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Data-driven stabilization of non-zero equilibrium for polynomial systems

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Abstract— **Most existing work on direct data-driven stabilization considers the equilibrium at the origin. When the desired equilibrium is not the origin, existing data-driven approaches often require performing coordinate transformation, or adding integrator action to the controller. As an alternative, this work addresses data-driven state feedback stabilization of any given assignable equilibrium via dissipativity theory. We show that for a polynomial system, if a data-driven stabilizer can be designed to render the origin globally asymptotically stable, then by modifying the stabilizer, we obtain a stabilizer for any given assignable equilibrium.**

I. Introduction

Direct data-driven control aims at synthesizing controllers directly from data without explicitly identifying a sufficiently accurate model. Indirect data-driven control approaches first perform system identification and then control the identified model. When the controlled system has complex dynamics, identifying a sufficiently accurate model can be difficult and time-consuming, and a direct data-driven control approach is preferable. Direct nonlinear data-driven control has been addressed using methods such as virtual reference feedback tuning (VRFT) [1], iterative feedback tuning [2], intelligent PID [3], [4]. More recently, Willems *et al.*'s fundamental lemma [5] has been used to describe the response of dynamical systems from input-output data and to design data-driven controllers for linear and nonlinear systems, as can be found in work such as [6]–[10].

For data-driven stabilization, almost all the existing work considers a special equilibrium that is the origin. In real-life applications, common problems such as set-point tracking require the system to be stabilized at a non-zero equilibrium. In the model-based setting, non-zero equilibrium stabilization can be achieved by first performing coordinate transformation on the system model such that the non-zero equilibrium is converted to the origin, and then designing stabilizers for the converted system. We note that this approach often requires redesigns of the controller for different equilibria. In this paper, we aim at developing a data-driven state feedback stabilizer that stabilizes any given assignable equilibrium directly from data without performing coordinate transformation.

Related work. Based on Willems *et al.*'s fundamental lemma [5], a data-driven closed-loop representation of discrete-time linear time-invariant systems was developed in [7] and used for data-driven stabilization of linear and nonlinear systems at the origin. For continuous-time polynomial systems, [10]–[12] wrote the systems into a linear-like form and achieved data-driven stabilization at the origin using Lyapunov's method. For stabilizing non-zero equilibrium directly using these existing methods, coordinate transformation is needed.

Dissipativity describes input-output properties of a dynamical system and it has been widely used for the analysis and control of single and interconnected nonlinear systems [13]–[16]. Equilibrium-independent dissipativity (EID) is a recent extension of the conventional dissipativity that does not reference to an explicit equilibrium, which makes it more advantageous in handling uncertain and interconnected systems [17]. The authors of [18] and [19] derived necessary and sufficient dissipativity-based conditions for state feedback stabilization of any given equilibrium for linear and nonlinear systems with known dynamics. Based on the linear parameter-varying (LPV) framework and EID, [20] proposed equilibrium-independent control of nonlinear systems. Recent work has also explored the conventional dissipativity properties in the data-driven setting. Using input-output or input-state data, most of recent work has been focused on data-driven verification of dissipativity, such as [21]–[25]. Conventional-dissipativity-based data-driven control was proposed in [26] for linear systems and in [27] for nonlinear systems. By adding an integrator, [28] and [29] achieved data-driven set-point tracking.

Contributions. In this work, we study direct data-driven state feedback stabilization of polynomial systems at non-zero equilibria. Using EID with a quadratic supply rate, we show that if a class of polynomial systems can be stabilized at the origin by a data-driven state feedback controller, then by modifying the controller, one can obtain a data-driven stabilizer for any given assignable equilibrium. The advantage of the proposed data-driven control approach is that the coordinate transformation is not needed, and a stabilizer can be obtained for any known equilibrium based on the stabilizer for the origin without redesigning the control gain. In addition, we show that if the polynomial system is data-driven asymptotically stabilized at the origin, it is EID with a specific fictitious output.

The rest of the paper is organized as follows. Section II presents the problem formulation and preliminaries on data-based closed-loop representation, dissipativity with quadratic supply rates, and sum of squares (SOS) polynomials. The connection between data-driven stabilization and dissipativity, as well as the data-driven stabilizer design for non-zero equilibria, are presented in Section III. Simulation results on Van der Pol oscillator are illustrated in Section IV. Conclusion remarks and possible future work are given

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finally in Section V.

Notation. Throughout the paper, $A \succ (\succeq) 0$ denotes that matrix A is positive (semi-)definite, and $A \prec (\preceq) 0$ denotes that matrix A is negative (semi-)definite. For a differentiable function $V : \mathbb{R}^n \to \mathbb{R}$, $\nabla V : \mathbb{R}^n \to \mathbb{R}^n$ is its gradient. The function V is strongly convex if $(\nabla V(x_1) - \nabla V(x_2))^\top (x_1 - x_2) \geq k(x_1, x_2)\|x_1 - x_2\|_2^2$ for all $x_1, x_2 \in \mathbb{R}^n$ and some function $k(x_1, x_2) > 0$ for all $x_1 \neq x_2$.

II. Problem formulation and preliminaries

A. Problem formulation

Consider an input-affine polynomial system

$$\dot{x} = AZ(x) + Bu \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input, and $m \leq n$. $A \in \mathbb{R}^{n \times N}$ and $B \in \mathbb{R}^{n \times m}$ are unknown constant matrices, and B has full-column rank, *i.e.*, $\operatorname{rank}(B) = m$. The function $Z : \mathbb{R}^n \to \mathbb{R}^N$ is a dictionary of monomials in x having degrees no smaller than 1. Therefore, it holds that $Z(0) = 0$, and thus the origin $(x, u) = (0, 0)$ is an equilibrium of (1). The dictionary $Z(x)$ can be obtained by prior knowledge on the dynamics, such as the physics of the system.

Denote $(x_e, u_e) \in \mathbb{R}^n \times \mathbb{R}^m$ as an equilibrium configuration of (1), that is

$$0 = AZ(x_e) + Bu_e. \tag{2}$$

As B has full-column rank, when $m = n$, (1) is fully actuated and for any equilibrium $x_e \in \mathbb{R}^n$, the associated equilibrium input can be found as $u_e = -B^{-1}AZ(x_e)$. When $m < n$, let $B^\perp \in \mathbb{R}^{(n-m) \times n}$ be such that $B^\perp B = 0$ and $\operatorname{rank}(B^\perp) = n - m$. Then, similar to [17], we define the set of assignable equilibrium points as

$$\mathcal{E} := \begin{cases} \mathbb{R}^n & \text{if } m = n \\ \{x_e \in \mathbb{R}^n | B^\perp AZ(x_e) = 0\} & \text{if } m < n \end{cases} \tag{3}$$

and the associated unique equilibrium input for each $x_e \in \mathcal{E}$ as $u_e = k_u(x_e) := -(B^\top B)^{-1} B^\top AZ(x_e)$.

In this work, we consider the case where the exact dynamics of (1) is not explicitly known. Instead, a dataset $\mathcal{DS} := \{(\dot{x}(t_k), x(t_k), u(t_k)), k = 0, 1, \ldots, T-1\}$ for some integer $T > 1$ collected from one or multiple experiment(s) is available for control design. The objective is to design a data-driven state feedback control law that stabilizes (1) at any *given* equilibrium point $x \in \mathcal{E}$ and $u_e = k_u(x_e)$.

Problem 1: Consider the system (1) with the set $\mathcal{E}$ of equilibrium points, and the dataset $\mathcal{DS}$. Given any (x_e, u_e) where $x_e \in \mathcal{E}$ and $u_e = k_u(x_e)$, use the dataset $\mathcal{DS}$ to design a state feedback control law $u = F(x, x_e, u_e)$, such that the equilibrium x_e is globally asymptotically stable for the closed-loop system.

Remark 1 (Known equilibrium configuration): The equilibrium configuration (x_e, u_e) is necessary for the proposed data-driven approach, which can be obtained via experiments. Nonetheless, it can be difficult to find the equilibrium input u_e when the system model is unknown. It is of our interest to develop a data-driven control approach that is independent of the equilibrium input u_e in the future.

B. Data-based closed-loop representation of polynomial systems

The direct data-driven stabilizer design presented in this work relies on the data-based closed-loop system representation of the polynomial system (1). Similar representations have been used in work such as [7] and [10] for direct data-driven control of linear and nonlinear polynomial systems.

Arrange the data in $\mathcal{DS}$ into the data matrices

$$X_0 := \begin{bmatrix} x(t_0) & x(t_1) & \cdots & x(t_{T-1}) \end{bmatrix} \in \mathbb{R}^{n \times T} \tag{4a}$$
$$X_1 := \begin{bmatrix} \dot{x}(t_0) & \dot{x}(t_1) & \cdots & \dot{x}(t_{T-1}) \end{bmatrix} \in \mathbb{R}^{n \times T} \tag{4b}$$
$$U_0 := \begin{bmatrix} u(t_0) & u(t_1) & \cdots & u(t_{T-1}) \end{bmatrix} \in \mathbb{R}^{m \times T}. \tag{4c}$$

Using the known vector $Z(x)$, we can evaluate the value of vector $Z(x)$ during the experiment as

$$Z_0 := \begin{bmatrix} Z(x(t_0)) & Z(x(t_1)) & \cdots & Z(x(t_{T-1})) \end{bmatrix} \in \mathbb{R}^{N \times T}.$$

For the dynamics of system (1), the data matrices satisfy that

$$X_1 = AZ_0 + BU_0. \tag{5}$$

Lemma 1 (Data-based closed-loop representation): Consider the polynomial system (1) and the dataset $\mathcal{DS}$. For any matrices K and G such that

$$\begin{bmatrix} K \\ I \end{bmatrix} = \begin{bmatrix} U_0 \\ Z_0 \end{bmatrix} G, \tag{6}$$

the closed-loop system under state feedback control law $u = KZ(x)$ can be written as

$$\dot{x} = X_1 G Z(x). \tag{7}$$

Proof: The closed-loop representation results from the equality that

$$A + BK = \begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} K \\ I_n \end{bmatrix} \overset{(6)}{=} \begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} U_0 \\ X_0 \end{bmatrix} G \overset{(5)}{=} X_1 G.$$

This completes the proof. ∎

We note that in the data-based closed-loop representation (7), the system matrix only depends on the data matrix X_1 and the design variable G. We will characterize stabilizers using the data-based system matrix.

Remark 2 (Noiseless data): In this work, we suppose that the collected data does not contain measurement noise. When the data is noisy, an uncertain data-based closed-loop representation can be derived and robust data-driven stabilizers can be designed, such as presented in [10]. The robust data-driven stabilization of non-zero equilibrium is of interest in our future work.

C. Dissipativity with quadratic supply rates

Dissipativity describes an input-output property of a system. Consider the system

$$\dot{x} = f(x) + g(x)u, \tag{8a}$$
$$y = h(x) \tag{8b}$$

where $y \in \mathbb{R}^p$ is the output and the function $h : \mathbb{R}^n \to \mathbb{R}^p$ is sufficiently smooth. Suppose that $f(0) = 0$ and $h(0) = 0$, which implies that $(x_e, u_e, y_e) = (0, 0, 0)$ is an equilibrium configuration of (8).

We focus on quadratic supply rates $w : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$ in the form of

$$w(u, y) = \begin{bmatrix} y \\ u \end{bmatrix}^\top \begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix} \tag{9}$$

where Q and R are symmetric matrices.

Following [18], we present the definition of strict QSR-dissipativity.

Definition 1 (Strict QSR-dissipativity): The system (8) is strictly QSR-dissipative with respect to the supply rate (9) if there exists a continuous differentiable storage function $V : \mathbb{R}^n \to \mathbb{R}$ such that $V(x) > 0 \ \forall x \neq 0$, $V(0) = 0$, and

$$\dot{V}(x) + \epsilon(x) \leq w(u, y) \tag{10}$$

for all admissible $u \in \mathbb{R}^m$, where $\epsilon(x) > 0$ for $x \neq 0$ and $\epsilon(0) = 0$.

The QSR-dissipativity refers to the equilibrium at the origin. When the desired equilibrium is not at the origin, simply shifting the storage function $V(x)$ does not work, and a new storage function $V_{x_e}(x)$ depending on the equilibrium x_e is needed for the dissipativity analysis [17]. EID describes the dissipativity of system (8) at any equilibrium $x_e \in \mathcal{E}$. We first define the unique equilibrium output y_e with respect to any $x_e \in \mathcal{E}$ as

$$y_e = k_y(x_e) = h(x_e).$$

Then, we can present the definition of EID.

Definition 2 (Strict EID [19]): The system (8) is strictly EID with supply rate $w : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$ if, for every equilibrium $x_e \in \mathcal{E}$, there exists a continuously differentiable storage function $V_{x_e} : \mathbb{R}^n \to \mathbb{R}$ such that $V_{x_e}(x) > 0$ for all $x \neq x_e$, $V_{x_e}(x_e) = 0$ and

$$\dot{V}_{x_e}(x) + \epsilon_{x_e}(x) \leq w(u - u_e, y - y_e) \tag{11}$$

for all admissible $u \in \mathbb{R}^m$, where $\epsilon_{x_e}(x) > 0$ for $x \neq x_e$ and $\epsilon_{x_e}(x_e) = 0$, $u_e = k_u(x_e)$, and $y_e = k_y(x_e)$.

In particular, we define the storage function $V_{x_e}(x)$ as

$$V_{x_e}(x) = V(x) - V(x_e) - \nabla V(x_e)^\top (x - x_e) \tag{12}$$

where $V(x)$ is as defined in Definition 1 and is strongly convex. By Bregman divergence properties [17, Lemma A.2], $V_{x_e}(x)$ is also strongly convex and radially unbounded.

Remark 3 (Strict dissipativity): Definitions 1 and 2 introduce strict dissipativity. If $\epsilon(x) = 0$ and $\epsilon_{x_e}(0) = 0$, one will get the definitions of QSR-dissipativity and EID as can be found in work such as [13]–[15], [17]. As this work addresses asymptotic stabilization of non-zero equilibrium, we will use strict dissipativity in the subsequent analysis. Moreover, we define the storage functions $V(x)$ and $V_{x_e}(x)$ to be radially unbounded for deriving global stabilization results, as they are also used as the Lyapunov candidate functions for the stability analysis.

D. SOS polynomials and SOS dissipativity

In this work, we focus on the polynomial system in the form of (1). To be able to derive computationally efficient conditions for characterizing data-driven state feedback stabilizers, we use the SOS technique, similar as our previous work [11] and [10]. Moreover, by the SOS decomposition, the relation of QSR-dissipativity and EID has been analyzed in [19], which will also be useful in the subsequent sections.

First, we introduce the definition and some important properties of SOS polynomial matrices.

Definition 3: (*SOS polynomial matrix* [30]) $M : \mathbb{R}^n \to \mathbb{R}^{\sigma \times \sigma}$ is an SOS polynomial matrix if there exist $M_1, \ldots, M_k : \mathbb{R}^n \to \mathbb{R}^{\sigma \times \sigma}$ such that

$$M(x) = \sum_{i=1}^{k} M_i(x)^\top M_i(x) \ \ \forall x \in \mathbb{R}^n. \tag{13}$$

Some important properties of SOS polynomial matrices are summarized in the following lemma.

Lemma 2: (*Properties of SOS polynomial matrices* [30]) For a polynomial matrix $M(x)$, consider the following conditions

(i) $M(x)$ is SOS;
(ii) $M(x) \succeq 0$ for all $x \in \mathbb{R}^n$;
(iii) the polynomial $y^\top M(x) y$ is SOS in the extended variable $\begin{bmatrix} x^\top & y^\top \end{bmatrix}^\top$ where $y \in \mathbb{R}^m$.

Then, (i) $\Rightarrow$ (ii) and (i) $\iff$ (iii).

When $\sigma = 1$, $M(x)$ is a scalar SOS polynomial. Now recall the definitions of strict QSR-dissipativity and strict EID. Using SOS polynomials to characterize the inequalities in Definitions 1 and 2 gives the definitions of SOS strict QSR dissipativity and SOS strict EID [19].

Definition 4 (SOS strict QSR-dissipativity): The system (8) is SOS strictly QSR-dissipative with respect to the supply rate (9) if there exists a continuous differentiable storage function $V : \mathbb{R}^n \to \mathbb{R}$ such that $V(x) > 0$ for all $x \neq 0$, $V(0) = 0$, and

$$-\dot{V}(x) - \epsilon(x) + w(u, y) \text{ is SOS} \tag{14}$$

for all admissible $u \in \mathbb{R}^m$, where $\epsilon(x) > 0$ for $x \neq 0$ and $\epsilon(0) = 0$.

Definition 5 (SOS strict EID): The system (8) is SOS strictly EID with supply rate $w : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$ if, for every equilibrium $x_e \in \mathcal{E}$, there exists a continuously differentiable storage function $V_{x_e} : \mathbb{R}^n \to \mathbb{R}$ such that $V_{x_e}(x) > 0$ for all $x \neq x_e$, $V_{x_e}(x_e) = 0$ and

$$-\dot{V}_{x_e}(x) - \epsilon_{x_e}(x) + w(u - u_e, y - y_e) \text{ is SOS} \tag{15}$$

for all admissible $u \in \mathbb{R}^m$, where $\epsilon_{x_e}(x) > 0$ for $x \neq x_e$ and $\epsilon_{x_e}(x_e) = 0$, $u_e = k_u(x_e)$, and $y_e = k_y(x_e)$.

By Lemma 2, a system is strict QSR-dissipative (EID) if it is SOS strict QSR-dissipative (EID), and the converse does not hold. Using SOS polynomials, we also present the definition of SOS asymptotically stable.

Definition 6 (SOS asymptotically stable): The closed-loop system (8) is SOS asymptotically stable under the state feedback $u = F(x)$, if there exist functions $V : \mathbb{R}^n \to \mathbb{R}$

and $\mu : \mathbb{R}^n \to \mathbb{R}$, such that $V(x) > 0$ for all $x \neq 0$, $V(0) = 0$, $\mu(x) > 0$ for all $x \neq 0$, $\mu(0) = 0$, and and the time-derivative of $V(x)$ satisfies that $-\dot{V}(x) - \mu(x)$ is SOS.

The following result from [19] reveals the connection between SOS asymptotic stability and the SOS strict QSR-dissipativity of a nonlinear system (8a).

Theorem 1: [19, Theorem 1] A dynamical system (8a) is SOS asymptotically stable by state feedback if and only if there exists an (fictitious) output variable (8b) such that the system (8) is SOS strictly QSR-dissipative with $R \succ 0$ and $\Delta = 0$, where $\Delta = SR^{-1}S^\top - Q$. A stabilizing state feedback is given by $u = -R^{-1}S^\top h(x)$.

Remark 4 (SOS asymptotic stabilization and dissipativity): A result similar to Theorem 1 was derived in [18], which shows that the closed-loop (8a) is asymptotically stable if and only if it is strict QSR-dissipative under the same conditions. In this work, we use Theorem 1 with the SOS definitions since it is essential for deriving the data-driven stabilizers for non-zero equilibrium as shown in Corollary 1 in Section III-A.

III. Data-driven stabilization of non-zero equilibrium

In this section, we present the data-driven stabilizer design for polynomial systems (1) at any given assignable non-zero equilibrium. We first show that if the system (1) is made SOS asymptotically stable at the origin using the dataset $\mathcal{DS}$, then it is SOS strictly QSR-dissipative with the matrices $R \succ 0$, and Q, S depending on data. Then, using SOS strictly QSR-dissipativity and SOS strictly EID, we can modify the data-driven stabilizer for the origin to obtain a stabilizer for any given equilibrium $x_e \in \mathcal{E}$.

A. Data-driven stabilization and dissipativity

In the following result, we show that if the closed-loop system (1) is globally SOS asymptotically stable at the origin, then it is SOS strictly QSR-dissipative with a fictitious output.

Theorem 2: Consider the system (1) and the dataset $\mathcal{DS}$. If there exist matrix $Y \in \mathbb{R}^{T\times N}$, positive definite matrix $P \in \mathbb{R}^{N\times N}$, and an SOS polynomial $\mu(x) > 0$ for all $x \neq 0$, such that

$$Z_0 Y = P, \tag{16a}$$

$$-\frac{\partial Z}{\partial x} X_1 Y - Y^\top X_1^\top \frac{\partial Z}{\partial x}^\top - \mu(x) I_N \text{ is SOS}, \tag{16b}$$

then the feedback control law $u = U_0 Y P^{-1} Z(x)$ renders the origin globally SOS asymptotically stable. Moreover, the system is SOS strictly QSR-dissipative with respect to the fictitious output $y = Z(x)$, the storage function $V(x) = Z(x)^\top P^{-1} Z(x)$, and the supply rate (9) with $R \succ 0$,

$$Q = P^{-1}(U_0Y)^\top R U_0 Y P^{-1}, S = -P^{-1}(U_0Y)^\top R. \tag{17}$$

Proof: Defining $G = YP^{-1}$, one can write (16a) into $Z_0 G = I$. Then, by the designed controller and (16a), we use Lemma 1 to obtain the data-based closed-loop representation, $\dot{x} = X_1 Y P^{-1} Z(x)$.

The time-derivative of $V(x) = Z(x)^\top P^{-1} Z(x)$ is

$$\dot{V}(x) = Z(x)^\top P^{-1} \left(\frac{\partial Z}{\partial x} X_1 Y + Y^\top X_1^\top \frac{\partial Z}{\partial x}^\top \right) P^{-1} Z(x).$$

It holds that

$$\begin{aligned} &- \dot{V}(x) - Z(x)^\top P^{-1} \cdot \mu(x) I_N \cdot P^{-1} Z(x) \\ &= Z(x)^\top P^{-1} \Big(-\frac{\partial Z}{\partial x} X_1 Y - \\ &\qquad Y^\top X_1^\top \frac{\partial Z}{\partial x}^\top - \mu(x) I_N \Big) P^{-1} Z(x). \end{aligned}$$

If (16b) is satisfied, by Lemma 2 ((i)→(iii)), $-\dot{V}(x) - Z(x)^\top P^{-1} \cdot \mu(x) I_N \cdot P^{-1} Z(x)$ is also an SOS polynomial. As the SOS polynomial $\mu(x) > 0$ for all $x \neq 0$, the system is SOS asymptotically stable.

Next, we apply Theorem 1 to prove the QSR-dissipativity of the system. For $R \succ 0$, by letting Q and S satisfy (17), we have that $u = U_0 Y P^{-1} Z(x) = -R^{-1} S^\top y$, and

$$\begin{aligned} \Delta = SR^{-1}S^\top - Q = P^{-1}(U_0Y)^\top R \cdot R^{-1} \cdot R U_0 Y P^{-1} \\ - P^{-1}(U_0Y)^\top R U_0 Y P^{-1} = 0. \end{aligned}$$

Therefore, by Theorem 1, the system is strictly QSR-dissipative with the output $y = Z(x)$ and the matrices Q, S, $R \succ 0$ given in (17). ■

For the polynomial system (8), one can prove that it is SOS strictly QSR-dissipative if and only if it is SOS strictly EID [19]. Therefore, Theorem 2 leads to the following result.

Corollary 1: Consider the system (1) and the dataset $\mathcal{DS}$. If there exist matrix $Y \in \mathbb{R}^{T\times N}$, positive definite matrix $P \in \mathbb{R}^{N\times N}$, and an SOS polynomial $\mu(x) > 0$ for all $x \neq 0$, such that (16) hold, then the system is SOS strictly EID with respect to the fictitious output $y = Z(x)$, the storage function $V_{x_e}(x)$ defined in (12), and the supply rate (9) with $R \succ 0$ and Q, S satisfying (17).

B. Data-driven stabilizer design for non-zero equilibrium

In the previous subsection, we have made the connection between data-driven SOS asymptotic stability of the origin with the SOS strict QSR-dissipativity and SOS strict EID. In what follows, we present the data-driven stabilizer design for any given equilibrium $x_e \in \mathcal{E}$ and $u_e = k_u(x_e)$.

First, we show the result on global asymptotic non-zero equilibrium stabilization via SOS strict EID.

Lemma 3: Consider polynomial system (1) that is SOS strictly EID with respect to the fictitious output $y = Z(x)$, the storage function (12), and the quadratic supply rate $w(u - u_e, y - y_e)$ in the form of (9). If there exists a matrix $K \in \mathbb{R}^{m\times N}$ such that

$$Q + 2SK + K^\top R K \preceq 0, \tag{18}$$

then the state feedback control law

$$u = K(Z(x) - Z(x_e)) + u_e \tag{19}$$

where $u_e = k_u(x_e)$, renders any equilibrium $x_e \in \mathcal{E}$ globally asymptotically stable.

Proof: Under the controller (19), one has that

$$u - u_e = K(Z(x) - Z(x_e)). \tag{20}$$

As the system is SOS strictly EID, for each $x_e \in \mathcal{E}$, the storage function $V_{x_e}(x)$ satisfies that

$$\begin{aligned}&\dot{V}_{x_e}(x) + \epsilon_{x_e}(x)\\ &\le (y - y_e)^\top Q(y - y_e) + 2(y - y_e)^\top S(u - u_e)\\ &\quad + (u - u_e)^\top R(u - u_e)\\ &= (Z(x) - Z(x_e))^\top \big(Q + 2SK + K^\top RK\big)(Z(x) - Z(x_e))\end{aligned}$$

where $\epsilon_{x_e}(x) > 0$ when $x \neq x_e$ and $\epsilon_{x_e}(x_e) = 0$. By condition (18), it holds that, for any $x \in \mathbb{R}^n$

$$\dot{V}_{x_e}(x) \le -\epsilon_{x_e}(x) \le 0. \tag{21}$$

As $\epsilon_{x_e}(x) > 0$ for all $x \neq x_e$, $\dot{V}_{x_e}(x) < 0$ for all $x \neq x_e$, and thus the equilibrium x_e is asymptotically stable. Recall that the storage function $V_{x_e}(x)$ is strictly convex and radially unbounded, so the controller renders the equilibrium x_e globally asymptotically stable. ■

The following result shows the data-driven controller design that globally asymptotically stabilizes any given equilibrium $x_e \in \mathcal{E}$ and $u_e = k_u(x_e)$.

Theorem 3: Consider the polynomial system (1) and the dataset $\mathcal{DS}$. If there exist matrix Y, positive definite matrix $P \in \mathbb{R}^{N\times N}$, and a SOS polynomial $\mu(x) > 0$ for all $x \neq 0$ such that the conditions in (16) hold, then the state feedback control law $u = U_0YP^{-1}(Z(x) - Z(x_e)) + u_e$ with $u_e = k_u(x_e)$ renders any equilibrium point $x_e \in \mathcal{E}$ globally asymptotically stable.

Proof: By Corollary 1, if the conditions in (16) are satisfied, the data-driven controller $u = KZ(x) = U_0YP^{-1}Z(x)$ renders the origin globally asymptotically stable, and the system is SOS strictly EID with $R \succ 0$ and Q and S such that (17) holds. Moreover, one has that

$$Q + 2SK + K^\top RK = K^\top RK - 2K^\top RK + K^\top RK = 0,$$

which satisfies (18). Then, by Lemma 3, the state feedback control law $u = U_0YP^{-1}(Z(x) - Z(x_e)) + u_e$ with $u_e = k_u(x_e)$ renders any equilibrium point $x_e \in \mathcal{E}$ asymptotically stable. ■

Theorem 3 shows that, if a data-driven state feedback control law can be designed to render the origin globally asymptotically stable for (1), one can modify the control law to obtain a new control law that renders any given equilibrium point $x_e \in \mathcal{E}$ globally asymptotically stable. The feature of this design approach is that the coordinate transformation is not needed, and the control gain K does not need to be redesigned for each equilibrium, as long as it is designed for stabilizing the origin.

IV. An Example

This section illustrates the simulation results of stabilizing a Van der Pol oscillator at any given assignable equilibrium via data. Specifically, we collect data from the system

$$\begin{aligned}\dot{x}_1 &= x_2,\\ \dot{x}_2 &= -x_1 + (1 - x_1^2)x_2 + u\end{aligned} \tag{22}$$

with the initial condition $x(0) = \begin{bmatrix}-0.1 & 0.1\end{bmatrix}^\top$ and the control input $u = \sin(t)$ from $t = 0$ to $t = 10s$. The data is collected with the sampling period $0.5s$ and the length of the data matrices is $T = 10$. It is known that the uncontrolled Van der Pol oscillator exhibits the limit cycle behavior, as shown in Fig. 1. The simulations are conducted in MATLAB using SOSTOOLS with the semidefinite programming solver MOSEK.

We write the unknown dynamics into the linear-like form (1) by selecting $Z(x)$ as

$$Z(x) = \begin{bmatrix}x_1 & x_2 & x_1x_2 & x_1^2 & x_2^2 & x_1^2x_2 & x_1x_2^2 & x_1^3 & x_2^3\end{bmatrix}^\top$$

which contains all monomials in x having degree from 1 to 3.

First, we stabilize the system at the origin. Applying Theorem 2 by setting $\epsilon(x) = 10^{-10}(x_1^2 + x_2^2)$ and minimizing the trace of P to enforce a rapid convergence rate towards the origin gives the state feedback control law as

$$\begin{aligned}u_1 = &-1.4146x_1^3 + 0.40755x_1^2x_2 - 0.64193x_1x_2^2\\ &-1.4184x_2^3 + 0.10161x_1^2 + 0.20035x_1x_2\\ &+0.26177x_2^2 + 0.78681x_1 - 10.482x_2.\end{aligned}$$

As illustrated in Fig 1, the control law u_1 renders the origin asymptotically stable.

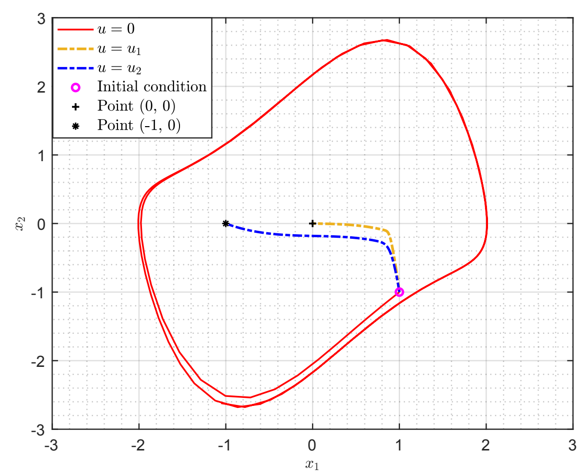

Fig. 1. Open-loop trajectory of the Van der Pol oscillator and closed-loop trajectories under the control laws u_1 and u_2.

Next, we show the stabilization of the equilibrium at $x_e = \begin{bmatrix}-1 & 0\end{bmatrix}^\top$. By applying Theorem 3, the controller u_2 stabilizing the system at the equilibrium point $[-1\ 0]^\top$ is designed as

$$u_2 = u_1 - 1.7294.$$

The blue curve in Fig. 1 shows that, under the designed control law u_2, the trajectory of the closed-loop system converges to the equilibrium at $x_e = \begin{bmatrix}-1 & 0\end{bmatrix}^\top$. Fig. 2 illustrates the phase portrait of the closed-loop system under u_2, which verifies the asymptotic stability of the equilibrium.

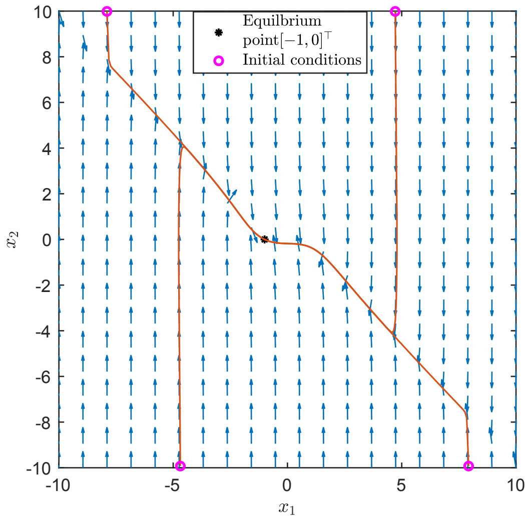

Fig. 2. Phase portrait of the closed-loop system under the controller u_2 designed using Theorem 3.

V. Conclusions and Future Works

This work considered the data-driven state feedback stabilization of any given assignable equilibrium for a class of polynomial systems. By writing the polynomial system into a linear-like form, we derived a data-based closed-loop representation with a state feedback controller using finite input-state data. We proved that if a controller can be designed based on the data-based closed-loop representation to globally SOS asymptotically stabilize the origin, then the system is SOS strictly QSR-dissipative and EID. Furthermore, given any admissible equilibrium configuration (x_e, u_e), modifying the designed controller for the origin can globally asymptotically stabilize any given assignable equilibrium. The proposed approach does not require an explicit model of the system, the coordinate transformation, or control gain redesign for stabilizing the non-zero equilibrium. Future work will consider more general classes of nonlinear dynamics, noisy measurements, as well as relaxing the restriction on the known equilibrium input u_e.

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