The far field radiated by a dipole above a substrate with a film

By

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Abstract

Dipoles are all around us and the extent to which it is possible to distinguish two dipoles next to each other is a very important criterion for optical resolution. This report proposes that the resolution can be enhanced by placing a substrate with a film underneath the dipoles. This report specifically looks at the effect on the near and far field using a coupling of the field radiated by the dipole and a guided mode in the film. In an integral over reflected plane waves, in case a guided wave occurs, the integrand has a singularity which has to be dealt with using the Cauchy Principle value technique. Other singularities are dealt with using multiple first order Taylor expansions. With the singularities dealt with, a Matlab function has been constructed that can compute the reflected field of a single dipole. The substrate underneath the dipole has a significant effect on the total field, both in the near and far field areas. Due to the reflection, the polarizability is represented more in the intensity and can be determined more easily. Although the influence of the film on enhancing the resolution requires to investigate to what extent two dipoles can be distinguished better when a film is present, due to lack of time, this report considers the computation of the field of a single dipole only. The extension to the case of two dipoles is in principle easy however.

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Introduction

From the physical dipole used in electromagnetism to the molecular dipole used in chemistry, dipoles are all around us and their use in scientific research has increased significantly. Also in quantum mechanics, dipoles play an important role, making dipoles a diverse physical phenomena.

Any source distribution can be written as superposition of dipoles. Hence a dipole is the basic ingredient of which all sources consist. This makes it interesting to study the field of dipoles.
An important criterium of an optical system is the resolution which is to what extent two close dipoles can be distinguished by measuring the emitted fields. The surroundings of the dipole influence the emitted field.

When a single dipole is excited by a given incident field, the far field is different for the case when the dipole is situated on a substrate than for the case when the dipole is in free space. In the case of a substrate, the far field consists of two contributions namely the field radiated directly and the field reflected by the substrate. In addition, the reflected field contributes to the excitation of the dipole, i.e. the dipole strength is different when a substrate is present. The latter effect is called the Purcell effect.

When more than one dipole is present, the dipoles influence each other. The dipole moment of a dipole is then not only determined by the incident field and the reflected field of this dipole but also by the fields emitted by all other dipoles. For many dipoles this leads to a large coupled system of equations for the dipole moments. However, this report will only look at the case of a single dipole. The aim is to quantify the change of the far field due to the presence of a substrate and in particular of a film on the substrate which supports one or several guided modes.

We shall use a scalar theory for simplicity, i.e. effects due to polarisation are neglected.

This report will take a look at the effect of a substrate underneath a single dipole, specifically the field in the near field and far field areas. The goal is to develop a theory that could be adapted to work for two or more dipoles. A Matlab script is provided that has been used to compute the results for the single dipole case.

This report consists of four chapters and two appendices. The first chapter introduces the problem and the configuration that is studied, as well as the background theory needed to solve the problem. The second chapter focusses on the derivation of the total field and uses the stationary phase method to simplify the expression for the far field. The third chapter will explain the Matlab script and show multiple results using the script. The final chapter will be the conclusion of my research. The appendices will contain a full derivation of the stationary phase and the Matlab script.

1. Theory

This chapter gives an overview of the problem which is studied in this report and will give some necessary background theory to solve this problem.

All fields that are considered are time-harmonic with frequency $\omega > 0$. We will use scalar theory and complex notation throughout the report. Real fields are typically written as:

$$\mathrm{Re}\left[U(x,y,z)e^{-i\omega t}\right], \tag{1.1}$$

where $U(x,y,z)$ is the complex field (i.e. without the time dependent factor). Complex fields will be used in all computations.

1.1. Two dipoles

The starting situation of this report is that of two dipoles next to each other in free space. When the space between the dipoles is small enough, one cannot conclusively state, based on the received fields of the dipoles at a detector, what the distance between them is. In other words, the distance between the dipoles is smaller than the Rayleigh Criterion. This limits the obtainable resolution of the system.

The next situation, as shown in figure 1 , can be described as follows: Two dipoles are a distance z_p above a substrate material consisting of a film of thickness d and refractive index n_2 on a half space with refractive index n_3. The half space above the substrate is filled with a material with refractive index n_1 (usually $n_1 = 1$). The refractive index n_1 is assumed to be real, but n_2 and/or n_3 may be complex numbers with a non-negative imaginary part.

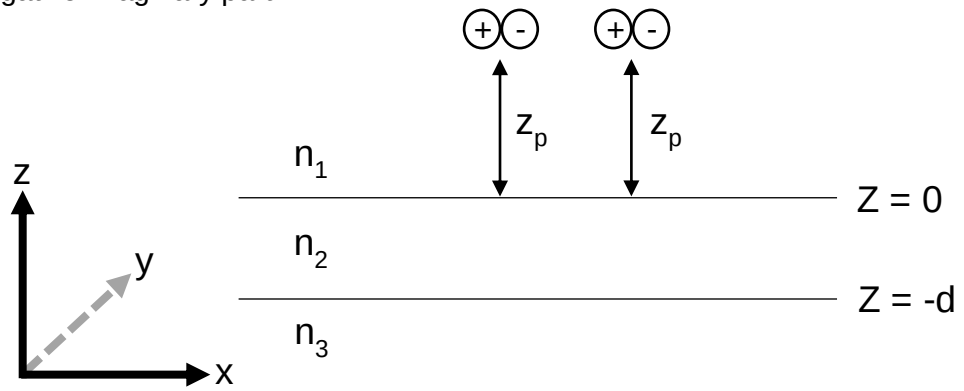

Figure 1: A sketch of two dipoles, depicted by the circles with a plus and a minus, a distance z_p above a substrate (with refractive index n_3) with a film of thickness d with refractive index n_2.

The idea is that the dipoles emit a field that hits the substrate which is then reflected back at the dipoles, interfering with the total field of the system. The goal of this setup is to find out if the reflection adds a usable interference to the total field that could potentially increase the resolution and thus allow for a more accurate measurement of the distance between the dipoles and the polarizability of the dipoles.

1.2. A single dipole

To determine the effect of the reflection on the total field of two dipoles, we first take a look at the effect of

the reflection on a single dipole. In this report the convention is used that the dipole is situated at (x_p, y_p, z_p).

Reflection on the substrate will always occur. However, we restrict ourselves to only consider the coupling of the radiated field of the dipole and a guided mode in the film. A guided mode is a wave propagating inside a so called wave guide which in this case is a film with thickness d, situated on top of the substrate [1]. Figure 2 shows a sketch of this situation. To obtain a guided mode inside the substrate, the wave has to be totally reflected inside the film. This only happens when the angle of incidence is larger than the critical angle of the material, which on itself can only happen when the index of refraction of the substrate, n_2, is larger than the indices of refraction of the surrounding materials n_1 and n_3. Furthermore, the interference of multiple reflected fields has to be constructive, meaning that after being reflected twice (once at $Z = 0$ and once at $Z = -d$) the phase of the field has to be increased by a integer multiple of 2π. As shown in figure 2, theoretically, the wave inside the substrate goes up and down multiple times. At every reflection, the field loses intensity since part of the field is transmitted into the substrate or into the half space above the film, indicated in figure 2 by dashed arrows.

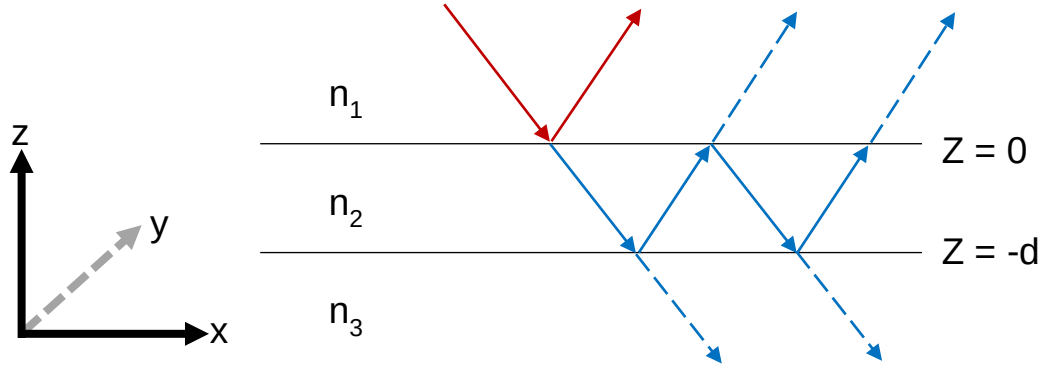

Figure 2: A sketch of a field hitting the film on top of the substrate. The arrows indicate the direction of the field. The blue arrows inside the film with refractive index n_2 are a reference to a guided mode. The dashed arrows indicate transmissions into the substrate and into the half space above the film. The red arrows are not part of a guided mode.

In order to determine the reflected field of a given incident plane wave from the half space $z > 0$, the reflection coefficient R for this particular situation is needed. The Fresnel reflection coefficient for a wave incident from a medium with refractive index n_i on an interface of a medium with refractive index n_j is:

$$r_{i,j} = \frac{k_{z,i} - k_{z,j}}{k_{z,i} + k_{z,j}}, \tag{1.2}$$

where $k_{z,i}$ is the z-component of the wavevector in the material with refractive index n_i

$$k_{z,i} = \sqrt{k_0^2 n_i^2 - k_\perp^2}, \tag{1.3}$$

$$k_\perp^2 = k_x^2 + k_y^2, \tag{1.4}$$

where k_0 is the wavevector in vacuum. The presence of the film gives rise to corrections of the reflection coefficient. These corrections are a transmission at the $n_1 - n_2$ interface, a reflection at the $n_2 - n_3$ interface, and lastly a transmission at the $n_2 - n_1$ interface. These corrections are only valid if, after one roundtrip, the particular wave exits the substrate with a phase difference of 2π or an integer multiple of 2π. However, part of the wave stays inside the substrate and allows for more reflections. The following equation shows how the reflection coefficient is dependent on the extra reflections.

$$R(k_\perp) = r_{1,2} + t_{2,1} r_{2,3} t_{1,2} e^{2ik_{z,2}d} \left\{ 1 + r_{2,3} r_{2,1} e^{2ik_{z,2}d} + \ldots \right\}, \tag{1.5}$$

where $t_{1,2}$ and $t_{2,1}$ are the Fresnel transmission coefficients

$$t_{i,j} = 1 + r_{i,j} = \frac{2k_{z,i}}{k_{z,i} + k_{z,j}}. \tag{1.6}$$

The exponent in (1.5) represents the change of phase due to one round rip of the wave inside the film. Since the field goes down and up once, the thickness d is traversed twice, explaining the 2 in the exponent. The wave loses amplitude at every reflection and in Eq. (1.5) only the contribution due to the first and second round trip is written (higher order round trips are represented by higher powers).
Equation (1.5), using (1.6), then becomes

$$R(k_\perp) = \frac{r_{1,2} + r_{2,3} e^{2ik_{z,2}d}}{1 + r_{2,3} r_{1,2} e^{2ik_{z,2}d}}. \tag{1.7}$$

In the derivation there has been used

$$t_{1,2} t_{2,1} = (1 + r_{1,2})(1 - r_{1,2}) = 1 - {r_{1,2}}^2. \tag{1.8}$$

We will use the following formulation for a dipole in a homogeneous space with refractive index n

$$G(\vec{r} - \vec{r}_p)\, p = -\frac{1}{4\pi} \frac{e^{ikn\left|\vec{r} - \vec{r}_p\right|}}{\left|\vec{r} - \vec{r}_p\right|} \frac{{k_0}^2}{\varepsilon_0} p, \tag{1.9}$$

where $\left|\vec{r} - \vec{r}_p\right|$ is given by

$$\left|\vec{r} - \vec{r}_p\right| = \sqrt{\left(x - x_p\right)^2 + \left(y - y_p\right)^2 + \left(z - z_p\right)^2}, \tag{1.10}$$

and ε_0 is the permittivity of free space.

2. The field of a single dipole

This chapter will take an in-depth look at the derivation of the reflected field of a single dipole as well as the total field in the near and far field areas.

2.1. The total field

The field incidence on the substrate in a point $\vec{r}=(x,y,z)$ will be indicated as $U^{In}(\vec{r})$. By taking the Fourier transform, one can determine the reflected waves that are reflected on the substrate for every incoming plane wave by multiplying the plane wave with the reflection coefficient. Summing over all the reflected waves will result in the reflected field $U^{r}(\vec{r})$. Adding the reflected field to the incidence field results in the total field in the upper half space $z>0$ without the contribution of the dipole:

$$U_{tot}^{In}(\vec{r})=U^{In}(\vec{r})+U^{r}(\vec{r}) \tag{2.1}$$
$$=U^{In}(\vec{r})+\frac{1}{4\pi^2}\iint R(k_\perp)\mathcal{F}\left(U^{In}\right)\left(\frac{k_x}{2\pi},\frac{k_y}{2\pi},z_p\right)e^{i(k_x x+k_y y-k_z(z+z_p))}dk_x dk_y.$$

Note that the integral in (2.1) is restricted to values of k_x and k_y for which the incident waves are propagating, i.e. for which $k_\perp<k_0 n_1$, so that $k_{z,1}$ is real. This implies in particular that no guided wave occurs in these integrals.

The total field emitted by the dipole consists of a part emitted in free space and a part due to the reflection by the surface

$$U_{dipole}(\vec{r})=G^{free}(\vec{r}-\vec{r}_p)\,p+G^{r}(\vec{r},\vec{r}_p)\,p, \tag{2.2}$$

where the first term in (2.2) is the field emitted in homogeneous space, given by (1.9), the second term is the field due to the reflection and p is the strength of the dipole. We assume here that the dipole strength is known. It will be explained below how p can be computed when the polarizability is known. The reflected field uses the same idea as the second term of (2.1): It is the inverse Fourier transform of the reflection coefficient times the Fourier transform of the field in homogeneous space and a phase factor due to propagation from z_p to z:

$$G^{r}(\vec{r},\vec{r}_p)=\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}R(k_\perp)\mathcal{F}\left(G^{free}\right)\left(\frac{k_x}{2\pi},\frac{k_y}{2\pi},z_p\right)e^{i(k_x(x-x_p)+k_y(y-y_p)+k_z z)}dk_x dk_y, \tag{2.3}$$

where

$$\mathcal{F}\left(G^{free}(\vec{r}-\vec{r}_p)\right)=\frac{e^{-i\left(k_x x_p+k_y y_p-k_z|z-z_p|\right)}}{2ik_z}. \tag{2.4}$$

In contrast to the integral in (2.2) , the integral in (2.3) extends over all real values of k_x and k_y.

The reflected field modifies the strength of the dipole, an effect called the Purcell effect. When the polarizability α is known, the dipole strength becomes

$$p=\varepsilon_0\alpha U_{tot}^{In}(\vec{r}_p)+\varepsilon_0\alpha G^{r}(\vec{r},\vec{r}_p)\,p. \tag{2.5}$$

Solving for p and substituting the result in (2.2) results in the following expression for the total field of the dipole

$$U_{dipole}(\vec{r}) = \left[G^{free}(\vec{r} - \vec{r}_p) + G^r(\vec{r}, \vec{r}_p) \right] \frac{\varepsilon_0 \alpha U_{tot}^{In}(\vec{r}_p)}{1 - \varepsilon_0 \alpha G^r(\vec{r}, \vec{r}_p)}. \tag{2.6}$$

2.2. Propagation constant β

For a guided mode, the reflection coefficient is infinitely large which will cause problems for the integral of (2.3) . Therefore, we first study guided modes. The situation as shown in figure 1 will be used to explain some theory about guided modes in a dielectric wave guide. Because of the rotational symmetry, it suffices to consider guided waves which do not depend on the y-coordinate and which propagate in the positive x-direction, i.e. we assume here that $k_y = 0$ and $k_x > 0$. Hence, we have $k_\perp = k_x$. Since scalar theory is used, only TE-polarised guided waves are considered here. We assume that all indices of refraction used in figure 1 are real and that $n_2 > n_1$ and $n_2 > n_3$. If one of these inequalities is violated, no guided mode exists.

The electric field of a TE-guided wave under the stated assumptions has only a y-component which is written as:

$$E_y(x, z) = u(z) e^{ik_\perp x}. \tag{2.7}$$

Maxwell's equations imply that the function $u(z)$ satisfies the following equations:

$$(k_0^2 n_1^2 - k_\perp^2) u + \frac{d^2 u}{dz^2} = 0, \text{ for } z > 0 \tag{2.8}$$

$$(k_0^2 n_2^2 - k_\perp^2) u + \frac{d^2 u}{dz^2} = 0, \text{ for } -d < z < 0 \tag{2.9}$$

$$(k_0^2 n_3^2 - k_\perp^2) u + \frac{d^2 u}{dz^2} = 0, \text{ for } z < -d \tag{2.10}$$

In this situation it is required that there is a guided mode inside the film. This means that the field outside the substrate, for $z > 0$ and $z < -d$, will decrease exponentially because the wave is totally reflected inside the film at the surfaces $z = 0$ and $z = -d$. This can only happen as long as $k_\perp^2 > k_0^2 n_1^2$ and $k_\perp^2 > k_0^2 n_3^2$. For (2.8) - (2.10) to satisfy the requirement that the field decreases exponentially outside the substrate, $u(z)$ is then given by

$$u(z) = A e^{-z\sqrt{k_\perp^2 - k_0^2 n_1^2}}, \quad \text{for} \quad z > 0 \tag{2.11}$$

$$u(z) = B e^{-iz\sqrt{k_0^2 n_2^2 - k_\perp^2}} + C e^{iz\sqrt{k_0^2 n_2^2 - k_\perp^2}}, \quad \text{for} \quad 0 > z > -d \tag{2.12}$$

$$u(z) = D e^{z\sqrt{k_\perp^2 - k_0^2 n_3^2}}, \quad \text{for} \quad z < -d \tag{2.13}$$

where A, B, C and D are amplitudes that have yet to be determined. To simplify upcoming equations, abbreviations are used

$$\alpha_1 = \sqrt{k_\perp^2 - k_0^2 n_1^2}, \tag{2.14}$$

$$\alpha_2 = \sqrt{k_0^2 n_2^2 - k_\perp^2}, \tag{2.15}$$

$$\alpha_3 = \sqrt{k_\perp^2 - k_0^2 n_3^2}. \tag{2.16}$$

The tangential electric and magnetic field components must be continuous at the interfaces $z=0$ and $z=-d$. This implies that E_y and $H_x=\left(\frac{-i}{\omega\mu_0}\right)\frac{\partial E_y}{\partial z}$ are continuous for $z=0$ and $z=-d$ which in turn is equivalent to the continuity of u and $\frac{du}{dz}$. By requiring this continuity, one can obtain the following equations

$$A-B-C=0, \tag{2.17}$$

$$-\alpha_1 A+i\alpha_2 B-i\alpha_2 C=0, \tag{2.18}$$

$$De^{-\alpha_3 d}-Be^{i\alpha_2 d}-Ce^{-i\alpha_2 d}=0, \tag{2.19}$$

$$\alpha_3 De^{-\alpha_3 d}+i\alpha_2 Be^{i\alpha_2 d}-i\alpha_2 Ce^{-i\alpha_2 d}=0. \tag{2.20}$$

The solution of A, B, C and D, (2.17) – (2.20) is only non-zero if the determinant of the system of equations is equal to zero. Setting this determinant equal to zero and rearranging terms leads to the following dispersion relation

$$\tan(\alpha_2 d)=\frac{\alpha_2\left(\alpha_1+\alpha_3\right)}{{\alpha_2}^2-\alpha_1\alpha_3}. \tag{2.21}$$

When ${\alpha_2}^2>{k_0}^2\min\left({n_2}^2-{n_1}^2,{n_2}^2-{n_3}^2\right)$, the right hand side becomes purely imaginary. Since the tangent does not have an imaginary part, (2.21) can only hold if

$${k_0}^2\max\left({n_1}^2,{n_3}^2\right)<{k_\perp}^2<{k_0}^2{n_2}^2. \tag{2.22}$$

Only when (2.21) is satisfied will there be a guided mode. The value of $k_\perp$ for which this is the case can be found by plotting the left hand side and right hand side of (2.21) and finding the intersections using a program like Matlab. The value of $k_\perp$ of these intersections will be the value of β. The amount of intersections is determined by the indices of refraction used. When $n_3\neq n_1$ and either n_2 or d is not large enough, it is possible that there are no guided modes available. When $n_1=n_3$ and $n_2>n_1$, there will always be at least one guided mode. We will explain how to handle multiple guided modes below.

As mentioned before, the reflection coefficient for a guided mode, $R(k_\perp)$, is infinitely large (i.e. for $k_\perp=\beta$). For this to happen, it is assumed that

$$R(k_\perp)=\frac{f(k_\perp)}{k_\perp-\beta}. \tag{2.23}$$

First of all, to find the function $f(k_\perp)$, we look at the denominator in (1.7) and denote it as $g(k_\perp)$:

$$g(k_\perp)=1+r_{2,3}r_{1,2}e^{2ik_{z,2}d}. \tag{2.24}$$

The next step is to use a first order Taylor expansion of $g(k_\perp)$ around $k_\perp=\beta$. β has to be such that $R(\beta)=\infty$, which means that, looking at (1.7) and (2.23), $g(\beta)=0$. This results in the following, first order Taylor expansion

$$g(k_\perp)=g(\beta)+g'(\beta)\left(k_\perp-\beta\right)=g'(\beta)\left(k_\perp-\beta\right), \tag{2.25}$$

where $g'(\beta)$ denotes the first order derivative of $g(k_\perp)$ evaluated at the point β. Finally, (2.25) is substituted in (1.7) to retrieve $f(k_\perp)$

$$R(k_\perp)=\frac{f(k_\perp)}{k_\perp-\beta}=\frac{r_{1,2}+r_{2,3}e^{2ik_{z,2}d}}{1+r_{2,3}r_{1,2}e^{2ik_{z,2}d}}=\frac{r_{1,2}+r_{2,3}e^{2ik_{z,2}d}}{g(\beta)}=\frac{r_{1,2}+r_{2,3}e^{2ik_{z,2}d}}{g'(\beta)}\frac{1}{(k_\perp-\beta)}. \tag{2.26}$$

Hence,

$$f(k_\perp)=\frac{r_{1,2}+r_{2,3}e^{2ik_{z,2}d}}{g'(\beta)}. \tag{2.27}$$

2.3. The reflected field

To derive the reflected field, it is easier to write (2.3) in polar coordinates. Substituting the Fourier transform (2.4) in (2.3) and writing the result in polar coordinates results in the following equation

$$G^r(\vec{r},\vec{r}_p)=\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{\infty}R(k_\perp)\frac{1}{2ik_{z,1}}e^{ik_\perp r_\perp\cos(\varphi-\psi)+ik_{z,1}\left(z+|z_p|\right)}k_\perp dk_\perp d\psi, \tag{2.28}$$

where we used polar coordinates defined such that:

$$k_x=k_\perp\cos(\psi), \tag{2.29}$$

$$k_y=k_\perp\sin(\psi), \tag{2.30}$$

$$x-x_p=r_\perp\cos(\varphi), \tag{2.31}$$

$$y-y_p=r_\perp\sin(\varphi). \tag{2.32}$$

The z-coordinate of the dipole z_p will always be positive and real, therefore the absolute value of z_p is z_p itself.

The integral over ψ is a well-known integral and results in a Bessel function [2]

$$\int_0^{2\pi}e^{ik_\perp r_\perp\cos(\varphi-\psi)}d\psi=2\pi J_0(k_\perp r_\perp). \tag{2.33}$$

Substituting (2.33) in (2.28) and using (2.23) for $R(k_\perp)$:

$$G^r(\vec{r},\vec{r}_p)=\frac{-i}{4\pi}\int_0^{\infty}\frac{f(k_\perp)}{k_\perp-\tilde{\beta}}J_0(k_\perp r_\perp)e^{ik_{z,1}(z+z_p)}\frac{k_\perp}{k_{z,1}}dk_\perp. \tag{2.34}$$

It is important to note two things about (2.34): First is the fact that it is an integral over all positive values of $k_\perp$. This is significant because of the restrictions on β by (2.22). Since the interval given by (2.22) is within the interval posed by (2.34), a singularity will occur when $k_\perp=\beta$ due to a division by zero. To cope with the singularity at $k_\perp=\beta$, the Cauchy Principal Value will be used [3]. The integral of (2.34) will be divided into four parts according to the Cauchy Principal Value. Let δ be a number such that $0<\delta<\beta$.

$$G^r(\vec{r},\vec{r}_p)=\frac{-i}{4\pi}\int_0^{\tilde{\beta}-\delta}\frac{f(k_\perp)}{k_\perp-\tilde{\beta}}J_0(k_\perp r_\perp)e^{ik_{z,1}(z+z_p)}\frac{k_\perp}{k_{z,1}}dk_\perp-\frac{i}{4\pi}\int_{\tilde{\beta}+\delta}^{\infty}\frac{f(k_\perp)}{k_\perp-\tilde{\beta}}J_0(k_\perp r_\perp)e^{ik_{z,1}(z+z_p)}\frac{k_\perp}{k_{z,1}}dk_\perp$$
$$-\frac{i}{4\pi}\int_{\tilde{\beta}-\delta}^{\tilde{\beta}+\delta}\frac{f(k_\perp)J_0(k_\perp r_\perp)e^{ik_{z,1}(z+z_p)}\frac{k_\perp}{k_{z,1}}-f(\tilde{\beta})J_0(\tilde{\beta}r_\perp)e^{ik_{z,1}(\tilde{\beta})(z+z_p)}\frac{\tilde{\beta}}{k_{z,1}(\tilde{\beta})}}{k_\perp-\tilde{\beta}}dk_\perp \tag{2.35}$$
$$-\frac{i}{4\pi}f(\tilde{\beta})J_0(\tilde{\beta}r_\perp)e^{ik_{z,1}(\tilde{\beta})(z+z_p)}\frac{\tilde{\beta}}{k_{z,1}(\tilde{\beta})}\int_{\tilde{\beta}-\delta}^{\tilde{\beta}+\delta}\frac{1}{k_\perp-\tilde{\beta}}dk_\perp.$$

The integral is divided into an integral just before the singularity, an integral after the singularity, followed by two integrals dealing with the singularity itself, respectively. For the last term the following equality holds

$$\lim_{\mathrm{Im}(\tilde{\beta})\to 0}\int_{\tilde{\beta}-\delta}^{\tilde{\beta}+\delta}\frac{1}{k_\perp-\tilde{\beta}}dk_\perp=\int_{\beta-\delta}^{\beta+\delta}\frac{1}{k_\perp-\beta}dk_\perp=0, \tag{2.36}$$

since the integrand is odd around $k_\perp=\beta$.

The second important thing to remark about the integral in (2.34) is that $\tilde{\beta}$ is the complex version of β. It is assumed that the film has absorption, meaning that

$$\varepsilon_2=n_2^{\,2}, \tag{2.37}$$

has a positive, imaginary part. This leads to $k_\perp=\tilde{\beta}$ also having a positive, imaginary part. Therefore $\tilde{\beta}$ is for a guided mode with absorption and is complex valued, while β is for a guided mode without absorption and is real valued. Taking the limit of $\tilde{\beta}$ to β (i.e. letting the absorption go to 0) and combining (2.35) and (2.36) gives:

$$G^r(\vec{r},\vec{r}_p)=\frac{-i}{4\pi}\int_0^{\beta-\delta}R(k_\perp)J_0(k_\perp r_\perp)e^{ik_{z,1}(z+z_p)}\frac{k_\perp}{k_{z,1}}dk_\perp-\frac{i}{4\pi}\int_{\beta+\delta}^{\infty}R(k_\perp)J_0(k_\perp r_\perp)e^{ik_{z,1}(z+z_p)}\frac{k_\perp}{k_{z,1}}dk_\perp$$
$$-\frac{i}{4\pi}\int_{\beta-\delta}^{\beta+\delta}\frac{f(k_\perp)J_0(k_\perp r_\perp)e^{ik_{z,1}(z+z_p)}\frac{k_\perp}{k_{z,1}}-f(\beta)J_0(\beta r_\perp)e^{ik_{z,1}(\beta)(z+z_p)}\frac{\beta}{k_{z,1}(\beta)}}{k_\perp-\beta}dk_\perp. \tag{2.38}$$

The first and last term of (2.38) both still have singularities.

It is possible for multiple guided modes to be available. Every guided mode has its own singularity and has to be dealt with using the same method as described above. This means that (2.38) will have additional integrals equal to the amount of guided modes available. This report will focus on the case where only one guided mode is available.

The next two parts will discuss a way to deal with the singularities of (2.38).

2.4. The first term of (2.38)

The first term of (2.38) has a singularity at $k_{z,1}=0$ which happens when $k_\perp^{\,2}=k_0^{\,2}n_1^{\,2}$. Such a singularity can be dealt with effectively by a change of the integration variable as will be explained now.

First, we split the integral in two parts as shown in (2.39):

$$\int_0^{\beta-\delta} R(k_\perp)J_0(k_\perp r_\perp)e^{ik_{z,1}(z+z_p)}\frac{k_\perp}{k_{z,1}}dk_\perp = \int_0^{k_0 n_1} R(k_\perp)J_0(k_\perp r_\perp)e^{ik_{z,1}(z+z_p)}\frac{k_\perp}{k_{z,1}}dk_\perp + \int_{k_0 n_1}^{\beta-\delta} R(k_\perp)J_0(k_\perp r_\perp)e^{ik_{z,1}(z+z_p)}\frac{k_\perp}{k_{z,1}}dk_\perp. \quad (2.39)$$

In the first integral we change the integration variable into $k_{z,1}$, using:

$$\frac{dk_{z,1}}{dk_\perp} = -\frac{k_\perp}{k_{z,1}}, \quad (2.40)$$

$$k_\perp = \sqrt{k_0{}^2 n_1{}^2 - k_{z,1}{}^2}. \quad (2.41)$$

This gives:

$$\int_0^{k_0 n_1} R(k_\perp)J_0(k_\perp r_\perp)e^{ik_{z,1}(z+z_p)}\frac{k_\perp}{k_{z,1}}dk_\perp = \int_0^{k_0 n_1} R(\sqrt{k_0{}^2 n_1{}^2 - k_{z,1}{}^2})J_0(r_\perp\sqrt{k_0{}^2 n_1{}^2 - k_{z,1}{}^2})e^{ik_{z,1}(z+z_p)}dk_{z,1}. \quad (2.42)$$

For the second integral at the right of (2.39) we use $s = \sqrt{k_\perp{}^2 - k_0{}^2 n_1{}^2}$ as integration variable . There holds:

$$k_{z,1} = is, \quad (2.43)$$

$$\frac{dk_{z,1}}{ds} = i, \quad (2.44)$$

$$k_\perp = \sqrt{k_0{}^2 n_1{}^2 + s^2}. \quad (2.45)$$

Using (2.43) – (2.45) , the second integral becomes

$$\int_{k_0 n_1}^{\beta-\delta} R(k_\perp)J_0(k_\perp r_\perp)e^{ik_{z,1}(z+z_p)}\frac{k_\perp}{k_{z,1}}dk_\perp = \int_0^{\sqrt{(\beta-\delta)^2 - k_0{}^2 n_1{}^2}} -iR(\sqrt{k_0{}^2 n_1{}^2 + s^2})J_0(\sqrt{k_0{}^2 n_1{}^2 + s^2})e^{-s(z+z_p)}ds. \quad (2.46)$$

This way, all singularities in the first term of (2.38) are dealt with and the integrals in (2.42) and (2.46) can be computed numerically without further problems.

2.5. The third term in (2.38)

The third term in (2.38) has a singularity at $k_\perp = \beta$ which, however, is removable. The integral can be rewritten as follows

$$\int_{\beta-\delta}^{\beta+\delta} \frac{f(k_\perp)J_0(k_\perp r_\perp)e^{ik_{z,1}(z+z_p)}\frac{k_\perp}{k_{z,1}} - f(\beta)J_0(\beta r_\perp)e^{ik_{z,1}(\beta)(z+z_p)}\frac{\beta}{k_{z,1}(\beta)}}{k_\perp - \beta}dk_\perp = \int_{\beta-\delta}^{\beta+\delta}\frac{j(k_\perp) - j(\beta)}{k_\perp - \beta}dk_\perp, \quad (2.47)$$

where

$$j(k_\perp) = f(k_\perp)J_0(k_\perp r_\perp)e^{ik_{z,1}(z+z_p)}\frac{k_\perp}{k_{z,1}(k_\perp)}. \quad (2.48)$$

In this form, a first order Taylor expansion around the point β will solve the issue with the singularity.

Using:

$$j(k_\perp) = j(\beta) + j'(\beta)(k_\perp - \beta), \tag{2.49}$$

we obtain

$$\int_{\beta-\delta}^{\beta+\delta} \frac{j(k_\perp) - j(\beta)}{k_\perp - \beta} dk_\perp = 2\delta j'(\beta), \tag{2.50}$$

where the apostrophe indicates the first order derivative with respect to $k_\perp$. Note that this is only an accurate approximation for a small enough δ. Using the Taylor expansion around β allows the circumvention of the singularity and only requires one to determine the derivative of (2.48) .

2.6. Far field computations

The previous formula are valid for an arbitrary point $\vec{r} = (x, y, z)$ in the upper half space, i.e. for $z > 0$. However, when z is very large (the far field), the plane wave expansion for the reflected field is not practical because the integrand is strongly oscillating. Since the axis are chosen as shown in figures 1 and 2, doing far field computations will require one to take the limit of the z coordinate being very large. Because the exponent in (2.3) is imaginary for the propagating plane waves, the integrand in the plane wave expansion of the reflected field oscillates rapidly as function of the integration variable $k_\perp$. However, it is still possible to compute the far field, namely using the method of stationary phase. Appendix A shows the full derivation. Here only the final result for the reflected field is given:

$$G^r(\vec{r}_\infty, \vec{r}_p) \approx -R\left(k_0 n_1 \sin(\theta)\right) \frac{1}{4\pi} \frac{e^{ik_0 n_1 \left|\vec{r}_\infty - \tilde{\vec{r}}_p\right|}}{\left|\vec{r}_\infty - \tilde{\vec{r}}_p\right|}, \tag{2.51}$$

where

$$\sin(\theta) = \frac{\sqrt{\left(x - x_p\right)^2 + \left(y - y_p\right)^2}}{\left|\vec{r}_\infty - \tilde{\vec{r}}_p\right|}. \tag{2.52}$$

and

$$\tilde{r}_p = (x_p, y_p, -z_p). \tag{2.53}$$

since the reflected field can be described as the field from a dipole at a distance z_p below the film.

The result of (2.51) is effectively the field of a dipole situated at $-z_p$ multiplied by a reflection coefficient that is dependent on the angle we are looking at.

Finally, the intensity in the far field is:

$$I_{scat} = \left| \left(G^{free}(\vec{r}_\infty - \vec{r})\, p + G^r(\vec{r}_\infty, \vec{r}) \right) p \right|^2. \tag{2.54}$$

3. Validation of the code and results

The derivation from the previous chapters is implemented numerically in a Matlab script, which is listed in Appendix B. The code consists of four parts:
1. The far field computations and plots
2. The near field computations and plots
3. A function to compute the reflected field
4. Extra functions.

The function to compute the reflected field consists of three parts: The first part computes the values of β, the second part computes the integral of (2.3) and the third part checks the code for the special case that the substrate consists of a perfect mirror. The code was initially tested for a symmetric situation, meaning $n_1 = n_3$ and $n_2 > n_1$. All values assigned in the code are in terms of the wavelength λ hence the unit of length is chosen such that the wavelength in vacuum is 1. Lastly, we assume that $\frac{k_0^{\,2}}{\varepsilon_0} p = 1$. A computation of the dipole strength as function of the distance z_p will be provided.

3.1. Computing β

To compute the values of β, the solutions of (2.21) have to be found, keeping (2.22) in mind. The shape of the plot of (2.21) heavily depends on the thickness d of the substrate. Due to (2.22), (2.21) has the following bounds

$$0 < a_2 d < k_0 \sqrt{\frac{2n_2^{\,2} - n_1^{\,2} - n_3^{\,2}}{2}}. \tag{3.1}$$

In the code, d has been chosen to be $\frac{\lambda}{4}$ because this value allows for a single intersection in (2.21) while not making the substrate extremely thin. Figure 3 shows a plot of (2.21) as function of $\alpha_2 d$ using $d = \frac{\lambda}{4}$.

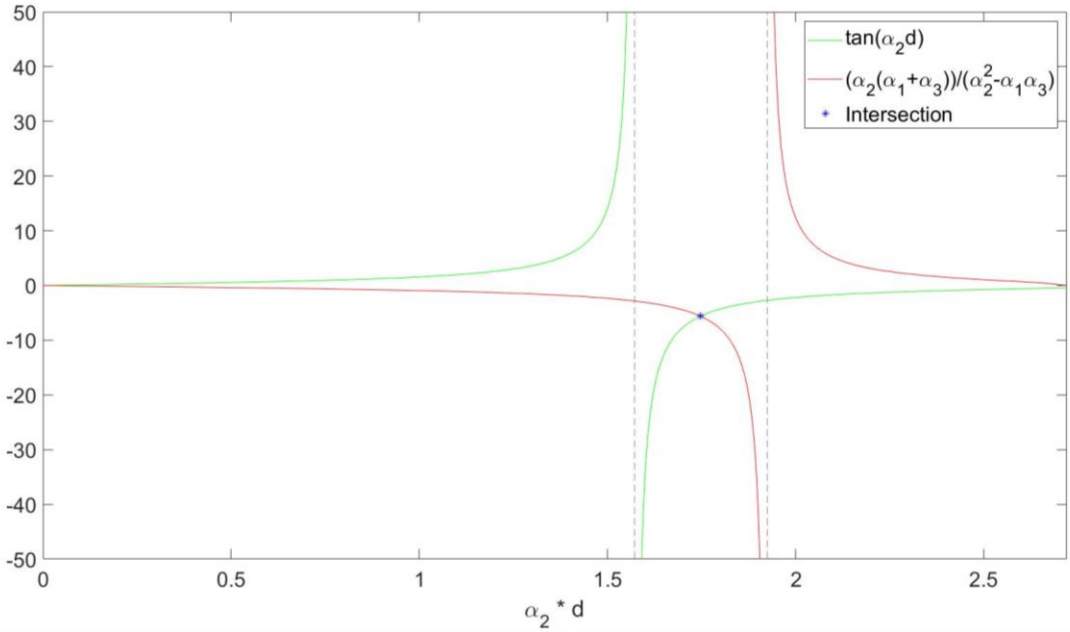

Figure 3: Graphical representation of (2.21). For the symmetric case where $n_1 = n_3 = 1$ and $n_2 = 2$.

The intersection of the left hand side and right hand side of (2.21) as can be seen in figure 3 corresponds to where $k_\perp = \beta$. To make sure there are solutions to (2.21), an *if-statement* around the code first checks if there are any solutions to (2.21) for the current settings and thus if there are any values for β.

3.2. Computing the reflected field

If the code has been able to compute a value for β, all parts of (2.38) are computed separately. The δ has been set to 0.1. After testing with different values, this value seemed to be small enough for the first order Taylor expansion to be accurate compared to the computation without the expansion. The integrals are computed using the *integral* function of Matlab. As a test, the code also computes the first integral of (2.38) without the substitution (2.40) using the *integral* function. The integral without the substitution results in a warning in the command window of Matlab, showing the necessity of the substitution. The output of the function computing the reflected field consists of a vector with five elements: The first element uses both substitutions (2.40) and (2.43) and the first order Taylor expansions. The second element also uses both substitutions (2.40) and (2.43) but computes the first integral of (2.38) using the *integral* function of Matlab. The third element does not use the substitutions but does use the first order Taylor expansions to compute (2.38). The fourth element does not use either the substitutions nor the first order Taylor expansions. The fifth and final element is the value computed for the field of a dipole in a homogeneous medium, (1.9).

3.3. Checking the code with a perfect mirror

If the substrate would be a prefect mirror, the reflected of the dipole can be described as the field of a dipole at the position that is on the opposite side of the interface $z = 0$ (i.e. $z = -z_p$) and with dipole strength equal but out of phase with respect to the original dipole. By choosing the values of n_2 and n_3 to be equal and large, the situation will mimic that of a perfect mirror. However, in this situation there is no value for β. Since the code heavily relies on values for β, an *if-statement* checks if there are any solutions for (2.21) . For the particular case when $n_2 = n_3$, the code assumes the substrate to be a perfect mirror if there are no values for β. When $n_2 = n_3$, $k_{z,2} = k_{z,3}$ and (1.7) becomes

$$R = r_{12} = \frac{k_{z,1} - k_{z,2}}{k_{z,1} + k_{z,2}}. \tag{3.2}$$

For the limit $n_2 >> n_1$, (3.2) becomes -1 for a large interval of values $k_\perp$, which represents a perfect mirror. Having no value for β means that there is no function $f(k_\perp)$ and thus no singularity at $k_\perp = \beta$, allowing for the integral of (2.34) to run from zero to infinity using the substitutions of (2.40) and (2.43). The computed reflected field is then compared to the theoretical field (1.9) for both the magnitude and the argument. The computed reflected field is plotted on a grid where x runs from -5λ to 5λ in 100 equidistant steps while the z-coordinate is between 0 and 5λ. The theoretical field is plotted on a grid where both x and z run from -5λ to 5λ in 100 equidistant steps. For these plots, the location of the dipole has been set to $(0, 0, \lambda)$. The plots are shown in figure 4 and figure 5.

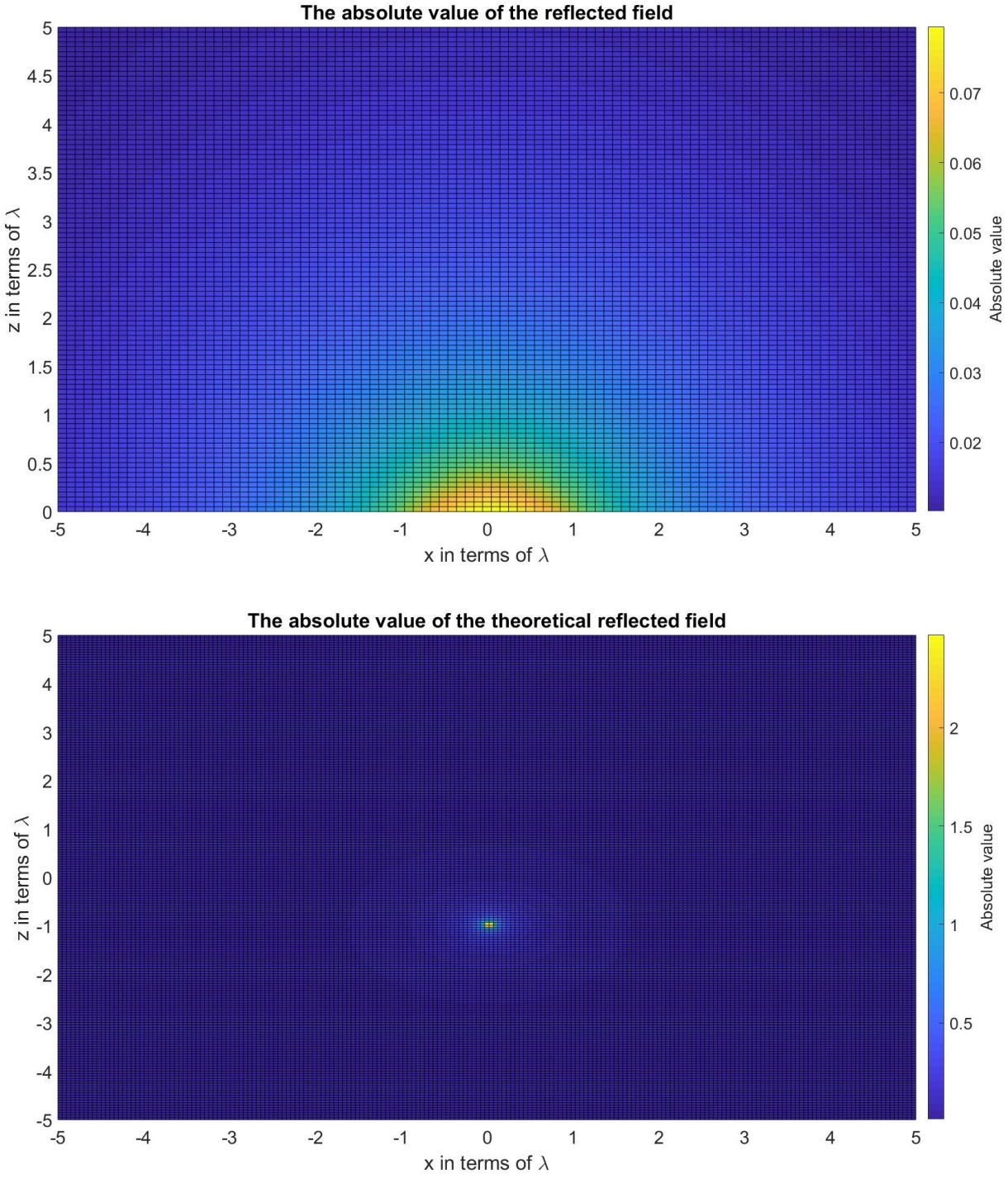

Figure 4: The absolute value of the computed reflected field and of the theoretical reflected field. We choose $n_2 = n_3 = 200$ and $n_1 = 1$.

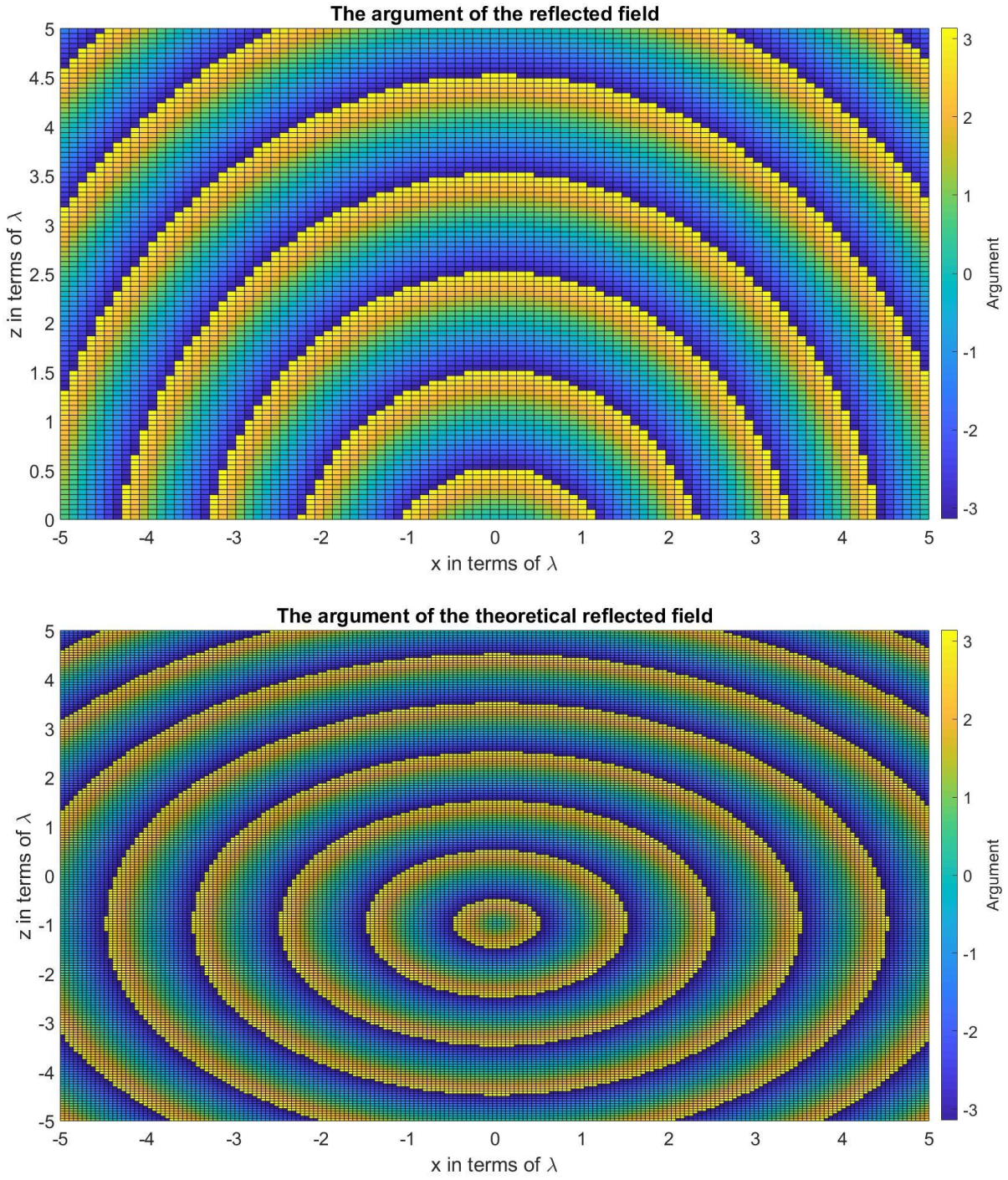

Figure 5: The argument of the computed reflected field and of the theoretical reflected field. We choose $n_2 = n_3 = 200$ and $n_1 = 1$.

3.4. Near field computations

The methods described in chapter 2 are all combined into a single function. The next step is to calculate the total field relatively close to the dipole itself, constructing a near field. For the near field, the dipole is situated at $(0,0,\lambda)$. The y-coordinate is kept at 0, the x-coordinate runs from -2λ to 2λ, and the z-coordinate from 0 to 2λ. We choose the refractive index $n_1 = 1$, for the film we choose $n_2 = 2$ and for the substrate $n_3 = 1$, creating a symmetric system. Figure 6 is a plot of the absolute value and figure 7 is a plot of the argument of the near field.

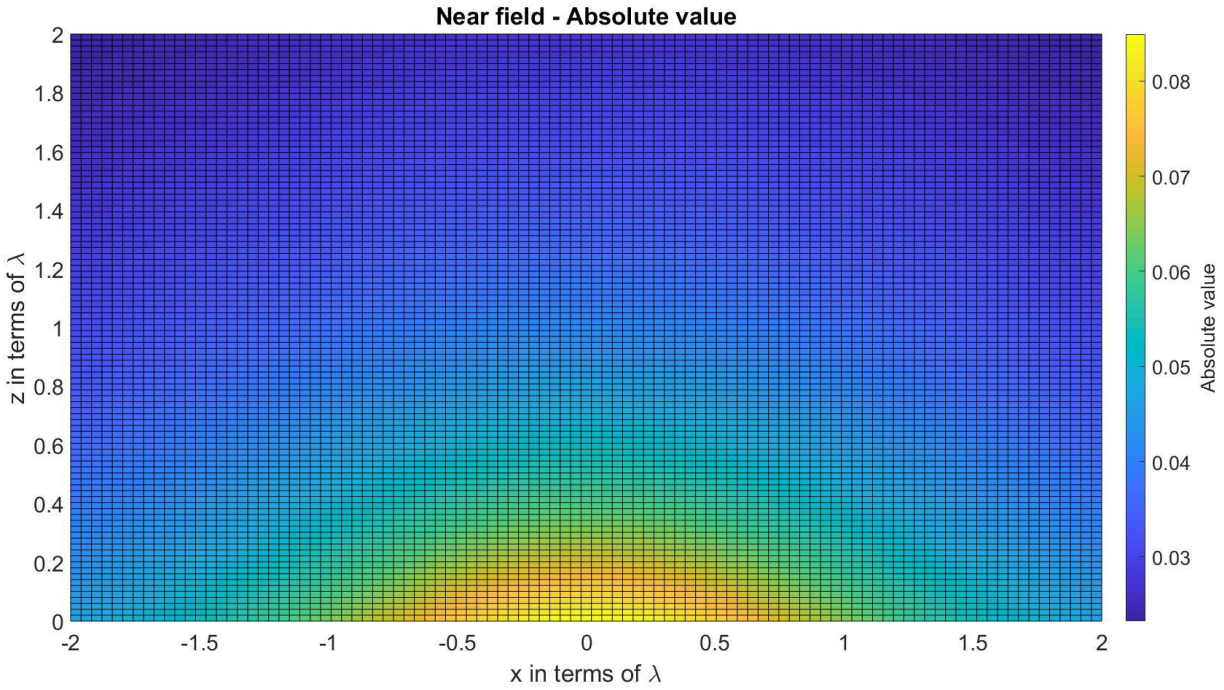

Figure 6: The absolute value of the near field.
We choose $n_1 = n_3 = 1$ and $n_2 = 2$.

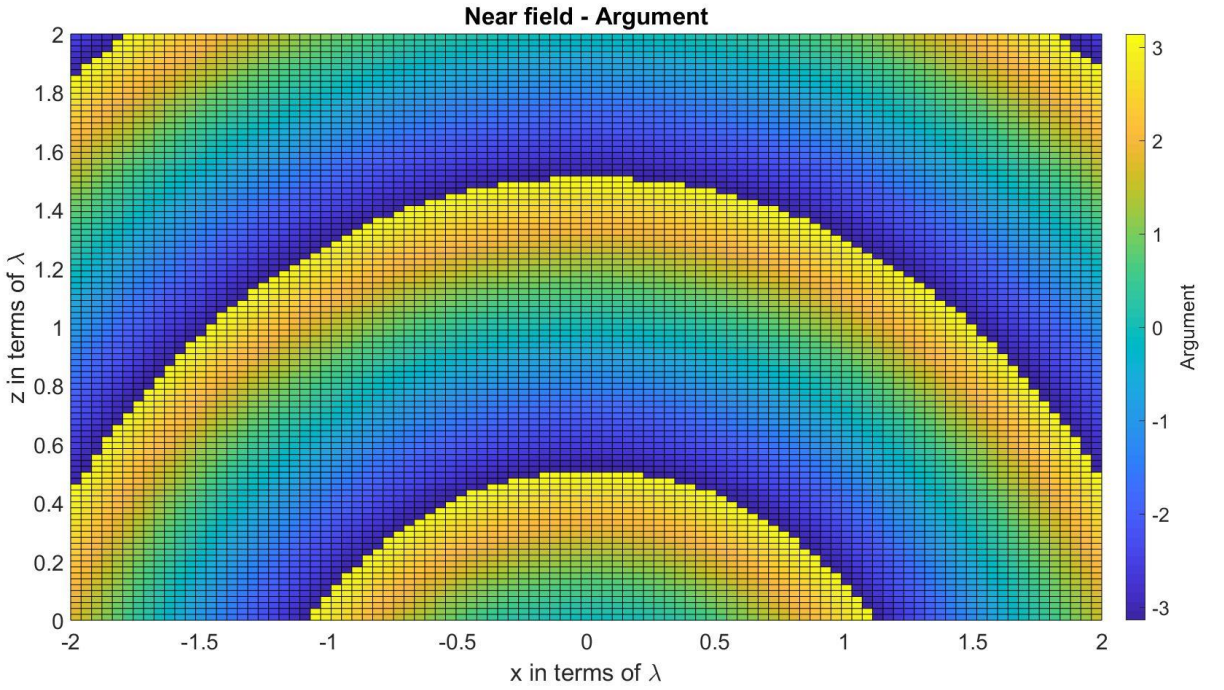

Figure 7: The argument of the near field.
We choose $n_1 = n_3 = 1$ and $n_2 = 2$.

3.5. Far field computations

The last step is to calculate the far field. This is done by using a *for-loop* that allows for the computation of the field and its intensity over a range of x-coordinates while keeping the z-coordinate at $z = 10^3$ and the dipole is situated at $(0,0,5\lambda)$. The plots are made using the angle θ, which for small angles relates to x and z in the following way

$$\theta = \frac{x}{z}. \tag{3.3}$$

Figures 8 and 9 have two subplots, one of the calculated far field for a perfect mirror and the other of a test field for comparison. The test plots in figures 8 and 9 plot the intensity and phase of

$$G^{test} = -\frac{1}{4\pi}\frac{e^{ikn\left|\vec{r}-\vec{r}_p\right|}}{\left|\vec{r}-\vec{r}_p\right|} - \frac{R}{4\pi}\frac{e^{ikn\left|\vec{r}-\tilde{\vec{r}}_p\right|}}{\left|\vec{r}-\tilde{\vec{r}}_p\right|}, \tag{3.4}$$

where, since we assume the system to be a perfect mirror (i.e. $R = -1$), $\left|\vec{r}-\vec{r}_p\right|$ is given by (1.10) and $\tilde{\vec{r}}_p$ is given by (2.51).

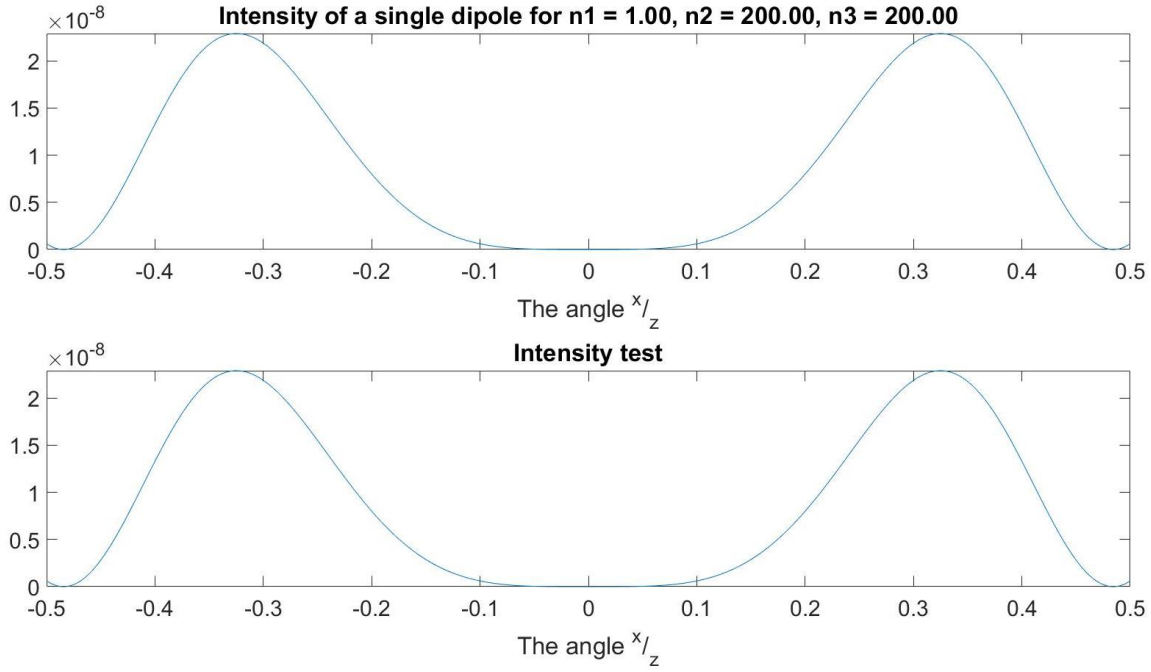

Figure 8: On the top: the intensity of a single dipole in the far field as calculated in the script. On the bottom: the intensity of a single dipole in the far field as calculated from (3.5).

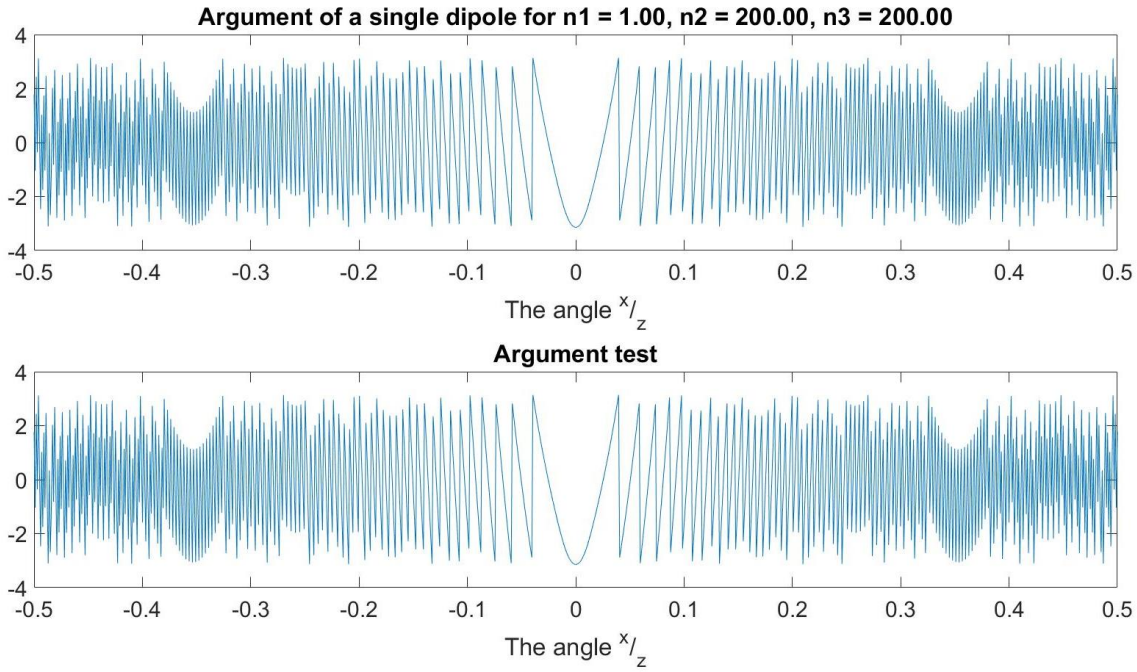

Figure 9: On the top: the argument of a single dipole in the far field as calculated in the script. On the bottom: the argument of a single dipole in the far field as calculated from (3.5).

Figure 10 shows the intensity plot of the far field of a single dipole using symmetric settings.

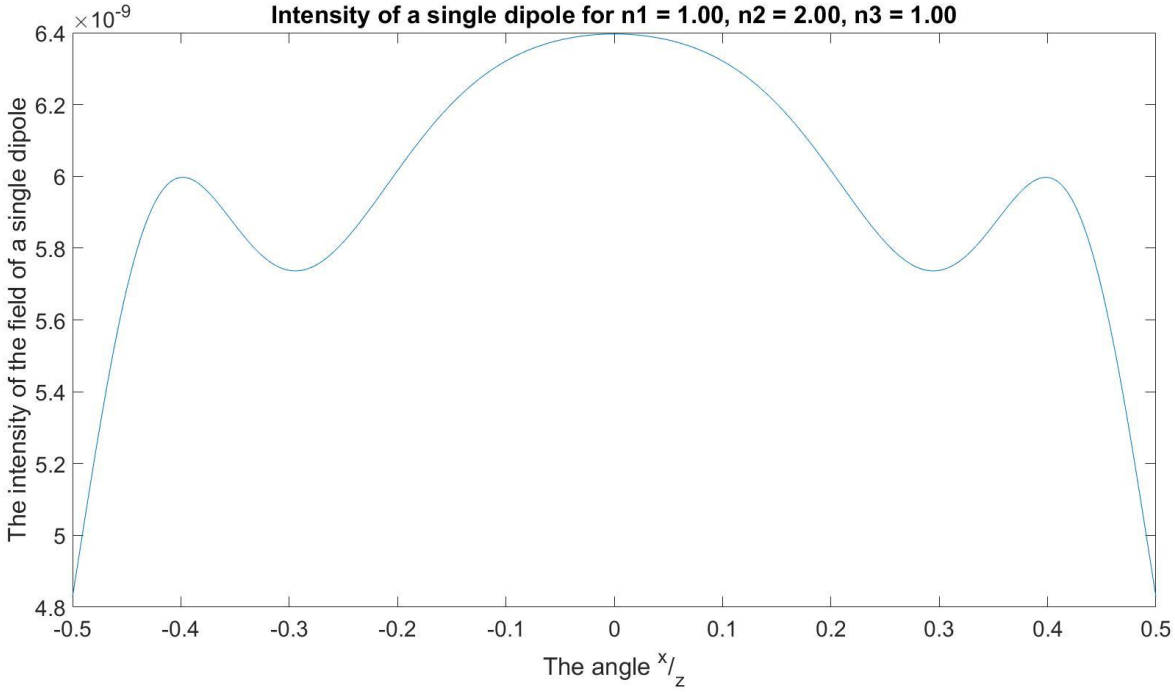

Figure 10: The intensity of a single dipole in the far field using symmetric settings (i.e. $n_1 = n_3$ and $n_2 > n_1$).

With these results, the code has been tested and can be used in further research to compute the reflected field of two dipoles.

3.6. Dipole strength *p*

With the function to compute the reflected field, we will take a look at the dipole strength p. We will look at the ratio between the dipole strength due to the reflection and the dipole strength due to the incident field,

which is given by:

$$\frac{p_{dipole}}{p_{free}} = \frac{\frac{\varepsilon_0 \alpha U_{Tot}^{In}}{1 - k_0^2 \alpha G^r(\vec{r}_p)}}{\varepsilon_0 \alpha U_{Tot}^{In}} = \frac{1}{1 - k_0^2 \alpha G^r(\vec{r}_p)}, \tag{3.5}$$

where α is the polarizability of the dipole which has been estimated to be $\alpha = \frac{\lambda^3}{10}$ since $\varepsilon_0 \alpha G^r(\vec{r}_p)$ can be estimated as

$$k_0^2 \alpha G^r(\vec{r}_p) \approx \alpha \frac{k_0^2}{z_p} \approx \frac{\alpha}{\lambda^3}, \tag{3.6}$$

which should not be equal to 1 since that will cause a singularity.

Figure 11 shows a plot of (3.5) as function of the distance z_p of the dipole above the film.

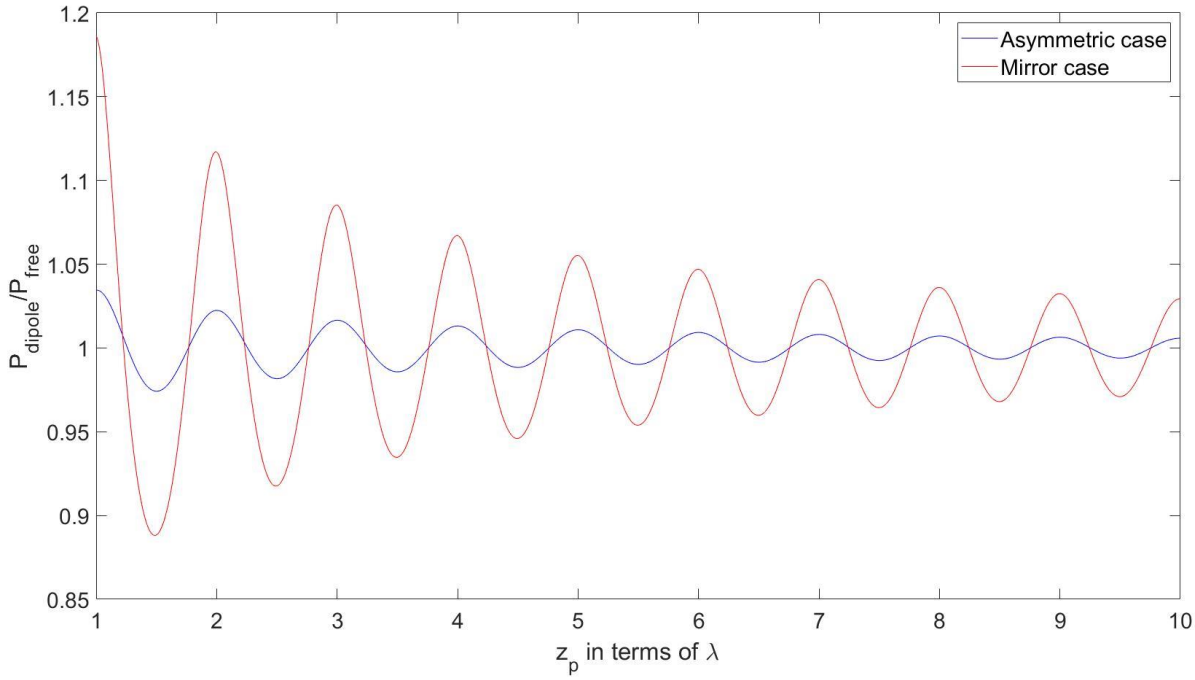

Figure 11: The dipole strength for two different cases: an asymmetric case (i.e. $n_1 \neq n_3$, $n_1 < n_3$ and $n_3 < n_2$) and a mirror case (i.e. $n_2 = n_3$ and $n_2 >> n_1$).

As can be seen from figure 11, the dipole strength is an oscillating function with a decreasing amplitude as function of the distance z_p above the film. For the asymmetric case the amplitude is lower than for the mirror case since the asymmetric case can have a guided mode which loses amplitude each round trip in the film.

4. Conclusion

In conclusion, a substrate underneath the dipole has a significant effect on the total field. As my report has shown, the influence on the total field will change as the indices of refraction of the different layers change. Not only the indices of refraction, but also the thickness of the substrate plays a role in the influence on the total field, being essential to the existence of a guided mode. When $n_1 = n_3$ and $n_2 > n_1$ (i.e. a symmetric case), there will always be at least one guided mode available but when $n_1 \neq n_3$, there will not necessarily be a guided mode and when $n_2 = n_3$, there is no film present and thus no guided mode.

In an integral over reflected plane waves, in case a guided wave occurs, the integrand has a singularity which has to be dealt with using the Cauchy Principle value technique. Other singularities are dealt with using multiple first order Taylor expansions. With the singularities dealt with, a Matlab function has been constructed that can compute the reflected field of a single dipole and near field in any given point in the half space $z > 0$. The far field (i.e. for large z) can be computed using the Matlab code as well only this time the method of stationary phase is used. The script has been positively tested using a perfect mirror as reference to what should happen to the field of the dipole.

With the computation of the reflected field as a function, it is possible to compute the near- and far field of two dipoles next to each other above the substrate once the influence of the reflection on their fields has been derived. The main influence of adding a second dipole will be on the dipole strength of both dipoles. It is then suggested that for further research this script is implemented to compute and further test the script for a two dipole system.

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Appendix A

This appendix will give a full the derivation of the stationary phase method as used for the far field computations.
The reflected field is first rewritten as

$$G^r(\vec{r},\vec{r}_p)=\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}q(k_x,k_y)e^{if(k_x,k_y)}dk_xdk_y, \tag{A.1}$$

where

$$q(k_x,k_y)=\frac{R(k_\perp)}{2ik_z}, \tag{A.2}$$

and

$$f(k_x,k_y)=k_x(x-x_p)+k_y(y-y_p)+k_z(z+z_p), \tag{A.3}$$

where $R(k_\perp)$ is the reflection coefficient. The integrand in (A.1) will oscillate very quickly as function of k_x and k_y when $\vec{r}$ is far from $\vec{r}_p$, i.e. when z is very large. The point of least fast oscillations, (k_{x0},k_{y0}), are the points where the phase is stationary, i.e.:

$$\frac{\delta f(k_x,k_y)}{\delta k_x}=\frac{\delta f(k_x,k_y)}{\delta k_y}=0. \tag{A.4}$$

Solving (A.4) gives the following results for k_{x0} and k_{y0}

$$k_{x0}=k_0n_1\frac{x-x_p}{\left|\vec{r}-\tilde{\tilde{r}}_p\right|}, \tag{A.5}$$

$$k_{y0}=k_0n_1\frac{y-y_p}{\left|\vec{r}-\tilde{\tilde{r}}_p\right|}, \tag{A.6}$$

$$\left|\vec{r}-\tilde{\tilde{r}}_p\right|^2=\left(x-x_p\right)^2+\left(y-y_p\right)^2+\left(z+z_p\right)^2, \tag{A.7}$$

and $k_\perp$ becomes

$$k_\perp=\sqrt{{k_{x0}}^2+{k_{y0}}^2}=k_0n_1\frac{\sqrt{\left(x-x_p\right)^2+\left(y-y_p\right)^2}}{\left|\vec{r}-\tilde{\tilde{r}}_p\right|}. \tag{A.8}$$

We have

$$k_{z0}=k_0n_1\frac{z+z_p}{\left|\vec{r}-\tilde{\tilde{r}}_p\right|}. \tag{A.9}$$

At the point of stationary phase, (A.3) can be written as

$$f(k_x,k_y)=f(k_{x0},k_{y0})+\frac{1}{2}a(k_x-k_{x0})^2+\frac{1}{2}b(k_y-k_{y0})^2+c(k_x-k_{x0})(k_y-k_{y0})+\ldots \tag{A.10}$$

where the coefficients a, b and c are given by

$$a=\left.\frac{\partial^2 f(k_x,k_y)}{\partial k_x^{\,2}}\right|_{k_x=k_{x0}}=-\frac{z+z_p}{k_{z0}}-\frac{k_{x0}^{\,2}}{k_{z0}^{\,3}}(z+z_p), \tag{A.11}$$

$$b=\left.\frac{\partial^2 f(k_x,k_y)}{\partial k_y^{\,2}}\right|_{k_y=k_{y0}}=-\frac{z+z_p}{k_{z0}}-\frac{k_{y0}^{\,2}}{k_{z0}^{\,3}}(z+z_p), \tag{A.12}$$

$$c=\left.\frac{\partial^2 f(k_x,k_y)}{\partial k_x\partial k_y}\right|_{k_x=k_{x0},k_y=k_{y0}}=-\frac{k_{x0}k_{y0}}{k_{z0}^{\,3}}(z+z_p). \tag{A.13}$$

Substituting (A.5) – (A.9) in (A.11) – (A.13) results in

$$a=-\frac{\left|\vec{r}-\tilde{\tilde{r}}_p\right|\left[\left(x-x_p\right)^2+\left(z+z_p\right)^2\right]}{k_0n_1\left(z+z_p\right)^2}, \tag{A.14}$$

$$b=-\frac{\left|\vec{r}-\tilde{\tilde{r}}_p\right|\left[\left(y-y_p\right)^2+\left(z+z_p\right)^2\right]}{k_0n_1\left(z+z_p\right)^2}, \tag{A.15}$$

$$c=-\frac{\left|\vec{r}-\tilde{\tilde{r}}_p\right|\left(x-x_p\right)\left(y-y_p\right)}{k_0n_1\left(z+z_p\right)^2}. \tag{A.16}$$

Rearranging the integration constants in (A.1) results in the following

$$G^r(\vec{r}_\infty,\vec{r}_p)\approx\frac{1}{2\pi}\frac{\sigma}{\sqrt{\left|ab-c^2\right|}}q(k_{x0},k_{y0})e^{if(k_{x0},k_{y0})}, \tag{A.17}$$

Where σ is given by

$$\sigma=\begin{cases} i & \text{, if } a>0 \text{ and } ab-c^2>0,\\ 1 & \text{, if } a>0 \text{ and } ab-c^2<0,\\ -i & \text{, if } a<0 \text{ and } ab-c^2>0,\\ 1 & \text{, if } a<0 \text{ and } ab-c^2<0, \end{cases} \tag{A.18}$$

and $\left|ab-c^2\right|$ can be calculated explicitly:

$$\left|ab-c^2\right|=\frac{\left|\vec{r}-\tilde{\tilde{r}}_p\right|^4}{k_0^{\,2}n_1^{\,2}\left(z+z_p\right)^2}. \tag{A.19}$$

Since (A.11) is always negative and (A.18) is always positive, σ is $-i$. Substituting (A.2), (A.3), (A.18) and (A.19) in (A.17) results in

$$G^r(\vec{r}_\infty, \vec{r}_p) \approx -R\left(k_0 n_1 \sin(\theta)\right) \frac{1}{4\pi} \frac{e^{ik_0 n_1 \left|\vec{r}_\infty - \tilde{\vec{r}}_p\right|}}{\left|\vec{r}_\infty - \tilde{\vec{r}}_p\right|}, \tag{A.20}$$

where

$$\sin(\theta) = \frac{\sqrt{\left(x - x_p\right)^2 + \left(y - y_p\right)^2}}{\left|\vec{r}_\infty - \tilde{\vec{r}}_p\right|}. \tag{A.21}$$

Appendix B

```
clear all
close all
clc

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Input
n1 = 1;                 %All calculations are made with n1 = 1
n2 = 2;                 %Index of refraction of the substrate
n3 = 1;                 %Index of refraction below the substrate
lambda = 1;             %Set to 1
d = lambda/4;           %d/lambda for 1 intersection
k0 = 2*pi;              %In vacuum and lambda = 1
rp = [0 0 1*lambda];    %x y z location of dipole in the near field, z = 0 is
the location of the substrate
r1 = [0 0 5*lambda];    %Location of the dipole in the far field
delta = 0.1;            %Small enough difference
testmode = 1;           %0 will give near and far field plots, 1 will give test
plots designed to test the code
Ntest = 10;             %Amount of grid points do to the calculations for the
half space case
Nnear = 10;             %Amount of points for the Near field grid
Nfar = 1e3;             %Amount of points for the Far field computations
Np = 10;                %Amount of points for the dipole strength calculations

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Near field, dipole at [0 0 1*lambda]
if testmode == 0
z = linspace(0,2*lambda,Nnear);
z(z==rp(3)) = NaN;                      %Remove the z-coordinate of the dipole
to prevent computational problems
z = rmmissing(z);
x = linspace(-2*lambda,2*lambda,Nnear);
x(x==rp(1)) = NaN;                      %Remove the x-coordinate of the dipole
to prevent computational problems
x = rmmissing(x);
Field = zeros(1,length(x));
figure('units','normalized','outerposition',[0 0 1 1])
figure('units','normalized','outerposition',[0 0 1 1])
for indz = 1:length(z)
    for indx = 1:length(x)
        r = [x(indx) 0 z(indz)];
        Gtotal = G_reflected(n1,n2,n3,lambda,k0,r,rp,delta,testmode,Ntest,d);
        Field(indz,indx) = Gtotal(1) + Gtotal(5);
    end
end
figure(1)
surface(x,z,abs(Field))
title('Near field - Absolute value','Fontsize',18)
xlabel('x in terms of \lambda','Fontsize',18)
ylabel('z in terms of \lambda','Fontsize',18)
cbabs = colorbar;
ylabel(cbabs,'Absolute value')
set(gca,'Fontsize',18)
figure(2)
surface(x,z,angle(Field))
```

```
title('Near field - Argument','Fontsize',18)
xlabel('x in terms of \lambda','Fontsize',18)
ylabel('z in terms of \lambda','Fontsize',18)
cbarg = colorbar;
ylabel(cbarg,'Argument')
set(gca,'Fontsize',18)

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Far field
intensity = zeros(Nfar,1);
free = intensity;
argument = intensity;
reflected = free;
inttest = reflected;
argtest = inttest;
count1 = 1;
z = 1e3*lambda;                         %Far field z
if n2 == n3 && n2 > 10*n1               %Mimics a mirror, adds a test field in the
plots
   for q = linspace(-0.5,0.5,Nfar)     %q = x/z for small angles
       rinf = [q*z 0 z];
       Gff1 = Ginf(k0,rinf,r1,n1,n2,n3,d);
       free(count1,1) = Gff1(1);
       reflected(count1,1) = Gff1(2);
       intensity(count1,1) = abs(Gff1(1)+Gff1(2))^2;
       argument(count1,1) = angle(Gff1(1)+Gff1(2));
       rminrp1 = sqrt((rinf(1)-r1(1))^2+(rinf(2)-r1(2))^2+(rinf(3)-r1(3))^2);
       rminrp2 = sqrt((rinf(1)-r1(1))^2+(rinf(2)-r1(2))^2+(rinf(3)+r1(3))^2);
       t1 = -
exp(1i*k0*n1*rminrp1)/(4*pi*rminrp1)+exp(1i*k0*n1*rminrp2)/(4*pi*rminrp2);
       inttest(count1,1) = abs(t1)^2;
       argtest(count1,1) = angle(t1);
       count1 = count1 + 1;
   end
   %Intensity
   figure('units','normalized','outerposition',[0 0 1 1])
   subplot(2,1,1)
   plot(linspace(-0.5,0.5,Nfar), intensity)
   xlabel('The angle ^{x}/_{z}','Fontsize',18)
   title(['Intensity of a single dipole for n1 = ',sprintf('%1.2f',n1), ...
       ', n2 = ',sprintf('%1.2f',n2),', n3 =
',sprintf('%1.2f',n3)],'Fontsize',18)
   set(gca,'Fontsize',18)
   %Test
   subplot(2,1,2)
   plot(linspace(-0.5,0.5,Nfar), inttest)
   title('Intensity test','Fontsize',18)
   xlabel('The angle ^{x}/_{z}','Fontsize',18)
   set(gca,'Fontsize',18)

   %Argument
   figure('units','normalized','outerposition',[0 0 1 1])
   subplot(2,1,1)
   plot(linspace(-0.5,0.5,Nfar), argument)
   xlabel('The angle ^{x}/_{z}','Fontsize',18)
   title(['Argument of a single dipole for n1 = ',sprintf('%1.2f',n1), ...
       ', n2 = ',sprintf('%1.2f',n2),', n3 =
',sprintf('%1.2f',n3)],'Fontsize',18)
```

```
    set(gca,'Fontsize',18)
    %Test
    subplot(2,1,2)
    plot(linspace(-0.5,0.5,Nfar), argtest)
    title('Argument test','Fontsize',18)
    xlabel('The angle ^{x}/_{z}','Fontsize',18)
    set(gca,'Fontsize',18)

    %Freespace
    figure('units','normalized','outerposition',[0 0 1 1])
    subplot(2,1,1)
    plot(linspace(-0.5*z,0.5*z,Nfar),real(free))
    title('Freespace field: real part','Fontsize',18)
    minp = islocalmin(real(free));
    maxp = islocalmax(real(free));
    xv = linspace(-0.5*z,0.5*z,Nfar);
    xlabel('x','Fontsize',18)
    hold on
    subplot(2,1,2)
    plot(linspace(-0.5*z,0.5*z,Nfar),abs(free))
    title('Freespace field: absolute value','Fontsize',18)
    xlabel('x','Fontsize',18)
    %Reflected
    figure('units','normalized','outerposition',[0 0 1 1])
    subplot(2,1,1)
    plot(linspace(-0.5*z,0.5*z,Nfar),real(reflected))
    title('Reflected field: real part','Fontsize',18)
    xlabel('x','Fontsize',18)
    hold on
    subplot(2,1,2)
    plot(linspace(-0.5*z,0.5*z,Nfar),abs(reflected))
    title('Reflected field: absolute value','Fontsize',18)
    xlabel('x','Fontsize',18)
else                                        %Non-mirror cases
     theta = linspace(-0.5,0.5,Nfar);
     for q = theta       %q = x/z for small angles
        rinf = [q*z 0 z];
        Gff1 = Ginf(k0,rinf,r1,n1,n2,n3,d);
        free(count1,1) = Gff1(1);
        reflected(count1,1) = Gff1(2);
        intensity(count1,1) = abs(Gff1(1)+Gff1(2))^2;
        count1 = count1 + 1;
    end
    %Intensity
    figure('units','normalized','outerposition',[0 0 1 1])
    plot(theta, intensity)
    ylabel('The intensity of the field of a single dipole','Fontsize',18)
    xlabel('The angle ^{x}/_{z}','Fontsize',18)
    title(['Intensity of a single dipole for n1 = ',sprintf('%1.2f',n1), ...
        ', n2 = ',sprintf('%1.2f',n2),', n3 =
',sprintf('%1.2f',n3)],'Fontsize',18)
    set(gca,'Fontsize',18)

    %Freespace
    figure('units','normalized','outerposition',[0 0 1 1])
    subplot(2,1,1)
    plot(linspace(-0.5*z,0.5*z,Nfar),real(free))
    title('Freespace field: real part','Fontsize',18)
```

```
    minp = islocalmin(real(free));
    maxp = islocalmax(real(free));
    xv = linspace(-0.5*z,0.5*z,Nfar);
    xlabel('x','Fontsize',18)
    hold on
    subplot(2,1,2)
    plot(linspace(-0.5*z,0.5*z,Nfar),abs(free))
    title('Freespace field: absolute value','Fontsize',18)
    xlabel('x','Fontsize',18)
    %Reflected
    figure('units','normalized','outerposition',[0 0 1 1])
    subplot(2,1,1)
    plot(linspace(-0.5*z,0.5*z,Nfar),real(reflected))
    title('Reflected field: real part','Fontsize',18)
    xlabel('x','Fontsize',18)
    hold on
    subplot(2,1,2)
    plot(linspace(-0.5*z,0.5*z,Nfar),abs(reflected))
    title('Reflected field: absolute value','Fontsize',18)
    xlabel('x','Fontsize',18)

    %Dipole strength p for 2 different cases
    zp = linspace(1*lambda,10*lambda,Np);
    p = zeros(length(zp),1);
    np2 = [2 200];
    np3 = [1.5 200];
    alpha = lambda^3/10;
    for o = 1:length(np2)
        for zi = 1:length(zp)
            rp = [0 0 zp(zi)];
            r = [0 0 1];
            Gr =
G_reflected(n1,np2(o),np3(o),lambda,k0,r,rp,delta,testmode,Ntest,d);
            p(zi,o) = 1/(1-k0^2*alpha*Gr(1));
        end
    end
    figure('units','normalized','outerposition',[0 0 1 1])
    plot(zp,real(p(:,1)),'b')
    hold on
    plot(zp,real(p(:,2)),'r')
    legend('Asymmetric case','Mirror case','Fontsize',18)
    set(gca,'Fontsize',18)
    xlabel('z_p in terms of \lambda')
    ylabel('P_d_i_p_o_l_e/P_f_r_e_e')
end
else
    r = [0 0 2*lambda];
    rp = [0 0 1*lambda];
    Gtest = G_reflected(n1,n2,n3,lambda,k0,r,rp,delta,testmode,Ntest,d);
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function output = G_reflected(n1,n2,n3,lambda,k0,r,rp,delta,testmode,N,d)
%This function will have four options for the output:
% Option 1 will give you an array with five outputs for the case when there
% a guided mode and one looks at a symmetric or asymmetric case with a slab
% of thickness d (for example when n1~n3 & n1<n2). The array has the
% following output:
```

```
%   1. The refelcted field using substitution and a first order Taylor
%   expansion around the singularity
%   2. The reflected field using substitution and Matlab around a
%   singularity
%   3. The reflected field using Matlab and a first order Taylor expansion
%   around the singularity
%   4. The reflected field using only Matlab
%   5. The theoretical incoming field
%
% Option 2 will give you a matrix containing the values for the reflected
% field for all values of a NxN grid, with x and z being the
% coordinates of the grid. This happens when the settings are such that the
% code looks at a halfspace (n2 = n3) and n2 is large enough compared to
% n1. This option also checks the code and will display if the difference
% between the code and the incoming field is negligibly small or not.
%
% Option 3 will give you an array with five outputs for looking at the case
% when there is no slab (n1 = n2 = n3). The output consists of the
% following:
%   1. The refelcted field using substitution and a first order Taylor
%   expansion around the singularity
%   2. The reflected field using substitution and Matlab around a
%   singularity
%   3. The reflected field using Matlab and a first order Taylor expansion
%   around the singularity
%   4. The reflected field using only Matlab
%   5. The theoretical incoming field
%
% Option 4 will give you an error message  if the used settings are not
% supported as of yet.

syms x kl
rl = sqrt((r(1)-rp(1))^2+(r(2)-rp(2))^2);
lhs = @(x)tan(x);
rhs = rhsfun(n1,n2,n3,k0,d);

%Test if there is a beta (= intersection of lhs and rhs)
xvalues = linspace(0,k0*sqrt((2*n2^2-n3^2-n1^2)/2)*d,800);
betatest = rhs(xvalues);
betatest(betatest<1e-10) = NaN;     %Replace small values by NaN
betatest = rmmissing(betatest);     %Remove NaN values

if isempty(betatest) == 0           %Now there is a beta value
    difflr = @(x) real(lhs(x)-rhs(x));
    zrocro = difflr(xvalues).*circshift(difflr(xvalues),[0 -2]);
    guess = find(zrocro <= 0);
    for k1 = 1:length(guess)
       intsct(k1) = fzero(difflr, xvalues(guess(k1)));
    end
    intsct = unique(round(intsct*10^6)./10^6);

    %Exclude certain intersection points
    intsct(intsct == 0) = NaN;                                      %If there
is an intersection at 0
    intsct(intsct >= k0*sqrt((2*n2^2-n3^2-n1^2)/2)*d) = NaN;        %All
intersections located at and past the boundary
    intsct(abs(intsct-d*k0*sqrt(1/2*(n2^2-n1^2))) <= 1e-6) = NaN;   %Asymptote
intersections
```

```
    for k2 = 1:length(intsct)
        intsct(abs(intsct-double(pi/2+(k2-1)*pi)) < 1e-6) = NaN;    %Asymptote
intersections
    end
    intsct = rmmissing(intsct);

    if testmode == 1
    %Plot rhs and lhs to find the intersection(s) --> beta value(s)
    figure('units','normalized','outerposition',[0 0 1 1])
    fplot(real(lhs(x)),[0 k0*sqrt((2*n2^2-n3^2-n1^2)/2)*d],'g')
    hold on
    fplot(real(rhs(x)),[0 k0*sqrt((2*n2^2-n3^2-n1^2)/2)*d],'r')
    xlabel('\alpha_2 * d','Fontsize',18)
    hold on
    plot(intsct, real(rhs(intsct)),'b*')
    ylim([-50 50])
    xlim([0 k0*sqrt((2*n2^2-n3^2-n1^2)/2)*d])
    legend('tan(\alpha_2d)','(\alpha_2(\alpha_1+\alpha_3))/(\alpha_2^2-
\alpha_1\alpha_3)','Intersection','Fontsize',18)
    set(gca,'Fontsize',18)
    hold off
    end

    %Put values of beta in an array
    beta{1,1} = sqrt(k0^2*n2^2 - (intsct(end)./d).^2);    %Only interested in
the largest beta

    %1st order Taylor version of the integral term
    bvalue = beta{1};
    f = ffun(k0,n1,n2,n3,d,bvalue);
    etermfun = efun(k0,n1,r(3),rp(3));
    ktermfun = kfun(k0,n1);
    J0fun = jfun(rl);
    j = @(kl)f(kl).*J0fun(kl).*etermfun(kl).*ktermfun(kl);
    dj = diff(j,kl);
    Gr3all{1,1} = vpa(subs(dj,kl,bvalue));
    Gr3Taylor(1) = 2*delta*double(Gr3all{1});

    %Integrals
    lower = bvalue-delta;   %Boundaries
    upper = bvalue+delta;
    syms kl
    R = Rfun(k0,n1,n2,n3,d);
    etermfun = efun(k0,n1,r(3),rp(3));
    ktermfun = kfun(k0,n1); J0fun = jfun(rl);
    intterms = @(kl) R(kl).*etermfun(kl).*ktermfun(kl).*J0fun(kl);
    Gr1full = integral(intterms, 0, lower);     %Gives an error due to the
1/kz1
    Gr2 = integral(intterms, upper, inf);
    f = ffun(k0,n1,n2,n3,d,bvalue);
    funGr3 = @(kl) (f(kl).*J0fun(kl).*etermfun(kl).*ktermfun(kl)-
double(j(bvalue)))/(kl-bvalue);
    Gr3int = integral(funGr3,lower,upper,'ArrayValued',true);

    %First integral with substitution to circumvent the singularity
    uppersubs1 = k0*n1;
    lowersubs1 = 0;     %New bounds due to kl*dkl = -kz1*dkz1 & kl =
sqrt(k0^2*n1^2 - kz1.^2)
```

```
    uppersubs2 = sqrt((lower).^2-k0^2*n1^2);
    lowersubs2 = 0;     %New bounds due to dkz1 = ids & kl = sqrt(k0^2*n1^2 +
s.^2)
    syms kz1 s
    integrandsubs1 = @(kz1)R(sqrt(k0^2*n1^2-kz1.^2)).*etermfun(sqrt(k0^2*n1^2-
kz1.^2)).*J0fun(sqrt(k0^2*n1^2-kz1.^2));
    Gr1subs1 = integral(integrandsubs1,lowersubs1,uppersubs1);
    integrandsubs2 = @(s) -
1i*R(sqrt(k0^2*n1^2+s.^2)).*etermfun(sqrt(k0^2*n1^2+s.^2)).*J0fun(sqrt(k0^2*n1^
2+s.^2));
    Gr1subs2 = integral(integrandsubs2,lowersubs2,uppersubs2);

    %The full integral
    Gr_parts_subs_Taylor = -1i/(4*pi)*(Gr1subs1 + Gr1subs2 + Gr2 + Gr3Taylor);
%Using substitution for Gr1 with Taylor integral
    Gr_parts_subs_int = -1i/(4*pi)*(Gr1subs1 + Gr1subs2 + Gr2 + Gr3int);
%Using substitution for Gr1 with Matlab integral
    Gr_parts_Taylor = -1i/(4*pi)*(Gr1full + Gr2 + Gr3Taylor);
%Using first order taylor approximation
    Gr_parts_Matlab = -1i/(4*pi)*(Gr1full + Gr2 + Gr3int);
%Using matlab around the singularity
    rminrp = sqrt(rl^2+(r(3)+rp(3))^2);
    Gtheor = 1/(4*pi)*exp(1i*k0*n1*rminrp)/rminrp;
    output = [Gr_parts_subs_Taylor; Gr_parts_subs_int; Gr_parts_Taylor;
Gr_parts_Matlab; Gtheor];

    if testmode == 1
    %Plots
    syms kl
    figure('units','normalized','outerposition',[0 0 1 1])
    subplot(2,1,1)
    fplot(real(intterms(kl)),[0 1.1*upper])
    title('Real part of the intergrand')
    xlabel('kl values')
    hold on
    xline(bvalue,'r--'); xline(k0*n1,'g--');
    legend('real(integrand)', '\beta','k0*n1')
    hold off
    subplot(2,1,2)
    fplot(imag(intterms(kl)),[0 1.1*upper])
    hold on
    title('Imaginary part of the integrand')
    xlabel('kl values')
    xline(bvalue,'r--'); xline(k0*n1,'g--');
    legend('imag(integrand)', '\beta','k0*n1')
    hold off
    figure('units','normalized','outerposition',[0 0 1 1])
    subplot(2,1,1)
    fplot(real(R(kl)),[0 1.1*upper])
    title(['Real part of R for n2 = ',sprintf('%1.2f',n2)])
    xlabel('kl values')
    hold on
    xline(bvalue,'r--'); xline(k0*n1,'g--');
    legend('real(R)', '\beta','k0*n1')
    hold off
    subplot(2,1,2)
    fplot(imag(R(kl)),[0 1.1*upper])
    title(['Imaginary part of R for n2 = ',sprintf('%1.2f',n2)])
```

```
    xlabel('kl values')
    hold on
    xline(bvalue,'r--'); xline(k0*n1,'g--');
    legend('imag(R)', '\beta','k0*n1')
    hold off
    end
elseif isempty(betatest) == 1 && n2 == n3 && n1 ~= n3     %Halfspace/mirror
    uppersubs1 = k0*n1;
    lowersubs1 = 0;     %New bounds due to kl*dkl = -kz1*dkz1 & kl =
sqrt(k0^2*n1^2 - kz1.^2)
    uppersubs2 = Inf;
    lowersubs2 = 0;     %New bounds due to dkz1 = ids & kl = sqrt(k0^2*n1^2 +
s.^2)
    syms kl kz1 s
    rl = sqrt((r(1)-rp(1))^2+(r(2)-rp(2))^2);
    R = -1;             %Force the reflection coefficient to be -1, this is in
approximation true
    etermfun = efun(k0,n1,r(3),rp(3));
    J0fun = jfun(rl);
    integrandsubs1 = @(kz1)R.*etermfun(sqrt(k0^2*n1^2-
kz1.^2)).*J0fun(sqrt(k0^2*n1^2-kz1.^2));
    Gr1subs1 = integral(integrandsubs1,lowersubs1,uppersubs1);
    integrandsubs2 = @(s) -
1i*R.*etermfun(sqrt(k0^2*n1^2+s.^2)).*J0fun(sqrt(k0^2*n1^2+s.^2));
    Gr1subs2 = integral(integrandsubs2,lowersubs2,uppersubs2);
    Gr = -1i/(4*pi)*(Gr1subs1 + Gr1subs2);
    rminrp = sqrt(rl^2+(r(3)+rp(3))^2);
    Gtheor = 1/(4*pi)*exp(1i*k0*n1*rminrp)/rminrp;
    output = [Gr; 0; 0; 0; Gtheor];

    if testmode == 1
    %Calculate the field for a grid of NxN
    Gr = zeros(N,N);
    clear x z
    z = linspace(0,5*lambda,N);
    x = linspace(-5*lambda,5*lambda,N);
    for indz = 1:N                %To plot for 0 < z < 10lambda
        for indx = 1:N            %To plot for -3lambda < x <3lambda
            %Full integral with substitution
            uppersubs1 = k0*n1;
            lowersubs1 = 0;     %New bounds due to kl*dkl = -kz1*dkz1 & kl =
sqrt(k0^2*n1^2 - kz1.^2)
            uppersubs2 = Inf;
            lowersubs2 = 0;     %New bounds due to dkz1 = ids & kl =
sqrt(k0^2*n1^2 + s.^2)
            syms kl kz1 s
            r = [x(indx) 0 z(indz)];
            rl = sqrt((r(1)-rp(1))^2+(r(2)-rp(2))^2);
            R = -1;             %Force the reflection coefficient to be -1
which is a valid approximation for large n2
            etermfun = efun(k0,n1,r(3),rp(3));
            J0fun = jfun(rl);
            integrandsubs1 = @(kz1)R.*etermfun(sqrt(k0^2*n1^2-
kz1.^2)).*J0fun(sqrt(k0^2*n1^2-kz1.^2));
            Gr1subs1 = integral(integrandsubs1,lowersubs1,uppersubs1);
            integrandsubs2 = @(s) -
1i*R.*etermfun(sqrt(k0^2*n1^2+s.^2)).*J0fun(sqrt(k0^2*n1^2+s.^2));
            Gr1subs2 = integral(integrandsubs2,lowersubs2,uppersubs2);
```

```
            Gr(indz,indx) = -1i/(4*pi)*(Gr1subs1 + Gr1subs2);
        end
    end
    Ntheor = 2*N;
    Gtheor = zeros(Ntheor,Ntheor);
    ztheor = linspace(-5*lambda,5*lambda,Ntheor);
    xtheor = ztheor;
    for indz = 1:Ntheor
        for indx = 1:Ntheor
            r = [xtheor(indx) 0 ztheor(indz)];
            rminrp = sqrt((r(1)-rp(1))^2+(r(2)-rp(2))^2+(r(3)+rp(3))^2);
            Gtheor(indz,indx) = 1/(4*pi)*exp(1i*k0*n1*rminrp)/rminrp;
        end
    end
        %Plots
        figure('units','normalized','outerposition',[0 0 1 1])
        surface(x,z,abs(Gr))
        cbabs = colorbar;
        title('The absolute value of the reflected field','Fontsize',18)
        xlabel('x in terms of \lambda','Fontsize',18)
        ylabel('z in terms of \lambda','Fontsize',18)
        ylabel(cbabs,'Absolute value')
        set(gca,'Fontsize',18)
        figure('units','normalized','outerposition',[0 0 1 1])
        surface(xtheor,ztheor,abs(Gtheor))
        cbabst = colorbar;
        title('The absolute value of the theoretical reflected
field','Fontsize',17)
        xlabel('x in terms of \lambda','Fontsize',18)
        ylabel('z in terms of \lambda','Fontsize',18)
        ylabel(cbabst,'Absolute value')
        set(gca,'Fontsize',18)

        figure('units','normalized','outerposition',[0 0 1 1])
        surface(x,z,angle(Gr))
        cbarg = colorbar;
        title('The argument of the reflected field','Fontsize',18)
        xlabel('x in terms of \lambda','Fontsize',18)
        ylabel('z in terms of \lambda','Fontsize',18)
        ylabel(cbarg,'Argument')
        set(gca,'Fontsize',18)
        figure('units','normalized','outerposition',[0 0 1 1])
        surface(xtheor,ztheor,angle(Gtheor))
        cbargt = colorbar;
        title('The argument of the theoretical reflected field','Fontsize',17)
        xlabel('x in terms of \lambda','Fontsize',18)
        ylabel('z in terms of \lambda','Fontsize',18)
        ylabel(cbargt,'Argument')
        set(gca,'Fontsize',18)

    end
elseif n1 == n2 && n3 == n1          %Free space (no substrate)
    rminrp = sqrt(rl^2+(r(3)+rp(3))^2);
    Gtheor = 1/(4*pi)*exp(1i*k0*n1*rminrp)/rminrp;
    output = [0; 0; 0; 0; Gtheor];
else
    output = [0 0 0 0 0];
    disp('Code is not yet able to process the used settings')
```

```
end
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Functions

%R, the reflection coefficient
function r = Rfun(k0,n1,n2,n3,d)
syms kl
kz1 = @(kl)sqrt(k0^2*n1.^2-kl.^2);
kz2 = @(kl)sqrt(k0^2*n2.^2-kl.^2);
kz3 = @(kl)sqrt(k0^2*n3.^2-kl.^2);
if n2 == n3 && n2 > 10*n1
    r12 = @(kl) -1;
else
    r12 = @(kl)(kz1(kl)-kz2(kl))./(kz1(kl)+kz2(kl));
end
r23 = @(kl)(kz2(kl)-kz3(kl))./(kz2(kl)+kz3(kl));
r = @(kl)(r12(kl) + r23(kl).*exp(2.*1i.*kz2(kl).*d))./(1 +
r23(kl).*r12(kl).*exp(2.*1i.*kz2(kl).*d));
end

%Exponent term in the integral
function e = efun(k0,n1,z,zp)
syms kl
kz1 = @(kl)sqrt(k0^2*n1.^2-kl.^2);
e = @(kl)exp(1i.*kz1(kl).*(z+abs(zp)));
end

%kl/k1z term in the integral
function k = kfun(k0,n1)
syms kl
kz1 = @(kl)sqrt(k0^2*n1.^2-kl.^2);
k = @(kl) kl./kz1(kl);
end

%Besselfunction as a function
function j = jfun(rl)
syms kl
j =  @(kl)besselj(0,kl.*rl);
end

%f-function from R
function F = ffun(k0,n1,n2,n3,d,bvalue)
syms kl
kz1 = @(kl)sqrt(k0^2*n1.^2-kl.^2);
kz2 = @(kl)sqrt(k0^2*n2.^2-kl.^2);
kz3 = @(kl)sqrt(k0^2*n3.^2-kl.^2);
if n2 == n3 && n2 > 10*n1
    r12 = @(kl)-1;
else
    r12 = @(kl)(kz1(kl)-kz2(kl))./(kz1(kl)+kz2(kl));
end
r23 = @(kl)(kz2(kl)-kz3(kl))./(kz2(kl)+kz3(kl));
gfun = @(kl) 1 + r23(kl).*r12(kl).*exp(2*.1i.*kz2(kl).*d);   dg =
diff(gfun(kl));
if dg ~= 0 &&  n2~=n3
    dgsubs = vpa(subs(dg,kl,bvalue));
```

```
    F = @(kl)1./(double(dgsubs)).*((r12(kl)+r23(kl).*exp(2.*1i.*kz2(kl).*d)));
else
    dgsubs = vpa(subs(dg,kl,bvalue));
    gsubs = gfun(bvalue);
    F = @(kl)1./(double(dgsubs)+gsubs/(kl-
bvalue)).*((r12(kl)+r23(kl).*exp(2.*1i.*kz2(kl).*d)));
end
end

%Rhs as a function of x = a2*d
function r = rhsfun(n1,n2,n3,k0,d)
syms x
a1 = @(x)sqrt(k0^2.*(n2^2-n1^2)-(x./d).^2);
a3 = @(x)sqrt(k0^2.*(n2^2-n3^2)-(x./d).^2);
r = @(x)((x./d).*(a1(x)+a3(x)))./((x./d).^2-a1(x).*a3(x));
end


%compute Ginf
function out = Ginf(k0,r,rp,n1,n2,n3,d)
R = Rfun(k0,n1,n2,n3,d);
rminrp1 = sqrt((r(1)-rp(1))^2+(r(2)-rp(2))^2+(r(3)-rp(3))^2);
rminrp2 = sqrt((r(1)-rp(1))^2+(r(2)-rp(2))^2+(r(3)+rp(3))^2);
out1 = -1/(4*pi)*exp(1i*k0*n1*rminrp1)/rminrp1;   %Incoming field
out2 = -1/(4*pi)*R(k0*n1*sqrt((r(1)-rp(1))^2+(r(2)-rp(2))^2)/rminrp2)...
    *exp(1i*k0*n1*rminrp2)/rminrp2;
out = [out1;out2];
end
```