omega}_k^{(2)}, \tilde{\omega}_l^{(2)} \right) \\ &= \frac{1}{2} \boldsymbol{\omega}^T F \boldsymbol{\omega}. \end{aligned}$$

(see appendix D, equation D.7). At the continuous level, the vorticity equation conserves the enstrophy integral in domains not subject to in- or outflow, a property which we strive to maintain in the numerical time integration. Let $\boldsymbol{\omega}^{n+\frac{1}{2}}$ be the value of $\boldsymbol{\omega}$ at time step $n+\frac{1}{2}$, $n \in \mathbb{Z}$. Then conservation of the enstrophy integral in the numerical scheme means that for all n,

$$\frac{1}{2} \left(\boldsymbol{\omega}^{n+\frac{1}{2}} \right)^T F \boldsymbol{\omega}^{n+\frac{1}{2}} = \frac{1}{2} \left(\boldsymbol{\omega}^{n-\frac{1}{2}} \right)^T F \boldsymbol{\omega}^{n-\frac{1}{2}}$$

or, similar to numerical conservation of kinetic energy,

$$\begin{aligned} 0 &= \frac{\left(\boldsymbol{\omega}^{n+\frac{1}{2}} \right)^T F \boldsymbol{\omega}^{n+\frac{1}{2}} - \left(\boldsymbol{\omega}^{n-\frac{1}{2}} \right)^T F \boldsymbol{\omega}^{n-\frac{1}{2}}}{2\,\Delta t} \\ &= \left(\frac{\boldsymbol{\omega}^{n+\frac{1}{2}} + \boldsymbol{\omega}^{n-\frac{1}{2}}}{2} \right)^T F \frac{\boldsymbol{\omega}^{n+\frac{1}{2}} - \boldsymbol{\omega}^{n-\frac{1}{2}}}{\Delta t}. \end{aligned}$$

Furthermore, if G is antisymmetric, then for any vector of the right dimension $\mathbf{b}$, $\mathbf{b}^T G \mathbf{b} = 0$. One particular choice for which this holds is

$$\mathbf{b} = \frac{\boldsymbol{\omega}^{n+\frac{1}{2}} + \boldsymbol{\omega}^{n-\frac{1}{2}}}{2}.$$

If G is antisymmetric, we may write the equality $0 = 0$ as

$$\left(\frac{\boldsymbol{\omega}^{n+\frac{1}{2}} + \boldsymbol{\omega}^{n-\frac{1}{2}}}{2}\right)^T F \frac{\boldsymbol{\omega}^{n+\frac{1}{2}} - \boldsymbol{\omega}^{n-\frac{1}{2}}}{\Delta t} = -\left(\frac{\boldsymbol{\omega}^{n+\frac{1}{2}} + \boldsymbol{\omega}^{n-\frac{1}{2}}}{2}\right)^T G \frac{\boldsymbol{\omega}^{n+\frac{1}{2}} + \boldsymbol{\omega}^{n-\frac{1}{2}}}{2}. \tag{4.11}$$

If we approximate

$$\dot{\boldsymbol{\omega}} \approx \frac{\boldsymbol{\omega}^{n+\frac{1}{2}} - \boldsymbol{\omega}^{n-\frac{1}{2}}}{\Delta t},$$
$$\boldsymbol{\omega} \approx \frac{\boldsymbol{\omega}^{n+\frac{1}{2}} + \boldsymbol{\omega}^{n-\frac{1}{2}}}{2},$$

then the semi-discrete vorticity equation $F\dot{\boldsymbol{\omega}} = -G(\boldsymbol{\alpha})\boldsymbol{\omega}$ yields the fully discrete vorticity equation

$$F\frac{\boldsymbol{\omega}^{n+\frac{1}{2}} - \boldsymbol{\omega}^{n-\frac{1}{2}}}{\Delta t} = -G\left(\boldsymbol{\alpha}^n\right)\frac{\boldsymbol{\omega}^{n+\frac{1}{2}} + \boldsymbol{\omega}^{n-\frac{1}{2}}}{2}.$$

When $\boldsymbol{\omega}^{m_1+\frac{1}{2}}$, $m_1 \in \mathbb{Z}$ is given, we may solve this equation $|m_2 - m_1|$ times to yield a solution $\boldsymbol{\omega}^{m_2+\frac{1}{2}}$, $m_2 \in \mathbb{Z}$, either by solving it for $\boldsymbol{\omega}^{n+\frac{1}{2}}$ for given $\boldsymbol{\omega}^{n-\frac{1}{2}}$ or the other way around. Each time we solve the fully discrete vorticity equation, the enstrophy integral is conserved if the matrix G is antisymmetric: pre-multiplying by the transpose of the approxmation of $\boldsymbol{\omega}$ yields equation 4.11.

We may rewrite the fully discrete vorticity equation as

$$\begin{aligned} F\frac{\boldsymbol{\omega}^{n+\frac{1}{2}} - \boldsymbol{\omega}^{n-\frac{1}{2}}}{\Delta t} &= -G\left(\boldsymbol{\alpha}^n\right)\frac{\boldsymbol{\omega}^{n+\frac{1}{2}} + \boldsymbol{\omega}^{n-\frac{1}{2}}}{2} \iff \\ 2F\left(\boldsymbol{\omega}^{n+\frac{1}{2}} - \boldsymbol{\omega}^{n-\frac{1}{2}}\right) &= -\Delta t G\left(\boldsymbol{\alpha}^n\right)\left(\boldsymbol{\omega}^{n+\frac{1}{2}} + \boldsymbol{\omega}^{n-\frac{1}{2}}\right) \iff \\ \left(2F + \Delta t G\left(\boldsymbol{\alpha}^n\right)\right)\boldsymbol{\omega}^{n+\frac{1}{2}} &= \left(2F - \Delta t G\left(\boldsymbol{\alpha}^n\right)\right)\boldsymbol{\omega}^{n-\frac{1}{2}}. \end{aligned} \tag{4.12}$$

Solving it for $\boldsymbol{\omega}^{n+\frac{1}{2}}$ for given Δt and $\boldsymbol{\omega}^{n-\frac{1}{2}}$ is possible if the matrix $\left(2F + \Delta t G\left(\boldsymbol{\alpha}^n\right)\right)$ is non-singular. Note that we use a staggered time grid: the velocity fields are computed at time steps n whereas the vorticity fields are determined at time steps $n + \frac{1}{2}$ because by discretizing time as such, we may apply the second order accurate midpoint rule.

Solving the fully discrete vorticity equation for $\boldsymbol{\omega}^{n-\frac{1}{2}}$ for given Δt and $\boldsymbol{\omega}^{n+\frac{1}{2}}$ may be done in a similar fashion.

Finally, the vorticity field approximation $\omega^{(2)}$ at time step $n + \frac{1}{2}$ is

$$\left(\omega^{(2)}\right)^{n+\frac{1}{2}} = \sum_{k=1}^{(N+1)^2}\left(\omega^{n+\frac{1}{2}}\right)^k \tilde{\omega}_k^{(2)}$$

and at each time step $n + \frac{1}{2}$, the enstrophy integral is the same if the matrix G is antisymmetric.

4.5.2 Conservation of vorticity

Each vorticity field approximation is constructed from a solution of the fully discrete vorticity equation 4.12,

$$(2F + \Delta t G(\boldsymbol{\alpha}^n))\,\boldsymbol{\omega}^{n+\frac{1}{2}} = (2F - \Delta t G(\boldsymbol{\alpha}^n))\,\boldsymbol{\omega}^{n-\frac{1}{2}},$$

which approximates the semi-discrete vorticity equation,

$$\sum_{k=1}^{(N+1)^2} \left(\tilde{\omega}_k^{(2)}, \tilde{\omega}_m^{(2)}\right) \frac{d\omega^k}{dt} = -\sum_{k=1}^{(N+1)^2} \sum_{l=0}^{(N+1)^2-1} \left(\left(\tilde{u}_l \frac{\partial}{\partial x}\left(\tilde{\omega}_k^{(2)}\right) + \tilde{v}_l \frac{\partial}{\partial y}\left(\tilde{\omega}_k^{(2)}\right)\right), \tilde{\omega}_m^{(2)}\right) \alpha^l \omega^k$$

(see equation 4.8). Instead of taking the inner product with $\tilde{\omega}_m^{(2)}$ for all m, we could have taken the inner product with $\tilde{\omega}_m^{(2)}$ for all m but one, and with

$$\sum_{i=1}^{N+1} \sum_{j=1}^{N+1} (\bar{x}_i - \bar{x}_{i-1})(\bar{y}_j - \bar{y}_{j-1})\,\tilde{\omega}_{i+(N+1)(j-1)}^{(2)} \tag{4.13}$$

and the same outcome for $\boldsymbol{\omega}^{n+\frac{1}{2}}$ would have been reached since the above is a linear combination of $\tilde{\omega}_m^{(2)}$ for all m. Let us now look into equation 4.13 in more detail:

$$\begin{aligned}
&\sum_{i=1}^{N+1} \sum_{j=1}^{N+1} (\bar{x}_i - \bar{x}_{i-1})(\bar{y}_j - \bar{y}_{j-1})\,\tilde{\omega}_{i+(N+1)(j-1)}^{(2)} \\
&= \sum_{i=1}^{N+1} \sum_{j=1}^{N+1} (\bar{x}_i - \bar{x}_{i-1})(\bar{y}_j - \bar{y}_{j-1})\, e_i(x) \wedge e_j(y) \\
&= \left(\sum_{i=1}^{N+1} (\bar{x}_i - \bar{x}_{i-1})\, e_i(x)\right) \wedge \left(\sum_{j=1}^{N+1} (\bar{y}_j - \bar{y}_{j-1})\, e_j(y)\right).
\end{aligned} \tag{4.14}$$

Consider the function $f(z) = z$, $z \in \mathbb{R}$. $f(z)$ can be represented as a linear combination of at least two Lagrange polynomials:

$$f(z) = z = \sum_{i=0}^{n} z_i h_i(z)$$

with $n \geq 1$ and for any set of points $\{z_i\}$ such that $z_i \neq z_j$ if $i \neq j$. The exterior derivative of $f(z)$ is

$$df(z) = d(z) = \frac{\partial z}{\partial z} dz = 1dz = dz$$

but also

$$df(z) = d\left(\sum_{i=0}^{n} z_i h_i(z)\right) = \sum_{i=0}^{n} z_i dh_i(z)$$
$$= \sum_{i=1}^{n} (z_i - z_{i-1})\, e_i(z)$$
$$= \sum_{i=1}^{n} (z_i - z_{i-1})\, \epsilon_i(z) dz$$

(see Appendix C). We find that if $(N+1) \geq 1$, which is always true,

$$dz = \sum_{i=1}^{N+1} (z_i - z_{i-1})\, e_i(z).$$

By applying this equality to equation 4.14, it follows that

$$\sum_{i=1}^{N+1} \sum_{j=1}^{N+1} (\bar{x}_i - \bar{x}_{i-1})\, (\bar{y}_j - \bar{y}_{j-1})\, \tilde{\omega}^{(2)}_{i+(N+1)(j-1)} = dx \wedge dy.$$

This means that the development of the solution $\boldsymbol{\omega}^{n+\frac{1}{2}}$ for all n satisfies the discretization of the inner product between equations 4.7 and 4.13,

$$\sum_{k=1}^{(N+1)^2} \left(\tilde{\omega}^{(2)}_k, dx \wedge dy\right) \frac{d\omega^k}{dt} = - \sum_{k=1}^{(N+1)^2} \sum_{l=0}^{(N+1)^2-1} \left(\left(\tilde{u}_l \frac{\partial}{\partial x}\left(\tilde{\omega}^{(2)}_k\right) + \tilde{v}_l \frac{\partial}{\partial y}\left(\tilde{\omega}^{(2)}_k\right)\right), dx \wedge dy\right) \alpha^l \omega^k,$$

in time as well. This is simply the integral of the vorticity equation, equation 4.1, over the square $[-1,1]\times[-1,1]$ where approximations 4.4 and 4.6 have been applied (see appendix D, equation D.8): our approximation satisfies equation 2.4. Therefore, if the development of the solution $\boldsymbol{\omega}^{n+\frac{1}{2}}$ conserves the integral of the enstrophy then it also conserves the integral of the vorticity over the square.

The above is valid for any choice of initial vector $\boldsymbol{\omega}^{m_1+\frac{1}{2}}$, and in the same way we may approximate the development over time of any 2-form $\alpha^{(2)}$ whose behaviour is described by the vorticity equation. In order to make it apply to the vorticity $\omega^{(2)}$ specifically, we need to somehow choose $\boldsymbol{\omega}^{m_1+\frac{1}{2}}$ for some m_1 based on all known $\boldsymbol{\alpha}^n$ as at the continuous level, vorticity and velocity are at any time related by

$$\omega(x,y) = \frac{\partial}{\partial x} v(x,y) - \frac{\partial}{\partial y} u(x,y) = -\Delta\psi(x,y).$$

The velocity field at time step $m_1 + \frac{1}{2}$ is approximately

$$\frac{1}{2}\left(\mathbf{v}_h^{m_1}+\mathbf{v}_h^{m_1+1}\right)=\frac{1}{2}\sum_{k=0}^{(N+1)^2-1}\left((\alpha^{m_1})^k\,\tilde{\mathbf{v}}_k(x,y)+\left(\alpha^{m_1+1}\right)^k\tilde{\mathbf{v}}_k(x,y)\right)$$
$$=\sum_{k=0}^{(N+1)^2-1}\left(\alpha^{m_1+\frac{1}{2}}\right)^k\tilde{\mathbf{v}}_k(x,y)$$
$$\doteq \mathbf{v}_h^{m_1+\frac{1}{2}}$$

where

$$\boldsymbol{\alpha}^{m_1+\frac{1}{2}}=\frac{\boldsymbol{\alpha}^{m_1}+\boldsymbol{\alpha}^{m_1+1}}{2}.$$

From this velocity field $\mathbf{v}_h^{m_1+\frac{1}{2}}$ we will now determine $\boldsymbol{\omega}^{m_1+\frac{1}{2}}$. Note that we could have approximated the velocity field at time step $m_1+\frac{1}{2}$ differently. A more crude approximation could have been $\mathbf{v}_h^{m_1+\frac{1}{2}}=\mathbf{v}_h^{m_1}$ and a more accurate approximation could have been made by applying, for example, higher order interpolation between the components of $\boldsymbol{\alpha}^n$ for more than two values of n. Remembering that $\omega^{i+(N+1)(j-1)}$ may be regarded to be the integral of the vorticity approximation over the rectangle $[\bar{x}_{i-1},\bar{x}_i]\times[\bar{y}_{j-1},\bar{y}_j]$, we find that the integral of the vorticity approximation over the square $[-1,1]\times[-1,1]$, V_h, is the sum of the components of $\boldsymbol{\omega}$. The stream function corresponding to $\mathbf{v}_h^{m_1+\frac{1}{2}}$ is

$$\psi_h^{m_1+\frac{1}{2}}(x,y)=\sum_{k=0}^{(N+1)^2-1}\left(\alpha^{m_1+\frac{1}{2}}\right)^k h_i(x)h_j(y),$$

and the corresponding vorticity is

$$\left(\omega_h^{(2)}\right)^{m_1+\frac{1}{2}}(x,y)=-\Delta\psi^{m_1+\frac{1}{2}}(x,y)dx\wedge dy$$
$$=-\sum_{i=0}^{N}\sum_{j=0}^{N}\left(\alpha^{m_1+\frac{1}{2}}\right)^{i+(N+1)j}\left(h_i''(x)h_j(y)+h_i(x)h_j''(y)\right)dx\wedge dy.$$

We now derive that the coefficients $\left(\omega^{m_1+\frac{1}{2}}\right)^{i+(N+1)(j-1)}$ of $\boldsymbol{\omega}^{m_1+\frac{1}{2}}$ are

$$\left(\omega^{m_1+\frac{1}{2}}\right)^{i+(N+1)(j-1)}=-\sum_{i=0}^{N}\sum_{j=0}^{N}\Bigg\{\left(\alpha^{m_1+\frac{1}{2}}\right)^{i+(N+1)j}\cdot$$
$$\left(\left[h_i'(\bar{x}_i)-h_i'(\bar{x}_{i-1})\right]\cdot\int_{\bar{y}_{j-1}}^{\bar{y}_j}h_j(y)dy+\int_{\bar{x}_{i-1}}^{\bar{x}_i}h_i(x)dx\cdot\left[h_j'(\bar{x}_j)-h_j'(\bar{y}_{j-1})\right]\right)\Bigg\}$$

(see appendix D, equation D.9). Note that the procedure of calculating $\boldsymbol{\omega}^{n+\frac{1}{2}}$ for $n \neq m_1$ does not guarantee that the numerical approximation $\left(\omega_h^{(2)}\right)^{m_1+\frac{1}{2}}(x, y)$ is the 2-form associated with the curl of the velocity field $\mathbf{v}_h^{n+\frac{1}{2}} \doteq \frac{1}{2}\left(\mathbf{v}_h^n + \mathbf{v}_h^{n+1}\right)$, as could be expected from the continuous relation $\boldsymbol{\omega} = \nabla \times \mathbf{v}$.

Chapter 5

Results

In this chapter we will discuss the results of a MATLAB implementation of the numerical method developed in the preceeding of this thesis.

For the discretization of the stream function, a primal mesh consisting of the $N+1$ Gauss-Lobatto-Legendre nodes in both directions is constructed. Using this mesh, a non-trivial initial stream function is constructed. A trivial stream function would be $\psi(x,y) = K$ for some $K \in \mathbb{R}$. The vorticity field is discretized on a dual mesh consisting of the N Gauss nodes supplemented by the end points $\bar{x}_0 = \bar{y}_0 = -1$ and $\bar{x}_{N+1} = \bar{y}_{N+1} = 1$.

Here we will discuss the results of flow simulations with different values of N of which the initial velocity field is such that $u = h_1(x)h_1'(y), v = h_1'(x)h_1(y)$ (see figure 5.1 for $N = 3$). The computations will be done with time going from $t = 0$ to $t = 50$ and we will choose $\Delta t = 0.01$. We say that $\boldsymbol{\alpha}^{n+1} = \left(\boldsymbol{\alpha}^{n+1}_{\text{temp}}\right)_{k+1}$ when the magnitude of each component of the vector $\left(\left(\boldsymbol{\alpha}^{n+1}_{\text{temp}}\right)_{k+1} - \left(\boldsymbol{\alpha}^{n+1}_{\text{temp}}\right)_{k}\right)$ is smaller than 10^{-13}. $\left(\boldsymbol{\alpha}^{n+1}_{\text{temp}}\right)_0$ is calculated by the explicit Euler method, from

$$A\frac{\left(\boldsymbol{\alpha}^{n+1}_{\text{temp}}\right)_0 - \boldsymbol{\alpha}^n}{\Delta t} = -B\left(\boldsymbol{\alpha}^n\right)\boldsymbol{\alpha}^n.$$

When $N = 3$, we find figures 5.2, 5.3 and 5.4. After 5000 time steps, representing 50 s, the absolute value of the deviation of the kinetic energy and the integrals of the vorticity and enstrophy from their initial values, are for any time step less than $7 \cdot 10^{-14}$, $8 \cdot 10^{-14}$ and $1.6 \cdot 10^{-12}$ respectively, whereas their initial values are of the order one. What is not depicted is the deviation from its initial value of the absolute value of the vorticity integral where the vorticity has been calculated by taking the curl of the velocity field, which never becomes more than $4 \cdot 10^{-13}$. Since numbers are only stored in the computer's memory with an accuracy of $2^{-52} \approx 2 \cdot 10^{-16}$ and since we use a convergence criterion for the calculation of the velocity field at each time step involving the number 10^{-13}, we consider the kinetic energy and integrals of the vorticity and enstrophy to remain constant in time. The matrix G in equation 4.9 may be considered antisymmetric: the

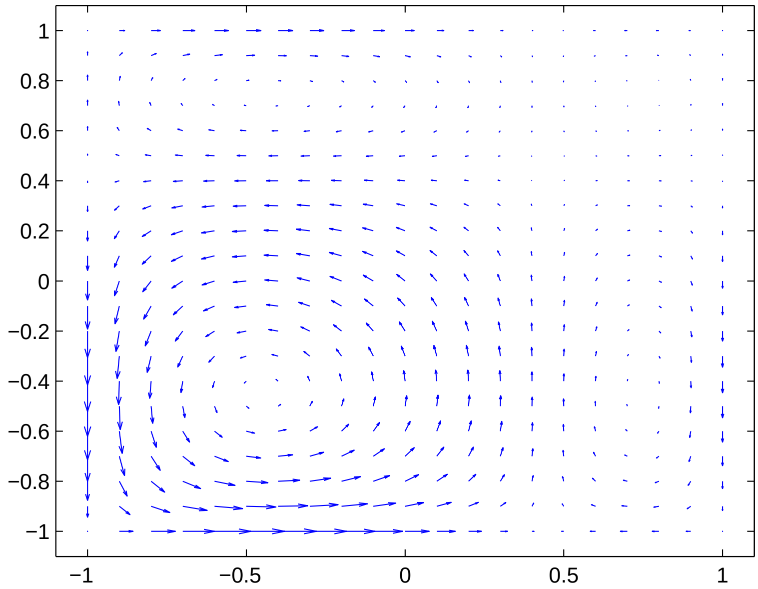

Figure 5.1: Initial velocity field for $N = 3$.

largest absolute value of a component of $G + G^T$ for any time step is less than $9 \cdot 10^{-14}$. Given the current mesh used, we see that the matrix H_k in equation 4.10 is antisymmetric for any k corresponding to a non-boundary node, that it is not antisymmetric for any k corresponding to a boundary node but also that for k corresponding to boundary nodes, $\sum_k H_k$ is antisymmetric. The latter corresponds to the situation where the stream function equals some constant K at any of the boundary nodes, or put differently, where the velocity is tangential to the boundary.

For $N = 8$, the absolute value of the deviation of the kinetic energy and the integrals of the vorticity and enstrophy from their initial values, respectively are less than $2 \cdot 10^{-13}$, $2.5 \cdot 10^{-12}$ and $1.2 \cdot 10^{-10}$ for any time step after 5000 time steps, whereas their initial values are again of the order one. Once more, the deviations are so small that we consider the kinetic energy and integrals of the vorticity and enstrophy to remain constant in time. Now, however, the deviation from its initial value of the absolute value of the vorticity integral where the vorticity has been calculated by taking the curl of the velocity field, becomes as large as 27, which should not be considered a deviation small enough to regard the vorticity integral thus calculated to be a conserved quantity. The largest absolute value of a component of $G + G^T$ for any time step is less than $5 \cdot 10^{-10}$, so again G may be considered antisymmetric. Note that the deviations from 0 are now larger than they were for $N = 3$. We believe this is because the numerical integration using Gauss quadrature becomes less accurate as the number of Gauss points used increases.

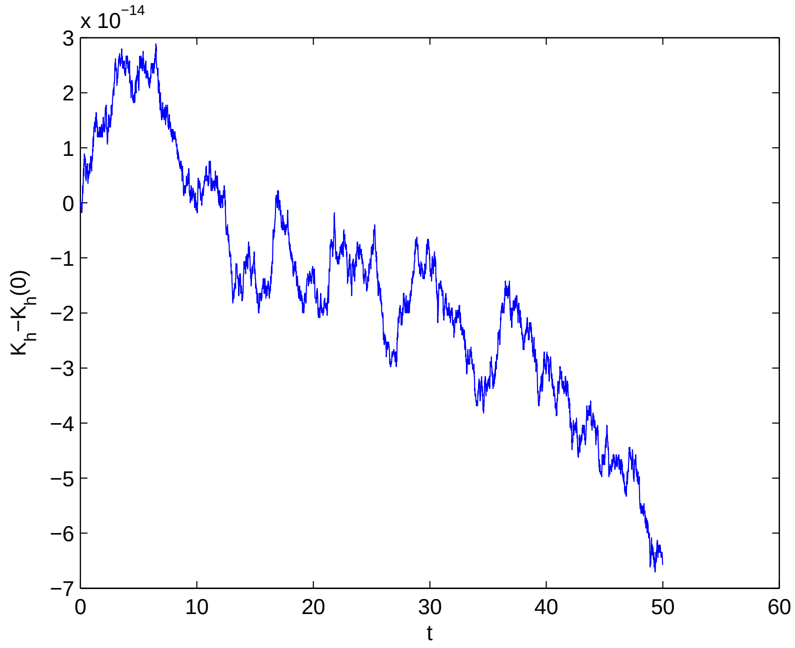

Figure 5.2: Deviation of the kinetic energy in the square from its initial value for $N = 3$.

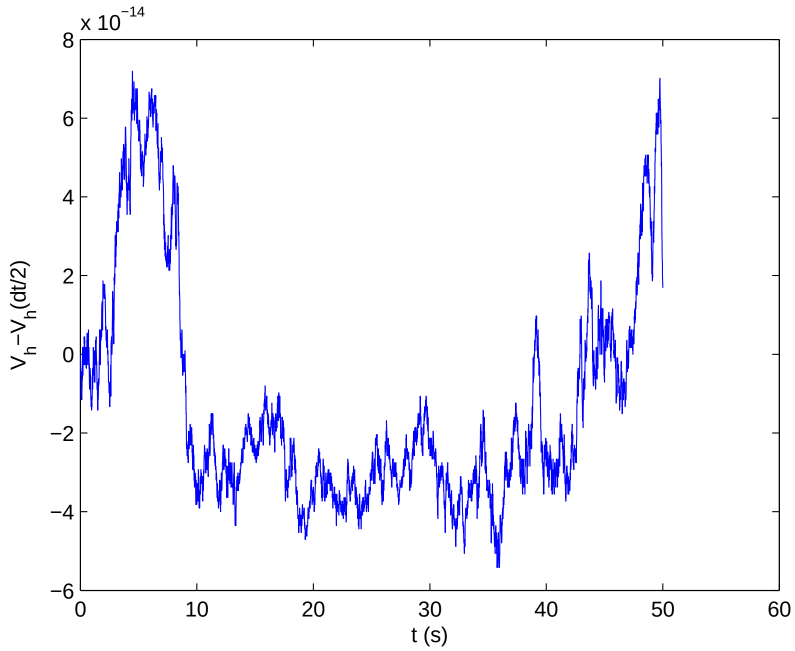

Figure 5.3: Deviation of the integral of the vorticity over the square from its initial value for $N = 3$.

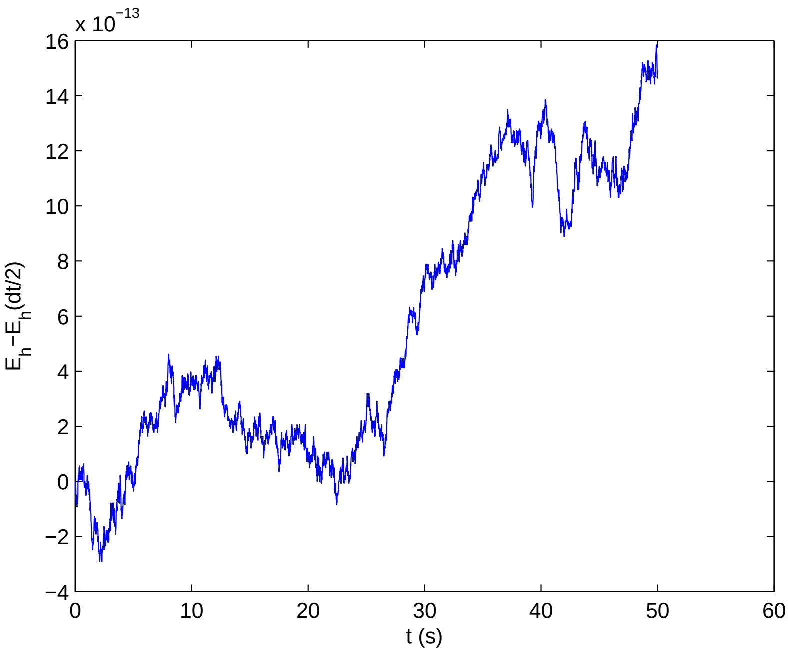

Figure 5.4: Deviation of the integral of the enstrophy in the square from its initial value for $N = 3$.

Chapter 6

Conclusions and recommendations

In this chapter we look back on the results obtained and interpret them from the perspective of the goal of this thesis. After that we suggest some directions for future work concerning the current topic.

6.1 Conclusions

The goal of our work was to develop a numerical method that conserves kinetic energy, the vorticity integral and the enstrophy integral for a flow which is governed by the Euler equations and which is confined to a domain that has no in- or outflow at the boundary. At the continuous level, such a flow conserves mentioned quantities.

The numerical tests performed indicate that we have succeeded. The method created and described in chapter 4 was tested in chapter 5 on a pair of meshes constructed using the points (x_i, y_i) and $(\bar{x}_j, \bar{y}_j)$ where x_i and y_i are the $N+1$ Gauss-Lobatto-Legendre nodes and where $\bar{x}_j$ and $\bar{y}_j$ are the N Gauss nodes supplemented by the end points $\bar{x}_0 = \bar{y}_0 = -1$ and $\bar{x}_{N+1} = \bar{y}_{N+1} = 1$. The calculations were performed for $N = 3$ and $N = 8$. The method indeed conserves kinetic energy, the vorticity integral and the enstrophy integral in the test cases: whereas the initial values themselves are of the order one, the deviations from their initial values that the quantities exhibit over time were never larger than $1.2 \cdot 10^{-10}$ and may be attributed to the accuracy of the implementation of the method instead of the method itself.

Conservation of the vorticity and enstrophy integrals can only be guaranteed if the matrix G in the fully discrete vorticity equation, $(2F + \Delta t G(\boldsymbol{\alpha}^n))\,\boldsymbol{\omega}^{n+\frac{1}{2}} = (2F - \Delta t G(\boldsymbol{\alpha}^n))\,\boldsymbol{\omega}^{n-\frac{1}{2}}$, is antisymmetric. This may be considered to be the case: for any time step, no component of $G + G^T$ was more than $5 \cdot 10^{-10}$ off from zero.

Conserving kinetic energy, the vorticity integral and the enstrophy integral comes at a price though. For the approximations of the velocity and vorticity fields provided by the method developed in this thesis, the identity $\boldsymbol{\omega} = \nabla \times \mathbf{v}$ does not hold exactly at the discrete level.

6.2 Recommendations

The method developed has been tested for a limited number of situations. It would be particularly interesting to see whether the matrix G in the fully discrete vorticity equation, $(2F + \Delta t G(\boldsymbol{\alpha}^n))\,\boldsymbol{\omega}^{n+\frac{1}{2}} = (2F - \Delta t G(\boldsymbol{\alpha}^n))\,\boldsymbol{\omega}^{n-\frac{1}{2}}$, is still antisymmetric if $\bar{x}_i \neq \bar{y}_i$ for at least one $i \in \{1, \ldots, N-1\}$ or if the number of nodes in the x-direction is different from that in the y-direction, both implying that $\epsilon_i^x(x) \neq \epsilon_i^y(x)$. If G is not antisymmetric, we cannot expect the enstrophy integral to be conserved in the numerical method.

In order to not have to check whether G is antisymmetric for each time step, it would be very welcome to know the conditions under which it is antisymmetric and under which it is not, especially when it is very easy to check whether these conditions are met.

The current research would not be complete without having found the null spaces of the matrices $2A \pm \Delta t B$ the fully discrete momentum equation, $(2A + \Delta t B)\boldsymbol{\alpha}^{n+1} = (2A - \Delta t B)\boldsymbol{\alpha}^n$, and $2F \pm \Delta t G$ in the fully discrete vorticity equation, $(2F + \Delta t G(\boldsymbol{\alpha}^n))\,\boldsymbol{\omega}^{n+\frac{1}{2}} = (2F - \Delta t G(\boldsymbol{\alpha}^n))\,\boldsymbol{\omega}^{n-\frac{1}{2}}$. As long as they are unknown, we can at most expect the boundary conditions to reduce the system of equations to one that has a unique solution.

Applying the method developed to a benchmark flow problem, the Taylor-Green vortex for example, would provide insight in how realistic the solution provided by our method is in areas other than conservation of kinetic energy and vorticity and enstrophy integrals.

So far, we have only looked at a single square element. It would be very interesting to check whether our numerical method also conserves kinetic energy, the vorticity integral and the enstrophy integral when multiple elements are used or when the element is deformed, or when both are the case.

After the method has been established to work well for modelling the Euler equations, a valuable extension of the method would be to include the viscous terms from the Navier-Stokes equations. Adding a non-zero viscosity would normally destroy all the conservation properties that our method has for the Euler equations, but that is what should happen since the now-conserved quantities are not conserved at the continuous level if there is dissipation in the flow.

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Appendix A

Gauss-Legendre quadrature rule

In this appendix we introduce the Gauss-Legendre quadrature rule and show two applications that are relevant for this thesis. A quadrature rule is an approximation of the definite integral of a function:

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} w_i f(x_i).$$

One particular quadrature rule is the n-point Gauss-Legendre quadrature rule. This is a quadrature rule that uses $[-1, 1]$ as the domain of integration and that yields an exact result for polynomials of degree at most $2n - 1$. The points x_i, $i \in \{1, \dots, n\}$ now are the Gauss nodes: they are the roots of $P_n(x)$, being the n-th degree Legendre polynomial normalized such that $P_n(1) = 1$. The weights are given by

$$w_i = \frac{2}{(1 - x_i^2)(P_n'(x_i))^2}.$$

In general, $a \neq -1$ or $b \neq 1$ so in order to apply the Gauss-Legendre quadrature rule on different domains of integration, the integral over $[a, b]$ should be changed into an integral over $[-1, 1]$, namely as such:

$$\begin{aligned}\int_a^b f(z)\,dz &= \frac{b-a}{2}\int_{-1}^{1} f\left(\frac{b-a}{2}x + \frac{a+b}{2}\right)dx \\ &\approx \frac{b-a}{2}\sum_{i=1}^{n} w_i f\left(\frac{b-a}{2}x_i + \frac{a+b}{2}\right)\end{aligned}$$

where x_i and w_i are the Gauss nodes and weights. If $\mathbb{P}^m$ is the space of all polynomials of degree at most m and $p_{2n-1}(z) \in \mathbb{P}^{2n-1}$ then

$$\int_a^b p_{2n-1}(z)\,dz = \sum_{i=1}^{n} w_i f\left(\frac{b-a}{2}x_i + \frac{a+b}{2}\right).$$

If $q_m(z) \in \mathbb{P}^m$ then the integral of $q_m(z)$ can be determined exactly using Gauss-Legendre quadrature if we use the roots of $P_n(x)$ where $2n-1 \geq m$ or $2n \geq m+1$ or $n \geq \lceil (m+1)/2 \rceil$, $\lceil x \rceil$ being the ceiling function of x. For the smallest possible n yielding exact integration, $n = \lceil (m+1)/2 \rceil$.

As an example, let us consider the inner products

$$\begin{aligned}\left(\tilde{\mathbf{v}}_{i_1+(N+1)j_1}, \tilde{\mathbf{v}}_{i_2+(N+1)j_2}\right) &= \int_{-1}^{1} h_{i_1}(x)h_{i_2}(x)\,dx \cdot \int_{-1}^{1} h'_{j_1}(y)h'_{j_2}(y)\,dy \\ &\quad + \int_{-1}^{1} h'_{i_1}(x)h'_{i_2}(x)\,dx \cdot \int_{-1}^{1} h_{j_1}(y)h_{j_2}(y)\,dy\end{aligned}$$

and

$$\begin{aligned}&\left(\left[\tilde{\mathbf{v}}_{i_1+(N+1)j_1}, \tilde{\mathbf{v}}_{i_2+(N+1)j_2}\right], \tilde{\mathbf{v}}_{i_3+(N+1)j_3}\right) \\ &= \int_{-1}^{1} h'_{i_1}(x)h_{i_2}(x)h_{i_3}(x)\,dx \cdot \int_{-1}^{1} \left\{h'_{j_1}(y)h'_{j_2}(y) + h_{j_1}(y)h''_{j_2}(y)\right\} h'_{j_3}(y)\,dy \\ &\quad - \int_{-1}^{1} h_{i_1}(x)h'_{i_2}(x)h_{i_3}(x)\,dx \cdot \int_{-1}^{1} \left\{h'_{j_1}(y)h'_{j_2}(y) + h''_{j_1}(y)h_{j_2}(y)\right\} h'_{j_3}(y)\,dy \\ &\quad + \int_{-1}^{1} \left\{h'_{i_1}(x)h'_{i_2}(x) + h''_{i_1}(x)h_{i_2}(x)\right\} h'_{i_3}(x)\,dx \cdot \int_{-1}^{1} h_{j_1}(y)h'_{j_2}(y)h_{j_3}(y)\,dy \\ &\quad - \int_{-1}^{1} \left\{h'_{i_1}(x)h'_{i_2}(x) + h_{i_1}(x)h''_{i_2}(x)\right\} h'_{i_3}(x)\,dx \cdot \int_{-1}^{1} h'_{j_1}(y)h_{j_2}(y)h_{j_3}(y)\,dy.\end{aligned}$$

First note that

$$h_i(x) \in \mathbb{P}^N, \qquad h'_i(x) \in \mathbb{P}^{N-1}, \qquad h''_{i_1}(x) \in \mathbb{P}^{N-2},$$

and that the degree of the product of two polynomials equals the sum of the degrees of the two polynomials, so

$$h_{i_1}(x)h_{i_2}(x) \in \mathbb{P}^{2N}, \qquad h'_{i_1}(x)h'_{i_2}(x) \in \mathbb{P}^{2N-2},$$
$$h'_{i_1}(x)h_{i_2}(x)h_{i_3}(x) \in \mathbb{P}^{3N-1}, \qquad \left\{h'_{i_1}(x)h'_{i_2}(x) + h_{i_1}(x)h''_{i_2}(x)\right\} h'_{i_3}(x) \in \mathbb{P}^{3N-3},$$

Furthermore, the degree of the Legendre polynomials from which the Gauss nodes and weights are derived in order to evaluate the integrals, are

$$\left\lceil \frac{2N+1}{2} \right\rceil = \left\lceil N + \frac{1}{2} \right\rceil = N+1, \qquad \left\lceil \frac{(2N-2)+1}{2} \right\rceil = \left\lceil N - \frac{1}{2} \right\rceil = N,$$
$$\left\lceil \frac{(3N-1)+1}{2} \right\rceil = \left\lceil \frac{3N}{2} \right\rceil \doteq A, \qquad \left\lceil \frac{(3N-3)+1}{2} \right\rceil = \left\lceil \frac{3N}{2} - 1 \right\rceil \doteq B.$$

We may now derive that

$$\begin{aligned}
\left(\tilde{\mathbf{v}}_{i_1+(N+1)j_1}, \tilde{\mathbf{v}}_{i_2+(N+1)j_2}\right) &= \int_{-1}^{1} h_{i_1}(x)h_{i_2}(x)\,dx \cdot \int_{-1}^{1} h'_{j_1}(y)h'_{j_2}(y)\,dy \\
&\quad + \int_{-1}^{1} h'_{i_1}(x)h'_{i_2}(x)\,dx \cdot \int_{-1}^{1} h_{j_1}(y)h_{j_2}(y)\,dy \\
&= \left(\sum_{i=1}^{N+1} w_i^{N+1} h_{i_1}(x_i^{N+1})h_{i_2}(x_i^{N+1})\right) \cdot \left(\sum_{i=1}^{N} w_i^{N} h'_{j_1}(y_i^{N})h'_{j_2}(y_i^{N})\right) \\
&\quad + \left(\sum_{i=1}^{N} w_i^{N} h'_{i_1}(x_i^{N})h'_{i_2}(x_i^{N})\right) \cdot \left(\sum_{i=1}^{N+1} w_i^{N+1} h_{j_1}(y_i^{N+1})h_{j_2}(y_i^{N+1})\right)
\end{aligned}$$

and

$$
\begin{aligned}
&\left(\left[\tilde{\mathbf{v}}_{i_1+(N+1)j_1}, \tilde{\mathbf{v}}_{i_2+(N+1)j_2}\right], \tilde{\mathbf{v}}_{i_3+(N+1)j_3}\right) \\
&= \int_{-1}^{1} h'_{i_1}(x)h_{i_2}(x)h_{i_3}(x)\,dx \cdot \int_{-1}^{1} \left\{h'_{j_1}(y)h'_{j_2}(y) + h_{j_1}(y)h''_{j_2}(y)\right\} h'_{j_3}(y)\,dy \\
&\quad - \int_{-1}^{1} h_{i_1}(x)h'_{i_2}(x)h_{i_3}(x)\,dx \cdot \int_{-1}^{1} \left\{h'_{j_1}(y)h'_{j_2}(y) + h''_{j_1}(y)h_{j_2}(y)\right\} h'_{j_3}(y)\,dy \\
&\quad + \int_{-1}^{1} \left\{h'_{i_1}(x)h'_{i_2}(x) + h''_{i_1}(x)h_{i_2}(x)\right\} h'_{i_3}(x)\,dx \cdot \int_{-1}^{1} h_{j_1}(y)h'_{j_2}(y)h_{j_3}(y)\,dy \\
&\quad - \int_{-1}^{1} \left\{h'_{i_1}(x)h'_{i_2}(x) + h_{i_1}(x)h''_{i_2}(x)\right\} h'_{i_3}(x)\,dx \cdot \int_{-1}^{1} h'_{j_1}(y)h_{j_2}(y)h_{j_3}(y)\,dy \\
&= \left(\sum_{i=1}^{A} w_i^A h'_{i_1}\left(x_i^A\right) h_{i_2}\left(x_i^A\right) h_{i_3}\left(x_i^A\right)\right) \\
&\qquad \cdot \left(\sum_{i=1}^{B} w_i^B \left\{h'_{j_1}\left(y_i^B\right) h'_{j_2}\left(y_i^B\right) + h_{j_1}\left(y_i^B\right) h''_{j_2}\left(y_i^B\right)\right\} h'_{j_3}\left(y_i^B\right)\right) \\
&\quad - \left(\sum_{i=1}^{A} w_i^A h_{i_1}\left(x_i^A\right) h'_{i_2}\left(x_i^A\right) h_{i_3}\left(x_i^A\right)\right) \\
&\qquad \cdot \left(\sum_{i=1}^{B} w_i^B \left\{h'_{j_1}\left(y_i^B\right) h'_{j_2}\left(y_i^B\right) + h''_{j_1}\left(y_i^B\right) h_{j_2}\left(y_i^B\right)\right\} h'_{j_3}\left(y_i^B\right)\right) \\
&\quad + \left(\sum_{i=1}^{B} w_i^B \left\{h'_{i_1}\left(x_i^B\right) h'_{i_2}\left(x_i^B\right) + h''_{i_1}\left(x_i^B\right) h_{i_2}\left(x_i^B\right)\right\} h'_{i_3}\left(x_i^B\right)\right) \\
&\qquad \cdot \left(\sum_{i=1}^{A} w_i^A h_{j_1}\left(y_i^A\right) h'_{j_2}\left(y_i^A\right) h_{j_3}\left(y_i^A\right)\right) \\
&\quad - \left(\sum_{i=1}^{B} w_i^B \left\{h'_{i_1}\left(x_i^B\right) h'_{i_2}\left(x_i^B\right) + h_{i_1}\left(x_i^B\right) h''_{i_2}\left(x_i^B\right)\right\} h'_{i_3}\left(x_i^B\right)\right) \\
&\qquad \cdot \left(\sum_{i=1}^{A} w_i^A h'_{j_1}\left(y_i^A\right) h_{j_2}\left(y_i^A\right) h_{j_3}\left(y_i^A\right)\right).
\end{aligned}
$$

Appendix B

Boundary conditions

This appendix elaborates on the boundary conditions that need to be imposed on the fully discrete momentum equation. We will show that this system of equations has a non-trivial null space and that it can be solved by prescribing one component of the unknown vector. We may write the fully discrete momentum equation as

$$(2A + \Delta t B)\boldsymbol{\alpha}^{n+1} = (2A - \Delta t B)\boldsymbol{\alpha}^{n}$$

(see equation 4.3) where component

$$A_{mk} = (\tilde{\mathbf{v}}_k, \tilde{\mathbf{v}}_m)$$

and where

$$B(\boldsymbol{\alpha}) = \sum_{k=0}^{(N+1)^2-1} \alpha^k C_k$$

with component ml of matrix C_k

$$(C_k)_{ml} = \left([\tilde{\mathbf{v}}_l, \tilde{\mathbf{v}}_m], \tilde{\mathbf{v}}_k\right).$$

Let us now consider the matrices A and B more carefully. In order to do so, let us first calculate

$$\begin{aligned}\sum_{k=0}^{(N+1)^2-1} \tilde{\mathbf{v}}_k(x,y) &= \sum_{i=0}^{N}\sum_{j=0}^{N} h_i(x)h_j'(y)\frac{\partial}{\partial x} - h_i'(x)h_j(y)\frac{\partial}{\partial y} \\ &= \left(\sum_{i=0}^{N} h_i(x)\right)\left(\sum_{j=0}^{N} h_j'(y)\right)\frac{\partial}{\partial x} - \left(\sum_{i=0}^{N} h_i'(x)\right)\left(\sum_{j=0}^{N} h_j(y)\right)\frac{\partial}{\partial y} \\ &= (1)(0)\frac{\partial}{\partial x} - (0)(1)\frac{\partial}{\partial y} \\ &= \mathbf{0}.\end{aligned}$$

Row m of A has row sum

$$\begin{aligned}\sum_{k=0}^{(N+1)^2-1} A_{mk} &= \sum_{k=0}^{(N+1)^2-1} (\tilde{\mathbf{v}}_k, \tilde{\mathbf{v}}_m) \\ &= \left(\sum_{k=0}^{(N+1)^2-1} \tilde{\mathbf{v}}_k, \tilde{\mathbf{v}}_m\right) \\ &= (\mathbf{0}, \tilde{\mathbf{v}}_m) \\ &= 0\end{aligned}$$

and row m of C_k has row sum

$$\begin{aligned}\sum_{l=0}^{(N+1)^2-1} (C_k)_{ml} &= \sum_{l=0}^{(N+1)^2-1} \left([\tilde{\mathbf{v}}_l, \tilde{\mathbf{v}}_m], \tilde{\mathbf{v}}_k\right) \\ &= \left(\left[\sum_{l=0}^{(N+1)^2-1} \tilde{\mathbf{v}}_l, \tilde{\mathbf{v}}_m\right], \tilde{\mathbf{v}}_k\right) \\ &= \left([\mathbf{0}, \tilde{\mathbf{v}}_m], \tilde{\mathbf{v}}_k\right) \\ &= (\mathbf{0}, \tilde{\mathbf{v}}_k) \\ &= 0\end{aligned}$$

from which follows that row m of B has row sum

$$\begin{aligned}
\sum_{l=0}^{(N+1)^2-1} (B(\boldsymbol{\alpha}))_{ml} &= \sum_{l=0}^{(N+1)^2-1} \sum_{k=0}^{(N+1)^2-1} \alpha^k (C_k)_{ml} \\
&= \sum_{k=0}^{(N+1)^2-1} \alpha^k \left(\sum_{l=0}^{(N+1)^2-1} (C_k)_{ml} \right) \\
&= \sum_{k=0}^{(N+1)^2-1} \alpha^k \cdot 0 \\
&= 0,
\end{aligned}$$

so also the row sum of row m of the matrices $2A \pm \Delta tB$ equals zero for all m. An element of the null space of $2A \pm \Delta tB$, $\mathcal{N}(2A \pm \Delta tB)$, is therefore $c(1, \ldots, 1)^T$, $c \in \mathbb{R}$, so if $\boldsymbol{\alpha}^{n+1} = \boldsymbol{\alpha}^*$ (or $\boldsymbol{\alpha}^n = \boldsymbol{\alpha}^*$) is a solution of the fully discrete momentum equation $(2A + \Delta tB)\boldsymbol{\alpha}^{n+1} = (2A - \Delta tB)\boldsymbol{\alpha}^n$, then so is $\boldsymbol{\alpha}^{n+1} = \boldsymbol{\alpha}^* + c$ (or $\boldsymbol{\alpha}^n = \boldsymbol{\alpha}^* + c$). If a solution exists, we need at least one additional constraint on $\boldsymbol{\alpha}^{n+1}$ (or $\boldsymbol{\alpha}^n$) in order to obtain a unique solution. We may, for example, prescribe one component of $\boldsymbol{\alpha}^{n+1}$ ($\boldsymbol{\alpha}^n$). Empirically, it turns out that prescribing exactly one component of $\boldsymbol{\alpha}^{n+1}$ ($\boldsymbol{\alpha}^n$) already yields a unique solution; $\mathcal{N}(2A \pm \Delta tB) = c(1, \ldots, 1)^T$, $c \in \mathbb{R}$. $(1, \ldots, 1)^T$ is an eigenvector of $2A \pm \Delta tB$ with corresponding eigenvalue $\lambda = 0$ and $2A \pm \Delta tB$ is not invertible.

Lemma 1. *Let C be an n-by-n matrix and let $\mathbf{a}$ and $\mathbf{b}$ be two n-dimensional column vectors. Then $(C^T\mathbf{a}, \mathbf{b}) = (\mathbf{a}, C\mathbf{b})$.*

Proof.

$$(C^T\mathbf{a}, \mathbf{b}) = \left(\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix}^T \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \right)$$

$$= \left(\begin{bmatrix} c_{11}a_1 + c_{21}a_2 + \cdots + c_{n1}a_n \\ c_{12}a_1 + c_{22}a_2 + \cdots + c_{n2}a_n \\ \vdots \\ c_{1n}a_1 + c_{2n}a_2 + \cdots + c_{nn}a_n \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \right)$$

$$\begin{aligned}
= &\quad (c_{11}a_1 + c_{21}a_2 + \cdots + c_{n1}a_n)\, b_1 \\
&+ (c_{12}a_1 + c_{22}a_2 + \cdots + c_{n2}a_n)\, b_2 \\
&+ \cdots \\
&+ (c_{1n}a_1 + c_{2n}a_2 + \cdots + c_{nn}a_n)\, b_n
\end{aligned}$$

(continued on next page)

$$
\begin{aligned}
= \quad & (c_{11}b_1 + c_{12}b_2 + \cdots + c_{1n}b_n)\, a_1 \\
& + (c_{21}b_1 + c_{22}b_2 + \cdots + c_{2n}b_n)\, a_2 \\
& + \cdots \\
& + (c_{n1}b_1 + c_{n2}b_2 + \cdots + c_{nn}b_n)\, a_n
\end{aligned}
$$

$$
= \left(\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}, \begin{bmatrix} c_{11}b_1 + c_{12}b_2 + \cdots + c_{1n}b_n \\ c_{21}b_1 + c_{22}b_2 + \cdots + c_{2n}b_n \\ \vdots \\ c_{n1}b_1 + c_{n2}b_2 + \cdots + c_{nn}b_n \end{bmatrix} \right)
$$

$$
= \left(\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}, \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \right)
$$

$$
= (\mathbf{a}, C\mathbf{b})
$$

□

Theorem 2. *If $\boldsymbol{b} \perp \mathcal{N}(A^T)$, then there exists a solution to the system $A\boldsymbol{x} = \boldsymbol{b}$.*

Proof. Let $\tilde{\boldsymbol{p}} \in \mathcal{N}(A^T)$, i.e. $A^T\tilde{\boldsymbol{p}} = \mathbf{0}$. Then for an arbitrary $\boldsymbol{q}$ we have that

$$
0 = (\mathbf{0}, \boldsymbol{q}) = (A^T\tilde{\boldsymbol{p}}, \boldsymbol{q}) = (\tilde{\boldsymbol{p}}, A\boldsymbol{q}) :
$$

any vector that is perpendicular to an element of $\mathcal{N}(A^T)$ can be written as $A\boldsymbol{q}$ for some $\boldsymbol{q}$. $\boldsymbol{b} \perp \mathcal{N}(A^T)$, so $\boldsymbol{b}$ can be written as $A\boldsymbol{q}$ for some $\boldsymbol{q}$, which is obviously the solution of $A\boldsymbol{x} = \boldsymbol{b}$. □

$\boldsymbol{b} \perp \mathcal{N}(A^T)$ is called the compatibility condition, which should be satisfied for a solution to exist.

Theorem 3. *If there exists a solution to the system $A\boldsymbol{x} = \boldsymbol{b}$, then $\boldsymbol{b} \perp \mathcal{N}(A^T)$.*

Proof. Since there exists a solution to the system $A\boldsymbol{x} = \boldsymbol{b}$, $\boldsymbol{b}$ can be written as $A\boldsymbol{q}$ where $\boldsymbol{q}$ is a solution to $A\boldsymbol{x} = \boldsymbol{b}$. Let $\tilde{\boldsymbol{p}} \in \mathcal{N}(A^T)$, i.e. $A^T\tilde{\boldsymbol{p}} = \mathbf{0}$. Then

$$
(\tilde{\boldsymbol{p}}, \boldsymbol{b}) = (\tilde{\boldsymbol{p}}, A\boldsymbol{q}) = (A^T\tilde{\boldsymbol{p}}, \boldsymbol{q}) = (\mathbf{0}, \boldsymbol{q}) = 0.
$$

□

Theorem 4. *Let $\boldsymbol{p} \in \mathcal{N}(A)$ and $\boldsymbol{x}^*$ is a solution to $A\boldsymbol{x} = \boldsymbol{b}$, then $\boldsymbol{x}^* + \boldsymbol{p}$ is also a solution.*

Proof.

$$
A\,(\boldsymbol{x}^* + \boldsymbol{p}) = A\boldsymbol{x}^* + A\boldsymbol{p} = \boldsymbol{b} + \mathbf{0} = \boldsymbol{b}
$$

□

Let us now show that if $\mathcal{N}(2A \pm \Delta t B) = c(1, \dots, 1)^T$, then, for a given $\boldsymbol{\alpha}^n$, a solution $\boldsymbol{\alpha}^{n+1} = \boldsymbol{\alpha}^*$ of the fully discrete momentum equation $(2A + \Delta t B)\boldsymbol{\alpha}^{n+1} = (2A - \Delta t B)\boldsymbol{\alpha}^n$ exists. In order to show this, it suffices, according to the above theorems, to show that if $\mathcal{N}(2A \pm \Delta t B) = c(1, \dots, 1)^T$, then $(2A - \Delta t B)\boldsymbol{\alpha}^n \perp \mathcal{N}((2A + \Delta t B)^T)$ or put differently, since $A = A^T$ and $B = -B^T$, $(2A - \Delta t B)\boldsymbol{\alpha}^n \perp \mathcal{N}((2A^T - \Delta t B^T)^T)$ or $(2A - \Delta t B)\boldsymbol{\alpha}^n \perp \mathcal{N}(2A - \Delta t B)$.

Since $A = A^T$ and $B = -B^T$, the column sum of $2A - \Delta t B$ equals the column sum of $2A^T + \Delta t B^T = (2A + \Delta t B)^T$, which equals the row sum of $2A + \Delta t B$, which we showed to equal zero. If entry ml of $2A - \Delta t B$ is D_{ml} then

$$(2A - \Delta t B)\boldsymbol{\alpha}^n = \begin{bmatrix} D_{11} & D_{12} & \cdots & D_{1k} \\ D_{21} & D_{22} & \cdots & D_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ D_{k1} & D_{k2} & \cdots & D_{kk} \end{bmatrix} \begin{bmatrix} (\alpha^n)^1 \\ (\alpha^n)^2 \\ \vdots \\ (\alpha^n)^k \end{bmatrix}$$

$$= \begin{bmatrix} D_{11}(\alpha^n)^1 + D_{12}(\alpha^n)^2 + \cdots + D_{1k}(\alpha^n)^k \\ D_{21}(\alpha^n)^1 + D_{22}(\alpha^n)^2 + \cdots + D_{2k}(\alpha^n)^k \\ \vdots \\ D_{k1}(\alpha^n)^1 + D_{k2}(\alpha^n)^2 + \cdots + D_{kk}(\alpha^n)^k \end{bmatrix}$$

$$= \begin{bmatrix} D_{11} \\ D_{21} \\ \vdots \\ D_{k1} \end{bmatrix} (\alpha^n)^1 + \begin{bmatrix} D_{12} \\ D_{22} \\ \vdots \\ D_{k2} \end{bmatrix} (\alpha^n)^2 + \cdots + \begin{bmatrix} D_{1k} \\ D_{2k} \\ \vdots \\ D_{kk} \end{bmatrix} (\alpha^n)^k$$

where $k = (N+1)^2$. The vectors $(D_{1i}, D_{2i}, \dots, D_{ki})^T$, $i \in \{1, \dots, k\}$ are the columns of $2A - \Delta t B$ and have column sum zero, so the column sum of $(2A - \Delta t B)\boldsymbol{\alpha}^n$ equals

$$0 \cdot (\alpha^n)^1 + 0 \cdot (\alpha^n)^2 + \cdots + 0 \cdot (\alpha^n)^k = 0,$$

or, put differently, if $\mathbf{b} = (2A - \Delta t B)\boldsymbol{\alpha}^n$ then $\sum_{i=1}^{k} b_i = 0$, from which follows that

$$\left((2A - \Delta t B)\boldsymbol{\alpha}^n, c \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \right) = \left(\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_k \end{bmatrix}, c \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \right) = c \cdot \sum_{i=1}^{k} b_i = c \cdot 0 = 0 :$$

the vector $c(1, \dots, 1)^T$ is perpendicular to $(2A - \Delta t B)\boldsymbol{\alpha}^n$. Therefore, if $\mathcal{N}(2A \pm \Delta t B) = c(1, \dots, 1)^T$, then $(2A - \Delta t B)\boldsymbol{\alpha}^n \perp \mathcal{N}(2A - \Delta t B)$ and a solution to the fully discrete momentum equation $(2A + \Delta t B)\boldsymbol{\alpha}^{n+1} = (2A - \Delta t B)\boldsymbol{\alpha}^n$ exists. Similarly, if $\mathcal{N}(2A \pm \Delta t B) = c(1, \dots, 1)^T$, then, for a given $\boldsymbol{\alpha}^{n+1}$, a solution $\boldsymbol{\alpha}^n = \boldsymbol{\alpha}^*$ of this equation exists.

Appendix C

Edge function

In this appendix we demonstrate that, given $N+1$ Lagrange polynomials $h_j(x)$, $j \in \{0,\ldots,N\}$, if $\alpha(x) = \sum_{i=0}^{N} a_i h_i(x)$ where $a_i = \alpha(x_i)$, then $d\alpha(x)$ may be written as $d\alpha(x) = \sum_{i=1}^{N} b_i e_i(x)$ where $b_i = a_i - a_{i-1}$ and

$$e_i(x) = -\sum_{j=0}^{i-1} dh_j(x),$$

which are the edge functions corrsponding to the Lagrange polynomials $h_j(x)$. The exterior derivative of $\alpha(x)$ is

$$\begin{aligned} d\alpha(x) &= d\left(\sum_{i=0}^{N} a_i h_i(x)\right) = \sum_{i=0}^{N} a_i dh_i(x) \\ &= a_0 dh_0(x) + \sum_{i=1}^{N-1} a_i \left[\sum_{j=0}^{i} dh_j(x) - \sum_{j=0}^{i-1} dh_j(x)\right] + a_N dh_N(x). \end{aligned}$$

Since $\sum_{i=0}^{N} h_i(x) = 1$, $\sum_{i=0}^{N} dh_i(x) = 0$ and therefore, $\sum_{i=0}^{N-1} dh_i(x) = -dh_N(x)$. We may now write

$$
\begin{aligned}
d\alpha(x) &= a_0 dh_0(x) + \sum_{i=1}^{N-1} a_i \left[\sum_{j=0}^{i} dh_j(x)\right] - \sum_{i=1}^{N-1} a_i \left[\sum_{j=0}^{i-1} dh_j(x)\right] + a_N \left[-\sum_{j=0}^{N-1} dh_j(x)\right] \\
&= -a_0 \left[-dh_0(x)\right] - \sum_{i=1}^{N-1} a_i \left[-\sum_{j=0}^{i} dh_j(x)\right] + \sum_{i=1}^{N-1} a_i \left[-\sum_{j=0}^{i-1} dh_j(x)\right] + a_N \left[-\sum_{j=0}^{N-1} dh_j(x)\right] \\
&= \sum_{i=0}^{N-1} -a_i \left[-\sum_{j=0}^{i} dh_j(x)\right] + \sum_{i=1}^{N} a_i \left[-\sum_{j=0}^{i-1} dh_j(x)\right] \\
&= \sum_{k=1}^{N} -a_{k-1} \left[-\sum_{j=0}^{k-1} dh_j(x)\right] + \sum_{i=1}^{N} a_i \left[-\sum_{j=0}^{i-1} dh_j(x)\right] \\
&= \sum_{i=1}^{N} \left[a_i - \alpha(x_{i-1})\right] \left[-\sum_{j=0}^{i-1} dh_j(x)\right].
\end{aligned}
$$

$d\alpha(x)$ is of the form $\sum_{i=1}^{N} b_i e_i(x)$ with $b_i = a_i - a_{i-1}$ and $e_i(x) = -\sum_{j=0}^{i-1} dh_j(x)$.

Appendix D

Large equations

This appendix contains the derivations of some identities, the demonstrations of which are too large to be included in the main body of this thesis.

$\left\langle d\phi^{(0)}, \mathbf{v} \right\rangle = 0$ since

$$\begin{aligned}
\left\langle d\phi^{(0)}, \mathbf{v} \right\rangle &= \int\limits_{M^n} \imath_{\mathbf{v}} d\phi^{(0)} \operatorname{vol}^n \\
&= \int\limits_{M^n} \imath_{\mathbf{v}} \left(d\phi^{(0)} \wedge \operatorname{vol}^n \right) + d\phi^{(0)} \wedge \imath_{\mathbf{v}} \operatorname{vol}^n \\
&= \int\limits_{M^n} \imath_{\mathbf{v}}(0) + d\left(\phi^{(0)} \wedge \imath_{\mathbf{v}} \operatorname{vol}^n \right) - \phi^{(0)} \wedge d\imath_{\mathbf{v}} \operatorname{vol}^n \\
&= \int\limits_{\partial M^n} \phi^{(0)} \wedge \imath_{\mathbf{v}} \operatorname{vol}^n - \int\limits_{M^n} \phi^{(0)} \wedge 0 \\
&= \int\limits_{\partial M^n} \phi^{(0)} \wedge d\psi^{(n-2)} \\
&= 0
\end{aligned} \tag{D.1}$$

where we have used equation 3.3, the fact that all coefficients of an $(n+1)$-form on an n-dimensional manifold equal zero, equation 3.2, Stokes' theorem and the fact that in divergence free flows, $d\imath_{\mathbf{v}} \operatorname{vol}^n = 0$.

The commutator of two basis vector fields of the velocity field approximation is

$$
\begin{aligned}
[\tilde{\mathbf{v}}_k, \tilde{\mathbf{v}}_l] &= \left\{ \tilde{u}_l \frac{\partial \tilde{u}_k}{\partial x} - \tilde{u}_k \frac{\partial \tilde{u}_l}{\partial x} + \tilde{v}_l \frac{\partial \tilde{u}_k}{\partial y} - \tilde{v}_k \frac{\partial \tilde{u}_l}{\partial y} \right\} \frac{\partial}{\partial x} \\
&\quad + \left\{ \tilde{u}_l \frac{\partial \tilde{v}_k}{\partial x} - \tilde{u}_k \frac{\partial \tilde{v}_l}{\partial x} + \tilde{v}_l \frac{\partial \tilde{v}_k}{\partial y} - \tilde{v}_k \frac{\partial \tilde{v}_l}{\partial y} \right\} \frac{\partial}{\partial y} \\
&= \left\{ \frac{\partial \tilde{\psi}_l}{\partial y} \frac{\partial^2 \tilde{\psi}_k}{\partial x \partial y} - \frac{\partial \tilde{\psi}_k}{\partial y} \frac{\partial^2 \tilde{\psi}_l}{\partial x \partial y} - \frac{\partial \tilde{\psi}_l}{\partial x} \frac{\partial^2 \tilde{\psi}_k}{\partial y^2} + \frac{\partial \tilde{\psi}_k}{\partial x} \frac{\partial^2 \tilde{\psi}_l}{\partial y^2} \right\} \frac{\partial}{\partial x} \\
&\quad + \left\{ -\frac{\partial \tilde{\psi}_l}{\partial y} \frac{\partial^2 \tilde{\psi}_k}{\partial x^2} + \frac{\partial \tilde{\psi}_k}{\partial y} \frac{\partial^2 \tilde{\psi}_l}{\partial x^2} + \frac{\partial \tilde{\psi}_l}{\partial x} \frac{\partial^2 \tilde{\psi}_k}{\partial x \partial y} - \frac{\partial \tilde{\psi}_k}{\partial x} \frac{\partial^2 \tilde{\psi}_l}{\partial x \partial y} \right\} \frac{\partial}{\partial y} \\
&= \frac{\partial}{\partial y} \left(\frac{\partial \tilde{\psi}_k}{\partial x} \frac{\partial \tilde{\psi}_l}{\partial y} - \frac{\partial \tilde{\psi}_k}{\partial y} \frac{\tilde{\psi}_l}{\partial x} \right) \frac{\partial}{\partial x} - \frac{\partial}{\partial x} \left(\frac{\partial \tilde{\psi}_k}{\partial x} \frac{\partial \tilde{\psi}_l}{\partial y} - \frac{\partial \tilde{\psi}_k}{\partial y} \frac{\tilde{\psi}_l}{\partial x} \right) \frac{\partial}{\partial y}.
\end{aligned}
\tag{D.2}
$$

The inner products $\left(\tilde{\mathbf{v}}_{i_1+(N+1)j_1}, \tilde{\mathbf{v}}_{i_2+(N+1)j_2}\right)$ and $\left(\left[\tilde{\mathbf{v}}_{i_1+(N+1)j_1}, \tilde{\mathbf{v}}_{i_2+(N+1)j_2}\right], \tilde{\mathbf{v}}_{i_3+(N+1)j_3}\right)$ are

$$
\begin{aligned}
\left(\tilde{\mathbf{v}}_{i_1+(N+1)j_1}, \tilde{\mathbf{v}}_{i_2+(N+1)j_2}\right) &= \int\limits_{M^2} \left\langle \tilde{\mathbf{v}}_{i_1+(N+1)j_1}, \tilde{\mathbf{v}}_{i_2+(N+1)j_2} \right\rangle \mathrm{vol}^2 \\
&= \int\limits_{-1}^{1} \int\limits_{-1}^{1} \tilde{u}_{i_1+(N+1)j_1} \tilde{u}_{i_2+(N+1)j_2} + \tilde{v}_{i_1+(N+1)j_1} \tilde{v}_{i_2+(N+1)j_2} \, dy \, dx \\
&= \int\limits_{-1}^{1} \int\limits_{-1}^{1} h_{i_1}(x) h'_{j_1}(y) h_{i_2}(x) h'_{j_2}(y) + h'_{i_1}(x) h_{j_1}(y) h'_{i_2}(x) h_{j_2}(y) \, dy \, dx \\
&= \int\limits_{-1}^{1} h_{i_1}(x) h_{i_2}(x) \, dx \cdot \int\limits_{-1}^{1} h'_{j_1}(y) h'_{j_2}(y) \, dy \\
&\quad + \int\limits_{-1}^{1} h'_{i_1}(x) h'_{i_2}(x) \, dx \cdot \int\limits_{-1}^{1} h_{j_1}(y) h_{j_2}(y) \, dy
\end{aligned}
\tag{D.3}
$$

and

$$
\begin{aligned}
&\left(\left[\tilde{\mathbf{v}}_{i_1+(N+1)j_1}, \tilde{\mathbf{v}}_{i_2+(N+1)j_2}\right], \tilde{\mathbf{v}}_{i_3+(N+1)j_3}\right) \\
&= \int_{M^2} \left\langle \tilde{\mathbf{v}}_{\left[\tilde{\mathbf{v}}_{i_1+(N+1)j_1}, \tilde{\mathbf{v}}_{i_2+(N+1)j_2}\right]}, \tilde{\mathbf{v}}_{i_3+(N+1)j_3} \right\rangle \mathrm{vol}^n \\
&= \int_{-1}^{1}\int_{-1}^{1} \tilde{u}_{\left[\tilde{\mathbf{v}}_{i_1+(N+1)j_1}, \tilde{\mathbf{v}}_{i_2+(N+1)j_2}\right]} \tilde{u}_{i_3+(N+1)j_3} + \tilde{v}_{\left[\tilde{\mathbf{v}}_{i_1+(N+1)j_1}, \tilde{\mathbf{v}}_{i_2+(N+1)j_2}\right]} \tilde{v}_{i_3+(N+1)j_3} \, dy \, dx \\
&= \int_{-1}^{1}\int_{-1}^{1} \Big[\big(h'_{i_1}(x) h_{i_2}(x) \left\{ h'_{j_1}(y) h'_{j_2}(y) + h_{j_1}(y) h''_{j_2}(y) \right\} \\
&\qquad - h_{i_1}(x) h'_{i_2}(x) \left\{ h'_{j_1}(y) h'_{j_2}(y) + h''_{j_1}(y) h_{j_2}(y) \right\} \big) h_{i_3}(x) h'_{j_3}(y) \\
&\quad + \big(-h_{j_1}(y) h'_{j_2}(y) \left\{ h'_{i_1}(x) h'_{i_2}(x) + h''_{i_1}(x) h_{i_2}(x) \right\} \\
&\qquad + h'_{j_1}(y) h_{j_2}(y) \left\{ h'_{i_1}(x) h'_{i_2}(x) + h_{i_1}(x) h''_{i_2}(x) \right\} \big) \left(-h'_{i_3}(x) h_{j_3}(y) \right) \Big] \, dy \, dx \\
&= \int_{-1}^{1}\int_{-1}^{1} h'_{i_1}(x) h_{i_2}(x) h_{i_3}(x) \left\{ h'_{j_1}(y) h'_{j_2}(y) + h_{j_1}(y) h''_{j_2}(y) \right\} h'_{j_3}(y) \, dy \, dx \\
&\quad - \int_{-1}^{1}\int_{-1}^{1} h_{i_1}(x) h'_{i_2}(x) h_{i_3}(x) \left\{ h'_{j_1}(y) h'_{j_2}(y) + h''_{j_1}(y) h_{j_2}(y) \right\} h'_{j_3}(y) \, dy \, dx \\
&\quad + \int_{-1}^{1}\int_{-1}^{1} \left\{ h'_{i_1}(x) h'_{i_2}(x) + h''_{i_1}(x) h_{i_2}(x) \right\} h'_{i_3}(x) h_{j_1}(y) h'_{j_2}(y) h_{j_3}(y) \, dy \, dx \\
&\quad - \int_{-1}^{1}\int_{-1}^{1} \left\{ h'_{i_1}(x) h'_{i_2}(x) + h_{i_1}(x) h''_{i_2}(x) \right\} h'_{i_3}(x) h'_{j_1}(y) h_{j_2}(y) h_{j_3}(y) \, dy \, dx \\
&= \int_{-1}^{1} h'_{i_1}(x) h_{i_2}(x) h_{i_3}(x) \, dx \cdot \int_{-1}^{1} \left\{ h'_{j_1}(y) h'_{j_2}(y) + h_{j_1}(y) h''_{j_2}(y) \right\} h'_{j_3}(y) \, dy \\
&\quad - \int_{-1}^{1} h_{i_1}(x) h'_{i_2}(x) h_{i_3}(x) \, dx \cdot \int_{-1}^{1} \left\{ h'_{j_1}(y) h'_{j_2}(y) + h''_{j_1}(y) h_{j_2}(y) \right\} h'_{j_3}(y) \, dy \\
&\quad + \int_{-1}^{1} \left\{ h'_{i_1}(x) h'_{i_2}(x) + h''_{i_1}(x) h_{i_2}(x) \right\} h'_{i_3}(x) \, dx \cdot \int_{-1}^{1} h_{j_1}(y) h'_{j_2}(y) h_{j_3}(y) \, dy \\
&\quad - \int_{-1}^{1} \left\{ h'_{i_1}(x) h'_{i_2}(x) + h_{i_1}(x) h''_{i_2}(x) \right\} h'_{i_3}(x) \, dx \cdot \int_{-1}^{1} h'_{j_1}(y) h_{j_2}(y) h_{j_3}(y) \, dy
\end{aligned}
\tag{D.4}
$$

and $\left(\tilde{\omega}_k^{(2)}, \tilde{\omega}_m^{(2)}\right)$ and $\left(\left(\tilde{u}_l \frac{\partial}{\partial x}\left(\tilde{\omega}_k^{(2)}\right) + \tilde{v}_l \frac{\partial}{\partial y}\left(\tilde{\omega}_k^{(2)}\right)\right), \tilde{\omega}_m^{(2)}\right)$ are

$$\left(\tilde{\omega}^{(2)}_{i_1+(N+1)(j_1-1)}, \tilde{\omega}^{(2)}_{i_2+(N+1)(j_2-1)}\right) = \int_{M^2} \left\langle \tilde{\omega}^{(2)}_{i_1+(N+1)(j_1-1)}, \tilde{\omega}^{(2)}_{i_2+(N+1)(j_2-1)} \right\rangle \mathrm{vol}^2$$
$$= \int_{M^2} \tilde{\omega}^{(2)}_{i_1+(N+1)(j_1-1)} \wedge *\tilde{\omega}^{(2)}_{i_2+(N+1)(j_2-1)}$$
$$= \int_{-1}^{1}\int_{-1}^{1} \epsilon_{i_1}(x)\epsilon_{j_1}(y)dx \wedge dy \wedge \epsilon_{i_2}(x)\epsilon_{j_2}(y)$$
$$= \int_{-1}^{1}\int_{-1}^{1} \epsilon_{i_1}(x)\epsilon_{j_1}(y)\epsilon_{i_2}(x)\epsilon_{j_2}(y)dx \wedge dy$$
$$= \int_{-1}^{1} \epsilon_{i_1}(x)\epsilon_{i_2}(x)dx \cdot \int_{-1}^{1} \epsilon_{j_1}(y)\epsilon_{j_2}(y)dy, \qquad \text{(D.5)}$$

and

$$\left(\left(\tilde{u}_{i_3+(N+1)j_3}\frac{\partial}{\partial x}\left(\tilde{\omega}^{(2)}_{i_1+(N+1)(j_1-1)}\right) + \tilde{v}_{i_3+(N+1)j_3}\frac{\partial}{\partial y}\left(\tilde{\omega}^{(2)}_{i_1+(N+1)(j_1-1)}\right)\right), \tilde{\omega}^{(2)}_{i_2+(N+1)(j_2-1)}\right)$$
$$= \int_{M^2} \left\langle \left(\tilde{u}_{i_3+(N+1)j_3}\frac{\partial}{\partial x}\left(\tilde{\omega}^{(2)}_{i_1+(N+1)(j_1-1)}\right) + \tilde{v}_{i_3+(N+1)j_3}\frac{\partial}{\partial y}\left(\tilde{\omega}^{(2)}_{i_1+(N+1)(j_1-1)}\right)\right), \tilde{\omega}^{(2)}_{i_2+(N+1)(j_2-1)} \right\rangle \mathrm{vol}^2$$
$$= \int_{M^2} \left(\tilde{u}_{i_3+(N+1)j_3}\frac{\partial}{\partial x}\left(\tilde{\omega}^{(2)}_{i_1+(N+1)(j_1-1)}\right) + \tilde{v}_{i_3+(N+1)j_3}\frac{\partial}{\partial y}\left(\tilde{\omega}^{(2)}_{i_1+(N+1)(j_1-1)}\right)\right) \wedge *\tilde{\omega}^{(2)}_{i_2+(N+1)(j_2-1)}$$
$$= \int_{-1}^{1}\int_{-1}^{1} \left(h_{i_3}(x)h'_{j_3}(y)\frac{\partial}{\partial x}\left(\epsilon_{i_1}(x)\epsilon_{j_1}(y)dx \wedge dy\right) - h'_{i_3}(x)h_{j_3}(y)\frac{\partial}{\partial y}\left(\epsilon_{i_1}(x)\epsilon_{j_1}(y)dx \wedge dy\right)\right) \wedge \epsilon_{i_2}(x)\epsilon_{j_2}(y)$$
$$= \int_{-1}^{1}\int_{-1}^{1} \left(h_{i_3}(x)h'_{j_3}(y)\epsilon'_{i_1}(x)\epsilon_{j_1}(y) - h'_{i_3}(x)h_{j_3}(y)\epsilon_{i_1}(x)\epsilon'_{j_1}(y)\right)\epsilon_{i_2}(x)\epsilon_{j_2}(y)dx \wedge dy$$
$$= \int_{-1}^{1} \epsilon'_{i_1}(x)\epsilon_{i_2}(x)h_{i_3}(x)dx \cdot \int_{-1}^{1} \epsilon_{j_1}(y)\epsilon_{j_2}(y)h'_{j_3}(y)dy - \int_{-1}^{1} \epsilon_{i_1}(x)\epsilon_{i_2}(x)h'_{i_3}(x)dx \cdot \int_{-1}^{1} \epsilon'_{j_1}(y)\epsilon_{j_2}(y)h_{j_3}(y)dy. \qquad \text{(D.6)}$$

The enstrophy integral is approximated by

$$E \approx E_h = \frac{1}{2}\int_{-1}^{1}\int_{-1}^{1}\left(\sum_{k=1}^{(N+1)^2}\omega^k\tilde{\omega}_k(x,y)\right)^2 dx\wedge dy$$
$$= \frac{1}{2}\sum_{k=1}^{(N+1)^2}\sum_{l=1}^{(N+1)^2}\int_{-1}^{1}\int_{-1}^{1}\omega^k\omega^l\tilde{\omega}_k(x,y)\tilde{\omega}_l(x,y)dx\wedge dy$$
$$= \frac{1}{2}\sum_{k=1}^{(N+1)^2}\sum_{l=1}^{(N+1)^2}\omega^k\omega^l\int_{-1}^{1}\int_{-1}^{1}\tilde{\omega}_k(x,y)\tilde{\omega}_l(x,y)dx\wedge dy$$
$$= \frac{1}{2}\sum_{k=1}^{(N+1)^2}\sum_{l=1}^{(N+1)^2}\omega^k\omega^l\left(\tilde{\omega}_k^{(2)},\tilde{\omega}_l^{(2)}\right). \tag{D.7}$$

The inner product between the vorticity equation approximation, equation 4.7, and equation 4.13, which has been shown to be the volume form $dx \wedge dy$, is the integral over the vorticity equation approximation:

$$\sum_{k=1}^{(N+1)^2}\left(\tilde{\omega}_k^{(2)},dx\wedge dy\right)\frac{d\omega^k}{dt}$$
$$= -\sum_{k=1}^{(N+1)^2}\sum_{l=0}^{(N+1)^2-1}\left(\left(\tilde{u}_l\frac{\partial}{\partial x}\left(\tilde{\omega}_k^{(2)}\right)+\tilde{v}_l\frac{\partial}{\partial y}\left(\tilde{\omega}_k^{(2)}\right)\right),dx\wedge dy\right)\alpha^l\omega^k \iff$$
$$\sum_{k=1}^{(N+1)^2}\int_{-1}^{1}\int_{-1}^{1}\tilde{\omega}_k^{(2)}\wedge *(dx\wedge dy)\frac{d\omega^k}{dt}$$
$$= -\sum_{k=1}^{(N+1)^2}\sum_{l=0}^{(N+1)^2-1}\int_{-1}^{1}\int_{-1}^{1}\left(\tilde{u}_l\frac{\partial}{\partial x}\left(\tilde{\omega}_k^{(2)}\right)+\tilde{v}_l\frac{\partial}{\partial y}\left(\tilde{\omega}_k^{(2)}\right)\right)\wedge *(dx\wedge dy)\alpha^l\omega^k \iff$$
$$\int_{-1}^{1}\int_{-1}^{1}\sum_{k=1}^{(N+1)^2}\frac{d\omega^k}{dt}\tilde{\omega}_k^{(2)}\wedge 1$$
$$= -\int_{-1}^{1}\int_{-1}^{1}\sum_{k=1}^{(N+1)^2}\sum_{l=0}^{(N+1)^2-1}\left(\alpha^l\tilde{u}_l\frac{\partial}{\partial x}\left(\omega^k\tilde{\omega}_k^{(2)}\right)+\alpha^l\tilde{v}_l\frac{\partial}{\partial y}\left(\omega^k\tilde{\omega}_k^{(2)}\right)\right)\wedge 1 \iff$$

(continued on next page)

$$\int_{-1}^{1}\int_{-1}^{1} \frac{\partial}{\partial t}\left(\sum_{k=1}^{(N+1)^2} \omega^k \tilde{\omega}_k^{(2)}\right)$$
$$= -\int_{-1}^{1}\int_{-1}^{1} \left(\left(\sum_{l=0}^{(N+1)^2-1} \alpha^l \tilde{u}_l\right) \frac{\partial}{\partial x}\left(\sum_{k=1}^{(N+1)^2} \omega^k \tilde{\omega}_k^{(2)}\right) + \left(\sum_{l=0}^{(N+1)^2-1} \alpha^l \tilde{v}_l\right) \frac{\partial}{\partial y}\left(\sum_{k=1}^{(N+1)^2} \omega^k \tilde{\omega}_k^{(2)}\right)\right)$$
$$= -\int_{-1}^{1}\int_{-1}^{1} u_h \frac{\partial \omega_h^{(2)}}{\partial x} + v_h \frac{\partial \omega_h^{(2)}}{\partial y}. \tag{D.8}$$

The coefficients $\left(\omega^{m_1+\frac{1}{2}}\right)^{i+(N+1)(j-1)}$ of $\boldsymbol{\omega}^{m_1+\frac{1}{2}}$, the initial condition on $\boldsymbol{\omega}$, are

$$\begin{aligned}
\left(\omega^{m_1+\frac{1}{2}}\right)^{i+(N+1)(j-1)} &= -\int_{\bar{x}_{i-1}}^{\bar{x}_i}\int_{\bar{y}_{j-1}}^{\bar{y}_j} \sum_{i=0}^{N}\sum_{j=0}^{N} \left(\alpha^{m_1+\frac{1}{2}}\right)^{i+(N+1)j} \left(h_i''(x)h_j(y) + h_i(x)h_j''(y)\right) dx \wedge dy \\
&= -\sum_{i=0}^{N}\sum_{j=0}^{N} \left(\alpha^{m_1+\frac{1}{2}}\right)^{i+(N+1)j} \int_{\bar{x}_{i-1}}^{\bar{x}_i}\int_{\bar{y}_{j-1}}^{\bar{y}_j} \left(h_i''(x)h_j(y) + h_i(x)h_j''(y)\right) dx \wedge dy \\
&= -\sum_{i=0}^{N}\sum_{j=0}^{N} \left\{ \left(\alpha^{m_1+\frac{1}{2}}\right)^{i+(N+1)j} \cdot \right. \\
&\left. \left(\int_{\bar{x}_{i-1}}^{\bar{x}_i} h_i''(x)dx \cdot \int_{\bar{y}_{j-1}}^{\bar{y}_j} h_j(y)dy + \int_{\bar{x}_{i-1}}^{\bar{x}_i} h_i(x)dx \cdot \int_{\bar{y}_{j-1}}^{\bar{y}_j} h_j''(y)dy \right) \right\} \\
&= -\sum_{i=0}^{N}\sum_{j=0}^{N} \left\{ \left(\alpha^{m_1+\frac{1}{2}}\right)^{i+(N+1)j} \cdot \right. \\
&\left. \left(\left[h_i'(\bar{x}_i) - h_i'(\bar{x}_{i-1})\right] \cdot \int_{\bar{y}_{j-1}}^{\bar{y}_j} h_j(y)dy + \int_{\bar{x}_{i-1}}^{\bar{x}_i} h_i(x)dx \cdot \left[h_j'(\bar{x}_j) - h_j'(\bar{y}_{j-1})\right] \right) \right\}.
\end{aligned} \tag{D.9}$$