APPLICATIONS OF THE ONE-FLUID AND THE TWO-FLUID MODEL IN MAGNETOHYDRODYNAMICS

PROEFSCHRIFT

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CHAPTER I

INTRODUCTION

The flow of electrically conducting media under the influence of magnetic and electric fields has received much attention during the last decennia.
Its behaviour was studied by scientists in many branches of physics and engineering. Perhaps the first workers in this field were astrophysicists, who studied phenomena of this kind in stellar space and on the sun.
Geophysicists were attracted to this field in studying the motion of liquid metals in the earth-core in the presence of the magnetic field of the earth.
Not only the representants of the very old science of astrophysics paid attention to the flow of conducting media, but also those who studied man's interplanetary travelling. For owing to the high temperature behind the shock wave that precedes the space-vehicle, when reentering in the atmosphere, the medium is ionized and hence conducting.
A strong impetus to the study of the motion of conducting media in the presence of electric and magnetic fields, was received from the challenging problem of nuclear fusion.
Here one attempts to heat a gas to such temperatures that light nuclei fuse. During such a fusion a certain amount of energy is set free, when one uses the right elements. Now for such a fusion temperatures of several millions degree Kelvin are required. To confine a gas at these temperatures, one cannot use material tubes, because it would be hard to find material walls able to suffer such temperatures, but above all because the heat would leak out through the wall. Therefore one takes advantage of the fact that an ionized gas is a good conductor and tries to confine the gas (in this fully ionized state it is called a plasma) by the action of externally applied electric and magnetic fields.
The behaviour of the gas under such circumstances is studied by plasma physics.
In engineering a most promising application is in the field of electric-power production. This is achieved by forcing an ionized gas through a magnetic field transverse to the flow. In the other direction transverse to the flow an electric field is induced which can be connected with an external load. In this generator the temperature is several thousands degree Kelvin and the conductivity is usually realized by seeding the gas with potassium or cesium, which have low ionization potentials.
Depending on the field where one encounters the flow of a conducting

fluid or gas, one meets the expressions plasma physics, magnetogasdynamics, magnetohydrodynamics, magnetofluiddynamics and many other epithets.

It will be not too difficult for the reader to label the applications mentioned with one of the names here above.

In the following we shall use the expression magnetohydrodynamics, without regard of the field of application and use the shorthand M.H.D. for this rather lengthy expression.

For the theorist there is a vast amount of new problems set by this extension of hydrodynamics and much work has already been done by hydrodynamicists, physicists, mathematicians and engineers.

Initially there were two different methods of approach to M. H. D. problems. The first is to add to the Navier-Stokes equations (or Euler-equations when viscosity is neglected) a term, representing the Lorentz force. Together with the continuity equation, Maxwell's equations and Ohm's law one then has a set of equations with which M.H.D. problems can be tackled. As in ordinary electromagnetic theory this set must be completed with statements about the permittivity, the permeability and the conductivity of the medium considered.

In order to simplify most authors take all these properties constant. Many problems have been treated on this base.

Experimentalists have been able to execute beautiful experiments with e.g. mercury, and have in some cases found good agreement with theory.

However, if we consider a fully or partially ionized gas, it is not very likely that results of this one-fluid or one-component theory apply to such gases, because the one-fluid theory ignores the particle character of the medium. Therefore a number of authors have again taken up the study of the motion of a single charged particle in a magnetic field. This is not a new problem; it was studied before in connection with the theory of diamagnetism. So we have two extrema. The continuum case, where the number of particles in a volume element is infinite and the case where there is only one particle, unaware of the presence of other ones. For both cases equations of motion are available. But the question is, to what extent do they apply to a gas, say consisting of equal amounts of protons and electrons.

It is evident that none of them apply completely.

The gas consists of particles, which is not incorporated in the continuum approximation, but these particles interact, which is disregarded in one-particle theory.

Therefore one has tried to derive M. H. D. equations directly, starting from the Boltzmann equation, which is the fundament of the kinetic theory of neutral gases.

For ordinary gases successive approximations to the solution of the

Boltzmann equation can be obtained by the methods given in the book by Chapman and Cowling [1].
The first approximation leads to the Euler equations of motion, the second approximation to the Navier-Stokes equations etc.
The physical picture, underlying the Boltzmann equation for neutral gases, is that the force on a particle results from external forces, if present, on the one side and from interactions, involving two particles only, on the other side.
In the case of an ionized gas one can try to apply the same method. One then can make a distinction between the forces resulting from macroscopic fields and currents and forces, arising from other interactions. But it is not clear how the latter must be described.
A description, involving only binary encounters under the influence of the electric Coulomb potential is not satisfactory, since actually there are also many multiple interactions. This fact manifests itself in the mathematical treatment as a diverging integral.
This difficulty can be circumvent when in the expression for the potential it is taken into account that at a certain distance from a charge its field is shielded by surrounding charges of opposite sign.
It is shown in chapter II that with use of this modified potential M.H.D. equations can be derived from the Boltzmann equation.
These equations are of restricted validity because certain assumptions about the state of the gas have to be made.
However, as far as now, it has not been possible, to derive M. H. D. equations from first principles without any additional assumptions.
When the gas is very dilute, one can hope that encounters are so rare, that the action on a particle can be described with macroscopic fields only. Under these circumstances the collision term in the Boltzmann equation can be omitted. The resulting equation is frequently called Vlasow equation or Boltzmann-Vlasow equation, widely studied in connection with electrostatic waves in a plasma.
However this equation cannot be a fundament for M. H. D. equations, because the Maxwell distribution is not obtained after a long time, when an arbitrary initial distribution is given.
Therefore a collision term is needed. The problem to find this collision term from first principles has not been solved yet.
In this work we shall use M.H.D. equations for a fully ionized gas consisting of protons and electrons, derived from the Boltzmann equation with help of a modified Coulomb potential.

Preceded by a discussion of the one-fluid model, the derivation of these equations is outlined in chapter II. The equations of this two-fluid model differ in several respects from those of the one-fluid model. It is the purpose of the present work to treat some M. H. D. problems, using both the one- and the two-fluid model and to study the influence of

the effects, incorporated in the two-fluid model, which are not present in the one-fluid model.
The point of view taken here is rather that of the hydrodynamicist than that of the plasma physicist, who is interested primarily in equilibrium conditions, where the gas is confined by magnetic walls. Here the problems considered are flow problems.

Chapter III deals with application of the models mentioned to the forced flow of an inviscid, conducting fluid between parallel plates, a magnetic field being present in the direction normal to the flow.
In chapter IV again such a flow is considered, but viscosity is taken into account there, the flow being engendered by the uniform motion of one of the plates in its own plane. This type of flow corresponds with so-called Couette flow in ordinary hydrodynamics.
In chapter V we consider the flow along a thin plate with a wavy profile on both sides. To this problem we are lead by the experience that in ordinary aerodynamics the study of the flow along such a configuration, has not only intrinsic interest, but also provides some insight in the flow around thin airfoils.
In chapter V we apply the one-fluid model to the flow of a conducting fluid along a thin plate, waveshaped on both sides, in the presence of a magnetic field parallel to the plate.
The governing equations are linearized and solutions are obtained for several values of the phase difference between upper and lower side of the plate. The validity of the linearization is inspected for each of these configurations.
Chapter V deals only with steady motion. In chapter VI we consider the unsteady phenomena occurring when the plate, moving at times $t < 0$ with the fluid in the direction of the magnetic field, is suddenly brought to rest at time $t = 0$.
In the subsequent and last chapter we apply the two-fluid model to one of the configurations treated in chapter V.

CHAPTER II

THE ONE- AND TWO-FLUIDS MODELS

1. General remarks.

In this chapter we shall briefly expose the ideas on which the M.H.D. equations, used in the subsequent chapters are based. It should be mentioned at the outset that the purpose of the present work is to apply several existing theories to M.H.D. problems and not to make a critical study of the M. H. D. equations properly.
Therefore the derivations of the equations given here are mainly meant to give an idea about the limitations of their applicability.
In section 2 we treat the one-component model, which is a coupling of the Navier-Stokes equations with Maxwell's equations.
In section 3 the two-component model is discussed in which the medium is considered as a mixture of two constituents, namely negatively and positively charged particles.

2. The one-fluid model.

The behaviour of a non-conducting viscous fluid is governed by the Navier-Stokes equations

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}.\nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{v} \quad {}^{*} \qquad (2.1)$$

and the continuity equation

$$\nabla . \mathbf{v} = 0, \qquad (2.2)$$

which is valid when the influence of pressure variations on the density is negligible.
In the following we shall assume that this condition is fulfilled.
In (2. 1) and (2. 2) the symbols have the following meaning

$\mathbf{v}$ = velocity,
ρ = density,
p = static pressure,
ν = kinematic viscosity,
t = time.

*) We use rationalized m.k.s. units throughout this work.

Because there is no coupling between p and ρ, the flow is completely determined by (2. 1) and (2. 2), when a sufficient number of initial and boundary conditions are available.

Let us now assume that the fluid has a scalar conductivity σ. Then electric currents can be transported by the fluid. When the medium is at rest the relation between electric current $\mathbf{j}$ and electric field strength $\mathbf{E}$ is given by Ohm's law

$$\mathbf{j} = \sigma \mathbf{E}. \tag{2.3}$$

When an element of the fluid moves with velocity $\mathbf{v}$ through a magnetic field $\mathbf{B}$, it experiences an electric field

$$\mathbf{E} \times \mathbf{v} \times \mathbf{B}. \tag{2.4}$$

Hence the appropriate form of Ohm's law in a moving medium is

$$\mathbf{j} - Q\mathbf{v} = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B}), \tag{2.5}$$

where Q is the charge density.

The current $\mathbf{j}$ exerts a body force on the fluid, the well-known Lorentz-force, which is given by $\mathbf{j} \times \mathbf{B}$. This force must be added to the pressure and to the viscous force in (2. 1), which reads now:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}.\nabla)\mathbf{v} = -\frac{\nabla p}{\rho} + \nu\nabla^2\mathbf{v} + \frac{\mathbf{j} \times \mathbf{B}}{\rho}. \tag{2.6}$$

Further the field quantities must obey Maxwell's laws

$$\nabla . \mathbf{D} = Q, \tag{2.7}$$

$$\nabla . \mathbf{B} = 0, \tag{2.8}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{2.9}$$

$$\nabla \times \mathbf{H} = \mathbf{j} + \frac{\partial \mathbf{D}}{\partial t}, \tag{2.10}$$

where

$\mathbf{D}$ = displacement,

$\mathbf{H}$ = magnetic field strength,

and the constitutive relations

$$\mathbf{D} = \varepsilon \mathbf{E}, \tag{2.11}$$

$$\mathbf{B} = \mu \mathbf{H}. \tag{2.12}$$

ε and μ are respectively the permittivity and permeability of the medium, coupled by the relation

$$\varepsilon\mu = \frac{1}{c^2}, \tag{2.13}$$

where c is the velocity of light in the medium.
The constitutive relations take the form (2. 11) and (2. 12) only when $\left(\frac{\epsilon\mu}{\epsilon_0\mu_0} - 1\right)$, ϵ_0 and μ_0 being the vacuum values of ϵ and μ, is negligibly small. When there is polarization, the moving dipoles cause a current loop and hence have the same effect as a magnetization. Therefore we assume henceforth that ϵ and μ are equal to the values in vacuo.
In M. H. D. problems the velocities are nearly always small with respect to c and under these circumstances eqs. (2. 5) and (2. 10) can be simplified.
Denoting the field in a moving frame of reference with $\mathbf{E}'$ we have, since for non-relativistic velocities the charge density is not affected by the state of motion of the observer.

$$\nabla \cdot \mathbf{E}' = Q/\epsilon \qquad (2.14)$$

From (2. 5), (2. 14) and the conservation of charge we have for a moving observer

$$\frac{\sigma Q}{\epsilon} + \frac{\partial Q}{\partial t} = 0. \qquad (2.15)$$

Hence a given charge density decays as $\exp{-\sigma/_\epsilon t}$, so that unless high-frequency oscillations are involved, Q is negligibly small because of the large value of $\frac{\sigma}{\epsilon}$.
Therefore we shall neglect $Q\mathbf{v}$ in (2. 5).
When L is a characteristic length and U a characteristic velocity, the term $\frac{\partial \mathbf{D}}{\partial t}$ is of the order $\frac{\epsilon U^2 B}{L}$, since in M. H. D. problems E is of the order UB. On account of (2. 12), $\nabla \times \mathbf{H}$ is of the order $\frac{B}{\mu L}$ which is $\frac{c^2}{U^2}$ times the magnitude of $\frac{\partial \mathbf{D}}{\partial t}$. Therefore we neglect the latter in (2. 10), which then becomes

$$\nabla \times \mathbf{B} = \mu \mathbf{j} \qquad (2.10)^a$$

The equations (2. 5) - (2. 12), together with the continuity equation, which takes the form (2. 2) when the influence of pressure variations on the density can be neglected, constitute the framework for one-fluid M.H.D. theory.

3. The two-fluid model.

Consider a mixture of two gases, one consisting of protons (mass m_i, charge e, number density n_i) and the other of electrons (mass m_e, charge -e, number density n_e).
We shall assume that both gases have the same temperature T and

that the gas as a whole is electrically neutral.
Locally a small charge density $e(n_i - n_e)$ may occur.
Consider the positive constituent. Let the distributionfunction, that is the number of particles at time t in an element of volume of the phase space, be given by f.
f is a function of time, of the position $\mathbf{r}$ in space and of the position $\mathbf{w}$ in velocity space.
The number density $n(\mathbf{r},t)$ is given by

$$n = \int f \, d^3 w, \qquad (2.16)$$

where $d^3 w$ means a volume-element, $dw_\alpha dw_\beta dw_\gamma$ say, in velocity space.
Now we write down the Boltzmann equation for f

$$\frac{\partial f}{\partial t} + (\mathbf{w}.\nabla) f + \frac{e}{m} \left\{ (\mathbf{E} + \mathbf{w} \times \mathbf{B}).\nabla_w \right\} f = \left(\frac{\partial f}{\partial t}\right)_{coll}. \qquad (2.17)$$

∇_w is the nabla operator referring to the velocity space.
The equation (2.17) expresses the conservation of particles in the $(\mathbf{r},\mathbf{w})$ or phase space.
The situation envisaged in writing down the Boltzmann equation is that one can distinguish between long-range forces and short-range forces. The long-range forces act in the same way on many neighbouring particles. They are caused by macroscopic fields and currents and are given in the third term on the left-hand side of (2. 17).
The short-range forces are assumed to be sufficiently described by two-particles encounters. The rate of change of f resulting from these binary encounters is expressed by the right-hand side of (2.17), where "coll" stands for "by collisions".
For a neutral gas and in the absence of external forces, the last term on the left-hand side of (2. 17) vanishes.
It is possible to derive transport equations for the macroscopic variables of the gas then by approximative solution of the remaining equation.
For dilute gases, for which the assumption, that only binary encounters are important, is justifiable, a method of solution is given by Chapman and Enskog. This method is exposed in the classic work of Chapman and Cowling 1).
A different method of solution of the Boltzmann equation for ordinary dilute gases, has been given by Grad 2).
Applying similar methods to (2. 17), one meets several serious difficulties as will be shown in the following.
Equations for the macroscopic variables n and $\mathbf{v}$ can be obtained by taking appropriate moments of the Boltzmann equation.
We consider a function φ of $\mathbf{w}$. $\varphi(\mathbf{w})$ may be a scalar or vector function of the position in velocity space.

Moments of (2.17) are obtained by multiplying (2.17) with φ and integrating over all possible velocities.
The first term on the left-hand side of (2.17) becomes upon multiplication with φ and integration over $\mathbf{w}$

$$\int \varphi \frac{\partial f}{\partial t} d^3w = \frac{\partial}{\partial t} \int f\varphi d^3w = \frac{\partial}{\partial t} n\overline{\varphi}, \qquad (2.18)$$

where the averaged value $\overline{\varphi}$ of φ is defined by

$$\overline{\varphi} = \frac{\int f\varphi d^3w}{\int fd^3w} = \frac{1}{n} \int f\varphi d^3w, \text{ on account of (2.16).}$$

A typical term obtained from the second term at the left-hand side of (2.17) is

$$\int \varphi w_\alpha \frac{\partial f}{\partial r_\alpha} d^3w = \frac{\partial}{\partial r_\alpha} \int (f\varphi w_\alpha) d^3w = \frac{\partial}{\partial r_\alpha} n \overline{w_\alpha \varphi},$$

where again the bar denotes the averaged value over all velocities. Taking all terms of this kind together, we obtain from the second term at the left-hand side of (2.17) upon multiplication with φ and integration over $\mathbf{w}$

$$\nabla \cdot (n \overline{\mathbf{w}\varphi}). \qquad (2.19)$$

Note that when φ is a vector function of $\mathbf{w}$, $(\mathbf{w}\varphi)$ is a dyad or tensor of the second rank.
Now consider the third term in (2.17). Upon multiplication with φ and integration over velocity, a typical element is, when $(\mathbf{E} + \mathbf{w} \times \mathbf{B}) = \mathbf{F}$,

$$\int \varphi F_\alpha \frac{\partial f}{\partial w_\alpha} \partial w_\alpha dw_\beta dw_\gamma =$$

$$\int \varphi F_\alpha f \Big|_{w_\alpha = -\infty}^{w_\alpha = +\infty} dw_\beta dw_\gamma - \int f\varphi \frac{\partial F_\alpha}{\partial w_\alpha} d^3w - \int fF_\alpha \frac{\partial \varphi}{\partial w_\alpha} d^3w. \qquad (2.20)$$

Now it is assumed that $\varphi F_\alpha f$ vanishes for $w_\alpha = \pm \infty$
Further it follows from the definition of $\mathbf{F}$ that $\frac{\partial F_\alpha}{\partial w_\alpha} = 0.$

Hence only the last term on the right-hand side of (2.20) remains, which can be written as

$$- n \overline{F_\alpha \frac{\partial \varphi}{\partial w_\alpha}}.$$

This, and similar terms give together, when the last term on the left-hand side of (2.17) is multiplied with φ and integrated over velocity

$$- \frac{ne}{m} \{ \overline{(\mathbf{E} + \mathbf{w} \times \mathbf{B}).\nabla_w \} \varphi}. \tag{2.21}$$

Summing the results (2.18), (2.19) and (2.21), it follows that multiplication of φ and integration over $\mathbf{w}$ transforms (2.17) in

$$\frac{\partial}{\partial t}(n\overline{\varphi}) + \nabla(n\overline{\mathbf{w}\varphi}) - \frac{ne}{m}\{\overline{(\mathbf{E} + \mathbf{w} \times \mathbf{B}).\nabla_w\}\varphi} = \int \varphi \left(\frac{\partial f}{\partial t}\right)_{coll} d^3w. \tag{2.22}$$

Upon labeling φ, $\mathbf{w}$, n and f with the subscript i, (2.22) holds for the protons. Changing i into e and the charge into -e gives the corresponding equation for the electrons.

Before defining φ we formulate the macroscopic variables of the gas. We define the density ρ by

$$\rho = \rho_i + \rho_e = n_i m_i + n_e m_e, \tag{2.23}$$

the mass-velocity $\mathbf{v}$ by

$$\rho \mathbf{v} = \rho_i \overline{\mathbf{w}_i} + \rho_e \overline{\mathbf{w}_e}, \tag{2.24}$$

and the electric current by

$$\mathbf{j} = en_i\overline{\mathbf{w}_i} - en_e\overline{\mathbf{w}_e} \tag{2.25}$$

In a simple gas the velocity of a particle can be written as

$$\mathbf{w}_g = \mathbf{v}_g + \mathbf{u}_g, \tag{2.26}$$

where $\mathbf{v}_g$ is the mean velocity and $\mathbf{u}_g$ the random or thermal velocity. As $\bar{\mathbf{u}}_g = 0$, we have

$$\overline{\mathbf{w}_g} = \overline{\mathbf{v}_g} \tag{2.27}$$

In the case of a gas-mixture it is convenient to refer the velocities to the mass-velocity $\mathbf{v}$.

Therefore we write

$$\mathbf{w}_i = \mathbf{v} + \mathbf{V}_i \tag{2.28}$$

and

$$\mathbf{w}_e = \mathbf{v} + \mathbf{V}_e \tag{2.29}$$

Because $\frac{m_e}{m_i} \simeq \frac{1}{1800} \ll 1$ and because $n_i \simeq n_e$, we deduce from the above relations,

$$\mathbf{v} = \frac{n_i m_i \mathbf{v}_i + n_e m_e \mathbf{v}_e}{n_i m_i + n_e m_e} \simeq \mathbf{v}_i . \qquad (2.30)$$

Then it follows from (2. 27), (2. 28) and (2. 30) that $\overline{\mathbf{V}}_i$ is small, so that the current

$$\mathbf{j} = e n_i \overline{\mathbf{w}_i} - e n_e \overline{\mathbf{w}_e} = e n_i \overline{\mathbf{V}}_i - e n_e \overline{\mathbf{V}}_e = e n_i \mathbf{v}_i - e n_e \mathbf{v}_e \simeq - e n_e \overline{\mathbf{V}}_e . \qquad (2.31)$$

From (2. 24), (2. 28) and (2. 29) we obtain the useful relation

$$\rho_i \overline{\mathbf{V}}_i + \rho_e \overline{\mathbf{V}}_e = 0. \qquad (2.32)$$

Let us now return to (2. 22) and insert $\varphi = 1$ for the ions.
Then the first term in (2. 22) becomes

$$\frac{\partial}{\partial t} (n_i \overline{\varphi}) = \frac{\partial n_i}{\partial t} ,$$

and the second one becomes on account of (2. 28)

$$\nabla . (n \overline{\mathbf{w} \varphi}) = \nabla . n_i \overline{\mathbf{w}_i} = \nabla . (n_i \mathbf{v} + n_i \overline{\mathbf{V}}_i) = \nabla . n_i \mathbf{v} + \nabla . n_i \overline{\mathbf{V}}_i .$$

Since $\nabla_w \varphi = 0$ and since the total number of ions is not changed by collisions, we obtain

$$\frac{\partial n_i}{\partial t} + \nabla . (n_i \mathbf{v}) + \nabla . (n_i \overline{\mathbf{V}}_i) = 0. \qquad (2.33)$$

For the electrons we obtain in the same way

$$\frac{\partial n_e}{\partial t} + \nabla . (n_e \mathbf{v}) + \nabla . (n_e \overline{\mathbf{V}}_e) = 0 \qquad (2.34)$$

Upon multiplying (2.33) with m_i and (2.34) with m_e we find by addition, using (2. 23) and (2. 32):

$$\frac{\partial \rho}{\partial t} + \nabla . (\rho \mathbf{v}) = 0, \qquad (2.35)$$

which is the continuity equation for the gas.
When multiplying both (2. 33) and (2. 34) with e and subtracting we get

$$\frac{\partial Q}{\partial t} + \nabla . (Q \mathbf{v}) + \nabla . \mathbf{j} = 0,$$

where Q is the charge density.
Since $e n_i \simeq e n_e$, this can in most cases be reduced to

$$\nabla . \mathbf{j} = 0 . \qquad (2.36)$$

Next we take $\varphi = m_i \mathbf{w}_i$. The first term in (2.22) gives

$$\frac{\partial}{\partial t}(n_i m_i \mathbf{v}) + \frac{\partial}{\partial t}(n_i m_i \bar{\mathbf{V}}_i), \tag{2.37}$$

the second term

$$\nabla(n_i m_i \overline{\mathbf{w}_i \mathbf{w}_i}) = \nabla(n_i m_i \mathbf{v}\mathbf{v}) + \nabla(n_i m_i \overline{\mathbf{V}_i \mathbf{V}_i}) + \nabla(n_i m_i \mathbf{v}\bar{\mathbf{V}}_i) + \nabla(n_i m_i \bar{\mathbf{V}}_i \mathbf{v}). \tag{2.38}$$

The third term in (2.22) becomes

$$- n_i e\,(\mathbf{E} + \overline{\mathbf{w}_i \times \mathbf{B}}) = - n_i e\,(\mathbf{E} + \mathbf{v} \times \mathbf{B} + \bar{\mathbf{V}}_i \times \mathbf{B}). \tag{2.39}$$

By addition of (2.37) - (2.39), we obtain from (2.22)

$$\frac{\partial}{\partial t}(n_i m_i \mathbf{v}) + \frac{\partial}{\partial t}(n_i m_i \bar{\mathbf{V}}_i) + \nabla(n_i m_i \mathbf{v}\mathbf{v}) + \nabla(n_i m_i \overline{\mathbf{V}_i \mathbf{V}_i})$$

$$+ \nabla(n_i m_i \mathbf{v}\bar{\mathbf{V}}_i) + \nabla(n_i m_i \bar{\mathbf{V}}_i \mathbf{v}) - n_i e\,(\mathbf{E} + \mathbf{v} \times \mathbf{B} + \bar{\mathbf{V}}_i \times \mathbf{B}) =$$

$$\int m_i \mathbf{w}_i \left(\frac{\partial f_i}{\partial t}\right)_{coll} d^3 w_i\,. \tag{2.40}$$

The corresponding equation for the electrons is obtained by changing the subscript i into e, and changing the sign of the last term on the left-hand side of (2.40).
Addition of the two equations yields with help of (2.23), (2.31) and (2.32)

$$\frac{\partial}{\partial t}(\rho\mathbf{v}) + \nabla(\rho\mathbf{v}\mathbf{v}) + \nabla(P_i + P_e) - \mathbf{j} \times \mathbf{B} = 0. \tag{2.41}$$

In (2.41) P_i and P_e are the stress-tensors for ions and electrons respectively, defined as

$$P_i = n_i m_i \overline{\mathbf{V}_i \mathbf{V}_i} \tag{2.42}$$

$$P_e = n_i m_i \overline{\mathbf{V}_e \mathbf{V}_e} \tag{2.43}$$

In obtaining (2.41) we have again used the fact that $en_i \simeq en_e$ and further the fact that the total momentum is not changed by collisions.
We can simplify (2.41) by using the continuity equation (2.35) and the identity

$$\nabla(\mathbf{ab}) = \mathbf{a}\nabla\cdot\mathbf{b} + (\mathbf{b}\cdot\nabla)\mathbf{a}. \tag{2.44}$$

Then (2.41) can be written as

$$\rho\frac{\partial\mathbf{v}}{\partial t} + (\rho\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla(P_i + P_e) - \mathbf{j} \times \mathbf{B} = 0 \tag{2.45}$$

Next we multiply (2.40) with $\frac{e}{m_i}$ and the corresponding equation for the electrons with $\frac{e}{m_e}$ and subtract.

With help of (2.23), (2.31), (2.32), (2.36), (2.44) and remembering that $\frac{m_e}{m_i} \ll 1$ and that $e n_i \simeq e n_e$, we obtain

$$\frac{\partial}{\partial t} \boldsymbol{j} + (\boldsymbol{v}.\nabla)\boldsymbol{j} + (\boldsymbol{j}.\nabla)\boldsymbol{v} + \boldsymbol{j}\nabla.\boldsymbol{v} +$$

$$- \frac{n_e e^2}{m_e}\left[\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B} - \frac{\boldsymbol{j} \times \boldsymbol{B}}{e n_e} + \frac{\nabla P_e}{e n_e}\right] = \frac{e}{m_i}\int m_i \boldsymbol{w}_i \left(\frac{\partial f_i}{\partial t}\right)_{coll} d^3 w_i +$$

$$- \frac{e}{m_e}\int m_e \boldsymbol{w}_e \left(\frac{\partial f_e}{\partial t}\right)_{coll} d^3 w_e . \qquad (2.46)$$

On account of Newton's third law the integrals in (2.46) have the same value, so that we have only to take the second one into account since $m_i \gg m_e$.

Rearranging terms we write (2.46) then as

$$\frac{m_e}{n_e e^2}\left[\frac{\partial \boldsymbol{j}}{\partial t} + (\boldsymbol{v}.\nabla)\boldsymbol{j} + (\boldsymbol{j}.\nabla)\boldsymbol{v} + \boldsymbol{j}\nabla.\boldsymbol{v}\right] =$$

$$\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B} - \frac{\boldsymbol{j} \times \boldsymbol{B}}{e n_e} + \frac{\nabla P_e}{e n_e} - \frac{1}{e n_e}\int m_e \boldsymbol{w}_e \left(\frac{\partial f_e}{\partial t}\right)_{coll} d^3 w_e . \qquad (2.47)$$

Equation (2.47) is sometimes called the "generalized Ohm's Law", because it takes the place of Ohm's law in the one-fluid model.
The terms still unknown in (2.45) and (2.47) are the stress-tensors and the collisionterm.
Let us first consider the collisionintegral. Suppose that there is a collisiontime τ, independent of w_e, and that there is only a small departure from equilibrium so that $\left(\frac{\partial f_e}{\partial t}\right)_{coll}$ can be approximated by

$$\frac{f_e^{\circ} - f_e}{\tau} ,$$

where f_e° is the Maxwell-Boltzmann distribution

$$f_e^{\circ} = n_e \left(\frac{m_e}{2\pi kT}\right)^{3/2} \exp - \frac{m_e V_e^2}{2kT} . \qquad (2.48)$$

Then $$\frac{1}{e n_e} \int m_e \mathbf{w}_e \left(\frac{\partial f_e}{\partial t}\right)_{coll} d^3 w_e =$$

$$\frac{1}{e n_e} \int m_e (\mathbf{v} + \mathbf{V}_e) \left(\frac{\partial f_e}{\partial t}\right)_{coll} d^3 w_e = \frac{1}{e n_e} \int m_e \mathbf{V}_e \frac{f_e^{\circ}}{\tau} d^3 w_e$$

$$- \frac{1}{e n_e} \int m_e \mathbf{V}_e \frac{f_e}{\tau} d^3 w_e = - \frac{m_e}{e^2 n_e \tau} n_e \bar{\mathbf{V}}_e = \frac{\mathbf{j}}{\sigma},$$

where we have used (2. 31) and where

$$\sigma = \frac{e^2 n_e \tau}{m_e} . \tag{2.49}$$

Of course this is not a real evaluation of the collision term, because, even when indeed the departure from equilibrium is small, we have not defined τ.

Let us therefore consider the action on a particle in the gas.

Part of the force on a particle is given by the third term on the left-hand side of the Boltzmann equation (2. 17). It is assumed that the rest of the interactions is describable in terms of binary encounters. During a collision with an other particle, a particle is deflected over a small angle. The deflection depends on the relative velocity, the mutual distance and last but not least on the potential for the central force between the particles. The total deflection during a certain time is given by integration over all possible velocities and distances.

In a gas as described here the potential is the Coulomb potential.

However, when performing the integration, mentioned above, it appears that the resulting integrals diverge (cf. e.g. Rose and Clark [3] p. 163), because the Coulomb potential falls off very slowly.

The reason for this difficulty is, that not all the interactions, not accounted for in the $\mathbf{E} + \mathbf{w} \times \mathbf{B}$ term of the Boltzmann equation, are binary Coulomb interactions. In an intermediate range the field of other particles plays a rôle too and the interactions are multiple. At this point it is useful to introduce the concept of the "shielded" or Debye-potential.

Consider an ion in the gas. Because of its positive charge it is surrounded by electrons, having in equilibrium the Maxwell-Boltzmann distribution.

$$n = n_o \exp \frac{e\Phi}{kT}, \tag{2.50}$$

where Φ is the electrostatic potential, n_o is the "neutral" number density and k is Boltzmann's constant.

When $e\Phi < kT$, we can in the expansion of (2.50) in a series restrict ourselves to the first two terms, obtaining for the charge density $e(n_o - n)$

$$-\frac{ne^2\Phi}{kT} \tag{2.51}$$

Hence we have

$$-\nabla^2\Phi = -\frac{ne^2\Phi}{\varepsilon_o kT} + \frac{e\delta(r)}{4\pi r^2\varepsilon_o}, \tag{2.52}$$

where $\frac{\delta(r)}{4\pi r^2}$ is Dirac's delta-function in spherical coordinates.
Without the first term on the right-hand side (2.52) has the "bare" Coulomb potential

$$\Phi = \frac{e}{4\pi\varepsilon_o r} \tag{2.53}$$

as solution.

The solution of (2.52) is the shielded potential

$$\Phi = \frac{e}{4\pi\varepsilon_o r} \exp - r/h, \tag{2.54}$$

where h is the Debye-length, defined by

$$h^2 = \frac{\varepsilon_o kT}{e^2 n_o}. \tag{2.55}$$

The potential is at small distances equal to the Coulomb potential, but at larger distances the ion potential is attenuated by the presence of the surrounding electrons. When $r > h$ the potential is practically zero. The shielded potential gives account of the fact that at distances intermediate between the particle diameter, which will be defined hereafter, and h the field of the electrons surrounding the ion plays a rôle in the electric interaction and therefore gives an improvement with respect to the bare Coulomb potential.

When using the shielded Coulomb potential in the Boltzmann collision integrals, no divergence difficulties occur and a collision time τ can be calculated, defined as the time during which a particle by many small deflections is deflected over 90°.

Another procedure is to use the ordinary Coulomb potential and to restrict the integration over all possible distances to h. Liboff [4)] has shown that the results are very nearly the same.

Although in this way a collisiontime can be defined the situation is unsatisfactory, for apart from the rather heuristic introduction of the shielded Coulomb potential, the treatment, outlined above, is inconsistent.

Upon recognizing that only short-range encounters are binary Coulomb interactions, but that at a larger distance the interactions are multiple, which is accounted for in the shielding factor in (2.54), still the binary collision methods are used to calculate τ.

It seems therefore better to leave the concept of binary collisions and to treat the multiple interactions as a diffusion process. The appropriate equation to use for this purpose is not the Boltzmann equation, but a Fokker-Planck equation. The derivation of the Fokker-Planck equation for a fully ionized gas is exposed e.g. by Kaufman in "The theory of neutral and ionized gases" [5].

It appears (see again Kaufman's contribution to reference 5), that the diffusion coefficients display the shielding effect and that the relaxation time, following from these coefficients, is equal to the collision time calculated with the Debye potential and the binary collision method. This agreement gives, in spite of the inconsistency mentioned above, confidence in the binary collision method.

In this connection it has been pointed out by Grad [6] that the agreement is not overly surprising, because, although the binary collision method refers to a hypothetical situation, the mathematical model is the same as in the Fokker-Planck philosophy. In the latter case one considers the effect of many simultaneous independent deflections, while in the Boltzmann case, one considers the deflection resulting from a sequence in time of many independent deflections of one particle.

Because the influence of this charge on the other charges is neglected, this particle is representative for the behaviour of the other charges. Therefore the mathematical model is the same, although the physical picture is quite different.

The calculation of τ either with the Fokker-Planck method or with the binary collision method, can be performed with various degrees of precision. The results do not differ much. Rose and Clark [3] find

$$\tau = \frac{32\sqrt{2\pi}\,\varepsilon_0^{\,2}\, m_e^{1/2}\,(kT)^{3/2}}{e^4 n_i \ \ln N} \qquad (2.56)$$

In (2.56) N is twice the ratio between h and the "particle diameter" or distance of closest approach. This distance is defined by

$$D = \frac{e^2}{6\pi\varepsilon_0 kT}\,, \qquad (2.57)$$

and thus we have

$$N = 2\,\frac{h}{D} \qquad (2.58)$$

Inserting (2. 56) in (2. 49) we obtain

$$\frac{1}{\sigma} = \frac{m_e^{1/2} e^2 \ln N}{32\sqrt{\pi}\epsilon_0^2 (kT)^{3/2}}, \tag{2.59}$$

or

$$\frac{1}{\sigma} = \frac{1.09 \times 10^2 \ln N}{T^{3/2}} \text{ ohm - m}, \tag{2.60}$$

where we have used the numerical values

$$\begin{aligned} m_e &= 9.10 \times 10^{-31} \text{ kg}, \\ e &= 1.6 \times 10^{-19} \text{ coulomb}, \\ \epsilon_0 &= 8.8 \times 10^{-12} \text{ farad/m}, \\ k &= 1.38 \times 10^{-23} \text{ joule/}^\circ\text{K}. \end{aligned}$$

Herdan and Liley [7] use the value

$$\frac{1}{\sigma} = \frac{1.29 \times 10^2 \ln N}{T^{3/2}} \text{ ohm - m}.$$

As we need only qualitative results we shall use, when necessary the value

$$\frac{1}{\sigma} = \frac{10^2 \ln N}{T^{3/2}} \text{ ohm - m}. \tag{2.61}$$

Ferraro and Plumpton [8] give also about this value.
Although the physical picture underlying (2. 61), is a rather rough approximation, it appears from the literature that the estimation (2. 61) is fairly good supported by experiment.
We mention in this connection the work of Lin, Resler and Kantrowitz [9], and that of Kino [10].
Now we want to know the consequences of the two important assumptions made in deriving the results given above. These are:
small departure from the Maxwell-Boltzmann equilibrium and the assumption that $e\Phi < kT$.
Let us consider the first one. In the absence of collisions the particles perform spiralling motions about the lines of magnetic induction. The number of revolutions per unit time is given by the cyclotron frequency which is for the ions

$$\omega_i = \frac{eB}{m_i} \tag{2.62}$$

and for the electrons

$$\omega_e = \frac{eB}{m_e}. \qquad (2.63)$$

It is well-known from the single-particle theory that under these circumstances the motion can be split up in the motion around the lines of induction and the motion along the induction. (cf. e. g. Spitzer [11].) These two motions are largely different, which leads to a strong anisotropy in the gas. Collisions can restore the isotropy, provided the collision frequency is of the order of magnitude of the largest cyclotron-frequency, which is ω_e.

We shall require

$$\omega_e \tau < 1, \qquad (2.64)$$

in which case the assumption of near-equilibrium can be justified.
The second assumption $e\Phi < kT$ amounts to the requirement

$$\frac{h}{D} > 1, \qquad (2.65)$$

which can be easily verified with help of (2. 54) and (2. 57).
In order to have an idea about the kind of gases for which (2.65) holds, we define some other lengths next to h and D, viz.

$$L = \text{characteristic macroscopic length}$$

$$d = n^{-1/3} = \text{interparticle distance}$$

$$l = \frac{1}{\pi n D^2} = \text{mean free path.}$$

l is the distance between two close-encounters, from which deflection of 90^o results.
From the definitions of these lengths we deduce the relations

$$\frac{l}{h} = 6\,\frac{h}{D}\ ;\ nh^3 = \frac{h^3}{d^3} = \frac{1}{6\pi}\,\frac{h}{D} \qquad (2.66)$$

From (2. 58) and (2. 66) we see that N is just nine-times the number of particles in a sphere with radius equal to h, and further we deduce from (2. 65) and (2. 66):

$$l > h > d > D. \qquad (2.67)$$

The situation given by (2. 67) is sketched in fig. 1, which is taken from Delcroix [12].

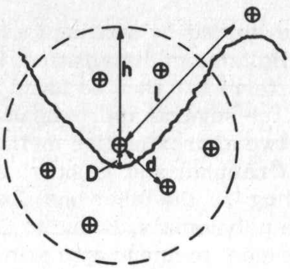

Fig. 1

Electron trajectory under circumstances, where (2.67) is obeyed.

As an example we take a gas with the following properties

$$n = 10^{24}/m^{3} \;;\; T = 10^{6}\ ^{o}K$$

Then we find

$$h = 6{,}9 \times 10^{-8}\ m$$

$$D = 1{,}1 \times 10^{-11}\ m$$

$$d = 10^{-8}\ m$$

$$l = 2{,}5 \times 10^{-8}\ m$$

With $N = 2\,\frac{h}{D} = 1{,}2 \times 10^{4}$ we find from (2. 61) and (2. 49) for τ

$$\tau = 0{,}35 \times 10^{-10}$$

If we choose $B = 0{,}1\ W/m^{2}$, we have from (2. 63)
$\omega_e = 1.76 \times 10^{10}$ and thus $\omega_e \tau = 0{,}62$.
Now that we found an appropriate expression for the collision integral in (2. 47), we turn our attention to the still unknown stress-tensors P_i and P_e.
We can find expressions for the rate of change of these quantities by taking higher moments of the Boltzmann equation. The resulting equations involve terms still one order higher than the stress-tensors etc.

Hence the chain of equations generated by taking moments of the Boltzmann equation is not closed. A closed set of equations can only be obtained if on a certain level of information the distribution function is known.

This difficulty can be solved by writing f as a series, the first term f_o being the Maxwell-Boltzmann distribution. With help of the moment-equations the second term can then be found and also expressions for the stress-tensors in the several approximations can be obtained. For simple neutral gases two approximative methods have been developed. The first one, due to Chapman and Enskog, is described in the book by Chapman and Cowling [1], the other one, based on development of f in a series of Hermite polynomals, is due to Grad[2]. The latter theory bears the name "thirteen moment approximation" because thirteen moments of the Boltzmann equation are considered.

The first approximation to the pressure is the scalar pressure.

$$p = n\,k\,T \tag{2.68}$$

The second approximation is

$$\mathbf{P} = p\,\mathbf{I} - 2\,\eta_o\,\mathbf{U}, \tag{2.69}$$

where $\mathbf{U}$ stands for the tensor

$$\mathbf{U} = 1/2\,(\nabla\mathbf{v} + \mathbf{v}\nabla) - 1/3\,(\nabla.\mathbf{v})\,\mathbf{I}, \tag{2.70}$$

$\mathbf{I}$ is the unit tensor and η_o the dynamic viscosity.

For the case of a fully ionized gas in the presence of a magnetic field, the same methods can be used, provided the departure from equilibrium is not too large, so that we must require that (2. 64) holds.

The first approximation is again (2. 68) for each component.

Chapman and Cowling give the results of the calculation of the second approximation, based on the methods described in their book.

Grad's thirteen moment approximation is used by Herdan and Liley [7]. A survey is recently given by Kaufman [13]. The results obtained by these authors are the same. If we assume the magnetic field to be in the x direction of a Cartesian frame, they find for the components of the nonhydrostatic part P' of the stress-tensor of the constituent considered

$$P'_{xx} = -\,2\eta\,U_{xx},$$

$$P'_{yy} = \frac{-2\eta}{1+4\omega^2\tau^2/\beta^2}\left[U_{yy} + 1/2(U_{yy}+U_{zz})\frac{4\omega^2\tau^2}{\beta^2} + U_{yz}\frac{2\omega\tau}{\beta}\right],$$

$$P'_{zz} = \frac{-2\eta}{1+4\omega^2\tau^2/\beta^2}\left[U_{zz} + 1/2(U_{yy}+U_{zz})\frac{4\omega^2\tau^2}{\beta^2} - U_{yz}\frac{2\omega\tau}{\beta}\right],$$

$$P'_{yz} = -\,\frac{2\eta}{1+4\omega^2\tau^2/\beta^2}\left[U_{yz} + 1/2(U_{zz}-U_{yy})\frac{2\omega\tau}{\beta}\right] = P'_{zy},$$

$$P'_{xy} = P'_{yx} = - \frac{2\eta}{1+4\omega^2\tau^2/\beta^2} \left[U_{xy} + \frac{\omega\tau}{\beta} U_{xz} \right], \qquad (2.71)$$

$$P'_{xz} = P'_{zx} = - \frac{2\eta}{1+4\omega^2\tau^2/\beta^2} \left[U_{xz} - \frac{\omega\tau}{\beta} U_{xy} \right].$$

In (2. 71) η is the coefficient of viscosity of the constituent considered, ω the cyclotron frequency and τ the "self collisiontime" that is the time for collisions between particles of the same kind.
The value of the constant β is of order one. The relation between η and τ is roughly

$$\tau = \frac{\eta}{p}. \qquad (2.72)$$

Now between the various collision times the following relations exist (cf. Rose and Clark [3] chapter 8).

$$\tau_{e\,i} \simeq \tau_{e\,e} = \left(\frac{m_e}{m_i}\right)^{1/2} \tau_{i\,i}. \qquad (2.73)$$

Combining (2. 73) with (2. 62) and (2. 63) we find

$$\omega_i\tau_{i\,i} = \left(\frac{m_e}{m_i}\right)^{1/2} \tau_{e\,i}\,\omega_e = \left(\frac{m_e}{m_i}\right)^{1/2} \omega\tau \simeq \frac{1}{43}\,\omega\tau. \; ^{*}$$

Since we assume $\omega\tau$ to be less than one, we can neglect the influence of the magnetic field on the stress-tensor P'_i. Further we deduce from (2.72) and (2.73) that $\eta_e < \eta_i$, so that we can neglect the nonhydrostatic part of the electron stress-tensor with respect to that of the ion stress-tensor (here we have used the fact that in gases, as considered here $p_i \simeq p_e$).
Hence we can write the second approximation to the total stress-tensor as $p_i + p_e + P'_i$, and putting $\omega\tau = 0$ in (2. 71) we see that the total stress-tensor reduces to (2. 63), with $p = p_i + p_e$, and $\eta = \eta_i$.
Thus the equation of motion (2. 45) becomes with help of (2. 70)

$$\rho \frac{\partial \mathbf{v}}{\partial t} + (\rho\,\mathbf{v}.\nabla)\mathbf{v} + \nabla p - \eta_i \nabla^2 \mathbf{v} - 1/3\; \eta_i \nabla\nabla.\mathbf{v} + \mathbf{j} \times \mathbf{B} = 0. \qquad (2.74)$$

(It is noteworthy that in this approximation there is no second viscosity coefficient).
Finally we consider the term $\frac{\nabla P_e}{e\,n}$ in (2. 47).
In the second approximation $P_e = p_e + P'_e$. Since $\omega\tau \simeq \omega\tau_{e\,e} < 1$, we deduce from (2. 70) and (2. 71) that P'_e is of the order $\eta_e \frac{V}{L}$, and

* In dealing with ω_e we henceforth omit the subscript e.

hence $\nabla P'_e$ of the order $\eta_e \frac{v}{L^2}$.
Using (2. 72) and the relation $p_e \simeq n_e m_e \frac{l^2}{\tau^2}$ we have

$$\frac{\nabla P_e}{e\,n} \approx \frac{n_e m_e}{e\,n_e} \cdot \frac{l^2}{\tau}\,\frac{v}{L^2} \tag{2.75}$$

Now let us estimate the order of magnitude of the term $\frac{\boldsymbol{j}}{\sigma}$ in (2. 47). From (2. 31) and (2. 49) we have

$$\frac{\boldsymbol{j}}{\sigma} \simeq \frac{e n_e \overline{\boldsymbol{V}}_e}{\sigma} = e\, n_e\, \overline{\boldsymbol{V}}_e\, \frac{m_e}{e^2 n_e \tau}. \tag{2.76}$$

From (2.75) and (2.76) we find that the ratio of $\frac{\nabla P'_e}{e\,n}$ to $\frac{\boldsymbol{j}}{\sigma}$ is of the order $\frac{l^2}{L^2} \cdot \frac{v}{V_e}$. Because this is a small quantity, we shall neglect the non-hydrostatic part of the electron stress-tensor in (2.47), which equation thus becomes:

$$\frac{m_e}{e^2 n_e}\left\{\frac{\partial \boldsymbol{j}}{\partial t} + (\boldsymbol{v}.\nabla)\,\boldsymbol{j} + (\boldsymbol{j}.\nabla)\,\boldsymbol{v} + j\,\nabla.\boldsymbol{v}\right\} =$$

$$\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B} - \frac{\boldsymbol{j} \times \boldsymbol{B}}{e n_e} + \frac{\nabla p_e}{e n_e} - \frac{\boldsymbol{j}}{\sigma}. \tag{2.77}$$

In the two-fluid or two-component model (2.77) takes the place of Ohm's Law in the one-fluid theory.
The Maxwell-equations are the same in both models.
We conclude this chapter with some remarks about (2. 77).
Let us consider a motion where the left-hand side of this equation is zero or negligibly small. This will be the case in our applications:

Denoting $\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B} - \frac{\nabla p_e}{e\,n}$ by $\boldsymbol{E}'$, (2. 77) reduces to

$$\boldsymbol{E}' = \frac{\boldsymbol{j}}{\sigma} + \frac{\boldsymbol{j} \times \boldsymbol{B}}{e\,n}. \tag{2.78}$$

When the induced magnetic fields can be neglected or do not play a rôle in (2.78), this equation can be considered as a relation between $\boldsymbol{j}$ and $\boldsymbol{E}$

$$\boldsymbol{E} = R\boldsymbol{j}. \tag{2.79}$$

where R is the resistivity. Clearly the resistivity is a tensor with array

$$\begin{pmatrix} \frac{1}{\sigma} & \frac{-\omega\tau}{\sigma} & 0 \\ \frac{\omega\tau}{\sigma} & \frac{1}{\sigma} & 0 \\ 0 & 0 & \frac{1}{\sigma} \end{pmatrix} \qquad (2.80)$$

Alternatively we can define the conductivity S by

$$\mathbf{j} = S\,\mathbf{E}.$$

The array of S can be found by inverting (2.80)

$$S = \begin{pmatrix} \frac{\sigma}{1+\omega^2\tau^2} & \frac{-\omega\tau\sigma}{1+\omega^2\tau^2} & 0 \\ \frac{\omega\tau\sigma}{1+\omega^2\tau^2} & \frac{\sigma}{1+\omega^2\tau^2} & 0 \\ 0 & 0 & \sigma \end{pmatrix}. \qquad (2.81)$$

When $\omega\tau < 1$, we can to a first approximation in $\omega\tau$, write S as

$$S = \begin{pmatrix} \sigma & -\omega\tau\sigma & 0 \\ \omega\tau\sigma & \sigma & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad (2.82)$$

We see that the term $\frac{\mathbf{j} \times \mathbf{B}}{e\,n}$, the Hall-term, in (2.77), which is not present in the one-fluid theory, causes the conductivity to be anisotropic. To what kind of phenomena this leads, will be investigated in the following chapters.

CHAPTER III

INVISCID FLOW BETWEEN PARALLEL PLATES

We start our applications of the theory developed in the preceding chapter with the following simple problem.[14)]

A fully ionized gas, as considered in chapter II, section 3, but without viscosity, flows between two infinite parallel plates, which we assume to be perfect conductors. A magnetic field B_o is externally applied in the direction normal to the plates. The gas is driven by a piston, situated between the plates and moving with constant velocity U. The situation is sketched in fig. 2. The distance between the plates is h.

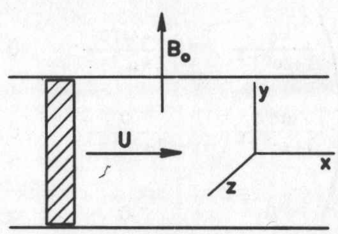

Fig. 2

Ionized gas, driven through
a magnetic field.

We choose a righthanded Cartesian frame of reference so, that the x-axis is in the direction of U, the y-axis in the direction of B_o, the x-z plane being halfway between the plates. We assume that the motion of the piston is slow enough to permit the neglect of density variations. When in this configuration, the piston is infinite in z direction, all quantities with the exception of the pressure, depend only on y.

Hence we deduce from the conservation of mass and from the condition that v_y must vanish at the plates, that

$$v_y = 0 , \ v_x = U . \tag{3.1}$$

From Maxwell's equation

$$\nabla \times \mathbf{B} = \mu \mathbf{j} \tag{3.2}$$

it follows on account of the independency on x and z that

$$j_y = 0 . \tag{3.3}$$

The electric field has no curl and hence

$$E_x = \text{const}\,;\; E_z = \text{const}. \tag{3.4}$$

Further it follows from $\nabla \cdot \mathbf{B} = 0$, that

$$B_y = \text{constant}. \tag{3.5}$$

In view of (3.1) and (3.3) the equation of motion (2.74) takes the simple form:

$$0 = -\nabla p + \mathbf{j} \times \mathbf{B}, \tag{3.6}$$

while (2.77) reads here

$$0 = \mathbf{E} + \mathbf{v} \times \mathbf{B} - \frac{\mathbf{j} \times \mathbf{B}}{e\,n} + \frac{\nabla p_e}{e\,n} - \frac{\mathbf{j}}{\sigma}. \tag{3.7}$$

We shall assume

$$\frac{dp}{dz} = 0. \tag{3.8}$$

∇p will have x and y components. It is convenient in this and in following problems to eliminate p_e from (3.7). When the charge density Q can be neglected the number density of protons and electrons is equal, therefore we have not labeled n. In chapter VII we will meet a situation where the charge density cannot be neglected everywhere. This is not the case here, so we put

$$p_i = p_e = n\,kT = 1/2\,p, \tag{3.9}$$

because we neglect viscosity and replace the stress-tensors P_i and P_e both by the first approximation, being the scalar pressures p_i and p_e. We can now eliminate p_e from (3.7) with help of (3.6) and (3.9), obtaining

$$0 = \mathbf{E} + \mathbf{v} \times \mathbf{B} - \frac{\mathbf{j} \times \mathbf{B}}{2\,e\,n} - \frac{\mathbf{j}}{\sigma}. \tag{3.10}$$

Before proceeding we examine the boundary conditions.
For the flow this is (3.1).
Of importance are now the electromagnetic boundary conditions which do not occur in ordinary fluid mechanics.
Because these boundary conditions are derived in any textbook on electromagnetism, (see for instance Stratton [15] or Panofsky and Phillips [16]), we give not a derivation, but only the results.
For the magnetic field it is necessary that the component normal to the boundary is always continuous.
Therefore as outside the plate the magnetic field has only the y-component B_o, we conclude on account of (3.5) that

$$B_y = B_o. \tag{3.11}$$

When finite conductivities (including zero conductivity) are involved only, the tangential components of the magnetic field strength must be continuous. However when one of the adjacent media has infinite conductivity, current sheets may be present, allowing a discontinuity in the tangential components. So in our present problem the perfect conducting plates may be the carriers of such current sheets and therefore we can allow for discontinuities in B_x and B_z at the plates.

Turning now to the electric field we require that the tangential components are always continuous. A discontinuity of the normal component however may occur when the boundary bears a surface charge. In the present case the plates are perfect conductors, so the tangential components of $\mathbf{E}$ are zero, since in a stationary perfect conductor no electric field can exist.

Thus (3. 4) becomes

$$E_x = E_z = 0 \tag{3.12}$$

The z-component of (3. 6) gives with help of (3. 3) and (3. 8)

$$j_x = 0. \tag{3.13}$$

or on account of (3. 2)

$$B_z = \text{const.}$$

The boundary condition for B_z is satisfied by

$$B_z = 0. \tag{3.14}$$

Using (3. 1) and (3. 11) - (3. 13) we find from the z-component of (3. 10)

$$j_z = \sigma U B_o\,. \tag{3.15}$$

Then from (3. 11) and (3. 15) and the x-components of (3. 6) we find

$$\frac{dp}{dx} = -\,\sigma U B_o{}^2. \tag{3.16}$$

The other quantities can now easily be derived from these results. We obtain

$$v_z = 1/2\ \omega\tau U, \tag{3.17}$$

$$B_x = -\,\sigma U B_o y, \tag{3.18}$$

$$\frac{dp}{dy} = -(\sigma U B_o)^2 y, \tag{3.19}$$

$$E_y = 0\,. \tag{3.20}$$

In (3. 17) $\omega = \dfrac{eB_o}{m_e}$ and τ is the collision time defined by (2. 56).

From (3.18) it follows that at the plates B_x has the value $\pm$ 1/2 $\sigma U B_o h$, + referring to the lower and - to the upper plate.
From the discussion of the boundary condition it follows that in each plate a current J_z flows equal to 1/2 $\sigma U B_o h$ in the negative z direction. Together they constitute the return current for the total current,

$$\int_{-h/2}^{+h/2} j_z \, dy = \sigma U B_o h, \text{ in the gas.}$$

In the one-fluid theory $\omega\tau = 0$, so from inspection of the above results, it appears that the only difference in the two-fluid model is a velocity in the direction tangential to the piston.
In the first order in $\omega\tau$ this drift velocity is proportional to U in the ratio 1/2 $\omega\tau$.
Electrons and protons obtain both this drift velocity, but in opposite direction. To the mass velocity only the positive charges contribute (this is a consequence of the fact that $\frac{m_e}{m_i}$ is neglected with respect to one), but both electrons and protons contribute to the electric current, which follows from (3.15) by observing that, since on account of (2.49) and (2.63)

$$\omega\tau = \frac{\sigma B_o}{e\,n}, \tag{3.21}$$

the current can be written as

$$j_z = 2\,e\,n\,v_z \,.$$

Due to this additional velocity component v_z, we see that there is an angle θ between the velocity of the fluid and that of the piston, given by

$$\text{tg}\,\theta = 1/2\;\omega\tau.$$

When we now construct a rectangular duct by inserting two walls in the gas parallel to $\boldsymbol{U}$ and $\boldsymbol{B}_o$, the result obtained above means that the pressure gradient now has a z component given by

$$\frac{dp}{dz} = -\,1/2\;\omega\tau\,\frac{dp}{dx} = 1/2\;\omega\tau\sigma U B_o{}^2 \,. \tag{3.22}$$

This effect is a mechanical analogue to the Hall-effect, which name is assigned to the phenomenon that in a conductor an electric field is set up in the direction $\boldsymbol{J} \times \boldsymbol{B}$, when a current $\boldsymbol{J}$ and a magnetic field $\boldsymbol{B}$ are present.

CHAPTER IV

MAGNETOHYDRODYNAMIC COUETTE FLOW

1. Introduction.

In order to study a flow, involving viscosity effects, we now consider magnetohydrodynamic Couette flow. This flow is as follows: A viscous fully ionized gas flows between two infinite parallel plates. One of the plates moves with a constant velocity U_o in its own plane, while the other is at rest. As in the preceding chapter, a magnetic field B_o is externally applied in the direction normal to the plates. Again the distance between the plates is h.

In ordinary fluid mechanics Couette flow has been extensively studied both for compressible and incompressible fluids, because it is one of the (few) problems in viscous fluid mechanics, where exact solution is possible. Also the M. H. D. Couette flow has received attention. The one-fluid model has been treated e. g. by Liepmann[17] (including compressibility effects, but without pressure gradient) and by Agarwal[18] (with pressure gradient, without compressibility effects). Till thus far Couette flow in the two-fluid theory has been considered by Peletier and van Wijngaarden[19]. We shall restrict ourselves to the case where there is only a pressure drop normal to the plates and where U_o is low enough to make compressibility effects negligible.
The behaviour of the flow is strongly affected by the electromagnetic properties of the plates. Therefore we study two cases. In the first one the lower plate (the one at rest) is a perfect conductor and the upper plate an insulator, in the second case both plates are insulators.

2. Equations for the two-fluid model.

It is convenient to introduce dimensionless variables here.
Denoting the physical quantities with asteriks, we locate the $x^* z^*$ plane of a right-handed Cartesian frame in the lower plate, x^* in the direction of U_o, and take B_o in the direction of the positive y^* axis.
We define dimensionless quantities by

$$x^* = xh \; ; \; y^* = yh \; ; \; z^* = zh \; ; \; \mathbf{v}^* = \mathbf{u} U_o \; ; \; \mathbf{B}^* = \mathbf{b} B_o \; ;$$

$$\mathbf{E}^* = \mathbf{E}\, U_o B_o \; ; \; \mathbf{j}^* = \mathbf{j}\, \sigma U_o B_o \; ; \; p^* = p \frac{\eta U_o}{h} . \qquad (4.1)$$

x

When compressibility effects can be neglected, as is assumed here, the continuity equation is

$$\nabla \cdot \boldsymbol{u} = 0. \qquad (4.2)$$

The conservation of charge requires

$$\nabla \cdot \boldsymbol{j} = 0. \qquad (4.3)$$

In this problem all quantities depend only on y, so that we have from Maxwell's equation (2.10) a

$$j_y = 0. \qquad (4.4)$$

Further we conclude from (4.2) and the condition that u_y must vanish at the plates

$$u_y = 0. \qquad (4.5)$$

With help of (4.1) - (4.5) we obtain from (2.74)

$$M^2 \, \boldsymbol{j} \times \boldsymbol{b} - \nabla p + \nabla^2 \boldsymbol{u} = 0, \qquad (4.6)$$

where M^2 is defined by

$$M^2 = \frac{B_0^2 h^2 \sigma}{\eta}. \qquad (4.7)$$

M is the Hartmann-number and represents the ratio between electromagnetic dissipation (joule heat) and viscous dissipation.
For gases, as considered here, reasonable values of σ are $10^3 - 10^4$ ohm^{-1} m^{-1} and of η $10^{-6} - 10^{-7}$ $\frac{kg}{m\,sec}$, so that from (4.7) it follows that M is large. When B_0 is 0,1 weber /m^2 and h = 10^{-1} m we find with the indicated ranges for σ and η

$$M \simeq 10^3.$$

The equation (2.77) reads here, on account of (4.1) - (4.5)

$$\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{b} - q \, \boldsymbol{j} \times \boldsymbol{b} + \frac{q}{M^2} \nabla p_e - \boldsymbol{j} = 0, \qquad (4.8)$$

where

$$q = \omega\tau. \qquad (4.9)$$

The dimensionless Maxwell equations are

$$\nabla \cdot \boldsymbol{b} = 0, \qquad (4.10)$$

$$\nabla \times \boldsymbol{E} = 0, \qquad (4.11)$$

$$\nabla \times \boldsymbol{b} = R \boldsymbol{j}. \qquad (4.12)$$

In (4.12) the magnetic Reynoldsnumber appears, defined by

$$R = \sigma \mu U_o h. \qquad (4.13)$$

In ordinary fluid mechanics the Reynoldsnumber Re, given by

$$R_e = \frac{\rho U_o h}{\eta}, \qquad (4.14)$$

represents the ratio between inertia forces and viscous forces.

In M.H.D. the quantity $(\mu\sigma)^{-1}$, having the dimension $\frac{m^2}{sec}$, plays the rôle of $\frac{\eta}{\rho} = \nu$, and is frequently called the "magnetic viscosity". From (4.13) and (4.14) we see that

$$R = \sigma \mu \nu Re.$$

R is the ratio between the work done by the Lorentz force and the joule dissipation.
From (4.10) it follows that

$$b_y = 1, \qquad (4.15)$$

and from (4.11) that

$$E_x = \text{constant}, \; E_z = \text{constant}. \qquad (4.16)$$

As already mentioned in the introduction, we exclude the presence of a pressure drop in the x and z directions. Then the x and z components of (4.6) are, using (4.4), (4.5) and (4.15)

$$- M^2 j_z + \frac{d^2 u_x}{dy^2} = 0, \qquad (4.17)$$

$$M^2 j_x + \frac{d^2 u_z}{dy^2} = 0, \qquad (4.18)$$

while the x and z components of (4.8) become

$$E_x - u_z + q j_z - j_x = 0, \qquad (4.19)$$

$$E_z + u_x - j_z - q j_x = 0. \qquad (4.20)$$

When the solution of (4.17) - (4.20) is known, the y components of (4.6) and (4.8) determine E_y and $\frac{dp}{dy}$.
The set (4.17) - (4.20) is not yet complete, since the electromagnetic properties of the plates have to be specified. In the next sections two different choices are made.

3. $E_x = E_z = 0.$

In this section we assume that the upper plate is insulating and that the lower plate is a perfect conductor. Because in the lower plate no electric field can exist, the continuity of the tangential components of $\mathbf{E}$ at the lower plate requires on account of (4. 16) that

$$E_x = E_z = 0.$$

Hence (4. 19) and (4. 20) reduce to

$$-u_z + q j_z - j_x = 0, \tag{4.21}$$

$$u_x - j_z - q j_x = 0. \tag{4.22}$$

In the one-fluid model the motion is two-dimensional and $j_x = 0$. Hence the term $q j_x$ in (4. 22) is of higher order than q and can be neglected, since we look for effects of the first order in q. Therefore (4. 22) yields in the present approximation

$$u_x = j_z \,. \tag{4.23}$$

Inserting this in (4. 17) gives

$$\frac{d^2 u_x}{dy^2} - M^2 u_x = 0. \tag{4.24}$$

Apparently there is no influence of q on u_x up to the second order in q. The boundary conditions for u_x are:

$$y = 0 \; : \; u_x = 0,$$
$$y = 1 \; : \; u_x = 1.$$

Solution of (4. 24) with these conditions gives

$$u_x = \frac{\sinh My}{\sinh M} \,. \tag{4.25}$$

From (4. 18), (4. 21) and (4. 23) it follows that

$$u_z = q u_x + \frac{1}{M^2} \frac{d^2 u_z}{dy^2} \,. \tag{4.26}$$

The driving force for the motion in z-direction is here the resulting shear stress working in x-direction on a volume element. The first term on the right-hand side of (4. 26) resembles the effect found in chapter III. The second term represents the shear stress in the z-direction.

The drift velocity u_z must be subjected to the boundary conditions

$$y = 0 \; : \; u_z = 0,$$
$$y = 1 \; : \; u_z = 0.$$

The solution of (4. 20), satisfying these conditions, is

$$u_z = \frac{Mq}{2\sinh^2 M}\left[\cosh M \sinh My - y \cosh My \sinh M\right]. \quad (4.27)$$

We have observed that usually M is large, so that we can put

$$\sinh M \simeq \cosh M \gg M \gg 1. \quad (4.28)$$

Using (4. 28) and introducing the variable $\xi = 1 - y$, we can simplify (4. 25) and (4. 27) to

$$u_x \simeq e^{-M\xi}, \quad (4.29)$$

and

$$u_z \simeq \frac{q M\xi}{2} e^{-M\xi}. \quad (4.30)$$

The flow in x-direction is restricted to a region of the order $\frac{1}{M}$ measured from the upper plate, from where u_x decreases rapidly towards the value zero at the lower plate. The velocity in z-direction starts at a value zero at the upper plate, reaches a maximum at $y = 1 - \frac{1}{M}$ and and decays from the maximum value $\frac{q}{2} e^{-1}$ to the value zero at the lower plate slower than u_x because of the factor M in (4. 30). At the point $y = 1 - \frac{2}{M}$ or $\xi = \frac{2}{M}$, we have $u_z = qu_x$, since there $\frac{d^2u_z}{dy^2} = 0$ (cf (4. 26)).

From (4. 23) and (4. 29) we obtain

$$j_z \simeq e^{-M\xi}, \quad (4.31)$$

and hence (4. 12) yields upon integration

$$b_x \simeq \frac{R}{M}(1 - e^{-M\xi}). \quad (4.32)$$

The integration constant has been adjusted to the condition that at the upper (insulating) plate $b_x = 0$.

Likewise we find for j_x and b_z :

$$j_x \simeq q/_2\, e^{-M\xi}(2 - M\xi),$$

$$b_z \simeq \frac{qR}{2M}\left\{1 - e^{-M\xi}(1 - M\xi)\right\}.$$

The values of b_z and b_x at the lower plate determine the return currents in the lower plate.
Quantities of interest are the forces exerted on the plates.
In hydrodynamics the stress τ_W at the wall is usually expressed in terms of $1/2\ \rho U_o^2$.

$$\tau_W = c_W \cdot 1/2\ \rho U_o^2 . \tag{4.33}$$

Without magnetic fields c_W is a function of the Reynoldsnumber, defined by (4.14).
In our case τ_W at the upper plate is given by $\frac{\eta U_o}{h} \left(\frac{du_x}{d\xi}\right)_{\xi = 0}$ in the x-direction and $\frac{\eta U_o}{h} \left(\frac{du_z}{d\xi}\right)_{\xi = 0}$ in the z-direction.

From (4.14), (4.29), (4.30) and (4.33) we find, that for the stress in the negative x-direction

$$c_{W_x} = \frac{2M}{R_e} , \tag{4.34}$$

and for the stress in the z-direction

$$c_{W_z} = \frac{qM}{R_e} . \tag{4.35}$$

The presence of a shear stress in z-direction is analogous to the pressure drop in z-direction, found in studying the problem of Chapter III.
It follows from (4.29) and (4.30) that the forces on the lower plate are of order e^{-M} and hence are negligibly small in the approximation formulated by (4.28).

4. Nonconductive plates.

We now return to eqs. (4.17) - (4.20) and consider the case where both plates are insulators. We assume that the conditions at infinite require that the current lines are closed in the gas:

$$\int_0^1 j_x\, dy = 0, \tag{4.36}$$

$$\int_0^1 j_z\, dy = 0. \tag{4.37}$$

With these conditions we obtain from (4.17) and (4.18) upon integration between 0 and 1

$$\left(\frac{du_x}{dy}\right)_{y=0} = \left(\frac{du_x}{dy}\right)_{y=1}, \tag{4.38}$$

$$\left(\frac{du_z}{dy}\right)_{y=0} = \left(\frac{du_z}{dy}\right)_{y=1}. \tag{4.39}$$

From (4.17) and (4.20) we obtain, again dropping the term qj_x in (4.20)

$$\frac{1}{M^2}\frac{d^2u_x}{dy^2} - u_x = E_z .$$

The solution that gives $u_x = 0$ for $y = 0$, $u_x = 1$ for $y = 1$ and satisfies (4.38) is:

$$u_x = \frac{1}{2}\left[1 + \frac{\sinh M\,(y - 1/2)}{\sinh \frac{M}{2}}\right], \tag{4.40}$$

$$E_z = -1/2. \tag{4.41}$$

The equation for u_z reads on account of (4.18) - (4.20) and (4.41)

$$u_z = q(u_x - 1/2) + \frac{1}{M^2}\frac{d^2u_z}{dy^2} + E_z .$$

The solution, satisfying (4.39) and vanishing both at the upper and at the lower plate, is

$$u_z = \frac{Mq}{4\sinh^2\frac{M}{2}}\left[\frac{1}{2}\cosh\frac{M}{2}\sinh M(y-1/2) - (y-1/2)\sinh\frac{M}{2}\cosh M(y-1/2)\right] \tag{4.42}$$

and

$$E_x = 0. \tag{4.43}$$

The solution of the problem of this section can be related with that of the foregoing one by the following transformation.
Let us move in the physical system of reference the x^* axis to $y^* = \frac{h}{2}$.
The relation between the new ordinate $y^{*\prime}$ and y^* is

$$y^{*\prime} = y^* - \frac{h}{2}.$$

When we refer to $\frac{h}{2}$ in the dimensionless coordinates and take $\zeta = \frac{y^{*\prime}}{\frac{h}{2}}$, then

$$y - 1/2 = \frac{y^*}{h} - 1/2 = \frac{\zeta}{2}. \tag{4.44}$$

With the transformation (4.44), (4.40) yields

$$u_x - 1/2 = 1/2 \frac{\sinh \frac{M}{2} \zeta}{\sinh \frac{M}{2}} . \qquad (4.45)$$

Comparison of this result with (4.25) learns, that in the upper half of the space between the plates, $0 \leqslant \zeta \leqslant 1$, $u_x - 1/2$ behaves just like u_x in the problem of section 3, when in that case we take the distance between the plates $\frac{h}{2}$ instead of h and the velocity of the upper plate $\frac{U_o}{2}$ instead of U_o.

The same holds for u_z, which becomes in terms of ζ

$$u_z = \frac{Mq}{8\sinh^2 \frac{M}{2}} \left[\cosh \frac{M}{2} \sinh \frac{M}{2}\zeta - \zeta \sinh \frac{M}{2} \cosh \frac{M}{2}\zeta \right] . \qquad (4.46)$$

This expression can be obtained either directly from (4.42) or from (4.27) by multiplying the right-hand side by 1/2 and changing M in $\frac{M}{2}$ and y in ζ.

We infer from (4.45) and (4.46) that $u_x - 1/2$ and u_z are antisymmetric with respect to $\zeta = 0$.

Since in cases of interest M is large, we can reduce (4.46) to

$$u_z = \frac{Mq}{4} e^{-\frac{M}{2}} \left[\sinh \frac{M}{2}\zeta - \zeta \cosh \frac{M}{2}\zeta \right] . \qquad (4.47)$$

From (4.17) and (4.45) we obtain for j_z

$$j_z = \frac{1}{2} \frac{\sinh \frac{M}{2} \zeta}{\sinh \frac{M}{2}} ,$$

and from (4.12) and this result, remembering that now b_x and b_z must vanish at both plates,

$$b_x = \frac{R}{2M} \left\{ 1 - \frac{\cosh \frac{M}{2} \zeta}{\sinh \frac{M}{2}} \right\} .$$

In the same way expressions for j_x and b_z can be obtained from (4.12), (4.18) and (4.46).

We observe that in the present configuration boundary layers of the type discussed in section 3 occur at both plates. Applying the scaling rules mentioned in the foregoing, we obtain from (4.29) and (4.30), that near the plates for large M

$$u_x - 1/2 \simeq \pm \frac{1}{2} \exp\left[- \frac{M}{2} (\mp \zeta + 1) \right], \qquad (4.48)$$

and

$$u_z \simeq \pm \frac{Mq}{8} (\mp \zeta + 1) \exp\left[- \frac{M}{2} (\mp \zeta + 1) \right]. \qquad (4.49)$$

The upper sign refers to the upper -, the lower sign to the lower plate. The forces on the plates are equal but opposite in sign.
From (4.34), (4.35) and the scaling rules we find

$$c_{w_x} = \frac{\tau_{w_x}}{1/2\, \rho U_o^2} = \frac{M}{Re} ,$$

and

$$c_{w_z} = \frac{\tau_{w_z}}{1/2\, \rho U_o^2} = \frac{qM}{2Re} .$$

5. Concluding remarks.

In the foregoing sections we have shown that in the Couette flow of a fully ionized gas, the motion of the upper plate engenders a transverse motion of the gas. Our results are also applicable to problems, involving the relative motion of two concentric cylinders, the magnetic field pointing radially outward.

When the difference between the radii is small with respect to both of them, the influence of curvature can be neglected and in that case the annulus between the cylinders can be considered as the space between two parallel plates.

The problem of section 3 corresponds for instance with the uniform translation of the outer cylinder in axial direction. The results of section 3 show that a torque is exerted on this cylinder, causing a rotation about its axis, when such a motion is not prevented by external means.

Regarding the problem of section 4, we can think of a constant angular velocity of the outer cylinder, the inner one being fixed. The results of section 4 learn us that the rotation of the gas is accompanied by a secondary motion in axial direction. This motion can be considered as a vortex motion, the vorticity given by the y-derivative of formula (4.42).

CHAPTER V

WAVY PLATE; ONE-FLUID MODEL; STEADY MOTION

1. Introduction.

In this and the following chapters we shall occupy ourselves with magnetohydrodynamic flow along a thin plate with a wavy profile on both sides.
The concept of flow along a wave shaped boundary is due to Ackeret[20], who published in 1928 a paper dealing with the motion of a gas along a wavy wall, the purpose being to study the effects of compressibility. When the amplitude is small with respect to the wave length the equation for the velocity potential (viscosity effects are ignored) can be linearized and due to this linearization solutions can easily be obtained.
The solution of the wavy wall problem gives some insight in the particular flow properties one wants to study and further it provides the means to arrive at solutions for more complicated problems such as the flow round thin airfoils. Since the theory is a linearized one, these solutions can be obtained from the wavy wall solutions by Fourier synthesis. Therefore it suggests itself to consider the wavy wall problem in M.H.D. The first paper on this subject was written by Sears and Resler[21] in 1959. They considered a perfectly conducting fluid, moving along the wavy boundary of an insulating medium, occupying the lower half plane, while the fluid moves in the upper half-plane. The influence of compressibility is neglected. They found that, when a magnetic field is present parallel to the undisturbed velocity, the flow is the same as in the nonmagnetic case, but due to the presence of the magnetic field current sheets occur at the boundary.
When the undisturbed magnetic field is perpendicular to the undisturbed velocity, the flow is largely different from the nonmagnetic case. The main part of this paper deals with this case.
In so far as thin airfoils are treated, it is indicated that for fluids of infinite conductivity the linearized theory is not valid for the flow around thin airfoils, when the undisturbed fields are parallel. No particular configuration however is treated.

The paper by Sears and Resler was followed by other ones, from this authors and their collaborators at Cornell University.
Mc.Cune[22] gave a theory for thin airfoils moving in a fluid with finite conductivity for the case where the undisturbed velocity and undisturbed magnetic field are perpendicular to each other.
In the book by Bershader[23], a survey of the work at Cornell was given

by Resler and Mc Cune. In this contribution compressibility effects are also taken into account, and attention is paid to the case where the undisturbed field is parallel to the flow and the conductivity is finite. No complete solution however is given.
Compressibility effects were also considered for the aligned fields case by Bhutani[24], assuming an infinite conductivity of the fluid. Following Sears and Resler[21], he treats the case where the wave shaped boundary separates the fluid from an infinite insulating medium.
Since the situation in the crossed fields case is fairly clear by now, we restrict ourselves to the case where the undisturbed fields are parallel. We shall deal with the flow of a fluid of finite conductivity over a thin plate of sinusoidal shape at both sides, because a complete solution for such a configuration has not been given in the cited references. We shall assume that compressibility effects can be neglected. In this chapter we shall deal with steady motion and use the one-fluid model. The study of time-dependent flow will be the subject of chapter VI, while we leave the application of the two-fluid model to chapter VII.

2. Development of equations.

Consider a thin plate with a wave shaped surface on both sides. The upper side is given by the real part of

$$y^* = \epsilon \exp i\lambda x^* . \tag{5.1}$$

Symbols, marked with * represent physical quantities. In the following we shall make use of dimensionless quantities. In (5.1) ϵ is the amplitude, $\frac{2\pi}{\lambda}$ the wave length of the upper side. They are in the present problem restricted in their magnitude by the condition

$$\epsilon\lambda \ll 1. \tag{5.2}$$

A conducting fluid moves parallel to this plate with a velocity, which is U_0 in the direction of the positive x^* axis, when the plate is absent. In the undisturbed state a magnetic induction B_0 is present parallel to U_0. The presence of the plate causes a disturbance of flow and magnetic field. We want to know the magnitude of these disturbances.
The lower side of the plate is sinusoidal too and has the same wave length and amplitude as the upper side, but there may be a difference in phase. We shall consider the cases where the phase difference is zero and where it is π. These configurations are sketched in fig. 3 (antisymmetric case) and fig. 4 (symmetric case). We shall assume that the material of the plate has zero conductivity and the same permeability as vacuum. In the fluid the permittivity and permeability are those of empty space. This is required by what is said about the

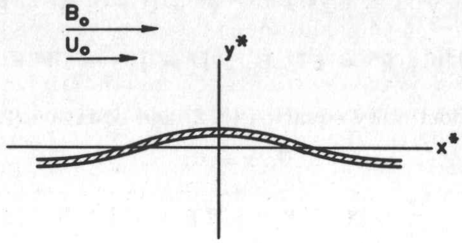

Fig. 3

Wavy plate; no phase difference between upper and lower side.

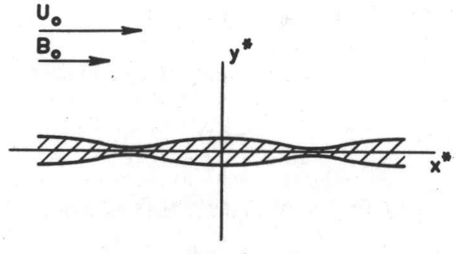

Fig. 4

Wavy plate; phase difference between upper and lower side equal to π

constitutive relations in chapter II.
We neglect viscosity effects and compressibility effects. Here, as in the foregoing chapter, it is convenient to write the equations in terms of dimensionless quantities. Therefore we define

$$x^* = \frac{x}{\lambda} \; ; \; y^* = \frac{y}{\lambda} \; ; \; z^* = \frac{z}{\lambda} \; ; \; \mathbf{v}^* = U_o \mathbf{v} \; ; \; \mathbf{B}^* = B_o \mathbf{B} \; ;$$
$$p^* = \rho U_o^2 p \; ; \; \mathbf{E}^* = \mathbf{E} U_o B_o \; ; \; \mathbf{j}^* = \mathbf{j} \frac{B_o \lambda}{\mu} \; ; \; t^* = \frac{t}{U_o \lambda} \, . \tag{5.3}$$

Using (5. 3) the continuity equation (2. 2) and equation (2. 6) become,

$$\nabla \cdot \mathbf{v} = 0, \tag{5.4}$$

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} . \nabla) \mathbf{v} = - \nabla p + \beta^2 \mathbf{j} \times \mathbf{B} \, . \tag{5.5}$$

In (5. 5) β^2 is given by

$$\beta^2 = \frac{B_o^2}{\rho \mu U_o^2} \, . \tag{5.6}$$

β^2 is the ratio between magnetic and kinetic energy. The quantity $\frac{B_o}{(\rho\mu)^{1/2}}$ is the speed of propagation of hydromagnetic waves and is called the Alfvén velocity. In aerodynamics one distinguishes between supersonic and subsonic flow. In M.H.D. a flow is in analogy herewith frequently called superalfvénic when $\beta < 1$, subalfvénic when $\beta > 1$.
From (2. 5) and (5. 3) we obtain, remembering that the convective current $Q\mathbf{v}$ can be neglected,

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \frac{1}{R} \mathbf{j} , \tag{5.7}$$

where R is the magnetic Reynoldsnumber, introduced in chapter IV and here given by

$$R = \frac{\sigma \mu U_o}{\lambda} \, . \tag{5.8}$$

Maxwell's equations (2. 8), (2. 9) and (2. 10)[a] become

$$\nabla \cdot \mathbf{B} = 0, \tag{5.9}$$

$$\nabla \times \mathbf{E} = - \frac{\partial \mathbf{B}}{\partial t} , \tag{5.10}$$

$$\nabla \times \mathbf{B} = \mathbf{j} \, . \tag{5.11}$$

Now we assume that the disturbances created by the plate, which is assumed to have infinite length and width, are small with respect to

the undisturbed fields.
Therefore we write, $\mathbf{k}$ being the unit vector in x-direction,

$$\mathbf{v} = \mathbf{k} + \mathbf{u}, \tag{5.12}$$

$$\mathbf{B} = \mathbf{k} + \mathbf{b}. \tag{5.13}$$

We suppose on account of (5.2) that u and b are small with respect to one. In the undisturbed state $\mathbf{j}$ is zero. Then if follows from (5.7) that also $\mathbf{E}$ is zero in the undisturbed state. We assume in view of the infinite width that all quantities depend only on x, y and t. Then in this one-fluid model the equations are satisfied by $u_z = b_z = 0$, so that we have only to deal with the components of $\mathbf{u}$ and $\mathbf{b}$ in the x-y plane. We introduce the streamfunction ψ by writing

$$u_x = \frac{\partial \psi}{\partial y}, \quad u_y = -\frac{\partial \psi}{\partial x}, \tag{5.14}$$

and the function A with

$$b_x = \frac{\partial A}{\partial y}, \quad b_y = -\frac{\partial A}{\partial x}. \tag{5.15}$$

A is the z-component of the vector potential. By the introduction of ψ and A, (5.4) and (5.9) are automatically satisfied.
Now we take the curl of (5.5) and obtain, using (5.9) - (5.15) and neglecting quantities of the second order in $\epsilon\lambda$

$$\nabla^2 \left\{ \frac{\partial \psi}{\partial t} + \frac{\partial \psi}{\partial x} - \beta^2 \frac{\partial A}{\partial x} \right\} = 0. \tag{5.16}$$

In the same way we obtain from (5.7)

$$\frac{\partial A}{\partial t} + \frac{\partial A}{\partial x} - \frac{\partial \psi}{\partial x} = \frac{1}{R} \nabla^2 A. \tag{5.17}$$

In deriving this equation from the curl of (5.7), it is assumed that far from the plate A and ψ are zero.
To solve (5.16) and (5.17) we need boundary and initial conditions.
Let us consider the antisymmetric case of fig. 3.
For $y \geqslant 0$ we have the conditions

$$y = 0 : u_y = -\frac{\partial \psi}{\partial x} = i\epsilon\lambda \exp(ix), \tag{5.18}$$

$$y = \infty : \psi = 0, \tag{5.19}$$

$$A = 0. \tag{5.20}$$

Anticipating the treatment of the unsteady motion in the next chapter, we require

$$t = 0 \; : \; \psi = \psi_o \, (x, y), \tag{5.21}$$

$$A = 0. \tag{5.22}$$

We do not consider boundary effects in x-direction, and take all quantities proportional to exp (i x).
Yet the problem is not determined, because a boundary condition for A is lacking.
Therefore we consider the lower half-plane. Let the stream function be given there by ψ' and the vector potential by A'. They must satisfy (5.16) and (5.17) and the conditions

$$y = 0 \; : \; u_y = - \frac{\partial \psi'}{\partial x} = i \, \varepsilon \, \lambda \, \exp \, (i \, x), \tag{5.23}$$

$$y = - \infty : \; \psi' = 0, \tag{5.24}$$

$$A' = 0. \tag{5.25}$$

$$t = 0 \; : \; \psi' = \psi_o \, (x, -y), \tag{5.26}$$

$$A' = 0. \tag{5.27}$$

In the problem in the lower half-plane another boundary condition is lacking. Both problems however turn into a single well-posed problem, when it is taken into account that both b_x and b_y must be continuous at the boundary. In the linearized theory this means that we must require

$$y = 0 \; : \; \frac{\partial A}{\partial x} = \frac{\partial A'}{\partial x}, \tag{5.28}$$

$$\frac{\partial A}{\partial y} = \frac{\partial A'}{\partial y}. \tag{5.29}$$

We state that the problem is solved by first looking for the solution of (5.16) - (5.22) with the additional condition

$$y = 0 \; : \; \frac{\partial A}{\partial y} = 0, \tag{5.30}$$

and then defining ψ' and A' by the relations

$$\psi' (t, x, y) = \psi (t, x, -y), \tag{5.31}$$

$$A' (t, x, y) = A (t, x, -y). \tag{5.32}$$

The proof of this is as follows. By (5.16) - (5.22) and (5.30) the solution in the upper half-plane is uniquely determined.
Since ψ and A satisfy (5.16) and (5.17) and since the operators working on ψ and A in these equations are even in y, ψ' and A', connected with ψ and A by (5.31) and (5.32), are also solutions of (5.16) and (5.17). Because ψ and A satisfy (5.18) - (5.22), we conclude on account of

(5.31) and (5.32), that ψ' and A' satisfy (5.23) - (5.27). Finally it follows from (5.30) - (5.32) that the conditions (5.28) and (5.29) are fulfilled. Streamfunction and vector potential, obtained in this way, satisfy all the equations, initial and boundary conditions, and hence constitute the unique solution of the problem.

Now we turn to the symmetric case of fig. 4. In the upper half-plane the conditions are the same as in the configuration of fig. 3. They are given by (5.18) - (5.22). In the lower half-plane, (5.24), (5.25) and (5.27) hold here too, but, owing to the phase difference, we must have

$$y = 0 \; : \; u_y = - \frac{\partial \psi'}{\partial x} = - i \, \varepsilon \, \lambda \exp(i\,x),$$

while in stead of (5.26) we must require here

$$t = 0 \; : \; \psi' = - \psi_0 \, (x, -y).$$

The condition, corresponding with (5.30), that determines in this case the problem in the upper half-plane, is

$$y = 0 \; : \; \frac{\partial A}{\partial x} = 0, \tag{5.33}$$

whilst here we have in the lower half-plane

$$\psi'(t, x, y) = - \psi(t, x, -y), \tag{5.34}$$

$$A'(t, x, y) = - A(t, x, -y). \tag{5.35}$$

These statements are easily proved by arguments analogous to those used in connection with the problem of fig. 3.

From the above consideration we conclude that in both cases, the problem in the upper half-plane is not determined by the available conditions. Consideration of the lower half-plane leads to the conclusion that the condition of continuity of the magnetic field at the plate is equivalent to the condition that the x-component of the magnetic induction must vanish at the plate in the case of fig. 3, the y-component in the case of fig. 4.

The physical reason for the fact that the problem in the upper half-plane is not determined, is that upper- and lower half-plane are separated by the plate only in a mechanical sense, not electromagnetically.

In the next sections we shall deal with the steady solutions for the configurations of fig. 3 and fig. 4.

In chapter VI a type of unsteady motion will be treated.

3. **Steady motion. Configuration of fig. 3.**

In the case of steady motion, (5.16) and (5.17) reduce to

$$\nabla^2 \left(\frac{\partial\psi}{\partial x} - \beta^2 \frac{\partial A}{\partial x}\right) = 0, \tag{5.36}$$

$$\frac{\partial A}{\partial x} - \frac{\partial\psi}{\partial x} = \frac{1}{R} \nabla^2 A. \tag{5.37}$$

Because ψ and A are periodic in x, we separate the variables by writing

$$\psi = f(y) \exp(i x), \tag{5.38}$$

$$A = g(y) \exp(i x). \tag{5.39}$$

Inserting these expressions in (5.36) yields

$$\left(\frac{d^2}{dy^2} - 1\right)(f - \beta^2 g) = 0, \tag{5.40}$$

whilst (5.37) becomes

$$\left(\frac{d^2}{dy^2} - 1\right) g = iR(g - f). \tag{5.41}$$

The boundary conditions (5.18) - (5.20) and (5.30) are in terms of f and g

$$y = 0 \ : \ f = -\varepsilon\lambda, \tag{5.42}$$

$$\frac{dg}{dy} = 0, \tag{5.43}$$

$$y = \infty \ : \ f = g = 0. \tag{5.44}$$

The independent solutions of (5.40) and (5.41) are $\exp \pm y$ and $\exp \pm \alpha y$, where

$$\alpha = [1 + iR(1 - \beta^2)]^{1/2}. \tag{5.45}$$

When $\beta = 1$, these solutions are identical and we have to find other solutions. This will be done in section 5. For the moment we assume $\beta \neq 1$.

On account of (5.44) we have to choose the - sign in both exponentials. Writing $g = k_1 \exp -y + k_2 \exp -\alpha y$, we obtain from (5.40) or (5.41)

$$f = k_1 \exp -y + \beta^2 k_2 \exp -\alpha y.$$

Using (5.42) and (5.43) we find for k_1 and k_2 :

$$k_1 = -\frac{\varepsilon\lambda\alpha}{\alpha - \beta^2},$$

$$k_2 = \frac{\varepsilon\lambda}{\alpha - \beta^2}.$$

Inserting these values in the expressions for f and g, yields with help of (5. 38) and (5. 39)

$$\psi = \frac{\varepsilon\lambda}{\alpha-\beta^2}\left[\beta^2 \exp - \alpha y - \alpha \exp - y\right] \exp (i\, x), \qquad (5.46)$$

$$A = \frac{\varepsilon\lambda}{\alpha-\beta^2}\left[\exp -\alpha y - \alpha \exp -y\right] \exp (i\, x). \qquad (5.47)$$

The corresponding expressions for the lower half-plane follow from (5. 31) and (5. 32)

$$\psi' = \frac{\varepsilon\lambda}{\alpha-\beta^2}\left[\beta^2 \exp \alpha y - \alpha \exp y\right] \exp (i\, x),$$

$$A' = \frac{\varepsilon\lambda}{\alpha-\beta^2}\left[\exp \alpha y - \alpha \exp y\right] \exp (i\, x).$$

When β^2 is zero, A is identically zero, and the nonvanishing part of ψ represents the nonmagnetic potential flow. The last term within the brackets in (5. 46) represents an irrotational flow of the same type as the nonmagnetic solution. The corresponding term in (5. 47) defines a magnetic field which is everywhere parallel and proportional to the flow. This field is irrotational too and therefore does not give rise to currents and Lorentz forces. Now the boundary conditions for flow and field are not identical, so that the solution for flow and field given by the irrotational terms is not complete. The first terms in the brackets in (5. 46) and (5. 47) represent the system of currents and vortices, required to adapt flow and field to these boundary conditions. The perturbations of the undisturbed flow and field, resulting from (5. 46) and (5. 47) are

$$u_x = \frac{\partial\psi}{\partial y} = \frac{\varepsilon\lambda}{\alpha-\beta^2}\left[\alpha \exp -y - \alpha\beta^2 \exp -\alpha y\right] \exp(ix), \qquad (5.48)$$

$$u_y = -\frac{\partial\psi}{\partial x} = \frac{i\varepsilon\lambda}{\alpha-\beta^2}\left[\alpha \exp -y - \beta^2 \exp -\alpha y\right] \exp(ix), \qquad (5.49)$$

$$b_x = \frac{\partial A}{\partial y} = \frac{\varepsilon\lambda}{\alpha-\beta^2}\left[\alpha \exp -y - \alpha \exp -\alpha y\right] \exp(ix), \qquad (5.50)$$

$$b_y = -\frac{\partial A}{\partial x} = \frac{i\varepsilon\lambda}{\alpha-\beta^2}\left[\alpha \exp -y - \exp -\alpha y\right] \exp(ix). \qquad (5.51)$$

As an illustration we have drawn in fig.5 the b_x lines for several values of y in the case of the flow of sodium along an antisymmetric wavy plate.

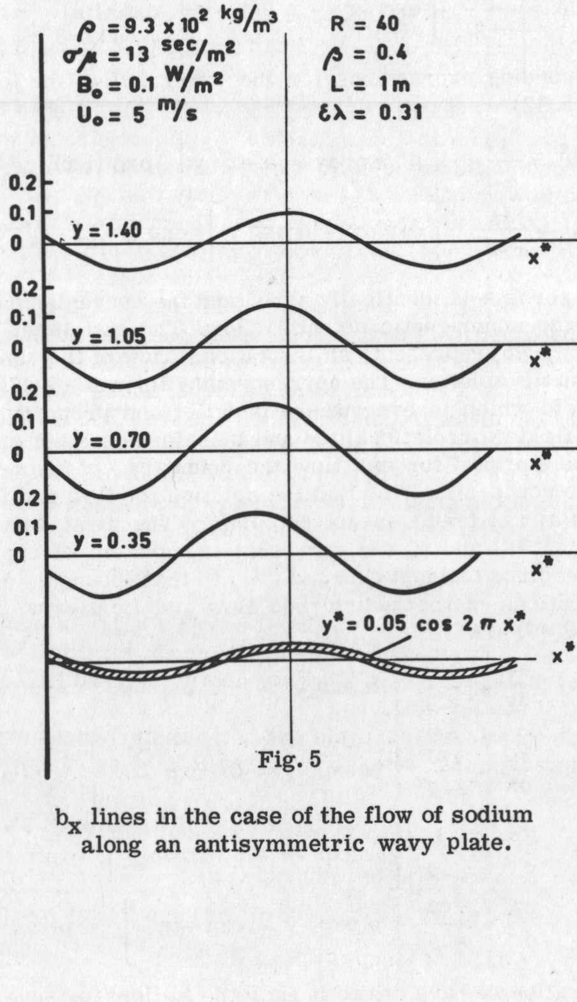

Fig. 5

b_x lines in the case of the flow of sodium along an antisymmetric wavy plate.

The values of the various quantities are given in the figure.
Note that, when y increases, the phase shifts in the downstream direction.
The direction of the phase shift depends on the sign of $1 - \beta^2$. This can be demonstrated in the easiest way when R is large.
Then it follows from (5.45) that we can write $\exp(ix) \exp -\alpha y$ as

$$\exp \left[-y \sqrt{\left(\frac{R\,|1-\beta^2|}{2}\right)} \right] \exp i \left\{ x \pm y \sqrt{\left(\frac{R\,|1-\beta^2|}{2}\right)} \right\}.$$

+ corresponding to sub -, - to superalfvénic flow. Hence the lines of constant phase point upstream when $\beta > 1$, downstream when $\beta < 1$.
We have presumed that the disturbances are small with respect to the undisturbed flow and field. Let us now verify this.
From (5.45) it follows that α is large when R is large or when β is large. When β is of order one, it follows from (5.48) - (5.51) that the disturbances are of order $\epsilon\lambda$ whatever the value of R.
In section 5 it will be shown that this holds for $\beta = 1$ too. Now let us consider the situation when R is of order one, but the magnetic field is very strong, resulting in a large value of β. Then from (5.45) we have $\alpha \simeq R^{1/2}\beta$. Inspection of (5.48) - (5.51) shows that, except in a thin layer at the plate, the disturbances are very small. The magnetic field is only slightly disturbed and the streamlines are tied to them. However at the plate, the fluid must follow its slope, and is compelled to move across the magnetic field lines. This causes a current in the layer near the plate. Denoting the vorticity $\nabla \times \boldsymbol{u}$ with $\boldsymbol{w}$, we have from (5.5)

$$\boldsymbol{w} = \beta^2 \boldsymbol{j}. \qquad (5.52)$$

Hence the current arising from the prescribed value of u_y at the plate, causes a large vorticity and large u_x. From (5.48) it follows that near the plate u_x is of order $\epsilon\lambda\beta$. Thus we obtain the condition $\epsilon\lambda\beta < 1$ for the validity of the linearization.
We conclude that apart from this case, the disturbances are always of order $\epsilon\lambda$. In particular we have for large R (β being of order one) from (5.48) - (5.51)

$$u_x = \epsilon\lambda\,(\exp -y \;-\; \beta^2 \exp -\alpha y) \exp (i\,x),$$

$$u_y = i\,\epsilon\lambda \exp(-y) \exp (i\,x),$$

$$b_x = \epsilon\lambda\,(\exp -y \;-\; \exp -\alpha y) \exp (i\,x),$$

$$b_y = i\,\epsilon\lambda \exp(-y) \exp (i\,x).$$

These relations show that although the disturbances are of order $\varepsilon\lambda$, terms neglected in the equation of motion, such as $u_y \frac{\partial u_x}{\partial y}$ or $b_y \frac{\partial b_x}{\partial y}$ are of order $(\varepsilon\lambda)^2\alpha$.

Since under the circumstances, envisaged here, α is large (cf. (5.45)), we have to inspect, whether this invalidates the linearization.
Therefore we insert the linearized solution in the full nonlinearized equations (5. 5), which can with help of (5. 4) be written in the form

$$\nabla (p + \beta^2/_2 \, B^2) + (\mathbf{v}.\nabla)\mathbf{v} - \beta^2(\mathbf{B}.\nabla)\mathbf{B} = 0.$$

The linearized version is

$$\nabla(p + \beta^2 b_x) + \frac{\partial \mathbf{u}}{\partial x} - \beta^2 \frac{\partial \mathbf{b}}{\partial x} = 0,$$

from which equation we obtain

$$p = p_o - u_x .^*$$

Since this expression for p is obtained from the linearized equation, we are free to add any terms of order $(\varepsilon\lambda)^2$.
It appears to be appropriate to correct p with the terms $\frac{\beta^2 b_x^2}{2} + \frac{\beta^2 b_y^2}{2}$, otherwise said to correct p with the second order terms in the "magnetic pressure" $\frac{\beta^2 B^2}{2}$.

When we take $p - p_o + \frac{\beta^2 b_x^2}{2} + \frac{\beta^2 b_y^2}{2} = - u_x$ and subsequently insert this together with the linearized solution for u_x, u_y, b_x and b_y in the complete equations, only terms of order $(\varepsilon\lambda)^2$ remain, terms of order $(\varepsilon\lambda)^2\alpha$ cancelling out. Hence the fact that some of the terms, neglected in the momentum equation are large when α is large, does not affect the validity of the linearization, but it must be borne in mind that the pressure as obtained from the linearized equation has to be corrected with the second order terms $\frac{\beta^2 b_x^2}{2}$ and $\frac{\beta^2 b_y^2}{2}$. In fact, the former suffices, because the derivatives of b_y are of order $\varepsilon\lambda$.

4. Steady motion; configuration of fig. 4.

The analysis of the configuration of fig. 4 runs parallel to that of section 3. The difference is that the constants k_1 and k_2 must now be deduced from (5. 42) and the continuity condition (5. 33).
Using (5. 39), this condition gives for g

* p_o is the undisturbed pressure.

$$y = 0 \; : \; g = 0. \tag{5.53}$$

Upon replacing (5. 43) by (5. 53) we obtain for k_1 and k_2

$$k_1 = \frac{-\epsilon\lambda}{1-\beta^2},$$

$$k_2 = \frac{\epsilon\lambda}{1-\beta^2}.$$

Substitution of these values into the expressions for g and f, leads with help of (5. 38) and (5. 39) to

$$\psi = \frac{\epsilon\lambda}{1-\beta^2}\left[\beta^2 \exp -\alpha y - \exp -y\right] \exp(i\,x), \tag{5.54}$$

$$A = \frac{\epsilon\lambda}{1-\beta^2}\left[\exp -\alpha y - \exp -y\right] \exp(i\,x). \tag{5.55}$$

In the lower half-plane we get on account of (5. 34) and (5. 35)

$$\psi' = -\frac{\epsilon\lambda}{1-\beta^2}\left[\beta^2 \exp \alpha y - \exp y\right] \exp(i\,x), \tag{5.56}$$

$$A' = -\frac{\epsilon\lambda}{1-\beta^2}\left[\exp \alpha y - \exp y\right] \exp(i\,x). \tag{5.57}$$

Also here we inspect the validity of the linearization.
As in the configuration, discussed in section 3, large disturbances result when $\beta \gg 1$ (R being of order one).
Another limitation presents itself in this symmetric configuration.
Consider the x-components of $\mathbf{u}$ and $\mathbf{b}$. These are

$$u_x = \frac{\epsilon\lambda}{1-\beta^2}\left[\exp \pm y - \alpha\beta^2 \exp \pm \alpha y\right] \exp(i\,x), \tag{5.58}$$

$$b_x = \frac{\epsilon\lambda}{1-\beta^2}\left[\exp \pm y - \alpha \exp \pm \alpha y\right] \exp(i\,x). \tag{5.59}$$

In the exponentials + refers to the lower, - to the upper half-plane. When α is large, the second terms in (5.58) and (5.59) are in the vicinity of the plate dominant. Remembering the definition (5. 45) of α, we deduce from (5. 58) and (5. 59) that in that case u_x and b_x are of order $\epsilon\lambda R^{1/2}$ near the plate. Therefore the linearization breaks down here. The same result will undoubtedly be valid for thin symmetric airfoils, as the solution for this problem could be obtained by Fourier synthesis from the present results. In the introduction to this chapter we men-

tioned the work of Sears and Resler[21] and Mc Cune[22] on thin airfoils moving in perpendicular fields at large and infinite conductivity of the ambient fluid. In that case no calamities occur. In parallel fields indiscriminate linearization is not possible.
In order to understand this difficulty, we consider the current densities. These are given by $-\nabla^2 A$ in the upper, $-\nabla^2 A'$ in the lower half-plane. The relation (5.32) shows that in the antisymmetric case (fig. 3) j_z and j_z' are equal, both in magnitude and direction.
However from (5. 35) it follows that they are equal but opposite in the symmetric case (fig. 4), which leads to $b_x \neq 0$ at the plate.
Now (5. 55) shows that when α is large, the currents are comprised in thin layers at both sides of the plate. The thickness of these boundary layers is of order $\frac{1}{\alpha}$.
In this linearized theory both sides of the plate are taken at $y = 0$. Hence the two systems of currents are brought infinitely close together. This causes a large value of b_x at the plate.
This discussion suggests to consider a plate with finite thickness in order to separate the currents. We undertake this in section 6, but we want to study first the configurations of this and the preceding section for the special case $\beta = 1$.

5. $\beta = 1$.

When $\beta = 1$, we have from (5. 6)

$$U_o = \frac{B_o}{(\rho\mu)^{1/2}} . \tag{5.60}$$

In this special case the equations (5. 40) and (5. 41) take the form

$$\left(\frac{d^2}{dy^2} - 1\right)(f - g) = 0, \tag{5.61}$$

$$\left(\frac{d^2}{dy^2} - 1\right)g = iR(g - f). \tag{5.62}$$

The independent solutions, vanishing at infinity, are now exp - y and y exp -y. For the antisymmetric case the remaining conditions are (5. 42) and (5. 43). Herewith we obtain

$$f = -\varepsilon\lambda \left[1 + \frac{yR}{R-2i}\right] \exp -y, \tag{5.63}$$

$$g = -\frac{\varepsilon\lambda R}{R-2i}\left[1 + y\right] \exp -y. \tag{5.64}$$

Inserting these results in (5. 38) and (5. 39), yields for ψ and A

$$\psi = - \varepsilon\lambda \left[1 + \frac{yR}{R-2i}\right] \exp\{ i(x+iy)\}, \tag{5.65}$$

$$A = - \frac{\varepsilon\lambda R}{R-2i} \left[1 + y\right] \exp\{ i(x+iy)\}. \tag{5.66}$$

Obviously the disturbances produced, are of order $\varepsilon\lambda$ for all values of R. There is no boundary layer, for the current densities and vortices occupy the whole region where the disturbances are nonzero.
It is worth while to remark that (5.63) and (5.64) can be directly obtained from (5. 46) and (5. 47) by taking the limit $\beta \to 1$.
With help of (5. 38) and (5. 39) we obtain for f and g, associated with (5. 46) and (5. 47)

$$f = - \frac{\varepsilon\lambda\alpha}{\alpha-\beta^2} \exp -y + \frac{\varepsilon\lambda\beta^2}{\alpha-\beta^2} \exp -\alpha y, \tag{5.67}$$

$$g = - \frac{\varepsilon\lambda\alpha}{\alpha-\beta^2} \exp -y + \frac{\varepsilon\lambda}{\alpha-\beta^2} \exp -\alpha y. \tag{5.68}$$

From the definition of α, (5. 45), we have to the first order in $(1-\beta^2)$

$$\alpha = 1 + \frac{iR}{2} (1 - \beta^2). \tag{5.69}$$

Using (5. 69) we obtain for the first term in (5. 67) to the first order in $(1 - \beta^2)$

$$\frac{-\varepsilon\lambda}{(1+\frac{iR}{2})(1-\beta^2)} \left[1 + \frac{iR}{2} (1 - \beta^2)\right] \exp -y, \tag{5.70}$$

and for the second term

$$\frac{\varepsilon\lambda\beta^2}{(1+\frac{iR}{2})(1-\beta^2)} \left[1 - \frac{iRy}{2} (1 - \beta^2)\right] \exp -y. \tag{5.71}$$

It is easily seen, that upon addition of (5. 70) and (5. 71), the factors $(1-\beta^2)$ cancel. Putting $\beta = 1$ in the remaining expression, gives again (5. 63).
The same procedure, applied to (5. 68), leads to (5. 64). We see here, that (5.67) and (5.68) are well-defined for $\beta = 1$, although at first sight one might be under the impression that difficulties arise in this case.
For the symmetric case we must find solutions of (5. 61) and (5. 62), vanishing at infinity and satisfying (5. 42) and (5. 53). We obtain

$$f = - \varepsilon\lambda \left[1 - \frac{yR}{2i} \right] \exp -y, \qquad (5.72)$$

$$g = \frac{\varepsilon\lambda Ry}{2i} \exp -y. \qquad (5.73)$$

From (5.38) and (5.39), we have in this case

$$\psi = - \varepsilon\lambda \left[1 - \frac{yR}{2i} \right] \exp\{ i(x + iy) \}, \qquad (5.74)$$

$$A = \frac{\varepsilon\lambda Ry}{2i} \exp\{ i(x + iy) \}. \qquad (5.75)$$

Again (5.74) and (5.75) can directly be found from (5.54) and (5.55) by expanding in terms of $(1-\beta^2)$ and taking the limit $\beta \to 1$.
As for the disturbances, (5.74) and (5.75) show that the situation is even more serious than in the case $\beta \neq 1$. For there we found that, when R is large, u_x and b_x are of order $\varepsilon\lambda R^{1/2}$ in the vicinity of the plate, whilst from (5.74) and (5.75) it follows that they are of order $\varepsilon\lambda R$ when $\beta = 1$ and R is large. At some larger distance of the plate, u_y and b_y also obtain values of this order.

6. Symmetric configuration with finite thickness.

Upon this interruption for the case $\beta = 1$, we continue the discussion of the symmetric case, starting from the point where we left it at the end of section 4.
In order to separate the currents, we take a finite thickness of the plate into account and therefore locate the upper surface at $y = \delta\lambda + \varepsilon\lambda \exp(ix)$ and the lower surface at $y = - \delta\lambda - \varepsilon\lambda \exp(ix)$.
Then we must also take the magnetic field in the plate into account. This field has no curl and divergence, since we have assumed that the plate is an insulator, and can therefore be described by a vector potential with z-component

$$\widetilde{A} = \exp(ix) \left[k_3 \sinh y + k_4 \cosh y \right], \qquad (5.76)$$

where k_3 and k_4 are constants.
In section 3 we found that the solutions of (5.40) and (5.41) which vanish at infinity, are in the upper half-plane

$$f = k_1 \exp -y + \beta^2 k_2 \exp -\alpha y, \qquad (5.77)$$

$$g = k_1 \exp -y + k_2 \exp -\alpha y. \qquad (5.78)$$

The corresponding functions in the lower half-plane follow from (5.34) and (5.35).

Since the magnetic field must be continuous at both sides of the plate, we conclude on account of (5.35) and (5.76) that $k_4 = 0$.
The remaining unknowns $k_1 - k_3$ are determined by the boundary condition for the flow

$$y = \delta\lambda \ : \ f = -\epsilon\lambda ,$$

and the continuity conditions for the magnetic field

$$y = \delta\lambda \ : \ g = k_3 \sinh \delta\lambda ,$$

$$\frac{dg}{dy} = k_3 \cosh \delta\lambda .$$

We obtain

$$k_1 = -\epsilon\lambda \exp \delta\lambda \left\{ \frac{2-(1-\alpha)(1-\exp -2\delta\lambda)}{2(1-\beta^2)-(1-\alpha)(1-\exp -2\delta\lambda)} \right\} ,$$

$$k_2 = \frac{2\,\epsilon\lambda \exp \alpha\delta\lambda}{2(1-\beta^2)-(1-\alpha)(1-\exp -2\delta\lambda)} ,$$

$$k_3 = \frac{-\epsilon\lambda(1-\alpha)}{(\beta^2-1)\cosh \delta\lambda + (\beta^2-\alpha)\sinh \delta\lambda} .$$

For large R, i.e. large α, this set reduces to

$$k_1 = -\epsilon\lambda \exp \delta\lambda ,$$

$$k_2 = \frac{2\epsilon\lambda \exp \alpha\delta\lambda}{\alpha(1-\exp -2\delta\lambda)} , \tag{5.79}$$

$$k_3 = -\frac{\epsilon\lambda}{\sinh \delta\lambda} .$$

Inserting (5.79) into (5.78), gives on account of (5.38) and (5.39)

$$\psi = -\epsilon\lambda \left\{ \exp(\delta\lambda - y) - \frac{2\beta^2 \exp \alpha(\delta\lambda - y)}{\alpha(1-\exp -2\delta\lambda)} \right\} \exp(ix), \tag{5.80}$$

$$A = -\epsilon\lambda \left\{ \exp(\delta\lambda - y) - \frac{2\exp \alpha(\delta\lambda - y)}{\alpha(1-\exp -2\delta\lambda)} \right\} \exp(ix). \tag{5.81}$$

From (5.76) and the value of k_3, given in (5.79), we obtain for $\widetilde{A}$, remembering that $k_4 = 0$:

$$\widetilde{A} = \frac{-\epsilon\lambda \sinh y \exp(ix)}{\sinh \delta\lambda} . \tag{5.82}$$

From (5.80) and (5.81) it follows, that outside a layer $0 < y - \delta\lambda < \frac{1}{\alpha}$, the stream function is that of the nonmagnetic potential flow and the magnetic field is given by

$$A = \psi. \tag{5.83}$$

Hence the field and velocity vectors are parallel in this region.
Inside the boundary layer we have from (5.80) and (5.81), neglecting terms of order $\frac{\varepsilon\lambda}{\alpha}$.

$$u_y = i\varepsilon\lambda \exp(\delta\lambda - y)\exp(ix), \tag{5.84}$$

$$u_x = \varepsilon\lambda \left[\exp(\delta\lambda - y) - \frac{2\beta^2 \exp\alpha(\delta\lambda - y)}{1 - \exp -2\delta\lambda}\right]\exp(ix), \tag{5.85}$$

$$b_y = i\varepsilon\lambda \exp(\delta\lambda - y)\exp(ix), \tag{5.86}$$

$$b_x = \varepsilon\lambda \left[\exp(\delta\lambda - y) - \frac{2\exp\alpha(\delta\lambda - y)}{1 - \exp -2\delta\lambda}\right]\exp(ix). \tag{5.87}$$

These expressions show that u_y and b_y in the boundary layer have the same ratio as they have outside. For the x-components this is not true, so that in the boundary layer the magnetic induction vector is not parallel to the velocity vector. Further it follows from (5.82), (5.84) and (5.86) that u_y and b_y are of order $\varepsilon\lambda$ everywhere. Outside the boundary layer u_x and b_x are also of this order of magnitude, but they have larger values near the plate and, so far as b_x is concerned, in the plate. In particular we have for $y = \delta\lambda$ from (5.82) or (5.87)

$$b_x = \varepsilon\lambda \frac{\cosh\delta\lambda}{\sinh\delta\lambda}\exp(ix).$$

Assuming $\delta\lambda < 1$, we can approximate this with

$$b_x = -\frac{\varepsilon}{\delta}\exp(ix) \tag{5.88}$$

This result shows that the linearization is valid when $\frac{\varepsilon}{\delta}$ is small with respect to one. For thin airfoils this ratio is of the order one rather than small, so in that case we may expect x-disturbances of flow and magnetic field, which are near the plate comparable in magnitude with the undisturbed fields.
Stewartson [25], considering the motion of a perfect conducting fluid along thin bodies, the undisturbed fields being parallel, came to the same result with respect to the validity of the linearization. His arguments, different from ours', are cited here.

Consider a symmetric thin body, given by

$$|y| = \theta S(x).$$

θ is a small dimensionless parameter, $S(x)$ determines the surface of the thin body.
On the body we require to the first order in θ

$$u_y = \theta \, U_0 \, \frac{dS(x)}{dx} \, \mathrm{sgn}\, y. \tag{5.89}$$

Taking the curl of Ohm's law (5. 7) for a steady motion leads to

$$\nabla \times (\mathbf{v} \times \mathbf{B}) = \frac{1}{R} \nabla^2 \mathbf{B}. \tag{5.90}$$

When R is allowed to become infinite, the right-hand side of (5. 90) vanishes, which leads to the conclusion that $\mathbf{v}$ and $\mathbf{B}$ are parallel everywhere.
Then we have at the body, on account of (5. 89):

$$b_y = \theta \, B_0 \, \frac{dS(x)}{dx} \, \mathrm{sgn}\, y. \tag{5.91}$$

In the body b_y = const. y in a first approximation.
The constant can be found by integration of $\frac{\partial b_y}{\partial y}$ through the body from lower side to upper side, resulting in

$$b_y = B_0 y \, \frac{dS(x)}{dx} \, / \, S(x).$$

Then $$b_x = - \int \frac{\partial b_y}{\partial y} \, dx = - B_0 \ln S(x) + \text{const},$$

which means that disturbances of order B_0 can be expected.
The results of our more close study of the similar problem in this section, showed that the fields are parallel outside the boundary layer, but not inside. It is easily verified, using (5. 81), that the right-hand side of (5. 90) does not vanish uniformly, but has a finite value in the boundary layer.
Nevertheless (5. 91) holds, since, as we showed previously in this section the y components have the same ratio everywhere, when the conductivity is sufficiently large.
Therefore Stewartson's qualitative result is confirmed by the present investigation.

7. Evacuated lower half-plane.

In order to compare results, we calculated also the stream function and vector potential for the case treated by Sears and Resler, where there is unbounded vacuum for $y < 0$.
Then the magnetic field in the lower half-plane is harmonic and since it must vanish for $y \to -\infty$, it can be represented by

$$\overline{A} = \epsilon\lambda k_5 \exp\{ i(x-iy) \}, \qquad (5.92)$$

k_5 being a constant.
The disturbances in the upper half-plane are again given by (5.77) and (5.78). The constants k_1, k_2 and k_5 are determined by

$$y = 0 \; : \; f = -\epsilon\lambda,$$
$$g = \epsilon\lambda k_5,$$
$$\frac{dg}{dy} = \epsilon\lambda k_5 .$$

We obtain

$$k_1 = -\frac{\epsilon\lambda(1+\alpha)}{1+\alpha-2\beta^2}, \qquad (5.93)$$

$$k_2 = \frac{2\epsilon\lambda}{1+\alpha-2\beta^2}, \qquad (5.94)$$

$$k_5 = \frac{(1-\alpha)}{1+\alpha-2\beta^2}. \qquad (5.95)$$

Substitution of (5.93) and (5.94) in (5.77) and (5.78), yields with help of (5.38) and (5.39)

$$\psi = \frac{\epsilon\lambda}{1+\alpha-2\beta^2}\left[2\beta^2 \exp -\alpha y - (1+\alpha)\exp -y \right] \exp(ix), \qquad (5.96)$$

$$A = \frac{\epsilon\lambda}{1+\alpha-2\beta^2}\left[2 \exp -\alpha y - (1+\alpha)\exp -y \right] \exp(ix). \qquad (5.97)$$

(5.92) together with (5.95) yields for the vector potential in the plate

$$\overline{A} = \frac{\epsilon\lambda(1-\alpha)}{1+\alpha-2\beta^2} \exp\{ i(x-iy) \}. \qquad (5.98)$$

The disturbances, following from (5.96) and (5.97) are for large α, in the fluid

$$u_x = \epsilon\lambda \, [\, \exp -y - 2\beta^2 \exp -\alpha y \,] \exp(ix),$$
$$u_y = i\epsilon\lambda \, \exp\{ i(x + iy) \},$$
$$b_x = \epsilon\lambda \, [\, \exp -y - 2 \exp -\alpha y \,] \exp(ix),$$
$$b_y = i\epsilon\lambda \, \exp\{ i(x + iy) \},$$

and in the lower half-plane

$$\bar{b}_x = - \epsilon\lambda \cdot \exp\{ i(x - iy) \},$$
$$\bar{b}_y = i\epsilon\lambda \, \exp\{ i(x - iy) \}.$$

Here again as in section 3, the disturbances remain of order $\epsilon\lambda$, when the conductivity is large. Linearization therefore is not suspect here . The above expressions differ from those obtained by Sears and Resler by the terms containing α, because these authors put R equal to ∞ à priori and got no rotational parts of ψ and A. Then a discontinuity of b_x at the boundary occurs. In order to account for this discontinuity Sears and Resler introduced a surface current,

$$J = - 2\epsilon\lambda \exp(ix). \qquad (5.99)$$

Now from (5.97), we have

$$j_z = - \nabla^2 A = - 2\alpha\epsilon\lambda \exp -\alpha y \exp(ix),$$

which results in a total current

$$\int_0^\infty j_z \, dy = - 2\epsilon\lambda \exp(ix).$$

This current has the value (5.99). Therefore the solution obtained by Sears and Resler is indeed the solution, valid when the conductivity is infinite.

CHAPTER VI

WAVY PLATE; ONE-FLUID MODEL; UNSTEADY MOTION

1. Introduction; Discussion of the initial conditions.

We have already mentioned Stewartson's paper [25] on the motion of a thin body through a perfect conducting fluid in the presence of a magnetic field, which is in the undisturbed state parallel to the velocity. In this paper the author develops a qualitative picture of the flow and conjectures that, when the perturbations are assumed to be small, the motion, set up from undisturbed conditions, cannot become ultimately steady. When a steady motion is assumed, the perturbations are no longer small. Although our investigation pertains to the somewhat artificial concept of an infinite plate, it certainly can be useful for the study of finite bodies. In the preceding chapter we showed that, when the body is symmetric and β^2 not too large, the breakdown of the linearization at large conductivity is due to the small thickness.
The work of Stewartson stimulated the present author to investigate, whether steady solutions as obtained in the foregoing chapter, can be realized, when the motion starts from undisturbed conditions.
The type of unsteady motion we choose for this purpose is an impulsive motion. Well-known in hydrodynamics is the Rayleigh problem. Rayleigh considered an infinite flat plate immersed in an incompressible viscous fluid. The plate is at time $t = 0$ set impulsively into motion with a velocity U in its own plane. For $t > 0$ this velocity remains constant. Rayleigh showed that the velocity in the fluid, which is parallel to U, is given by

$$U \operatorname{erfc}\left(\frac{y}{2\sqrt{\nu t}}\right), \tag{6.1}$$

where y is the distance from the plate and ν the kinematic viscosity. With erfc(x) the complementary error function $\frac{2}{\sqrt{\pi}}\int_x^\infty e^{-v^2}\,dv$ is denoted.

(6.1) shows how viscous effects diffuse outwards. This type of motion has been extended to compressible flow also. A survey is given recently by Stewartson [26].
In our case the influence of viscosity is neglected, but diffusion takes place through the action of the magnetic viscosity $\frac{1}{\mu\sigma}$.
In analogy to the Rayleigh problem, we shall consider the following

problem. A wavy plate of the type considered in chapter V moves when $t < 0$ with a constant velocity U_0 parallel to its own plane in a conducting fluid which has the same velocity. A magnetic field B_0 is present parallel to U_0. At time $t = 0$, the plate is suddenly brought to rest. We ask for the subsequent behaviour of fluid and magnetic field. We are in particular interested in the behaviour for $t \to \infty$, when the conductivity of the fluid is large. The solution for this type of impulsive motion will be obtained by means of Laplace transformations. The available methods to obtain asymptotic solutions, valid when a certain time has elapsed, will enable us to find expressions, representing the ultimate behaviour of fluid and magnetic field.
Before dealing with the magnetic case, we treat the case where there is no magnetic field, but where compressibility is taken into account. The use of this problem for our investigation is two-fold. First it enables us to determine the initial conditions for the magnetic problem and further it gives opportunity to expose the way in which solutions with help of Laplace transformations can be obtained.
When there is no magnetic field, the flows in upper and lower half-plane are independent, so we shall deal with the upper half-plane only.
Using the same variables as in the preceding chapter and taking in addition $\rho^* = \rho_0 \rho$, where ρ_0 is the density of the undisturbed gas, the linearized continuity equation is

$$\frac{\partial \rho}{\partial t} + \nabla . \boldsymbol{u} + \frac{\partial \rho}{\partial x} = 0.$$

The momentum equation is in the acoustic approximation

$$\frac{\partial \boldsymbol{u}}{\partial t} + \frac{\partial \boldsymbol{u}}{\partial x} = - \frac{1}{M^2} \nabla \rho ,$$

where $M = \dfrac{U_0}{a_0}$. a_0 is the velocity of sound in the undisturbed gas and M is here the well-known Mach number. The undisturbed motion is irrotational. Hence the flow remains free of vorticity and we write therefore:

$\boldsymbol{u} = \nabla \Phi$, where Φ is the velocity potential.

Then we eliminate ρ, obtaining

$$\frac{\partial^2 \Phi}{\partial t^2} + 2 \frac{\partial^2 \Phi}{\partial x \partial t} + \frac{\partial^2 \Phi}{\partial x^2} - \frac{1}{M^2} \nabla^2 \Phi = 0. \tag{6.2}$$

The boundary conditions are

$$\begin{matrix} y = 0 \\ t > 0 \end{matrix} \quad : \frac{\partial \Phi}{\partial y} = u_y = i \varepsilon \lambda \exp(i x), \tag{6.3}$$

$$y \geqslant 0 \;:\; \Phi = \frac{\partial \Phi}{\partial t} = 0, \qquad t = 0 \tag{6.4}$$

$$y \to \infty \;:\; \Phi = 0. \qquad t \neq \infty \tag{6.5}$$

The last condition amounts to the requirement that disturbances originate from the wall only and not from infinity.
Now we separate the variables by writing

$$\Phi = \varphi(y,t)\exp(ix), \tag{6.6}$$

and introduce the Laplace transform

$$\overline{\varphi} = \int_0^\infty e^{-st}\,\varphi(y,t)\,dt. \tag{6.7}$$

In (6.7) s is a positive variable. We introduce (6.6) and (6.7) in (6.2) and transform the equation (6.2) with help of (6.4).*
The transformed equation becomes

$$\frac{1}{M^2}\frac{d^2\overline{\varphi}}{dy^2} - \overline{\varphi}\left\{(s+i)^2 + \frac{1}{M^2}\right\} = 0.$$

The solution satisfying (6.5) is

$\overline{\varphi} = a(s)\exp -yM\sqrt{\left\{(s+i)^2 + \frac{1}{M^2}\right\}}$, where a(s) is a function of s alone.

From transformation of (6.3), we obtain for a

$$a = -\frac{i\epsilon\lambda}{Ms\sqrt{\left\{(s+i)^2 + \frac{1}{M^2}\right\}}}.$$

Hence

$$\overline{\varphi} = -\frac{i\epsilon\lambda}{M}\,\frac{\exp -yM\sqrt{\left\{(s+i)^2 + \frac{1}{M^2}\right\}}}{s\sqrt{\left\{(s+i)^2 + \frac{1}{M^2}\right\}}} \tag{6.8}$$

Then we have from the Inversion Theorem*

* For the general theory of Laplace transformations, see Carslaw and Jaeger[27]) or Mc Lachlan[28]).

$$\varphi = - \frac{i \varepsilon \lambda}{2 \pi i} \int_{c-i\infty}^{c+i\infty} \frac{\exp \left[st - yM \sqrt{\{(s+i)^2 + \frac{1}{M^2}\}} \right]}{s \sqrt{\{(s+i)^2 + \frac{1}{M^2}\}}} ds. \qquad (6.9)$$

In (6.9) s is complex and the integration is to be performed along a straight line parallel to the imaginary axis at the right of the singularities of the integrand. It is noteworthy that this problem is from a mathematical point of view the same as a problem in the theory of electric transmission lines.

In references 27) and 28) the problem of a uniform semi-infinite transmission line with zero initial current and potential is considered. When at $t = 0$ a constant e.m.f. is applied, the potential for $t > 0$ is represented by an integral of the type (6.9).

The singularities of the integrand in (6.9) are a pole in $s = 0$ and branch points in $s = - i \pm \frac{i}{M}$.

As is usual in evaluating integrals following from the Inversion Theorem, we deform the path of integration in a closed contour by addition of a large semi-circle to the straight path, introducing suitable barriers from the branch points to infinity.

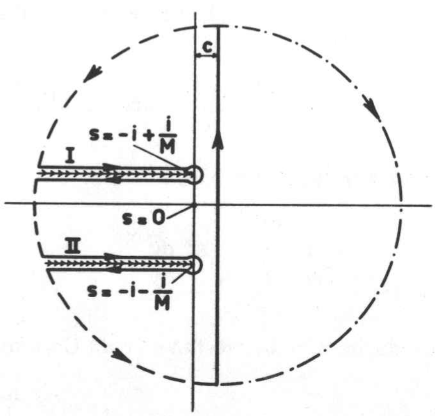

Fig. 6

Contourintegration in complex s plane.

In fig. 6 two possible contours are sketched. Now it follows from (6.9) that the integral along the large semi-circle in the right half-plane tends to zero, when the radius tends to ∞, provided $t < My$ or (from (5.3)) $y^* > a_o t^*$.
Since the integrand in (6.9) is analytic in the right half-plane, it follows from Cauchy's theorem that the integral along the closed contour in the right half-plane is zero. Hence Φ is zero as long as $t^* < \frac{y^*}{a_o}$.
When $t > yM$, the integral along the semi-circle in the left half-plane vanishes.
Then, applying Cauchy's theorem, we find, due to the singularities, a nonzero Φ. The physical picture is that a wave front propagates with the velocity of sound from the plate into the gas. At a point y^* the gas is at rest till it is reached by the wave.
From reference 27) p. 200 we obtain for $t > yM$

$$\varphi = - i\varepsilon\lambda M \int_{yM}^{t} e^{-i\tau} J_o \left\{ \frac{1}{M} \sqrt{(\tau^2 - y^2M^2)} \right\} d\tau,$$

where J_o is the Bessel function of zeroth order.

For further reference it is of interest to consider the case where $a_o \to \infty$ and hence $M \to 0$, and to look at the ultimate behaviour of the flow. The Laplace transform of u_y is simpler than that of φ and therefore we consider

$$\frac{d\bar{\varphi}}{dy} = \frac{i\varepsilon\lambda \exp -yM \sqrt{\{(s+i)^2 + {}^1/M^2\}}}{s}. \qquad (6.10)$$

We ask for the value for large time of

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} \frac{d\bar{\varphi}}{dy} ds. \qquad (6.11)$$

Evaluating the residu in $s = 0$, we have from Cauchy's theorem

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} = i\varepsilon\lambda \exp\{-y\sqrt{(1-M^2)}\} - \frac{1}{2\pi i}\int_{I} - \frac{1}{2\pi i}\int_{II}. \qquad (6.12)$$

The transient part is given by the integrals around the cuts I and II in fig. 6.

To evaluate the integral along I we shift the origin to $s = i\left(\frac{1}{M} - 1\right)$ and write

$$s = \frac{i}{M} - i + \zeta .$$

Then, neglecting unity with respect to $\frac{1}{M}$, we have from (6.10) and (6.11)

$$\frac{1}{2\pi i}\int_{I} = \frac{\epsilon\lambda\, e^{it/M}}{2\pi}\int_{I}\frac{e^{\zeta t - yM\sqrt{\frac{2i\zeta}{M}\left(1-\frac{iM\zeta}{2}\right)}}}{1/M + \zeta}\, d\zeta .$$

When t is large, only the neighbourhood of $\zeta = 0$ gives a contribution. Expanding the integrand, we obtain for the first term that gives a contribution

$$-\frac{yM^2}{i}\sqrt{\frac{2i}{M}}\,\zeta^{1/2}\, e^{\zeta t} .$$

Therefore we calculate $\frac{1}{2\pi i}\int_{I} e^{\zeta t}\,\zeta^{1/2}\, d\zeta$.

Along the upper side of the cut I we have $\zeta = xe^{i\pi}$, along the lower side $\zeta = xe^{-i\pi}$. On the upper side x runs from ∞ to 0, on the lower side from 0 to ∞.

Together we obtain

$$\frac{1}{\pi}\int_0^\infty e^{-xt}\, x^{1/2}\, dx = \frac{1}{2\sqrt{\pi}}\, t^{-3/2} .$$

Taking all factors into account we obtain for the leading term in the asymptotic expansion of the integral along I

$$\frac{2^{1/2}\,\epsilon\lambda\, e^{it/M + i\pi/4}\, y\, M^{3/2}}{2it^{3/2}\sqrt{\pi}} .$$

(The integral along the small circle around $\zeta = 0$ gives no contribution when the radius tends to zero).
To obtain the result of the integral along the path II, we change i into -i in this expression. Then addition of the two results gives for the transient part

$$\epsilon\lambda\,\sin(t/M + \pi/4)\; y\left(\frac{2M^3}{\pi t^3}\right)^{1/2} . \qquad (6.13)$$

The next term in the expansion appears to be proportional to $\frac{yM^{5/2}}{t^{5/2}}$,
so that for sufficiently large t the transient part is adequately described by (6. 13).
From (6. 12) and (6. 13) it follows that, when $a_o \to \infty$ and hence $M \to 0$, u_y is, for any finite y and t, given by the residu in (6. 12), which gives with help of (6. 6)

$$u_y = i\varepsilon\lambda \exp i(x+iy).$$

The stream function connected herewith is

$$\psi_o = -\varepsilon\lambda \exp i(x+iy). \tag{6.14}$$

In the actual problem we want to investigate, there is a magnetic field present. However the velocities involved in the interaction between the flow and the magnetic field, are the Alfvén-velocity given by (5. 60) and the diffusion velocity arising from the magnetic viscosity $\frac{1}{\mu\sigma}$. Both these velocities are finite.

Since in our approximation, where the effects of pressure variations on the density have been neglected, the velocity of sound can be considered as infinite, we assume that at time $t = 0^+$ the nonmagnetic flow is already present, but the magnetic field is still undisturbed.
Therefore the appropriate initial conditions, to apply in the following, are to require that at $t = 0^+$ the streamfunction is given by (6. 14) and that the vector potential is zero.
Inspection of Ohm's law (5. 7) shows that at $t = 0^+$ an electric field is present, given by the y-component of the nonmagnetic velocity. The velocity involved in the establishment of this electric field is the velocity of light, so there is a short period in which the displacement current cannot be neglected.
The velocity of light however is still larger than the velocity of sound and therefore in the present approximation ought to be considered as infinite too. Then it is consistent to neglect the displacement current and assume that at $t = 0^+$ the electric field, mentioned above, suddenly arises.

2. General solution in terms of Laplace transforms.

The initial conditions given in the preceding section are chosen already in chapter V without further reference, (cf. (5. 21) and (5. 22)). Hence we can use the results of the symmetry considerations, given in chapter V, and therefore have only to deal with the upper half-plane. In view of the fact that at $t = 0^+$ the stream function is given by (6. 14), we shall denote the actual stream function with $\hat{\psi}$ and the difference

between $\hat{\psi}$ and ψ_0 with ψ.
Hence

$$\hat{\psi} = \psi - \varepsilon\lambda \exp(ix)\exp -y. \tag{6.15}$$

Introducing this, the equations (5.16) and (5.17) become

$$\nabla^2 \left\{ \frac{\partial\psi}{\partial t} + \frac{\partial\psi}{\partial x} - \beta^2 \frac{\partial A}{\partial x} \right\} = 0, \tag{6.16}$$

$$\frac{\partial A}{\partial t} + \frac{\partial A}{\partial x} - \frac{\partial\psi}{\partial x} - \frac{1}{R}\nabla^2 A + i\varepsilon\lambda \exp(ix)\exp -y = 0. \tag{6.17}$$

At $y = 0$ we must have $u_y = -\frac{\partial\hat{\psi}}{\partial x} = i\varepsilon\lambda \exp(ix)$.
Then from (6.15) we must require for all t

$$y = 0 \quad : \quad \psi = 0. \tag{6.18}$$

The other conditions for ψ and A, independent of the situation in the lower half-plane, are

$$\begin{array}{ll} y \to \infty & : \ \psi = 0, \\ t \neq \infty & \end{array} \tag{6.19}$$

$$A = 0. \tag{6.20}$$

$$t = 0^+ : \ \psi = 0, \tag{6.21}$$

$$A = 0. \tag{6.22}$$

With help of (5.38) and (5.39), f and g here being functions of y and t, (6.16) yields upon integration

$$\frac{\partial f}{\partial t} + if - i\beta^2 g - K(t)\exp -y = 0, \tag{6.23}$$

where K(t) is a still unknown function of t, while (6.17) becomes in terms of f and g

$$\frac{\partial g}{\partial t} + ig - if - \frac{1}{R}\left(\frac{\partial^2 g}{\partial y^2} - g\right) - i\varepsilon\lambda \exp -y = 0. \tag{6.24}$$

Let now the Laplace transforms of f, g, and K be F, G and $\overline{K}$.
Then (6.23) and (6.24) become upon transformation and with help of (6.21) and (6.22)

$$(s+i)F - i\beta^2 G - \overline{K}\exp -y = 0, \tag{6.25}$$

$$(s+i)G - iF - \frac{1}{R}\left(\frac{d^2G}{dy^2} - G\right) + \frac{i\varepsilon\lambda \exp -y}{s} = 0. \tag{6.26}$$

Elimination of F between (6.25) and (6.26), gives for G

$$(s+i)\frac{d^2G}{dy^2} - G\{R(s+i)^2 + (s+i) + R\beta^2\} + \frac{iR}{s}\{s\bar{K}(s) - (s+i)\varepsilon\lambda \exp -y\} = 0. \tag{6.27}$$

The general solution of (6. 27), satisfaying (6. 20), is

$$G = i\,\frac{s\bar{K}(s) - (s+i)\varepsilon\lambda}{s\{\beta^2 + (s+i)^2\}} \exp -y + M(s) \exp -yC. \tag{6.28}$$

Here M(s) is a function of s alone and C is defined by

$$C = \left[\frac{R(s+i)^2 + (s+i) + R\beta^2}{s+i}\right]^{1/2}. \tag{6.29}$$

When G is known, F can be found from (6. 25).
From (6. 18) we deduce that at $y = 0$, F must be zero.
This means for G, using (6. 25)

$$y = 0 \;:\; G = \frac{i\bar{K}(s)}{\beta^2}. \tag{6.30}$$

Another condition for G, necessary to determine $\bar{K}(s)$ and M(s) follows from the results of the symmetry conditions given in chapter V.
In the following we shall treat first the antisymmetric case without thickness and subsequently the symmetric case with thickness.

3. Antisymmetric case.

Here the additional relation for the magnetic field is the condition (5. 43), which is in terms of G

$$y = 0 \;:\; \frac{dG}{dy} = 0. \tag{6.31}$$

From (6. 28), (6. 30) and (6. 31), we obtain

$$G = \frac{i\varepsilon\lambda(s+i)}{s\{\beta^2 + C(s+i)^2\}}\left[\exp -Cy - C\exp -y\right]. \tag{6.32}$$

Then from (6. 25) and (6. 32) we get

$$F = \frac{\varepsilon\lambda\beta^2}{s\{\beta^2 + C(s+i)^2\}}\left[\exp -y - \exp -Cy\right]. \tag{6.33}$$

This solution for F satisfies the condition (6. 19).
Obviously F and G behave similarly. Therefore we shall restrict ourselves to the magnetic field and try to deduce the behaviour of the magnetic field, when t is large, from (6.32) and the Inversion Theorem. We shall assume that the electrical conductivity and hence R is large .

From the Inversion Theorem it follows that we must evaluate

$$\frac{1}{2\pi i}\int e^{st}\, G(y,s)\,ds, \tag{6.34}$$

for large t.
The path of integration in (6. 34) is a straight line in the complex s plane to the right of the singularities of G.
It is convenient to shift the origin to $s = -i$. Therefore we put $s = -i + z$. Then (6. 32), (6. 29) and (6. 34) become

$$G = \frac{i\epsilon\lambda z}{(z-i)(\beta^2 + Cz^2)}\left[\exp -Cy - C\exp -y\right], \tag{6.35}$$

$$C = \left[\frac{Rz^2 + z + R\beta^2}{z}\right]^{1/2}, \tag{6.36}$$

$$g = \frac{1}{2\pi i}\, e^{-it}\int e^{zt}\, G(y,z)\,dz. \tag{6.37}$$

G has a simple pole in $z = i$ and when R is large, it follows from (6.35) and (6. 36) that the other singularities are in the vicinity of $z = 0$ and near the points where $C = 0$. When $z \to 0$, G behaves on account of (6. 35) and (6. 36) as

$$G_1 = \frac{\epsilon\lambda z}{\beta^2 + R^{1/2} z^{3/2}\beta}\left[z^{-1/2} R^{1/2}\beta\exp -y - \exp(-yR^{1/2}\beta z^{-1/2})\right]. \tag{6.38}$$

The singularities of (6.38) are: a branchpoint in $z = 0$ and simple poles in

$$z = \frac{\beta^{2/3}}{2R^{1/3}}(-1 \pm i\sqrt{3}), \tag{6.39}$$

which are indeed near $z = 0$ when R is large.
C vanishes when

$$z = -\frac{1}{2R} \pm i\left[\beta^2 - \frac{1}{4R^2}\right]^{1/2}. \tag{6.40}$$

These points are associated with hydromagnetic or Alfvén waves.
In a perfect conducting fluid, hydromagnetic waves would travel with the Alfvén velocity $\frac{B_0}{(\rho\mu)^{1/2}}$, upstream and downstream with respect to the undisturbed flow. When there is a finite conductivity the waves are damped and the propagation speed is modified.
When R is large we can replace the square root in (6. 40) by

$$\beta\,(1 - \frac{1}{8\beta^2 R^2}).$$

We shall assume that R is large enough to allow the neglect of $(\beta R)^{-2}$ with respect to one and shall only take the damping into account. Therefore we situate the relevant branch points in

$$z = -\frac{1}{2R} \pm i\beta. \tag{6.41}$$

The real part of these branchpoints is larger than that of (6.39), so that we shall ignore the poles (6.39) in calculating the asymptotic behaviour.

Now we close the path of integration in the complex z plane as shown in fig. 7.

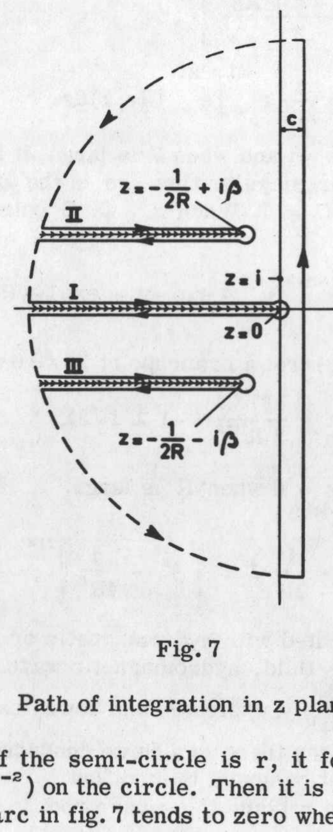

Fig. 7

Path of integration in z plane.

When the radius of the semi-circle is r, it follows from (6.35) and (6.36) that $G = 0(r^{-2})$ on the circle. Then it is easily verified that the integral along the arc in fig. 7 tends to zero when $r \to \infty$.

Therefore, applying Cauchy's theorem, we find that (6. 37) equals the residu in $z = i$ minus the contribution of the contours I, II and III. The residu in $z = i$ is, when $\beta \neq 1$,

$$\frac{\varepsilon\lambda}{\{1+iR(1-\beta^2)\}^{1/2}-\beta^2}\left[\exp -y\{1+iR(1-\beta^2)\}^{1/2}-\{1+iR(1-\beta^2)\}^{1/2}\exp -y\right], \qquad (6.42)$$

and

$$\frac{\varepsilon\lambda R}{2i-R}\left[y\exp -y + \exp -y\right], \qquad (6.43)$$

when $\beta = 1$.

These expressions give the steady solutions found in chapter V, sections 3 and 5, (cf (5. 47) and (5. 64)).

The transient part of the magnetic field is

$$-\frac{1}{2\pi i}\exp(-it)\left[\int_{I} e^{zt}\, G(y,z)dz + \int_{II} e^{zt}\, G(y,z)dz + \int_{III} e^{zt}\, G(y,z)dz\right]. \qquad (6.44)$$

The dominant contribution to the asymptotic value of the time-dependent part, comes from the singularity with largest real part. In order however to obtain some knowledge about the contribution of the Alfvén-waves, we also take the branchpoints (6. 41) into account.

Let us first consider the integral along the contour I.

When t is large, only the neighbourhood of $z = 0$ tributes to the first integral in (6. 44).

Therefore we replace in the integrand G by (6. 38) and consider

$$g_1 = -\frac{e^{-it}}{2\pi i}\int_{I} G_1\,(y,z)\, e^{zt}\, dz.$$

Using (6. 38) we write

$$g_1 = g_{1a} + g_{1b} = -\frac{\exp -it}{2\pi i}\int_{I}\frac{\varepsilon\lambda R^{1/2} z^{1/2}\beta\, e^{zt-y}}{\beta^2 + R^{1/2}z^{3/2}\beta}\,dz + \frac{\exp -it}{2\pi i}\int_{I}\frac{\varepsilon\lambda z\, e^{zt-yR^{1/2}\beta z^{-1/2}}}{\beta^2 + R^{1/2}z^{3/2}\beta}\,dz. \qquad (6.45)$$

An integral of the type g_{1a} is discussed by Mc Lachlan[28] (p. 113 and seq.). From this reference we obtain that for large t

$$g_{1a} \simeq \varepsilon\lambda\,\left(\frac{R}{4\pi\beta^2 t^3}\right)^{1/2}\exp -(y + it). \qquad (6.46)$$

In obtaining this expression the poles (6. 39) are ignored.
The contribution g_{1b} is small with respect to g_{1a} .

Proof:

Along the upper side of the contour I we have $z = x e^{i\pi}$, and thus the integral along this side is

$$\frac{\varepsilon \lambda \exp -it}{2\pi i \beta^2} \int_{\infty}^{0} \frac{x\, e^{(iyR^{1/2}\beta x^{-1/2} - xt)}}{1 - i \frac{R^{1/2}}{\beta} x^{3/2}} dx.$$

Likewise we obtain for the lower side, where $z = x e^{-i\pi}$

$$\frac{\varepsilon \lambda \exp -it}{2\pi i \beta^2} \int_{0}^{\infty} \frac{x\, e^{(-iyR^{1/2}\beta x^{-1/2} - xt)}}{1 + i \frac{R^{1/2}}{\beta} x^{3/2}} dx.$$

Together

$$\frac{\varepsilon \lambda \exp -it}{\pi \beta^2} \left[\int_{0}^{\infty} e^{-xt} \frac{\frac{R^{1/2}}{\beta} x^{5/2} \cos(yR^{1/2}\beta x^{-1/2})}{1 + \frac{R}{\beta^2} x^3} dx + \right.$$

$$\left. - \int_{0}^{\infty} e^{-xt} \frac{x \sin(yR^{1/2}\beta x^{-1/2})}{1 + \frac{R}{\beta^2} x^3} dx \right].$$

The first integral in [ ] is less than

$$\frac{R^{1/2}}{\beta} \int_{0}^{\infty} e^{-xt} x^{5/2} dx = \frac{15}{8} \sqrt{\frac{\pi R}{t^7 \beta^2}} ,$$

and the second one is less than $\int_{0}^{\infty} e^{-xt} x\, dx = \frac{1}{t^2}$.

On account of (6. 46) both are small with respect to g_{1a} when t is large.
To evaluate the integral along the small circle around the origin, put $z = r e^{i\theta}$.

Then the absolute value of the integral around the origin is

$$\left| \frac{1}{2\pi} \int_{-\pi}^{+\pi} \varepsilon \lambda r^2 \frac{e^{\{-yR^{1/2}\beta r^{-1/2}(\cos\theta/2 - i\sin\theta/2) + rt(\cos\theta + i\sin\theta + 2i\theta)\}}}{\beta^2 + R^{1/2} r^{3/2}(\cos 3\theta/2 + i\sin 3\theta/2)} d\theta \right|$$

$$< \int_{-\pi}^{+\pi} \varepsilon \lambda r^2 \frac{e^{(-yR^{1/2}\beta r^{-1/2}\cos\theta/2 + rt\cos\theta)}}{(\beta^4 + Rr^3 + 2\beta^2 R^{1/2} r^{3/2}\cos 3\theta/2)^{1/2}} d\theta .$$

This integral tends to zero when $r \to 0$.
Hence the main contribution to the transient part associated with the integral along I is given by (6.46).
To obtain the contributions of the branchpoints (6.41), we apply the method used in the introductory example. This method is based on the following theorem in the theory of Laplace transformations (see Carslaw and Jaeger [27] p. 280). Let s_o be the singularity with largest real part of a Laplace transform L(s). Let L(s) possess near $s = s_o$ a series expansion, convergent for $|s-s_o| < \gamma$,

$$\sum_{n=0}^{\infty} a_n (s-s_o)^{n-1} + (s-s_o)^{b-1} \sum_{n=0}^{\infty} d_n (s-s_o)^n, \text{ with } 0 < b < 1. \qquad (6.47)$$

Then the original has for large t the asymptotic expansion

$$e^{s_o t} \left\{ a_o + \frac{\sin \pi b}{\pi} \sum_{n=0}^{\infty} (-1)^n d_n \Gamma(b+n) t^{-b-n} \right\}. \qquad (6.48)$$

When $s = \xi + i\eta$ and $s_o = \xi_o + i\eta_o$, it is required that $L(s) \to o$ for $\eta \to \pm \infty$, uniformly in ξ when $\xi_o - \delta \leqslant \xi \leqslant c$.
Here c is the distance from the path of integration prescribed by the Inversion Theorem (see fig. 8) to the imaginary axis, and δ is smaller than the radius of convergence γ of the series (6.47).

Further it is required that $\int |L(s)| d\eta$ converges for $\eta = \pm \infty$, when $\xi = \xi_o - \delta$.

These conditions are satisfied by the Laplace transforms we have to deal with.

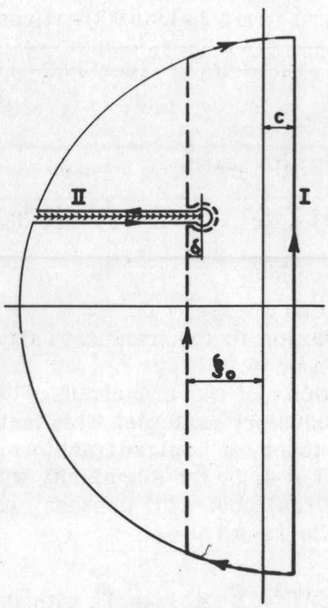

Fig. 8

Deformations of path of integration,
when an asymptotic solution is required.

Another way to arrive at the asymptotic expansion (6.48) is adopted by Mc Lachlan [28]. Let us for convenience assume that s_o is the only singularity, so that, provided that the integral along the large semi-circle vanishes when the radius tends to ∞, the path I in fig. 8 can be replaced by the path II. When $t \to \infty$, then, due to the factor $e^{\xi t}$, the main contribution comes from the neighbourhood of s_o, in which point ξ has its maximum value. When in

$$\int_{II} e^{st} L(s)\, ds,$$

L(s) can be expanded in a series convergent in a neighbourhood of s_o, then term by term integration gives (6.48).
Now it can be shown that both methods are equivalent. For it is clear that when the integral along the semi-circle vanishes and to the integral

along II only the neighbourhood of s_0 contributes, the path II + semicircle can be replaced by the broken path of integration in fig. 8. Such a path is envisaged by Carslaw and Jaeger in discussing (6.48).
When several singularities are present, it is enough to consider the singularity with largest real part. In our problem the singularity with largest real part is $z = 0$, but we take also the branchpoints (6.41) into account, because we want to know of what kind the transient effects, arising from these points, are.

Considering the branchpoint $z = -\frac{1}{2R} + i\beta$, we write

$$z = -\frac{1}{2R} + i\beta + \zeta .$$

Introducing this into (6.35) and (6.36), we expand G in a series, which is convergent for $|\zeta| < \frac{1}{2R}$.

The first term in the expansion is

$$G_2 = -\frac{\varepsilon\lambda}{\beta} \frac{\sqrt{(2R\zeta)}(1-y-\exp -y)}{i(\beta-1) - \frac{1}{2R}} .$$

In deriving this expression we have there, where this is harmless, neglected $\frac{1}{2R}$ with respect to β.
Then by applying (6.48) or by integration of (from (6.44))

$$\frac{-\exp -\left\{\frac{1}{2R} + i(1-\beta)\right\} t}{2\pi i} \int_{II} e^{\zeta t} \; G_2 \, d\zeta,$$

along the path II in fig. 7, we obtain for large t

$$g_2 \simeq \varepsilon\lambda\left(\frac{R}{2\pi\beta^2 t^3}\right)^{1/2} \left[\frac{(1-y-\exp -y)\exp -\left\{\frac{1}{2R} + i(1-\beta)\right\} t}{i(\beta-1) - \frac{1}{2R}}\right]. \quad (6.49)$$

The point $z = -\frac{1}{2R} - i\beta$, yields in the same way

$$g_3 \simeq -\varepsilon\lambda\left(\frac{R}{2\pi\beta^2 t^3}\right)^{1/2} \left[\frac{(1-y-\exp -y)\exp -\left\{\frac{1}{2R} + i(1+\beta)\right\} t}{-i(\beta+1) - \frac{1}{2R}}\right]. \quad (6.50)$$

With help of (5.39), we obtain, taking (6.46), (6.49) and (6.50) together, for the transient part of A

$$A_t \simeq \varepsilon\lambda\left(\frac{R}{4\pi\beta^2 t^3}\right)^{1/2} \exp\{-y+i(x-t)\} + \varepsilon\lambda(1-y-\exp -y)\left(\frac{R}{2\pi\beta^2 t^3}\right)^{1/2} \exp\left(-\frac{t}{2R}\right)\cdot \tag{6.51}$$

$$\left[\frac{\exp i\{x-(1-\beta)t\}}{i(\beta-1)-\frac{1}{2R}} + \frac{\exp i\{x-(1+\beta)t\}}{i(\beta+1)+\frac{1}{2R}}\right].$$

The first term in (6.51) is an irrotational disturbance convected with the fluid. The other terms constitute two systems of Alfvén waves, travelling upstream and downstream with respect to the undisturbed fluid. Eventually A_t vanishes, the time necessary to arrive at the steady state being determined by the quantity $\left(\frac{R}{\beta^2 t^3}\right)^{1/2}$. The Alfvén waves decay faster due to the additional factor $\exp -\frac{t}{2R}$.

In general there are two systems of Alfvén waves. However when the Alfvén velocity equals U_o, i.e. $\beta = 1$, the second term on the right-hand side of (6.51) is no longer periodic in time, and only the downstream wave remains. The other one is aperiodically damped and tends to zero as $\left(\frac{2R^3}{\pi\beta^2 t^3}\right)^{1/2} \exp -\frac{t}{2R}$, which is appreciably slower than in the case $\beta \neq 1$, due to the extra factor R.
Further we remark that (6.49) and (6.50) vanish when $y = 0$. This means that at the plate the asymptotic behaviour of the Alfvén waves is described by further terms of the asymptotic expansion.

4. Symmetric case.

The results of chapter V, section 4, indicate that we cannot, as in the antisymmetric case, neglect the thickness of the plate.
Therefore we take here a finite thickness into account and locate again the upper surface at

$$y = \delta\lambda + \varepsilon\lambda \exp(ix).$$

Hence the appropriate form of (6.28) is here

$$G = i\,\frac{s\overline{K}(s) - (s+i)\varepsilon\lambda}{s\{\beta^2 + (s+i)^2\}} \exp -(y-\delta\lambda) + M(s)\exp -C(y-\delta\lambda). \tag{6.52}$$

Inside the plate there is a harmonic potential $\widetilde{A}$, which is, as has been shown in chapter V, section 6, odd with respect to y. Therefore we write

$$\widetilde{A} = \varepsilon\lambda\, h(t) \exp(ix) \sinh y. \tag{6.53}$$

Let the Laplace transform of h(t) be H(s). Then we have for H(s), $\overline{\overline{K}}(s)$ and M(s), the relation (6.30), which is here

$$y = \delta\lambda \quad : \quad G = \frac{i\,\overline{\overline{K}}(s)}{\beta^2},$$

and from (6.52) and (6.53) the continuity conditions

$$y = \delta\lambda \quad : \quad G = H(s) \sinh \delta\lambda,$$

$$\frac{dG}{dy} = H(s) \cosh \delta\lambda.$$

After some algebra, we find

$$H(s) = -\frac{i}{s} \frac{(C-1)(s+i)\varepsilon\lambda}{\{(s+i)^2+\beta^2\} \cosh \delta\lambda + \{\beta^2+C(s+i)^2\} \sinh \delta\lambda}, \tag{6.54}$$

$$\overline{\overline{K}}(s) = -\frac{1}{s} \frac{\beta^2(C-1)(s+i)\varepsilon\lambda \sinh \delta\lambda}{\{(s+i)^2+\beta^2\} \cosh \delta\lambda + \{\beta^2+C(s+i)^2\} \sinh \delta\lambda}, \tag{6.55}$$

$$M(s) = \frac{i}{s} \frac{(s+i)\varepsilon\lambda \exp \delta\lambda}{\{(s+i)^2+\beta^2\} \cosh \delta\lambda + \{\beta^2+C(s+i)^2\} \sinh \delta\lambda}. \tag{6.56}$$

With (6.55) and (6.56) we can write down G and once G is obtained, F follows from (6.55) and (6.25). These are complicated expressions and therefore we shall restrict ourselves here to H(s). For if we can show that $\widetilde{A}$ becomes ultimately steady, the same will be true for A and ψ. We consider

$$\widetilde{A} = \frac{\varepsilon\lambda \exp(ix)}{2\pi i} \sinh y \int_{c-i\infty}^{c+i\infty} e^{st}\, H(s)\, ds, \tag{6.57}$$

where H(s) is given by (6.54) and (6.29).
H(s) has a simple pole in s = 0, where the residu is

$$\frac{\varepsilon\lambda\, [\{1 + iR(1-\beta^2)\}^{1/2} - 1]}{(\beta^2-1) \cosh \delta\lambda + [\beta^2 - \{1+iR(1-\beta^2)\}^{1/2}] \sinh \delta\lambda}. \tag{6.58}$$

From (5.45), (6.53) and (6.58), we find the steady solution

$$\tilde{A} = \frac{\varepsilon\lambda(\alpha-1)\sinh y}{(\beta^2-1)\cosh\delta\lambda + (\beta^2-\alpha)\sinh\delta\lambda}\exp(ix). \qquad (6.59)$$

In dealing with the steady motion, we have already discussed (6.59) and shown that when $\varepsilon \approx \delta$, the disturbance in x direction becomes of order one in the plate.

To investigate the transient effects, we consider on account of the results of the foregoing section, the points $s+i = 0$ and $s+i = -\frac{1}{2R} \pm i\beta$.

We write again $z = s+i$. Then $H(z)$ becomes

$$H(z) = -\frac{i\varepsilon\lambda}{(z-i)}\,\frac{(C-1)\,z}{(\beta^2+z^2)\cosh\delta\lambda + (\beta^2+Cz^2)\sinh\delta\lambda}, \qquad (6.60)$$

C being given by (6.36).

Expanding $H(z)$ for $z \to 0$, we obtain for the first term $\frac{R^{1/2}z^{1/2}}{\beta}$, which gives upon transformation and remembering that $z = s+i$,

$$h_1 \simeq \varepsilon\lambda(\exp -it)\left(\frac{R}{4\pi\beta^2t^3}\right)^{1/2}, \qquad (6.61)$$

an expression similar to (6.46) and independent of δ.

Higher terms, containing δ, are proportional to $\frac{\sinh\delta\lambda}{\cosh\delta\lambda + \sinh\delta\lambda}$ or positive powers of this quantity.

Therefore we conclude, that difficulties, arising when δ is small, must, if present, be due to other singularities.

Consider the singularity $z = -\frac{1}{2R} + i\beta$. We introduce $\zeta = z + \frac{1}{2R} - i\beta$.

For small ζ and small $\delta\lambda$ we approximate (6.60) with

$$H_2(\zeta) = \frac{i\varepsilon\lambda}{\delta\lambda\beta(\beta-1)}\,\frac{1-\sqrt{(2R\zeta)}}{1-\sqrt{(2R\zeta)}+\frac{2i\zeta}{\beta\lambda\delta}} \qquad (6.62)$$

Here we have taken $\beta \neq 1$, which permits us to neglect the term $-\frac{1}{2R}$ with respect to $(1-\beta)$.

The contribution to the asymptotic behaviour of $h(t)$ for large t follows from evaluation for large t of

$$e^{-t/2R + i(\beta-1)t}\int H_2(\zeta)\,e^{\zeta t}\,d\zeta$$

along the path II in fig. 7.

We obtain, omitting numerical factors,

$$\left|h_2(t)\right| \simeq \frac{\varepsilon\lambda}{\beta\,|\beta-1|\,\delta^2\lambda^2}\left(\frac{R}{\beta^2t^5}\right)^{1/2} e^{-t/2R} \qquad (6.63)$$

From (6. 63) it follows that for every finite δ there is a time t large enough to render $h_2(t)$ small. Therefore we conclude that a steady state is attained for every finite δ.
We showed on the other hand, that when $\delta \approx \varepsilon$ and R is large, the disturbance of the magnetic field in x-direction, resulting from the steady solution (6. 59), is of order one, which invalidates the linearization. Therefore we inspect the behaviour for small times. Since b_x and $\widetilde{b}_x$ are zero at $t = 0$, there will be a period during which the linearization is valid, whatever the value of $\lambda\delta$ and R.
Because the discussion will also involve the magnetic field in the fluid, we first write (6. 52) with help of (6. 55) and (6. 56), thereby assuming $\delta\lambda$ to be small, as

$$G = -\frac{i}{s}\,\frac{\varepsilon\lambda(s+i)(C\delta\lambda+1)}{\{(s+i)^2+\beta^2\}+\delta\lambda\{\beta^2+C(s+i)^2\}}\exp-(y-\delta\lambda) + \tag{6.64}$$
$$+\frac{i}{s}\,\frac{\varepsilon\lambda(s+i)(1+\delta\lambda)}{\{(s+i)^2+\beta^2\}+\delta\lambda\{\beta^2+C(s+i)^2\}}\exp-C(y-\delta\lambda).$$

From the theory of Laplace transformations it follows (Carslaw and Jaeger [27] p. 255) that

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sL\{f(t)\}^{*}.$$

Therefore

$$\lim_{t\to 0}\frac{\partial b_x}{\partial t} = \lim_{s\to\infty} s^2\frac{dG}{dy}\exp(ix),$$

and

$$\lim_{t\to 0}\frac{\partial b_y}{\partial t} = -i\lim_{s\to\infty} s^2 G\exp(ix).$$

Using these relations and (6. 29), we obtain for $t = 0$ and $y = \delta\lambda$ from (6. 64)

$$\frac{\partial b_x}{\partial t} = i\varepsilon\lambda\exp(ix) - \frac{i\varepsilon\lambda(1+\delta\lambda)}{\lambda\delta}\exp(ix), \tag{6.65}$$

$$\frac{\partial b_y}{\partial t} = -\varepsilon\lambda\exp(ix). \tag{6.66}$$

In the same way we obtain from (6. 53) and (6. 54) that for $t = 0$ and $y = \delta\lambda$ we have in the plate

* Here L{ } means: "Laplace transform of { }"

$$\frac{\partial \tilde{b}_x}{\partial t} = - i \frac{\varepsilon}{\delta} \exp(ix), \tag{6.67}$$

$$\frac{\partial \tilde{b}_y}{\partial t} = - \varepsilon \lambda \exp(ix). \tag{6.68}$$

We recall that at $t = 0$ the current is zero and the velocity is given by the nonmagnetic streamfunction

$$\psi_o = - \varepsilon \lambda \exp(ix) \exp -(y - \delta\lambda). \tag{6.69}$$

The electric field, present at $t = 0$, then follows from (6.69) and (5.7)

$$E_z^o = i \varepsilon \lambda \exp(ix) \exp -(y - \delta\lambda). \tag{6.70}$$

Likewise in the lower half-plane

$$E_z'^o = - i \varepsilon \lambda \exp(ix) \exp(y + \delta\lambda). \tag{6.71}$$

On account of the continuity of the electric field at $y = \pm \delta\lambda$, we have in the plate at $t = 0$

$$\tilde{E}_z^o = \frac{i \varepsilon y \exp(ix)}{\delta}. \tag{6.72}$$

From Maxwell's equations

$$\frac{\partial E_z}{\partial x} = \frac{\partial b_y}{\partial t}, \tag{6.73}$$

$$\frac{\partial E_z}{\partial y} = - \frac{\partial b_x}{\partial t}, \tag{6.74}$$

it follows that the electric field causes a change of the magnetic field.

At $y = \delta\lambda$, $\frac{\partial E_z^o}{\partial x}$ and $\frac{\partial \tilde{E}_z^o}{\partial x}$, calculated from (6.70) and (6.72) are both equal to $-\varepsilon \lambda \exp(ix)$.

Comparison with (6.66) and (6.68) then shows that the change of b_y and $\tilde{b}_y$ is initially accounted for by the electric field present at $t = 0$.

However $\frac{\partial E_z^o}{\partial y}$ and $\frac{\partial \tilde{E}_z^o}{\partial y}$ are different at $y = \delta\lambda$.

In the fluid we have from (6.70)

$$\frac{\partial E_z^o}{\partial y} (y = \delta\lambda) = - i \varepsilon \lambda \exp(ix),$$

whilst in the plate, from (6.72),

$$\frac{\partial \widetilde{E}_z^o}{\partial y} (y=\delta\lambda) = - i \frac{\epsilon}{\delta} \exp(ix).$$

Then we conclude on account of (6. 74) and the corresponding relation for the fields in the plate, that the change of the x component of the magnetic field, due to the initial electric field, is given by (6. 67) for the plate and by the first term on the right-hand side of (6. 65) for the fluid.

The second term on the right-hand side of (6. 65) is apparently due to the Alfvén waves, set up to ensure the continuity of b_x.

The sum of the two terms at the right-hand side of (6. 65) is equal to the right-hand side of (6. 67).

In order to find the development of the disturbance in x direction, we try to find an expression for $\widetilde{b}_x$, valid when t is small. The theory of Laplace transformations learns that a series representation of f(t) can be found from a series development of $L\{f(t)\}$ for large s. From (6.29) it follows that we can approximate C for large s by

$$C = R^{1/2} (s+i)^{1/2} \qquad (6.75)$$

Using (6. 53), (6. 54) and (6. 75), we obtain for large s

$$L\{\widetilde{b}_x\} = - \frac{i\epsilon}{\delta} \frac{\exp(ix)}{s(s+i)} \left[1 - \frac{1}{1 + \delta\lambda R^{1/2}(s+i)^{1/2}} \right]. \qquad (6.76)$$

The fraction in the brackets in (6.76) can be expanded in a series convergent when $|(s+i)^{1/2}| > \frac{1}{\delta\lambda R^{1/2}}$.

Then we obtain

$$L\{\widetilde{b}_x\} = - \frac{i\epsilon}{\delta} \frac{1}{s(s+i)} \left[1 - \frac{1}{\delta\lambda R^{1/2}(s+i)^{1/2}} + \dots \right] \exp(ix). \qquad (6.77)$$

By inversion of (6. 77) a series representation for $\widetilde{b}_x$ is obtained. Because (6. 76) represents $L\{\widetilde{b}_x\}$ only for large s this series is asymptotic.

Inversion of (6. 77) yields

$$\widetilde{b}_x \simeq - \frac{i\epsilon}{\delta} \left[\int_0^t e^{-i\tau} d\tau - \frac{1}{\delta\lambda R^{1/2} \Gamma(3/2)} \int_0^t e^{-i\tau} \tau^{1/2} d\tau \right] \exp(ix).$$

For $t < 1$ this can be written as

$$\widetilde{b}_x \simeq - \frac{i\epsilon}{\delta} \left[t - \frac{t^{3/2}}{\delta\lambda R^{1/2} \Gamma(5/2)} + \dots \right] \exp(ix). \qquad (6.78)$$

The first term on the right-hand side of (6.78) has already been discussed.
To understand the second term we note that we find from (6.64) for the first term of the asymptotic expansion of the current $j_z = -\nabla^2 A$

$$j_z \simeq -\frac{i\varepsilon R^{1/2} t^{1/2}}{\delta\,\Gamma(3/2)} \exp(ix), \text{ at } y = \delta\lambda. \tag{6.79}$$

The z component of (5.7) is

$$E_z = u_y - b_y + \frac{j_z}{R}.$$

To the order $t^{1/2}$, $u_y = -\frac{\partial\psi_0}{\partial x}$ and $b_y = 0$, so that we have at $y = \delta\lambda$ to the order $t^{1/2}$

$$E_z = i\varepsilon\lambda \exp(ix) - \frac{i\varepsilon t^{1/2}}{\delta R^{1/2}\,\Gamma(3/2)} \exp(ix). \tag{6.80}$$

The Alfvén waves are rotational and currents arise given by (6.79). Consequently the electric field decreases according to (6.80). From the continuity of E_z it follows that in the plate

$$\tilde{E}_z = \frac{i\varepsilon y \exp(ix)}{\delta} - \frac{i\varepsilon y t^{1/2}}{\lambda\delta^2 R^{1/2}\,\Gamma(3/2)} \exp(ix). \tag{6.81}$$

Then by applying (6.74) to the fields in the plate we find upon integration the expression (6.78). When $\delta\lambda R^{1/2} > 1$, the first term on the right-hand side of (6.78) suffices up till $t \simeq 1$. If in addition $\frac{\varepsilon}{\delta} = 0(1)$, we conclude from (6.78) that $\tilde{b}_x$ becomes of order one during the time in which only the first term at the right-hand side of (6.78) needs to be considered. This causes the breakdown of the linearization. The physical reason is that because of the large conductivity the electric fields decreases slowly.
When the conductivity is lower so that $\delta\lambda R^{1/2} < 1$, soon after $t = 0$ the second and higher terms at the right-hand side of (6.78) become important and $\tilde{b}_x$ stays small during the time for which the first term is enough.
After a time determined by the magnitude of $\delta\lambda$ and R the asymptotic solution for large t can be used. This solution given by (6.59), (6.61) and (6.63) shows that eventually a steady state is attained.
From (6.59) it follows that when $\delta\lambda$ is small, we have in the plate in the steady state.

$$\tilde{b}_x = \frac{\varepsilon\lambda(\alpha - 1)}{(\beta^2 - 1) + \delta\lambda(\beta^2 - \alpha)} \exp(ix). \tag{6.82}$$

From (5.45) we obtain $\alpha \simeq R^{1/2}$, when R is large and β is of order one but not equal to one.
Then (6.82) shows that for $\delta\lambda R^{1/2} < 1$ and $\frac{\varepsilon}{\delta} = 0(1)$, $\tilde{b}_x$ is in the steady state of the order $\varepsilon \lambda R^{1/2} \simeq \lambda \delta R^{1/2} < 1$.
We conclude that for small $\delta\lambda$ the linearized theory is in the symmetric case valid both for small and large time, provided $\delta\lambda R^{1/2} < 1$.
In general the range of applicability of the asymptotic expansion for large t will not overlap that of the expansion for small t.
A convergent expansion for $\tilde{b}_x$ valid for all t can be obtained by expanding $L\{\tilde{b}_x\}$, using the full expression (6.54).
This is not done here, because it is in view of the satisfactory results obtained both for small and large t, when $\delta\lambda R^{1/2} < 1$, not likely that large disturbances occur at intermediate values of t.

CHAPTER VII

ANTISYMMETRIC WAVY PLATE; TWO-FLUID MODEL; STEADY MOTION

1. Introduction.

In this chapter we shall apply the two-fluid model to the steady flow of a fully ionized gas along the antisymmetric wavy plate, considered in chapter V, section 3.
Work along the same lines has been done by Sonnerup[29]) for the Sears-Resler configuration, discussed in chapter V, section 7.
This author considered the cases where the magnetic Reynoldsnumber R is either very large or very small. In the latter case this is accompanied by the assumption that $q = \omega\tau$ is large.
The two-fluid theory, as developed in chapter II, section 3, and used essentially in that form by Sonnerup, holds only for small values of q. From some numerical examples, given in this chapter, it follows further, that in cases of interest R has moderate values.
We shall therefore develop a two-component theory for the wavy wall problem, assuming q to be small, but leaving other parameters free. Again however we shall neglect compressibility effects. The governing equations of the two-fluid model are, as in chapter V for the one-fluid model, solved by means of a linearization process.

2. Equations for the two-fluid model.

The equation of motion in the two-fluid model is the same as in the one-fluid model. From (5.5) we obtain for steady motion, using (5.11) - (5. 13) and neglecting quantities of order $(\varepsilon\lambda)^2$.

$$\frac{\partial \mathbf{u}}{\partial x} = - \nabla p + \beta^2 \left(\frac{\partial \mathbf{b}}{\partial x} - \nabla b_x\right). \tag{7.1}$$

We expect that now $\mathbf{u}$ and $\mathbf{b}$ also have components in z-direction. These are not present in the one-fluid theory and, to the first approximation in q, will be of order $q\varepsilon\lambda$. We remember the reader that q stands for the product of the electron cyclotronfrequency ω and the ion-electron collisiontime τ.
We assume that

$$(\varepsilon\lambda)^2 < q\varepsilon\lambda < \varepsilon\lambda. \tag{7.2}$$

Instead of Ohm's law (5. 7), we must use now the relation (2. 77).
In the left-hand side only the second term remains upon linearization.

We write the remaining equation in dimensionless form by means of the relations (5.3).
Then we obtain with help of (2.49), (2.63) and (5.8)

$$R(\mathbf{E} + \mathbf{v} \times \mathbf{B}) = \mathbf{j} + \lambda U_o \tau \frac{\partial \mathbf{j}}{\partial x} + q\mathbf{j} \times \mathbf{B} - q\nabla p_e. \qquad (7.3)$$

The quantity $\lambda U_o \tau$ in the second term at the right-hand side of (7.3) measures the number of crests of the wavy profile passed by the fluid during a collisiontime.
Since this number is very small we shall omit the corresponding term in (7.3).
We assume that the kinetic energy is small with respect to the thermal energy. Since $m_i \gg m_e$, this amounts to the assumption

$$m_i U_o^2 \ll kT. \qquad (7.4)$$

Then we can neglect the influence of pressure variation on the density and take $n_i = n_e = n_o$.
We can replace now p_e in (7.3) by $\frac{1}{2}$ p.
Elimination of p between (7.1) and (7.3) yields

$$R(\mathbf{E} + \mathbf{v} \times \mathbf{B}) = \nabla \times \mathbf{b} + \frac{q}{2\beta^2}\frac{\partial \mathbf{u}}{\partial x} + \frac{q}{2}\left(\frac{\partial \mathbf{b}}{\partial x} - \nabla b_x\right). \qquad (7.5)$$

In (7.3) $\mathbf{j}$ has been replaced by $\nabla \times \mathbf{b}$, and the $\mathbf{j} \times \mathbf{B}$ term has been linearized. Just as in the one-fluid model we take $E_z = 0$ and $\frac{\partial p}{\partial z} = 0$.
The components of (7.1) are

$$\frac{\partial u_x}{\partial x} = -\frac{\partial p}{\partial x}, \qquad (7.6)$$

$$\frac{\partial u_y}{\partial x} = -\frac{\partial p}{\partial y} + \beta^2\left(\frac{\partial b_y}{\partial x} - \frac{\partial b_x}{\partial y}\right), \qquad (7.7)$$

$$\frac{\partial u_z}{\partial x} = \beta^2 \frac{\partial b_z}{\partial x}, \qquad (7.8)$$

while the components of (7.5) become, linearizing also the $\mathbf{v} \times \mathbf{B}$ term,

$$RE_x = \frac{\partial b_z}{\partial y} + \frac{q}{2\beta^2}\frac{\partial u_x}{\partial x}, \qquad (7.9)$$

$$R(E_y + u_z - b_z) = -\frac{\partial b_z}{\partial x} + \frac{q}{2\beta^2}\frac{\partial u_y}{\partial x} + \frac{q}{2}\left(\frac{\partial b_y}{\partial x} - \frac{\partial b_x}{\partial y}\right), \qquad (7.10)$$

$$R(b_y - u_y) = \left(\frac{\partial b_y}{\partial x} - \frac{\partial b_x}{\partial y}\right) + \frac{q}{2\beta^2}\frac{\partial u_z}{\partial x} + \frac{q}{2}\frac{\partial b_z}{\partial x}. \qquad (7.11)$$

With help of (7. 8) we can reduce (7. 11) to

$$R(b_y - u_y) = (\frac{\partial b_y}{\partial x} - \frac{\partial b_x}{\partial y}) + q\,\frac{\partial b_z}{\partial x}. \qquad (7.12)$$

In the one-fluid model u_z and b_z are zero. We are investigating phenomena of the first order in q. Therefore we can neglect the term $q\,\frac{\partial b_z}{\partial x}$ in (7. 13), since this term will be of the second order in q.
Doing this, (7. 12) becomes

$$R(b_y - u_y) = (\frac{\partial b_y}{\partial x} - \frac{\partial b_x}{\partial y}). \qquad (7.13)$$

The equations (7.6), (7.7) and (7.13) form together with (5.4) the same set of equations for p and the disturbances in the x, y plane, as in the one-fluid model.
Apparently the Hall-effect and the electron pressure, taken into account in the two-fluid theory, don't affect the components of flow and field in the x, y plane up to the second order in q.
In chapter IV a similar effect was noticed with respect to the Couette flow.
Hence the components of velocity and magnetic field can be obtained from the solution in the one-fluid model.
The expressions (5. 46) and (5. 47) for respectively the streamfunction and the vector potential in the upper half-plane, are repeated here for convenience

$$\psi = \frac{\varepsilon\lambda}{\alpha-\beta^2}\;[\,\beta^2 \exp -\alpha y - \alpha \exp -y\,]\; \exp(ix), \qquad (7.14)$$

$$A = \frac{\varepsilon\lambda}{\alpha-\beta^2}\;[\,\exp -\alpha y - \alpha \exp -y\,]\; \exp(ix). \qquad (7.15)$$

We recall that in the lower half-plane

$$\psi'(x,y) = \psi(x,-y), \qquad (7.16)$$

$$A'(x,y) = A(x,-y). \qquad (7.17)$$

3. Determination of u_z, b_z, E_x and E_y.

The components in the direction of the z-axis of the disturbances and the electric field components are determined by (7. 8) - (7. 10) and the z-component of Maxwell's equation (5.10), which reads in the steady state

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = 0. \qquad (7.18)$$

This equation is satisfied by the scalar potential $\Phi(x,y)$ defined by

$$\mathbf{E} = -\nabla\Phi. \tag{7.19}$$

Elimination of the electric field between (7.9) and (7.10) with help of (7.18), gives using (5.11), (5.52) and (7.8),

$$\nabla^2 b_z + R(\beta^2-1)\frac{\partial b_z}{\partial x} = q\frac{\partial j_z}{\partial x}. \tag{7.20}$$

Now we must determine the boundary conditions for b_z. In the first place we require that $b_z = 0$ for $y = \infty$.
A second condition follows from the fact that the plate is nonconductive. Therefore $j_y = -\frac{\partial b_z}{\partial x}$ vanishes at the plate and hence the second condition is in view of the periodicity in x direction

$$y = 0 \ : \ b_z = 0.$$

The right-hand side of (7.20) follows from (7.15) through the relation

$$j_z = -\nabla^2 A.$$

Upon evaluating $\frac{\partial j_z}{\partial x}$ with help of (7.15), the solution of (7.20), that vanishes at $y = 0$ and $y = \infty$, appears to be

$$b_z = \frac{iq\varepsilon\lambda(\alpha^2-1)}{2\alpha(\alpha-\beta^2)}\, y \exp -\alpha y \exp(ix). \tag{7.21}$$

Then we obtain from (7.8) and (7.21)

$$u_z = \frac{iq\varepsilon\lambda\beta^2(\alpha^2-1)}{2\alpha(\alpha-\beta^2)}\, y \exp -\alpha y \exp(ix). \tag{7.22}$$

For comparison we list again the x and y components of the disturbances, given in chapter V, eqs. (5.48) - (5.51).

$$u_x = \frac{\varepsilon\lambda \exp(ix)}{\alpha-\beta^2}\,[\,\alpha\exp -y - \alpha\beta^2\exp -\alpha y\,], \tag{7.23}$$

$$u_y = \frac{i\varepsilon\lambda \exp(ix)}{\alpha-\beta^2}\,[\,\alpha\exp -y - \beta^2\exp -\alpha y\,], \tag{7.24}$$

$$b_x = \frac{\varepsilon\lambda \exp(ix)}{\alpha-\beta^2}\,[\,\alpha\exp -y - \alpha\exp -\alpha y\,], \tag{7.25}$$

$$b_y = \frac{i\varepsilon\lambda \exp(ix)}{\alpha-\beta^2}\,[\,\alpha\exp -y - \exp -\alpha y\,]. \tag{7.26}$$

At this point it is worth while to give for cases of interest the numerical values of the various parameters.
We take $\lambda = 1\ m^{-1}$; $\epsilon\lambda = 10^{-2}$.

Example 1.

We choose $T = 10^4\ {}^\circ K$. From (2.55) and (2.57) we obtain $\frac{h}{D} = 10^7\ \frac{T^{1/2}}{n^{1/2}}$.
For $T = 10^4$, we deduce from this relation and (2. 65) that n is not allowed to exceed the value $10^{26}/m^3$.
We take $n = 10^{22}/m^3$. Then $\frac{h}{D} = 100$ and from (2. 58) it follows that $N = 200$. Since $\rho = n m_i$ (cf 2. 23) and $m_i = 1.7 \times 10^{-27}$ kg, we have $\rho = 1.7 \times 10^{-5}\ kg/m^3$.
The relation (7. 4) requires $U_o \ll (\frac{kT}{m_i})^{1/2} \simeq 10^4$.
We take $U_o = 5 \times 10^2$ m/s. Further we choose $B_o = 5 \times 10^{-2}\ \frac{weber}{m^2}$.
From these values of U_o, B_o, ρ and the value 1.3×10^{-6} for μ, we obtain, using (5. 6)

$$\beta = 21.$$

Note that this relatively large value of β is due to the low density of the gas.
Computation of σ from (2. 61) gives: $\sigma = 0.2 \times 10^4\ ohm^{-1}m^{-1}$.
Then the magnetic Reynoldsnumber becomes (cf (5. 8)),

$$R = 1.3\ .$$

Finally the relation $q = \frac{\sigma B_o}{e n}$ yields

$$q = 0.06,$$

while from (5. 45) we obtain

$$|\alpha| \simeq R^{1/2}\beta = 24.$$

In discussing (7. 23) - (7. 26) in chapter V, section 3, we observed that u_x becomes large in the vicinity of the plate, when β is large. From this example it follows that such situations can occur here, due to the low density of the gas.
In our example u_x is of the order $\epsilon\lambda R^{1/2}\beta = 0.24$ in the neighbourhood of the plates. This value can still be accepted.

Example 2.

We take now

$$T = 10^5\ {}^\circ K;\ U_o = 10^3\ m/s;\ n = 10^{23}/m^3;\ B_o = 10^{-1}\ \frac{weber}{m^2}.$$

Using the relations, mentioned in discussing the foregoing example, we obtain here

$$2\frac{h}{D} = N = 2000;\ \rho = 1.7 \times 10^{-4}\ \frac{kg}{m^3};\ \sigma = 0.4 \times 10^5\ \mathrm{ohm}^{-1}\mathrm{m}^{-1}$$

and thus

$$\beta = 6.7$$

$$R = 18$$

$$q = 0.25$$

$$|\alpha| \simeq \beta R^{1/2} = 28.$$

From these examples it follows that in general R has moderate values but β can become large.

In order that u_x remains small and hence the linearization is valid we must require for large β

$$\epsilon \lambda R^{1/2} \beta < 1. \qquad (7.27)$$

Since in most cases $\alpha > 1$, we can use the concept of a magneto-hydrodynamic boundary layer, separating the plate from the region, where flow and field are irrotational.

From (7.21) and (7.22) it follows that u_z and b_z are nonzero only in the boundary layer.

The maximum value of u_z occurs at $y = \frac{1}{\alpha}$ and is

$$|u_{z,max}| \simeq \frac{q \epsilon \lambda \beta^2 (\alpha^2 - 1)}{4\alpha^2(\alpha - \beta^2)}. \qquad (7.28)$$

This expression shows that u_z remains small when β is large.

Therefore the condition (7.27) for the validity of the linearization is solely due to the fact that u_x becomes large when β is large.

In the one-fluid theory we found that the electric field has no components in x and y-direction.

In the two-fluid theory these components are nonzero. From (7.9) we obtain, using (7.19), (7.21) and (7.23)

$$\Phi = \frac{q \epsilon \lambda}{2R(\alpha - \beta^2)} \left[\frac{\exp - \alpha y}{\alpha} + (\alpha^2 - 1)\, y \exp - \alpha y - \frac{\alpha}{\beta^2} \exp - y \right] \exp(ix). \qquad (7.29)$$

This expression gives the electric field in the upper half-plane.

The relation, valid in the lower half-plane, can be obtained in the following way:

Because $j_z = -\nabla^2 A$, it follows from (7.17) and (7.20), that b_z keeps its sign in passing $y = 0$.

Then $\frac{\partial b_z}{\partial y}$ changes in sign in passing $y = 0$ and the same is on account of (5. 14) and (7. 16) true for $\frac{\partial u_x}{\partial x}$.

The sum of these two expressions forms the right-hand side of (7. 9), so that E_x and also Φ change in sign at $y = 0$.

Therefore we have for $y \leqslant 0$

$$\Phi' = \frac{q \varepsilon \lambda}{2R(\alpha-\beta^2)} \left[\frac{\alpha}{\beta^2} \exp y + (\alpha^2-1)y \exp \alpha y - \frac{\exp \alpha y}{\alpha} \right] \exp(ix). \quad (7.30)$$

The electric field being known, we can calculate its sources, the charge density Q, from (2. 7), which relation reads here (cf (5. 3))

$$U_0 B_0 \lambda \nabla . \boldsymbol{E} = \frac{Q}{\varepsilon_0}.^{*} \quad (7.31)$$

We take the divergence of (7. 5) and obtain, expressing $\nabla^2 b_x$ with help of the divergence of (7. 1) in terms of $\nabla^2 p$,

$$\nabla . \boldsymbol{E} = \frac{q}{2R\beta^2} \nabla^2 p - \nabla . (\boldsymbol{v} \times \boldsymbol{B}). \quad (7.32)$$

We denote the "neutral" density with n_0. Remembering that we can write $\frac{\sigma B_0}{e n}$ for q (cf (3.21)), and using the definitions (5.6) and (5.8) for R and β^2, we deduce from (7. 31) and (7. 32) that the first term on the right-hand side of (7. 32) corresponds with a charge density

$$Q_1 = \frac{\lambda^2 m_i U_0^2}{2e} \nabla^2 p. \quad (7.33)$$

Q_1 arises from the density variations caused by the pressure variations. The density variations are neglected in our theory on account of (7. 4). Since we have replaced p_e by 1/2 p in the "generalized" Ohm's law (2. 77), the variations in p cause a small depart from neutrality in the negative constituent. The corresponding charge density is given by (7.33). We recall that we have neglected the convective current $Q\boldsymbol{v}$ with respect to $\boldsymbol{j}$.

Since $\nabla^2 p = - \nabla^2 u_x$ is of the order of magnitude $\varepsilon\lambda$ and the current density of the order $\frac{\varepsilon \lambda B_0 \lambda}{\mu}$, it follows that the ratio between the current density and $Q_1 \boldsymbol{v}$ is of the order

* Since it is customary to use the symbol ε_0 for the permittivity of free space, we have not chosen another symbol, in spite of the fact that the amplitude of the wavy plate is indicated with ε. No confusion can arise however, owing to the use of the subscript $_0$ when the permittivity is meant.

$$\frac{\lambda U_o}{\omega_i} \cdot \frac{U_o^2}{c^2}, \tag{7.34}$$

where we have used (2. 13) and (2. 62).
Both fractions in (7. 34) are very small, so that the neglect of $Q_1 v$ is certainly permitted.
It is useful for further reference to write (7. 33) with help of (2. 55) as

$$\frac{Q_1}{e\, n_o} = \frac{\lambda^2 h^2 m_i U_o^2}{2kT} \nabla^2 p. \tag{7.35}$$

The second term on the right-hand side of (7. 32) is in linearized form $-\frac{\partial u_z}{\partial y} + \frac{\partial b_z}{\partial y}$ or, with help of (5. 11) and (7. 9), $(1-\beta^2) j_x$.
Then it follows from (5. 3) and (7. 31) that the charge density connected with this expression is

$$Q_2 = \frac{(1-\beta^2) j_x^* U_o}{c^2}. \tag{7.36}$$

Q_2 represents a relativistic effect.
Relativistically the charge density is the fourth component of a four - vector, the other three components being the current densities. When changing the frame of reference, this four-vector must be transformed according to the Lorentz-transformation (see Panofsky and Phillips[16]).
Then it appears that the charge density depends on the state of motion of the observer. Since we are dealing with nonrelativistic velocities, we neglect Q_2.
Inspection of (7. 29) and (7. 30) shows that the potential Φ displays a discontinuity at $y = 0$.
From the point of view of potential theory the plate is a singular surface and the discontinuity can be ascribed to a dipole distribution.
In the next section we shall discuss in a first approximation the charge distribution at the plate.

4. The electric field in the vicinity of the plate.

Well-known in plasmaphysics is the concept of Debye sheaths, i. e. regions near solid walls where appreciable charge separation occurs and where the component of the electric field normal to the wall is large.
The order of magnitude of the thickness of these regions is the Debye length h. In most textbooks (e.g. Spitzer[11] p.17) this is made plausible on the basis of the Debye potential, discussed in chapter II, section 3.
From that discussion it follows that the Debye length is a measure for the distance over which an appreciable depart from neutrality can exist

in a plasma at rest.
The concept of Debye sheaths has been proved to be a useful tool in dealing with surface phenomena. A description in terms of the Debye potential is necessarily approximative, since the variation of the macroscopic electric field is considered over distances of the order of magnitude h.
It is difficult to say to what extent this can be justified and consequently how good the approximation is. With this in mind, we shall apply the concept of Debye sheaths to our problem.
Doing this, we assume that there are on both sides of the plate thin layers, in which the electric field and the charge density vary as $\exp -\frac{y}{\lambda h}$.
We expect that the charge density Q_3 will be large with respect to Q_1.
In the following discussion we shall use physical variables and suppress for convenience the asteriks.
To avoid confusion, we shall explicitly announce the return to dimensionless variables.
Then considering the Debye sheath on the upper side of the plate, we write in this region

$$p_i = n_o (1 + \gamma)kT, \tag{7.37}$$

$$p_e = n_o (1 + \gamma')kT. \tag{7.38}$$

We assume that both γ and γ' are small with respect to one.
We shall prove now, that under the circumstances envisaged here

$$\gamma + \gamma' = 0. \tag{7.39}$$

For this purpose we consider the y-component of the momentum equation (2.40) for the ions.
We replace the stress-tensor P_i by (7.37) and recall that the motion is steady.
When the potential Φ varies as exp $-y/h$, then the term

$$- n_i e E_y = e n_o (1 + \gamma) \frac{\partial \Phi}{\partial y},$$

is large near the plate.
The only other term, which can become large is the term

$$\frac{\partial P_i}{\partial y} = n_o kT \frac{\partial \gamma}{\partial y}. \text{ (cf (7.37)).}$$

Therefore these terms must balance each other.
Hence

$$- e n_o (1+\gamma) \frac{\partial \Phi}{\partial y} = n_o kT \frac{\partial \gamma}{\partial y}. \tag{7.40}$$

Note that when U_o is large ($m_i U_o^2 \gg kT$), also the acceleration term can compete with these terms.
A case, to which this remark pertains, has been studied by Yoshihara[30].
For the electrons we obtain in the same way

$$e\, n_o\, (1+\gamma')\frac{\partial \Phi}{\partial y} = n_o kT\, \frac{\partial \gamma'}{\partial y}. \tag{7.41}$$

Elimination of Φ between (7.40) and (7.41) yields, bearing in mind that γ and γ' are assumed to be small with respect to one

$$\frac{\partial}{\partial y} \ln(1+\gamma+\gamma') = 0.$$

or

$$\gamma + \gamma' = 0.$$

Using this result, we write (7.38) as

$$p_e = n_o\,(1-\gamma)kT. \tag{7.42}$$

Since $p = p_e + p_i = 2n_o kT$, an alternative expression is

$$p_e = 1/2\ p - n_o \gamma kT. \tag{7.43}$$

In evaluating the pressure term in (2.77) we have till thus far considered only the first term on the right-hand side of (7.43).
The contribution of the second term to the pressure term in (2.77) is

$$-\frac{kT}{e}\nabla \gamma. \tag{7.44}$$

On the other hand we have, when the charge density Q_1 can be neglected

$$\nabla . \mathbf{E} = \frac{2n_o e\gamma}{\varepsilon_o}. \tag{7.45}$$

Therefore we can write (7.44) as

$-\dfrac{\varepsilon_o kT}{2e^2 n_o}\nabla^2 \mathbf{E}$, which is on account of (2.55) equal to

$$-\frac{h^2}{2}\nabla^2 \mathbf{E}. \tag{7.46}$$

The expression (7.46) is of importance only in the Debye sheath.
Outside this layer the potential Φ is given by (7.29).
(7.46) gives in the dimensionless equation (7.5) a term

$$-\frac{h^2\lambda^2}{2} R\, \nabla^2 \mathbf{E}, \tag{7.47}$$

which must be added to the right-hand side of (7.5).

In (7.47) $\mathbf{E}$ is the dimensionless electric field.
Taking (7.47) into account, the x-component of (7.5) becomes

$$R(E_x - \frac{\lambda^2 h^2}{2} \nabla^2 E_x) = \frac{\partial b_z}{\partial y} + \frac{q}{2\beta^2}\frac{\partial u_x}{\partial x}. \qquad (7.48)$$

Since $\lambda h \ll 1$, we can for the right-hand side of (7.48) take the value of $\frac{\partial b_z}{\partial y} + \frac{q}{2\beta^2}\frac{\partial u_x}{\partial x}$ at $y = 0$. This value is equal to RE_x at $y = 0$, when E_x is calculated from (7.29).
Doing this we obtain from (7.48) and (7.29)

$$\Phi^i - \frac{\lambda^2 h^2}{2} \nabla^2 \Phi^i = \frac{q\,\epsilon\,\lambda\,(\beta^2-\alpha^2)}{2R(\alpha-\beta^2)\alpha\beta^2} \exp(ix), \qquad (7.49)$$

where the potential inside the Debye sheath is distinguished from that outside by means of a superscript i.
In the Debye sheath at the lower side of the plate an equation similar to (7.48) holds for the electric field E_x', the difference being that compared with (7.48) the right-hand side is of opposite sign.
Therefore we have in the sheath at the lower side

$$\Phi^{i'} - \frac{\lambda^2 h^2}{2} \nabla^2 \Phi^{i'} = - \frac{q\,\epsilon\,\lambda\,(\beta^2-\alpha^2)}{2R(\alpha-\beta^2)\alpha\beta^2} \exp(ix). \qquad (7.50)$$

We assume now that there is no affinity of the plate for either ions or electrons, so that there is no potential difference between gas and plate in the undisturbed state.
Then the electric field is solely caused by the motion of the gas.
Under these circumstances Φ^i is in view of the antisymmetry zero at the plate.
The appropriate solutions of (7.49) and (7.50) then are

$$\Phi^i = \frac{q\,\epsilon\,\lambda\,(\beta^2-\alpha^2)}{2R(\alpha-\beta^2)\alpha\beta^2} \left[1 - \exp\left(- \frac{y\sqrt{2}}{\lambda h}\right) \right] \exp(ix). \qquad (7.51)$$

and

$$\Phi^{i'} = \frac{q\,\epsilon\,\lambda\,(\beta^2-\alpha^2)}{2R(\alpha-\beta^2)\alpha\beta^2} \left[\exp\left(\frac{y\sqrt{2}}{\lambda h}\right) - 1 \right] \exp(ix). \qquad (7.52)$$

The charge density $Q_3 = - \epsilon_0 U_0 B_0 \lambda \nabla^2 \Phi^i$ is equal to $2n_0 e\gamma$ (cf. (7.45)).

For the charge density Q_1, associated with (7.29) through

$$Q_1 = - \varepsilon_o U_o B_o \lambda \nabla^2 \Phi \ ,$$

we found the relation (7. 35).
Comparison of (7.29) with (7.51) shows that Q_s is roughly $\frac{2}{\lambda^2 h^2}$ times Q_1. Then it follows from (7. 35) that

$$\frac{Q_s}{e n_o} = 2\gamma \simeq \frac{m_i U_o^2}{kT} \nabla^2 p.$$

We conclude on account of (7. 4) that γ is indeed small with respect to one. The assumptions, used in the analysis of this section, that Q_s is large with respect to Q_1 and that γ is small with respect to one, are therefore justified by the results.
It follows further from (7. 51) and (7. 52) that the charge densities in the Debye sheaths are at the same x equal in magnitude but opposite in sign.
The plate behaves therefore from a macroscopic point of view as a dipole layer.

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OVERZICHT

De stroming van electrisch geleidende vloeistoffen en gassen door electromagnetische velden vormt het onderwerp van beschouwing in de magnetohydrodynamica.
In dit proefschrift worden een aantal magnetohydrodynamische problemen geanalyseerd met behulp van de theorie van het één-vloeistof (one-fluid) en van het twee-vloeistoffen (two-fluid) model.
In het eerstgenoemde model is het medium een continuum, waaraan, naast andere, uit de gewone hydrodynamica bekende, eigenschappen, een constant geleidingsvermogen wordt toegekend.
Een medium dat goed aan het één-vloeistof model beantwoordt is b.v. kwik.
De vergelijkingen van het één-vloeistof model bestaan uit de, met de Lorentzkracht uitgebreide, vergelijkingen van Navier-Stokes, de vergelijkingen van Maxwell en de Wet van Ohm.
De vergelijkingen van het één-vloeistof model worden gegeven in hoofdstuk II, paragraaf 2.
Een voor de practijk belangrijk geleidend medium is een geioniseerd gas (kernfusie, magnetohydrodynamische generator). Een dergelijk gas bestaat uit geladen deeltjes en beantwoordt niet of onvolledig aan de voorstelling, gemaakt in het één-vloeistof model, omdat daarin het deeltjeskarakter niet is verdisconteerd.
Het twee-vloeistoffen model dient om de magnetohydrodynamica tot volledig geioniseerde gassen uit te breiden. (In het geval van onvolledig geioniseerde gassen moet ook een derde, neutrale, component worden beschouwd).
In het twee-vloeistoffen model beschouwt men een mengsel van een gas, bestaande uit negatief geladen deeltjes, en een gas, bestaande uit positief geladen deeltjes.
In dit proefschrift wordt voor de eenvoud verondersteld, dat de positief geladen deeltjes protonen zijn.
Om voor een dergelijk gas transportvergelijkingen op te stellen, gaat men uit van de Boltzmann-vergelijkingen voor de distributiefuncties van de beide componenten. Voor een gewoon, verdund, gas kunnen n.l. de transportvergelijkingen uit de Boltzmann-vergelijking worden afgeleid. Het blijkt dat analoge methodes voor een volledig geioniseerd gas niet zonder meer opgaan. De oorzaak hiervoor is, dat, terwijl voor een gewoon gas, bij voldoend lage dichtheid, de onderlinge wisselwerking tussen de deeltjes beschouwd kan worden als te bestaan uit binaire botsingen, dit bij een geioniseerd gas niet het geval is, vanwege het lange-afstands karakter van de Coulombkrachten. In hoofdstuk II, paragraaf 3, wordt in grote lijnen geschetst, hoe de hierdoor ontstane moeilijkheden kunnen worden ondervangen door het invoeren van een

gewijzigde Coulombpotentiaal en hoe onder gebruikmaking van deze zogenaamde Debye-potentiaal, transportvergelijkingen kunnen worden gevonden.

Het is nodig hierbij enige veronderstellingen te maken, waarvan de consequentie is, dat de gassen waarvoor de resulterende vergelijkingen gelden, moeten voldoen aan verschillende eisen. Met name moet het product van de cyclotronfrequentie van electronen en de botsingstijd voor botsingen tussen electronen en protonen klein zijn ten opzichte van één.

Het voornaamste verschil tussen de vergelijkingen van het één-vloeistof model en het twee-vloeistoffen model is gelegen in de Wet van Ohm, die in het twee-vloeistoffen model met enige termen is uitgebreid en in de aldus ontstane vorm wel de "gegeneraliseerde Wet van Ohm" wordt genoemd.

Tengevolge van deze uitbreiding van de Wet van Ohm, is de geleidbaarheid niet langer isotroop.

Na een algemene inleiding in hoofdstuk I en een uiteenzetting van het één-vloeistof model en van het twee-vloeistoffen model in hoofdstuk II, worden in de volgende hoofdstukken enige magnetohydrodynamische problemen behandeld, met de theorie van beide modellen, teneinde na te gaan op welke wijze de tussen de vergelijkingen aanwezige verschillen in de resultaten tot uiting komen. Compressibiliteitseffecten worden buiten beschouwing gelaten.

In hoofdstuk III wordt de stroming beschouwd van een niet-visceus volledig geioniseerd gas tussen twee evenwijdige platen. Er is een magneetveld aanwezig, gericht volgens de normaal op de platen. Het gas wordt voortgedreven door een zuiger, die zich met constante snelheid beweegt. Het blijkt dat het gas zich in het kader van de twee-vloeistoffen theorie scheef ten opzichte van de zuiger beweegt.

Hoofdstuk IV behandelt magnetohydrodynamische Couette stroming. Hierbij bevindt het gas, waarvan de viscositeit thans niet buiten beschouwing wordt gelaten, zich eveneens tussen twee evenwijdige platen. Een van de platen wordt met constante snelheid voortbewogen, terwijl de andere in rust is. Ook hier is een magneetveld, gericht volgens de normaal op de platen, aanwezig.

Beschouwd wordt het geval, waarin de bewegende plaat een isolator is, terwijl de stilstaande plaat een ideale geleider is, en het geval, waarin beide platen isolatoren zijn.

Ook hier levert de theorie van het twee-vloeistoffen model een component van de gassnelheid, loodrecht op het vlak door de snelheidsvector van de bewegende plaat en de normaal op de platen. Tevens oefent het gas in deze richting krachten uit op de platen. Deze effecten worden niet gevonden in het één-vloeistof model.

De volgende hoofdstukken handelen over de stroming van een geleidend medium langs een dunne niet geleidende plaat, waarvan boven- en onderzijde golfvormig zijn. De amplitude van het golfprofiel wordt aangeduid met ϵ, de golflengte met L. Er is een magneetveld aanwezig, dat ver van de plaat evenwijdig is aan de snelheidsvector van de vloeistof.

In hoofdstuk V wordt de theorie van het één-vloeistof model toegepast op stationnaire stroming langs de gegolfde plaat.

In verband met het feit dat $\epsilon\lambda$, waarbij $\lambda = \frac{2\pi}{L}$, klein is verondersteld, worden de vergelijkingen gelineariseerd. In paragraaf 2 worden de gelineariseerde vergelijkingen en randvoorwaarden opgesteld voor de gevallen waarin het faseverschil tussen onder- en bovenzijde nul, resp. π is. Hierbij wordt de dikte van de plaat verwaarloosd.

In paragraaf 3 wordt de oplossing van de gelineariseerde vergelijkingen voor het eerstgenoemde (antisymmetrische) geval gegeven, in paragraaf 4 die voor het tweede (symmetrische).

Het blijkt dat in het laatste geval de verstoringen van snelheid en magneetveld, veroorzaakt door de plaat, van de orde $\epsilon\lambda R^{1/2}$ worden, wanneer het magnetisch Reynoldsgetal $R(R = \frac{\sigma\mu U_0}{\lambda}$; σ = geleidingsvermogen; μ = permeabiliteit; U_0 = snelheid van ongestoorde vloeistof) groot is. Linearisering is dan niet meer geoorloofd.

Nadat in paragraaf 5 het speciale geval, waarin de ongestoorde snelheid gelijk is aan de voortplantingssnelheid van magnetohydrodynamische golven, is behandeld, wordt in paragraaf 6 opnieuw het symmetrische geval beschouwd, maar nu met eindige dikte 2δ.

Het blijkt dat wanneer $\delta\lambda R^{1/2} \gg 1$, storingen van de orde $\frac{\epsilon}{\delta}$ voorkomen, zodat bij groot geleidingsvermogen linearisering niet geoorloofd is voor dunne vleugels ($\epsilon \approx \delta$).

In paragraaf 7 wordt nog het geval beschouwd, waarin de vloeistof stroomt langs de golfvormige begrenzing van een oneindig uitgestrekt vacuum.

In hoofdstuk VI wordt instationnaire stroming langs een gegolfde plaat beschouwd, eveneens in het kader van het één-vloeistof model. De plaat beweegt voor tijden $t < 0$ met de vloeistof mee, terwijl een magneetveld, als in hoofdstuk V, aanwezig is, en wordt ten tijde $t = 0$ plotseling tot rust gebracht.

In paragraaf 1 wordt als inleiding de niet-magnetische compressibele versie van dit probleem beschouwd, op grond waarvan de beginvoorwaarden voor het magnetohydrodynamische probleem kunnen worden bepaald.

In paragraaf 2 worden de gelineariseerde instationnaire vergelijkingen getransformeerd volgens de transformatie van Laplace. In paragraaf 3

worden asymptotische oplossingen, geldig voor $t \to \infty$, verkregen voor het antisymmetrische geval, zonder dikte, in paragraaf 4 voor het symmetrische geval, met eindige dikte.
In beide gevallen worden voor $t \to \infty$ de stationnaire oplossingen van hoofdstuk V gevonden. Voor het symmetrische geval wordt, op grond van een analyse van de verschijnselen kort na $t = 0$, verklaard waarom de gelineariseerde theorie niet meer geldig is bij kleine dikte van de plaat en groot geleidingsvermogen van de vloeistof.
Hoofdstuk VII handelt over de twee-vloeistoffen theorie van de stationnaire stroming om de antisymmetrische plaat. De snelheid en de magnetische inductie blijken behalve de componenten, gevonden in hoofdstuk V, tevens loodrecht op het vlak daarvan componenten te hebben.
Terwijl in hoofdstuk V geen electrisch veld aanwezig is in een referentiesysteem, verbonden aan de plaat, is er thans een electrisch veld, waarvan de veldsterkte-vector ligt in het vlak door de ongestoorde snelheid en de normaal op de plaat.
Wanneer ladingsscheiding wordt verwaarloosd, is de electrische potentiaal discontinu op de plaat.
Aan het slot van dit hoofdstuk wordt de ladingsscheiding in de buurt van de plaat niet langer verwaarloosd en wordt bovengenoemde discontinuiteit in verband gebracht met zogenaamde Debye-lagen ter weerszijden van de plaat.

STELLINGEN

1. De oplossingen, verkregen door Sonnerup, voor de verstoringen van snelheid en magneetveld, optredende bij de stroming van een electrisch geleidend gas langs een gegolfde wand in de aanwezigheid van een magneetveld, dat ver van de wand evenwijdig is aan de snelheid van het gas, blijven, in tegenstelling tot wat Sonnerup meent, geldig wanneer de geleidbaarheid van het gas groot is.

   B. Sonnerup, Journal of the Aerospace Sciences, 28, 8, 1961.

2. Voor een neutraal gas kan uit de Boltzmann vergelijking een formele betrekking worden afgeleid voor de verandering van de, over alle snelheden gemiddelde, waarde van een bepaalde eigenschap. Deze betrekking kan niet zonder meer, zoals Ferraro en Plumpton hebben gedaan, worden gebruikt voor een geioniseerd gas, omdat hierbij in het algemeen de op de deeltjes uitgeoefende krachten van de snelheid afhangen.

   V. C. A. Ferraro and C. Plumpton, An Introduction to Magnetofluiddynamics, Oxford University Press, 1961, p. 123.

3. Het is bekend dat uit de wet van Ohm volgt, dat in het kader van de één-vloeistof theorie van de magnetohydrodynamica de vloeistof met het magneetveld meebeweegt, wanneer de geleidbaarheid groot is. De conclusie van Tayler en van Ware, dat op analoge wijze uit de, in de twee-vloeistoffen theorie geldende, "gegeneraliseerde wet van Ohm" volgt, dat, wanneer de electron cyclotronfrequentie groot is t. o. v. de botsingsfrequentie voor botsingen tussen electronen en ionen, de electronen met het magneetveld meebewegen, is onjuist.

   A.A.Ware, Conference on Plasma Physics and Controlled Nuclear Fusion Research, CN-20/47, Salzburg 1961.

   R.J.Tayler, Conference on Plasma Physics and Controlled Nuclear Fusion Research, CN-10/63, Salzburg 1961.

4. De door Phillips opgestelde uitdrukking voor het rendement van magnetohydrodynamische voortstuwing van schepen is onjuist. De gecorrigeerde formule voor het rendement laat zien, dat vooral bij gebruik van sterke magneetvelden een aanzienlijk hoger rendement kan worden verkregen dan uit de formule van Phillips volgt.

   Owen M. Phillips, Journal of Ship Research, 5, 4, 1962.

5. Van het ontstaan van de circulatie om een tweedimensionaal draagvlak, wordt in meerdere leerboeken een inconsistente verklaring gegeven.

   J. C. Hunsaker and B. G. Rightmire, Engineering Applications of Fluid Mechanics, McGraw-Hill, 1947, p. 240-241.

   L. A. v. d. Putte, Dictaat Technische Stromingsleer, Delft, 1958, p. 110.

   H. Schlichting und E. Truckenbrodt, Aerodynamik des Flugzeuges, Band I, Springer Verlag 1959, p. 371-374.

6. De door Whittaker en Watson in hun boek "A Course of Modern Analysis" gegeven vergelijking voor de analytische voortzetting van de hypergeometrische functie is niet geheel juist. De argumenten van de in deze vergelijking voorkomende gammafuncties moeten, voor zover die argumenten uit twee termen bestaan, met min één worden vermenigvuldigd.

7. De integraal $\int_0^\infty J_0(at)\, J_0(bt)\, e^{-t}\, dt$ kan tot een elliptische integraal van eenvoudiger gedaante dan die, door A. van Wijngaarden gevonden, worden herleid.

   A. van Wijngaarden, Enige toepassingen van Fourrier-integralen op elastische problemen Proefschrift Delft, 1945, p. 56-57.

   Wiskundige opgaven met de oplossingen, uitgegeven door het Wiskundig Genootschap te Amsterdam, deel 21 no. 2, 1961.
   oplossing door L. van Wijngaarden van opgave 50.

8. In Markus 5 vers 13 is geen enkele reden te vinden om, zoals Benjamin en Lighthill hebben gedaan, aan bepaalde oplossingen de voor zwaartekrachtsgolven geldende vergelijkingen de naam van "Gadareense oplossingen" te verbinden.

T. B. Benjamin and M. J. Lighthill,
Proc. of the Royal Society of London,
A 224, 1954, p. 454.

9. Het door S. Vestdijk in zijn essay "De geheimen van Wuthering Heights" gevoerde betoog is niet bij machte de uitspraak van A. Roland Holst "Heathcliff is een element" te weerleggen.

S. Vestdijk, De Poolse Ruiter,
Bert Bakker/ Daamen N. V., 1958.

10. Door Euwe wordt in de vierpaardenvariant van de Siciliaanse verdediging na 1. e4, c5 2. Pf3, e6 3. d4, cd4 4. Pd4, Pf6 5. Pc3, Pc6, de voortzetting 6. Lg5 als minder goed gekwalificeerd vanwege het antwoord 6 .. Db6. Deze opvatting is aanvechtbaar, daar wit na bv. 7. Pb3, Lb4 8. Lf6, gf6 9. Df3, Ke7 10. 0-0-0 goed staat.

Dr. M. Euwe, Theorie der Schaakopeningen,
deel 9, Half-open spelen, 1953, p. 119-120.

11. In voetbalcompetities en voetbaltournooien wordt de onderlinge rangschikking van teams, wier aantal wedstrijdpunten hetzelfde is, bepaald door het zogenaamde doelgemiddelde.
Deze maatstaf beloont aan de top van de ranglijst verdedigen meer dan aanvallen en kan daardoor tot de factoren worden gerekend, die, met name in tournooien zoals het tournooi om het wereldkampioenschap, verdedigend spel in de hand werken. Een beter criterium zou het verschil tussen doelpunten vóór en doelpunten tegen zijn.