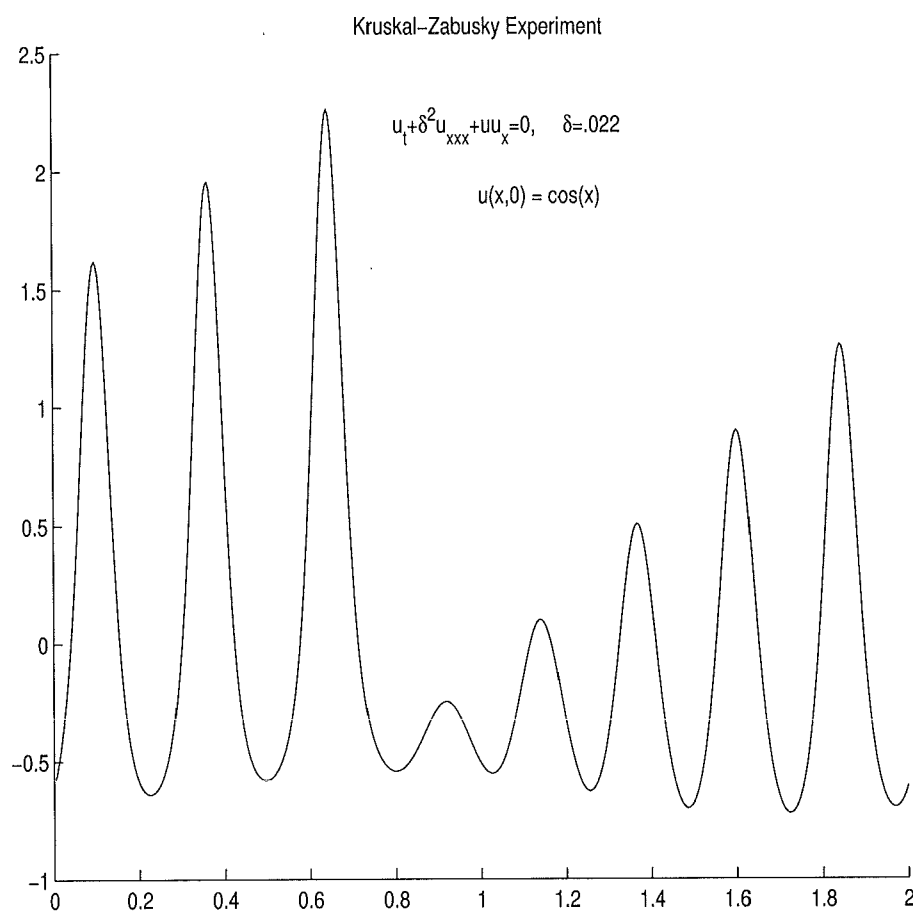# Solitons & Nonlinear Dispersive Waves

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Kruskal–Zabusky Experiment

$$u_t + \delta^2 u_{xxx} + u u_x = 0, \quad \delta = .022$$

$$u(x,0) = \cos(x)$$

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Chapter 1

The Korteweg deVries Equation

1.1 The Solitary Wave

The origins of the subject of solitons date to detailed observations and experiments by John Scott Russell, F.R.S. Edinburgh. Russell initially observed a solitary wave in a barge canal and, for a period of over ten years, made extensive observations and experiments on these waves. These scientific studies were reported to the British Association for the Advancement of Science in 1844; the following passage from his paper is quoted very often [38]:

I believe I shall best introduce this phenomenon by describing the circumstances of my own first acquaintance with it. I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped . . . not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined head of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on a a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular

and beautiful phenomenon which I have called the Wave of Translation, a name which it now very generally bears; which I have since found to be an important element in almost every case of fluid resistance, and ascertained to be the type of that great moving elevation of the sea, which, with the regularity of a planet, ascends our rivers and rolls along our shores.

Russell constructed a device for generating waves,

a long narrow channel or box a foot wide, eight or nine inches deep, and twenty or thirty feet long, filled with water to the height of say four inches

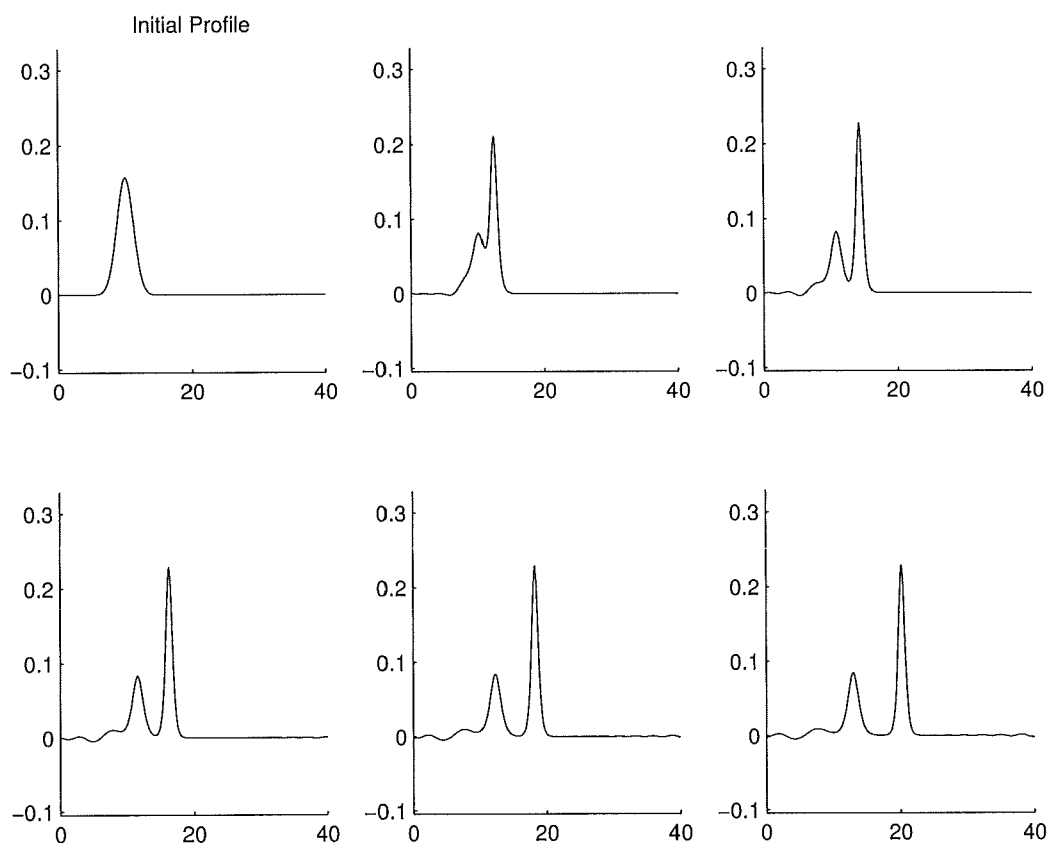and carried out extensive experiments:

Figure 1.1: Solution of the Kortweg-deVries equation with a Gaussian initial pulse. $\delta = .05$; $u(x, 0) = e^{-2(x-20)^2}$

If the volume of the generating fluid considerably exceed the length of the wave of a height equal to that of the fluid, the wave will assume its usual form W notwithstanding, and will pass forward with its usual volume and height. It will free itself from the redundant matter w by which it is accompanied, leaving it behind, and this residuary wave, w_2, will follow after it, only with a less velocity, so that although the two waves were at first united in the compound wave, they advance afterwards separate, . . . and are more and more apart the further they travel.

Disintegration of large Wave Masses.– . . . The existence of a moving heap of water of any arbitrary shape or magnitude is not sufficient to entitle it to designation of a wave of the first order. If such a heap be by any means forced into existence, it will rapidly fall to pieces and become disintegrated and resolved into a series of different waves, which do not move forward in company with each other, but move on separately, each with a velocity of its own, and each of course continuing to depart from the other. Thus a large compound heap or wave becomes resolved into the principal and resiudary waves by a species of spontaneous analysis.

The phenomenon observed experimentally by Russell for water waves is mimicked by the evolution of an initial Gaussian pulse for the Kortweg-deVries equation, which is the main character in our play. In Chapter §5 I have given a Matlab code which animates this phenomenon.

Modern researches have focused on the solitary wave of elevation, that which Russell termed postive. But Russell also described waves of depression in the fluid, as well as compound waves consisting of a positive and negative wave. Russell also described the dissipation in the waves, which is not modeled in the Euler equations or in the KdV approximation to them.

He was particularly concerned with the factors determining the velocity of the wave, and made extensive measurements. He arrived at the following heuristic formula, which gave very good agreement with experimental observation:

$$v = \sqrt{g(h+k)}$$

where g is the acceleration due to gravity, h is the depth of the fluid "in repose", and "k is the height of the crest of the wave above the plane of repose". Russell produced a table of experimental results comparing this formula with observed speeds (Table II, p. 327).

I have not found the phaeonenon, which I have called the wave of the first order, or the great solitary wave of translation, described in any observations, nor predicted in any theory of hydrodynamics.

After publication of his first observations of a solitary wave, a theoretical study was taken up by Kelland in the Edinburgh Philosophical Transactions. A theoretical formula for the velocity of the wave, based on the general equations of fluid dynamics was obtained; but the result did not agree very accurately with observation.

Airy took up the same study, in an extensive paper published in the 'Encyclopedia Metropolitana', and obtained a modification of Kelland's formula.

Mr. Airy has obtained for the velocity of a wave, an expression of a form closely representing that which Mr. Kelland had previously obtained. From the resemblance of this form of expression to the form previously obtained by Mr. Kelland, we are prepared for the conclusion that Mr. Airy has advanced in this direction little beyond his predecessor. . . . As however Mr. Airy appears to intimate to his readers that his own formula is as close an approximation to my experiments as the nature of these experiments will warrant, I have thought it necessary to make a complete re-examination of my experiments . . .

The result of the whole is, that there is an irresistible body of evidence in favour of the conclusion that Mr. Airy's formulae do not present anything like even a plausible representation of the velocity of the wave of the first order, and that the formula I have adopted does as accruately represent them as the inevitable imperfections of all observations will admit.

Russell's subsequent experiments stimulated great interest in the subject of water waves, and his discoveries were immediately taken up by Airy [3] and Stokes [44]. Stokes computed the Fourier series of the formal approximations to second order in the case of finite depth, and to third order in the case of infinite depth.

I have proceeded to a third approximation in the particular case in which the depth of the fluid is very great . . . This term gives an increase in the velocity of propagation depending on the square of the ratio of the height of the waves to their length.

Among the many conclusions of his investigations, Stokes concluded, er-

roneously, as it turns out, "that it is only an indefinite series of waves which possesses the property of being propagated with a uniform velocity." On the other hand,

> *There is one result of a second approximation which may possibly be of practical importance. It appears that the forward motion of the particles is not altogether compensated by their backward motion; so that, in addition to their motion of oscillation, the particles have a progressive motion in the direction of the propagation of the waves. . . . Now when a ship at sea is overtaken by a storm, and the sky remains overcast, so as to prevent astronomical observations, there is nothing to trust to for finding the ship's place but the dead reckoning. But the estimated velocity and direction of the motion of the ship are her velocity and direction of motion relatively to the water. If then the whole of the water near the surface be moving in the direction of the waves, it is evident that the ship's estimated place will be erroneous. If, however, the velocity of the water can be expressed in terms of the length and height of the waves, both which can be observed approximately from the ship, the motion of the water can be allowed for in the dead reckoning.*

1.2 The Euler Equations

The mathematical analysis of wave motion on the surface of an incompressible, inviscid fluid with irrotational flow is based on the equations

$$\Delta \varphi = 0 \qquad 0 \le y \le h + \eta,$$

$$\eta_t + \varphi_x \eta_x = \varphi_y \qquad \text{on } y = h + \eta(x, t)$$

$$\rho \left(\varphi_t + \frac{1}{2} |\nabla \varphi|^2 + g\eta \right) = 0 \qquad \text{on } y = h + \eta(x, t)$$

$$\varphi_y = 0 \qquad \text{on } y = 0.$$

Here g is the acceleration due to gravity, and ρ is the density. The function φ is the velocity potential; that is, the fluid velocity $\vec{u}$ is given by the gradient of φ:

$$\vec{u} = (u, v) = \nabla \varphi$$

The function η is the height of the free surface of the fluid above the equilibrium level $y = h$. Surface tension is ignored, and the solutions are called gravity waves, since there are no other forces acting on the fluid.

The mathematical difficulty surrounding the discovery of the solitary wave was due precisely to the problem of correctly balancing the effects of dispersion and nonlinearity in the asymptotic perturbation series, which are highly singular. If dispersion is ignored, one obtains the shallow water equations, of which the equation

$$u_t + uu_x = 0$$

is a simple prototype. This equation produces shocks, or breaking of waves. At the other limit, if the nonlinear effects are ignored, one obtains essentially the Airy equation

$$u_t + u_x + u_{xxx} = 0$$

which exhibits only a dispersive decay of the waves, and supports neither periodic wave trains, nor solitary waves.

There are three relevant length scales in the theory, h, the depth of the fluid; l, the length of the wave; and a, the amplitude of the wave. Accordingly, there are two dimensionless parameters,

$$\varepsilon = \frac{h}{l}, \qquad \mu = \frac{a}{h}.$$

The Korteweg-deVries (KdV) equation,

$$u_t + uu_x + u_{xxx} = 0, \tag{1.1}$$

is presumed to be valid in a regime where $\mu = O(\varepsilon^2)$; and this is where the solitary wave occurs.

The Korteweg deVries equation may be solved explicitly and exhibits both periodic wave trains (the cnoidal waves), and, in the limit of infinite period, the solitary wave:

$$u(x, t) = 3c \operatorname{sech}^2 \tfrac{1}{2}\sqrt{c}(x - ct).$$

We note that for the solitary wave of the KdV equation, the amplitude is directly proportional to the speed.

The KdV equation, named for Korteweg and deVries [21] was in fact first found by Boussinesq [8]; and, moreover, referring specifically to Russell's observations, he found the exact solution for the solitary wave of the nonlinear

equation Boussinesq's treatise was, however, 680 pages long, and much of his work was not fully appreciated.

Rigorous proofs of the the existence of periodic wave trains were finally given[1] by Levi-Civita [27] in the case of infinitely deep water and a year later by Struik [45] for periodic wave trains in water of finite depth. Levi-Civita used a conformal mapping technique to map the unknown domain conformally into a half-plane. Friedrichs and Hyers [16] gave a rigorous construction of the solitary wave. Walter Craig [11] has given a rigorous justification of the validity of the KdV approximation over long time intervals of order ε^{-3}. The formal derivation of the KdV equation water waves is considerably involved.

1.3 The Fermi-Pasta-Ulam Experiment

The remarkable mathematical structure of the KdV equation came to light not from the theory of water waves, but as a fall-out from a famous experiment by Fermi,Pasta, and Ulam [13] in 1955. Fermi, Pasta, and Ulam were attempting to use the computational power of the new computers to observe *thermalization* of energy in a nonlinear dynamical system with a large number (in this case, 64) degrees of freedom. They took as a simple model a coupled chain of 64 masses coupled by nonlinear springs. The equations of motion of such a system are

$$m\ddot{y}_n = f(y_{n+1} - y_n) - f(y_n - y_{n-1}), \qquad (1.2)$$

where f is a nonlinear restoring force, *e.g.* $f(y) = y + \alpha y^2$.

Instead of thermalization of the energy, that is, a tendency toward a stationary distribution of energy among the modes, they observed a quasiperiodic exchange of energy between the modes. This was unexpected, and led to a number of computer experiments on such nonlinear systems with a large number of degrees of freedom.

A continuum limit of this model is obtained formally as follows. We assume that $y_n(t) \doteq y(nh, t)$, where $y(x, t)$ is a smooth function. By Taylor's theorem,

$$y_{n\pm1} = y_n \pm hDy_n + \frac{h^2}{2!}D^2y_n \pm \frac{h^3}{3!}D^3y_n + \frac{h^4}{4!}D^4y_n \pm \ldots$$

---

[1]Geometers take note!

where $Dy_n = y_x(nh, t)$, etc. If we take $y = y_n$, substitute these expansions into (1.2), and expand in powers of h, we find

$$my_{tt} = h^2 y_x x + 2\alpha h^3 y_x y_{xx} + \frac{h^4}{4!} y_{xxxx} + \dots,$$

where the dots denote higher order terms in h. Now consider the formal limit of this equation as the spacing $h \to 0$. We must assume that the masses also decrease, and in fact, that $m = h^2/c^2$. Then, dropping terms of order m^2 and higher, the equation becomes

$$\frac{1}{c^2} y_{tt} = y_{xx} + 2\alpha \sqrt{m} y_x y_{xx} + \frac{m}{4!} y_{xxxx} \qquad (1.3)$$

This equation is similar to the Boussinesq equation that arises in the theory of water waves. The equation was shown to be integrable by the inverse scattering method by V.E. Zakharov [54]; I will discuss Zakharov's paper in Chapter (3.3).

An account of this experiment and subsequent investigations of Kruskal and Zabusky [23] is given by Cercignani [10] and Palais [33].

1.4   The Kruskal-Zabusky Experiments

A very simple derivation of the KdV equation is given by Kruskal [22]. He begins with the weakly nonlinear dispersive wave equation

$$u_{tt} = u_{xx}(1 + \varepsilon u_x) + \alpha u_{xxxx},$$

where ε, $\alpha \ll 1$ are small parameters measuring nonlinearity and dispersion respectively. The linear equation, with $\varepsilon = 0$, has the dispersion relation

$$\omega^2 = k^2 - \alpha k^4.$$

For $\alpha = \varepsilon = 0$ the equation reduces to the linear wave equation, which has as a general solution left and right progressing wave-forms. We look for a solution of the full equation which is right progressing, with only a slow variation in time. To make this precise, we look for a solution of the form

$$u(x, t, \varepsilon, \alpha) = w(\xi, \tau) \qquad \xi = x - t, \qquad \tau = \varepsilon t.$$
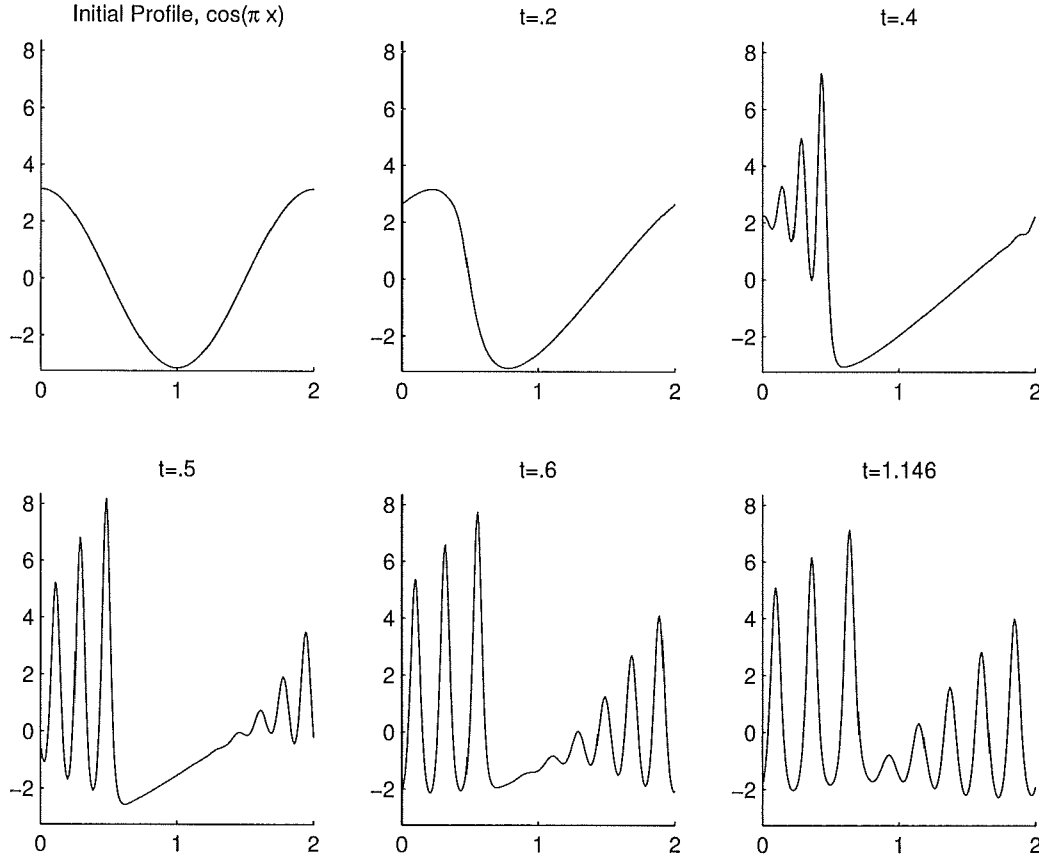
Figure 1.2: Steepening and development of oscillations by the Korteweg-deVries equation with intial data cos πx; one of Kruskal and Zabusky's original experiments.

By the chain rule

$$\frac{\partial u}{\partial t} = \varepsilon \frac{\partial w}{\partial \tau} - \frac{\partial w}{\partial \xi}, \qquad \frac{\partial u}{\partial x} = \frac{\partial w}{\partial \xi};$$

and, in the variables ξ, τ the equation becomes

$$\varepsilon^2 w_{\tau\tau} - 2\varepsilon w_{\xi\tau} + w_{\xi\xi} = w_{\xi\xi}(1 + \varepsilon w_\xi) + \alpha w_{\xi\xi\xi\xi}.$$

We drop the quadratic term in ε since it is second order. Putting $U = w_\xi/2$ we obtain the KdV equation for U:

$$U_\tau + UU_\xi + \frac{\alpha}{2\varepsilon}U_{\xi\xi\xi} = 0.$$

This simple derivation indicates two of the primary ingredients in the KdV approximation: First, one must specialize to a unidirectional frame; and second, one must scale the time variable in an appropriate way. These two features also appear in the more complicated derivation of the KdV approximation to the plasma and Euler equations.

The KdV equation has as a special solution the solitary wave,

$$u(x,t) = 12\omega^2 \mathrm{sech}^2[\omega(x - 4\omega^2 t)]$$

These waves move to the right with speed $4\omega^2$. Note that their amplitude depends on the wave speed, and that larger waves travel faster. One could choose as initial data two solitons separated by a distance great enough so that their interaction was extremely small, since they decay exponentially in either direction. Suppose the soliton to the left is larger. As time evolves, the larger soliton will overtake the smaller one. Since the equation is nonlinear they will react in a nonlinear way. After a period of time the two solitons again separate, the larger one moving ahead to the right and regaining its original shape. For large time, the two solitons are perturbed only by a phase shift: they are not quite where they would be had they been purely solitary waves. These facts were discovered by computational experiments by Kruskal and Zabusky in the early 60's [23].

Moreover, the same thing happens when the initial data consists of several solitons, separated originally into distinguishable solitary waves. As time progresses, the faster solitons overtake the slower ones, and as time goes to infinity, the solution evolves into separated solitons, each with its own original amplitude and speed, but with slightly displaced phase.

Even though the Korteweg-deVries equation is nonlinear, there is a closed formula for the n-soliton solution:

$$u(x,t) = 12\frac{d^2}{dx^2} \log \det \left\| \delta_{jk} + \frac{e^{-(\theta_j + \theta_k)}}{\omega_j + \omega_k} \right\| \tag{1.4}$$

$$\theta_j = \omega_j(x - \alpha_j - 4\omega_j^2 t). \tag{1.5}$$

1.5   Pseudospectral codes

A survey of numerical methods for weakly nonlinear dispersive wave equations appears in the book by Drazin and Johnson [35]. Comparative studies
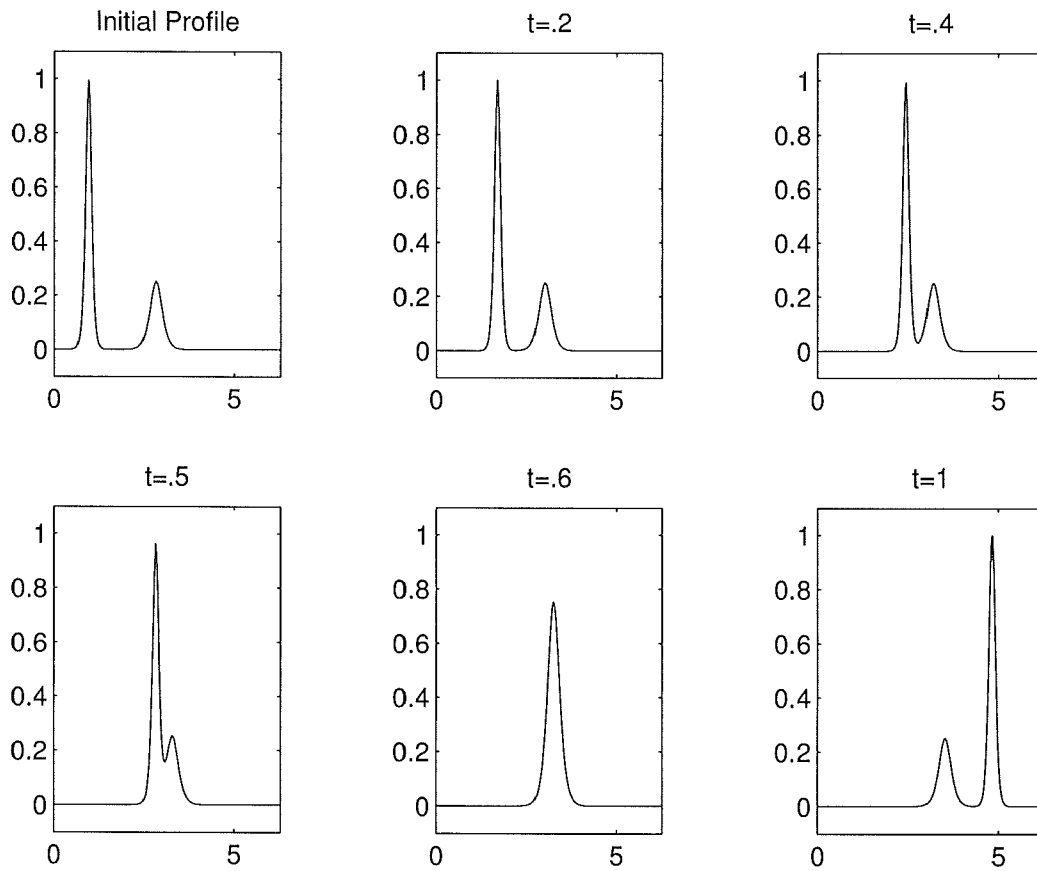
Figure 1.3: The interaction of 2 solitary waves in the exact solution. Two solitary waves are pictured in the first frame. As time progresses the two solitary waves interact and separate. Note the dip in the larger wave as they interact, indicating clearly that the interaction is nonlinear and not a simple superpostion. After the interaction they have regained their shape, but are displaced from where they would have been had there been no interaction.

of various methods for the nonlinear Schrödinger equation have been carried out by Taha and Ablowitz [46], [47]. The split-step method introduced by F. Tappert [48] has been discussed by R.S. Palais [33].

Fornberg and Whitham [15] used a leap-frog method with an explicit time step to solve numerically the KdV equation. To achieve any kind of accuracy with an explicit method, one must take very small time steps. In the case of the third order operator D^3 one must have $\Delta t = O((\Delta x)^3)$.

Y. Li, a postdoctoral fellow at Minnesota, and I wrote some simple codes in Matlab to numerically integrate the KdV equation and animate the solutions. We chose an implicit pseudo-spectral scheme, based on the method discussed in [51], for its simplicity, speed and versatility. A number of these codes are given in Chapter 5. We reconstructed one of Kruskal and Zabusky's original experiments, [23] taking as initial data $\cos \pi x$ for the KdV equation. A sequence of frames is given in Figure 1.4.

The codes use a pseduo-spectral method with an implicit method for the time step. This leads to a nonlinear equation to solve at each time step, and the scheme uses a simple iteration at each stage to solve the nonlinear equation by successive approximations. An implicit method which uses the solution u at the preceeding step is generally unstable. We got much better numerical results by averaging the nonlinear term over the previous and current time step, and incorporating an iteration on the nonlinear term in the scheme. The size of the time step is determined by the requirement that the iteration scheme converge sufficiently rapidly. Wineberg et. al. [51] remark that they found it sufficient to simply carry out two iterations at each step.

We summarize here the derivation of the scheme for the KdV equation:

$$u_t + D^3 u + D \left(\frac{u^2}{2} \right) = 0.$$

We first illustrate the trapezoid method with the simple equation

$$u_t + Du = 0.$$

Write the equation in the form

$$u_t = -Du,$$

and integrate from t to $t + \Delta t$:

$$u(x, t + \Delta t) - u(x, t) = - \int_t^{t+\Delta t} Du \, dt.$$

Now approximate the integral on the right by the trapezoid rule to obtain

$$u(x, t + \Delta t) - u(x, t) = - \frac{Du(x, t + \Delta t) + Du(x, t)}{2} \Delta t.$$

This equation may be written in the form

$$(I + \frac{1}{2}\Delta t D)u(x, t + \Delta t) = (I - \frac{1}{2}\Delta t D)u(x, t),$$

or symbolically, by

$$u(x, t + \Delta t) = Uu(x, t), \qquad U = \frac{I - \frac{1}{2}\Delta t D}{I + \frac{1}{2}\Delta t D}$$

There is no problem in inverting $I + \frac{1}{2}\Delta t D$ since D is skew adjoint and its eigenvalues are imaginary. The operator U is unitary[2], and its L^2 norm is 1, *regardless of the size of* Δt.

Applying the 'trapezoid' argument to the full KdV equation, we get

$$u(t + \Delta t) = Uu(t) - B(u^2(t + \Delta t) + u^2(t)),$$

where

$$U = \frac{I - \frac{1}{2}\Delta t D^3}{I + \frac{1}{2}\Delta t D^3}, \qquad B = \frac{.25\Delta t D}{1 + .5\Delta t D^3}. \tag{1.6}$$

The operator U is unitary. U and B are evaluated using the fast Fourier transform algorithm. The fast Fourier transform in Matlab is called by 'fft'. The inverse fast Fourier transform is called up by 'ifft'. Let us set

$$v = \text{fft}(u), \qquad u = \text{ifft}(v).$$

Then the above equation can be written in the form

$$v(t + \Delta t) = Uv(t) - B\text{fft}(u^2(t) + u^2(t + \Delta t)).$$

The nonlinear terms are best computed in the spatial representation, so we transform back to the original spatial picture, carry out the multiplication, which is pointwise on the x side, and then transform back. This is a nonlinear, implicit scheme, since it is nonlinear in $u(t + \Delta t)$.

The final step in the procedure is to solve this nonlinear scheme by successive approximations.[3] That is, we write a subroutine to carry out the iterations

$$v_{j+1}(t + \Delta t) = Uv(t) - B\text{fft}(u^2(t) + u_j^2(t + \Delta t)),$$

---

[2]U is the Cayley transform of $\frac{1}{2}D$; cf. Riesz and Nagy [37]

[3]One could try to set up a Newton iteration scheme, which converges quadratically, but this is complicated by use of the Fourier transform, since then one has to invert a full matrix.

$$u_j = \text{ifft}(v_j); \qquad v_j = \text{fft}(u_j(t)).$$

This scheme is quite robust and gives extremely good accuracy for N sufficiently large (say $N = 512$) and the time steps sufficiently small (say $\Delta t = .005$). We compared the computed two soliton solution with the exact solution and found the difference was negligible after the interaction. A simple routine which carries out this comparision is given in the code kdv-comp.m

The calculations are carried out on 2π periodic functions, since we are using the finite Fourier transform. However, in order to place two solitary waves in the interval $[0, 2\pi]$ one finds that the velocities c_1 and c_2 must be relatively large (of the order of 3,5) in order that the solitons be sufficiently narrow. The solitary waveform for the equation $u_t + u_{xxx} + uu_x = 0$ is

$$12o^2\text{sech}^2\omega x$$

so its height is proportional to ω^2; and this leads to numerical instability due to the nonlinear terms. With $\omega = 5$ the amplitude of the solitary wave is then 300, and this large amplitude causes instability in the numerical scheme, necessitating very small time steps.

It is therefore necessary to rescale the problem to an interval $0 \le y \le L$. We let w satisfy

$$w_t + w_{yyy} + ww_y = 0, \qquad 0 \le y \le L,$$

and define

$$u(x,t) = \frac{1}{a}w(ax,t), \qquad a = \frac{L}{2\pi}.$$

Then u satisfies the equation

$$u_t + \frac{1}{a^3}u_{xxx} + uu_x = 0 \qquad 0 \le x \le 2\pi.$$

We found it sufficient to take $L = 40$. We have also included a coefficient of dispersion, disp, in the code.

We can also see from this numerical scheme the mechanism by which the dispersive term smooths out the shocks developed by the nonlinear term. The operator B defined in (1.6) acts as a low-pass filter for the Fourier modes. To see this, note that on the spectral side B is multiplication by

$$\frac{.25i\Delta tk}{1 - .5i\Delta t(disp)a^{-3}k^3}.$$

The graph of the spectral properties of the operator B not only makes clear how dispersion and nonlinearity compete, but it also suggests why this numerical algorithm is so effective for nonlinear systems with strong dispersion. The nonlinear terms push energy into the higher modes; but it is in turn filtered out by the high frequency cut-off. This cut-off is characterisitic of weak nonlinearity, since the operator B will always consist of lower order differential operators divided by higher order operators.
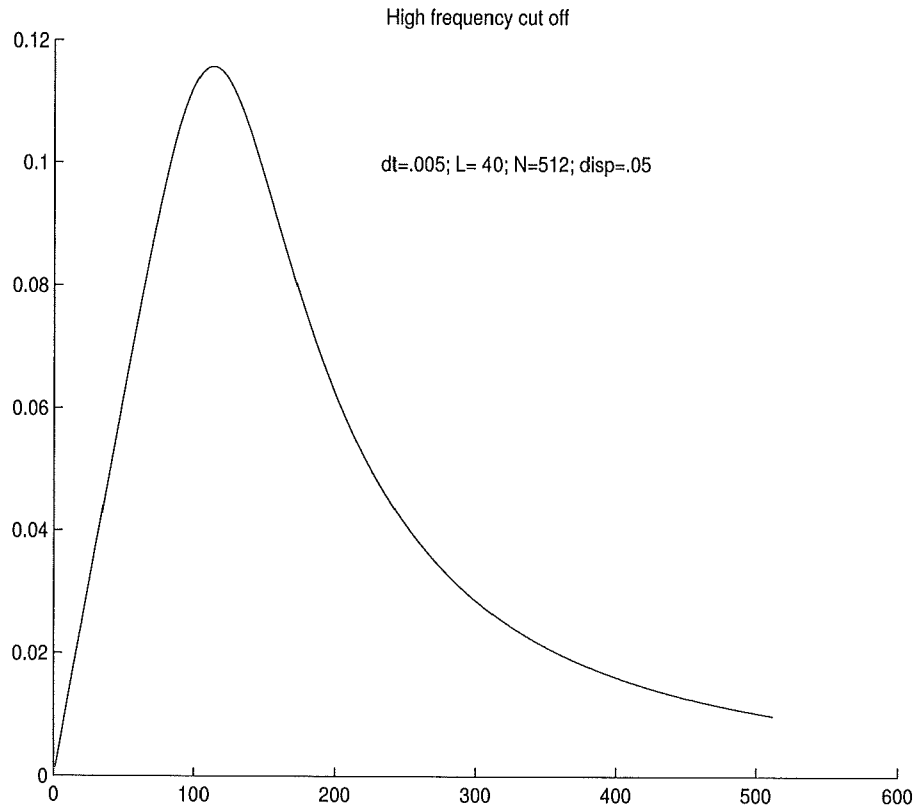

Figure 1.4: High frequency cut-off due to the dispersion acting on the nonlinear terms. $\Delta t = .05$, disp=.05, $L = 40$, $N = 256$.

Chapter 2

The Plasma Equations

2.1 The KdV Approximation

The formal derivation of the KdV approximation for the plasma equations is considerably easier than in the case of water waves [50], [24]. We carry out the details here with an emphasis on the fundamental role played by the Galilean and scaling groups in the approximation.

A plasma consists of negatively charged electrons and positively charged ions. The electrons are treated as a gas and equations of motion for the ions are derived. The ion density is denoted by n, the electron density by n_e, the electric force field by E and the velocity of the ions by v. The equations of the plasma may be written in the following form [24], [43]

$$n_t + (nv)_x = 0, \qquad v_t + vv_x = E,$$

$$E + (\log n_e)_x = 0, \qquad E_x + n_e = n,$$

where n is the ion density, n_e is the electron density, v is the ion velocity, and E is the electric field. We eliminate n_e from the equation by defining $\varphi = \log n_e$, φ being the electric potential, and the equations reduce to three equations in three unknowns

$$n_t + (nv)_x = 0, \qquad v_t + \left(\frac{v^2}{2} + \varphi\right)_x = 0, \qquad \varphi_{xx} - e^\varphi + n = 0. \qquad (2.1)$$

First we determine the dispersion relation from a formal perturbation for small disturbances of the equilibrium states. We look for small perturbations

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about the quiescent state $n = 1$, $v = c$, $\varphi = 0$:

$$n = 1 + \varepsilon n_1 + \varepsilon^2 n_2 + \ldots, \qquad \varphi = \varepsilon \varphi_1 + \varepsilon^2 \varphi_2 + \ldots, \qquad v = c + \varepsilon v_1 + \varepsilon^2 v_2 + \ldots.$$

Substituting these expansions into (2.1) we obtain, at lowest order, the linear equations for small disturbances

$$n_{1,t} + (v_1 + c n_1)_x = 0,$$

$$v_{1,t} + (c v_1 + \varphi_1)_x = 0,$$

$$\varphi_{1,xx} - \varphi_1 + n_1 = 0.$$

The dispersion relation for this linear system is obtained by looking at the Fourier modes

$$\begin{pmatrix} n_1 \\ v_1 \\ \varphi_1 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} e^{i(kx - \omega t)}.$$

This leads to the linear algebraic system

$$\begin{pmatrix} \omega - ck & -k & 0 \\ 0 & \omega - ck & -k \\ -1 & 0 & 1 + k^2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = 0.$$

The system has a nontrivial solution iff the determinant of the above matrix vanishes, and this leads to the condition $(\omega - ck)^2(1 + k^2) - k^2 = 0$, or, solving for ω,

$$\omega = ck \pm \frac{k}{\sqrt{1 + k^2}}.$$

First we consider the case $c = 0$:

$$\omega = \pm \frac{k}{\sqrt{1 + k^2}}..$$

Waves traveling to the right are then obtained by taking the positive sign, since then the phase velocity is

$$\frac{\omega}{k} = \frac{1}{\sqrt{k^2 + 1}}.$$

The Taylor expansion of the dispersion relation for general c is then $\omega = (c+1)k - \frac{1}{2}k^3 + O(k^5)$. This is a good approximation for small k, that is, for long waves. The leading term $(c+1)k$ in this approximation corresponds to a dispersionless system with a wave speed $c + 1$. By taking $c = -1$ we remove the linear term in the dispersion relation.

The plasma equations are Galilean invariant; that is, they are unchanged under the one-parameter group of transformations

$$x' = x - ct, \qquad t' = t$$

$$n'(x', t') = n(x, t), \qquad \varphi'(x', t') = \varphi(x, t), \qquad v'(x', t') = v(x, t) - c.$$

This means that the equations are the same in any Galilean frame, and we can shift to a moving frame of reference simply by subtracting the speed of the moving frame from the velocity v. In the moving frame, the velocity v' tends to $-c$ at infinity. In particular, the collection of Galilean frames is labeled by the values of v at infinity. This amounts to expanding about the quiescent state $n = 1, v = -c, \varphi = 0$, where c is the velocity of the reference frame.

In particular, $v = -1$ at infinity in a Galilean frame with speed 1; hence the dispersion relation is

$$\omega = -k + \frac{k}{\sqrt{1 + k^2}} = k(-1 + [1 - \frac{1}{2}k^2 + \ldots]);$$

and we obtain, in the long wave approximation,

$$\omega \doteq -\frac{1}{2}k^3.$$

The partial differential operator associated with this dispersion relation is

$$\frac{\partial}{\partial t} + \frac{1}{2}\frac{\partial^3}{\partial x^3},$$

for which the natural scaling is $x' = \varepsilon x$, $t' = \varepsilon^3 t$. If we introduce this scaling into the equations (2.1) we obtain (after division by ε)

$$\varepsilon^2 n_{t'} + (nv)_{x'} = 0,$$

$$\varepsilon^2 v_{t'} + \left(\frac{v^2}{2} + \varphi\right)_{x'} = 0,$$

$$-\varepsilon^2 \varphi_{x'x'} + e^\varphi = n.$$

This perturbation scheme is singular, since the character of the equations is changed when $\varepsilon = 0$. Since only ε^2 appears in these equations, we formally expand all quantities in powers of ε^2:

$$n = 1 + \varepsilon^2 n_1 + \ldots, \qquad v = -1 + \varepsilon^2 v_1 + \ldots, \qquad \varphi = \varepsilon^2 \varphi_1 + \ldots.$$

When we do this, substitute the expansions into the above equations, and collect terms, we get at order ε^2:

$$(-n_1 + v_1)_{x'} = 0, \qquad (-v_1 + \varphi_1)_{x'} = 0, \qquad \varphi_1 = n_1.$$

Since all quantities tend to zero as $x \to \infty$ we have

$$n_1 = v_1 = \varphi_1.$$

At next order we obtain

$$n_{1,t'} + (n_1 v_1)_{x'} + (v_2 - n_2)_{x'} = 0,$$

$$v_{1,t'} + \left(\frac{v_1^2}{2} - v_2 + \varphi_2 \right)_{x'} = 0,$$

$$-\varphi_{1,x'x'} + \varphi_2 + \frac{1}{2}\varphi_1^2 = n_2.$$

The second order quantities n_2, v_2, and φ_2 may be eliminated from this system; and, dropping the primes, one obtains the Korteweg-de Vries equation for v_1:

$$v_{1,t} + v_1 v_{1,x} + \frac{1}{2}v_{1,xxx} = 0. \tag{2.2}$$

2.2 The two soliton collision

The derivation of the KdV approximation in the preceding section was, of course, formal. To what extent is it a valid approximation? For example, which of the various phenomena of the KdV equation hold also for the full set of plasma equations? In particular, does the elastic collision of a pair of solitons for the KdV equation hold also, in some restricted sense, for the plasma equations?

Jürgen Moser and Robert Sachs looked at this question for the Euler equations of water waves, but were unable to prove that the two soliton solution of the KdV equation could be extended to the full Euler equations. The KdV approximation is a singular approximation to the Euler equations, as well as to the Plasma equations.

Y. Li, a postdoc at Minnesota, and I carried out a numerical experiment on the plasma equations to see the interaction of two solitary waves [28]. We first showed that the plasma equations have solitary waves that travel at speeds proportional to their amplitude, and then superposed two solitary waves of different amplitudes, and numerically computed their interaction.

In equations (2.1) we replace n by $1 + n$, where now $n \to 0$ at infinity. Then the equations for a traveling wave with speed c in the Galilean frame moving with the same speed are

$$((1+n)v)_x = 0, \qquad \left(\frac{v^2}{2} + \varphi\right)_x = 0, \qquad \varphi'' - e^\varphi + 1 + n = 0.$$

Moreover, $v \to -c$ as $x \to \pm\infty$. We therefore obtain

$$1 + n = -\frac{c}{v}, \qquad v = -\sqrt{c^2 - 2\varphi}, \tag{2.3}$$

and, therefore,

$$\varphi'' = e^\varphi - \frac{c}{\sqrt{c^2 - 2\varphi}}. \tag{2.4}$$

Equation (2.4) is easily analyzed by standard phase plane methods cf. [28]. One may then superpose two solitary waves, suitably separated.

We should not expect that two solitary waves of the plasma equations interact cleanly and leave no trace of the their interaction in the form of dispersive tails, as happens for the KdV equation. Rather, we should expect that as the amplitude of the solitary waves vanishes, the collision becomes more and more elastic.

Numerically, it is impractical to study the interaction of very small amplitude waves. As the amplitude decreases, the spread of the wave increases. The solitary waves decrease exponentially from their maximum, but slower waves have a smaller rate of decrease, and so one must set the solitons on a larger interval and integrate over a longer time. This results in longer and longer computation times as the amplitude diminishes.

Despite these obvious *caveats*, when we ran the experiment, we saw that the collisions were virtually elastic. Here is a sequence of frames of the interaction:
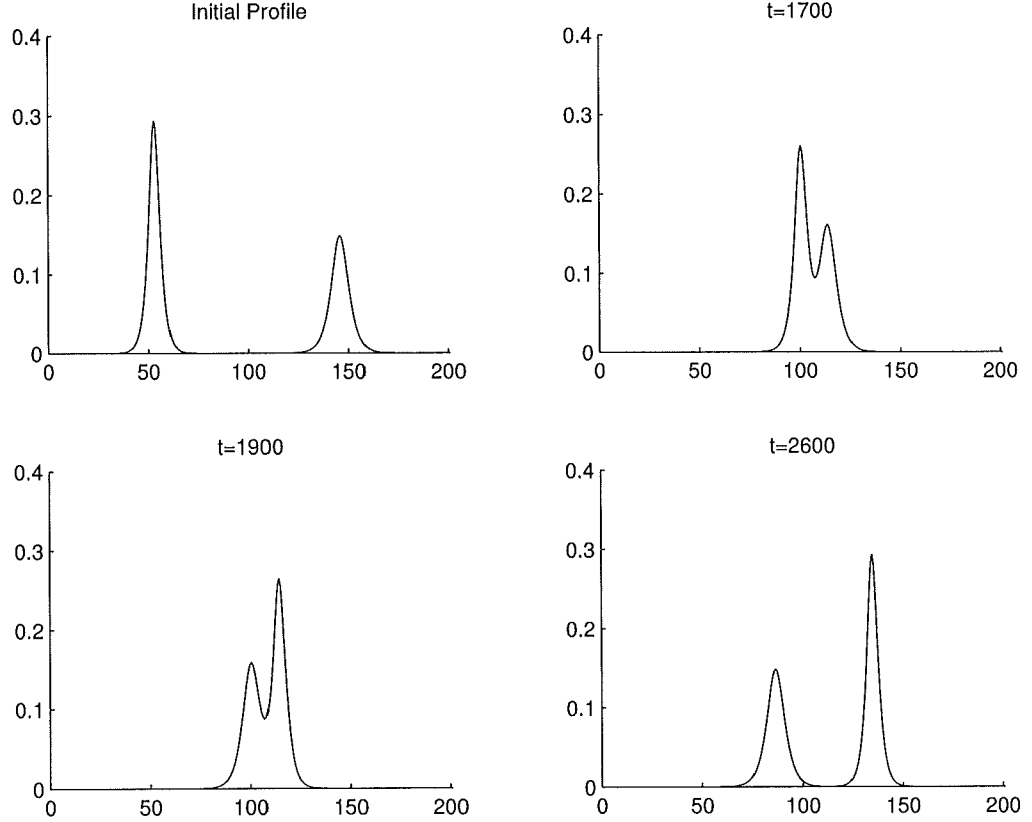

Figure 2.1: Interaction of two solitary waves for the ion acoustic plasma equations, by a pseudo-spectral method. The time step was $dt = .008$, $N = 2^{13} = 8192$ Fourier modes. The computation is done in a moving frame moving at speed c=1.07. The speed of the slower wave is 1.5; while that of the larger wave is 1.1. The sequence indicates a nearly elastic collision.

The code integrates the equations in a moving frame. The speed of the moving frame is 1.07. The speed of the larger wave is 1.1; that of the smaller wave is 1.05. The speed of the maximum wave is approximately 1.5852. In the moving frame, the smaller wave drops back, while the larger wave advances. This has the advantage of keeping both waves within a fixed

interval throughout the interaction.

Our first attempts to apply this numerical method broke down because of very small 'discontinuities' at the endpoints of the interval. Though the solitary wave decays exponentially fast, and is of the order of 10^{-6} at the endpoints of the interval, there is nevertheless a small jump at the endpoints, due to the fact that we used a solitary wave, rather than a periodic wave.

The small discrepancies at the endpoints contribute energy to the high frequency modes. These are cascaded into the higher modes by the nonlinear terms, and show up as highly oscillatory noise at the endpoints. These low amplitude, high frequency oscillations propagate into the interior from the boundaries, and eventually cause a break-down in the computation.

To deal with this problem we did two things: 1. Rather than computing the solitary wave, which is a homoclinic orbit, we instead computed a periodic wave very close to the homoclinic orbit; 2. we introduced a very mild filtering into the scheme.

Numerical error introduces high frequency "noise". For example, the fact that in any numerical process the solution is essentially a piecewise linear function introduces certain errors, albeit very small, into the process. For the KdV equation this high frequency noise causes no problems in the computation, since the dispersion relation of the KdV equation grows like k^3, it acts like a high frequency cut-off and filters out high frequency noise automatically, as I explained in §1.5.

But the plasma equations have much weaker dispersion, namely

$$\omega = \frac{k}{\sqrt{1+k^2}}(\sqrt{1+k^2} - 1) \sim k,$$

and high frequency energies are not attenuated.[1] Moreover, nonlinear terms transfer energy from lower to higher modes. The result is that the high frequency data corrupts the calculations over time. Because of the scaling involved, the code must be integrated over a very long time scale in order to see the interaction of the two solitary waves, in this case, to T=2600. We actually carried the calculations further, to $T = 3800$ in order to see the waves separate completely.

To compensate for this problem, we introduced a very mild filtering into the scheme. The filtration is accomplished by multiplying the Fourier transform of the functions by a function which "cuts off" the high frequencies; a

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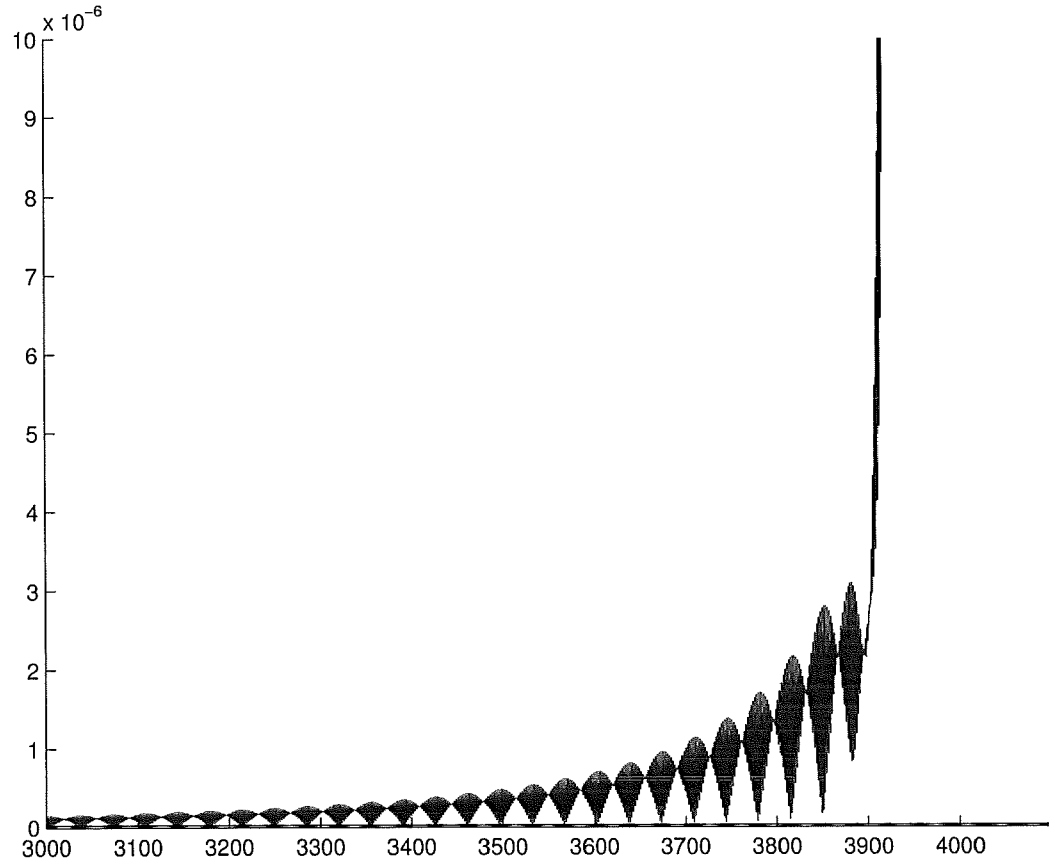[1] The dispersion of the Euler equations is even weaker.

Figure 2.2: Filtered Fourier transform of the initial data, with the high frequency modes magnified. The unfiltered data shows up as a highly oscillatory function; while the filtered data appears as the line $y = 0$.

graph is shown in Figure 6. We filtered the initial data, and then filtered the solution at every time interval of 50 units. Since the time step was .008, this means that the solution was filtered once in every 6250 time steps.

The filter we used is the 'sharpened raised cosine' (cf. [9], p. 248):

$$\sigma(\theta) = \sigma_0^4(35 - 84\sigma_0 + 70\sigma_o^2 - 20\sigma_0^3), \qquad \sigma_0 = \frac{1}{2}(1 + \cos\theta).$$

The filtering is extremely mild. There is no apparent difference between the initial data and its filtration on the spatial side; but the suppression of the high frequency noise is apparent when the Fourier transforms of the initial data and the filtered data are enlarged, as in Figure 2.2.
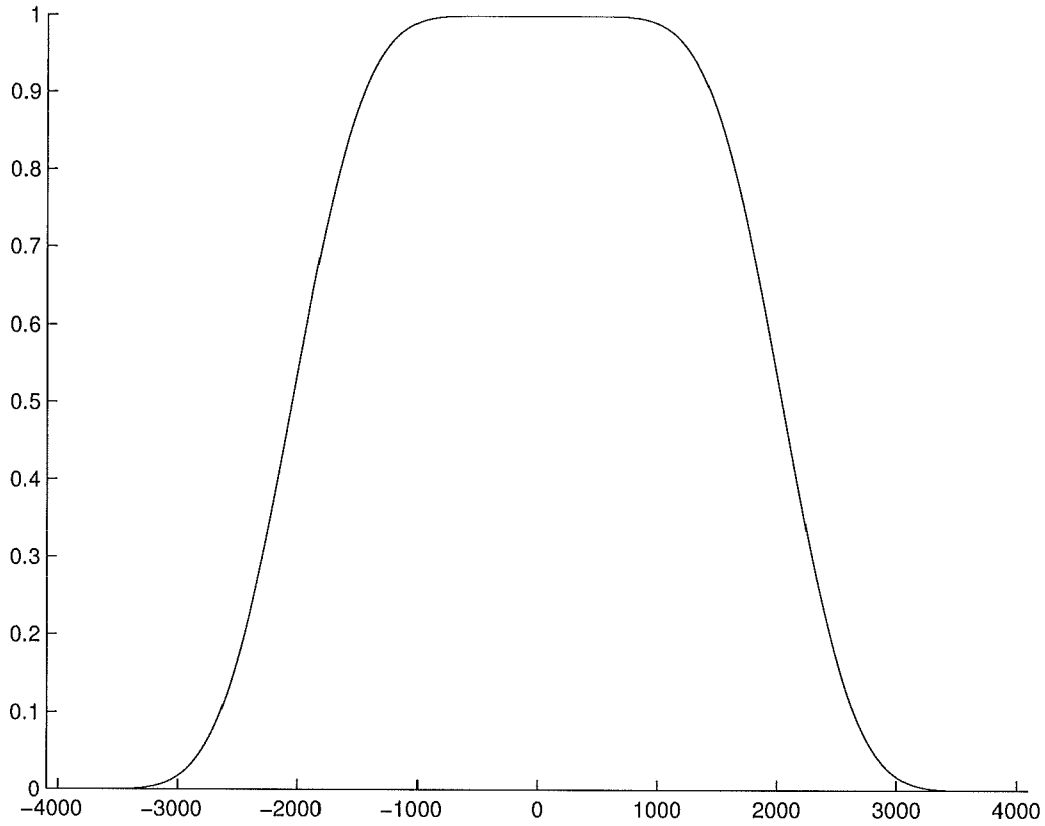
Figure 2.3: Sharpened Raised Cosine low pass filter function used in the numerical computations, from *Spectral Methods in Fluid Mechanics*, Canuto, Hussaini, Quarteroni, & Zang [9].

The filtering, of course, removes energy from the system; but we calculated the energy and momentum,

$$\mathcal{E} = \int \frac{1}{2}(\varphi_x)^2 + e^\varphi - 1 - (\varphi + v^2/2)(1 + n) - cnv \, dx, \qquad P = \int nv \, dx,$$

over the course of the interaction. They deviated from their original values by .37% and .2%, respectively, showing that the filtering is indeed very mild.

2.3 Comparision with KdV

Let us compare the numerical data with a suitably chosen exact two soliton solution of the Korteweg deVries equation. This procedure will introduce us to some of the more subtle properties of the two soliton solution of the KdV equation, namely the scattering of solitons during the interaction. First note that there is a factor of 1/2 in the KdV approximation (2.2). This is easily accounted for by a simple rescaling:

$$v(x,t) = \frac{1}{2}u\left(x, \frac{1}{2}t\right),$$

where u satisfies the KdV equation

$$u_t + uu_x + u_{xxx} = 0.$$

By expanding the determinant in (1.4), and renaming the phase shifts α_1, α_2, the two-soliton solution of (2.2) can be written

$$v = 6\frac{d^2}{dx^2}\log\tau(\theta_1, \theta_2),$$

where

$$\tau = 1 + e^{-2\theta_1} + e^{-2\theta_2} + e^{-2(\theta_1+\theta_2+\alpha)};$$

$$\theta_1 = \omega_1(x - \alpha_1 - (c + 2\omega_1^2)t), \qquad \theta_2 = \omega_2(x - \alpha_2 - (c + 2\omega_2^2)t),$$

$$\alpha = \log\frac{\omega_2 + \omega_1}{\omega_2 - \omega_1}.$$

The parameter c is the relative speed of the reference frame, $c = 1.07$.

The two soliton solutions form a four parameter family, $\alpha_1, \alpha_2, \omega_1, \omega_2$. The linearized KdV equation at the 2 soliton solution therefore has a four dimensional null space, obtained by differentiating the equation with respect to the four parameters. In [20] a Fredholm alternative for the time dependent operator was proved, and a formal perturbation scheme was described by which one could construct a series solution of the full Euler (or in this case plasma) equations whose leading term was a two-soliton solution of KdV.

In that perturbation series, the four parameters $\alpha_1, \alpha_2, \omega_1, \omega_2$ must be allowed to vary in order to eliminate the resonance terms which lie in the
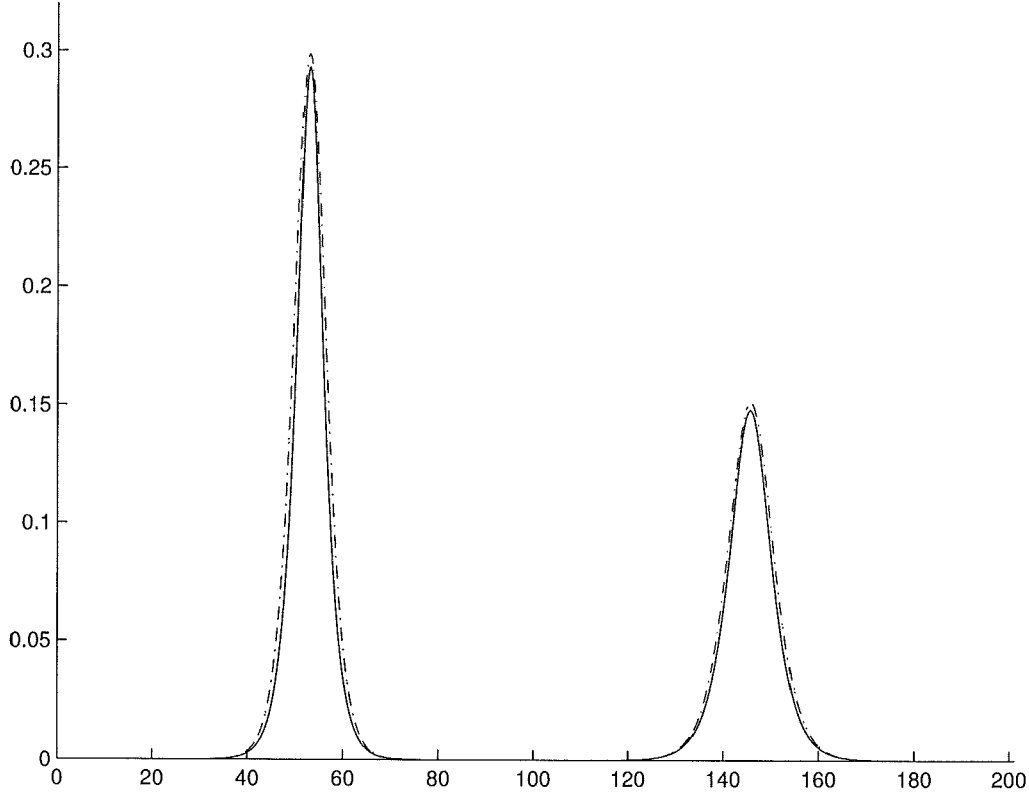
Figure 2.4: Fit of the two-soliton KdV solution to the initial plasma data at time $t = 0$.

null space of the linearized operator. Such arguments have been used in [39] in the study of the stability of traveling waves of parabolic systems, and more specifically by Pego and Weinstein [34] in their study of the stability of solitary waves of generalized KdV type equations.

In the perturbation scheme, the four parameters depend on the small parameter of the expansion, and are determined in the course of the perturbation series, as in a bifurcation problem. For now, we simply determine them by fitting the two soliton solution to the numerical data. The locations of the large and small waves before and after the interaction lead to four equations. Moreover, the waves are sufficiently separated before and after the interaction that we can apply the known formulae for the phase shifts incurred in the interaction [30]. We will discuss the scattering shifts later in

these lectures.

Lemma 2.3.1 *The two soliton solution has the asymptotic behavior*

$$u = 6\frac{d^2}{dx^2}\log\tau \sim 6\omega_1^2\mathrm{sech}^2(\theta_1 + \alpha) + 6\omega_2^2\mathrm{sech}^2\theta_2, \qquad t \to \infty;$$

and

$$u \sim 6\omega_1^2\mathrm{sech}^2\theta_1 + 6\omega_2^2\mathrm{sech}^2(\theta_2 + \alpha), \qquad t \to -\infty.$$

The proof of Lemma 2.3.1 will be given in §4.3. The peaks of the two solitary waves occur at $\theta_1 = 0$, $\theta_2 + \alpha = 0$ at time $t = 0$. Therefore, we take our matching conditions to be

$$t = 0: \qquad \theta_1 = 0, \qquad \theta_2 + \alpha = 0;$$

$$t = T: \qquad \theta_1 + \alpha = 0, \qquad \theta_2 = 0.$$

These four conditions lead to the equations

$$\alpha_1 = x_1^-,$$

$$\alpha_2 = x_2^- + \frac{1}{\omega_2}\log\frac{\omega_2 + \omega_1}{\omega_2 - \omega_1},$$

$$\alpha_1 = x_1^+ - (c + 2\omega_1^2)T + \frac{1}{\omega_1}\log\frac{\omega_2 + \omega_1}{\omega_2 - \omega_1},$$

$$a_2 = x_2^+ - (c + 2\omega_2^2)T.$$

Here $x_j^\pm$ denote the locations of the jth wave at times $t = 0$ and $t = T$, the total elapsed time, and c is the relative speed of the computation frame to the Galilean frame.

The phase constants can be eliminated from these equations, and we obtain two equations in ω_1 and ω_2:

$$2\omega_1^2 T - \frac{1}{\omega_1}\log\frac{\omega_2 + \omega_1}{\omega_2 - \omega_1} = \Delta x_1 - cT \qquad\qquad (2.5)$$

$$2\omega_2^2 T + \frac{1}{\omega_2}\log\frac{\omega_2 + \omega_1}{\omega_2 - \omega_1} = \Delta x_2 - cT, \qquad\qquad (2.6)$$
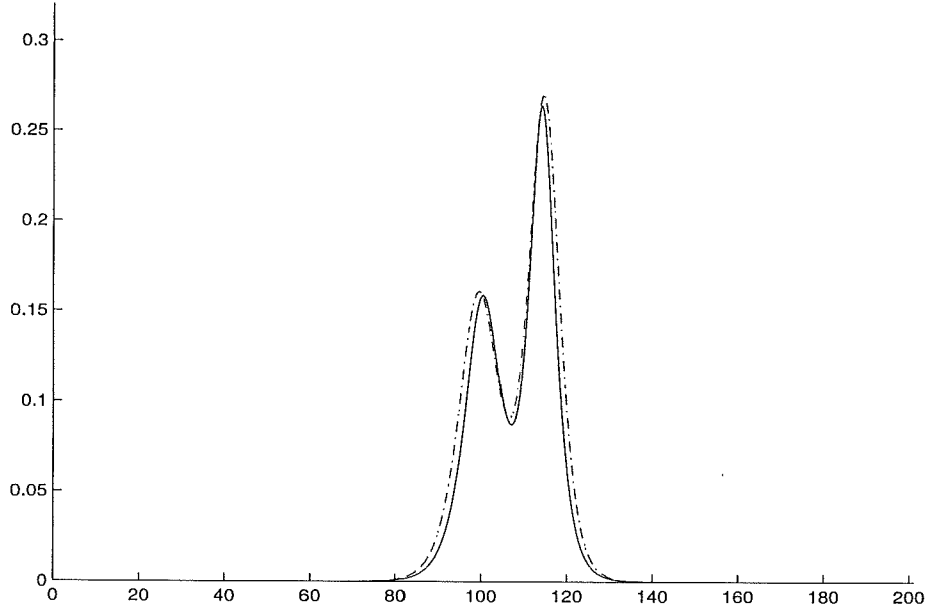
Figure 2.5: Comparison of the two soliton KdV solution (dashed) with the plasma data (solid) during the interaction, $t = 1900$.

where Δx_j is the total distance traversed by the jth wave.

These equations are easily solved by successive approximations for ω_1, ω_2 iteratively. As an initial guess we took the values obtained by matching the speeds of the two KdV waves exactly with the speeds of the soliton waves. The speed of the solitary wave $6\omega^2\text{sech}^2(\omega(x - 2\omega^2 t)$ is $2\omega^2$. The speeds of the two solitary plasma waves (relative to the Galilean frame) are .05 and .1. Therefore, as a first approximation, we take

$$\omega_1 = \sqrt{.05/2} = .1581; \qquad \omega_2 = \sqrt{.1/2} = .2236.$$

Our data are

$$T = 3800; \qquad c = -.07; \qquad x_1^- = 147.7349; \qquad x_2^- = 50.0468;$$

$$\Delta x_1 = -85.2439; \qquad \Delta x_2 = 120.4166$$

We obtained

$$\omega_1 = .1589; \qquad \omega_2 = .2231;$$

The relative phases α_1 and α_2 are then determined by any of the four equations above.

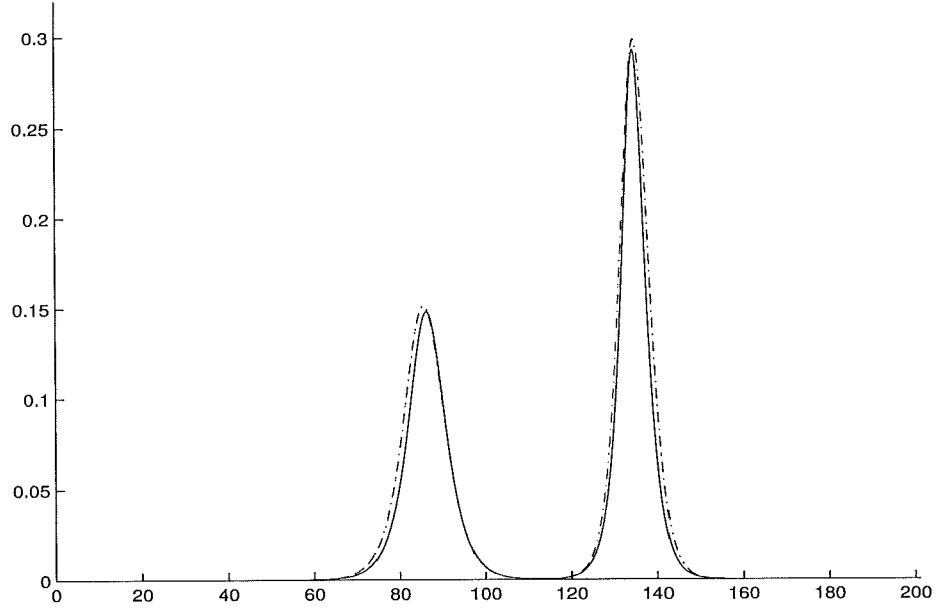Here we see the interaction of the two waves compared with the two soliton KdV solution:

Figure 2.6: Comparison of the two soliton KdV solution with the plasma data after the interaction, $t = 2600$.

Chapter 3

Hierarchies of Commuting Flows

3.1 The KdV Hierarchy

The computational discovery of the highly unusual behavior of solutions of the KdV equation prompted an intense, and, as it turned out, highly fruitful theoretical investigation of the KdV equation. The original theoretical breakthrough was made by Gardner, Greene, Kruskal, and Miura [17]. An account of the early developments in the subject is given by Cercignani [10] and Palais [33]

Later researchers clarified and simplified their arguments, and ultimately constructed myriad further examples of such special systems. One of the early papers which has had a fundamental influence on the development of the subject was the 1968 paper by Peter Lax [25]. Gardner, Greene, Kruskal, and Miura [17] had found that the eigenvalues of the Schrödinger operator

$$ L = D^2 + \frac{1}{6}u, \quad D = \frac{d}{dx}, $$

are constant in time if u evolves according to the KdV equation.

Lax simplified and clarified the situation conceptually by casting the situation in what is known as the Heisenberg picture in quantum mechanics. Suppose that the family of operators $\{L(t)\}$ is unitarily equivalent under the flow. Assume U is a one-parameter family of unitary operators:

$$ UU^* = I, \quad U_t = BU, \quad U^*L(t)U = L(0), $$

where B is a skew-adjoint operator. Differentiating the third equation with respect to time we get

$$U^*B^*L(t)U + U^*L_tU + U^*LBU = 0,$$

hence

$$L_t + B^*L + LB = 0.$$

In these calculations we interpret L_t as the operation of multiplication by the function u_t. Since $B^* = -B$ this equation reduces to the *Lax equation*

$$L_t = [B, L] \qquad\qquad (3.1)$$

where $[B, L]$ is the commutator $BL - LB$. The pair of operators L and B is called a *Lax pair*. Equation (3.1) is none other than the Heisenberg picture of the Korteweg-deVries equation.

Since L_t is a multiplication operator, the commutator $[B, L]$ must also be a pure multiplication operator. For example, taking $B = D$ we find

$$[D, L] = \frac{1}{6}u_x, \qquad L_t = \frac{1}{6}u_t$$

and the Lax equation is $u_t = u_x$. This equation generates the one parameter family of translations, $u(x, t) = u_0(x + t)$; and so, of course, L is unitarily equivalent under the flow.

The KdV equation itself is obtained by taking a third order skew adjoint operator

$$B = -4D^3 - \frac{1}{2}(uD + Du)$$

The details of the calculation are left as an exercise.

In all these calculations we may replace L by $L + k^2$, so the KdV equation is formally obtained as a consistency condition for the overdetermined system of partial differential equations

$$(D^2 + \frac{1}{6}u + k^2)\psi = 0, \qquad \frac{\partial \psi}{\partial t} = B\psi.$$

The two isospectral flows

$$u_t = u_x, \qquad u_t + u_{xxx} + uu_x = 0$$

are only two flows in an infinite hierarchy of commuting Hamiltonian flows. this hierarchy of flows is generated by a recursion relation, namely [26]

$$DF_{j+1} = (-D^3 - \frac{1}{3}(uD + Du))F_j, \qquad F_1 = u. \qquad (3.2)$$

The jth flow is then given by

$$u_t = DF_j.$$

This recursion relation was first proposed by A. Lenard; Peter Lax [26] showed that each F_j in the recursion relation is a differential polynomial in u and, furthermore, is the gradient of a functional, $\mathcal{H}_j$, so that the flows have the form

$$u_t = D\frac{\delta \mathcal{H}_j}{\delta u}.$$

For example, the first two terms in this recursion relation and their corresponding functionals and flows are

$$F_1 = u, \qquad \mathcal{H}_1 = \int_{-\infty}^{\infty} \frac{1}{2}u^2\, dx, \qquad u_t = u_x;$$

$$F_2 = -D^2 u - \frac{u^2}{2}, \qquad \mathcal{H}_1 = \int_{-\infty}^{\infty} \frac{1}{2}u_x^2 - \frac{u^3}{6}\, dx, \qquad u_t = -u_{xxx} - uu_x.$$

The operators D, $-(D^3 + \frac{1}{3}(uD + Du))$ are an example of a *bi-Hamiltonian* pair. Bi-Hamiltonian pairs of operators can be used to generate hierarchies of commuting Hamiltonian flows: cf. Magri, [29], Olver and Rosenau, [32], Fokas and Fuchsteiner [14]; although one drawback of the method is that it does not produce a Lax pair for the equations obtained.

3.2 The AKNS hierarchy

After the remarkable properties of the Korteweg deVries equation were discovered, it was first thought that this was a rather unusual case and would not be repeated. It was soon discovered, however, that there were other, indeed many other, examples of such remarkable equations. In particular, it was discovered that the sine Gordon equation [1]

$$u_{xt} = \sin u$$

and the nonlinear Schrödinger equation [52]

$$iu_t = u_{xx} + 2|u|^2 u \tag{3.3}$$

are also examples of completely integrable systems. These equations are derived as compatibility equations for a pair of first order differential operators, as follows: Let

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

and define

$$D_x = \frac{\partial}{\partial x} - iz\sigma_3 - p\sigma_+ - q\sigma_-, \tag{3.4}$$

$$D_t = \frac{\partial}{\partial t} - A\sigma_3 - B\sigma_+ - C\sigma_- \tag{3.5}$$

Set $p = u$, $q = -\bar{u}$, $A = -2iz^2 + i|u|^2$, $B = -2zu + iu_x$, and $C = 2z\bar{u} + i\bar{u}_x$. Then the equation

$$[D_x, D_t] = 0$$

leads to the nonlinear Schrödinger equation (3.3). This is sometimes called the "zero-curvature" condition, since it formally expresses the fact that the connection with components D_x and D_t is flat. The nonlinear Schrödinger equation had been studied for many years, but was first obtained in this way by Zakharov and Shabat [52]; they developed a scattering theory for the operator D_x and showed how to solve the nonlinear Schrödinger equation by the inverse scattering method.

The sine-Gordon equation is obtained by setting $p = -q = u_x/2$, $A = -i\cos u/4z$, and $B = C = i\sin u/4z$. Then it is easily seen that

$$[D_x, D_t] = (1/2)(u_{xt} - \sin u)(\sigma_+ - \sigma_-).$$

Hence the sine-Gordon equation also arises as a zero curvature equation. The sine-Gordon equation was already known in the nineteenth century; it arose as the equation for the embedding of a surface of constant negative curvature in $\mathbb{R}^3$.

The general theory of integrable systems based on zero curvature conditions for 2×2 matrix differential operators was initiated by Zakharov and Shabat for the nonlinear Schrödinger equation and extended in a significant

way by Ablowitz, Kaup, Newell, and Segur [2]. They constructed not one equation, but an infinite hierarchy of equations, just as the KdV equation is only one equation in an infinite set of commuting flows.

Ablowitz *et. al.* worked only for 2×2 systems, but their ideas are easily extended to the $n \times n$ case, as follows [7], [4], [40]. We construct an infinite hierarchy of commuting flows generated by the first order $n \times n$ operator

$$D_x = \frac{\partial}{\partial x} - zJ - q \qquad (3.6)$$

where

$$J = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}, \qquad q(x) = \begin{pmatrix} 0 & q_{12} & \cdots & q_{1n} \\ q_{21} & 0 & \cdots & q_{2n} \\ & & \ddots & \\ q_{n1} & \cdots & & 0 \end{pmatrix}.$$

We look for wave functions ψ satisfying $D_x \psi = 0$ of the form $\psi = m e^{xzJ}$. Then m satisfies the system of differential equations

$$m_x = z[J, m] + qm \qquad (3.7)$$

Lemma 3.2.1 *There exist solutions of* (3.7) *with the following properties:*

1. $m(x, z)$ is sectionally meromorphic as a function of z in the domain

$$\Omega = \left\{ z : \Re \, z(\lambda_j - \lambda_k) \neq 0 \right\}.$$

2. $m(x, z) \to I$ as $x \to -\infty$

3. $\sup_x |m(x, z)| < +\infty$ for regular values of z.

The solutions of (3.7) *are uniquely determined by items 2,3. If $q \in \mathcal{S}$ then m has an asymptotic expansion*

$$m \sim \sum_{j=0}^{\infty} m_j(x) z^{-j}, \qquad m_0 = I$$

uniformly valid as z tends to infinity in each sector of Ω.

A proof of this result may be found in the fundamental paper by Beals and Coifman on the inverse scattering problem for the operator D_x [6].

Let μ be a diagonal matrix with $\operatorname{tr} \mu = 0$ and set $F = m\mu m^{-1}$. Then

$$F_x = m_x \mu m^{-1} - m\mu m^{-1} m_x m^{-1} = [m_x m^{-1}, F]$$

$$= [zJ - zmJm^{-1} + q, F] = [zJ + q, F]. \tag{3.8}$$

If $q \in \mathcal{S}$ then F has an asymptotic expansion in each sector of Ω:

$$F \sim \sum_{j=0}^{\infty} F_j z^{-j}, \qquad F_0 = \mu.$$

Substituting this series into the equation for F we obtain the recursion relations

$$[J, F_{j+1}] = [\frac{d}{dx} - q, F_j] = \frac{\partial F_j}{\partial x} - [q, F_j]. \tag{3.9}$$

These are the analog of the Lenard recursion relations for the KdV hierarchy.

Now define

$$D_t = \frac{\partial}{\partial t} - (z^k F)_+, \qquad (z^k F)_+ = \sum_{j=0}^{k} F_j z^{k-j}.$$

Then it is a simple consequence of the recursion relations (3.9) that:

$$[D_x, D_t] = q_t - [\frac{d}{dx} - q, F_k] = q_t - [J, F_{k+1}]$$

Hence $\{D_x, D_t\}$ is a flat connection if and only if q satisfies the nonlinear evolution equation

$$q_t = [J, F_{k+1}] = [\frac{d}{dx} - q, F_k] \tag{3.10}$$

Given the first order differential operator D_x it is therefore straightforward, though somewhat tedious, to calculate the flows in the hierarchy. It was shown in [40] that F_k is a polynomial in q and its derivatives up to order $k-1$, so that (3.10) is *local*, i.e. a system of partial differential equations.

As an example of this procedure, we obtain the modified KdV equation. This was first shown to be integrable by the inverse scattering method by Wadati [49]. We take

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad q = \begin{pmatrix} 0 & u \\ -u & 0 \end{pmatrix}.$$

Since μ must be a traceless diagonal matrix, we must take μ to be a multiple of J; the simplest thing to do is to take $\mu = J$. Then $F_0 = \mu$. Then

$$[J, F_1] = [\partial_x - q, F_0] = -[q, J].$$

The general solution of this equation is

$$F_1 = q + c_1 J,$$

where c_1 is a general function of x and t. Then

$$[J, F_2] = [\partial_x - q, F_1] = [\partial_x - q, q + c_1 J]$$

$$= q_x + \partial_x c_1 J + c_1 [J, q].$$

The linear equation $[J, F] = G$ is in general not solvable; in fact, a direct calculation shows that for and matrix F, $[J, F]$ has zeroes on the diagonal. Therefore, the above equation for F_2 is solvable only if all the diagonal entries of the right side of the equation vanish. This leads to a condition on c_1, namely, $c_{1,x} = 0$. Thus c_1 is a constant; and we may take $c_1 = 0$. This gives $[J, F_2] = q_x$; the general solution of this equation is

$$F_2 = \frac{1}{2} \begin{pmatrix} 0 & u_x \\ u_x & 0 \end{pmatrix} + c_2 J.$$

The equation for F_3 reduces to

$$[J, F_3] = [\partial_x - q, F_2]$$

$$= \left(\frac{1}{2} u_{xx} + 2c_2 u \right) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{\partial}{\partial x} \left(c_2 - \frac{1}{2} u^2 \right) J$$

To solve for F_3 the diagonal entries on the right side must vanish; this means we must take

$$c_2 = \frac{1}{2} u^2.$$

We then find

$$[J, F_3] = \left(\frac{1}{2}u_{xx} + u^3\right)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad F_3 = \left(\frac{1}{2}u_{xx} + u^3\right)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + c_3 J.$$

We can continue in this way as far as we want. To solve for F_k at each stage, we must set the diagonal entries of $[\partial_x - q, F_{k-1}]$ equal to zero. At each stage $F_{k-1} = O_{k-1} + c_{k-1}J$, where O_{k-1} is an off-diagonal matrix and c_{k-1} is a coefficient to be determined. The solvability condition therefore reduces to an equation of the form

$$\partial_x c_{k-1} = \ldots$$

where the $\ldots$ denote known terms which depend on u and its derivatives up to order $k-1$. If $c_{k-1}(x)$ is to be a local function of u, that is, to depend only on the values of u and its derivatives at x, then the $\ldots$ must be an exact derivative.

That this is always the case, even for $n \times n$ AKNS systems, was proven in [40].

Theorem 3.2.2 *Each matrix function F_n obtained from the recursion relation (3.9) is a function of q and its derivatives up to order $n - 1$.*

To compute c_3 we must set the diagonal entries of $[\partial_x - q, F_3]$ equal to zero. But the diagonal entries of this expression simply turn out to be $c_{3,x}J$; so we may take $c_3 \equiv 0$, and

$$F_3 = \left(\frac{1}{4}u_{xx} + \frac{1}{2}u^3\right)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

The MKdV flow is then given by[1]

$$q_t = [J, F_4] = [\partial_x - q, F_3]$$

$$= \left(\frac{1}{4}u_{xxx} + \frac{3}{2}u^2 u_x\right)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

hence

$$u_t = \frac{1}{4}u_{xxx} + \frac{3}{2}u^2 u_x.$$

---

[1]Note that $[q, F_3] = 0$

This equation is of third order in x. A simple check shows that there are no flows at even order. For example, at second order in x the equation

$$q_t = [J, F_3],$$

is incompatible, since

$$q_t = u_t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

whereas

$$[J, F_3] = 2 \left(\frac{1}{4} u_{xx} + \frac{1}{2} u^3 \right) \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

3.3 The Gel'fand-Dikii Flows

The Lax pair for the KdV equation can be obtained in the following way, [18], [19]. We first construct a formal square root of L, call it M:

$$M^2 = D^2 + u, \qquad M = D + w_0 + w_1 D^{-1} + w_2 D^{-2} + \dots. \tag{3.11}$$

The operators D^{-k} are formal inverses of the differential operator D. M is considered to be a *pseudodifferential* operator. The Lax equation for the KdV equation is then [42]

$$\dot{L} = [(M^3)_+, L] \tag{3.12}$$

where $(M^3)_+$ denotes the differential part of M^3. That is, formally truncate M^3 by throwing away all the negative powers of D. In fact, there is an entire hierarchy of equations given by

$$\dot{L} = [(M^k)_+, L], \qquad k = 1, 3, 5, \dots. \tag{3.13}$$

The details are outlined in a series of exercises below.

This method extends to the case where L is an nth order scalar differential operator:

$$L = \sum_{j=0}^{n} u_j D^{n-j}, \tag{3.14}$$

where $u_j = u_j(x,t)$, $u_0 = 1$, and $u_1 = 0$. (By a simple transformation we may always transform away the coefficient of D^{n-1}.) The flows now are

$$\dot{L} = [L_+^{k/n}, L] \tag{3.15}$$

where $k \neq 0 \bmod n$, $L_+^{k/n}$ denotes the differential part of $L^{k/n}$ considered as a pseudodifferential operator, and

$$\dot{L} = \sum_{j=2}^{n} \dot{u}_j D^{n-j}.$$

This time $\dot{L}$ is a differential operator of order $n - 2$, so the commutator $[L_+^{k/n}, L]$ must also be a differential operator of the same order.

For each n there is an entire hierarchy of nonlinear evolution equations. We leave it as an exercise to show that $L_+^{k/n}$ is a differential operator of order k whose coefficients are differential polynomials in $u_2, \ldots, u_n$. The flows (3.15) are known as the *Gel'fand-Dikii flows*.

The Gel'fand-Dikii flows can be formally integrated by the inverse scattering theory for n^{th} order ordinary differential operators. This theory has been worked out in detail by Beals, Deift, and Tomei [5].

The equation (1.3), which arose in the continuum limit of the Fermi-Pasta-Ulam model, is an example of a Gel'fand-Dikii flow. In 1973, V.E. Zakharov [54] gave a Lax pair for this equation. Zakharov took the equation in the form:

$$y_{tt} = y_{xx} + (y_x^2)_x + \frac{1}{4} y_{xxxx}.$$

Set $u = y_x$ and differentiate with respect to x:

$$u_{tt} = u_{xx} + (u^2)_{xx} + \frac{1}{4} u_{xxxx}.$$

Now introduce a new function Φ by

$$u_t = \Phi_{xx}, \qquad \Phi_t = u + u^2 + \frac{1}{4} u_{xx}.$$

Then a Lax pair for this system is given by

$$L = i[D^2 + uD + Du + D] - \sqrt{\frac{1}{2}} \Phi_x, \qquad A = \sqrt{\frac{3}{4}} D^2 + \sqrt{\frac{4}{3}} u.$$

The system above is Hamiltonian, with Hamiltonian

$$\mathcal{H} = \int \frac{1}{2}u^2 + \frac{1}{2}\Phi_x^2 + \frac{1}{3}u^2 - \frac{1}{8}u_x^2,$$

and Zakharov's construction of a Lax pair shows that it is completely integrable. In fact, Zakharov shows how to construct an infinite hierarchy of conservation laws. The fact that the continuum model (1.3) is completely integrable suggests why quasiperiodic motions were observed in the computational experiments. Zakharov points out, however, that

It must be emphasized from the very outset, however, that this explanation can only be qualitative, since we consider equation [(1.3)] with periodic boundary conditions, whereas in the numerical experiments the ends of the chains were regarded as fixed.

One really wants to find isospectral deformations of the operator L with boundary conditions appropriate to the fourth order partial differential equation for u or u. Isospectral deformations of Sturm-Liouville (second order self-adjoint) operators on finite intervals have been discussed in detail in the book by Pöschel and Trubowitz; but, to my knowledge, no work has been done on deformations of higher order operators on finite intervals. Thus the issue raised by Zakharov represents an apparently open field of research.

3.4 Notes, Exercises, and Remarks

1. Show by direct calculation that

$$[B, L] = -\frac{1}{6}(u_{xxx} + uu_x),$$

where $L = D^2 + \frac{1}{6}u$ and $B = -4D^3 - \frac{1}{2}(uD + Du)$.

2. a) Show that $Df = f' + fD$. b) Show that formally

$$D^{-1}f = fD^{-1} - f'D^{-2} + f''D^{-3} + \cdots = \sum_{j=0}^{\infty}(-1)^j(D^j f)D^{-(j+1)}$$

c) Show that $M^2 = L$ determines a sequence of recursion relations for the coefficients w_j, where M is given in (3.11), and that w_j is a differential

polynomial in u. d) Compute w_0, w_1, w_2 and $(M^3)_+$; show that a Lax pair for a KdV equation with a different time scale is obtained in this way.

Ans. $(M^3)_+ = D^3 + (3/4)(uD + Du)$.

3. Prove that

$$D^k f = \sum_{j=0}^{k} \binom{k}{j} D^j f D^{k-j}, \qquad k > 0 \tag{3.16}$$

$$D^{-k} f = \sum_{j=0}^{\infty} \binom{-k}{j} D^j f D^{-k-j}, \qquad k > 0. \tag{3.17}$$

where the negative binomial coefficients are:

$$\binom{-k}{j} = (-1)^j \binom{k+j-1}{j}.$$

3. Show that $L_+^{k/n}$ is a kth order differential operator. Its coefficients are polynomials in the x derivatives of the coefficients of the nth order scalar operator L in (3.14). cf. [19]

5. Let $L = D^3 + pD + q$; calculate the Gel'fand-Dikii flow at order $k = 2$. This leads to [5] $B = D^2 + (2/3)p$; $p_t = 2q_x - p_{xx}$, $q_t = q_{xx} - (2/3)(p_{xxx} + pp_x)$. Eliminating q we obtain

$$p_{tt} = \frac{1}{3} \left(p_{xxxx} - (p^2)_{xx} \right).$$

This differs from the equation (1.3) that arises as the continuum limit in the Fermi-Pasta-Ulam experiment. It is ill-posed and does not have global solutions for smooth initial data [5], [7]. On the other hand, the iso-spectral operator L given by Zakharov for (1.3) is self adjoint; and Beals, Deift and Tomei have shown that, on the real line, the corresponding scattering problem is invertible, so that global solutions are guaranteed by the inverse scattering method.

Solution to 1:

We write down the calculation for the Lax pair for the KdV equation. We take

$$L = D^2 + \frac{u}{6}, \qquad B = -4D^3 - \frac{1}{2}(uD + Du).$$

The associated KdV equation is

$$u_t + u_{xxx} + uu_x = 0.$$

This equation is written in the form

$$L_t = \frac{1}{6}u_t = [B, L].$$

We have

$$
\begin{aligned}
[B, L] &= -[4D^3 + \frac{1}{2}(uD + Du), D^2 + \frac{u}{6}] \\
&= -\frac{2}{3}[D^3, u] - \frac{1}{2}[uD + Du, D^2] - \frac{1}{12}[uD + Du, u].
\end{aligned}
$$

Now

$$[D^3, u] = 3u_x D^2 + 3u_{xx} D + u_{xxx}, \qquad [uD + Du, u] = 2uu_x$$

$$[uD + Du, D^2] = -u_{xxx} - 4(u_{xx}D + u_x D^2).$$

Hence

$$[B, L] = -\frac{1}{6}(u_{xxx} + uu_x),$$

and $L_t = [B, L]$ gives the KdV equation above.

Chapter 4

Scattering Theory: Schrödinger operator

4.1 The Gel'fand-Levitan Equation

Consider the eigenvalue problem for the Schrödinger equation

$$(D^2 + k^2)\psi = q\psi, \tag{4.1}$$

where q is real and lies in the Schwartz class $\mathcal{S}(\mathbb{R})$: all C^∞ functions q on the real line for which

$$\sup_x |x^m D^n q| < +\infty$$

for all non-negative integers m and n. It follows that all derivatives of q tend to zero as $x \to \pm\infty$ faster than any power of x. Such functions are said to be rapidly decreasing.

Equation (4.1) can be converted to a Volterra integral equation, for example:

$$\psi_+(x, k) = e^{ikx} - \int_x^\infty \frac{\sin k(x - y)}{k} q\psi_+ dy.$$

It is convenient to consider the reduced wave functions; these are defined by $m_+(x, k) = e^{-ikx}\psi_+(x, k)$. The reduced wave functions satisfy following Volterra integral equation for m_+:

$$m(x, k) = 1 - \int_x^\infty \frac{1 - e^{-2ik(x-y)}}{2ik} q\, m\, dy.$$

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Since

$$\frac{1 - e^{-2ik(x-y)}}{2ik}$$

is uniformly bounded on the interval of integration for Im $k \geq 0$, this integral equation can be solved by succesive approximations when $q \in L_1$. Its solution m_+ is analytic in the upper half k plane, continous onto the real axis at $k \neq 0$ and tends to 1 as x tends to $+\infty$ or as k tends to infinity in Im $k > 0$.

Letting k tend to zero we obtain the integral equation for $m(x, 0)$:

$$m(x, 0) = 1 - \int_x^\infty (x - y)q\, m\, dy.$$

This equation may be solved by successive approximations if

$$\int_{-\infty}^\infty (1 + |y|)q\, dy < +\infty.$$

Thus a slightly stronger condition integrability condition on the potential is needed in order to construct the wave functions for $k = 0$. This is thus the minimal condition on the potential needed to construct the wave functions, and hence to solve the forward scattering problem. For later purposes, we shall always assume the much stronger condition $q \in \mathcal{S}$.

The wave function $\psi_+ = m_+ e^{ikx}$ is analytic in the upper half k plane and is asymptotic to e^{ikx} as $x \to \infty$. Similarly, by constructing other Volterra integral equations for the wave functions of (4.1) we also find solutions ψ_-, $\phi_\pm$ to (4.1) which are analytic in the half planes $\pm\Im k \geq 0$ and have the asymptotic behavior

$$\phi_\pm \sim e^{\mp ikx}, \quad x \to -\infty; \qquad \psi_\pm \sim e^{\pm ikx}, \quad x \to \infty.$$

From the asymptotic behavior of these wave functions, it is clear that for real k

$$\overline{\phi_+(x, k)} = \phi_-(x, k); \qquad \overline{\psi_+(x, k)} = \psi_-(x, k).$$

Furthermore, the asymptotic behavior of ψ_+ and ψ_- shows that they must be linearly independent, so $\phi_\pm$ may be expressed as linear combinations of $\psi_\pm$. We leave it to the reader to prove that for real k there exist constants a and b such that

$$\phi_+ = a(k)\psi_- + b(k)\psi_+; \qquad \phi_- = \overline{b(k)}\psi_- + \overline{a(k)}\psi_+. \tag{4.2}$$

The Wronskian of two functions f and g is given by $W(f, g) = fg' - f'g$. It is a simple matter to show that the Wronskian of two solutions of (4.1) is independent of x. Therefore one may evaluate Wronskians of the wave functions as $x \to \pm\infty$ and verify the following relations

$$W(\phi_+, \psi_+) = 2ika(k), \qquad W(\psi_-, \phi_+) = 2ikb(k) \qquad (4.3)$$

It follows immediately that $a(k)$ is analytic in the upper half k plane; furthermore, using the Wronskians one can show

$$\overline{a(k)} = a(-k); \quad \overline{b(k)} = b(-k); \quad |a(k)|^2 - |b(k)|^2 = 1 \qquad (4.4)$$

The zeroes of a in the upper half plane are bound states of L; at a zero k_j of a we have

$$\phi_+(x, k_j) = c_j\psi_+(x, k_j)$$

for some constant c_j. Since ψ_+ and ϕ_+ decay as $x \to \pm\infty$ respectively, they each in fact decay at both ends, and so constitute a bound state for L.

Lemma 4.1.1 *The zeroes of a in the upper half plane lie on the imaginary axis, the corresponding eigenfunctions of L are real, and the zeroes of a are simple. In fact,*

$$ic_j a'(k_j) = \int_{-\infty}^{\infty} \phi_j^2 dx$$

where ϕ_j is the eigenfunction associated with the zero k_j of a.

Proof: For a proof see [30], [41].
Define the transmission and reflection coefficients by

$$r(k) = \frac{b(k)}{a(k)}; \qquad t(k) = \frac{1}{a(k)}.$$

Then $|t(k)|^2 + |r(k)|^2 = 1$. The scattering data for L is the set

$$\{r(k), k_j, c_j\}.$$

We shall see below that this data is sufficient to reconstruct the potential q. We shall restrict our discussion to the case where there are only a finite number of bound states k_j, $j = 1, \ldots, N$.

Theorem 4.1.2 *Under the KdV flow the scattering data evolves as follows: a is constant,*

$$r(k,t) = e^{8ik^3 t} r(k,0), \qquad k_j = const. \qquad c_j(t) = c_j(0)e^{8ik_j^3 t}$$

This is the remarkable fact discovered by Gardner, Greene, Kruskal, and Zabusky [17]: the scattering data evolves linearly even though the evolution equation for q is nonlinear. The transformation to scattering data thus *linearizes* the flow.

Proof: Let $\phi_+(x,t,k)$ be the wave function of L which is asymptotic to e^{-ikx} as $x \to -\infty$. Since $q = -u/6$, it satisfies the KdV equation $q_t - 6qq_x + q_{xxx} = 0$, and we have

$$0 = (\partial_t - B)(D^2 + k^2 - q)\phi_+ = (D^2 + k^2 - q)(\partial_t - B)\phi_+$$

hence $(\partial_t - B)\phi_+$ is also a wave function for L. We may assume that q lies in the Schwartz class for all time. (This fact may be proved rigorously using the infinite sequence of conservations laws for solutions of the KdV equation; cf §3.1) Now

$$(\partial_t - B)\phi_+ \sim 4D^3 e^{-ikx} = 4ik^3 e^{-ikx} \qquad x \to -\infty.$$

Since the wave functions are uniquely determined by their asymptotic behavior at infinity,

$$(\partial_t - B)\phi_+ = 4ik^3\phi_+.$$

On the other hand, $\phi_+ \sim a(k,t)e^{-ikx} + b(k,t)e^{ikx}$ as $x \to +\infty$, so

$$(\partial_t - B)\phi_+ \sim (\dot{a} + 4ik^3 a)e^{-ikx} + (\dot{b} - 4ik^3 b)e^{ikx}$$
$$= 4ik^3(ae^{-ikx} + be^{ikx}),$$

and it follows that $\dot{a} = 0$, $\dot{b} = 8ik^3 b$. This establishes that a is constant and r evolves as given above.

Since a is constant its zeroes are fixed, so the k_j are constant. The evolution of the coupling coefficients is derived by a similar argument. □

We now derive the Gel'fand-Levitan-Marcenko integral equation. The Fourier transform and its inverse for a function in $L_2(\mathbb{R})$ is

$$\widetilde{f}(s) = \int_{-\infty}^{\infty} e^{-iks} f(k)dk, \qquad f(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iks}\widetilde{f}(s)ds.$$

Let $K(x, s)$ denote the Fourier transform of $m_+(x, k) - 1$ with respect to the variable k:

$$K(x, s) = \int_{-\infty}^{\infty} e^{-iks}(m_+(x, k) - 1)\, dk.$$

Since $m_+ - 1$ is analytic in the upper half k-plane and tends to 0 as k tends to infinity, $K(x, s) = 0$ for $s < 0$ by the Paley-Wiener theorem. [1] Hence

$$m_+(x, k) - 1 = \frac{1}{2\pi} \int_0^{\infty} e^{iks} K(x, s) ds.$$

and

$$\psi_+(x, k) = e^{ikx} + \int_x^{\infty} e^{iks} G(x, s) ds, \qquad (4.5)$$

where $G(x, s) = (2\pi)^{-1} K(x, s - x)$. From the symmetry

$$\overline{\psi_+(x, k)} = \psi_+(x, -k), \qquad k \in \mathbb{R},$$

we find that K and G are real, hence

$$\psi_-(x, k) = e^{-ikx} + \int_x^{\infty} e^{-iks} G(x, s) ds.$$

Theorem 4.1.3 *Let $r(k)$, k_j, c_j, $j = 1, \ldots, N$ be the scattering data for the operator $L = -D^2 + q$ and let $G(x, s)$ be the Fourier transform of ψ_+ as given in (4.5). Then G satisfies the Gel'fand-Levitan-Marcenko integral equation*

$$G(x, s) + f(x + s) + \int_x^{\infty} G(x, t) f(t + s) dt = 0, \quad s > x, \qquad (4.6)$$

where

$$f(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iks} r(k) dk + \sum_{j=1}^{N} \frac{c_j}{ia'(k_j)} e^{ik_j s}. \qquad (4.7)$$

Moreover G satisfies the hyperbolic equation

$$G_{xx} - G_{ss} - q(x) G(x, s) = 0, \qquad q(x) = -2\frac{d}{dx} G(x, x). \qquad (4.8)$$

---

[1] This can be shown by closing the contour in the upper half plane when $s < 0$.

Proof: Using the representations for $\psi_\pm$ write the first equation of (4.2) as:

$$\frac{\phi_+}{a} - e^{-ikx} = \int_x^\infty e^{-iks} G(x,s)ds + re^{ikx} + r\int_x^\infty e^{iks}G(x,s)ds.$$

The Gel'fand-Levitan-Marcenko equation is the obtained by taking the inverse Fourier transform of this equation. Write

$$f_1(s) = \frac{1}{2\pi}\int_{-\infty}^\infty r(k)e^{iks}dk.$$

The inverse transforms of the terms on the right side above are

$$G(x,s) = \frac{1}{2\pi}\int_{-\infty}^\infty e^{iks}\int_x^\infty e^{-ikt}G(x,t)dtdk,$$

$$f_1(x+s) = \frac{1}{2\pi}\int_{-\infty}^\infty e^{iks}e^{ikx}r(k)dk,$$

and

$$\frac{1}{2\pi}\int_{-\infty}^\infty e^{iks}r(k)\int_x^\infty e^{ikt}G(x,t)dtdk = \int_x^\infty G(x,t)\frac{1}{2\pi}\int_{-\infty}^\infty e^{ik(s+t)}r(k)dkdt$$

$$= \int_x^\infty G(x,t)f_1(s+t)dt.$$

The inverse Fourier transform of the left side can be evaluated by closing the contour in the upper half plane for $s > x$ and using the residue theorem:

$$\frac{1}{2\pi}\int_{-\infty}^\infty e^{iks}\left[\frac{\phi_+(x,k)}{a} - e^{-ikx}\right]dk$$

$$= \frac{1}{2\pi}\int_{-\infty}^\infty e^{ik(s-x)}\left[\frac{\phi_+(x,k)e^{ikx}}{a} - 1\right]dk$$

$$= i\sum_{j=1}^N \frac{\phi_j e^{ik_j s}}{a'(k_j)} = -\sum_{j=1}^N \frac{c_j\psi_j e^{ik_j s}}{ia'(k_j)}$$

$$= -\sum_{j=1}^N \frac{c_j}{ia'(k_j)}\left[e^{ik_j(x+s)} + \int_x^\infty e^{ik_j(s+t)}G(x,t)dt\right]$$

$$= -f_2(x+s) - \int_x^\infty f_2(s+t)G(x,t)dt,$$

where

$$f_2(s) = \sum_{j=1}^{N} \frac{c_j}{ia'(k_j)} e^{ik_j s}.$$

Putting these together, we get (2.1.10), with $f = f_1 + f_2$.

To establish (4.8) we apply $D^2 + k^2 - q$ to (4.5) to get

$$0 = \psi_{xx} + (k^2 - q)\psi = \int_x^\infty \left(G_{xx}(x, s) + (k^2 - q(x))G(x, s) \right) e^{iks} ds$$

$$- e^{ikx} \left(G_x(x, x) + \frac{d}{dx}G(x, x) + ikG(x, x) + q(x) \right).$$

Now

$$\int_x^\infty k^2 e^{iks} G(x, s) ds = - \int_x^\infty \frac{\partial^2 e^{iks}}{\partial s^2} G(x, s) ds$$

$$= ikG(x, x)e^{ikx} - G_s(x, x)e^{ikx} - \int_x^\infty G_{ss}(x, s)e^{iks} ds.$$

Combining these two calculations and multiplying by e^{-ikx} we get

$$\int_x^\infty (G_{xx} - G_{ss} - qG)(x, s)e^{ik(s-x)} ds - \left(2\frac{d}{dx}G(x, x) + q(x) \right) = 0.$$

This holds for $\Im k \geq 0$. Letting k tend to infinity in the upper half plane, we see that the integral tends to zero. Since the other term does not depend on k it must vanish identically. But then the integral must also vanish identically; and, by the uniqueness of the Fourier transform, the integrand must also vanish identically, thus proving the result. $\square$

4.2 The inverse problem

We now consider the inverse scattering problem, that of reconstructing q from the scattering data $\{r, k_j, c_j\}$. We shall do this by solving the Gel'fand-Levitan-Marcenko equation. First note that while a is not given as part of the scattering data, it appears, albeit inconspicuously, in the construction of f in (4.7). This dependence could be finessed, for example, by defining the scattering data to be f. But such a step occurs in some of the more complicated inverse problems, and so we carry it out here as an illustration.

We shall derive the following representation for a:

$$\log a(k) = \sum_{j=1}^{N} \log \frac{k - k_j}{k + k_j} + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\log |a|^2}{t - k} dt, \quad \Im\, k > 0 \qquad (4.9)$$

The derivation of (4.9) is carried out as follows. We know $|a|^2$ from r and the relation $|t|^2 + |r|^2 = 1$, and we know the location of the zeroes of a in the upper half plane since we are given the k_j. If we define $\tilde{a}$ by

$$a(k) = \tilde{a}(k) \prod_{j=1}^{N} \frac{k - k_j}{k + k_j}$$

then $\tilde{a}$ is analytic in the upper half plane, tends to 1 as k tends to infinity, and $|\tilde{a}| = |a|$ on the real axis. Therefore $\log \tilde{a}$ is analytic in the upper half k plane and tends to zero at infinity.

The function A defined by

$$A(k) = \begin{cases} \log \tilde{a}(k), & \Im\, k > 0; \\ -\log \overline{a(\bar{k})} & \Im\, k < 0. \end{cases}$$

is sectionally holomorphic in $\Im\, k \neq 0$ and tends to zero as k tends to infinity. Its jump across the real axis is

$$A(k + i0) - A(k - i0) = [A] = \log \tilde{a}(k) + \log \overline{\tilde{a}(k)} = \log |\tilde{a}|^2 = \log |a|^2.$$

These properties uniquely determine A. We claim that

$$A = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\log |a|^2}{t - k} dt, \qquad \Im\, k \neq 0$$

In fact, the expression on the right is sectionally holomorphic and tends to zero as k tends to infinity. By the Plemelj formulae,

$$\lim_{z \to k \pm i0} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\log |a|^2}{t - z} dt = \pm \frac{1}{2} \log |a|^2 + \frac{P}{2\pi i} \int_{-\infty}^{\infty} \frac{\log |a|^2}{t - k} dt.$$

Hence the sectionally holomorphic function defined by the Cauchy integral has the same jump across the real axis as A.

The next step is to solve the GLM equation using the Fredholm theory of integral equations. We fix x and consider (4.6) as an integral equation on the half line (x, ∞). It involves the integral operator defined by

$$(F_x K)(s) = \int_x^\infty f(s+t)K(t)dt, \qquad s \geq x.$$

We first observe that if $f(s)$ decays sufficiently rapidly as s tends to $+\infty$, then F_x is a Hilbert-Schmidt operator, i.e.

$$\int_x^\infty \int_x^\infty |f(s+t)|^2 ds\, dt < +\infty.$$

In fact, the simple change of variables $y = s + t$, $v = s$ shows that this integral is equal to

$$\int_{2x}^\infty (u - 2x)|f(u)|^2 du \tag{4.10}$$

so F_x is a Hilbert-Schmidt operator if $|u|^{1/2} f$ is in $L_2(2x, \infty)$.

Now look at (4.7). The discrete sum decays exponentially as $s \to +\infty$. The integral term decays if r is differentiable. (This is a standard fact about Fourier transforms). For example, the second derivative of r is in L_1 then f decays like s^{-2}. In fact, if $q \in \mathcal{S}$ then so does r and hence so is f. (cf. exercise 2 above)

Therefore, for each x the GLM equation is a Fredholm integral equation. The Fredholm alternative states that the integral operator $I + F_x$ has a bounded inverse if and only if the homogeneous equation $(I + F_x)G = 0$ has only the trivial solution. (Note that F_x is a symmetric integral operator. Therefore, to prove the existence of a solution of (4.6) it is enough to prove that $\ker(I + F_x)$ is trivial.

Suppose

$$G(s) + \int_x^\infty f(s+t)G(t)dt = 0, \qquad s \geq x.$$

Since G is real,

$$\int_x^\infty |G(s)|^2 ds + \int_x^\infty \int_x^\infty f(s+t)G(t)G(s)ds\,dt = 0.$$

But $f = f_1 + f_2$, where

$$f_2 = \sum_{j=1}^N d_j e^{-\omega_j s}, \qquad d_j > 0, \ k_j = i\omega_j$$

Therefore

$$\int_x^\infty \int_x^\infty f_2(s+t)G(t)G(s)\,ds\,dt$$

$$= \sum_{j=1}^N d_j \int_x^\infty \int_x^\infty e^{-\omega_j t} e^{-\omega_j s} G(t)G(s)\,ds\,dt$$

$$= \sum_{j=1}^N d_j \left| \int_x^\infty e^{-\omega_j t} G(t)\,dt \right|^2 > 0$$

Similarly,

$$\int_x^\infty \int_x^\infty f_1(s+t)G(t)G(s)\,ds\,dt$$

$$= \int_x^\infty \int_x^\infty \frac{1}{2\pi} \int_{-\infty}^\infty r(k) e^{ik(s+t)} G(t)G(s)\,dk\,ds\,dt$$

$$= \int_{-\infty}^\infty r(k)(\widetilde{G}(k))^2\,dk$$

where

$$\widetilde{G}(k) = \frac{1}{2\pi} \int_x^\infty G(t)e^{ikt}\,dt.$$

This latter quantity is real, since both f and G are real; hence

$$0 = ((I+F_x)G, G) \geq \int_{-\infty}^\infty (|\widetilde{G}|^2 + r(k)(\widetilde{G}(k))^2\,dk \geq \int_{-\infty}^\infty |\widetilde{G}|^2 (1 - |r(k)|)\,dk,$$

where we have used the Plancherel theorem for the Fourier transform. Since $|t|^2 + |r|^2 = 1$, $|r(k)| = 1$ implies $t = 0$. In that case $a = t^{-1}$ is infinite; and this can happen only at $k = 0$. Therefore $\widetilde{G}$ and hence G must vanish identically.

We have proved:

Theorem 4.2.1 *The Gel'fand-Levitan-Marcenko equation is uniquely solvable whenever the integral in (4.10) is finite.*

The GLM equation can be solved explicitly in the so-called *reflectionless potential* case where $r(k) \equiv 0$, and $k_j = i\omega_j$; the choice

$$f(s) = \sum_{j=1}^{n} d_j e^{-\omega_j s}, \qquad d_j(t) = e^{8\omega_j^3 t + \log d_j(0)} > 0,$$

leads to the *multisoliton solutions*.

We look for a solution of the GLM equation of the form

$$G(x, y; t) = \sum_{j=1}^{n} g_j(x, t) e^{-\omega_j y}.$$

This *Ansatz* leads to the linear system

$$g_j + d_j e^{-\omega_j x} + \sum_{k=1}^{n} \frac{g_k d_j e^{-(\omega_j + \omega_k)x}}{\omega_k + \omega_j} = 0.$$

Writing $g_j = d_j^{1/2} h_j$, we obtain the system

$$e^{-\theta_j} + h_j + \sum_{k=1}^{n} h_k \frac{e^{-(\theta_j + \theta_k)}}{\omega_j + \omega_k} = 0, \tag{4.11}$$

where

$$\theta_j(x, t) = \omega_j(x - 4\omega_j^2 t - \alpha_j), \qquad \alpha_j = \frac{1}{2\omega_j} \log d_j(0).$$

The kernel G is then given by

$$G(x, y, t) = \sum_{j=1}^{n} h_j(x, t) e^{-\theta_j(y, t)}$$

where the h_j are the solutions to (4.11).

The solution to equation (4.11) is obtained in closed form as follows. Let

$$D_{jk} = \delta_{jk} + \frac{e^{-(\theta_j + \theta_k)}}{\omega_j + \omega_k}, \qquad E = \begin{pmatrix} e^{-\theta_1} \\ \vdots \\ e^{-\theta_n} \end{pmatrix},$$

and let C_k denote the kth column vector of the matrix $||D_{jk}||$. By Cramer's rule,

$$h_k = -\frac{D_k}{D},$$

where $D = D(x,t)$ is the determinant of the matrix D_{jk}, and

$$D_k = \det ||C_1, \ldots, E, \ldots, C_n||.$$

Hence

$$G(x,y;t) = -\sum_{j=1}^{n} \frac{D_j(x,t)e^{-\theta_j(y,t)}}{D}.$$

Now note that $C_k' = -Ee^{-\theta_k}$, so that

$$\frac{d}{dx}D = \det ||C_1', \ldots, C_n|| + \cdots + \det ||C_1, \ldots, C_n'||$$

$$= -\sum_{j=1}^{n} D_j(x,t)e^{-\theta_j(x,t)} = G(x,x;t)D,$$

or

$$G(x,x;t) = \frac{d}{dx}\log D. \tag{4.12}$$

From (4.8), and the fact that $q = -u/6$ we find

$$u(x,t) = 12\frac{d^2}{dx^2}\log\det \left|\left|\delta_{jk} + \frac{e^{-(\theta_j+\theta_k)}}{\omega_j+\omega_k}\right|\right|. \tag{4.13}$$

4.3   Elastic scattering of solitons

In this section we derive the formulae for the scattering shifts of the solitons in the multi-soliton equations. A formal theory, based on the wave functions, is given in [30]. We describe here[2] the method in [20]; this method shows also the uniform decay of the multi-soliton solutions in the regions between the solitons.

---

[2]We have incorporated the phase constants into the definition of θ_j here; but the results are equivalent

Theorem 4.3.1 *The n-soliton solution of the KdV equation can be written in the form*

$$u(x,t) = 12 \sum_{j=1}^{n} \omega_j^2 \text{sech}^2(\theta_j + \gamma_j) + 12 \frac{d^2}{dx^2} \log(1+R), \qquad (4.14)$$

where

$$\theta_j = \omega_j(x - 4\omega_j^2 t) + \frac{1}{2} \log 2\omega_j,$$

$$\gamma_j = \sum_{k=j+1}^{n} \log \left(\frac{\omega_k + \omega_j}{\omega_k - \omega_j} \right), \quad 1 \leq j \leq n-1,$$

and

$$\sup_{x,t>0} |\cosh(ax)R(x,t)| \leq Ce^{-bt}, \qquad (4.15)$$

for some $a, b > 0$ and some positive constant C.

A similar result is true as $t \to -\infty$, but with different phase shifts γ_j. This theorem shows that the n-soliton solution is asymptotic to a sum of n travelling solitary waves plus a remainder term that decays exponentially fast to zero as $t \to \infty$, uniformly in x.

The general result is reduced to the 2-soliton case by induction. In the case of 2-solitons, the tau function takes the simple form

$$\tau = 1 + e^{-2\theta_1} + e^{-2\theta_2} + e^{-2(\theta_1+\theta_2+\gamma)}$$

where

$$\gamma = \log \frac{\omega_2 + \omega_1}{\omega_2 - \omega_1}.$$

We factor τ as

$$\tau = 2e^{-\theta_2} \cosh \theta_2 \, \tau_1,$$

where

$$\tau_1 = 1 + e^{-2\theta_1} \frac{e^{\theta_2}}{2\cosh \theta_2} + e^{-2(\theta_1+\gamma)} \frac{e^{-\theta_2}}{2\cosh \theta_2}$$

$$= 1 + \frac{1 + \tanh \theta_2}{2} e^{-2\theta_1} + \frac{1 - \tanh \theta_2}{2} e^{-2(\theta_1+\gamma)}.$$

We in turn factor τ_1 as

$$\tau_1 = e^{-(\theta_1+\gamma)} \left[e^{\theta_1+\gamma} + \frac{1+\tanh\theta_2}{2} e^{\gamma-\theta_1} + \frac{1-\tanh\theta_2}{2} e^{-(\theta_1+\gamma)} \right]$$

$$= 2\cosh(\theta_1+\gamma)e^{-(\theta_1+\gamma)} \left(1 + \sinh\gamma \frac{e^{-\theta_1}}{\cosh(\theta_1+\gamma)} \frac{1+\tanh\theta_2}{2} \right).$$

This leads, ultimately, to the factorization

$$\tau = 4e^{-(\theta_1+\theta_2+\gamma)} \cosh(\theta_2)\cosh(\theta_1+\gamma)(1+R),$$

with

$$R = 1 + \left(\frac{1+\tanh\theta_2}{2} \right) \left(\frac{1-\tanh(\theta_1+\gamma)}{2} \right) \left(e^{2\gamma} - 1 \right). \qquad (4.16)$$

As $t \to \infty$,

$$\frac{1+\tanh\theta_2}{2} \to 1, \qquad \frac{1-\tanh(\theta_1+\gamma)}{2} \to 0; \qquad (4.17)$$

hence $R \to 1$, and its derivatives tend to zero, uniformly on compact subsets of $\mathbb{R}$ as $t \to \infty$, and

$$u(x,t) \sim 12\omega_1^2 \text{sech}^2(\theta_1+\gamma) + 12\omega_2^2 \text{sech}^2\theta_2, \qquad t \to \infty.$$

Similarly, by reversing the roles of θ_1 and θ_2 in the factorizations, we find that

$$u(x,t) \sim 12\omega_1^2 \text{sech}^2\theta_1 + 12\omega_2^2 \text{sech}^2(\theta_2+\gamma), \qquad t \to -\infty.$$

Since $\gamma > 0$, the faster wave is shifted forward in the course of the interaction, while the slower wave is shifted back.

4.4 Fredholm determinants

This formula for the multisoliton potentials can be extended to the general solution of the GLM equation by the method of Fredholm determinants [31], [36]. If the integral in (4.10) is finite for all x then a solution to the GLM equation is given by

$$G(x,s) = -\frac{D(x,s)}{D(x)} \qquad (4.18)$$

where

$$D(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_x^{\infty} \cdots \int_x^{\infty} f \begin{pmatrix} t_1 & t_2 & \cdots & t_n \\ t_1 & t_2 & \cdots & t_n \end{pmatrix}) dt_1 \ldots dt_n$$

$$D(x,s) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_x^{\infty} \cdots \int_x^{\infty} f \begin{pmatrix} x & t_1 & t_2 & \cdots & t_n \\ s & t_1 & t_2 & \cdots & t_n \end{pmatrix}) dt_1 \ldots dt_n$$

and

$$f \begin{pmatrix} s_1 & s_2 & \cdots & s_n \\ t_1 & t_2 & \cdots & t_n \end{pmatrix} = \det \; ||f(s_j + t_k)||.$$

The relation (4.18) is an infinite dimensional version of Cramer's rule. The series $D(x)$ and $D(x, s)$ are respectively called the Fredholm determinant and Fredholm minor for the integral operator F_x. For a discussion of Fredholm's method, see Riesz and Nagy [37]; but it is not hard to give a direct verification that (4.18) is a solution of the GLM equation, provided the series converge. The following result is due to Dyson and Jost

Lemma 4.4.1 *The potential in the Schrödinger operator is given by*

$$q(x) = -2 \frac{d^2}{dx^2} \log \det(I + F_x)$$

where here det denotes the Fredholm determinant.

Proof: The result follows from (4.8) and

$$G(x,x) = \frac{d}{dx} \log \; D(x). \tag{4.19}$$

which we now prove. The derivative of the nth term in the series for $D(x)$ is

$$\frac{d}{dx} \frac{1}{n!} \int_x^{\infty} \cdots \int_x^{\infty} f \begin{pmatrix} t_1 & t_2 & \cdots & t_n \\ t_1 & t_2 & \cdots & t_n \end{pmatrix} dt_1 \ldots dt_n$$

$$= -\frac{1}{n!} \sum_{j=1}^{n} \int_x^{\infty} \cdots \int_x^{\infty} f \begin{pmatrix} t_1 & \cdots & x & \cdots & t_n \\ t_1 & \cdots & x & \cdots & t_n \end{pmatrix} dt_1 \ldots \hat{dt_j} \ldots dt_n$$

where $\hat{dt_j}$ means that the integration with respect to t_j is omitted. Noting that

$$f \begin{pmatrix} s_1 & s_2 & \cdots & s_n \\ t_1 & t_2 & \cdots & t_n \end{pmatrix}$$

is unchanged under the permutation of its columns we can rewrite this as

$$= -\frac{1}{(n-1)!}\int_x^\infty \cdots \int_x^\infty f\begin{pmatrix} x & t_1 & \cdots & t_{n-1} \\ x & t_1 & \cdots & t_{n-1} \end{pmatrix} dt_1 \ldots \ldots dt_{n-1}.$$

Hence

$$\frac{d}{dx}D(x) = -D(x,x),$$

and (4.19) follows. □

A somewhat different proof was given by Dyson [12], who credits the result to unpublished lecture notes of Jost, written in 1954. This result was discovered independently and applied to the KdV equation by Oishi and Pöppe.

4.5   EXERCISES

1. Show that the integral equation for m_+ can be solved by successive approximations. (Hint:

$$m_{j+1} = 1 + \int_x^\infty K(x,y)m_j(y)dy, \quad \text{where } K(x,y) = \frac{1 - e^{2ik(x-y)}}{2ik}.$$

Show by induction that

$$|m_{j+1} - m_j| \le \frac{Q^{(j+1)}}{(j+1)!},$$

where

$$Q(x) = \int_x^\infty |q|dy.)$$

If q is infinitely differentiable, show that m_+ has an asymptotic expansion

$$m_+ \sim \sum_{j=0}^\infty \frac{m_j(x)}{(2ik)^j}, \qquad m_0 = 1$$

Find a recursion relation for the coefficients m_j. What is m_1?

2. Show that $a \to 1$ in the upper half k plane. (Hint: Show that $\phi_+ \sim e^{-ikx}$ and $\psi_+ \sim e^{ikx}$ as $k \to \infty$ in the upper half plane.) Show that the map $u \to f$,

where f is given by (4.7) linearizes the KdV flow, and find the corresponding linear partial differential equation for f.

3. Let $K_\pm$ be the Volterra integral operators

$$K_+\psi(x) = \int_x^\infty K_+(x,s)\psi(s)ds \qquad K_-\psi(x) = \int_{-\infty}^x K_-(x,s)\psi(s)ds.$$

Suppose that $K_\pm$ both "dress" the operator D^2 to $L = D^2 - q$ in the sense that:

$$(D^2 - q)(I + K_\pm) = (I + K_\pm)D^2,$$

where I is the identity operator. Show that $K_\pm(x,s)$ satisfy (4.8) and that $(I + K_\pm)e^{\pm ikx}$ are wave functions for L whenever they are well-defined. This is the basis of the *dressing method* for the Schrödinger operator. [53].

4. Let k tend to zero in the integral equation for $m(x,k)$ and find the equation for $m(x,0)$. Prove this equation is solvable by successive approximations if $(1 + |x|)u \in L_1$. The reflection coefficient r is in general not defined at $k = 0$ unless this condition is satisfied.

5. Prove that the Fredholm determinant representation (4.18) formally satisfies the GLM equation. *Hint:*

$$f\begin{pmatrix} x & t_1 & t_2 & \dots & t_n \\ s & t_1 & t_2 & \dots & t_n \end{pmatrix} = f(x+s)f\begin{pmatrix} t_1 & t_2 & \dots & t_n \\ t_1 & t_2 & \dots & t_n \end{pmatrix}$$

$$- f(s+t_1)f\begin{pmatrix} x & t_2 & \dots & t_n \\ t_1 & t_2 & \dots & t_n \end{pmatrix} - f(s+t_2)f\begin{pmatrix} x & t_1 & t_3 & \dots & t_n \\ t_2 & t_1 & t_3 & \dots & t_n \end{pmatrix}$$

$$\dots - f(s+t_n)f\begin{pmatrix} x & t_1 & \dots & t_{n-1} \\ t_n & t_1 & \dots & t_{n-1} \end{pmatrix}$$

6. Obtain the solitary wave from the Fredholm determinant and Theorem 4.4.1 when $f = d_1 e^{-\omega s}$. Calculate the Fredholm determinant when

$$f = d_1 e^{-\omega_1 x} + d_2 e^{-\omega_2 x}.$$

A series expansion for the Fredholm determinant (called the Hirota series) for the N soliton solution is given by Pöppe [36].

7. Show that for any matrix A

$$\det(I + A) = \prod_j (1 + \lambda_j)$$

If the norm of A is less than 1 then

$$\det(I+A) = e^{\operatorname{tr}\,\log(I+A)} = \exp\left(\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\operatorname{tr}\,A^n\right).$$

8. Let K be the integral operator on $L_2(a,b)$ with kernel $K(x,y)$. Assume K is compact and symmetric and has eigenvalues λ_j and eigenfunctions $\phi_j(x)$. Then

$$K(x,y) = \sum_j \lambda_j \phi_j(x)\bar\phi_j(y)$$

Assume that $K(x,x)$ is continuous and that the series

$$K(x,x) = \sum_j \lambda_j |\phi_j(x)|^2$$

converges uniformly. Prove that

$$\frac{1}{n!}\int_a^b \cdots \int_a^b K\begin{pmatrix} t_1 & t_2 & \cdots & t_n \\ t_1 & t_2 & \cdots & t_n \end{pmatrix} dt_1 \ldots dt_n = \sum_{\lambda_{j_1}<\cdots<\lambda_{j_n}} \lambda_{j_1}\cdots\lambda_{j_n}.$$

Hint: use Warings formula:

$$e_n = \frac{1}{n!}\sum_{\pi\in S_n}(-1)^\pi P_\pi$$

where

$$e_n = \sum_{j_1<\cdots<j_n}\lambda_{j_1}\cdots\lambda_{j_n},$$

and

$$P_\pi = \prod_{j=1}^n p_j^{\nu_j}, \qquad p_k = \sum_j \lambda_j^k$$

Here π has ν_j cycles of length j, with $\nu_1 + 2\nu_2 + \cdots + n\nu_n = n$.

Chapter 5

Matlab codes

The following are a sequence of Matlab codes that numerically integrate the kdV equation and animate the results:

This code animates the breakup of a Gaussian pulse into a sequence of solitons. This mimicks actual experimental observations by Russell in 1834.

```
%kdvdis.m: isospectral integration of u_t+ uu_x +disp*u_xxx =0;
%disintegration of an initial Gaussian pulse into solitons.
clear;
N=512;
L = 30;
a=L/(2*pi);
T=10;
h=2*pi/N;
dt=.005;
x =(h:h:2*pi)';
y=a*x;
u= exp(-2*(y-.25*L).^2)/a;
k=[(0:N/2)';(1-N/2:-1)'];
disp = 0.01;  %disp=input('disp=?');   %0.05 works well
m=disp*a^(-3)*.5*dt*k.^3;
d1=(1+i*m)./(1-i*m);
d2= -.5*i*dt*k./(1-i*m);
d3=.5*d2;
sol= plot(y,a*u,'Erasemode', 'background');
axis([ 2 13 -.1 1.4]);
```

```
zoom;
title('KdV: Disintegration of a Gaussian Pulse'); drawnow;
t=0;
while t<T;
        v=real(ifft(d1.*fft(u)+d3.*fft(u.^2 )));
        w=real(ifft(d1.*fft(u)+d2.*fft(u.^2)));
        for n=1:3;
            w=v+real(ifft(d3.*fft(w.^2)));
        end
        u=w;
t=t+dt;
        set(sol,'ydata',a*u);
end
```

This code gives the 2 soliton interaction:

```
%kdv2sol.m: isospectral integration of u_t+ uu_x +u_xxx =0
clear;
N=512;
L = 50;
a=L/(2*pi);
T=11;
h=2*pi/N;
dt=.005;
x =(h:h:2*pi)';
y=a*x;
c1=.8;c2=.5;
u=(12/a)*(c1^2*sech(c1*(y-.1*L)).^2+c2^2*sech(c2*(y-.3*L)).^2);
k=[(0:N/2)';(1-N/2:-1)'];
m=a^(-3)*.5*dt*k.^3;
d1=(1+i*m)./(1-i*m);
d2= -.5*i*dt*k./(1-i*m);
d3=.5*d2;
sol= plot(y,a*u,'Erasemode','background');
axis([ 0 L -.1 8]);
title('KdV: 2 soliton interaction');drawnow
t=0;
```

```
while t<T;
    fftu = fft(u); fftuu = fft(u.^2);
           v=real(ifft(d1.*fftu+0.5*d2.*fftuu));
           w=real(ifft(d1.*fftu+d2.*fftuu));
         for n=1:3;
             w=v+real(ifft(d3.*fft(w.^2)));
         end
         u=w;
    t=t+dt;
         set(sol,'ydata',a*u);
end
```

This code compares the exact two soliton solution with the numerically computed solution:

```
%kdvcomp.m: comparison of numerical computation with
%exact 2 soliton solution of kdv equation
clear;
N=512; %input('N=');
L=40; %input('L=')
a=L/(2*pi);
T=5; %input('T=');
h=2*pi/N;
dt=.005;
x =(h:h:2*pi)';
y=a*x;
c1=1.2;c2=.8; % solitary wave speeds;


%Computation of Exact 2 soliton solution;


A=(c1-c2)^2/(4*c1*c2*(c1+c2)^2);
t=0;
th1=c1*(y-7-4*c1^2*t);
th2=c2*(y-15-4*c2^2*t);
tau=1+exp(-2*th1)/(2*c1)+
        exp(-2*th2)/(2*c2)+A*exp(-2*(th1+th2));
```

```
ww1=diff(log(tau))/(a*h);
ww2=diff(ww1)/(a*h);
u0=12*[0;ww2;0];
clear th1 th2 ww1 ww2 tau;
t=T;
th1=c1*(y-7-4*c1^2*t);
th2=c2*(y-15-4*c2^2*t);
tau=1+exp(-2*th1)/(2*c1)+exp(-2*th2)/(2*c2)
        +A*exp(-2*(th1+th2));
ww1=diff(log(tau))/(a*h);
ww2=diff(ww1)/(a*h);
uT=12*[0;ww2;0];

plot(y,u0,'r');
box off;
title('2 soliton solution, initial value');
drawnow;


%Numerical computation;

k=[(0:N/2)';(1-N/2:-1)'];
m=a^(-3)*.5*dt*k.^3;
d1=(1+i*m)./(1-i*m);
d2= -.5*i*dt*k./(1-i*m);
d3=.5*d2;

u=u0/a; t=0;

while t<T;
        fftu = fft(u); fftuu = fft(u.^2);
        v=real(ifft(d1.*fftu+0.5*d2.*fftuu));
        w=real(ifft(d1.*fftu+d2.*fftuu));

            for n=1:3;
                w=v+real(ifft(d3.*fft(w.^2)));
            end
```

```
   u=w;
   t=t+dt;
end

figure;
plot(y,uT/a,'r',y,u,'b');
zoom;
title('Two Soliton Solution: Numerical vs. Exact');
text(2, 2.5, 'dt=.005; N=512;');
text(2, 2.2, 'Exact Red: Numerical Blue; zoom on');
```

This code reproduces one of the original experiments of Kruskal and Zabusky:

```
%kz.m: Experiment of Kruskal and Zabusky;
clear;
N=512;
L=2; a=L/(2*pi);
T=1.146;
h=2*pi/N;
dt=.005;
x =(h:h:2*pi)';
y=a*x;
u=cos(pi*y)/a;
k=[(0:N/2)';(1-N/2:-1)'];
disp=.000484; %input('disp=');
m=disp*a^(-3)*.5*dt*k.^3;
d1=(1+i*m)./(1-i*m);
d2= -.5*i*dt*k./(1-i*m);
d3=.5*d2;
sol= plot(y,a*u,'Erasemode','background');
axis([ 0 L -1.5 3]);
box off
title('Kruskal-Zabusky Experiment');
text(.8, 2.2,'u_t+\delta^2u_{xxx}+uu_x=0,     \delta=.022')
```

```
text(1.0,1.9,'u(x,0) = cos(x)');
drawnow;
t=0;

while t<T;
   fftu = fft(u); fftuu = fft(u.^2);
         v=real(ifft(d1.*fftu+d3.*fftuu));
         w=real(ifft(d1.*fftu+d2.*fftuu));
            for n=1:5;
            w=v+real(ifft(d3.*fft(w.^2)));
            end
         u=w;
  t=t+dt;
         set(sol,'ydata',a*u);
end
```

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