Schur Multipliers of Divided Differences and Multilinear Harmonic Analysis

Thesis Report

by

Jesse Reimann

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Student number:	5635144
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Faculty:	Electrical Engineering, Mathematics, and Computer Science
Thesis committee:	Dr. M. P. T. Caspers
	Dr. F. M. de Oliveira Filho
	Dr. S. E. Zegers

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TU Delft

Abstract

It was first shown by D. Potapov and F. Sukochev in 2009 that Lipschitz functions are also operator-Lipschitz on Schatten class operators S^p, $1 < p < \infty$, which is related to a conjecture by M. Krein. Their proof combined Schur multiplication, a generalisation of component-wise matrix multiplication, with the so-called first order divided difference of a function, an approximation of its derivative. Showing that the Schur multipier associated with a divided difference function is bounded relies on a so-called transference technique, the boundedness of certain Schur multipliers can be inferred from the boundedness of associated Fourier multipliers. Soon after, this boundedness result was extended by D. Potapov, A. Skripka, and F. Sukochev to multilinear Schur multipliers of divided differences of arbitrary order, i.e. approximations of higher derivatives.

In this thesis, we offer an alternative boundedness proof for bilinear Schur multipliers of second order divided differences, in which we use recent results of multilinear harmonic analysis towards a multilinear transference proof, as well as recently found sufficient conditions for the boundedness of linear Schur multipliers which cannot be studied by transference. These methods were not known at the time Potapov, Skripka, and Sukochev proved their result.

Moreover, we show that this new proof improves the growth of the bound on the norm of the considered Schur multiplier for $p \to \infty$ significantly. Finally, we give an outlook on further steps towards an alternative boundedness proof of multilinear Schur multipliers of divided differences of arbitrary order.

Preface

Towards the end of my BSc studies, it became clear that I wanted to continue studying functional analysis, yet I was unsure in which field to specialise. This motivated me to enrol in the MSc Applied Mathematics programme in Delft, as it gave me the freedom to study various related subjects such as PDEs, harmonic analysis, and operator algebras.

After my first year of studies here, it was clear to me that a project with Martijn Caspers on the intersection of operator algebras and harmonic analysis would be a great match for my mathematical interests. In fact, it turns out that the proposed topic was not wholly unrelated to my previous studies – while I was not aware of the terminology at the time, Schur multipliers have already made an appearance in my BSc thesis! I want to sincerely thank Martijn for all the support and feedback, for our weekly meetings, and for encouraging me to attend conferences while I was working on my thesis. I am very much looking forward to my PhD studies under his supervision.

I was very fortunate to have had the opportunity to spend nearly a full year abroad during my MSc studies. Huge thanks to Jukka Kohonen and Kalle Kytölä for introducing me to the delightful world of formalisation during my internship at Aalto University, to Mamoru Okamoto for welcoming me into his group at Osaka University, to Ichiro Hasuo for the wonderful time I had during my internship at the National Institute of Informatics in Tokyo, and to Mark Veraar for assisting me in going to Japan in the first place.

Finally, I want to thank the German Academic Exchange Service for their generous funding of my second year of studies, Fernando de Oliveira Filho and Sophie Zegers for agreeing to be part of my thesis committee, Martijn Caspers and Frederik Thomsen for partially supporting my coffee habit, the bands Deviloof and Set to Stun for creating the soundtrack to which most of this thesis was written, and my friends and family for their support over the years.

Jesse Reimann
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Contents

1 Introduction

This thesis is heavily based on the work of D. Potapov and F. Sukochev in [36], as well as the subsequent work of D. Potapov, A. Skripka, and F. Sukochev in [35]. Both papers solved conjectures related to the functional calculus, which we will briefly introduce here.

In [36], the following conjecture by M. Krein (see [25, fourth lecture]) is addressed.

Conjecture 1.1. Let F_p denote the class of functions $f : \mathbb{R} \to \mathbb{C}$ such that $f(a) - f(b) \in S_p$ for any self-adjoint a and b with $a - b \in S_p$, and such that

$$\|f\|_{F_p} := \sup_{a,b} \frac{\|f(a) - f(b)\|_{S_p}}{\|a - b\|_{S_p}} < \infty.$$

Then $f' \in L^\infty(\mathbb{R})$ is sufficient for $f \in F_1$.

This conjecture was shown to be false by Yu. Farforovskaya in [13]; however, the study of the classes F_p, $p \in [1, \infty]$ continued, see e.g. [8, 9, 29]. In [36], the authors show that in fact for $p \in (1, \infty)$, F_p is given by the Lipschitz continuous functions on $\mathbb{R}$. In particular, this implies that Conjecture 1.1 does indeed hold for $p \in (1, \infty)$. Their proof, of which we will give a sketch in Section 1.3, relies on a so-called *Schur multiplier of divided differences*, which we will introduce shortly.

In [35], an open conjecture related to perturbation theory and noncommutative geometry was solved, namely the existence of so-called *spectral shift functions* η_n, for which the expression

$$\tau\left(f(H+V) - \sum_{k=0}^{n-1} \frac{1}{k!} \frac{d^k}{dt^k} f(H+tV)\Big|_{t=0}\right) = \int_{\mathbb{R}} f^{(n)}(t)\eta_n(t)dt \tag{1.1}$$

is well-defined for a trace τ, a self-adjoint operator H, and for V a suitable compact perturbation of H. Here again, the boundedness of Schur multipliers of divided differences was key.

Since this boundedness result was proven over a decade ago, significant advances have been made in the theory of Schur multipliers. This motivates our re-examination of the boundedness of Schur multipliers of divided differences, as we investigate whether modern proof methods offer new insights into these objects.

The aim of this section is to give a high-level overview over the central topics in this thesis; rigorous definitions will be deferred to Section 2. Here, we merely assume the reader has knowledge equivalent to a first course in functional analysis. We will first separately introduce Schur multipliers in Section 1.1 and divided differences in Section 1.2, before discussing the role Schur multipliers of divided differences play in the results in [36] and [35] in Section 1.3. In Section 1.4, we introduce some recent proof methods that we will use towards showing the boundedness of such Schur multipliers. In particular, this section introduces the link between Schur multipliers and harmonic analysis. In Section 1.5, we give an overview over the boundedness proof we offer for bilinear Schur multipliers of divided differences, and compare it to the original proof in [35]. Finally, we give an overview over the structure of the thesis in Section 1.6.

1.1. Schur multipliers on Schatten spaces

Schur multipliers are the central objects studied in this thesis. We shall first introduce them for finite-dimensional matrices, before introducing the idea behind their generalisation to compact operators, the context in which this thesis is written.

Consider two matrices $A = (a_{ij})_{ij}$, $B = (b_{ij})_{ij} \in \mathbb{C}^{n\times n}$, $n \in \mathbb{N}$. As opposed to regular matrix multiplication, we consider a more "naive" product, defined by element-wise multiplication as

$$A \cdot_S B := (a_{ij}b_{ij})_{ij} \in \mathbb{C}^{n\times n}.$$

This product is known as the *Schur product* of two matrices. By fixing one of the matrices, we can define a linear operator

$$M_A : \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}, \quad M_A(B) := A \cdot_S B. \tag{1.2}$$

Such an operator is called a *Schur multiplier*[1]. As for many concepts from linear algebra, there exists an infinite-dimensional generalisation. The following exposition is adapted from [20, Chapter 5.4b], see also Sections 2.3–2.4 of this thesis for rigorous definitions.

Let H be a separable Hilbert space and let $(p_\lambda)_{\lambda\in\Lambda}$ be a countable family of orthogonal projections on H such that $\sum_{\lambda\in\Lambda} p_\lambda x = x$ for all $x \in B(H)$, where $B(H)$ denotes the bounded linear operators on H. Such a family can be constructed from an orthonormal basis $(h_\lambda)_{\lambda\in\Lambda}$ of H by setting $p_\lambda := (\cdot, h_\lambda)h_\lambda$, where $(\cdot, \cdot)$ denotes the inner product of H. For a function $m : \Lambda^2 \to \mathbb{C}$, we may now formally define the Schur multiplier M_m on $B(H)$ as

$$M_m x := \sum_{\lambda,\mu\in\Lambda} m(\lambda, \mu)p_\lambda x p_\mu. \tag{1.3}$$

The function m is often referred to as the *symbol* of M_m. For $H = \mathbb{C}^n$ and $\Lambda = \{1, \dots, n\}$, this definition indeed coincides with (1.2), using as projections the diagonal matrices $p_\lambda = (\delta_{\lambda,\mu}\delta_{\lambda,\nu})_{\mu\nu}$ and letting $m(\lambda, \mu) = a_{\lambda,\mu}$. For infinite-dimensional Hilbert spaces, more care is needed to ensure that the sum in (1.3) converges.

A suitable choice for the domains of Schur multipliers are the so-called *Schatten spaces* S_p, defined as the compact operators on a (separable) Hilbert space H such that their singular value sequences form an ℓ^p-sequence. This property of the singular values gives rise to a norm, under which S_p-spaces are in fact Banach spaces. In particular, the Schatten class S_1 coincides with the trace class operators, and S_2 with the Hilbert-Schmidt operators. In fact, in analogy with the tracial definition of these spaces we have

$$\|x\|_{S_p}^p = \tau(|x|^p),$$

where τ denotes the trace and the modulus of x is defined via functional calculus. Note that the definition of S_p is reminiscent of the definition of the L^p-norm of measurable functions. Indeed, this construction is a special case of the construction of so-called *noncommutative L^p-spaces* associated with semifinite von Neumann algebras. See [33] for an introduction to this topic.

A well-understood class of Schur multipliers are the so-called *Toeplitz form* Schur multipliers, associated with a symbol of the form $(\lambda, \mu) \mapsto m(\lambda - \mu)$. They are closely related to the associated *Fourier multipliers*

$$\widehat{T_m f} := m\widehat{f},$$

and in fact it is known [31] that

$$\|M_{(\lambda,\mu)\mapsto m(\lambda-\mu)} : S_p \to S_p\| \leq \|T_m : L^p(\mathbb{T}, S_p) \to L^p(\mathbb{T}, S_p)\|, \quad p \in (1, \infty).$$

This method of studying Schur multipliers is known as *transference*, and we will introduce it further in Section 1.4. For Schur multipliers that are not of Toeplitz form, the literature was much more limited until recently; we will also return to this point in Section 1.4. In the following sections, we will restrict ourselves to considering $p \in (1, \infty)$; see [26, Theorem 1.7] for a characterisation of Schur multipliers for $p = 1$, and [1] for some results for $p \in (0, 1)$.

[1] Not to be confused with the Schur multiplier of a group in homological algebra [40].

$$
\begin{array}{llllll}
\lambda_0 & f(\lambda_0) = f^{[0]}(\lambda_0) & & & & \\
& & f^{[1]}(\lambda_0, \lambda_1) & & & \\
\lambda_1 & f(\lambda_1) = f^{[0]}(\lambda_1) & & f^{[2]}(\lambda_0, \lambda_1, \lambda_2) & & \\
& & f^{[1]}(\lambda_1, \lambda_2) & & f^{[3]}(\lambda_0, \lambda_1, \lambda_2, \lambda_3) & \\
\lambda_2 & f(\lambda_2) = f^{[0]}(\lambda_2) & & f^{[2]}(\lambda_1, \lambda_2, \lambda_3) & & \\
& & f^{[1]}(\lambda_2, \lambda_3) & & & \\
\lambda_3 & f(\lambda_3) = f^{[0]}(\lambda_3) & & & &
\end{array}
$$

Figure 1.1: Schematic representation of the inductive calculation method for divided differences (see Definition 2.1). Higher order divided differences can be calculated from the difference of two lower order divided differences immediately to its left in the schematic, divided by the difference of the corresponding inputs.

In this thesis, we are particularly interested in multilinear Schur multipliers on Schatten spaces. As in commutative L^p-spaces, a Hölder-type inequality holds for Schatten spaces, i.e. for $p_1, p_2, p \in [1, \infty)$ such that $1/p_1 + 1/p_2 = 1/p$ we have

$$\|xy\|_{S_p} \leq \|x\|_{S_{p_1}} \|y\|_{S_{p_2}}.$$

This allows us to make sense of multilinear Schur multipliers as maps $S_{p_1} \times \cdots \times S_{p_n} \to S_p$ for $p_1, \ldots, p_n, p \in (1, \infty)$ such that $1/p_1 + \ldots + 1/p_n = 1/p$, given by

$$M_m(x_1, \ldots, x_n) := \sum_{\lambda_0, \ldots, \lambda_n} m(\lambda_0, \ldots, \lambda_n) p_{\lambda_0} x_1 p_{\lambda_1} x_2 \ldots p_{\lambda_{n-1}} x_n p_{\lambda_n}.$$

Note that there are further generalisations of Schur multipliers that are not considered in this thesis. For instance, one may consider a σ-finite measure space (Ω, μ) giving rise to the Hilbert space $L^2(\Omega, \mu)$, which is not necessarily separable. One can then define Schur multipliers with symbol $m : \Omega \times \Omega \to \mathbb{C}$ acting on a suitable subspace of $B(L^2(\Omega, \mu))$, such as operators that can be represented by an integral kernel or approximated by such. See e.g. [26] for further reading. Furthermore, multilinear Schur multipliers can be seen as a special case of so-called *multiple operator integrals*, which appear in perturbation theory and noncommutative geometry. See [39] for an introduction.

1.2. Divided differences

Divided differences were introduced in Isaac Newton's *Principia Mathematica* "to find a curve line of the parabolic kind which shall pass through any given number of points" [32, Book III, Lemma V, Case ii], or in other words, to solve an *interpolation problem* with given data, assumed to be of the form $f(\lambda_0), \ldots, f(\lambda_n)$ for some continuous function f and points $\lambda_0, \ldots, \lambda_n$. Divided differences are constructed inductively; this process is frequently visualised as in Figure 1.1. For example, for given pairwise distinct points $\lambda_0, \lambda_1, \lambda_2 \in \mathbb{R}$, the second order divided difference of a function f is given by

$$f^{[2]}(\lambda_0, \lambda_1, \lambda_2) = \frac{f^{[1]}(\lambda_0, \lambda_1) - f^{[1]}(\lambda_1, \lambda_2)}{\lambda_0 - \lambda_2} = \frac{\frac{f(\lambda_0) - f(\lambda_1)}{\lambda_0 - \lambda_1} - \frac{f(\lambda_1) - f(\lambda_2)}{\lambda_1 - \lambda_2}}{\lambda_0 - \lambda_2}.$$

In cases where several points coincide, i.e. $\lambda_k = \cdots = \lambda_{k+r}$ for some $k, r \in \mathbb{N}$, a divided difference may still be defined for a sufficiently smooth function f by setting

$$f^{[r]}(\lambda_k, \ldots, \lambda_{k+r}) := \frac{f^{(r)}(\lambda_k)}{r!}.$$

We will see in Section 2.2 that this choice is justified.

Notably, divided differences provide a method of constructing the (unique) solution to the *Hermite interpolation problem*: Given a function $f \in C^n([a, b])$ and points $\Lambda = \{\lambda_0, \ldots, \lambda_n\} \subset [a, b]$, such that r of those points coincide in a point ζ, then the unique polynomial $p_{f,X}$ of degree $\leq n$ that interpolates f in ζ up to the $(r-1)$-th derivative is given by

$$\begin{aligned} p_{f,X}(\lambda) = f(\lambda_0) + (\lambda - \lambda_0)(f^{[1]}(\lambda_0, \lambda_1) \\ + (\lambda - \lambda_1)(f^{[2]}(\lambda_0, \lambda_1, \lambda_2) + \ldots + (\lambda - \lambda_{n-1}) f^{[n]}(\lambda_0, \ldots, \lambda_n)) \ldots)). \end{aligned}$$

See e.g. [38, Chapter 8.6] for details. In this thesis we will however consider divided differences outside the context of numerical analysis, as will be explained in the next section.

1.3. Schur multipliers of divided differences

In this section, we give an brief overview over the use of Schur multipliers (and multiple operator integrals) of divided differences in the work of D. Potapov and F. Sukochev in [36], and of D. Potapov, A. Skripka, and F. Sukochev in [35].

In [36], the following statement was shown.

Theorem 1.2. *Let f be a Lipschitz function, and let $p \in (1, \infty)$. Then*

$$\|f\|_{F_p} := \sup_{a,b} \frac{\|f(a) - f(b)\|_{S_p}}{\|a - b\|_{S_p}} < \infty,$$

where the supremum is over self-adoint a and b such that $a - b \in S_p$.

We will sketch the role of Schur multipliers and divided differences in the proof here, in the special case where $a, b \in S_p$.

Let $[\cdot, \cdot]$ denote the commutator $[a, b] := ab - ba$ of two operators, then by using the definition of the S_p-norm, one can show that

$$2^{1/p}\|f(a) - f(b)\|_{S_p} = \|[f(u), v]\|_{S_p} \quad \text{for} \quad u := \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix},\ v := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Furthermore, for u compact and self-adjoint one can show

$$[f(u), v] = M_{f^{[1]}}[u, v]. \tag{1.4}$$

It remains to show that the linear Schur multiplier $M_{f^{[1]}}$ is bounded by a constant $C_{f,p} \leq C_p\|f\|_{\mathrm{Lip}}$, which then implies

$$\|f(a) - f(b)\|_{S_p} = 2^{-1/p}\|M_{f^{[1]}}[u, v]\|_{S_p} \lesssim_p \|M_{f^{[1]}}\|_{S_p \to S_p}\|[u, v]\|_{S_p} \lesssim_p \|f\|_{\mathrm{Lip}}\|a - b\|_{S_p}.$$

If one removes the restriction $a, b \in S_p$, the Schur multiplier in (1.4) is replaced by a multiple operator integral. Its boundedness follows from the boundedness of the Schur multiplier $M_{f^{[1]}}$ by an approximation argument.

In a subsequent paper, the result of [36] on the boundedness of $M_{f^{[1]}}$ was extended to show that multilinear Schur multipliers of higher order divided differences are bounded. This was in fact achieved by proving boundedness of a more general class of Schur multipliers; we give the result specialised to divided differences here.

Theorem 1.3 ([35, Theorem 5.3 for divided differences])**.** *Let $p_1, \ldots, p_n, p \in (1, \infty)$ be such that $1/p_1 + \ldots + 1/p_n = 1/p$, and let $f \in C^n(\mathbb{R})$. Then the Schur multiplier*

$$M_{f^{[n]}} : S_{p_1} \times \ldots \times S_{p_n} \to S_p$$

is bounded with

$$\|M_{f^{[n]}}\| \leq C_{p_1,\cdots,p_n}\|f^{(n)}\|_\infty.$$

As previously mentioned, the motivation for this extension was to show the existence of so-called higher order spectral shift functions, i.e. a function $\eta_n = \eta_{n,V,H}$ such that (1.1) holds. Here, n-linear Schur multipliers appear by the key relation

$$\frac{d^n}{dt^n} f(H + tV) = n! T_{t,f^{[n]}}(V, \ldots, V), \tag{1.5}$$

where $T_{t,f^{[n]}}$ denotes a multiple operator integral, which coincides with an n-linear Schur multiplier if $H + tV$ has discrete spectrum. Again, the boundedness of the multiple operator integral follows by approximation from the corresponding bound on the Schur multiplier in Theorem 1.3. Once the boundedness of the multiple operator integral has been established, this relationship shows that the left hand side of (1.1) a well-defined bounded linear functional on the continuous functions with compact support. The existence of η_n then follows by approximation and the Riesz representation theorem.

Let us give an example to show how (1.5) holds with a Schur multiplier on the right hand side in the finite dimensional case. Let $f(z) = \sum_{k=0}^{N} c_k z^k$ be a polynomial, let $\lambda_1 \neq \lambda_2$, and let

$$H = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We can calculate the first derivative of $f(H+tV)$ in $t=0$ as

$$\begin{aligned}
\frac{d}{dt}\Big|_{t=0} f(H+tV) &= \frac{d}{dt}\Big|_{t=0} \sum_{k=0}^{N} c_k \begin{pmatrix} \lambda_1 & t \\ t & \lambda_2 \end{pmatrix}^k \\
&= \sum_{k=0}^{N} c_k \sum_{j=0}^{k-1} \begin{pmatrix} \lambda_1 & t \\ t & \lambda_2 \end{pmatrix}^j \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \lambda_1 & t \\ t & \lambda_2 \end{pmatrix}^{k-1-j} \Big|_{t=0} \\
&= \sum_{k=0}^{N} c_k \sum_{j=0}^{k-1} \begin{pmatrix} \lambda_1^j & 0 \\ 0 & \lambda_2^j \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \lambda_1^{k-1-j} & 0 \\ 0 & \lambda_2^{k-1-j} \end{pmatrix} \\
&= \sum_{k=0}^{N} c_k \sum_{j=0}^{k-1} \begin{pmatrix} 0 & \lambda_1^j \\ \lambda_2^j & 0 \end{pmatrix} \begin{pmatrix} \lambda_1^{k-1-j} & 0 \\ 0 & \lambda_2^{k-1-j} \end{pmatrix} \\
&= \sum_{k=0}^{N} c_k \sum_{j=0}^{k-1} \begin{pmatrix} 0 & \lambda_1^j \lambda_2^{k-1-j} \\ \lambda_2^j \lambda_1^{k-1-j} & 0 \end{pmatrix}.
\end{aligned}$$

Using

$$\lambda_1^k - \lambda_2^k = (\lambda_1 - \lambda_2)(\sum_{j=0}^{k-1} \lambda_1^j \lambda_2^{k-1-j}),$$

we conclude

$$\begin{aligned}
\frac{d}{dt}\Big|_{t=0} f(H+tV) &= \sum_{k=0}^{N} c_k \begin{pmatrix} 0 & \frac{\lambda_1^k - \lambda_2^k}{\lambda_1 - \lambda_2} \\ \frac{\lambda_2^k - \lambda_1^k}{\lambda_2 - \lambda_1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & f^{[1]}(\lambda_1, \lambda_2) \\ f^{[1]}(\lambda_2, \lambda_1) & 0 \end{pmatrix} \\
&= \begin{pmatrix} f^{[1]}(\lambda_1, \lambda_1) \cdot 0 & f^{[1]}(\lambda_1, \lambda_2) \cdot 1 \\ f^{[1]}(\lambda_2, \lambda_1) \cdot 1 & f^{[1]}(\lambda_2, \lambda_2) \cdot 0 \end{pmatrix} = M_{f^{[1]}} V,
\end{aligned}$$

which coincides with (1.5). Here, the Schur multiplier has index set $\Lambda = \{\lambda_1, \lambda_2\}$ and projections $p_{\lambda_j} = (\delta_{j\mu}\delta_{j\nu})_{\mu,\nu} \in \mathbb{C}^{2\times 2}$.

1.4. Recent proof methods for boundedness of Schur multipliers

The goal of this thesis is to investigate whether Theorem 1.3 can be proven using recent methods that were not yet known at the time the original proof in [35] was discovered. In this section, we will give an overview over the methods we will use for this purpose, the majority of which has been developed after the publication of [35, 36].

Multilinear Transference Transference between linear Schur multipliers and Fourier multipliers, as already mentioned in Section 1.1, is the main ingredient of the boundedness proofs in [35, 36]. We will give an informal introduction to the transference method in the linear case, before discussing its recent extension to multilinear multipliers.

Consider a Schur multiplier M_m acting on S_p, $p \in (1, \infty)$. Key to this method is finding a suitable embedding ι of the Schatten space into an L^p-space of functions with values in S_p. Ideally, this embedding is in fact an isometry such that $\iota \circ T_{\tilde{m}} = M_m \circ \iota$, where $T_{\tilde{m}}$ is a suitable Fourier multiplier. If M_m is of Toeplitz form, then usually $\tilde{m}$ is such that $m(\lambda, \mu) = \tilde{m}(\lambda - \mu)$. We give the linear transference proof used in [35, 36] for the symbol $m(\lambda, \mu) = (\lambda - \mu)^{is}$, $s \in \mathbb{R}$ in detail in the proof of Theorem 6.3.

More recently, transference between Schur multipliers and Fourier multipliers has been extended to multilinear multipliers [4, 5]. Rather than embedding $S_{p_1} \times \ldots \times S_{p_n}$ into an L^p-space directly, each Schatten space S_{p_k} is separately embedded into an L^{p_k}-space, and a multilinear Fourier multiplier on the product of those new L^p-spaces is considered. This allows us to treat multilinear Schur multipliers of Toeplitz form in a significantly more efficient manner, as we will demonstrate in Section 5.

L^p-extension of multilinear Calderón-Zygmund operators The transference method for studying Schur multipliers hinges on the idea that the boundedness of a Fourier multiplier may be easier to show than the boundedness of a Schur multiplier. Indeed, while Fourier multipliers on scalar-valued L^p-spaces have not been fully classified for $p \in (1,\infty) \setminus \{2\}$, several sufficient conditions for their boundedness are known, see e.g. [15, 20].

However, these results do not automatically apply in the vector-valued case. This is the so-called *L^p-extension problem*: If (Ω, μ) is a measure space, T is a bounded operator on $L^p(\Omega, \mathbb{C})$ and X is a Banach space, when is

$$T \otimes I_X : f \otimes x \mapsto Tf \otimes x$$

a bounded operator on $L^p(\Omega, X)$? Examples of operators for which there exist choices of p and X such that their extension onto $L^p(\Omega, X)$ is not bounded are the Fourier transform or the Hilbert transform, see [20, Chapter 2.1.3] for details.

It has become evident that for the extension of harmonic analysis to the vector-valued setting, so-called *UMD spaces* are the correct class of Banach spaces to consider. While their definition stems from martingale theory, X being a UMD space is in fact equivalent to the boundedness of the Hilbert transform on $L^p(\mathbb{R}, X)$ for $p \in (1,\infty)$, which in turn provides tools from classical harmonic analysis such as the Mihlin multiplier theorem or the Littlewood-Paley decomposition on $L^p(\mathbb{R}, X)$. See [20, Chapters 4–5] for a detailed introduction.

One class of operators that appears to be rather well-behaved under extension to UMD space-valued L^p-spaces are the so-called *Calderón-Zygmund operators* (see e.g. [22, Chapters 11–12]), which are integral operators with a sufficiently well-behaved singular kernel. In particular, Calderón-Zygmund operators can be represented as a sum of two particular types of operators, *dyadic shifts* and *paraproducts*, simplifying boundedness and extension arguments [19].

For linear Calderón-Zygmund operators bounded on $L^p(\mathbb{R}, \mathbb{C})$, it is known that they can be extended to $L^p(\mathbb{R}, X)$ for any UMD space X, see [22, Theorem 12.3.1]. For the extension of bounded multilinear Calderón-Zygmund operators, more care is needed to ensure compatibility of the target UMD spaces. The Schatten spaces we consider are in fact UMD spaces for $p > 1$, and in [12], it was shown that Schatten spaces (and more generally, noncommutative L^p-spaces) are such compatible UMD spaces if their exponents are in Hölder combination, i.e. $1/p_1 + \ldots + 1/p_n = 1/p$ with $p_1, \ldots, p_n, p \in (1,\infty)$. In Section 5, We will use the following bilinear theorem from [12], specialised to Schatten spaces.

Theorem 1.4 (Special case of [12, Theorem 1.1])**.** *Let T be a bilinear Calderón-Zygmund operator on $\mathbb{R}$. Then the bilinear operator*

$$T_{\text{ext}}(\sum_{j=1}^{J} f_j \otimes y_j, \sum_{k=1}^{K} g_k \otimes z_k) := \sum_{j,k} T(f_j, g_k) \otimes y_j z_k$$

with $f_j, g_k \in L_c^\infty(\mathbb{R})$, $y_j \in S_{p_1}$, $z_k \in S_{p_2}$, extends to a bounded operator

$$T_{\text{ext}} : L^{p_1}(\mathbb{R}, S_{p_1}) \times L^{p_2}(\mathbb{R}, S_{p_2}) \to L^p(\mathbb{R}, S_p) \tag{1.6}$$

for $p_1, p_2, p \in (1,\infty)$ such that $1/p_1 + 1/p_2 = 1/p$.

Hörmander-Mihlin condition for linear Schur multipliers For non-Toeplitz form Schur multipliers, the transference method is generally difficult to apply, if at all possible. However, a recent result on the boundedness of linear Schur multipliers, including those of non-Toeplitz form, gives a rather simple sufficient condition for their boundedness.

For Fourier multipliers, the Hörmander-Mihlin theorem [15, Theorem 6.2.7] states that a Fourier multiplier T_m with a bounded and sufficiently smooth symbol m is bounded on $L^p(\mathbb{R}^d, \mathbb{C})$ if

$$\sum_{|\gamma| \leq \lfloor \frac{d}{2} \rfloor + 1} \| \xi \mapsto |\xi|^{|\gamma|} |\partial^\gamma m(\xi)| \|_\infty < \infty, \tag{1.7}$$

where $\gamma \in \mathbb{N}_0^d$ is a multi-index. By linear transference, condition (1.7) was already known to be sufficient for the boundedness of the associated Toeplitz form Schur multiplier on S_p for $p \in (1,\infty)$. In [7], it was shown that condition (1.7) is indeed also sufficient for boundedness of the Schur multipliers M_m if the symbol m is not of Toeplitz form. In fact, a weaker version of (1.7) is sufficient, as mixed derivatives need not be considered.

Theorem 1.5 ([7, Theorem A]). *Let* $m \in C^{\lfloor \frac{d}{2} \rfloor + 1}(\mathbb{R}^{2d} \setminus \{\lambda = \mu\})$, $p \in (1, \infty)$, *and let* M_m *be the Schur multiplier associated with* m. *Then*

$$\|M_m\|_{S_p \to S_p} \lesssim \frac{p^2}{p-1} |||m|||_{\mathrm{HMS}}$$

with $|||m|||_{\mathrm{HMS}} := \sum_{|\gamma| \leq \lfloor \frac{d}{2} \rfloor + 1} \|(\lambda, \mu) \mapsto |\lambda - \mu|^{|\gamma|}(|\partial_\lambda^\gamma m(\lambda, \mu)| + |\partial_\mu^\gamma m(\lambda, \mu)|)\|_\infty$.

As will be shown in Section 4, boundedness of the Schur multiplier $M_{f^{[1]}}$ follows as a corollary from Theorem 1.5.

1.5. Comparison of boundedness proofs in the bilinear case

The main result of this thesis is that it is indeed possible to prove the bilinear case of Theorem 1.3 using the methods introduced in the previous section. Here we will give a high-level overview over the differences between the proof of [35] and our proof developed in Sections 3–5.

Both proofs are based on a similar observation – it is possible to decompose a second order divided difference into Toeplitz form symbols and divided differences of the form $(\lambda, \mu) \mapsto f^{[2]}(\lambda, \lambda, \mu)$. It is possible to do this in such a manner that the bilinear Schur multiplier $M_{f^{[2]}}$ can then also be decomposed into bilinear Toeplitz form Schur multipliers and linear Schur multipliers of divided differences. Theorem 1.5 allows us to show the boundedness of the linear Schur multiplier in an efficient manner, whereas in [35], its boundedness was shown by an induction argument based on linear transference.

The bilinear Toeplitz form Schur multipliers in both proofs are of the form $(\lambda_0, \lambda_1, \lambda_2) \mapsto \frac{\lambda_0 - \lambda_1}{\lambda_0 - \lambda_2}$. Note however that these symbols are not bounded and hence not suitable for transference to Fourier multipliers – as we will see in Section 5.1, the transference procedure would yield a Fourier multiplier with an unbounded symbol, whereas $m \in L^\infty$ is a necessary condition for the L^p-boundedness of the Fourier multiplier T_m. In both [35] and this thesis, this problem is addressed by strategically restricting the Schatten spaces to either upper or lower triangular operators. For the symbol $(\lambda_0, \lambda_1, \lambda_2) \mapsto \frac{\lambda_0 - \lambda_1}{\lambda_0 - \lambda_2}$, this corresponds to restricting its domain such that $|\lambda_0 - \lambda_1| \leq |\lambda_0 - \lambda_2|$ for all considered $(\lambda_0, \lambda_1, \lambda_2)$. In [35], boundedness of the resulting Schur multiplier is then shown by using linear transference separately on the numerator and denominator. In this thesis, we show that bilinear transference yields a Fourier multiplier that is also a Calderón-Zygmund operator, allowing us to apply Theorem 1.4.

We further compare the two proof methods by comparing the growth rate in p of the resulting upper bound on the norm of $\|M_{f^{[2]}} : S_p \times S_p \to S_{p/2}\|$ for $p \to \infty$. This bound was not explicitly stated in [35], hence we infer it from the proof of Theorem 1.3 in Section 6.1. This in particular allows us to compare the growth rate of the boundedness constant of the bilinear Toeplitz form Schur multipliers and the linear Schur multipliers separately. We find that due to the absence of paraproducts in the decomposition of the Calderón-Zygmund operator, the bilinear transference method improves the boundedness constant of the bilinear Toeplitz form Schur multiplier in p by three orders of magnitude. Furthermore, the use of Theorem 1.5 improves the bound on the linear Schur multiplier to linear growth in p, whereas the the bound achieved in [35] by linear transference has cubical growth in p.

1.6. Outline of the thesis

In Section 2, we give the necessary preliminaries that will be used in the following sections. Besides properties of Schur multipliers and divided differences, dyadic constructions from harmonic analysis will be also introduced, as this will be necessary for the discussion of Calderón-Zygmund operators.

We present our proof of the bilinear version of Theorem 1.3 in three steps. In Section 3, we introduce the decomposition of the bilinear Schur multiplier $M_{f^{[2]}}$ into a composition of bilinear Toeplitz form Schur multipliers and linear Schur multipliers. In Section 4, the boundedness of the resulting linear Schur multiplier is shown using Theorem 1.5. In Section 5, we use bilinear transference to prove the boundedness of the bilinear Toeplitz form Schur multiplier using Theorem 1.4.

We investigate the growth rate of the boundedness constant in p in the bilinear case in Section 6, where we give an explicit upper bound on the constants in Theorem 1.3 and Theorem 1.4. Finally, we give an outlook on the remaining steps towards proving the full version of Theorem 1.3 using multilinear transference in Section 7.

2

Preliminaries

In this section, we introduce some definitions and related results that we will use throughout this thesis. We assume that the reader has knowledge equivalent to a first course on functional analysis (see e.g. [30]) and is familiar with basic notions of measure theoretic probability theory.

2.1. Notations and assumptions

In this section, we fix some general notation for the remainder of this thesis.

- We use $\mathbb{N}$ to refer to the set of positive integers, the set of non-negative integers is denoted by $\mathbb{N}_0$.
- By $\mathbb{T}^d$, we refer to the d-dimensional torus $\mathbb{T}^d = [0, 2\pi]^d$.
- For two functions $f, g : \mathbb{R}^d \to \mathbb{C}$, $d \in \mathbb{N}$, we define $\langle f, g \rangle := \int_{\mathbb{R}^d} fg dx$.
- For $f : \mathbb{R}^d \to \mathbb{C}$ and $Q \subset \mathbb{R}^d$ bounded, we define $\langle f \rangle_Q := |Q|^{-1} \int_Q f dx$.
- We denote the expectation, i.e. the integral over a probability space, by $\mathbb{E}$, usually without reference to the underlying probability space.
- The Fourier transform of a distribution f is denoted by $\hat{f}$. For Schwartz functions, we use the convention
$$\hat{f}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^d} f(x) e^{-ix\cdot\xi} dx$$
to normalise the Fourier transform. The inverse Fourier transform of f is denoted by $\check{f}$.
- The n-th derivative of a function f may be denoted by $f^{(n)}$.
- We write $A \lesssim B$ when there exists a constant $C > 0$ such that $A \leq CB$. If C depends on a parameter p relevant to the discussion, we write $A \lesssim_p B$.
- To describe the growth rate of a constant c_p in its parameter p, we write $c_p = O(p^k)$, $k \in \mathbb{N}$, if there exists a constant C independent of p and some p_0 such that $c_p \leq Cp^3$ for all $p \geq p_0$.
- For $p \in (1, \infty)$, p' is defined by the relation $1/p + 1/p' = 1$.
- For a set S, the indicator function of that set is denoted by 1_S.
- Unless stated otherwise, all Hilbert spaces H are assumed to be separable. The inner product on H, denoted by $(\cdot, \cdot)$, is linear in the first argument.
- Throughout this thesis, $(p_\lambda)_\lambda$ refers to a countable orthonormal set of projections with $\sum_\lambda p_\lambda = 1$.

2.2. Divided differences

After introducing divided differences in Section 1.2, we will now give the full definition of divided differences and introduce some of their basic properties.

Definition 2.1 (Divided differences, [38, Section 8.6])**.** Let $f \in C^n(\mathbb{R})$, $n \in \mathbb{N}_0$. We define the *n-th order divided difference* $f^{[n]}$ of f inductively as follows. The first order divided difference is constructed as

$$f^{[0]}(\lambda_0) := f(\lambda_0), \tag{2.1}$$

$$f^{[1]}(\lambda_0, \lambda_1) := \begin{cases} \frac{f^{[0]}(\lambda_0) - f^{[0]}(\lambda_1)}{\lambda_0 - \lambda_1}, & \lambda_0 \neq \lambda_1, \\ f'(\lambda_0), & \lambda_0 = \lambda_1. \end{cases} \tag{2.2}$$

$$f^{[n]}(\lambda_0, ..., \lambda_n) := \begin{cases} \frac{f^{[n-1]}(\lambda_0,...,\lambda_{j-1},\lambda_{j+1},...,\lambda_n) - f^{[n-1]}(\lambda_0,...,\lambda_{i-1},\lambda_{i+1},...,\lambda_n)}{\lambda_i - \lambda_j}, & \text{if } \lambda_i \neq \lambda_j \text{ for some } i \neq j, \\ \frac{f^{(n)}(\lambda_0)}{n!}, & \lambda_0 = \ldots = \lambda_n, \end{cases} \tag{2.3}$$

where $\lambda_0, \ldots, \lambda_n \in \mathbb{R}$ and $\lambda_0 \neq \lambda_n$.

We will see in Lemma 2.3 that the divided differences are invariant under permutation of their variables, hence the n-th order divided difference $f^{[n]}$ is well-defined on $\mathbb{R}^{n+1}$ for $f \in C^n(\mathbb{R})$, even if there are multiple possible choices for i, j in (2.3). We will often deal with divided differences with many coinciding arguments, therefore we introduce the following notation for convenience.

Notation 2.2. For $\lambda, \mu \in \mathbb{R}$, we set

$$f^{[n]}(\lambda^{(k)}, \mu^{(n+1-k)}) := f^{[n]}(\underbrace{\lambda, \ldots, \lambda}_{k \text{ times}}, \underbrace{\mu, \ldots, \mu}_{\substack{n+1-k \\ \text{times}}}).$$

In the following lemma, we collect some straightforward consequences of Definition 2.1.

Lemma 2.3 (Properties of divided differences)**.** Let $f : \mathbb{R} \to \mathbb{R}$ and let $n \in \mathbb{N}$.

1. $f^{[n]}$ is invariant under permutation of its variables.
2. For $f \in C^n(\mathbb{R})$, we have the estimate

$$\|f^{[n]}\|_\infty \leq \frac{\|f^{(n)}\|_\infty}{n!}.$$

Proof.

1. We prove this by induction. For $n = 0$ this is trivially fulfilled, for $n = 1$ it holds by symmetry of (2.2). Now let $n > 1$; we prove the statement in three steps, where we define $f^{[n]}(\lambda_0, \ldots, \lambda_n)$ as in (2.3) for $\lambda_0 \neq \lambda_n$. We shall always assume that the permuted variables λ_i and λ_j are not equal.

   - Permutation of λ_0 and λ_n: We use induction for the second equality to see

$$\begin{aligned} f^{[n]}(\lambda_n, \lambda_1, \ldots, \lambda_{n-1}, \lambda_0) &= \frac{f^{[n-1]}(\lambda_n, \lambda_1, \ldots, \lambda_{n-1}) - f^{[n-1]}(\lambda_1, \ldots, \lambda_{n-1}, \lambda_0)}{\lambda_n - \lambda_0} \\ &= \frac{f^{[n-1]}(\lambda_1, \ldots, \lambda_{n-1}, \lambda_n) - f^{[n-1]}(\lambda_0, \lambda_1, \ldots, \lambda_{n-1})}{\lambda_n - \lambda_0} \\ &= f^{[n]}(\lambda_0, \ldots, \lambda_n). \end{aligned}$$

   - For permutation of λ_i, λ_j, $i < j$ and $i, j \notin \{0, n\}$, we see by induction that

$$\begin{aligned} & f^{[n]}(\lambda_0, \ldots, \lambda_j, \ldots, \lambda_i, \ldots, \lambda_n) \\ =& \frac{f^{[n-1]}(\lambda_0, \ldots, \lambda_j, \ldots, \lambda_i, \ldots, \lambda_{n-1}) - f^{[n-1]}(\lambda_1, \ldots, \lambda_j, \ldots, \lambda_i, \ldots, \lambda_n)}{\lambda_0 - \lambda_n} \\ =& \frac{f^{[n-1]}(\lambda_0, \ldots, \lambda_i, \ldots, \lambda_j, \ldots, \lambda_{n-1}) - f^{[n-1]}(\lambda_1, \ldots, \lambda_i, \ldots, \lambda_j, \ldots, \lambda_n)}{\lambda_0 - \lambda_n} \\ =& f^{[n]}(\lambda_0, \ldots, \lambda_n). \end{aligned}$$

- For permutations of λ_i, λ_j with $i \in \{0, n\}$ and $j \notin \{0, n\}$, we will without loss of generality assume $i = 0$, $j = 1$. We obtain by induction

$$\begin{aligned}
&f^{[n]}(\lambda_1, \lambda_0, \lambda_2, \dots, \lambda_n) - f^{[n]}(\lambda_0, \lambda_1, \lambda_2, \dots, \lambda_n)\\
=&\frac{f^{[n-1]}(\lambda_1, \lambda_0, \lambda_2, \dots, \lambda_{n-1}) - f^{[n-1]}(\lambda_0, \lambda_2, \dots, \lambda_n)}{\lambda_1 - \lambda_n}\\
&\qquad - \frac{f^{[n-1]}(\lambda_0, \lambda_1, \lambda_2, \dots, \lambda_{n-1}) - f^{[n-1]}(\lambda_1, \lambda_2, \dots, \lambda_n)}{\lambda_0 - \lambda_n}\\
=&\left(\frac{1}{\lambda_1 - \lambda_n} - \frac{1}{\lambda_0 - \lambda_n}\right) f^{[n-1]}(\lambda_0, \lambda_1, \lambda_2, \dots, \lambda_{n-1})\\
&\qquad - \frac{f^{[n-1]}(\lambda_0, \lambda_2, \dots, \lambda_n)}{\lambda_1 - \lambda_n} + \frac{f^{[n-1]}(\lambda_1, \lambda_2, \dots, \lambda_n)}{\lambda_0 - \lambda_n}.
\end{aligned}$$

By further decomposing the second and third divided difference, we have

$$\begin{aligned}
&\frac{1}{(\lambda_0 - \lambda_n)(\lambda_1 - \lambda_n)} \Big((\lambda_0 - \lambda_1) f^{[n-1]}(\lambda_0, \dots, \lambda_{n-1}, \lambda_1)\\
&\qquad - (f^{[n-2]}(\lambda_0, \lambda_2, \dots, \lambda_{n-1}) - f^{[n-2]}(\lambda_2, \dots, \lambda_n))\\
&\qquad + (f^{[n-2]}(\lambda_1, \lambda_2, \dots, \lambda_{n-1}) - f^{[n-2]}(\lambda_2, \dots, \lambda_n))\Big)\\
=&\frac{1}{(\lambda_0 - \lambda_n)(\lambda_1 - \lambda_n)} \Big((\lambda_0 - \lambda_1) f^{[n-1]}(\lambda_0, \dots, \lambda_{n-1}, \lambda_1)\\
&\qquad - (f^{[n-2]}(\lambda_0, \lambda_2, \dots, \lambda_{n-1}) - f^{[n-2]}(\lambda_1, \lambda_2, \dots, \lambda_{n-1})\Big).
\end{aligned}$$

The second two divided differences can be combined as

$$f^{[n-2]}(\lambda_1, \lambda_2, \dots, \lambda_{n-1}) - f^{[n-2]}(\lambda_0, \lambda_2, \dots, \lambda_{n-1}) = (\lambda_0 - \lambda_1) f^{[n-1]}(\lambda_0, \lambda_1, \dots, \lambda_{n-1})$$

by induction, from which we conclude

$$f^{[n]}(\lambda_1, \lambda_0, \lambda_2, \dots, \lambda_n) - f^{[n]}(\lambda_0, \lambda_1, \lambda_2, \dots, \lambda_n) = 0.$$

2. Following [35], we can express $f^{[n]}$ in integral form as follows. Define

$$\begin{aligned}
\mathcal{S}_n &:= \{(s_0, \dots, s_n) \in \mathbb{R}^{n+1} \mid \sum_{j=0}^{n} s_j = 1,\ s_j \geq 0,\ 0 \leq j \leq n\},\\
\mathcal{R}_n &:= \{(s_0, \dots, s_{n-1}) \in \mathbb{R}^{n} \mid \sum_{j=0}^{n-1} s_j \leq 1,\ s_j \geq 0,\ 0 \leq j \leq n-1\},\\
\int_{\mathcal{S}_n} \phi(s_0, \dots, s_n) d\sigma_n &:= \int_{\mathcal{R}_n} \phi(s_0, \dots, s_{n-1}, 1 - \sum_{j=0}^{n-1} s_j) ds,
\end{aligned}$$

where ds denotes the Lebesgue measure on $\mathbb{R}^n$. With these definitions, we can now express $f^{[n]}$ as

$$f^{[n]}(\lambda_0, \dots, \lambda_n) = \int_{\mathcal{S}_n} f^{(n)}\left(\sum_{j=0}^{n} s_j \lambda_j\right) d\sigma_n.$$

For a proof see [35, Lemma 5.1]. Hence we conclude

$$\|f^{[n]}\|_\infty \leq \|f^{(n)}\|_\infty \int_{\mathcal{S}_n} d\sigma_n = \frac{\|f^{(n)}\|_\infty}{n!}.$$

□

Finally, we state another useful representation of the divided differences; for a proof see [10, Chapter 4, (7.12)].

Lemma 2.4. If $f \in C^n(\mathbb{R})$ and $\lambda_0, \ldots, \lambda_n \in \mathbb{R}$ such that not all of them coincide, then

$$f^{[n]}(\lambda_0, \ldots, \lambda_n) = \int_0^1 dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}} f^{(n)}(\lambda_n + (\lambda_{n-1} - \lambda_n)t_1 + \ldots + (\lambda_0 - \lambda_1)t_n)dt_n.$$

2.3. Schatten spaces

We have formally introduced Schatten spaces in Section 1.1. We will now define them properly, starting with the definition of the singular value decomposition.

Definition 2.5 (Singular value decomposition [30, Definition 14.14])**.** Let x be a compact operator on a Hilbert space.

- The *singular values* of x are the nonzero eigenvalues of its modulus $|x| := \sqrt{x^*x}$. Since $|x|$ is a positive operator, the singular values are positive real numbers.
- The *singular value sequence* $(\mu_n(x))_n$ of x is defined as the non-increasing sequence of singular values of x, repeated according to their multiplicity.
- The *singular value decomposition* consists of two orthonormal families $(f_n)_n$ and $(g_n)_n$ such that

$$x = \sum_n \mu_n(x)(\cdot, f_n)g_n.$$

Every compact operator admits a (not necessarily unique) singular value decomposition, see [30, Theorem 9.2]. We can now define Schatten spaces as follows.

Definition 2.6 (Schatten space, [20, Definition D.1.5])**.** Let H be a separable Hilbert space, and let $p \in [1, \infty)$. Then the *Schatten space* $S_p := S_p(H)$ is defined as

$$S_p := \{x \in B(H) \mid x \text{ compact}, \|(\mu_n(x))_n\|_{\ell^p} < \infty\},$$

where $B(H)$ denotes the bounded linear operators on H. These spaces are Banach spaces with the norm

$$\|x\|_{S_p} := \|(\mu_n(x))_n\|_{\ell^p}.$$

In this thesis, the underlying Hilbert space H is not relevant (apart from its separability), hence we suppress H from the notation. From now on, by saying that x is a Schatten space element, we will refer to $x \in S_p$ for some $p \in [1, \infty)$.

Let us now give some elementary properties of the Schatten spaces.

Lemma 2.7 (Schatten space properties)**.**

1. (Hölder inequality, [20, Corollary D.2.4]) Let $(p_1, p_2, p) \in [1, \infty]$ be such that $1/p_1 + 1/p_2 = 1/p$. Let $x \in S_{p_1}$ and $y \in S_{p_2}$, where we set S_∞ to be the space of compact operators with the operator norm. Then $xy \in S_p$ with $\|xy\|_{S_p} \leq \|x\|_{S_{p_1}} \|y\|_{S_{p_2}}$.
2. (Complex interpolation, [20, Proposition D.3.1]) Let $(p_1, p_2, p) \in [1, \infty)$ and $\theta \in (0, 1)$ be such that $\theta/p_1 + (1-\theta)/p_2 = 1/p$. Then for all linear operators T that are bounded on S_{p_1} and S_{p_2}, we have

$$\|T\|_{S_p \to S_p} \leq \|T\|^{\theta}_{S_{p_1} \to S_{p_1}} \|T\|^{1-\theta}_{S_{p_2} \to S_{p_2}}.$$

3. Let $u \in B(H)$ unitary, $x \in S_p$ for $p \in [1, \infty)$, then $\|u^*xu\|_{S_p} = \|x\|_{S_p}$.
4. For $x \in S_p$, we have $\|x\|_{S_p} = \|x^*\|_{S_p}$.

Proof. For the proofs of 1. and 2., see the references provided. We shall prove 3. and 4.

Let the singular value decomposition of x be given by $(f_n)_n$ and $(g_n)_n$. Then for $h \in H$,

$$(u^*xu)(h) = u^* \sum_n \mu_n(x)(uh, f_n)g_n = \sum_n \mu_n(x)(h, u^*f_n)u^*g_n.$$

Since u^* is unitary, the families $(u^* f_n)_n$ and $(u^* g_n)_n$ are again orthonormal, hence the formula above is a singular value decomposition of u^*xu. This implies that u^*xu and x have the same singular value sequence, which in turn implies that their Schatten space norms are equal for all $p \in [1, \infty)$.

The fourth statement follows from the fact that x and x^* have the same singular value sequence. Indeed, for x as above and arbitrary $h_1, h_2 \in H$ we have

$$(x^* v, w) = (v, xw) = \sum_n \mu_n(x)(f_n, w)(v, g_n) = \left(\sum_n \mu_n(x)(v, g_n) f_n, w \right),$$

hence $x^* = \sum_n \mu_n(x)(\cdot, g_n) f_n$. This is a singular value representation, from which the norm equality follows as in the previous proof. ☐

As discussed in Section 1.1, one can characterise Schatten spaces via the trace on $B(H)$, which we now introduce.

Definition 2.8 (Operator trace, [30, Definition 14.17])**.** Let x be a compact operator on a Hilbert space H, and let $(h_n)_n$ be an orthonormal basis of H. The *trace* of x is defined as

$$\tau(x) := \sum_n (Th_n, h_n).$$

This definition is independent of the chosen orthonormal basis.

Before giving the tracial form of the Schatten space norms, let us give some useful properties of the trace. For proofs, see the provided references.

Lemma 2.9 (Properties of the trace)**.**

1. (Singular value form, [30, Theorem 14.15(1)]) Let $x \in B(H)$ be compact with singular value sequence $(\mu_n(x))_n$. Then $\tau(|x|) = \sum_n \mu_n(x)$.
2. (Permutation property, [30, Proposition 14.27]) For $x_1, \dots, x_n$ Schatten space elements, $\tau(x_1 \dots x_n) = \tau(x_n x_1 \dots x_{n-1}) = \tau(x_2 \dots x_n x_1)$.

Lemma 2.10. Let $x \in S_p$, $p \in (1, \infty)$. Then

$$\|x\|_{S_p}^p = \tau(|x|^p) = \left(\sup_{\|y\|_{S_q}=1} \tau(xy) \right)^p.$$

Proof. Since the singular values are eigenvalues of $|x|$, its singular value decomposition can be expressed as $|x| = \sum_n \mu_n(x)(\cdot, h_n)h_n$ for some orthonormal family $(h_n)_n$. This in particular implies

$$|x|^p = \sum_n \mu_n(x)^p(\cdot, h_n)h_n,$$

hence by Definition 2.8,

$$\tau(|x|^p) = \sum_n (|x|^p h_n, h_n) = \sum_n \mu_n(x)^p = \|x\|_{S_p}^p.$$

For the second equality, see the proof of [20, Theorem D.2.6]. ☐

Let us now introduce the so-called triangular truncations. Acting on finite-dimensional matrices, these operators map a matrix to its upper- or lower diagonal part. In Section 3, we will apply them strategically in order to decompose Schur multipliers of divided differences in a manner suitable for our proof strategy.

Definition 2.11 (Triangular truncations [3])**.** Let $p \in (1, \infty)$ and let $(p_\lambda)_\lambda$ be a countable family of orthogonal projections such that $\sum_\lambda p_\lambda = 1$. We define the following *triangular truncation* operators on S_p.

$$T_{\triangle_{\text{upper}}} : x \mapsto \sum_{\lambda \geq \mu} p_\lambda x p_\mu, \qquad T_{\triangle^{\text{off}}_{\text{upper}}} : x \mapsto \sum_{\lambda > \mu} p_\lambda x p_\mu,$$

$$T_{\triangle_{\text{lower}}} : x \mapsto \sum_{\lambda \leq \mu} p_\lambda x p_\mu, \qquad T_{\triangle^{\text{off}}_{\text{lower}}} : x \mapsto \sum_{\lambda < \mu} p_\lambda x p_\mu.$$

The operator $T_{\triangle_{\text{upper}}}$ (resp. $T_{\triangle_{\text{lower}}}$) is called *upper* (resp. *lower*) *triangular truncation*. The operators $T_{\triangle^{\text{off}}_{\text{upper}}}$ and $T_{\triangle^{\text{off}}_{\text{lower}}}$ denote *off-diagonal* triangular truncations.

To denote the restriction of an operator to its diagonal, we further define

$$T_{\text{diag}} \;:=\; T_{\triangle_{\text{upper}}} - T_{\triangle^{\text{off}}_{\text{upper}}} \;=\; T_{\triangle_{\text{lower}}} - T_{\triangle^{\text{off}}_{\text{lower}}}.$$

These operators are in fact bounded operators, for a proof see the provided reference.

Theorem 2.12 (Boundedness of triangular truncations, [3, Corollary 19])**.** *All triangular truncation operators $T_\triangle \in \{T_{\triangle_{\text{upper}}}, T_{\triangle_{\text{lower}}}, T_{\triangle^{\text{off}}_{\text{upper}}}, T_{\triangle^{\text{off}}_{\text{lower}}}\}$ are bounded operators on S_p for $p \in (1,\infty)$, with the common bound*

$$\|T_\triangle\|_{S_p\to S_p} \le C_{\triangle,p} = C\frac{p^2}{p-1}.$$

Within a trace, these operators can be transformed into one another as follows.

Lemma 2.13 (Trace of product of triangular truncations)**.** Let $p \in (1,\infty)$ and let $x \in S_p$, $y \in S_{p'}$. Then

$$\begin{aligned}\tau((T_{\triangle_{\text{upper}}}x)y) &= \tau((T_{\triangle_{\text{upper}}}x)(T_{\triangle_{\text{lower}}}y)) = \tau(x(T_{\triangle_{\text{lower}}}y)),\\ \tau((T_{\triangle^{\text{off}}_{\text{upper}}}x)y) &= \tau((T_{\triangle^{\text{off}}_{\text{upper}}}x)(T_{\triangle^{\text{off}}_{\text{lower}}}y)) = \tau(x(T_{\triangle^{\text{off}}_{\text{lower}}}y)).\end{aligned}$$

Proof. By writing out the triangular truncations, we have

$$\tau((T_{\triangle_{\text{upper}}}x)y) = \sum_{\lambda\ge\mu}\tau(p_\lambda x p_\mu y),.$$

Using $p_\lambda^2 = p_\lambda$ and the permutation property of the trace yields

$$\sum_{\lambda\ge\mu}\tau(p_\lambda x p_\mu y) = \sum_{\lambda\ge\mu}\tau(p_\lambda x p_\mu y p_\lambda).$$

By orthogonality of the projections,

$$\sum_{\lambda\ge\mu}\tau(p_\lambda x p_\mu y p_\lambda) = \tau\left(\left(\sum_{\lambda_1\ge\mu_1} p_{\lambda_1} x p_{\mu_1}\right)\left(\sum_{\lambda_2\ge\mu_2} p_{\mu_2} y p_{\lambda_2}\right)\right) = \tau((T_{\triangle_{\text{upper}}}x)(T_{\triangle_{\text{lower}}}y))$$

and furthermore,

$$\sum_{\lambda\ge\mu}\tau(p_\lambda x p_\mu y) = \sum_{\lambda\ge\mu}\tau(x p_\mu y p_\lambda) = \tau(x(T_{\triangle_{\text{lower}}}y)).$$

The statement for $T_{\triangle^{\text{off}}_{\text{upper}}}$ and $T_{\triangle^{\text{off}}_{\text{lower}}}$ follows in the same manner. □

2.4. Schur multipliers

After formally introducing Schur multipliers in Section 1.1, we now give their definition. We first define them on finite-rank operators, which ensures that all sums are finite.

Definition 2.14. Let $m : \mathbb{R}^{n+1} \to \mathbb{C}$. The associated *Schur multiplier* M_m is defined as the n-linear operator acting on finite-rank operators $x_1, \dots, x_n$ as

$$M_m(x_1,\dots,x_n) := \sum_{\lambda_0,\dots,\lambda_n} m(\lambda_0,\dots,\lambda_n) p_{\lambda_0} x_1 p_{\lambda_1} \dots p_{\lambda_{n-1}} x_n p_{\lambda_n},$$

where $(p_{\lambda_n})_n$ is a countable family of orthogonal projections such that $\sum_\lambda p_\lambda = 1$. The function m is called *symbol* of the Schur multiplier M_m.

We call M_m a *Toeplitz form Schur multiplier* if m is of the form

$$m(\lambda_0,\dots,\lambda_n) = \tilde m(\lambda_0-\lambda_1, \lambda_1-\lambda_2, \dots, \lambda_{n-1}-\lambda_n)$$

for some $\tilde m : \mathbb{R}^n \to \mathbb{C}$.

Since finite-rank operators are dense in the Schatten spaces, we immediately have that if

$$\|M_m(x_1,\dots,x_n)\|_{S_p} \lesssim \|x_1\|_{S_{p_1}} \dots \|x_n\|_{S_{p_n}},$$

the Schur multiplier extends to a bounded operator $M_m : S_{p_1} \times \ldots \times S_{p_n} \to S_p$.

In the following lemma we collect some properties of Schur multipliers.

Lemma 2.15 (Properties of Schur multipliers, [35, Lemma 3.2])**.** Let $m : \mathbb{R}^{n+1} \to \mathbb{C}$ and let $p_1,\dots,p_n,p \in (1,\infty)$ be such that $1/p_1 + \ldots + 1/p_n = 1/p$. Let $x_j \in S_{p_j}$, $j = 1,\dots,n$.

1. If there exist $m_1, m_2 : \mathbb{R}^{n+1} \to \mathbb{C}$ such that $m = m_1 + m_2$, then the associated Schur multiplier M_m can be decomposed as $M_m = M_{m_1} + M_{m_2}$.
2. If there exist $m_1 : \mathbb{R}^{(n+1)-(k-1)} \to \mathbb{C}$ and $m_2 : \mathbb{R}^{k+1} \to \mathbb{C}$ such that

$$m(\lambda_0,\dots,\lambda_n) = m_1(\lambda_0,\dots,\lambda_j,\lambda_{j+k},\dots,\lambda_n)m_2(\lambda_j,\dots,\lambda_{j+k})$$

for some $j \in \{0,\dots,n-1\}$, $k \in \{1,\dots,n-j\}$, then

$$M_m(x_1,\dots,x_n) = M_{m_1}(x_1,\dots,x_j,M_{m_2}(x_{j+1},\dots,x_{j+k}),x_{j+k+1},\dots,x_n).$$

3. Let $\tilde{m}(\lambda_0,\dots,\lambda_n) := \overline{m(\lambda_n,\dots,\lambda_0)}$. Then

$$\|M_{\tilde{m}} : S_{p_1} \times \ldots \times S_{p_n} \to S_p\| = \|M_m : S_{p_n} \times \ldots \times S_{p_1} \to S_p\|.$$

4. Let $y \in S_{p'}$ and let $m_*(\lambda_0,\dots,\lambda_n) := m(\lambda_n,\lambda_0,\dots,\lambda_{n-1})$. Then

$$\tau(M_m(x_1,\dots,x_n)y) = \tau(x_1 M_{m_*}(x_2,\dots,x_n,y)),$$

and hence $\|M_m : S_{p_1} \times \ldots \times S_{p_n} \to S_p\| = \|M_{m_*} : S_{p_2} \times \ldots \times S_{p_n} \times S_{p'} \to S_{p_1'}\|$.

Proof.

1. This follows directly from Definition 2.14.
2. By writing out the definition of the Schur multiplier and using the orthogonality of the projections, we have

$$\begin{aligned}
&M_m(x_1,\dots,x_n)\\
&= \sum_{\lambda_0,\dots,\lambda_n} m_1(\lambda_0,\dots,\lambda_j,\lambda_{j+k},\dots,\lambda_n)m_2(\lambda_j,\dots,\lambda_{j+k})p_{\lambda_0}x_1p_{\lambda_1}\cdots p_{\lambda_{n-1}}x_np_{\lambda_n}\\
&= \sum_{\substack{\lambda_0,\dots,\lambda_j,\\ \lambda_{j+k},\dots,\lambda_n}} m_1(\lambda_0,\dots,\lambda_j,\lambda_{j+k},\dots,\lambda_n)p_{\lambda_0}x_1p_{\lambda_1}\cdots p_{\lambda_{j-1}}x_jp_{\lambda_j}\\
&\quad\times\left(\sum_{\mu_1,\mu_2}\sum_{\lambda_{j+1},\dots,\lambda_{j+k-1}} m_2(\mu_1,\lambda_{j+1}\dots,\lambda_{j+k-1},\mu_2)p_{\mu_1}x_{j+1}p_{\lambda_{j+1}}\cdots p_{\lambda_{j+k-1}}x_{j+k}p_{\mu_2}\right)\\
&\quad\times p_{\lambda_{j+k}}x_{j+k+1}p_{\lambda_{j+k+1}}\cdots p_{\lambda_{n-1}}x_np_{\lambda_n}.
\end{aligned}$$

The expression in brackets is again a Schur multiplier, hence we have

$$\begin{aligned}
&M_m(x_1,\dots,x_n)\\
&= \sum_{\substack{\lambda_0,\dots,\lambda_j,\\ \lambda_{j+k},\dots,\lambda_n}} m_1(\lambda_0,\dots,\lambda_j,\lambda_{j+k},\dots,\lambda_n)p_{\lambda_0}x_1p_{\lambda_1}\cdots\\
&\qquad\qquad\times p_{\lambda_{j-1}}x_jp_{\lambda_j}M_{m_2}(x_{j+1},\dots,x_{j+k})p_{\lambda_{j+k}}x_{j+k+1}p_{\lambda_{j+k+1}}\cdots p_{\lambda_{n-1}}x_np_{\lambda_n}\\
&=M_{m_1}(x_1,\dots,x_j,M_{m_2}(x_{j+1},\dots,x_{j+k}),x_{j+k+1},\dots,x_n).
\end{aligned}$$

3. We have $M_{\tilde{m}}(x_1,\dots,x_n) = M_m(x_n^*,\dots,x_1^*)^*$. Indeed,

$$\begin{aligned} M_{\tilde{m}}(x_1,\dots,x_n) &= \sum_{\lambda_0,\dots,\lambda_n} \tilde{m}(\lambda_0,\dots,\lambda_n) p_{\lambda_0} x_1 p_{\lambda_1} \dots p_{\lambda_{n-1}} x_n p_{\lambda_n} \\ &= \sum_{\lambda_0,\dots,\lambda_n} \overline{m(\lambda_n,\dots,\lambda_0)} (p_{\lambda_n} x_n^* p_{\lambda_{n-1}} \dots p_{\lambda_1} x_1^* p_{\lambda_0})^* \\ &= (\sum_{\lambda_n,\dots,\lambda_0} m(\lambda_n,\dots,\lambda_0) p_{\lambda_n} x_n^* p_{\lambda_{n-1}} \dots p_{\lambda_1} x_1^* p_{\lambda_0})^* \\ &= M_m(x_n^*,\dots,x_1^*)^*. \end{aligned}$$

Since the S_p-norms are invariant under adjoints, we have

$$\begin{aligned} \|M_{\tilde{m}}\| &= \sup_{\|x_1\|_{S_{p_1}}=1} \dots \sup_{\|x_n\|_{S_{p_n}}=1} \|M_{\tilde{m}}(x_1,\dots,x_n)\|_{S_p} \\ &= \sup_{\|x_1\|_{S_{p_1}}=1} \dots \sup_{\|x_n\|_{S_{p_n}}=1} \|M_m(x_n^*,\dots,x_1^*)^*\|_{S_p} \\ &= \sup_{\|x_1^*\|_{S_{p_1}}=1} \dots \sup_{\|x_n^*\|_{S_{p_n}}=1} \|M_m(x_n^*,\dots,x_1^*)\|_{S_p} \\ &= \|M_m\|. \end{aligned}$$

4. By writing out the operator and using the permutation property of the trace, we see that

$$\begin{aligned} \tau(M_m(x_1,\dots,x_n)y) &= \tau\left(\sum_{\lambda_0,\dots,\lambda_n} m(\lambda_0,\dots,\lambda_n) p_{\lambda_0} x_1 p_{\lambda_1} \dots p_{\lambda_{n-1}} x_n p_{\lambda_n} y\right) \\ &= \sum_{\lambda_0,\dots,\lambda_n} m(\lambda_0,\dots,\lambda_n) \tau(x_1 p_{\lambda_1} x_2 \dots p_{\lambda_{n-1}} x_n p_{\lambda_n} y p_{\lambda_0}) \\ &= \tau\left(x_1 \sum_{\lambda_0,\dots,\lambda_n} m(\lambda_0,\dots,\lambda_n) p_{\lambda_1} x_2 \dots p_{\lambda_{n-1}} x_n p_{\lambda_n} y p_{\lambda_0}\right). \end{aligned}$$

By renumbering the indices we have

$$\begin{aligned} &\tau(x_1 \sum_{\lambda_0,\dots,\lambda_n} m(\lambda_0,\dots,\lambda_n) p_{\lambda_1} x_2 \dots p_{\lambda_{n-1}} x_n p_{\lambda_n} y p_{\lambda_0}) \\ &\qquad = \tau(x_1 \sum_{\lambda_0,\dots,\lambda_n} m(\lambda_n,\lambda_0,\dots,\lambda_{n-1}) p_{\lambda_0} x_2 \dots p_{\lambda_{n-2}} x_n p_{\lambda_{n-1}} y p_{\lambda_n}) \end{aligned}$$

and hence

$$\tau(M_m(x_1,\dots,x_n)y) = \tau(x_1 M_{m_*}(x_2,\dots,x_n,y)).$$

By the tracial definition of the S_p-norms, we now have

$$\begin{aligned} \|M_m : S_{p_1} \times \dots \times S_{p_n} \to S_p\| &= \sup_{\|x_1\|_{S_{p_1}}=1} \dots \sup_{\|x_n\|_{S_{p_n}}=1} \|M_m(x_1,\dots,x_n)\|_{S_p} \\ &= \sup_{\|x_1\|_{S_{p_1}}=1} \dots \sup_{\|x_n\|_{S_{p_n}}=1} \sup_{\|y\|_{S_{p'}}=1} \tau(M_m(x_1,\dots,x_n)y) \\ &= \sup_{\|x_1\|_{S_{p_1}}=1} \dots \sup_{\|x_n\|_{S_{p_n}}=1} \sup_{\|y\|_{S_{p'}}=1} \tau(x_1 M_{m_*}(x_2,\dots,x_n,y)) \\ &= \sup_{\|x_2\|_{S_{p_2}}=1} \dots \sup_{\|x_n\|_{S_{p_n}}=1} \sup_{\|y\|_{S_{p'}}=1} \|M_{m_*}(x_2,\dots,x_n,y)\|_{S_{p_1'}} \\ &= \|M_{m_*} : S_{p_2} \times \dots \times S_{p_n} \times S_{p'} \to S_{p_1'}\|. \end{aligned}$$

□

Next, we introduce some special settings in which the norm of a linear Schur multiplier is easily estimated.

Lemma 2.16 (Boundedness on S_2)**.** Let $m : \mathbb{R}^2 \to \mathbb{C}$, and let M_m be the associated Schur multiplier. Then

$$\|M_m : S_2 \to S_2\| \leq \|m\|_\infty.$$

Proof. Let $x \in S_2$. Using the tracial form of the S_2-norm and $|x|^2 = x^*x$, we have

$$\begin{aligned}\|M_m x\|_{S_2}^2 &= \tau(|M_m x|^2)\\ &= \tau\left((M_m x)^* M_m x\right)\\ &= \sum_{j,k,l,m} \tau\left((m(j,k)p_j x p_k)^* m(l,m) p_l x p_m\right)\\ &= \sum_{j,k,l,m} \overline{m(j,k)} m(l,m) \tau\left(p_k x^* p_j p_l x p_m\right).\end{aligned}$$

By the orthogonality of the projections $(p_j)_j$ and the permutation property of the trace, we have

$$\begin{aligned}\sum_{j,k,l,m} \overline{m(j,k)} m(l,m) \tau\left(p_k x^* p_j p_l x p_m\right) &= \sum_{j,k,m} \overline{m(j,k)} m(j,m) \tau\left(p_k x^* p_j p_j x p_m\right)\\ &= \sum_{j,k} \overline{m(j,k)} m(j,k) \tau\left(p_k x^* p_j p_j x p_k\right)\\ &= \sum_{j,k} |m(j,k)|^2 \tau\left((p_j x p_k)^* p_j x p_k\right).\end{aligned}$$

Since $\tau\left((p_j x p_k)^* p_j x p_k\right) = \tau(|p_j x p_k|^2) = \|p_j x p_k\|_{S_2}^2 \geq 0$, we can estimate each $|m(j,k)|$ by the supremum of m and obtain

$$\sum_{j,k} |m(j,k)|^2 \tau\left((p_j x p_k)^* p_j x p_k\right) \leq \|m\|_\infty \sum_{j,k} \tau\left((p_j x p_k)^* p_j x p_k\right).$$

By reverting our calculations from before, we can rewrite the remaining sum as

$$\sum_{j,k} \tau\left((p_j x p_k)^* p_j x p_k\right) = \tau\left((\sum_{j,k} p_j x p_k)^* (\sum_{l,m} p_l x p_m) \right) = \tau(x^*x) = \tau(|x|^2) = \|x\|_{S_2}^2,$$

which concludes the proof. □

Lemma 2.17 (Boundedness by duality)**.** Let $M_m : S_p \to S_p$ be a bounded Schur multiplier $p \in (1, \infty)$. Then $M_{\overline{m}}$ is a bounded Schur multiplier on S_p' and

$$\|M_{\overline{m}} : S_{p'} \to S_{p'}\| = \|M_m : S_p \to S_p\|.$$

Proof. By Lemma 2.15(4) for $n = 1$, we have

$$\|M_m : S_p \to S_p\| = \|M_{m_*} : S_p' \to S_p'\|,$$

where $m_*(\lambda, \mu) := m_(\mu, \lambda)$. Furthermore, by Lemma 2.15(3),

$$\|M_{m_*} : S_p' \to S_p'\| = \|M_{\overline{m}} : S_p' \to S_p'\|.$$

□

Finally, note that Schur multipliers are trivially bounded on diagonal Schatten space elements.

Lemma 2.18 (Boundedness on the diagonal)**.** Let $m : \mathbb{R}^{n+1} \to \mathbb{C}$ and let $p_1, \ldots, p_n, p \in (1, \infty)$ be such that $1/p_1 + \ldots + 1/p_n = 1/p$. For $j = 1, \ldots, n$, let x_j be a diagonal element of S_{p_j}, i.e. $x_j = \sum_\lambda p_\lambda x_j p_\lambda$. Then

$$\|M_m(x_1, \ldots, x_n)\|_{S_p} \leq \|m\|_\infty \|x_1\|_{S_{p_1}} \cdots \|x_n\|_{S_{p_n}}.$$

Proof. By applying the Hölder-inequality repeatedly, we have

$$\begin{aligned} \|M_m(x_1, \ldots, x_n)\|_{S_p} &= \left\| \sum_\lambda m(\lambda, \ldots, \lambda) p_\lambda x_1 p_\lambda \ldots p_\lambda x_n p_\lambda \right\|_{S_p} \\ &\leq \left\| \sum_\lambda m(\lambda, \ldots, \lambda) p_\lambda \right\|_{S_\infty} \left\| \sum_\lambda p_\lambda x_1 p_\lambda \ldots p_\lambda x_n p_\lambda \right\|_{S_p} \\ &\leq \ldots \\ &\leq \left\| \sum_\lambda m(\lambda, \ldots, \lambda) p_\lambda \right\|_{S_\infty} \|x_1\|_{S_{p_1}} \cdots \|x_n\|_{S_{p_n}}. \end{aligned}$$

Since the projections p_λ are positive operators, we have

$$\sum_\lambda m(\lambda, \ldots, \lambda) p_\lambda \leq \|m\|_\infty \sum_\lambda p_\lambda = \|m\|_\infty \mathrm{id}_H,$$

and hence

$$\left\| \sum_\lambda m(\lambda, \ldots, \lambda) p_\lambda \right\|_{S_\infty} \leq \|m\|_\infty,$$

which concludes the proof. □

2.5. Dyadic constructions

Here, we introduce some constructions from harmonic analysis, on which we will build in subsequent sections. While all concepts in this section are well-defined on $\mathbb{R}^d$, only $d = 1$ will be relevant in this thesis, hence we only give the definitions in this special case to simplify notation. We will refer to the elements of the dyadic grid defined below as *cubes* as is standard in the literature. Note however that one-dimensional cubes are line segments, hence concepts such as volume or side length coincide. Unless noted otherwise, all definitions are from [12, Section 2.2].

Definition 2.19 (Dyadic grid)**.**

1. The *standard dyadic grid* on $\mathbb{R}$ is defined as
$$\mathcal{D}_0 \ := \ \{2^{-k}([0, 1) + m) \mid k, m \in \mathbb{Z}\}.$$
2. Let $\Omega = \{0, 1\}^{\mathbb{Z}}$, and equip Ω with a probability measure such that its coordinates are independent and uniformly distributed on $\{0, 1\}$. Let $\omega = (\omega_k)_{k\in\mathbb{Z}} \in \Omega$. The *random dyadic grid* on $\mathbb{R}$ associated with ω is defined by
$$\begin{aligned} \mathcal{D}_\omega \ &:= \ \{Q + \omega \mid Q \in \mathcal{D}_0\}, \\ Q + \omega \ &:= \ Q + \sum_{\substack{k\in\mathbb{Z} \\ 2^{-k}<|Q|}} 2^{-k}\omega_k, \end{aligned}$$
where $|Q|$ denotes the length of the cube Q.
3. By a *dyadic grid* $\mathcal{D}$ we refer to $\mathcal{D} = \mathcal{D}_\omega$ for some $\omega \in \Omega$.
4. For $Q \in \mathcal{D}$, $\mathcal{D}$ dyadic grid, define $Q^{(k)}$ as the cube $R \in \mathcal{D}$ such that $Q \subset R$ and $2^k|Q| = |R|$.
5. For $Q \in \mathcal{D}$, $\mathcal{D}$ dyadic grid, define
$$\mathrm{ch}_{\mathcal{D}}(Q) := \{Q' \in \mathcal{D} \mid Q' \subsetneq Q \text{ and there exists no } Q'' \in \mathcal{D} \text{ such that } Q' \subsetneq Q'' \subsetneq Q\}.$$
This set is called the set of *children* of Q in $\mathcal{D}$. The index denoting the dyadic grid may be omitted.

Next, we introduce the so-called *Haar functions*. This family of functions is particularly well-behaved on UMD spaces, as we will see in the next section, and plays a key role in the representation theory of Calderón-Zygmund operators (see Section 2.8).

Definition 2.20 (Haar functions on $\mathbb{R}$)**.** Let $\mathcal{D}$ be a dyadic grid on $\mathbb{R}$ and let $Q \in \mathcal{D}$. Let Q_{left} (resp. Q_{right}) denote the left (resp. right) half of Q. For $\eta \in \{0, 1\}$, we define the *Haar function*

$$h_Q^\eta := \begin{cases} |Q|^{-1/2} 1_Q, & \eta = 0, \\ |Q|^{-1/2}(1_{Q_{\text{left}}} - 1_{Q_{\text{right}}}), & \eta = 1. \end{cases}$$

To simplify the notation, we set $h_Q := h_Q^1$. Note that for $\eta = 1$, it holds that $\int_{\mathbb{R}} h_Q(x)dx = 0$, hence we refer to h_Q as a *cancellative Haar function.*

A construction using Haar functions that will be used multiple times in Section 6.2 is the following.

Lemma 2.21. Let $\mathcal{D}$ be a dyadic grid, let $Q \in \mathcal{D}$, and let b be a locally integrable function on $\mathbb{R}$. Define

$$D_Q b := \langle b \rangle_{Q_{\text{left}}} 1_{Q_{\text{left}}} + \langle b \rangle_{Q_{\text{right}}} 1_{Q_{\text{right}}} - \langle b \rangle_Q 1_Q.$$

Then

$$D_Q b = \langle b, h_Q \rangle h_Q.$$

Proof. Using $|Q_{\text{left}}| = |Q_{\text{right}}| = |Q|/2$ we have

$$D_Q(b) = \frac{1}{|Q|}(2\langle b, 1_{Q_{\text{left}}}\rangle 1_{Q_{\text{left}}} + 2\langle b, 1_{Q_{\text{right}}}\rangle 1_{Q_{\text{right}}} - \langle b, 1_Q \rangle 1_Q)$$

Note that $1_{Q_{\text{left}}} = \frac{1}{2}|Q|^{1/2}(h_Q^0 + h_Q)$ and $1_{Q_{\text{right}}} = \frac{1}{2}|Q|^{1/2}(h_Q^0 - h_Q)$. Hence,

$$\begin{aligned} D_Q(b) &= \frac{1}{|Q|^{1/2}}(\langle b, h_Q^0 + h_Q \rangle 1_{Q_{\text{left}}} + \langle b, h_Q^0 - h_Q \rangle 1_{Q_{\text{right}}} - \langle b, h_Q^0 \rangle 1_Q) \\ &= \frac{1}{|Q|^{1/2}}(\langle b, h_Q \rangle 1_{Q_{\text{left}}} - \langle b, h_Q \rangle 1_{Q_{\text{right}}}) \\ &= \langle b, h_Q \rangle h_Q. \end{aligned}$$

□

2.6. UMD spaces

Section 6.2 heavily relies on the theory of *UMD spaces*. We shall merely give a brief overview over the properties of UMD spaces used here, for an extensive introduction to martingales and UMD spaces see [20]. In particular, despite martingales being at the heart of UMD space theory, giving a proper definition of martingales is beyond the scope of this thesis. This is not a limitation however, since we will only apply the statements in this section to one particular type of martingale difference.

Example 2.22 (Dyadic martingale difference [20, Proof of Theorem 4.2.13])**.** For a locally integrable function b on $\mathbb{R}$ and a dyadic grid $\mathcal{D}$ on $\mathbb{R}$, the sequence $(D_Q)_Q$, defined in Lemma 2.21 by

$$D_Q b = \langle b \rangle_{Q_{\text{left}}} 1_{Q_{\text{left}}} + \langle b \rangle_{Q_{\text{right}}} 1_{Q_{\text{right}}} - \langle b \rangle_Q 1_Q = \langle b, h_Q \rangle h_Q,$$

is a martingale difference with respect to a σ-finite filtration generated by $\mathcal{D}$.

Definition 2.23 (UMD Space, [20, Definition 4.2.1])**.** A Banach space X is called a *UMD space* if for all $p \in (1, \infty)$ there exists a constant $\beta_{p,X} \geq 0$ such that for all X-valued L^p-martingale differences $(df_n)_{n=0}^N$ with respect to a σ-finite filtration $(\mathcal{F}_n)_{n=0}^N$ on a σ-finite measure space (Ω, μ), and for all scalars $(\varepsilon_n)_{n=0}^N$ with $|\varepsilon_n| = 1$, $0 \leq n \leq N$, we have

$$\| \sum_{n=1}^N \varepsilon_n df_n \|_{L^p(\Omega, X)} \leq \beta_{p,X} \| \sum_{n=1}^N df_n \|_{L^p(\Omega, X)}.$$

The constant $\beta_{p,X}$ is called the *UMD constant* of X.

Next, we introduce a special type of random variable that will play a role in characterising UMD spaces.

Definition 2.24 (Rademacher variable [20, Definition 3.2.9]). A *Rademacher variable* is a random variable $\varepsilon : \Omega \to \mathbb{K}$ on some probability space Ω that is uniformly distributed over $\{z \in \mathbb{K} \mid |z| = 1\}$. Here, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. A *Rademacher sequence* is a sequence $(\varepsilon_i)_{i \in I}$ of independent Rademacher variables.

The following lemma collects properties of Banach spaces that are equivalent to the UMD property.

Lemma 2.25 (Equivalent properties of UMD spaces). Let X be a Banach space and $p \in (1, \infty)$. Then the following are equivalent:

- X is a UMD space.
- (Randomised UMD property, [20, Proposition 4.2.3]) There exist constants $\beta^{\pm}_{p,X} \in (0, \infty)$ such that for all L^p-martingale differences $(df_n)_{n=0}^N$ on a σ-finite measure space (Ω, μ) and all Rademacher sequences $(\varepsilon_n)_{n=0}^N$ we have

$$\frac{1}{\beta^-_{p,X}} \| \sum_{n=1}^N df_n \|_{L^p(\Omega,X)} \leq (\mathbb{E} \| \sum_{n=1}^N \varepsilon_n df_n \|^p_{L^p(\Omega,X)})^{1/p} \leq \beta^+_{p,X} \| \sum_{n=1}^N df_n \|_{L^p(\Omega,X)}.$$

  It holds that $\max(\beta^-_{p,X}, \beta^+_{p,X}) \leq \beta_{p,X} \leq \beta^-_{p,X} \beta^+_{p,X}$.
- (Hilbert transform, [20, Corollary 5.7.7]) The Hilbert transform, given by the Fourier multiplier $H := T_m$ with symbol $m(\xi) = -i\,\mathrm{sgn}(\xi)$, is a bounded linear operator on either $L^p(\mathbb{T}, X)$ or $L^p(\mathbb{R}, X)$. In this case, the norms satisfy $\|H\|_{L^p(\mathbb{T},X)} = \|H\|_{L^p(\mathbb{R},X)}$, hence boundedness of the Hilbert transform on one of these spaces is equivalent to boundedness on the other. This norm is denoted by $\hbar_{p,X}$.
- (Haar decomposition, [20, Theorem 4.2.13]) Let $\mathcal{D}$ be a dyadic grid and let $(\varepsilon_Q)_{Q \in \mathcal{D}}$ be a Rademacher sequence. Then

$$\frac{1}{\beta^-_{p,X}} \|f\|_{L^p(\mathbb{R},X)} \leq (\mathbb{E} \| \sum_{Q \in \mathcal{D}} \varepsilon_Q \langle f, h_Q \rangle h_Q \|^p_{L^p(\mathbb{R},X)})^{1/p} \leq \beta^+_{p,X} \|f\|_{L^p(\mathbb{R},X)}.$$

The only UMD spaces relevant in this thesis are the Schatten spaces. The following estimates for the constants introduced in Lemma 2.25 are known.

Lemma 2.26. The Schatten space S_p is a UMD space for $p \in (1, \infty)$. Its constants can be estimated as follows:

- ([3, Corollary 17]) $\hbar_{p,S_p} \lesssim \frac{p^2}{p-1}$,
- ([37, Theorem 4.3]) $\beta_{p,S_p} \lesssim \frac{p^2}{p-1}$. By the inequalities in Lemma 2.25, this implies $\beta^{\pm}_{p,S_p} \lesssim \frac{p^2}{p-1}$.

2.7. Fourier multipliers

Having defined UMD spaces, we can now give a brief introduction to Fourier multipliers on vector-valued L^p-spaces. We will restrict ourselves to scalar-valued symbols; for further reading, see e.g. [15, 20].

For a Banach space X, define

$$\check{L}^1(\mathbb{R}^d, X) := \{g \in L^\infty(\mathbb{R}^d, X) \mid g = \check{f} \text{ for some } f \in L^1(\mathbb{R}^d, X)\}.$$

Following [20, Lemma 2.4.7], $L^p(\mathbb{R}^d, X) \cap \check{L}^1(\mathbb{R}^d, X)$ is a dense subspace of $L^p(\mathbb{R}^d, X)$, which will be used in the following definition.

Definition 2.27 (Fourier multiplier on $\mathbb{R}^d$, [20, Definition 5.3.3]). Let X be a Banach space and let $m \in L^\infty(\mathbb{R}^d, \mathbb{C})$. If the associated operator T_m, defined on $L^p(\mathbb{R}^d, X) \cap \check{L}^1(\mathbb{R}^d, X)$ for $p \in [1, \infty)$ as

$$T_m f(x) := \frac{1}{\sqrt{2\pi}^d} \int_{\mathbb{R}^d} m(\xi) \hat{f}(\xi) e^{i\xi \cdot x} d\xi,$$

extends to a bounded operator $T_m : L^p(\mathbb{R}^d, X) \to L^p(\mathbb{R}^d, X)$, we call T_m an *L^p-Fourier multiplier* with *symbol m*.

Note our terminology is somewhat non-standard in order to match the terminology of Schur multipliers – in the literature, one often refers to m as the Fourier multiplier and to T_m as the Fourier multiplier operator.

We will also consider Fourier multipliers on the torus, for which the definition of the multiplier simplifies as follows.

Definition 2.28 (Fourier multiplier on $\mathbb{T}^d$, [20, Section 5.7])**.** Let X be a Banach space and let $(m_k)_{k\in\mathbb{Z}^d} \in \ell^\infty(\mathbb{Z}^d,\mathbb{C})$. If the associated operator, defined on trigonometric polynomials

$$f : \mathbb{Z}^d \to X, \quad f(\mu) = \frac{1}{\sqrt{2\pi}^d} \sum_{\lambda\in\mathbb{Z}^d} \hat{f}(\lambda) e^{i\lambda\cdot\mu}$$

as

$$T_{(m_k)_k} f(\mu) := \frac{1}{\sqrt{2\pi}^d} \sum_{\lambda\in\mathbb{Z}^d} m(\lambda)\hat{f}(\lambda) e^{i\lambda\cdot\mu}$$

extends to a bounded operator $T_{(m_k)_k} : L^p(\mathbb{T}^d, X) \to L^p(\mathbb{T}^d, X)$ for $p \in [1,\infty)$, we call $T_{(m_k)_k}$ an *L^p-Fourier multiplier* with *symbol* $(m_k)_k$.

Next, we give two sufficient conditions for the boundedness of such Fourier multipliers on UMD-valued spaces, which will be used in Section 6.1.

Theorem 2.29 (Marcinkiewicz multiplier theorem, [3, Theorem 4])**.** *Let X be a complex Banach space and let $m \in L^\infty(\mathbb{T},\mathbb{C})$ be such that*

$$\sup_{k\in\mathbb{Z}} \sum_{2^{k-2}\leq j\leq 2^{k-1}} |m(j+1) - m(j)|$$

In this case,

$$\|T_m : L^p(\mathbb{T}, X) \to L^p(\mathbb{T}, X)\| \lesssim \hbar_{p,X}(\beta_{p,X})^2 \left(\sum_{|n|=2^k}^{2^{k+1}} |m(n+1) - m(n)| \right),$$

where $\hbar_{p,X}$ and $\beta_{p,X}$ are from Lemma 2.25.

Theorem 2.30 (Mihlin multiplier theorem on $\mathbb{R}$, [20, Theorem 5.3.18])**.** *Let X be a complex UMD spaces and let $p \in (1,\infty)$. Let $m \in L^\infty(\mathbb{R},\mathbb{C})$. Then T_m is an L^p-Fourier multiplier if*

$$\|m\|_{\mathcal{M}} := \sup_{\xi\in\mathbb{R}\setminus\{0,\pm 2^k\}_{k\in\mathbb{Z}}} |m(\xi)| + \sup_{\xi\in\mathbb{R}\setminus\{0,\pm 2^k\}_{k\in\mathbb{Z}}} |\xi m'(\xi)| < \infty.$$

In this case,

$$\|T_m : L^p(\mathbb{R}, X) \to L^p(\mathbb{R}, X)\| \lesssim \hbar_{p,X}(\beta_{p,X})^2 \|m\|_{\mathcal{M}},$$

where $\hbar_{p,X}$ and $\beta_{p,X}$ are from Lemma 2.25.

It is possible to show that the boundedness of a Fourier multiplier m on $\mathbb{R}^d$ implies that its restriction to $(m(k))_{k\in\mathbb{Z}^d}$ is a bounded Fourier multiplier on $\mathbb{T}^d$. Such theorems are known as *transference* or *de Leeuw restriction theorems*, and we present a linear theorem here.

Theorem 2.31 (Analysis in Banach spaces I, Prop 5.7.1)**.** *Let X be a Banach space, let $m \in L^\infty(\mathbb{R}^d,\mathbb{C})$ be the symbol of a Fourier multiplier on $L^p(\mathbb{R}^d, X)$, $p \in (1,\infty)$. Suppose that all $k \in \mathbb{Z}^d$ are Lebesgue points of m. Then $T_{(m(k))_{k\in\mathbb{Z}^d}}$ is a Fourier multiplier on $L^p(\mathbb{T}^d, X)$ with $\|T_{(m(k))_k}\| \leq \|T_m\|$.*

In Section 5.1, we will return to the bilinear case of this theorem. For this purpose, let us briefly define multilinear Fourier multipliers. In order to avoid considerations about vector space compatibility, we give the definitions immediately on L^p-spaces with values in Schatten spaces.

Definition 2.32 (Multilinear Fourier multipliers, [16, Section 5])**.**

1. Let $m \in L^\infty((\mathbb{R}^d)^n, \mathbb{C})$ and let $p_1, \ldots, p_n, p \in (1, \infty)$ such that $1/p_1 + \ldots + 1/p_n = 1/p$. The Fourier multiplier $T_m : L^{p_1}(\mathbb{R}^d, S_{p_1}) \times \ldots \times L^{p_n}(\mathbb{R}^d, S_{p_n}) \to L^p(\mathbb{R}^d, S_p)$ is defined as
$$T_m(f_1, \ldots, f_n)(x) := \frac{1}{\sqrt{2\pi}^{dn}} \int_{\mathbb{R}^{dn}} m(\xi_1, \ldots, \xi_n) \hat{f}_1(\xi_1) \ldots \hat{f}_n(\xi_n) e^{i(\xi_1 + \ldots + \xi_n) \cdot x} d\xi.$$
2. Let $(m_k)_k \in \ell^\infty((\mathbb{Z}^d)^n, \mathbb{C})$ and let $p_1, \ldots, p_n, p \in (1, \infty)$ such that $1/p_1 + \ldots + 1/p_n = 1/p$. The Fourier multiplier $T_m : L^{p_1}(\mathbb{T}^d, S_{p_1}) \times \ldots \times L^{p_n}(\mathbb{T}^d, S_{p_n}) \to L^p(\mathbb{T}^d, S_p)$ is defined on trigonomic polynomials $f_1, \ldots, f_n$ as
$$T_{(m_k)_k}(f_1, \ldots, f_n)(\mu) := \frac{1}{\sqrt{2\pi}^{dn}} \sum_{(\lambda_1, \ldots, \lambda_n) \in (\mathbb{Z}^d)^n} m(\lambda_1, \ldots, \lambda_n) \hat{f}_1(\lambda_1) \ldots \hat{f}_n(\lambda_n) e^{i(\lambda_1 + \ldots + \lambda_n) \cdot \mu}.$$

2.8. Calderón-Zygmund operators and dyadic model operators

In this section, we introduce Calderón-Zygmund operators and their decomposition into so-called *dyadic model operators*, namely dyadic shifts and paraproducts. All definitions are taken from [12, Section 2.4].

Definition 2.33 (Calderón-Zygmund operator)**.** Let T be an n-linear operator defined by an integral kernel on a suitable function space, i.e. for $\Delta := \{x \in \mathbb{R}^{n+1} \mid x_1 = \ldots = x_{n+1}\}$ there exists a function $K : \mathbb{R}^{n+1} \setminus \Delta \to \mathbb{C}$ such that
$$\langle T(f_1, \ldots, f_n), f_{n+1} \rangle = \int_{\mathbb{R}^{n+1}} K(x_{n+1}, x_1, \ldots, x_n) \prod_{j=1}^{n+1} f_j(x_j) dx$$
whenever $\mathrm{supp} f_i \cap \mathrm{supp} f_j = \emptyset$ for some $i \neq j$. Such an operator T is called a *Calderón-Zygmund operator* if there exists some $\alpha \in (0, 1]$ and $C_K > 0$ such that the following conditions hold:

- (Size condition) for all $x \in \mathbb{R}^{n+1} \setminus \Delta$,
$$|K(x)| \leq \frac{C_K}{(\sum_{m=2}^{n+1} |x_1 - x_m|)^n},$$
- (Smoothness condition) for all $1 \leq j \leq n+1$,
$$|K(x) - K(x')| \leq \frac{C_K |x_j - x_j'|^\alpha}{(\sum_{m=2}^{n+1} |x_1 - x_m|)^{n+\alpha}}$$
holds whenever $x \in \mathbb{R}^{n+1} \setminus \Delta$ and $x' = (x_1, \ldots, x_{j-1}, x_j', x_{j+1}, \ldots, x_{n+1}) \in \mathbb{R}^{n+1}$ are such that $2|x_j - x_j'| \leq \max_{2 \leq m \leq n+1} |x_1 - x_m|$,
- (Boundedness) for some (equivalently, for all) exponents $p_1, \ldots, p_n \in (1, \infty)$ and $q_{n+1} \in (1/n, \infty)$ such that $1/p_1 + \ldots + 1/p_n = 1/q_{n+1}$,
$$\|T(f_1, \ldots, f_n)\|_{L^{q_{n+1}}(\mathbb{R})} \lesssim \prod_{m=1}^n \|f_m\|_{L^{p_m}(\mathbb{R})}.$$

The definitions of dyadic shifts and paraproducts rely on the dyadic constructions from Section 2.5. We will again give the definitions only for $d = 1$, however note that these notions are well-defined on $\mathbb{R}^d$ for $d \in \mathbb{N}$.

Definition 2.34 (n-linear dyadic shift)**.** Let X be a Banach space, let $k = (k_1, \ldots, k_{n+1}) \in \mathbb{N}_0^{n+1}$, and let $\mathcal{D}$ be a dyadic grid on $\mathbb{R}$. The *n-linear dyadic shift* S^k of complexity k is defined for $f_1, \ldots, f_n \in L_c^\infty(\mathbb{R}, X)$ as
$$S^k(f_1, \ldots, f_n) := \sum_{Q \in \mathcal{D}} A_Q^k(f_1, \ldots, f_n),$$
$$A_Q^k(f_1, \ldots, f_n) := \sum_{\substack{I_1, \ldots, I_{n+1} \subseteq Q \\ |I_j| = 2^{-k_j}|Q|}} \alpha_{I_1, \ldots, I_{n+1}, Q} \prod_{j=1}^n \langle f_j, \tilde{h}_{I_j} \rangle \tilde{h}_{I_{n+1}},$$

where there exist two indices $j_0, j_1 \in \{1, \dots, n+1\}$ such that $\tilde{h}_{I_{j_1}} = h_{I_{j_1}}$, $\tilde{h}_{I_{j_2}} = h_{I_{j_2}}$, and $\tilde{h}_{I_j} = h^0_{I_j}$ for all $j \neq j_1, j_2$. Furthermore,

$$|\alpha_{I_1,\dots,I_{n+1},Q}| \leq \frac{1}{|Q|^2} \prod_{j=1}^{n+1} |I_j|^{1/2}.$$

Definition 2.35 (n-linear paraproduct)**.** Let $\mathcal{D}$ be a dyadic grid and let $(a_Q)_{Q\in\mathcal{D}}$ be scalars such that

$$\sup_{Q_0\in\mathcal{D}} \left(\frac{1}{|Q_0|} \sum_{\substack{Q\in\mathcal{D}\\ Q\subset Q_0}} |a_Q|^2 \right)^{1/2} \leq 1.$$

The *n-linear paraproduct* is given by

$$\pi(f_1, \dots, f_n) \ := \ \sum_{Q\in\mathcal{D}} a_Q \prod_{j=1}^{n} \langle f_j, \tilde{h}_{j,Q}\rangle \tilde{h}_{n+1,Q},$$

where $(\tilde{h}_{1,Q}, \dots, \tilde{h}_{n+1,Q})$ are such that there is one $j_0 \in \{1, \dots, n+1\}$ such that for all $Q \in \mathcal{D}$, $\tilde{h}_{j_0,Q} = h_Q$ and $\tilde{h}_{j,Q} = 1_Q/|Q|$ for all $j \neq j_0$. In particular, $\langle f_j, \tilde{h}_{j,Q}\rangle = \langle f_j\rangle_Q$ for $j \neq j_0$.

We now present the representation theorem for Calderón-Zygmund operators as stated in [12].

Theorem 2.36 (Representation theorem)**.** *Let T be an n-linear Calderón-Zygmund operator. Then T can be decomposed as*

$$\langle T(f_1, \dots, f_n), f_{n+1}\rangle = C_T \mathbb{E}_\omega \sum_{k\in\mathbb{N}_0^{n+1}} \sum_u 2^{-\max_i k_i \alpha/2} \langle U^k_{\mathcal{D}_\omega,u}(f_1, \dots, f_n), f_{n+1}\rangle, \tag{2.4}$$

where C_T is a constant depending only on T, the sum over u is finite, α is the Hölder-continuity parameter from Definition 2.33, and $\mathcal{D}_\omega$ is a random dyadic grid (see Definition 2.19). Furthermore, for $\max_j k_j > 0$*, $U^k_{\mathcal{D}_\omega,u}$ denotes an n-linear dyadic shift of complexity k, whereas for* $\max_j k_j = 0$*, $U^k_{\mathcal{D}_\omega,u}$ denotes either an n-linear dyadic shift of complexity* 0 *or an n-linear paraproduct.*

Remark 2.37. While we do not want to give the full construction of the representation theorem here, it is important to note that the paraproducts are constructed such that their scalar sequence $(a_Q)_Q$ is (up to considering partial adjoints of T) of the form

$$a_Q = \langle T(1, \dots, 1), h_Q\rangle \tilde{a}_Q,$$

where $\tilde{a}_Q$ again denotes a scalar sequence, and $T(1, \dots, 1)$ is to be understood as a suitable approximation of the application of T to functions with constant value 1, see [11, Section 6.4] or [27, Section 4.2]. This in particular implies that the paraproducts in Theorem 2.36 vanish if $\langle T(1, \dots, 1), h_Q\rangle = 0$ for all considered dyadic cubes Q, see e.g. [22, Corollary 12.4.13] for a proof in the linear case. We will return to this point in Section 6.2.

2.9. BMO functions and the John-Nirenberg inequality

In Section 6.2, an alternative definition of linear paraproducts will be introduced. For this purpose, we briefly introduce BMO-spaces; see e.g. [14] for a more in-depth introduction.

Definition 2.38 (BMO-space [14, Definition 7.1.1])**.** Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$. Define

$$\|f\|_{\mathrm{BMO}} \ := \ \sup_{\substack{Q\subset\mathbb{R}^d\\ Q \text{ cube}}} \frac{1}{|Q|} \int_Q |f(x) - \langle f\rangle_Q| dx.$$

This defines a seminorm, and we define the BMO-*space* as

$$\mathrm{BMO}(\mathbb{R}^d) \ := \ \{f \in L^1_{\mathrm{loc}}(\mathbb{R}^d) \mid \|f\|_{\mathrm{BMO}} < \infty\}.$$

The John-Nirenberg inequality provides an L^p-characterisation of the BMO-space.

Theorem 2.39 (John-Nirenberg inequality [14, Corollary 7.1.8–7.1.9])**.** *Let $p \in (0,\infty)$. Define*

$$\|f\|_{\mathrm{BMO}_p} := \sup_{\substack{Q \subset \mathbb{R}^d \\ Q \text{ cube}}} \left(\frac{1}{|Q|} \int_Q |f(x) - \langle f \rangle_Q|^p dx \right)^{1/p}.$$

Then

$$\|f\|_{\mathrm{BMO}_p} \leq C_{\mathrm{BMO}_{p,d}} \|f\|_{\mathrm{BMO}},$$

where $C_{\mathrm{BMO}_{p,d}} := 2^d e (ep\Gamma(p))^{1/p}$ and Γ denotes the Gamma function. Furthermore, for $p \in (1,\infty)$ we have

$$\|f\|_{\mathrm{BMO}} \leq \|f\|_{\mathrm{BMO}_p}.$$

3

Decomposition of $M_{f^{[2]}}$

As discussed in Section 1, multilinear non-Toeplitz form Schur multipliers are generally not well understood. It is therefore necessary to decompose a general multilinear Schur multiplier into better understood multipliers, in particular linear Schur multipliers and Toeplitz form Schur multipliers. In this section, we demonstrate such a decomposition for the bilinear Schur multipliers $M_{f^{[2]}}$. Throughout this section, we will assume that $f \in C^2(\mathbb{R})$ with $\|f''\|_\infty < \infty$ unless stated otherwise.

In Section 3.1, we present a key lemma for decomposing divided differences of any order in a suitable manner. Using this lemma, we can already achieve our desired decomposition for $M_{f^{[2]}}$ on a particular domain, namely on lower triagonal operators. In order to find a suitable decomposition on the full domain, we express the triangular truncations introduced in Section 2.3 as Schur multipliers, which is discussed in Section 3.2. Finally, we strategically apply triangular truncations and present our decomposition of $M_{f^{[2]}}$ on the full domain in Section **??**.

3.1. Decomposition on lower triangular operators

In the following lemma, we demonstrate a method for decomposing a divided difference into a sum of two terms. Both terms of this sum are of the same form, namely they are products of a fraction in Toeplitz form and a divided difference of the same order as the original one.

Lemma 3.1. Let $f \in C^n(\mathbb{R})$, $n \geq 1$, and let $\lambda_0, \dots, \lambda_n \in \mathbb{R}$. Let $i, j \in \{0, \dots, n\}$ be such that $\lambda_i \neq \lambda_j$. Let $\mu \in \mathbb{R}$. Then

$$f^{[n]}(\lambda_0, \dots, \lambda_n) = \frac{\lambda_i - \mu}{\lambda_i - \lambda_j} f^{[n]}(\lambda_0, \dots, \lambda_{j-1}, \mu, \lambda_{j+1}, \dots, \lambda_n) + \frac{\mu - \lambda_j}{\lambda_i - \lambda_j} f^{[n]}(\lambda_0, \dots, \lambda_{i-1}, \mu, \lambda_{i+1}, \dots, \lambda_n).$$

Proof. Since $f^{[n]}$ is invariant under permutation of its variables (see Lemma 2.3), we assume without loss of generality $(i, j) = (0, 1)$. It follows for $\mu \neq \lambda_i$, $i = 0, 1$, that

$$\begin{aligned} f^{[n]}(\lambda_0, \lambda_1, \lambda_2, \dots, \lambda_n) &= \frac{1}{\lambda_0 - \lambda_1} \Big(f^{[n-1]}(\lambda_0, \lambda_2, \lambda_3, \dots, \lambda_n) - f^{[n-1]}(\lambda_1, \lambda_2, \dots, \lambda_n) \Big) \\ &= \frac{1}{\lambda_0 - \lambda_1} \Big(f^{[n-1]}(\lambda_0, \lambda_2, \lambda_3, \dots, \lambda_n) - f^{[n-1]}(\mu, \lambda_2, \lambda_3, \dots, \lambda_n) \Big) \\ &\quad + \frac{1}{\lambda_0 - \lambda_1} \Big(f^{[n-1]}(\mu, \lambda_2, \lambda_3, \dots, \lambda_n) - f^{[n-1]}(\lambda_1, \lambda_2, \dots, \lambda_n) \Big) \\ &= \frac{\lambda_0 - \mu}{\lambda_0 - \lambda_1} f^{[n]}(\lambda_0, \mu, \lambda_2, \lambda_3, \dots, \lambda_n) + \frac{\mu - \lambda_1}{\lambda_0 - \lambda_1} f^{[n]}(\mu, \lambda_1, \lambda_2, \dots, \lambda_n). \end{aligned}$$

Note the same formula holds for $\lambda_0 = \mu$ or $\lambda_1 = \mu$ as long as $\lambda_0 \neq \lambda_1$: Assume (without loss of generality) $\lambda_0 = \mu \neq \lambda_1$, then

$$\underbrace{\frac{\lambda_0 - \mu}{\lambda_0 - \lambda_1}}_{=0} f^{[n]}(\lambda_0, \mu, \lambda_2, \lambda_3, \ldots, \lambda_n) + \underbrace{\frac{\mu - \lambda_1}{\lambda_0 - \lambda_1}}_{=1} f^{[n]}(\mu, \lambda_1, \lambda_2, \ldots, \lambda_n) = f^{[n]}(\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_n).$$

□

Using this lemma, we can already achieve our desired decomposition of $M_{f^{[2]}}$ into Toeplitz form Schur multipliers and linear Schur multipliers when the domain of $M_{f^{[2]}}$ is restricted to lower triangular operators.

Example 3.2. Let x, y be lower triangular Schatten space elements, i.e. we have $x = \sum_{i \geq j} p_i x p_j$ and $y = \sum_{i \geq j} p_i y p_j$. Our Schur multiplier $M_{f^{[2]}}$ reduces to

$$M_{f^{[2]}}(x, y) = \sum_{\lambda_0 \geq \lambda_1 \geq \lambda_2} f^{[2]}(\lambda_0, \lambda_1, \lambda_2) p_{\lambda_0} x p_{\lambda_1} y p_{\lambda_2}.$$

On the diagonal, i.e. for $\lambda_0 = \lambda_1 = \lambda_2$, the Schur multiplier is bounded by Lemma 2.18. We discuss the off-diagonal part separately by setting

$$M_{f^{[2]}}(x, y) = M_{f^{[2]},\mathrm{diag}}(x, y) + \sum_{\substack{\lambda_0 \geq \lambda_1 \geq \lambda_2 \\ \lambda_0 > \lambda_2}} f^{[2]}(\lambda_0, \lambda_1, \lambda_2) p_{\lambda_0} x p_{\lambda_1} y p_{\lambda_2}.$$

For $\lambda_0 \neq \lambda_2$, we can then use Lemma 3.1 to obtain

$$f^{[2]}(\lambda_0, \lambda_1, \lambda_2) = \frac{\lambda_0 - \lambda_1}{\lambda_0 - \lambda_2} f^{[2]}(\lambda_0, \lambda_1, \lambda_1) + \frac{\lambda_1 - \lambda_2}{\lambda_0 - \lambda_2} f^{[2]}(\lambda_1, \lambda_1, \lambda_2), \tag{3.1}$$

where the fractions are bounded due to the restriction $\lambda_0 \geq \lambda_1 \geq \lambda_2$, $\lambda_0 \neq \lambda_2$. Define the following functions.

$$\psi : \{(\lambda_0, \lambda_1, \lambda_2) \in \mathbb{R}^3 \mid \lambda_0 \geq \lambda_1 \geq \lambda_2\} \to \mathbb{C},\ \psi(\lambda_0, \lambda_1, \lambda_2) := \begin{cases} \frac{\lambda_0 - \lambda_1}{\lambda_0 - \lambda_2}, & \lambda_0 \neq \lambda_2 \\ 1, & \lambda_0 = \lambda_1 = \lambda_2. \end{cases}$$

$$\phi : \mathbb{R}^2 \to \mathbb{C},\ \phi(\lambda, \mu) := f^{[2]}(\lambda, \mu, \mu),$$
$$\tilde{\phi} : \mathbb{R}^2 \to \mathbb{C},\ \tilde{\phi}(\lambda, \mu) := \phi(\mu, \lambda) = f^{[2]}(\lambda, \lambda, \mu).$$

We use this and Lemma 2.15 to decompose $M_{f^{[2]}}$ into

$$\begin{aligned}
&M_{f^{[2]}}(x, y) \\
=&M_{f^{[2]},\mathrm{diag}}(x, y) + \sum_{\substack{\lambda_0 \geq \lambda_1 \geq \lambda_2 \\ \lambda_0 > \lambda_2}} \frac{\lambda_0 - \lambda_1}{\lambda_0 - \lambda_2} f^{[2]}(\lambda_0, \lambda_1, \lambda_1) p_{\lambda_0} x p_{\lambda_1} y p_{\lambda_2} \\
&+ \sum_{\substack{\lambda_0 \geq \lambda_1 \geq \lambda_2 \\ \lambda_0 > \lambda_2}} \frac{\lambda_1 - \lambda_2}{\lambda_0 - \lambda_2} f^{[2]}(\lambda_1, \lambda_1, \lambda_2) p_{\lambda_0} x p_{\lambda_1} y p_{\lambda_2} \\
=&M_{f^{[2]},\mathrm{diag}}(x, y) + \sum_{\substack{\lambda_0 \geq \lambda_1 \geq \lambda_2 \\ \lambda_0 > \lambda_2}} \frac{\lambda_0 - \lambda_1}{\lambda_0 - \lambda_2} p_{\lambda_0} (M_\phi x) p_{\lambda_1} y p_{\lambda_2} + \sum_{\substack{\lambda_0 \geq \lambda_1 \geq \lambda_2 \\ \lambda_0 > \lambda_2}} \frac{\lambda_1 - \lambda_2}{\lambda_0 - \lambda_2} p_{\lambda_0} x p_{\lambda_1} (M_{\tilde{\phi}} y) p_{\lambda_2} \\
=&M_{f^{[2]},\mathrm{diag}}(x, y) + M_\psi(M_\phi x, y) + M_{1-\psi}(x, M_{\tilde{\phi}} y).
\end{aligned}$$

3.2. Triangular truncations as Schur multipliers

Define the sets $U := \{(\lambda_1, \lambda_2) \in \mathbb{R}^2 \mid \lambda_1 \geq \lambda_2\}$, $L := \{(\lambda_1, \lambda_2) \in \mathbb{R}^2 \mid \lambda_1 < \lambda_2\}$. The triangular truncations $T_{\triangle^{\mathrm{off}}_{\mathrm{upper}}}$, $T_{\triangle_{\mathrm{lower}}}$ are then precisely the Schur multipliers associated with the indicator functions M_{1_U}, M_{1_L}, hence in particular $M_{1_U} + M_{1_L}$ is the identity on S_p, $1 < p < \infty$. This allows us to decompose a bilinear Schur multiplier M_m as

$$M_m(x, y) = \sum_{j=1,2,3} \sum_{D_j = U,L} M_{1_{D_1}}(M_m(M_{1_{D_2}} x, M_{1_{D_3}} y)). \tag{3.2}$$

	$D_1 = L$			$D_1 = U$	
	$D_2 = L$	$D_2 = U$		$D_2 = L$	$D_2 = U$
$D_3 = L$	$\{\lambda_0 \geq \lambda_1 \geq \lambda_2\}$	$\{\lambda_1 > \lambda_0 \geq \lambda_2\}$	$D_3 = L$	$\emptyset$	$\{\lambda_1 \geq \lambda_2 > \lambda_0\}$
$D_3 = U$	$\{\lambda_0 \geq \lambda_2 > \lambda_1\}$	$\emptyset$	$D_3 = U$	$\{\lambda_2 > \lambda_0 \geq \lambda_1\}$	$\{\lambda_2 > \lambda_1 > \lambda_0\}$

Table 3.1: All domains $D := D_{D_1 D_2 D_3}$ defined in (3.3), arising from the application of triangular truncations as in (3.2).

By Lemma 2.15, composing M_m with triangular truncations in this manner corresponds to constructing the bilinear Schur multiplier with symbol $1_D m$, where

$$D := D_{D_1 D_2 D_3} := \{(\lambda_0, \lambda_1, \lambda_2) \in \mathbb{R}^3 \mid (\lambda_0, \lambda_2) \in D_1,\ (\lambda_0, \lambda_1) \in D_2,\ (\lambda_1, \lambda_2) \in D_3\} \tag{3.3}$$

with $D_j \in \{U, L\}$, $j = 1, 2, 3$, i.e. to setting $m = 0$ outside of the specified domain D. These domains correspond to fixing the order of the values of λ_j, $j = 0, 1, 2$, as summarised in Table 3.1. The cases $(D_1, D_2, D_3) \in \{(L, U, U), (U, L, L)\}$ correspond to incompatible conditions on the order of the λ_i, thus the associated Schur multiplier sends all matrix elements to zero. For all other cases, we can find a permutation $\rho_D : \{0, 1, 2\} \to \{0, 1, 2\}$ such that for $(\lambda_0, \lambda_1, \lambda_2) \in D$ we have $\lambda_{\rho_D(0)} \geq \lambda_{\rho_D(1)} \geq \lambda_{\rho_D(2)}$.

4

Boundedness of linear Schur multipliers M_ϕ

In this section, we show the boundedness of the linear Schur multipliers M_ϕ defined in Section 3. Note that while the majority of this thesis is concerned with second order divided differences, we will prove the results in this section for general n-th order divided differences; we return to this point in Section 7.

We want to apply Theorem 1.5 to multipliers with symbol $\phi(\lambda,\mu) = f^{[n]}(\lambda^{(k)},\mu^{(n+1-k)})$ for some $1 \le k \le n$. Here, we use the notation introduced in Section 2.2. We need the following two lemmas.

Lemma 4.1. Let $n \geq 1$, $0 \leq k \leq n+1$, and let $f \in C^{n+1}(\mathbb{R})$. Then the partial derivatives of $(\lambda,\mu) \mapsto f^{[n]}(\lambda^{(k)},\mu^{(n+1-k)})$ are given by

$$\begin{aligned}\partial_\lambda f^{[n]}(\lambda^{(k)},\mu^{(n+1-k)}) &= k f^{[n+1]}(\lambda^{(k+1)},\mu^{(n+1-k)}),\\ \partial_\mu f^{[n]}(\lambda^{(k)},\mu^{(n+1-k)}) &= (n+1-k) f^{[n+1]}(\lambda^{(k)},\mu^{(n+2-k)}).\end{aligned}$$

Furthermore, $\big((\lambda,\mu) \mapsto f^{[n]}(\lambda^{(k)},\mu^{(n+1-k)})\big) \in C^1(\mathbb{R}^2 \setminus \{\lambda = \mu\})$.

Proof. Since $f^{[n]}$ is invariant under permutation of its variables (see Lemma 2.3), it is sufficient to calculate the partial derivatives in λ. For $n = 1$, there are three cases to consider:

- $k = 0$: $\partial_\lambda f^{[1]}(\mu,\mu) = 0$.
- $k = 2$: $\partial_\lambda f^{[1]}(\lambda,\lambda) = \partial_\lambda f'(\lambda) = f''(\lambda) = 2 f^{[2]}(\lambda,\lambda,\lambda)$, where we used (2.3).
- $k = 1$: We use the product rule to show

$$\partial_\lambda f^{[1]}(\lambda,\mu) = \partial_\lambda \frac{f(\lambda)-f(\mu)}{\lambda-\mu} = \frac{f'(\lambda)}{\lambda-\mu} - \frac{f(\lambda)-f(\mu)}{(\lambda-\mu)^2} = \frac{f^{[1]}(\lambda,\lambda)-f^{[1]}(\lambda,\mu)}{\lambda-\mu} = f^{[2]}(\lambda,\lambda,\mu).$$

By definition, continuity of $f^{[1]}$ follows from continuity of f. Furthermore, its derivatives are continuous in $\lambda \neq \mu$ by continuity of f'' and $f^{[1]}$.

Now let $n \in \mathbb{N}$. For $k = 0$, the statement is immediate. For $0 < k \leq n+1$, we use the product rule and induction to show

$$\begin{aligned}&\partial_\lambda f^{[n]}(\lambda^{(k)},\mu^{(n+1-k)})\\ =&\frac{\partial_\lambda (f^{[n-1]}(\lambda^{(k)},\mu^{(n-k)}) - f^{[n-1]}(\lambda^{(k-1)},\mu^{(n+1-k)}))}{\lambda-\mu} - \frac{f^{[n-1]}(\lambda^{(k)},\mu^{(n-k)}) - f^{[n-1]}(\lambda^{(k-1)},\mu^{(n+1-k)})}{(\lambda-\mu)^2}\\ =&\frac{k f^{[n]}(\lambda^{(k+1)},\mu^{(n-k)}) - (k-1) f^{[n]}(\lambda^{(k)},\mu^{(n+1-k)}) - f^{[n]}(\lambda^{(k)},\mu^{(n+1-k)})}{\lambda-\mu}\\ =&k f^{[n+1]}(\lambda^{(k+1)},\mu^{(n+1-k)}).\end{aligned}$$

Continuity of $(\lambda,\mu) \mapsto f^{[n]}(\lambda^{(k)},\mu^{(n+1-k)})$ in $\lambda \neq \mu$ follows by induction from continuity of the corresponding $f^{[n-1]}$-terms. As in the base case, continuity of its first derivatives in $\lambda \neq \mu$ follows from continuity of $f^{(n+1)}$ and $f^{[n]}$. □

Lemma 4.2. For $n \in \mathbb{N}$, $0 \leq k \leq n+1$, $0 \leq \gamma \leq \min\{k, n+1-k\}$, and $(\lambda, \mu) \in \mathbb{R}^2 \setminus \{\lambda = \mu\}$,

$$|\lambda - \mu|^\gamma |\partial_\lambda^\gamma f^{[n]}(\lambda^{(k)}, \mu^{(n+1-k)})| \leq 2^\gamma \frac{(k+\gamma-1)!}{(k-1)!} \frac{\|f^{(n)}\|_\infty}{n!}.$$

Proof. For $\gamma = 0$, this statement is immediate from Lemma 2.3. Let now $0 < \gamma \leq \min\{k, n+1-k\}$. By repeatedly applying Lemma 4.1, we obtain

$$\partial_\lambda^\gamma f^{[n]}(\lambda^{(k)}, \mu^{(n+1-k)}) = \frac{(k+\gamma-1)!}{(k-1)!} f^{[n+\gamma]}(\lambda^{(k+\gamma)}, \mu^{(n+1-k)}).$$

We now decompose $f^{[n+\gamma]}$ by applying the definition of divided differences multiple times and have

$$\begin{aligned} f^{[n+\gamma]}(\lambda^{(k+\gamma)}, \mu^{(n+1-k)}) &= \frac{1}{\lambda - \mu} \left(f^{[n+\gamma-1]}(\lambda^{(k+\gamma)}, \mu^{(n-k)}) - f^{[n+\gamma-1]}(\lambda^{(k+\gamma-1)}, \mu^{(n+1-k)}) \right) \\ &= \dots \\ &= \frac{1}{(\lambda-\mu)^\gamma} \sum_{j=0}^{\gamma} (-1)^j \binom{\gamma}{j} f^{[n]}(\lambda^{(k+\gamma-j)}, \mu^{(n+1-k-(\gamma-j))}). \end{aligned}$$

Using the estimate $\|f^{[n]}\|_\infty \leq \frac{\|f^{(n)}\|_\infty}{n!}$ from Lemma 2.3, we conclude

$$\begin{aligned} |\lambda - \mu|^\gamma |\partial_\lambda^\gamma f^{[n]}(\lambda^{(k)}, \mu^{(n+1-k)})| &\leq \sum_{j=0}^{\gamma} \binom{\gamma}{j} |f^{[n]}(\lambda^{(k+\gamma-j)}, \mu^{(n+1-k-(\gamma-j))})| \\ &\leq \sum_{j=0}^{\gamma} \binom{\gamma}{j} \frac{\|f^{(n)}\|_\infty}{n!} = 2^\gamma \frac{\|f^{(n)}\|_\infty}{n!}. \end{aligned}$$

□

Altogether we can now show

Theorem 4.3. *Let $n \in \mathbb{N}$, $f \in C^n(\mathbb{R})$, $1 \leq k \leq n$, and $p \in (1, \infty)$. Set $\phi(\lambda, \mu) := f^{[n]}(\lambda^{(k)}, \mu^{(n+1-k)})$. Then*

$$\|M_\phi\|_{S_p \to S_p} \lesssim \frac{2n+3}{n!} \frac{p^2}{p-1} \|f^{(n)}\|_\infty.$$

Proof. We can apply Theorem 1.5, since $\phi \in C^1(\mathbb{R}^2 \setminus \{\lambda = \mu\})$ by Lemma 4.1. From Lemma 4.2, we conclude

$$\begin{aligned} |||\phi|||_{\mathrm{HMS}} &\leq \|\phi\|_\infty + \|(\lambda, \mu) \mapsto |\lambda - \mu| \partial_\lambda \phi(\lambda, \mu)\|_\infty + \|(\lambda, \mu) \mapsto |\lambda - \mu| \partial_\mu \phi(\lambda, \mu)\|_\infty \\ &\leq (1 + 2k + 2(n+1-k)) \frac{\|f^{(n)}\|_\infty}{n!} = \frac{2n+3}{n!} \|f^{(n)}\|_\infty. \end{aligned}$$

□

5

Boundedness of multilinear Schur multipliers M_{ψ_j}

In this section, we demonstrate the bilinear transference approach towards the boundedness of T_{ψ_j}, $j = 0, 1, 2$, as defined in Section **??**. In Section 5.1, we set up the bilinear transference proof. The Fourier multiplier that is key to the transference approach is then constructed in Section 5.2, and the proof is concluded by showing the boundedness of the Fourier multiplier in Section 5.3.

5.1. Bilinear transference

The bilinear transference proof is illustrated in Figure 5.1. In this section, we discuss the relevant constructions.

The key idea behind transference is to find an isometry ι and a Fourier multiplier T_m, such that for a given Schur multiplier $M_{\tilde{m}}$ we have $\iota \circ M_{\tilde{m}} = T_m \circ \iota$. This implies $\|M_{\tilde{m}}\| = \|T_m\|$ for suitably chosen norms. In the following lemma, we construct such an isometry.

Lemma 5.1. Let $e_\lambda(t) := e^{-i\lambda t}$. Define $u \in L^p(\mathbb{T}) \otimes S_p$,

$$u(t) := \sum_{\lambda \in \mathbb{Z}} e_\lambda \otimes p_\lambda,$$

where $(p_\lambda)_\lambda$ is a family of orthogonal projections such that $\sum_\lambda p_\lambda = 1$. Then

$$\begin{aligned} &\iota_p : S_p \to L^p(\mathbb{T}) \otimes S_p \simeq L^p(\mathbb{T}, S_p), \\ &\iota_p(x) := u(1 \otimes x)u^* = \sum_{\lambda, \mu \in \mathbb{Z}} e^{i(\lambda-\mu)\cdot} \otimes p_\lambda x p_\mu \end{aligned}$$

is an isometry for $p \in (1, \infty)$.

Proof. Let $x \in S_p$, then

$$\|u(1 \otimes x)u^*\|^p_{L^p(\mathbb{T}, S_p)} = \int_{\mathbb{T}} \|u(t)xu^*(t)\|^p_{S_p} dt.$$

For each $t \in \mathbb{T}$, $u(t)$ is unitary, since

$$u^*u(t) = \sum_{\lambda,\mu} e^{i(\lambda-\mu)t} \otimes p_\lambda p_\mu = \sum_\lambda 1 \otimes p_\lambda = \mathrm{id}_{L^p(\mathbb{T})} \otimes id_{S_p},$$

and $uu^* = \mathrm{id}_{L^p(\mathbb{T})} \otimes id_{S_p}$ follows by the same calculation. Hence by Lemma 2.7,

$$\int_{\mathbb{T}} \|u(t)xu^*(t)\|^p_{S_p} dt = \int_{\mathbb{T}} \|x\|^p_{S_p} dt = \|x\|^p_{S_p}.$$

□

In the next lemma, we investigate how a Fourier multiplier T_m acts on a function with values in $\iota_p(S_p)$.

Lemma 5.2. Let $x \in S_p$ and let T_m be a bounded Fourier multiplier on $L^p(\mathbb{T}, S_p)$. Let u be as in Lemma 5.1. Then

$$T_m(u^*xu) = \sum_{\lambda,\mu} m(\lambda - \mu)e^{i(\lambda-\mu)\cdot} \otimes p_\lambda x p_\mu.$$

Proof. Define $e_{\lambda,\mu} : \mathbb{T} \to \mathbb{C}$, $t \mapsto e^{i(\lambda-\mu)t}$. We can express $e_{\lambda,\mu}$ as a Fourier series

$$e_{\lambda,\mu} = \frac{1}{\sqrt{2\pi}} \sum_{\nu\in\mathbb{Z}} \widehat{e_{\lambda,\mu}}(\nu)e^{i\nu\cdot},$$

where its Fourier coefficients are given by $\widehat{e_{\lambda,\mu}}(\nu) = \sqrt{2\pi}\delta_{\lambda-\mu}(\nu)$. Hence by Definition 2.28,

$$T_m e_{\lambda,\mu} = \frac{1}{\sqrt{2\pi}} \sum_{\nu\in\mathbb{Z}} m(\nu)\widehat{e_{\lambda,\mu}}(\nu)e^{i\nu\cdot} = m(\lambda-\mu)e^{i(\lambda-\mu)\cdot} = m(\lambda-\mu)e_{\lambda,\mu}.$$

We may now first assume that u^*xu is given by a finite sum. Then by linearity,

$$T_m(u^*xu) = \sum_{\lambda,\mu} T_m(e^{i(\lambda-\mu)\cdot}) \otimes p_\lambda x p_\mu = \sum_{\lambda,\mu} m(\lambda-\mu)e^{i(\lambda-\mu)\cdot} \otimes p_\lambda x p_\mu.$$

Since T_m is bounded by assumption, the statement follows for infinite sums by continuity. □

Finally, we need a bilinear version of Theorem 2.31, which can be found in [2, Theorem 2.4]. Here we give the proof explicitly for $d = 1$, as the constant in this statement will be relevant in Section 6.

Lemma 5.3. Let $(p_1, p_2, p) \in (1, \infty)$ be such that $1/p_1 + 1/p_2 = 1/p$, and let $m \in L^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ be the symbol of a Fourier multiplier

$$T_m : L^{p_1}(\mathbb{R}^d, S_{p_1}) \times L^{p_2}(\mathbb{R}^d, S_{p_2}) \to L^p(\mathbb{R}^d, S_p).$$

Suppose that all $(j, k) \in \mathbb{Z}^d \times \mathbb{Z}^d$ are Lebesgue points of m. Then $(m(j, k))_{j,k\in\mathbb{Z}^d}$ gives rise to a bounded Fourier multiplier

$$T_{(m(j,k))_{j,k}} : L^{p_1}(\mathbb{T}^d, S_{p_1}) \times L^{p_2}(\mathbb{T}^d, S_{p_2}) \to L^p(\mathbb{T}^d, S_p)$$

with $\|T_{(m(j,k))_{j,k}}\| \leq \frac{1}{\sqrt{2\pi}}\|T_m\|$.

Proof $(d = 1)$. We first show that for $\phi_p(x) := \exp(-\frac{\pi x^2}{p})$, we have

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^2} \int_{\mathbb{R}^2} m(\xi_1, \xi_2)\widehat{\phi}_{p_1}\left(\frac{\xi_1 - j}{\varepsilon}\right)\widehat{\phi}_{p_2}\left(\frac{\xi_2 - k}{\varepsilon}\right)\check{\phi}_q\left(\frac{\xi_1 + \xi_2 - l}{\varepsilon}\right) d\xi = \sqrt{2\pi}m(j, k)\delta_{j+k}(l),$$

where q is such that $1/p + 1/q = 1$. Indeed, we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^2} \int_{\mathbb{R}^2} m(\xi_1, \xi_2)\widehat{\phi}_{p_1}\left(\frac{\xi_1 - j}{\varepsilon}\right)\widehat{\phi}_{p_2}\left(\frac{\xi_2 - k}{\varepsilon}\right)\check{\phi}_q\left(\frac{\xi_1 + \xi_2 - l}{\varepsilon}\right) d\xi \\
=&\lim_{\varepsilon\to 0} \int_{\mathbb{R}^2} m(j + \varepsilon\eta_1, k + \varepsilon\eta_2)\widehat{\phi}_{p_1}(\eta_1)\,\widehat{\phi}_{p_2}(\eta_2)\,\check{\phi}_q\left(\eta_1 + \eta_2 + \frac{j + k - l}{\varepsilon}\right) d\eta.
\end{aligned}$$

Since all integer points are Lebesgue points of m, we have

$$\lim_{\varepsilon\to 0} m(j + \varepsilon\eta_1, k + \varepsilon\eta_2)\check{\phi}_q\left(\eta_1 + \eta_2 + \frac{j + k - l}{\varepsilon}\right) = m(j, k)\delta_{j+k}(l)\check{\phi}_q(\eta_1 + \eta_2),$$

and hence by dominated convergence

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^2} \int_{\mathbb{R}^2} m(\xi_1, \xi_2)\widehat{\phi}_{p_1}\left(\frac{\xi_1 - j}{\varepsilon}\right)\widehat{\phi}_{p_2}\left(\frac{\xi_2 - k}{\varepsilon}\right)\check{\phi}_q\left(\frac{\xi_1 + \xi_2 - l}{\varepsilon}\right) d\xi \\
=&m(j, k)\delta_{j+k}(l) \int_{\mathbb{R}^2} \widehat{\phi}_{p_1}(\eta_1)\,\widehat{\phi}_{p_2}(\eta_2)\,\check{\phi}_q(\eta_1 + \eta_2)\, d\eta \\
=&m(j, k)\delta_{j+k}(l)\sqrt{\frac{p_1}{2\pi}}\sqrt{\frac{p_2}{2\pi}}\sqrt{\frac{q}{2\pi}} \int_{\mathbb{R}^2} \exp\left(-p_1\frac{\eta_1^2}{4\pi}\right)\exp\left(-p_2\frac{\eta_2^2}{4\pi}\right)\exp\left(-q\frac{(\eta_1 + \eta_2)^2}{4\pi}\right) d\eta.
\end{aligned}$$

We can calculate the integral as

$$\begin{aligned}
&\sqrt{\frac{p_1p_2q}{(2\pi)^3}}\int_{\mathbb{R}^2}\exp\left(-p_1\frac{\eta_1^2}{4\pi}\right)\exp\left(-p_2\frac{\eta_2^2}{4\pi}\right)\exp\left(-q\frac{(\eta_1+\eta_2)^2}{4\pi}\right)d\eta\\
=&\sqrt{\frac{p_1p_2q}{(2\pi)^3}}\int_{\mathbb{R}^2}\exp\left(-\frac{p_1+q}{4\pi}\eta_1^2\right)\exp\left(-\frac{p_2+q}{4\pi}\eta_2^2\right)\exp\left(-\frac{2q\eta_1\eta_2}{4\pi}\right)d\eta\\
=&\sqrt{\frac{p_1p_2q}{(2\pi)^3}}\int_{\mathbb{R}^2}\exp\left(-\frac{p_1+q}{4\pi}(\eta_1^2+2\tfrac{q\eta_2}{p_1+q}\eta_1)\right)\exp\left(-\frac{p_2+q}{4\pi}\eta_2^2\right)d\eta\\
=&\sqrt{\frac{p_1p_2q}{(2\pi)^3}}\int_{\mathbb{R}^2}\exp\left(-\frac{p_1+q}{4\pi}(\eta_1+\tfrac{q\eta_2}{p_1+q})^2\right)\exp\left(\frac{p_1+q}{4\pi}\frac{q^2}{(p_1+q)^2}\eta_2^2\right)\exp\left(-\frac{p_2+q}{4\pi}\eta_2^2\right)d\eta\\
=&\sqrt{\frac{p_1p_2q}{(2\pi)^3}}\sqrt{\frac{4\pi^2}{p_1+q}}\int_{\mathbb{R}}\exp\left(-\frac{1}{4\pi}(p_2+q-\frac{q^2}{p_1+q})\eta_2^2\right)d\eta_2\\
=&\sqrt{\frac{p_1p_2q}{(2\pi)^3}}\sqrt{\frac{4\pi^2}{p_1+q}}\sqrt{\frac{4\pi^2}{p_2+q-\frac{q^2}{p_1+q}}}\\
=&\sqrt{2\pi p_1p_2q}\sqrt{\frac{1}{(p_1+q)(p_2+q)-q^2}}\\
=&\sqrt{2\pi}\sqrt{\frac{p_1p_2q}{p_1p_2+(p_1+p_2)q}}\\
=&\sqrt{2\pi}\sqrt{\frac{q}{1+\frac{p_1+p_2}{p_1p_2}q}}.
\end{aligned}$$

Note that

$$\frac{p_1+p_2}{p_1p_2}=\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p},$$

hence

$$\frac{q}{1+\frac{p_1+p_2}{p_1p_2}q}=\frac{q}{1+\frac{q}{p}}=\frac{1}{\frac{1}{q}+\frac{1}{p}}=1.$$

Thus altogether,

$$\lim_{\varepsilon\to0}\frac{1}{\varepsilon^2}\int_{\mathbb{R}^2}\widehat{\phi}_{p_1}\left(\frac{\xi_1-j}{\varepsilon}\right)\widehat{\phi}_{p_2}\left(\frac{\xi_2-k}{\varepsilon}\right)\check{\phi}_q\left(\frac{\xi_1+\xi_2-l}{\varepsilon}\right)m(\xi_1,\xi_2)d\xi=\sqrt{2\pi}m(j,k)\delta_{j+k}(l).$$

We can use this to show for trigonomic polynomials $f\in L^{p_1}(\mathbb{T},S_{p_1})$, $g\in L^{p_2}(\mathbb{T},S_{p_2})$, $h\in L^q(\mathbb{T},S_q)$, with $e_k:=\sqrt{2\pi}^{-1}e^{ik\cdot}$ that

$$\begin{aligned}
&\langle T_{(m(j,k))_{j,k}}(f,g),h\rangle\\
=&\sum_{j,k,l}m(j,k)\widehat{f}(j)\widehat{g}(k)\widehat{h}(l)\langle e_je_k,e_l\rangle\\
=&\frac{1}{\sqrt{2\pi}}\sum_{j,k}m(j,k)\widehat{f}(j)\widehat{g}(k)\widehat{h}(-j-k)\\
=&\frac{1}{2\pi}\sum_{j,k,l}\lim_{\varepsilon\to0}\frac{1}{\varepsilon^2}\int_{\mathbb{R}^2}\widehat{\phi}_{p_1}\left(\frac{\xi_1-j}{\varepsilon}\right)\widehat{\phi}_{p_2}\left(\frac{\xi_2-k}{\varepsilon}\right)\check{\phi}_q\left(\frac{\xi_1+\xi_2-l}{\varepsilon}\right)m(\xi_1,\xi_2)d\xi\widehat{f}(j)\widehat{g}(k)\widehat{h}(-l).
\end{aligned}$$

Note that by substituting $y=\varepsilon x$, we have

$$\begin{aligned}
\widehat{\phi_p(\varepsilon\cdot)e_k}(\xi)&=\frac{1}{2\pi}\int_{\mathbb{R}}\exp(-\tfrac{\pi}{p}(\varepsilon x)^2)\exp(-i(\xi-k)x)dx=\frac{1}{2\pi\varepsilon}\int_{\mathbb{R}}\exp(-\tfrac{\pi}{p}y^2)\exp(-i\tfrac{\xi-k}{\varepsilon}y)dy\\
&=\frac{1}{\sqrt{2\pi}\varepsilon}\widehat{\phi}_p\left(\frac{\xi-k}{\varepsilon}\right),
\end{aligned}$$

$$\begin{array}{ccc} S_{p_1} \times S_{p_2} & \xrightarrow{T_{\triangle_{1,j}} \circ M_{\psi_j} \circ (T_{\triangle_{2,j}} \times T_{\triangle_{3,j}})} & S_p \\ \Big\downarrow{\scriptstyle \iota_{p_1} \times \iota_{p_2}} & & \Big\downarrow{\scriptstyle \iota_p} \\ L^{p_1}(\mathbb{T}, S_{p_1}) \times L^{p_2}(\mathbb{T}, S_{p_2}) & \xrightarrow{T_{\triangle_{1,j}} \circ T_{m_j} \circ (T_{\triangle_{2,j}} \times T_{\triangle_{3,j}})} & L^p(\mathbb{T}, S_p) \end{array}$$

Figure 5.1: Bilinear transference as discussed in Section 5. See Lemma 5.1 for the definition of ι_p, Section 3 for the construction of M_{ψ_j} and the associated triangular truncations, and Section 5.2 for the definition of T_{m_j}. To simplify the notation, we identify operators T on S_p and $(1 \otimes T)$ on $L^p(\mathbb{T}, S_p)$.

and by the same calculation

$$\widecheck{\phi_p(\varepsilon\cdot)e_{-l}}(\xi_1 + \xi_2) = \frac{1}{\sqrt{2\pi}\varepsilon} \widecheck{\phi_p} \left(\frac{\xi_1 + \xi_2 - l}{\varepsilon} \right),$$

hence

$$\begin{aligned} &\frac{1}{2\pi} \sum_{j,k,l} \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \int_{\mathbb{R}^2} \widehat{\phi}_{p_1} \left(\frac{\xi_1 - j}{\varepsilon} \right) \widehat{\phi}_{p_2} \left(\frac{\xi_2 - k}{\varepsilon} \right) \widecheck{\phi}_q \left(\frac{\xi_1 + \xi_2 - l}{\varepsilon} \right) m(\xi_1, \xi_2) d\xi \widehat{f}(j) \widehat{g}(k) \widehat{h}(-l) \\ =& \sqrt{2\pi} \lim_{\varepsilon \to 0} \varepsilon \left\langle m\mathcal{F} \left(\phi_{p_1}(\varepsilon\cdot) \sum_j \widehat{f}(j) e_j \right) \mathcal{F} \left(\phi_{p_2} \sum_k \widehat{g}(k) e_k \right), \mathcal{F}^{-1} \left(\phi_q \sum_l \widehat{h}(l) e_l \right) \right\rangle \\ =& \sqrt{2\pi} \lim_{\varepsilon \to 0} \varepsilon \langle T_m(\phi_{p_1}(\varepsilon\cdot) f, \phi_{p_2}(\varepsilon\cdot) g), \phi_q(\varepsilon\cdot) h \rangle, \end{aligned}$$

where $\mathcal{F}$ denotes the Fourier transform. Following [20, Lemma 5.7.3], we have

$$\lim_{\varepsilon \to 0} \varepsilon^{1/p} \|\phi_p(\varepsilon\cdot) f\|_{L^p(\mathbb{R}, S_p)} = \frac{1}{(2\pi)^{1/p}} \|\phi_p\|_{L^p(\mathbb{R}, \mathbb{C})} \|f\|_{L^p(\mathbb{T}, S_p)},$$

thus altogether

$$\begin{aligned} &|\langle T_{(m(j,k))_{j,k}}(f, g), h \rangle| \\ &= \sqrt{2\pi} \lim_{\varepsilon \to 0} \varepsilon |\langle T_m(\phi_{p_1}(\varepsilon\cdot) f, \phi_{p_2}(\varepsilon\cdot) g), \phi_q(\varepsilon\cdot) h \rangle| \\ &\leq \frac{1}{\sqrt{2\pi}} \lim_{\varepsilon \to 0} \|T_m\| (2\pi\varepsilon)^{1/p_1} \|\phi_{p_1}(\varepsilon\cdot) f\|_{L^{p_1}(\mathbb{R}, S_{p_1})} (2\pi\varepsilon)^{1/p_2} \|\phi_{p_2}(\varepsilon\cdot) g\|_{L^{p_2}(\mathbb{R}, S_{p_2})} (2\pi\varepsilon)^{1/q} \|\phi_q(\varepsilon\cdot) h\|_{L^q(\mathbb{R}, S_q)} \\ &= \frac{1}{\sqrt{2\pi}} \|T_m\| \|\phi_{p_1}\|_{L^{p_1}(\mathbb{R}, \mathbb{C})} \|f\|_{L^{p_1}(\mathbb{T}, S_{p_1})} \|\phi_{p_2}\|_{L^{p_2}(\mathbb{R}, \mathbb{C})} \|g\|_{L^{p_2}(\mathbb{T}, S_{p_2})} \|\phi_q\|_{L^q(\mathbb{R}, \mathbb{C})} \|h\|_{L^q(\mathbb{T}, S_q)} \\ &= \frac{1}{\sqrt{2\pi}} \|T_m\| \|f\|_{L^{p_1}(\mathbb{T}, S_{p_1})} \|g\|_{L^{p_2}(\mathbb{T}, S_{p_2})} \|h\|_{L^q(\mathbb{T}, S_q)}. \end{aligned}$$

□

5.2. Construction of Fourier multipliers T_{m_j}

We now construct candidates for the symbols of the Fourier multipliers T_{m_j} in Figure 5.1, $j = 0, 1, 2$. For this, we will make the triangular truncations that are part of the definition of ψ_j explicit as in (**??**). The key property we are looking to achieve is the commutativity of the diagram in Figure 5.1, i.e. we are looking for m_j such that

$$\iota_p \circ M_{\psi_j} = T_{\triangle_{1,j}} \circ T_{m_j} \circ (T_{\triangle_{2,j}} \times T_{\triangle_{3,j}}) \circ (\iota_{p_1} \times \iota_{p_2}),$$

where $(T_{\triangle_{1,j}}, T_{\triangle_{2,j}}, T_{\triangle_{3,j}})$ denotes one of the two triples of triangular truncations associated with ψ_j, see Section **??**. We will give the argument in full detail for $j = 1$, i.e. the case where the associated permutation ρ_1 is trivial (see (**??**)); for $j = 0, 2$ the symbol m_j can be derived in the same manner.

For $j = 1$, we are considering the triples of triangular truncations

$$(T_{\triangle_{1,1}}, T_{\triangle_{2,1}}, T_{\triangle_{3,1}}) \in \{(T_{\triangle^{\mathrm{off}}_{\mathrm{upper}}}, T_{\triangle^{\mathrm{off}}_{\mathrm{upper}}}, T_{\triangle^{\mathrm{off}}_{\mathrm{upper}}}), (T_{\triangle_{\mathrm{lower}}}, T_{\triangle_{\mathrm{lower}}}, T_{\triangle_{\mathrm{lower}}})\},$$

see (**??**). We will demonstrate the calculation for the second triple.

Let $x \in S_{p_1}$ and $y \in S_{p_2}$ be lower triangular. By Lemma 2.18, we may choose them such that xy is off-diagonal lower triangular. We calculate

$$\begin{aligned}
\iota_p(M_{\psi_1}(x,y)) &= \sum_{\mu_0 \geq \mu_3} e^{i(\mu_0-\mu_3)\cdot} \otimes p_{\mu_0} \left(\sum_{\substack{\lambda_0 \geq \lambda_1 \geq \lambda_2 \\ \lambda_0 > \lambda_2}} \frac{\lambda_0 - \lambda_1}{\lambda_0 - \lambda_2} p_{\lambda_0} x p_{\lambda_1} y p_{\lambda_2} \right) p_{\mu_3} \\
&= \sum_{\substack{\lambda_0 \geq \lambda_1 \geq \lambda_2 \\ \lambda_0 > \lambda_2}} e^{i(\lambda_0-\lambda_2)\cdot} \frac{\lambda_0 - \lambda_1}{\lambda_0 - \lambda_2} \otimes p_{\lambda_0} x p_{\lambda_1} y p_{\lambda_2} \\
&= \sum_{\substack{\lambda_0 \geq \lambda_1 \geq \lambda_2 \\ \lambda_0 > \lambda_2}} e^{i((\lambda_0-\lambda_1)+(\lambda_1-\lambda_2))\cdot} \frac{\lambda_0 - \lambda_1}{(\lambda_0 - \lambda_1) + (\lambda_1 - \lambda_2)} \otimes p_{\lambda_0} x p_{\lambda_1} y p_{\lambda_2}.
\end{aligned}$$

Define $m_1(\lambda, \mu) := \frac{\lambda}{\lambda+\mu}$ to see

$$\begin{aligned}
\iota_p(M_{\psi_1}(x,y)) &= \sum_{\substack{\lambda_0 \geq \lambda_1 \geq \lambda_2 \\ \lambda_0 > \lambda_2}} e^{i((\lambda_0-\lambda_1)+(\lambda_1-\lambda_2))\cdot} m_1(\lambda_0 - \lambda_1, \lambda_1 - \lambda_2) \otimes p_{\lambda_0} x p_{\lambda_1} y p_{\lambda_2} \\
&= \sum_{\lambda_0 \geq \lambda_1} \sum_{\substack{\mu_0 \geq \mu_1 \\ \lambda_0 > \mu_1}} m_1(\lambda_0 - \lambda_1, \lambda_1 - \lambda_2) e^{i(\lambda_0-\lambda_1)\cdot} e^{i(\mu_0-\mu_1)\cdot} \otimes (p_{\lambda_0} x p_{\lambda_1})(p_{\mu_0} y p_{\mu_1}) \\
&= \sum_{\lambda > \mu} m_1(\lambda, \mu) \sum_{\lambda_0 \geq \lambda_1} \sum_{\mu_0 \geq \mu_1} \delta_{\lambda_0 - \lambda_1}(\lambda) \delta_{\mu_0 - \mu_1}(\mu) e^{i(\lambda+\mu)\cdot} \otimes (p_{\lambda_0} x p_{\lambda_1})(p_{\mu_0} y p_{\mu_1}).
\end{aligned}$$

Using Lemma 5.2, we now have

$$\begin{aligned}
\iota_p(M_{\psi_1}(x,y)) &= \frac{1}{2\pi} \sum_{\lambda > \mu} m_1(\lambda, \mu) \sum_{\lambda_0 \geq \lambda_1} \sum_{\mu_0 \geq \mu_1} \widehat{e}_{\lambda_0, \lambda_1}(\lambda) \widehat{e}_{\mu_0, \mu_1}(\mu) e^{i(\lambda+\mu)\cdot} \otimes (p_{\lambda_0} x p_{\lambda_1})(p_{\mu_0} y p_{\mu_1}) \\
&= (T_{\triangle^{\mathrm{off}}_{\mathrm{lower}}} \circ T_{m_1} \circ (T_{\triangle_{\mathrm{lower}}} \times T_{\triangle_{\mathrm{lower}}}) \circ (\iota_{p_1} \times \iota_{p_2}))(x, y).
\end{aligned}$$

By the same calculation,

$$\iota_p(M_{\psi_1}(x,y)) = (T_{\triangle^{\mathrm{off}}_{\mathrm{upper}}} \circ T_{m_1} \circ (T_{\triangle^{\mathrm{off}}_{\mathrm{upper}}} \times T_{\triangle^{\mathrm{off}}_{\mathrm{upper}}}) \circ (\iota_{p_1} \times \iota_{p_2}))(x, y)$$

holds for x,y off-diagonal upper triangular. A similar calculation yields the candidates $m_0(\lambda, \mu) := -\frac{\lambda}{\mu}$, $m_2(\lambda, \mu) := \frac{\lambda}{\mu}$ for the transference of the Schur multipliers with symbols ψ_0 and ψ_2. Note however that boundedness of m is a necessary condition for the boundedness of L^p-Fourier multipliers (see [15, Section 2.5.5]), hence we need to refine our symbols in a suitable way.

Set $m_1(\lambda, \mu) := \frac{\lambda}{\lambda+\mu}$ on the domain $\{(\lambda, \mu) \in \mathbb{R}^2 \setminus \{0\} \mid |\lambda| \leq |\lambda + \mu|\}$. Outside of this domain, we extend m_1 onto $\mathbb{R}^2 \setminus \{0\}$ such that it remains homogeneous of degree 0 (i.e. $m_1(s\lambda, s\mu) = m_1(\lambda, \mu)$ for $s \neq 0$), such that it is smooth away from the origin, and such that its integral along the unit circle vanishes. We extend m_0 and m_2 in the same manner, with initial domain $\{(\lambda, \mu) \in \mathbb{R}^2 \setminus \{0\} \mid |\lambda| \leq |\mu|\}$. Note that for $(\lambda_0, \lambda_1, \lambda_2) \in \Delta_j$ as defined in (**??**), we have that $(\lambda_0 - \lambda_1, \lambda_1 - \lambda_2)$ is an element of the initial domain of m_j, $j = 0, 1, 2$. This domain Δ_j was constructed such that it corresponds to the triangular truncations applied in Figure 5.1, hence the diagram commutes for the extended symbol m_j by precisely the same calculation as above.

5.3. T_m as a Calderón-Zygmund operator

For $p \neq 2$, boundedness of the symbol m is a necessary but not sufficient condition for the boundedness of the Fourier multiplier T_m. In order to show the boundedness of T_{m_j}, $j = 0, 1, 2$, with m_j as defined in Section 5.2, we will use Theorem 1.4. By Lemma 5.3, this theorem implies boundedness of $T_{m_j} : L^{p_1}(\mathbb{T}, S_{p_1}) \times L^{p_2}(\mathbb{T}, S_{p_2}) \to L^p(\mathbb{T}, S_p)$. It hence remains to show that the Fourier multipliers $T_{m_j} : L^{p_1}(\mathbb{R}, \mathbb{C}) \times L^{p_2}(\mathbb{R}, \mathbb{C}) \to L^p(\mathbb{R}, \mathbb{C})$, $j = 0, 1, 2$, are Calderón-Zygmund operators.

Recall from Section 5.2 that all m_j were constructed such that they are homogeneous of degree 0, smooth away from the origin, and have vanishing integral along the unit circle. Following the arguments in the proof of [6, Lemma 4.3], such functions can be expressed in polar coordinates as

$$\lambda + i\mu = re^{i\theta} \Rightarrow m(\lambda, \mu) = \sum_{0 \neq k \in \mathbb{Z}} a_k e^{ik\theta}, \tag{5.1}$$

where the coefficients $(a_k)_k$ decrease faster than any power of k. Note that since m is homogeneous, we have

$$\lambda + i\mu = re^{i\theta} \Rightarrow \sum_{0 \neq k \in \mathbb{Z}} (-1)^k a_k e^{ik\theta} = m(-\lambda, -\mu) = m(\lambda, \mu) = \sum_{0 \neq k \in \mathbb{Z}} a_k e^{ik\theta},$$

where by orthogonality of $(\theta \mapsto e^{ik\theta})_{k \neq 0}$ we have uniqueness of $(a_k)_k$ and thus $a_k = 0$ for k odd.

This allows us to treat all cases $j = 0, 1, 2$ at once by showing that a generic bilinear Fourier multiplier with symbol m as in (5.1) is a Calderón-Zygmund operator as defined in Definition 2.33. We first rewrite the integral expression for the Fourier multiplier T_m in order to construct its integral kernel K. In this calculation, the Fourier transform of m will be important. Following the proof of [6, Lemma 4.3], it is given by

$$\xi_1 + i\xi_2 = re^{i\theta} \Rightarrow \widehat{m}(\xi_1, \xi_2) = \frac{1}{|r|^2} \sum_{0 \neq k \in \mathbb{Z}} \frac{|k|}{2\pi i^k} a_k e^{ik\theta} = \frac{1}{|r|^2} \sum_{\substack{k \text{ even} \\ k \neq 0}} \frac{(-1)^{k/2}|k|}{2\pi} a_k e^{ik\theta}. \tag{5.2}$$

Note that for the remainder of this section, x, y, z will denote real numbers rather than operators. We calculate

$$\begin{aligned}
T_m(f, g)(x) &= \frac{1}{2\pi} \int_{\mathbb{R}^2} m(\xi_1, \xi_2) \widehat{f_1}(\xi_1) \widehat{f_2}(\xi_2) e^{i(\xi_1 + \xi_2)x} d\xi \\
&= \frac{1}{2\pi} \int_{\mathbb{R}^2} m(\xi_1, \xi_2)(e^{i\xi_1 x} \widehat{f_1}(\xi_1))(e^{i\xi_2 x} \widehat{f_2}(\xi_2)) d\xi \\
&= \frac{1}{2\pi} \int_{\mathbb{R}^2} m(\xi_1, \xi_2) \widehat{f(\cdot + x)}(\xi_1) \widehat{g(\cdot + x)}(\xi_2) d\xi \\
&= \frac{1}{2\pi} \int_{\mathbb{R}^2} \widehat{m}(\xi_1, \xi_2) f(\xi_1 + x) g(\xi_2 + x) d\xi \\
&= \frac{1}{2\pi} \int_{\mathbb{R}^2} \widehat{m}(-\xi_1, -\xi_2) f(x - \xi_1) g(x - \xi_2) d\xi,
\end{aligned}$$

where by substituting $y = x - \xi_1$, $z = x - \xi_2$ and using the homogeneity of $\widehat{m}$, we obtain

$$T_m(f, g)(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \widehat{m}(x - y, x - z) f(y) g(z) dy dz = \int_{\mathbb{R}^2} K(x, y, z) f(y) g(z) dy dz$$

with

$$K(x, y, z) := \frac{1}{2\pi} \widehat{m}(x - y, x - z) = \frac{1}{(x - y)^2 + (x - z)^2} \sum_{\substack{k \text{ even} \\ k \neq 0}} \frac{(-1)^{k/2}|k|}{4\pi^2} a_k \left(\frac{(x - y) + i(x - z)}{|(x - y) + i(x - z)|} \right)^k. \tag{5.3}$$

In the following lemmas, we will show that K satisfies the properties in Definition 2.33. As in Definition 2.33, set $\Delta := \{(x, y, z) \in \mathbb{R}^3 \mid x = y = z\}$.

Lemma 5.4 (Size condition)**.** For all $(x, y, z) \in \mathbb{R}^3 \setminus \Delta$,

$$|K(x, y, z)| \leq \frac{C_K}{(|x - y| + |x - z|)^2}$$

for some constant $C_K > 0$.

Proof. Recall that $(a_k)_k$ decays faster than any polynomial in k. Furthermore, by using polar coordinates we have

$$(x-y)+i(x-z)=re^{i\theta} \Rightarrow \left|\left(\frac{(x-y)+i(x-z)}{|(x-y)+i(x-z)|}\right)^k\right| = |e^{ik\theta}| = 1.$$

Hence we conclude

$$\begin{aligned}
|K(x,y,z)| &= \left|\frac{1}{(x-y)^2+(x-z)^2}\sum_{\substack{k \text{ even}\\ k\neq 0}}\frac{(-1)^{k/2}|k|}{4\pi^2}a_k\left(\frac{(x-y)+i(x-z)}{|(x-y)+i(x-z)|}\right)^k\right| \\
&\leq \frac{1}{|x-y|^2+|x-z|^2}\sum_{\substack{k \text{ even}\\ k\neq 0}}\frac{|ka_k|}{4\pi^2} \\
&\leq \frac{C}{|x-y|^2+|x-z|^2} \\
&\leq \frac{2C}{(|x-y|+|x-z|)^2},
\end{aligned}$$

where in the last inequality we used $(\lambda+\mu)^2 \leq 2(\lambda^2+\mu^2)$ for $\lambda, \mu \geq 0$. □

Lemma 5.5 (Smoothness condition)**.** Let $(x_1,x_2,x_3) \in \mathbb{R}^3 \setminus \Delta$. For any $j = 1,2,3$, choose $\tilde{x}_j$ such that $|x_j - \tilde{x}_j| \leq \frac{1}{2}\max(|x_1-x_2|,|x_1-x_3|)$. Furthermore, set $\tilde{x}_k := x_k$ for all $k \neq j$. Then there exist $C_K < \infty$ and $\alpha \in (0,1]$ such that

$$|K(x_1,x_2,x_3) - K(\tilde{x}_1,\tilde{x}_2,\tilde{x}_3)| \leq \frac{C_K|x_j-\tilde{x}_j|^\alpha}{(|x_1-x_2|+|x_1-x_3|)^{2+\alpha}}.$$

Proof. We will show this estimate for $\alpha = 1$. For clearer notation, we rename (x_1,x_2,x_3) to (x,y,z); accordingly, the cases $j = 1,2,3$ will be referred to as the x-, y-, or z-case. We will give the proof explicitly for the z-case and then discuss the necessary modifications for the x- and y-cases.

Fix x and y. For $k \in \mathbb{Z}$, define the map

$$f_k : \mathbb{R} \to \mathbb{C},\ f_k(z) := \frac{((x-y)+i(x-z))^k}{|(x-y)+i(x-z)|^{k+2}},$$

or in polar coordinates

$$(x-y)+i(x-z) = re^{i\theta} \Rightarrow f_k(z) = \frac{e^{ik\theta}}{r^2}. \tag{5.4}$$

We can now rewrite $|K(x,y,z) - K(x,y,\tilde{z})|$ as

$$\begin{aligned}
|K(x,y,z)-K(x,y,\tilde{z})| &= \left|\sum_{\substack{k \text{ even}\\ k\neq 0}}\frac{(-1)^{k/2}|k|}{4\pi^2}a_k\left(\frac{((x-y)+i(x-z))^k}{|(x-y)+i(x-z)|^{k+2}} - \frac{((x-y)+i(x-\tilde{z}))^k}{|(x-y)+i(x-\tilde{z})|^{k+2}}\right)\right| \\
&\leq \sum_{\substack{k \text{ even}\\ k\neq 0}}\frac{|ka_k|}{4\pi^2}|f_k(z)-f_k(\tilde{z})|.
\end{aligned}$$

Denote by g_k the real part of f_k and by h_k its imaginary part. We will show that there exist a constant C and a polynomial p, both independent of x and y, such that

$$\max\left(|g_k(z)-g_k(\tilde{z})|,|h_k(z)-h_k(\tilde{z})|\right) \leq C\frac{p(k)|z-\tilde{z}|}{(|x-y|+|x-z|)^3} \tag{5.5}$$

for $|z-\tilde{z}| \leq \frac{1}{2}\max(|x-y|,|x-z|)$.

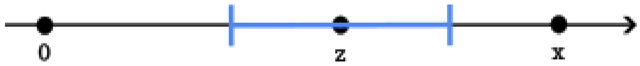

Figure 5.2: Relationship between x, z, and $\tilde{z}$ in the z-case with $x = y$ in the proof of Lemma 5.5 for $0 < z < x$. The blue lines denote the interval $[z - \frac{|x-z|}{2}, z + \frac{|x-z|}{2}]$ in which $\tilde{z}$ can lie.

We first consider the case $x = y$, for which (5.4) implies

$$f_k(z) = \frac{(i\ \text{sgn}(x-z))^k}{(x-z)^2} = \frac{(-1)^{k/2}}{(x-z)^2} = g_k(z), \quad h_k(z) = 0,$$

since we are only considering even k. By a slight abuse of notation, we will let $[z, \tilde{z}]$ denote the closed interval between z and $\tilde{z}$, even if $\tilde{z} < z$. The mean value theorem yields a $\zeta \in [z, \tilde{z}]$ such that

$$|f_k(z) - f_k(\tilde{z})| = \left| \frac{1}{(x-z)^2} - \frac{1}{(x-\tilde{z})^2} \right| = \frac{|z-\tilde{z}|}{|x-\zeta|^3}.$$

Here, we used that $\text{sgn}(x - z) = \text{sgn}(x - \tilde{z})$, since $\tilde{z} \in [z - \frac{|x-z|}{2}, z + \frac{|x-z|}{2}]$; see Figure 5.2 for an illustration. By construction, we have the estimate

$$|x-\zeta| \geq \min_{\zeta' \in [z,\tilde{z}]} |x-\zeta'| \geq \min_{\zeta' \in [z - \frac{|x-z|}{2}, z + \frac{|x-z|}{2}]} |x-\zeta'| = \frac{1}{2}|x-z|,$$

hence altogether,

$$|f_k(z) - f_k(\tilde{z})| \leq 8\frac{|z-\tilde{z}|}{|x-z|^3} = \frac{8|z-\tilde{z}|}{(|x-y| + |x-z|)^3}.$$

Now let $x \neq y$. We will prove (5.5) explicitly for g_k, the estimate for h_k follows by the same arguments. By fixing $x \neq y$ and using (5.4), we have

$$g_k(z) = \frac{\cos(k\theta(z))}{r(z)^2},$$

where the polar coordinates are given by

$$r(z) = \sqrt{(x-y)^2 + (x-z)^2}, \quad \theta(z) = \begin{cases} \arccos\left(\frac{x-y}{r(z)}\right), & x - z \geq 0, \\ 2\pi - \arccos\left(\frac{x-y}{r(z)}\right), & x - z < 0. \end{cases}$$

We apply the mean value theorem to find a $\zeta \in [z, \tilde{z}]$ such that

$$|g_k(z) - g_k(\tilde{z})| = |g_k'(\zeta)||z-\tilde{z}|,$$

hence it remains to estimate g_k'. We calculate the derivative as

$$\frac{d}{dz}g_k(z) = \frac{d}{dz}\frac{\cos(k\theta(z))}{r(z)^2} = \frac{-k\sin(k\theta(z))\theta'(z)}{r(z)^2} - \frac{2r'(z)\cos(k\theta(z))}{r(z)^3}.$$

We further calculate

$$\begin{aligned}
r'(z) &= \frac{d}{dz}\sqrt{(x-y)^2+(x-z)^2} = \frac{-(x-z)}{\sqrt{(x-y)^2+(x-z)^2}} = -\frac{x-z}{r(z)},\\
\theta'(z) &= \operatorname{sgn}(x-z)\frac{d}{dz}\arccos\left(\frac{x-y}{r(z)}\right)\\
&= \operatorname{sgn}(x-z)\frac{-1}{\sqrt{1-\left(\frac{x-y}{r(z)}\right)^2}}\frac{-(x-y)}{r(z)^2}r'(z)\\
&= -\operatorname{sgn}(x-z)\sqrt{\frac{(x-y)^2+(x-z)^2}{(x-z)^2}}\frac{x-y}{r(z)^2}\frac{x-z}{r(z)}\\
&= -\frac{|x-z|(x-y)}{|x-z|r(z)^2}\\
&= -\frac{x-y}{r(z)^2}.
\end{aligned}$$

Hence altogether,

$$\frac{d}{dz}g_k(z) = \frac{k\sin(k\theta(z))(x-y)+2\cos(k\theta(z))(x-z)}{r(z)^4}.$$

We can thus estimate $g_k'(\zeta)$ by

$$|g_k'(\zeta)| \le (|k|+2)\frac{|x-y|+|x-\zeta|}{(|x-y|^2+|x-\zeta|^2)^2}.$$

Now we distinguish two cases:

1. $|x-z|\le|x-y|$: In this case, we have

$$|x-\zeta| \le |x-z|+|z-\zeta| \le |x-y|+|z-\zeta| \le |x-y|+|z-\tilde{z}| \le 2|x-y|.$$

   Hence,

$$\frac{|x-y|+|x-\zeta|}{(|x-y|^2+|x-\zeta|^2)^2} \le \frac{3|x-y|}{(|x-y|^2+|x-\zeta|^2)^2} \le 3\frac{|x-y|}{|x-y|^4} = \frac{3}{|x-y|^3} \le \frac{24}{(|x-y|+|x-z|)^3},$$

   where in the last inequality we used our assumption $|x-z|\le|x-y|$ to obtain the estimate $|x-y|^3 \ge \left(\frac{|x-y|+|x-z|}{2}\right)^3$.

2. $|x-y|\le|x-z|$: This implies $|z-\tilde{z}|\le\frac{1}{2}|x-z|$, hence we have by construction that

$$\begin{aligned}
|x-\zeta| &\le |x-z|+|z-\zeta| \le |x-z|+|z-\tilde{z}| \le \frac{3}{2}|x-z|,\\
|x-\zeta| &\ge \min_{\zeta'\in[z,\tilde{z}]}|x-\zeta'| \ge \min_{\zeta'\in[z-\frac{|x-z|}{2},z+\frac{|x-z|}{2}]}|x-\zeta'| = \frac{1}{2}|x-z|.
\end{aligned}$$

   Hence by similar estimates as in the previous case we have

$$\frac{|x-y|+|x-\zeta|}{(|x-y|^2+|x-\zeta|^2)^2} \le \frac{\frac{5}{2}|x-z|}{(\frac{1}{4}|x-z|^2)^2} = \frac{40}{|x-z|^3} \le \frac{320}{(|x-y|+|x-z|)^3}.$$

Altogether, we thus have

$$|g_k(z)-g_k(\tilde{z})| = |g_k'(\zeta)||z-\tilde{z}| \le C(|k|+2)\frac{|z-\tilde{z}|}{(|x-y|+|x-z|)^3}.$$

For the imaginary part

$$h_k(z) = \frac{\sin(k\theta(z))}{r(z)^2},$$

the estimate follows in the same manner. Hence,

$$\begin{aligned}
|K(x,y,z) - K(x,y,\tilde{z})| &\leq \sum_{0\neq k\in\mathbb{Z}} \frac{|ka_k|}{4\pi^2} |f_k(x-y, x-z) - f_k(x-y, x-\tilde{z})| \\
&\leq \sum_{0\neq k\in\mathbb{Z}} \frac{|ka_k|}{4\pi^2} (|g_k(z) - g_k(\tilde{z})| + |h_k(z) - h_k(\tilde{z})|) \\
&\leq 2C \sum_{0\neq k\in\mathbb{Z}} \frac{|k|(|k|+2)|a_k|}{4\pi^2} \frac{|z-\tilde{z}|}{(|x-y|+|x-z|)^3} \\
&\leq C_K \frac{|z-\tilde{z}|}{(|x-y|+|x-z|)^3}.
\end{aligned}$$

We now sketch the proof in the x-case and y-case. In the y-case, the proof proceeds in the same manner. In particular, we define the polar coordinate functions $r(y)$ and $\theta(y)$ exactly as above. For the derivatives, we have

$$r'(y) = -\frac{x-y}{r(y)}$$

immediately by symmetry. The derivative of θ can be calculated for $x \neq y$ as

$$\begin{aligned}
\theta'(y) &= \operatorname{sgn}(x-z) \frac{d}{dy} \arccos\left(\frac{x-y}{r(y)}\right) \\
&= \operatorname{sgn}(x-z) \frac{-1}{\sqrt{1-\left(\frac{x-y}{r(y)}\right)^2}} \frac{d}{dy} \frac{x-y}{r(y)} \\
&= \operatorname{sgn}(x-z) \frac{r(y)}{|x-z|} \frac{r(y) + r'(y)(x-y)}{r(y)^2} \\
&= \operatorname{sgn}(x-z) \frac{1}{|x-z|} \left(1 + \frac{r'(y)(x-y)}{r(y)}\right) \\
&= \operatorname{sgn}(x-z) \frac{1}{|x-z|} \left(1 - \frac{(x-y)^2}{r(y)^2}\right) \\
&= \operatorname{sgn}(x-z) \frac{(x-z)^2}{|x-z| r(y)^2} \\
&= \frac{x-z}{r(y)^2}.
\end{aligned}$$

Hence the proof in the y-case proceeds in the same manner as in the z-case.

In the x-case, first let $y = z$. Then

$$f_k(x) = \frac{e^{ik\theta(x)}}{r(x)^2} = \begin{cases} \dfrac{e^{ik\pi/4}}{2(x-y)^2}, & x-y>0 \\ \dfrac{e^{5ik\pi/4}}{2(x-y)^2}, & x-y<0. \end{cases}$$

Hence the proof reduces to applying the mean value theorem to $x \mapsto (x-y)^{-2}$ as in the other cases. In the remaining case $y \neq z$, we find

$$r'(x) = \frac{(x-y)+(x-z)}{r(x)}, \quad \theta'(x) = \frac{(x-y)-(x-z)}{r(x)^2},$$

allowing us to finish the proof as in the z-case. □

It remains to show the last condition of Definition 2.33, namely the boundedness condition. For this, we use the following theorem.

Theorem 5.6 (Bilinear version of [24, Theorem 8])**.** *Let $p_1, p_2 \in (1, \infty)$ and let p be such that $1/p_1 + 1/p_2 = 1/p$, Let k be an integral kernel in $\mathbb{R}^2$ that is homogeneous of degree -2, smooth away from the origin, and such that its integral along the circle vanishes. Then the integral operator $T_k : L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R}) \to L^p(\mathbb{R})$,*

$$T_k(f_1, f_2)(x) := \int_{\mathbb{R}^2} k(y_1, y_2) f_1(x - y_1) f_2(x - y_2) dy_1 dy_2$$

is a bounded operator.

Lemma 5.7 (Boundedness)**.** For all $p_1, p_2 \in (1, \infty)$ and $p \in (1/2, \infty)$ such that $1/p_1 + 1/p_2 = 1/p$, and m as in (5.1) we have

$$\|T_m(f, g)\|_{L^p(\mathbb{R})} \lesssim \|f\|_{L^{p_1}(\mathbb{R})} \|g\|_{L^{p_2}(\mathbb{R})}.$$

Proof. When deriving (5.3), we saw that

$$T_m(f, g)(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \widehat{m}(\xi_1, \xi_2) f(x - \xi_1) g(x - \xi_2) d\xi.$$

It is immediate from (5.2) that $\widehat{m}$ satisfies the conditions of Theorem 5.6 — it is smooth away from the origin by construction, the integral along the unit circle vanishes for every $(\theta \mapsto e^{ik\theta})_{k \neq 0}$ and hence for $\widehat{m}$, and for $s \in \mathbb{R} \setminus \{0\}$, the homogeneity property holds since

$$\xi_1 + i\xi_2 = re^{i\theta} \Rightarrow \widehat{m}(s\xi_1, s\xi_2) = \frac{1}{|sr|^2} \sum_{\substack{k \text{ even} \\ k \neq 0}} \frac{(-1)^{k/2}|k|}{2\pi} \operatorname{sgn}(s)^k a_k e^{ik\theta} = s^{-2} m(\xi_1, \xi_2).$$

□

6

p-dependence of the constant in Theorem 1.3 for $n = 2$

In Sections 3–5, we have presented an alternative proof of Theorem 1.3 based on bilinear transference. In this section, we will compare our proof to the original proof by Potapov, Skripka, and Sukochev in [35] by comparing the growth rate in p of the constant $C_{p_1,\cdots,p_n}$ in the special case of the multiplier $M_{f^{[2]}} : S_p \times S_p \to S_{p/2}$ for $p \in (1,\infty)$, i.e. in the case $n = 2$ and $p_1 = p_2$.

Since not all relevant constants are explicitly stated in the literature, we first repeat proofs where necessary, while keeping track of the p-dependent constants. We first focus on the proof of Theorem 1.3 in [35] in Section 6.1. Our presentation is mostly self-contained, merely a few proof steps that are not relevant to the p-dependence of the constant are omitted. In order to discuss the constant in our alternative proof, it remains to determine the constant in Theorem 1.4. This is done in Section 6.2, which is again mostly self-contained up to some steps that are not relevant to the constant. Finally, we compare and discuss the constant yielded by both proof methods in Section 6.3.

6.1. Original proof by Potapov, Skripka, and Sukochev

As in Section 3, we first compose the Schur multiplier $M_{f^{[2]}}$ with triangular truncations such that we can decompose the resulting truncated terms into bilinear Toeplitz form multipliers and linear multipliers. All cases as listed in Table 3.1 arise. In this section we demonstrate the proof for the $\lambda_0 \geq \lambda_1 \geq \lambda_2$ term.

Similar to our work in Section 3, the function $f^{[2]}$ is decomposed on this domain as

$$f^{[2]}(\lambda_0,\lambda_1,\lambda_2) = \frac{\lambda_0-\lambda_1}{\lambda_0-\lambda_2}(\phi_{1,f''}(\lambda_1,\lambda_0)-\phi_{2,f''}(\lambda_1,\lambda_0)) + \frac{\lambda_1-\lambda_2}{\lambda_0-\lambda_2}(\phi_{1,f''}(\lambda_1,\lambda_2)-\phi_{2,f''}(\lambda_1,\lambda_2)), \quad (6.1)$$

for $\lambda_0 > \lambda_2$, where for $m \in \mathbb{N}$ and h a continuous and bounded function we define

$$\phi_{m,h}(\lambda,\mu) := \int_0^1 t^{m-1} h(\lambda + (\mu-\lambda)t)dt. \quad (6.2)$$

Note that this decomposition of $f^{[2]}$ is equal to the decomposition (3.1) we used in Section 3: One can show using Lemma 2.4 and Lemma 4.1 that $\phi_{1,f''}(\lambda,\mu) = f'^{[1]}(\lambda,\mu)$ and $\phi_{2,f''}(\lambda,\mu) = f^{[2]}(\lambda,\mu,\mu)$, thus

$$\phi_{1,f''}(\lambda,\mu)-\phi_{2,f''}(\lambda,\mu) = \frac{f'(\lambda)-f'(\mu)-f^{[1]}(\lambda,\mu)+f'(\mu)}{\lambda-\mu} = \frac{f'(\lambda)-f^{[1]}(\lambda,\mu)}{\lambda-\mu} = f^{[2]}(\lambda,\lambda,\mu).$$

The difference between the proof by Potapov, Skripka, and Sukochev in [35] and the method we demonstrated in Sections 3–5 lies in the fact that the proof in [35] heavily relies on a *linear* transference argument, where the Fourier multiplier considered is a complex power. This multiplier is indeed bounded.

Theorem 6.1. *Let $p \in (1,\infty)$ and $s \in \mathbb{R}$. Define $\varphi_s : \mathbb{R} \to \mathbb{C}$, $\varphi_s(\xi) := \xi^{is}$. Then the associated periodic Fourier multiplier T_{φ_s} is bounded on $L^p(\mathbb{T}, S_p)$ with*

$$\|T_{\varphi_s} f\|_{L^p(\mathbb{T},S_p)} \leq c_p(1+|s|)\|f\|_{L^p(\mathbb{T},S_p)}.$$

We shall sketch two proofs for this theorem here. In both proofs, the constant c_p has the upper bound

$$c_p \lesssim \hbar_{p.S_p}(\beta_{p,S_p})^2.$$

where the constants β_{p,S_p} and $\hbar_{p.S_p}$ are from Lemma 2.25.

Proof 1 [20, Corollary 5.3.19]. This proof uses the Mihlin multiplier theorem (Theorem 2.30). Applied to our multiplier $\varphi_s(\xi) = |\xi|^{is}$, it states that

$$\|T_{\varphi_s}\|_{L^p(\mathbb{R},S_p)} \leq c_p \left(\sup_{\xi \in \mathbb{R}\setminus\{0\}} |\varphi_s(\xi)| + \sup_{\xi \in \mathbb{R}\setminus\{0\}} |\xi \varphi_s'(\xi)|) \right),$$

with c_p bounded as stated above, from which we deduce

$$\|T_{\varphi_s}\|_{L^p(\mathbb{R},S_p)} \leq c_p \left(\sup_{\xi \in \mathbb{R}\setminus\{0\}} ||\xi|^{is}| + \sup_{\xi \in \mathbb{R}\setminus\{0\}} |\xi \frac{is}{\xi} |\xi|^{is}| \right) = c_p(1+|s|).$$

By Theorem 2.31, $\|T_{\varphi_s}\|_{L^p(\mathbb{T},S_p)} \leq \|T_{\varphi_s}\|_{L^p(\mathbb{R},S_p)}$, which completes the proof. □

Proof 2 [36, Lemma 5]. This proof uses the vector-valued Marcinkiewicz multiplier theorem (Theorem 2.29), which gives the bound

$$\|T_{\varphi_s}\|_{L^2(\mathbb{T},X)} \leq c_p \left(\sup_k |\varphi_s(k)| + \sup_k \sum_{2^{k-2} \leq j \leq 2^{k-1}} |\varphi_s(j+1) - \varphi_s(j)| \right).$$

The required estimates hold for the sequence $(k^{is})_{k\in\mathbb{N}}$, since $|k^{is}| = 1$ for all k and

$$\begin{aligned} \sum_{2^{k-2} \leq j \leq 2^{k-1}} |(j+1)^{is} - j^{is}| &= \sum_{2^{k-2} \leq j \leq 2^{k-1}} | \int_j^{j+1} \frac{d}{dx} x^{is} dx| \\ &= \sum_{2^{k-2} \leq j \leq 2^{k-1}} | \int_j^{j+1} is \frac{x^{is}}{x} dx| \\ &\leq \sum_{2^{k-2} \leq j \leq 2^{k-1}} \frac{|s|}{|j|} \leq |s| \end{aligned}$$

by the fundamental theorem of calculus.

□

These Fourier multipliers appear by linear transference, where the multilinear Schur multiplier with symbol $(\lambda_0, \lambda_1, \lambda_2) \mapsto \frac{\lambda_0 - \lambda_1}{\lambda_0 - \lambda_2}$ is first decomposed into two linear multipliers by using the following lemma.

Lemma 6.2 ([36], Lemma 6)**.** There exists a function $g : \mathbb{R} \to \mathbb{C}$ such that

$$\int_{\mathbb{R}} |s|^n |g(s)| ds < \infty$$

for all $n \in \mathbb{N}$, and such that for every $\lambda, \mu > 0$ with $0 \leq \frac{\lambda}{\mu} \leq 2$,

$$\frac{\lambda}{\mu} = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} g(s) \lambda^{is} \mu^{-is} ds.$$

Proof. We set $g = \hat{f}$, where f is a C^∞-function such that 1) $f \geq 0$, 2) $f(t) = 0$ if $t \geq 1 + \log 2$, and 3) $f(t) = e^t$ if $t \leq \log 2$. For this function,

$$\int_{\mathbb{R}} |s|^n |g(s)| ds = \int_{\mathbb{R}} |s^n \hat{f}(s)| ds = C \int_{\mathbb{R}} |\widehat{f^{(n)}}(s)| ds \leq \sqrt{2} C (\|\widehat{f^{(n)}}\|_{L^2} + \|\widehat{f^{(n+1)}}\|_{L^2}),$$

for all $n \in \mathbb{N}_0$, where the last estimate is from [34, Lemma 7]. For all derivatives of f, it holds that

$$\|f^{(n)}\|_{L^2}^2 = \int_{-\infty}^{\log 2} e^{2s} ds + \int_{\log 2}^{1+\log 2} |f^{(n)}(s)|^2 ds \leq 4 + \sup_{s \in [\log 2, 1+\log 2]} |f^{(n)}(s)| < \infty,$$

hence $\int_{\mathbb{R}} |s|^n |g(s)| ds < \infty$ for all $n \in \mathbb{N}_0$ by Plancherel's theorem. Furthermore, for $t \leq \log 2$ it holds by the inverse Fourier transform that

$$e^t = f(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} g(s) e^{its} ds,$$

hence for $t = \log \frac{\lambda}{\mu}$ the second statement follows. □

Using this decomposition, we can now prove the boundedness of the Schur multiplier with symbol $(\lambda_0, \lambda_1, \lambda_2) \mapsto \frac{\lambda_0 - \lambda_1}{\lambda_0 - \lambda_2}$ on the domain where $\lambda_0 \geq \lambda_1 \geq \lambda_2$. This proof can be found in the proof of Lemma 4.6 in [35].

Theorem 6.3. *Let $p_1, p_2, p \in (1, \infty)$ such that $1/p_1 + 1/p_2 = 1/p$. Let $x \in S_{p_1}$ and $y \in S_{p_2}$ be lower triangular operators. Define $\psi : \{(\lambda_0, \lambda_1, \lambda_2) \in \mathbb{Z}^3 \mid \lambda_0 \geq \lambda_1 \geq \lambda_2\} \to \mathbb{R}$, $(\lambda_0, \lambda_1, \lambda_2) \mapsto \frac{\lambda_0 - \lambda_1}{\lambda_0 - \lambda_2}$, where we set $\psi(\lambda, \lambda, \lambda) := 1$. Then the associated Schur multiplier M_ψ is bounded with*

$$\|M_\psi(x, y)\|_{S_p} \leq c_{p_1, p_2} \|x\|_{S_{p_1}} \|y\|_{S_{p_2}}.$$

Proof. We split the multiplier into its diagonal and off-diagonal part.

$$M_\psi(x, y) = \sum_{\lambda_0 \geq \lambda_1 \geq \lambda_2} \frac{\lambda_0 - \lambda_1}{\lambda_0 - \lambda_2} p_{\lambda_0} x p_{\lambda_1} y p_{\lambda_2} = \sum_{\lambda} p_\lambda x p_\lambda y p_\lambda + \sum_{\substack{\lambda_0 \geq \lambda_1 \geq \lambda_2 \\ \lambda_0 > \lambda_2}} \frac{\lambda_0 - \lambda_1}{\lambda_0 - \lambda_2} p_{\lambda_0} x p_{\lambda_1} y p_{\lambda_2}.$$

Note that the since $\psi = 1$ on the diagonal, the first sum can be expressed as a truncation of x and y to their diagonals, hence by Theorem 2.12,

$$\begin{aligned}\|\sum_\lambda p_\lambda x p_\lambda y p_\lambda\|_{S_p} &= \|(\sum_\lambda p_\lambda x p_\lambda)(\sum_\mu p_\mu y p_\mu)\|_{S_p} \leq \|\sum_\lambda p_\lambda x p_\lambda\|_{S_{p_1}} \|\sum_\mu p_\mu y p_\mu\|_{S_{p_2}} \\ &= \|T_{\mathrm{diag}} x\|_{S_{p_1}} \|T_{\mathrm{diag}} y\|_{S_{p_2}} \leq 4 C_{\triangle, p_1} C_{\triangle, p_2} \|x\|_{S_{p_1}} \|y\|_{S_{p_2}}.\end{aligned}$$

We now focus on the off-diagonal sum. Define $\varphi_s(\lambda, \mu) := (\lambda - \mu)^{is}$, $s \in \mathbb{R}$. Assume that x and y can be expressed as finite sums $x = \sum_{\lambda_0, \lambda_1} p_{\lambda_0} x p_{\lambda_1}$, $y = \sum_{\lambda_1, \lambda_2} p_{\lambda_1} y p_{\lambda_2}$; the general case then follows by continuity. Using Lemma 6.2, we can rewrite the Schur multiplier as follows.

$$\begin{aligned}M_\psi(x, y) &= \sum_{\substack{\lambda_0 \geq \lambda_1 \geq \lambda_2 \\ \lambda_0 > \lambda_2}} \frac{\lambda_0 - \lambda_1}{\lambda_0 - \lambda_2} p_{\lambda_0} x p_{\lambda_1} y p_{\lambda_2} \\ &= \frac{1}{\sqrt{2\pi}} \sum_{\substack{\lambda_0 \geq \lambda_1 \geq \lambda_2 \\ \lambda_0 > \lambda_2}} \int_{\mathbb{R}} g(s) (\lambda_0 - \lambda_1)^{is} (\lambda_0 - \lambda_2)^{-is} ds p_{\lambda_0} x p_{\lambda_1} y p_{\lambda_2} \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} g(s) \sum_{\substack{\lambda_0 \geq \lambda_1 \geq \lambda_2 \\ \lambda_0 > \lambda_2}} (\lambda_0 - \lambda_2)^{-is} p_{\lambda_0} (\sum_{\mu_0 \geq \mu_1} (\mu_0 - \mu_1)^{is} p_{\mu_0} x p_{\mu_1}) p_{\lambda_1} y p_{\lambda_2} ds \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} g(s) \sum_{\substack{\lambda_0 \geq \lambda_1 \geq \lambda_2 \\ \lambda_0 > \lambda_2}} (\lambda_0 - \lambda_2)^{-is} p_{\lambda_0} (M_{\varphi_s} x) p_{\lambda_1} y p_{\lambda_2} ds \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} g(s) \sum_{\lambda_0 > \lambda_2} (\lambda_0 - \lambda_2)^{-is} p_{\lambda_0} (M_{\varphi_s} x) y p_{\lambda_2} ds \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} g(s) M_{\varphi_{-s}}((M_{\varphi_s} x) y) ds.\end{aligned}$$

$$\begin{array}{ccc} S_p & \xrightarrow{M_{\varphi_s}} & S_p \\ \big\downarrow \iota & & \big\downarrow \iota \\ L^p(\mathbb{T}, S_p) & \xrightarrow{T_{\varphi_s}} & L^p(\mathbb{T}, S_p) \end{array}$$

Figure 6.1: Transference argument in the proof of Theorem 6.3 illustrated as a commutative diagram. Here, $\iota : x \mapsto u^*(1 \otimes x)u$ is the isometry defined in Lemma 5.1 with $u(t) = \sum_\lambda e^{i\lambda t} \otimes p_\lambda$.

It is hence sufficient to that $M_{\varphi_s} : S_p \to S_p$ is bounded for all $p \in (1, \infty)$ and $s \in \mathbb{R}$, and that the bound grows polynomially in s. We can show this using Theorem 6.1 and a linear transference argument similar to Section 5.1, see Figure 6.1. The diagram in Figure 6.1 indeed commutes, since for $x \in S_p$, $t \in [0, 2\pi]$,

$$\begin{aligned} [(\iota \circ M_{\varphi_s})x](t) &= \sum_{\lambda,\mu} e^{i(\lambda-\mu)t} \otimes p_\lambda (M_{\varphi_s} x) p_\mu = \sum_{\lambda,\mu} e^{i(\lambda-\mu)t} (\lambda - \mu)^{is} \otimes p_\lambda x p_\mu \\ &= T_{\varphi_s}\Big(\sum_{\lambda,\mu} e^{i(\lambda-\mu)\cdot} \otimes p_\lambda x p_\mu\Big)(t) = [(T_{\varphi_s} \circ \iota)x](t), \end{aligned}$$

where the third equality was shown in Lemma 5.2. Boundedness of T_{φ_s} was shown in Theorem 6.1, hence $\|M_{\varphi_s}\|_{S_p \to S_p} = \|T_{\varphi_s}\|_{L^p(\mathbb{T},S_p) \to L^p(\mathbb{T},S_p)}$ and thus for the off-diagonal operator,

$$\begin{aligned} \|M_\psi(x,y)\|_{S_p} &\leq \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |g(s)| \|M_{\varphi_{-s}}((M_{\varphi_s} x)y)\|_{S_p} ds \\ &\leq \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |g(s)| \|M_{\varphi_{-s}}\|_{S_p \to S_p} \|M_{\varphi_s}\|_{S_{p_1} \to S_{p_1}} ds \|x\|_{S_{p_1}} \|y\|_{S_{p_2}} \\ &\lesssim \hbar_{p.S_p} (\beta^+_{p,S_p})^2 \hbar_{p_1,S_{p_1}} (\beta^+_{p_1,S_{p_1}})^2 \int_{\mathbb{R}} |g(s)| (1 + |s|)^2 ds \|x\|_{S_{p_1}} \|y\|_{S_{p_2}} \\ &\lesssim \hbar_{p.S_p} (\beta^+_{p,S_p})^2 \hbar_{p_1,S_{p_1}} (\beta^+_{p_1,S_{p_1}})^2 \|x\|_{S_{p_1}} \|y\|_{S_{p_2}}. \end{aligned}$$

□

Remark 6.4. Following [23, Corollary 5.4], the L^p-estimate of the multiplier T_{φ_s} can be improved significantly in p, with

$$\|T_{\varphi_s}\|_{L^p \to L^p} \lesssim \max(p, \frac{1}{p-1})(|s|^{-1} e^{\pi|s|})^{|1/2 - 1/p|}.$$

Note however that this constant grows exponentially in s. Therefore, the integral in the proof of Theorem 6.3 would diverge if this constant was used.

To prove Theorem 1.3 as in [35] for $n = 2$, it remains to show the boundedness of the Schur multiplier with symbol $\phi_{m,h}$. In this proof, linear transference again plays a key role.

Theorem 6.5 ([35, Theorem 5.6])**.** *Let $m \in \mathbb{N}$ and $h \in C_b(\mathbb{R})$. Let $\phi_{m,h}$ be as defined in (6.2). Then the associated Schur multiplier $M_{\phi_{m,h}}$ is bounded on S_p, $p \in (1, \infty)$, with*

$$\|M_{\phi_{m,h}}\|_{S_p \to S_p} \leq c_{p,m} \|h\|_\infty.$$

Proof. Let $x \in S_p$ and $y \in S_q$ be such that $1/p + 1/q = 1$, $p, q \in (1, \infty)$, and $\|x\|_p = \|y\|_q = 1$. Assume $p > 2$; the case $1 < p < 2$ then follows by duality (see Lemma 2.17) and the case $p = 2$ from Lemma 2.16.

We shall consider $\tau(y M_{\phi_{m,h}} x)$ to estimate the norm of $M_{\phi_{m,h}}$. By Lemma 2.18, we may assume that x is off-diagonal. Furthermore, we decompose x as

$$x = T_{\triangle^{\text{off}}_{\text{upper}}} x + T_{\triangle^{\text{off}}_{\text{lower}}} x.$$

Without loss of generality, we first assume that $x = T_{\triangle^{\text{off}}_{\text{upper}}} x$ and return to this decomposition at the end of the proof; the same estimate will hold for both $T_{\triangle^{\text{off}}_{\text{upper}}} x$ and $T_{\triangle^{\text{off}}_{\text{lower}}} x$. By Lemma 2.13, we may in this case assume that y is off-diagonal lower triangular.

By [28, Theorem 4.3] we may for any $\varepsilon > 0$ decompose y as $y = ab$, where $a \in S_2$ and $b \in S_r$ with $1/r + 1/2 = 1/q$ are both lower triangular, and $1 \leq \|a\|_2\|b\|_r \leq 1 + \varepsilon$. By the permutation property of the trace (see Lemma 2.9), we have

$$\begin{aligned}
\tau(yM_{\phi_{m,h}}x) = \tau(abM_{\phi_{m,h}}x) &= \sum_{\substack{\lambda_0 \geq \lambda_1 \geq \lambda_2 \\ \lambda_0 > \lambda_2}} \tau(p_{\lambda_0}ap_{\lambda_1}bp_{\lambda_2}M_{\phi_{m,h}}x) \\
&= \sum_{\substack{\lambda_0 \geq \lambda_1 \geq \lambda_2 \\ \lambda_0 > \lambda_2}} \tau(p_{\lambda_0}ap_{\lambda_1}bp_{\lambda_2}M_{\phi_{m,h}}xp_{\lambda_0}) \\
&= \sum_{\substack{\lambda_0 \geq \lambda_1 \geq \lambda_2 \\ \lambda_0 > \lambda_2}} \sum_{\mu_1,\mu_2} \phi_{m,h}(\mu_1,\mu_2)\tau(p_{\lambda_0}ap_{\lambda_1}bp_{\lambda_2}p_{\mu_1}xp_{\mu_2}p_{\lambda_0}) \\
&= \sum_{\substack{\lambda_0 \geq \lambda_1 \geq \lambda_2 \\ \lambda_0 > \lambda_2}} \phi_{m,h}(\lambda_2,\lambda_0)\tau(p_{\lambda_0}ap_{\lambda_1}bp_{\lambda_2}xp_{\lambda_0}).
\end{aligned}$$

Following [35, Lemma 5.7], we can decompose $\phi_{m,h}(\lambda,\mu)$ for $\lambda < \mu$ as

$$\phi_{m,h}(\lambda,\mu) = (\frac{\lambda-\xi}{\lambda-\mu})^m\phi_{m,h}(\lambda,\xi) + (\frac{\xi-\mu}{\lambda-\mu})^m\phi_{m,h}(\xi,\mu) + \sum_{l=1}^{m-1} C_{m-1}^{l-1}(\frac{\lambda-\xi}{\lambda-\mu})^{m-l}(\frac{\xi-\mu}{\lambda-\mu})^l\phi_{l,h}(\xi,\mu),$$

where C_{m-1}^{l-1} is a scalar constant and $\lambda \leq \xi \leq \mu$. Hence we have

$$\begin{aligned}
\tau(yM_{\phi_{m,h}}x) = &\sum_{\substack{\lambda_0 \geq \lambda_1 \geq \lambda_2 \\ \lambda_0 > \lambda_2}} (\frac{\lambda_1-\lambda_2}{\lambda_0-\lambda_2})^m\phi_{m,h}(\lambda_2,\lambda_1)\tau(p_{\lambda_0}ap_{\lambda_1}bp_{\lambda_2}xp_{\lambda_0}) \\
&+ \sum_{\substack{\lambda_0 \geq \lambda_1 \geq \lambda_2 \\ \lambda_0 > \lambda_2}} (\frac{\lambda_0-\lambda_1}{\lambda_0-\lambda_2})^m\phi_{m,h}(\lambda_1,\lambda_0)\tau(p_{\lambda_0}ap_{\lambda_1}bp_{\lambda_2}xp_{\lambda_0}) \\
&+ \sum_{\substack{\lambda_0 \geq \lambda_1 \geq \lambda_2 \\ \lambda_0 > \lambda_2}} \sum_{l=1}^{m-1} (\frac{\lambda_0-\lambda_1}{\lambda_0-\lambda_2})^l(\frac{\lambda_1-\lambda_2}{\lambda_0-\lambda_2})^{m-l}\phi_{l,h}(\lambda_1,\lambda_0)\tau(p_{\lambda_0}ap_{\lambda_1}bp_{\lambda_2}xp_{\lambda_0}).
\end{aligned}$$

Define $\widetilde{\phi}_{m,h}(\lambda,\mu) := \phi_{m,h}(\mu,\lambda)$. By using Lemma 6.2 with $\varphi_s(\xi) = \xi^{is}$, this yields

$$\begin{aligned}
\tau(yM_{\phi_{m,h}}x) = &\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} g(s)\tau(a((M_{\varphi_{ms}}M_{\widetilde{\phi}_{m,h}}b)M_{\varphi_{-ms}}x)ds \\
&+ \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} g(s)\tau((M_{\varphi_{ms}}M_{\widetilde{\phi}_{m,h}}a)bM_{\varphi_{-ms}}x)ds \\
&+ \sum_{l=1}^{m-1}\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} g(s)\tau((M_{\varphi_{(m-l)s}}a)((M_{\varphi_{ls}}M_{\widetilde{\phi}_{l,h}}b)M_{\varphi_{-ms}}x)ds.
\end{aligned}$$

We apply Theorem 6.3 to estimate the $M_{\varphi_{ms}}$ terms and obtain

$$\begin{aligned}
|\tau(yM_{\phi_{m,h}}x)| \leq &\, c_2c_p\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} g(s)(1+|ms|)^2ds\|M_{\widetilde{\phi}_{m,h}}a\|_{S_2}\|b\|_{S_r}\|x\|_{S_p} \\
&+ c_rc_p\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} g(s)(1+|ms|)^2ds\|a\|_{S_2}\|M_{\widetilde{\phi}_{m,h}}b\|_{S_r}\|x\|_{S_p} \\
&+ \sum_{l=1}^{m-1} c_2c_rc_p\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} g(s)(1+|(m-l)s|)(1+|ls|)(1+|ms|)ds\|a\|_{S_2}\|M_{\widetilde{\phi}_{l,h}}b\|_{S_r}\|x\|_{S_p} \\
&\lesssim c_p(c_2\|M_{\widetilde{\phi}_{m,h}}a\|_{S_2}\|b\|_{S_r} + c_r\|a\|_{S_2}\|M_{\widetilde{\phi}_{m,h}}b\|_{S_r} + \sum_{l=1}^{m-1} c_2c_r\|a\|_{S_2}\|M_{\widetilde{\phi}_{l,h}}b\|_{S_r})\|x\|_{S_p},
\end{aligned}$$

where c_α denotes the constant from Theorem 6.1, $\alpha = 2, r, p$. Note that we may use the estimate $c_2 \leq 1$ by Lemma 2.16. This lemma also implies

$$\|M_{\widetilde{\phi}_{m,h}} a\|_{S_2} \leq \|\widetilde{\phi}_{m,h}\|_\infty \|a\|_{S_2}, \tag{6.3}$$

hence by definition of $\widetilde{\phi}_{m,h}$ we have

$$\begin{aligned}|\tau(yM_{\phi_{m,h}}x)| &\lesssim c_p(\|h\|_\infty \|a\|_{S_2}\|b\|_{S_r} + c_r\|a\|_{S_2}\|M_{\widetilde{\phi}_{m,h}} b\|_{S_r} + \sum_{l=1}^{m-1} c_r\|a\|_{S_2}\|M_{\widetilde{\phi}_{l,h}} b\|_{S_r})\|x\|_{S_p} \\ &\lesssim c_p(\|h\|_\infty + c_r\|M_{\widetilde{\phi}_{m,h}}\|_{S_r\to S_r} + \sum_{l=1}^{m-1} c_r\|M_{\widetilde{\phi}_{l,h}}\|_{S_r\to S_r})(1+\varepsilon)\|x\|_{S_p}.\end{aligned}$$

Note that by Lemma 2.15, $\|M_{\widetilde{\phi}_{m,h}}\|_{S_p\to S_p} = \|M_{\phi_{m,h}}\|_{S_p\to S_p}$ for all $p \in (1,\infty)$. The same estimate as above follows for $x = T_{\triangle^{\text{off}}_{\text{lower}}} x$, hence in total we have for general $x \in S_p$ (letting $\varepsilon \to 0$)

$$\begin{aligned}|\tau(yM_{\phi_{m,h}}x)| &\leq |\tau(yM_{\phi_{m,h}}T_{\triangle^{\text{off}}_{\text{upper}}}x)| + |\tau(yM_{\phi_{m,h}}T_{\triangle^{\text{off}}_{\text{lower}}}x)| + |\tau(yM_{\phi_{m,h}}T_{\text{diag}}x)| \\ &\lesssim C_{\triangle,p}c_p(\|h\|_\infty + c_r\|M_{\phi_{m,h}}\|_{S_r\to S_r} + \sum_{l=1}^{m-1} c_r\|M_{\phi_{l,h}}\|_{S_r\to S_r})\|x\|_{S_p}.\end{aligned}$$

The statement of the theorem now follows by induction on m. For the purpose of keeping track of the constants in our particular case, we restrict the following discussion to the cases $m = 1, 2$ appearing in (6.1); the proof for higher m follows the same inductive approach.

Case 1: $m = 1$. We now have the estimate

$$|\tau(yM_{\phi_{1,h}}x)| \lesssim C_{\triangle,p}c_p(\|h\|_\infty + c_r\|M_{\phi_{1,h}}\|_{S_r\to S_r})\|x\|_{S_p}.$$

We first let $p > 4$, where $1/p + 1/r = 1/2$ implies $2 < r < p$ by construction. Thus there exists a $\theta \in (0,1)$ such that $\theta/p + (1-\theta)/2 = 1/r$. By complex interpolation (see Lemma 2.7) and (6.3), it follows that

$$\|M_{\phi_{1,h}}\|_{S_r\to S_r} \leq \|M_{\phi_{1,h}}\|^\theta_{S_p\to S_p}\|M_{\phi_{1,h}}\|^{1-\theta}_{S_2\to S_2} \leq \|M_{\phi_{1,h}}\|^\theta_{S_p\to S_p}\|h\|^{1-\theta}_\infty. \tag{6.4}$$

We may assume that $\|h\|_\infty \leq 1$ (the general case then follows by scaling), hence

$$\begin{aligned}\|M_{\phi_{1,h}}\|_{S_p\to S_p} &= \sup_{\|x\|_{S_p}=1}\ \sup_{\|y\|_{S_q}=1} |\tau(yM_{\phi_{1,h}}x)| \\ &\leq \underbrace{CC_{\triangle,p}c_p\max(c_r,1)}_{=:\tilde{C}}(1 + \|M_{\phi_{1,h}}\|^\theta_{S_p\to S_p}).\end{aligned}$$

From this, our statement follows from the following observation: If $\lambda \leq 1$ then $1 + \lambda^\theta \leq 2$, thus $\tilde{C} \geq \frac{\lambda}{1+\lambda^\theta} \geq \frac{\lambda}{2}$. If $\lambda \geq 1$, then $\tilde{C} \geq \frac{\lambda}{1+\lambda^\theta} \geq \frac{\lambda}{2\lambda^\theta} = \frac{\lambda^{1-\theta}}{2}$. Hence we conclude for $p > 4$,

$$\|M_{\phi_{1,h}}\|_{S_p\to S_p} \lesssim \max(C_{\triangle,p}c_p\max(c_r,1), (C_{\triangle,p}c_p\max(c_r,1))^{1/1-\theta}) =: C_{p,1}.$$

Boundedness for $2 < p < 4$ follows from (6.4): Let r' such that $1/p + 1/r' = 1/2$, then

$$\|M_{\phi_{1,h}}\|_{S_p\to S_p} \leq \max(1, \|M_{\phi_{1,h}}\|_{S_{r'}\to S_{r'}}).$$

Boundedness for $p = 4$ may thus be concluded by interpolation.

Case 2: $m = 2$. Using the previous case and assuming $\|h\|_\infty \leq 1$, we now have the estimate

$$|\tau(yM_{\phi_{2,h}}x)| \lesssim C_{\triangle,p}c_p(1 + c_r\|M_{\phi_{2,h}}\|_{S_r\to S_r} + c_r\|M_{\phi_{1,h}}\|_{S_r\to S_r})\|x\|_{S_p}.$$

As in the previous case, we use interpolation to show

$$\|M_{\phi_{2,h}}\|_{S_p\to S_p} \lesssim C_{\triangle,p}c_p(1 + c_r\|M_{\phi_{1,h}}\|_{S_r\to S_r} + c_r\|M_{\phi_{2,h}}\|^{\theta}_{S_p\to S_p})$$

for large p, and thus for $p > 4$ we have

$$\|M_{\phi_{2,h}}\|_{S_p\to S_p} \lesssim \max(C_{\triangle,p}c_p \max(c_r, 1 + c_r\|M_{\phi_{1,h}}\|_{S_r\to S_r}), \\ (C_{\triangle,p}c_p \max(c_r, 1 + c_r\|M_{\phi_{1,h}}\|_{S_r\to S_r}))^{1/1-\theta}).$$

Boundedness for $2 < p \leq 4$ follows as in the previous case. □

6.2. Alternative proof based on bilinear transference

In this section, we collect the p-dependent constants of our proof method from Sections 3–5. We restrict our attention to the constant in Theorem 1.4, since the p-dependence of the constants of the other theorems used in our proof (Theorem 2.12, Theorem 1.5, and Lemma 5.3) has already been recorded in their respective statements.

Key to the proof of Theorem 1.4 is the decomposition of Calderón-Zygmund operators (see Definition 2.33) into dyadic model operators. Recall from Section 2.8 that a bilinear Calderón-Zygmund operator T can be decomposed as

$$\langle T(f,g), h\rangle = C_T\mathbb{E}_\omega \sum_{k\in\mathbb{N}_0^3}\sum_u 2^{-\max_i k_i\alpha/2}\langle U^k_{\mathcal{D}_\omega,u}(f,g), h\rangle, \tag{6.5}$$

where $f, g, h \in L^\infty_c(\mathbb{R})$, C_T is a constant depending only on T and $U^k_{\mathcal{D}_\omega,u}$ denote dyadic model operators. Following [12], $U^k_{\mathcal{D}_\omega,u}$ denotes a sum of cancellative dyadic shifts and their adjoints for $\max_i k_i > 0$, whereas for $\max_i k_i = 0$ it either denotes a cancellative dyadic shift or a bilinear paraproduct.

For $1/p_1 + 1/p_2 + 1/p_3 = 1$, we can extend the trilinear form (6.5) associated with T to a trilinear form $(L^\infty_c(\mathbb{R}) \otimes S_{p_1}) \times (L^\infty_c(\mathbb{R}) \otimes S_{p_2}) \times (L^\infty_c(\mathbb{R}) \otimes S_{p_3}) \to \mathbb{C}$ by setting

$$\langle T_{\text{ext}}(\sum_{j=1}^N f_j \otimes x_j, \sum_{k=1}^M g_k \otimes y_k), \sum_{l=1}^L h_l \otimes z_l\rangle \;:=\; \sum_{j=1}^N\sum_{k=1}^M\sum_{l=1}^L \langle T(f_j, g_k), h_l\rangle\tau(x_j y_k z_l).$$

To show that for $1/p_1 + 1/p_2 = 1/p$, T_{ext} is a bounded map $L^{p_1}(\mathbb{R}, S_{p_1}) \times L^{p_2}(\mathbb{R}, S_{p_2}) \to L^p(\mathbb{R}, S_p)$, we need a uniform bound of the form

$$|\langle T_{\text{ext}}(f,g), h\rangle| \lesssim \|f\|_{L^{p_1}(\mathbb{R},S_{p_1})}\|g\|_{L^{p_2}(\mathbb{R},S_{p_2})}\|h\|_{L^{p_3}(\mathbb{R},S_{p_3})}$$

for all $f \in L^\infty_c(\mathbb{R})\otimes S_{p_1}$, $g \in L^\infty_c(\mathbb{R})\otimes S_{p_2}$, $h \in L^\infty_c(\mathbb{R})\otimes S_{p_3}$. By the decomposition (6.5), it is sufficient to show such a bound for all dyadic model operators. In fact, we will show that the decomposition of our Calderón-Zygmund operator from Section 5.3 only consists of dyadic shifts and does not contain paraproducts. However, we will give the boundedness proof for bilinear paraproducts regardless, as this demonstrates how vanishing paraproducts impact the p-dependence of the boundedness constant of Theorem 1.4.

The following theorems are a collection of estimates that are used in the boundedness proofs. Many of them rely on the theory of UMD spaces, see Section 2.6 for a very brief overview.

Theorem 6.6 (Decoupling Inequality [18, Theorem 6])**.** *Let $p \in (1,\infty)$, let X be a UMD space with UMD constant $\beta_{p,X}$, and let $\mathcal{D}$ be a dyadic grid. Further define the following:*

- $\mathcal{D}_{j,k} := \{Q \in \mathcal{D} \mid |Q| = 2^{m(k+1)+j}$ *for some* $m \in \mathbb{Z}\}$ *for* $j, k \in \mathbb{Z}$ *fixed,*
- $\mathcal{V}_Q := (Q, \operatorname{Leb}(Q), \lambda_Q)$ *is a probability space, where* $\operatorname{Leb}(Q)$ *denotes the Lebesgue measurable subsets of* Q *and* λ_Q *is the normalised restriction of the Lebesgue measure to* Q*,*
- $\mathcal{V} := \prod_{Q\in\mathcal{D}} \mathcal{V}_Q$ *is a product probability space with measure* ν *and elements* $y = (y_Q)_{Q\in\mathcal{D}}$*.*

Let $(\varepsilon_Q)_{Q\in\mathcal{D}}$ be a Rademacher sequence. Let $(f_Q)_{Q\in\mathcal{D}}$ be a sequence of functions such that for all $Q \in \mathcal{D}$, f_Q is 1) supported on Q, 2) constant on every $Q' \in \mathrm{ch}_{\mathcal{D}}(Q)$, and 3) $\langle f_Q\rangle_Q = 0$ holds. Then

$$\frac{1}{\beta^p_{p,S_p}}\mathbb{E}\int_{\mathbb{R}}\int_{\mathcal{V}}\|\sum_{Q\in\mathcal{D}_{j,k}}\varepsilon_Q 1_Q(x) f_Q(y_Q)\|^p_{S_p}d\nu(y)dx$$
$$\leq \int_{\mathbb{R}}\|\sum_{Q\in\mathcal{D}_{j,k}} f_Q(x)\|^p_{S_p}dx \leq \beta^p_{p,S_p}\mathbb{E}\int_{\mathbb{R}}\int_{\mathcal{V}}\|\sum_{Q\in\mathcal{D}_{j,k}}\varepsilon_Q 1_Q(x) f_Q(y_Q)\|^p_{S_p}d\nu(y)dx. \quad (6.6)$$

This inequality also holds when replacing $\mathcal{D}_{j,k}$ with $\mathcal{D}$.

Following [12], the decoupling inequality is applied in this section with

$$f_Q := \Delta^l_Q f := \sum_{\substack{R\in\mathcal{D}\\ R^{(l)}=Q}} D_R f = \sum_{\substack{R\in\mathcal{D}\\ R^{(l)}=Q}}\sum_{R'\in\mathrm{ch}_{\mathcal{D}}(R)}(\langle f\rangle_{R'} - \langle f\rangle_R)1_{R'} \quad (6.7)$$

for $0 \leq l \leq k$. For the relevant notation regarding dyadic cubes, see Section 2.5. Note that $(\Delta^l_Q f)_{Q\in\mathcal{D}_{j,k}}$ indeed satisfies the required properties of Theorem 6.6:

1. By construction, $R' \subset Q$ for all R' considered in (6.7), hence
$$\mathrm{supp}\,\Delta^l_Q f \subseteq \bigcup_{\substack{R\in\mathcal{D}\\ R^{(l)}=Q}}\bigcup_{R'\in\mathrm{ch}_{\mathcal{D}}(R)} R' \subseteq Q.$$
2. Let $Q' \in \mathrm{ch}_{\mathcal{D}_{j,k}}(Q)$. Note that here we are considering the children of Q in $\mathcal{D}_{j,k}$, hence if $|Q| = 2^{m(k+1)+j}$ for some $m \in \mathbb{Z}$, then $|Q'| = 2^{(m-1)(k+1)+j}$. Note further that by construction $R'^{(l+1)} = Q$ for all R' considered in (6.7), i.e.
$$|R'| = 2^{-(l+1)}|Q| = 2^{m(k+1)+j-(l+1)}.$$
Therefore, $R' \in \mathcal{D}_{j,k}$ if and only if $l = k$ (in this case, $R' \in \mathrm{ch}_{\mathcal{D}_{j,k}}(Q)$). In either case, there exists exactly one R'_* such that $R'^{(l+1)}_* = Q$ and $Q' \subseteq R'_*$. Hence by construction,
$$(\Delta^l_Q f)1_{Q'} = (\langle f\rangle_{R'_*} - \langle f\rangle_{R_*})1_{R'_*\cap Q'} = (\langle f\rangle_{R'_*} - \langle f\rangle_{R_*})1_{Q'},$$
thus $\Delta^l_Q f$ is constant on Q'.
3. Note that $D_R f = \langle f, h_R\rangle h_R$ by Lemma 2.21. Hence,
$$\langle\Delta^l_Q f\rangle_Q = \sum_{\substack{R\in\mathcal{D}\\ R^{(l)}=Q}}\langle f, h_R\rangle\langle h_R\rangle_Q = 0,$$
since $R \subseteq Q$ and the Haar functions h_R are cancellative.

Theorem 6.7 (Kahane-Khintchine inequality [20, Theorem 3.2.23])**.** *Let $(\varepsilon_n)_n$ be a Rademacher sequence on a probability space Ω, and let X be a Banach space. Then for all $p, q \in (0,\infty)$, there exists $\kappa_{p,q} < \infty$ such that*

$$\|\sum_{n=1}^N \varepsilon_n x_n\|_{L^p(\Omega,X)} \leq \kappa_{p,q}\|\sum_{n=1}^N \varepsilon_n x_n\|_{L^q(\Omega,X)}$$

.

Remark 6.8. Relevant in this section is the constant $\kappa_{2,q}$, where the optimal constant has been determined in [17] as $\kappa_{2,q} = 1$ for $q \in [2,\infty)$. For $q \in (0,2)$, it follows from more general estimates that $\kappa_{2,q} \leq \sqrt{\frac{2}{q}}$, see [21, Theorem 6.6.5]. In particular, $\kappa_{2,q} \leq \sqrt{2}$ for all $q \in (1,\infty)$.

Theorem 6.9 (Kahane contraction principle [20, Proposition 3.2.10])**.** *Let $(\varepsilon_n)_n$ be a Rademacher sequence on a probability space Ω, $(a_n)_n$ a finite scalar sequence, and $(x_n)_n$ a finite sequence in a Banach space X. Let $1 \le p \le \infty$. Then*

$$\|\sum_{n=1}^{N} a_n \varepsilon_n x_n\|_{L^p(\Omega;X)} \le \max_n |a_n| \|\sum_{n=1}^{N} \varepsilon_n x_n\|_{L^p(\Omega;X)}.$$

The following theorem has been specialised to our dyadic setting.

Theorem 6.10 (Stein's inequality, adapted from [20, Theorem 4.2.23])**.** *Let X be a UMD space, let $(f_Q \in L^1_{\mathrm{loc}}(X))_Q$ be such that* $\mathrm{supp} f_Q \subseteq Q$ *and such that the sum below is finite, and $p \in (1, \infty)$. Then*

$$\mathbb{E}\|\sum_{Q\in\mathcal{D}} \varepsilon_Q \langle f_Q\rangle_Q 1_Q\|_{L^p(\mathbb{R},X)} \le \beta^+_{p,X} \mathbb{E}\|\sum_{Q\in\mathcal{D}} \varepsilon_Q f_Q\|_{L^p(\mathbb{R},X)}.$$

Lemma 6.11 (Special case of [12, Lemma 4.1])**.** Let $K \in \mathbb{N}$ and let $p_1, p_2, p \in (1, \infty)$ be such that $1/p_1 + 1/p_2 = 1/p$. For each $k = 1, \dots, K$, let a_k be a scalar such that $|a_k| \le 1$, and for each $j = 1, 2$ let $x_{j,k} \in S_{p_j}$. Then

$$\|\sum_{k=1}^{K} a_k x_{1,k} x_{2,k}\|_{S_p} \le \prod_{j=1}^{2} \|(x_{j,k})_{k=1}^K\|_{\mathrm{Rad}(S_{p_j})}. \tag{6.8}$$

Here,

$$\|(x_k)_{k=1}^K\|_{\mathrm{Rad}(S_p)} := \left(\mathbb{E}\|\sum_{k=1}^{K} \varepsilon_k x_k\|^2_{S_p}\right)^{1/2},$$

where $(\varepsilon_k)_k$ denotes a Rademacher sequence.

Let us further recall Jensen's inequality.

Lemma 6.12 (Jensen's inequality [20, Proposition 1.2.11])**.** Let (Ω, μ) be a probability space and let X be a Banach space. Let $f : \Omega \to X$ be a Bochner integrable function and let $\phi : X \to \mathbb{R}$ be a convex and lower semi-continuous function. If $\phi \circ f$ is integrable, then

$$\phi\left(\int_\Omega f d\mu\right) \le \int_\Omega \phi \circ f d\mu.$$

6.2.1. Dyadic shifts

We can now prove the boundedness of bilinear dyadic shifts, following [12, Section 4].

Theorem 6.13. *Let $p_1, p_2, p_3 \in (1, \infty)$ such that $1/p_1 + 1/p_2 + 1/p_3 = 1$. Let S^k be a bilinear dyadic shift of complexity k as in Definition 2.34 and let $f_j \in L^\infty_c(\mathbb{R}, S_{p_j})$, $j = 1, 2, 3$. It then holds that*

$$|\langle S^k(f_1, f_2), f_3\rangle| \lesssim C^{\mathrm{shift}}_{p_1,p_2} \prod_{j=1}^{3} \|f_j\|_{L^{p_j}(\mathbb{R},S_{p_j})}. \tag{6.9}$$

Proof. The dyadic shift is first rewritten as

$$\langle S^k(f_1, f_2), f_3\rangle = \sum_{i=0}^{\kappa} \langle S^k_i(f_1, f_2), f_3\rangle, \tag{6.10}$$

$$\langle S^k_i(f_1, f_2), f_3\rangle = \sum_{K\in\mathcal{D}_{i,\kappa}} \sum_{\substack{L_1,L_2,L_3\in\mathcal{D}\\ L_j^{(l_j)}=K}} b_{L_1,L_2,L_3,K} \tau\left(\prod_{j=1}^{3} \langle f_j, h'_{L_j}\rangle\right), \tag{6.11}$$

$$b_{L_1,L_2,L_3,K} = \sum_{\substack{Q_1,Q_2,Q_3\in\mathcal{D}\\ Q_j^{(k_j-l_j)}=L_j}} a_{Q_1,Q_2,Q_3,K} \prod_{j=1}^{3} \frac{|Q_j|^{1/2}}{|L_j|^{1/2}}, \tag{6.12}$$

where $0 \leq l_j \leq k_j$ and $\kappa = \max k_j$. This is a new shift operator with $h'_{L_j} \in \{h^0_{L_j}, h_{L_j}\}$ such that there may be more than two indices j such that their associated Haar functions are cancellative, whereas in Definition 2.34, the Haar functions are cancellative for exactly two indices. Furthermore, the construction is such that if h'_{L_j} is not cancellative, then $l_j = 0$. For details on how to construct this new shift, see [12].

The proof now proceeds as follows. First, boundedness is shown in the case where all Haar functions h'_{L_j} are cancellative. In the second case, where not all Haar functions are cancellative, the fact $h'_{L_j} = h^0_{L_j} \Rightarrow l_j = 0$ allows us to reduce the trilinear form (6.11) to a bilinear form with only cancellative Haar functions. For this new bilinear form, boundedness follows by the same proof method as in the first case.

Case 1. Let $0 \leq i \leq \kappa$ be such that all associated Haar functions in (6.11) are cancellative. Note that for $L_3^{(l_3)} \in \mathcal{D}_{i,\kappa}$, we have

$$\sum_{K \in \mathcal{D}_{i,\kappa}} \Delta_K^{l_3} h_{L_3} = \sum_{K \in \mathcal{D}_{i,\kappa}} \sum_{\substack{L \in \mathcal{D} \\ L^{(l_3)} = K}} D_L h_{L_3} = \sum_{K \in \mathcal{D}_{i,\kappa}} \sum_{\substack{L \in \mathcal{D} \\ L^{(l_3)} = K}} \langle h_{L_3}, h_L \rangle h_L = h_{L_3}$$

by Lemma 2.21 and the orthogonality of the Haar functions. Using the decoupling inequality from Theorem 6.6, we thus have

$$\|S_i^k(f_1, f_2)\|_{L^p(\mathbb{R}, S_p)} = \Big\| \sum_{K \in \mathcal{D}_{i,\kappa}} \sum_{\substack{L_1, L_2, L_3 \in \mathcal{D} \\ L_j^{(l_j)} = K}} b_{L_1,L_2,L_3,K} \langle f_1, h_{L_1} \rangle \langle f_2, h_{L_2} \rangle h_{L_3} \Big\|_{L^p(\mathbb{R}, S_p)}$$

$$\leq \beta_{p,S_p} \left(\mathbb{E} \int_{\mathbb{R}} \int_{\mathcal{V}} \Big\| \sum_{K \in \mathcal{D}_{i,\kappa}} \varepsilon_K 1_K(x) \sum_{\substack{L_1, L_2, L_3 \in \mathcal{D} \\ L_j^{(l_j)} = K}} b_{L_1,L_2,L_3,K} \prod_{j=1}^{2} \langle f_j, h_{L_j} \rangle h_{L_3}(y_K) \Big\|_{S_p}^p d\nu(y) dx \right)^{1/p} .$$

We can rewrite the inner sum in the integral by using $\langle f_j, h_{L_j} \rangle = \langle \Delta_K^{l_j} f_j, h_{L_j} \rangle$. Indeed,

$$\langle \Delta_K^{l_j} f_j, h_{L_j} \rangle = \sum_{\substack{L \in \mathcal{D} \\ L^{(l_j)} = K}} \langle D_L f_j, h_{L_j} \rangle = \sum_{\substack{L \in \mathcal{D} \\ L^{(l_j)} = K}} \langle f_j, h_L \rangle \langle h_L, h_{L_j} \rangle = \langle f_j, h_{L_j} \rangle.$$

Hence we can write

$$\begin{aligned}
&\sum_{\substack{L_1, L_2, L_3 \in \mathcal{D} \\ L_j^{(l_j)} = K}} b_{L_1,L_2,L_3,K} \prod_{j=1}^{2} \langle f_j, h_{L_j} \rangle h_{L_3}(y_K) \\
&= \sum_{\substack{L_1, L_2, L_3 \in \mathcal{D} \\ L_j^{(l_j)} = K}} b_{L_1,L_2,L_3,K} \prod_{j=1}^{2} \langle \Delta_K^{l_j} f_j, h_{L_j} \rangle h_{L_3}(y_K) \\
&= \int_{K^2} \sum_{\substack{L_1, L_2, L_3 \in \mathcal{D} \\ L_j^{(l_j)} = K}} b_{L_1,L_2,L_3,K} \prod_{j=1}^{2} \Delta_K^{l_j} f_j(z_j) h_{L_j}(z_j) h_{L_3}(y_K) dz \\
&= \frac{1}{|K|^2} \int_{K^2} b_K(y_K, z) \prod_{j=1}^{2} \Delta_K^{l_j} f_j(z_j) dz = \int_{\mathcal{V}^2} b_K(y_K, z_K) \prod_{j=1}^{2} \Delta_K^{l_j} f_j(z_{j,K}) d\nu(z),
\end{aligned}$$

where $\mathcal{V}$ and ν are as defined in Theorem 6.6, and

$$b_K(y_K, z_K) = |K|^2 \sum_{\substack{L_1, L_2, L_3 \in \mathcal{D} \\ L_j^{(l_j)} = K}} b_{L_1,L_2,L_3,K} \prod_{j=1}^{2} h_{L_j}(z_{j,K}) h_{L_3}(y_K).$$

Since $\mathcal{V}^2$ is a probability space, we can use monotonicity of the integral and Jensen's inequality (see Lemma 6.12) to show

$$\begin{aligned}
&\|S_i^k(f_1,f_2)\|_{L^p(\mathbb{R},S_p)} \\
&\le \beta_{p,S_p}\left(\mathbb{E}\int_{\mathbb{R}}\int_{\mathcal{V}}\|\int_{\mathcal{V}^2}\sum_{K\in\mathcal{D}_{i,\kappa}}\varepsilon_K 1_K(x)b_K(y_K,z_K)\prod_{j=1}^{2}\Delta_K^{l_j}f_j(z_{j,K})d\nu(z)\|_{S_p}^p d\nu(y)dx\right)^{1/p} \\
&\le \beta_{p,S_p}\left(\mathbb{E}\int_{\mathbb{R}}\int_{\mathcal{V}}\left(\int_{\mathcal{V}^2}\|\sum_{K\in\mathcal{D}_{i,\kappa}}\varepsilon_K 1_K(x)b_K(y_K,z_K)\prod_{j=1}^{2}\Delta_K^{l_j}f_j(z_{j,K})\|_{S_p}d\nu(z)\right)^p d\nu(y)dx\right)^{1/p} \\
&\le \beta_{p,S_p}\left(\mathbb{E}\int_{\mathbb{R}}\int_{\mathcal{V}}\int_{\mathcal{V}^2}\|\sum_{K\in\mathcal{D}_{i,\kappa}}\varepsilon_K 1_K(x)b_K(y_K,z_K)\prod_{j=1}^{2}\Delta_K^{l_j}f_j(z_{j,K})\|_{S_p}^p d\nu(z)d\nu(y)dx\right)^{1/p}.
\end{aligned}$$

Note that by construction (see Definition 2.34),

$$\begin{aligned}
|b_K(y_K,z_K)| &\le |K|^2\sum_{\substack{L_1,L_2,L_3\in\mathcal{D}\\ L_j^{(l_j)}=K}}|b_{L_1,L_2,L_3,K}|\prod_{j=1}^{2}|h_{L_j}(z_{j,K})||h_{L_3}(y_K)| \\
&\le |K|^2\sum_{\substack{L_1,L_2,L_3\in\mathcal{D}\\ L_j^{(l_j)}=K}}\sum_{\substack{Q_1,Q_2,Q_3\in\mathcal{D}\\ Q_j^{(k_j-l_j)}=L_j}}|a_{Q_1,Q_2,Q_3,K}|\prod_{j=1}^{3}\frac{|Q_j|^{1/2}}{|L_j|^{1/2}}\frac{1_{L_1}(z_{1,K})}{|L_1|^{1/2}}\frac{1_{L_2}(z_{2,K})}{|L_2|^{1/2}}\frac{1_{L_3}(y_K)}{|L_3|^{1/2}} \\
&\le \sum_{\substack{L_1,L_2,L_3\in\mathcal{D}\\ L_j^{(l_j)}=K}}\sum_{\substack{Q_1,Q_2,Q_3\in\mathcal{D}\\ Q_j^{(k_j-l_j)}=L_j}}\prod_{l=1}^{3}|Q_l|^{1/2}\prod_{j=1}^{3}\frac{|Q_j|^{1/2}}{|L_j|}1_{L_1}(z_{1,K})1_{L_2}(z_{2,K})1_{L_3}(y_K) \\
&= \sum_{\substack{L_1,L_2,L_3\in\mathcal{D}\\ L_j^{(l_j)}=K}}\sum_{\substack{Q_1,Q_2,Q_3\in\mathcal{D}\\ Q_j^{(k_j-l_j)}=L_j}}\prod_{j=1}^{3}\frac{|Q_j|}{|L_j|}1_{L_1}(z_{1,K})1_{L_2}(z_{2,K})1_{L_3}(y_K) \\
&= \sum_{\substack{L_1,L_2,L_3\in\mathcal{D}\\ L_j^{(l_j)}=K}}\sum_{\substack{Q_1,Q_2,Q_3\in\mathcal{D}\\ Q_j^{(k_j-l_j)}=L_j}}\prod_{j=1}^{3}2^{l_j-k_j}1_{L_1}(z_{1,K})1_{L_2}(z_{2,K})1_{L_3}(y_K) \\
&= \sum_{\substack{L_1,L_2,L_3\in\mathcal{D}\\ L_j^{(l_j)}=K}}\prod_{j=1}^{3}2^{k_j-l_j}2^{l_j-k_j}1_{L_1}(z_{1,K})1_{L_2}(z_{2,K})1_{L_3}(y_K) \\
&= \sum_{\substack{L_1,L_2,L_3\in\mathcal{D}\\ L_j^{(l_j)}=K}}1_{L_1}(z_{1,K})1_{L_2}(z_{2,K})1_{L_3}(y_K) \\
&= 1_K(z_{1,K})1_K(z_{2,K})1_K(y_K) \\
&\le 1.
\end{aligned}$$

We can now finish the proof of this case with

$$
\begin{aligned}
&\|S_i^k(f_1,f_2)\|_{L^p(\mathbb{R},S_p)} \\
&\le \beta_{p,S_p}\left(\mathbb{E}\int_{\mathbb{R}}\int_{\mathcal{V}}\int_{\mathcal{V}^2}\|\sum_{K\in\mathcal{D}_{i,\kappa}}\varepsilon_K 1_K(x) b_K(y_K,z_K)\prod_{j=1}^2 \Delta_K^{l_j} f_j(z_{j,K})\|_{S_p}^p d\nu(z)d\nu(y)dx\right)^{1/p} \\
&\le \beta_{p,S_p}\left(\int_{\mathbb{R}}\int_{\mathcal{V}}\int_{\mathcal{V}^2}\prod_{j=1}^2 \|(1_K(x)\Delta_K^{l_j} f_j(z_{j,K}))_{K\in\mathcal{D}_{i,\kappa}}\|_{\mathrm{Rad}(S_{p_j})}^p d\nu(z)d\nu(y)dx\right)^{1/p} && \text{Lemma 6.11} \\
&= \beta_{p,S_p}\left(\int_{\mathbb{R}}\int_{\mathcal{V}^2}\prod_{j=1}^2 \|(1_K(x)\Delta_K^{l_j} f_j(z_{j,K}))_{K\in\mathcal{D}_{i,\kappa}}\|_{\mathrm{Rad}(S_{p_j})}^p d\nu(z)dx\right)^{1/p} && \mathcal{V}\text{ probability space} \\
&\le \beta_{p,S_p}\prod_{j=1}^2\left(\int_{\mathbb{R}}\int_{\mathcal{V}^2} \|(1_K(x)\Delta_K^{l_j} f_j(z_{j,K}))_{K\in\mathcal{D}_{i,\kappa}}\|_{\mathrm{Rad}(S_{p_j})}^{p_j} d\nu(z)dx\right)^{1/p_j} && \text{Hölder's inequality} \\
&= \beta_{p,S_p}\prod_{j=1}^2\left(\int_{\mathbb{R}}\int_{\mathcal{V}^2} (\mathbb{E}\|\sum_{K\in\mathcal{D}_{i,\kappa}}\varepsilon_K 1_K(x)\Delta_K^{l_j} f_j(z_{j,K})\|_{S_{p_j}}^2)^{p_j/2} d\nu(z)dx\right)^{1/p_j} && \text{definition } \|\cdot\|_{\mathrm{Rad}} \\
&\le \beta_{p,S_p}\prod_{j=1}^2\kappa_{2,p_j}\left(\int_{\mathbb{R}}\int_{\mathcal{V}^2} \mathbb{E}\|\sum_{K\in\mathcal{D}_{i,\kappa}}\varepsilon_K 1_K(x)\Delta_K^{l_j} f_j(z_{j,K})\|_{S_{p_j}}^{p_j} d\nu(z)dx\right)^{1/p_j} && \text{Theorem 6.7} \\
&= \beta_{p,S_p}\prod_{j=1}^2\kappa_{2,p_j}\left(\mathbb{E}\int_{\mathbb{R}}\int_{\mathcal{V}^2} \|\sum_{K\in\mathcal{D}_{i,\kappa}}\varepsilon_K 1_K(x)\Delta_K^{l_j} f_j(z_{j,K})\|_{S_{p_j}}^{p_j} d\nu(z)dx\right)^{1/p_j} && \text{Fubini's theorem} \\
&\le \beta_{p,S_p}\prod_{j=1}^2\kappa_{2,p_j}\beta_{p_j,S_{p_j}}\|f_j\|_{L^{p_j}(\mathbb{R},S_{p_j})}. && \text{decoupling estimate}
\end{aligned}
$$

Case 2. Let $0 \le i \le \kappa$ be such that one Haar function in (6.11) is not cancellative. We assume that $h'_{L_2} = h^0_{L_2}$ and $h'_{L_j} = h_{L_j}$, $j = 1, 3$; the estimates for the other cases follow in the same manner. Note that (6.11) has been constructed such that this implies $l_2 = 0$, hence $L_2 = K$; see [12] for details. We use the decoupling estimate (Theorem 6.6) to estimate

$$
\begin{aligned}
\|S_i^k(f_1,f_2)\|_{L^p(\mathbb{R},S_p)} &= \|\sum_{K\in\mathcal{D}_{i,\kappa}}\sum_{\substack{L_1,L_3\in\mathcal{D}\\ L_j^{(l_j)}=K}} b_{L_1,L_3,K}\langle f_1,h_{L_1}\rangle |K|^{1/2}\langle f_2\rangle_K h_{L_3}\|_{L^p(\mathbb{R},S_p)} \\
&\le \beta_{p,S_p}\left(\mathbb{E}\int_{\mathbb{R}}\int_{\mathcal{V}}\|\sum_{K\in\mathcal{D}_{i,\kappa}}\varepsilon_K 1_K(x)\langle\varphi_{K,y}\rangle_K\|_{S_p}^p d\nu(y)dx\right)^{1/p},
\end{aligned}
$$

where the function $\varphi_{K,y} : \mathbb{R} \to S_p$ is defined as

$$
\varphi_{K,y}(x) \ := \ |K|^{1/2}\sum_{\substack{L_1,L_3\in\mathcal{D}\\ L_j^{(l_j)}=K}} b_{L_1,L_3,K}\langle f_1,h_{L_1}\rangle f_2(x) h_{L_3}(y_K).
$$

We can now apply Stein's inequality (Theorem 6.10) with respect to $x \in \mathbb{R}$ to obtain

$$
\|S_i^k(f_1,f_2)\|_{L^p(\mathbb{R},S_p)} \le \beta_{p,S_p}\beta^+_{p,S_p}\left(\mathbb{E}\int_{\mathbb{R}}\int_{\mathcal{V}}\|\sum_{K\in\mathcal{D}_{i,\kappa}}\varepsilon_K 1_K(x)\varphi_{K,y}(x)\|_{S_p}^p d\nu(y)dx\right)^{1/p}.
$$

By Lemma 2.7 and Hölder's inequality we can further estimate

$$
\begin{aligned}
&\|S_i^k(f_1,f_2)\|_{L^p(\mathbb{R},S_p)} \\
&\le \beta_{p,S_p}\beta^+_{p,S_p} \\
&\quad\times (\mathbb{E}\int_{\mathbb{R}}\int_{\mathcal{V}} \|\sum_{K\in\mathcal{D}_{i,\kappa}} \varepsilon_K 1_K(x)|K|^{1/2} \sum_{\substack{L_1,L_3\in\mathcal{D}\\ L_j^{(l_j)}=K}} b_{L_1,L_3,K}\langle f_1,h_{L_1}\rangle h_{L_3}(y_K)\|^p_{S_{p_1}} \|f_2(x)\|^p_{S_{p_2}} d\nu(y)dx)^{1/p} \\
&\le \beta_{p,S_p}\beta^+_{p,S_p}(\mathbb{E}\int_{\mathbb{R}}\int_{\mathcal{V}} \|\sum_{K\in\mathcal{D}_{i,\kappa}} \varepsilon_K 1_K(x)|K|^{1/2} \sum_{\substack{L_1,L_3\in\mathcal{D}\\ L_j^{(l_j)}=K}} b_{L_1,L_3,K}\langle f_1,h_{L_1}\rangle h_{L_3}(y_K)\|^{p_1}_{S_{p_1}} d\nu(y)dx)^{1/p_1} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\times \|f_2\|_{L^{p_2}(\mathbb{R},S_{p_2})}.
\end{aligned}
$$

We now proceed as in Case 1 to estimate the remaining term. We use

$$
|K|^{1/2} \sum_{\substack{L_1,L_3\in\mathcal{D}\\ L_j^{(l_j)}=K}} b_{L_1,L_3,K}\langle f_1,h_{L_1}\rangle h_{L_3}(y_K) = \int_{\mathcal{V}} b_K(y_k,z_K)\Delta_K^{l_1} f_1(z_K)d\nu(z),
$$

$$
b_K(y_k,z_K) = |K|^{3/2} \sum_{\substack{L_1,L_3\in\mathcal{D}\\ L_j^{(l_j)}=K}} b_{L_1,L_3,K} h_{L_1}(z) h_{L_3}(y_K)
$$

and estimate the remaining integral as

$$
\begin{aligned}
&\left(\mathbb{E}\int_{\mathbb{R}}\int_{\mathcal{V}} \|\sum_{K\in\mathcal{D}_{i,\kappa}} \varepsilon_K 1_K(x)|K|^{1/2} \sum_{\substack{L_1,L_3\in\mathcal{D}\\ L_j^{(l_j)}=K}} b_{L_1,L_3,K}\langle f_1,h_{L_1}\rangle h_{L_3}(y_K)\|^{p_1}_{S_{p_1}} d\nu(y)dx\right)^{1/p_1} \\
&= \left(\mathbb{E}\int_{\mathbb{R}}\int_{\mathcal{V}} \|\sum_{K\in\mathcal{D}_{i,\kappa}} \varepsilon_K 1_K(x) \int_{\mathcal{V}} b_K(y_k,z_K)\Delta_K^{l_1} f_1(z_K)d\nu(z)\|^{p_1}_{S_{p_1}} d\nu(y)dx\right)^{1/p_1} \\
&\le \left(\mathbb{E}\int_{\mathbb{R}}\int_{\mathcal{V}}\int_{\mathcal{V}} \|\sum_{K\in\mathcal{D}_{i,\kappa}} \varepsilon_K 1_K(x) b_K(y_k,z_K)\Delta_K^{l_1} f_1(z_K)\|^{p_1}_{S_{p_1}} d\nu(z)d\nu(y)dx\right)^{1/p_1}
\end{aligned}
$$

Using Fubini's theorem and the Kahane contraction principle (Theorem 6.9) we further have

$$
\begin{aligned}
&\left(\mathbb{E}\int_{\mathbb{R}}\int_{\mathcal{V}}\int_{\mathcal{V}} \|\sum_{K\in\mathcal{D}_{i,\kappa}} \varepsilon_K 1_K(x) b_K(y_k,z_K)\Delta_K^{l_1} f_1(z_K)\|^{p_1}_{S_{p_1}} d\nu(z)d\nu(y)dx\right)^{1/p_1} \\
&= \left(\int_{\mathbb{R}}\int_{\mathcal{V}}\int_{\mathcal{V}} \mathbb{E}\|\sum_{K\in\mathcal{D}_{i,\kappa}} \varepsilon_K 1_K(x) b_K(y_k,z_K)\Delta_K^{l_1} f_1(z_K)\|^{p_1}_{S_{p_1}} d\nu(z)d\nu(y)dx\right)^{1/p_1} \\
&\le \left(\int_{\mathbb{R}}\int_{\mathcal{V}}\int_{\mathcal{V}} \max_{K\in\mathcal{D}_{i,\kappa}} |b_K(y_k,z_K)| \mathbb{E}\|\sum_{K\in\mathcal{D}_{i,\kappa}} \varepsilon_K 1_K(x)\Delta_K^{l_1} f_1(z_K)\|^{p_1}_{S_{p_1}} d\nu(z)d\nu(y)dx\right)^{1/p_1}.
\end{aligned}
$$

As in Case 1, we have the pointwise estimate $|b_K(y_k, z_K)| \leq 1$, since

$$
\begin{aligned}
|b_K(y_k, z_K)| &\leq |K|^{3/2} \sum_{\substack{L_1, L_3 \in \mathcal{D} \\ L_j^{(l_j)} = K}} |b_{L_1,L_3,K}||h_{L_1}(z)||h_{L_3}(y_K)| \\
&= |K|^{3/2} \sum_{\substack{L_1, L_3 \in \mathcal{D} \\ L_j^{(l_j)} = K}} |b_{L_1,L_3,K}| \frac{1_{L_1}(z)}{|L_1|^{1/2}} \frac{1_{L_3}(y_K)}{|L_3|^{1/2}} \\
&\leq |K|^{3/2} \sum_{\substack{L_1, L_3 \in \mathcal{D} \\ L_j^{(l_j)} = K}} \sum_{\substack{Q_1, Q_2, Q_3 \in \mathcal{D} \\ Q_j^{(k_j - l_j)} = L_j}} |a_{Q_1,Q_2,Q_3,K}| \prod_{j=1}^{3} \frac{|Q_j|^{1/2}}{|L_j|^{1/2}} \frac{1_{L_1}(z)}{|L_1|^{1/2}} \frac{1_{L_3}(y_K)}{|L_3|^{1/2}} \\
&\leq \sum_{\substack{L_1, L_3 \in \mathcal{D} \\ L_j^{(l_j)} = K}} \sum_{\substack{Q_1, Q_2, Q_3 \in \mathcal{D} \\ Q_j^{(k_j - l_j)} = L_j}} \prod_{j=1}^{3} \frac{|Q_j|}{|L_j|} 1_{L_1}(z) 1_{L_3}(y_K) \\
&\leq 1.
\end{aligned}
$$

Using the decoupling estimate (Theorem 6.6) we thus conclude

$$
\begin{aligned}
&\left(\int_{\mathbb{R}} \int_{\mathcal{V}} \int_{\mathcal{V}} \max_{K \in \mathcal{D}_{i,\kappa}} |b_K(y_k, z_K)| \mathbb{E} \| \sum_{K \in \mathcal{D}_{i,\kappa}} \varepsilon_K 1_K(x) \Delta_K^{l_1} f_1(z_K) \|_{S_{p_1}}^{p_1} d\nu(z) d\nu(y) dx \right)^{1/p_1} \\
&\leq \left(\int_{\mathbb{R}} \int_{\mathcal{V}} \int_{\mathcal{V}} \mathbb{E} \| \sum_{K \in \mathcal{D}_{i,\kappa}} \varepsilon_K 1_K(x) \Delta_K^{l_1} f_1(z_K) \|_{S_{p_1}}^{p_1} d\nu(z) d\nu(y) dx \right)^{1/p_1} \\
&= \left(\mathbb{E} \int_{\mathbb{R}} \int_{\mathcal{V}} \| \sum_{K \in \mathcal{D}_{i,\kappa}} \varepsilon_K 1_K(x) \Delta_K^{l_1} f_1(z_K) \|_{S_{p_1}}^{p_1} d\nu(z) dx \right)^{1/p_1} \\
&\leq \beta_{p_1, S_{p_1}} \int_{\mathbb{R}} \|f_1\|_{S_p}^p dx.
\end{aligned}
$$

Combining all cases, we conclude (using $\kappa_{2,q} \leq \sqrt{2}$, see Remark 6.8)

$$
C_{p_1,p_2}^{\text{shift}} \lesssim \beta_{p,S_p} \beta_{p_1,S_{p_1}} \beta_{p_2,S_{p_2}} + \beta_{p,S_p} \beta_{p,S_p}^+ \beta_{p_1,S_{p_1}} + \beta_{p_1,S_{p_1}} \beta_{p_1,S_{p_1}}^+ \beta_{p_2,S_{p_2}} + \beta_{p_2,S_{p_2}} \beta_{p_2,S_{p_2}}^+ \beta_{p,S_p}.
$$

□

6.2.2. Paraproducts

It remains to show the boundedness of bilinear paraproducts. For this, we first need the boundedness of the *linear* paraproduct.

Theorem 6.14. *Let $\mathcal{D}$ be a dyadic grid and let $(a_Q)_{Q \in \mathcal{D}}$ be scalars satisfying*

$$
\sup_{Q_0 \in \mathcal{D}} \left(\frac{1}{|Q_0|} \sum_{\substack{Q \in \mathcal{D} \\ Q \subset Q_0}} |a_Q|^2 \right)^{1/2} \leq 1.
$$

Then the linear paraproduct

$$
\pi(f) := \sum_{Q \in \mathcal{D}} a_Q \langle f \rangle_Q h_Q,
$$

where h_Q denotes a cancellative Haar function, is a bounded map on $L^p(\mathbb{R}, S_p)$, $p \in (1, \infty)$, with

$$
\|\pi f\|_{L^p(\mathbb{R}, S_p)} \lesssim pp' C_{\mathrm{BMO}_p} \beta_{p,S_p}^2 \beta_{p,\mathbb{R}} \|f\|_{L^p(\mathbb{R}, S_p)}.
$$

A proof with explicit constants was given in [18, Theorem 3] for a more general definition of the paraproduct via BMO-functions (see Section 2.9):

Theorem 6.15 ([18, Theorem 3])**.** *Let X be a Banach space and let $\mathcal{T}$ be a UMD subspace of $B(X)$, the bounded linear operators on X. Let $b \in \mathrm{BMO}_p(\mathbb{R}, \mathcal{T})$ and $f \in L^p(\mathbb{R}, X)$. The paraproduct is defined as*

$$\Pi_b f \ := \sum_{Q \in \mathcal{D}} D_Q b \langle f \rangle_Q.$$

This paraproduct is a bounded operator on $L^p(\mathbb{R}, X)$ with

$$\|\Pi_b f\|_{L^p(\mathbb{R},X)} \lesssim pp' \beta_{p,X}^2 \beta_{p,\mathcal{T}} \|b\|_{\mathrm{BMO}_p} \|f\|_{L^p(\mathbb{R},X)}.$$

Note that since $D_Q b = \langle b, h_Q \rangle h_Q$ by Lemma 2.21, this definition coincides for scalar-valued b with the definition of the paraproduct in Theorem 6.14 for $X = S_p$, $p \in (1, \infty)$, and $a_Q := \langle b, h_Q \rangle$, up to the boundedness constant of $(a_Q)_Q$. On the other hand, given $(a_Q)_Q$ as in Definition 2.35, setting $b := \sum_{Q \in \mathcal{D}} a_Q h_Q$ yields $D_Q b = \sum_{Q'} a_{Q'} \langle h_{Q'}, h_Q \rangle h_Q = a_Q h_Q$. Hence the definition of the paraproduct used in [18] is equivalent to the one used in this section.

Proof of Theorem 6.14. Using Theorem 6.15 with $\mathcal{T} \simeq \mathbb{K}$, $\mathbb{K}$ scalar field, it remains to show that for $b := \sum_{Q \in \mathcal{D}} a_Q h_Q$, $\|b\|_{\mathrm{BMO}_p}$ is bounded. By the John-Nirenberg inequality (Theorem 2.39), we have

$$\|b\|_{\mathrm{BMO}_p} \le C_{\mathrm{BMO}_p} \|b\|_{\mathrm{BMO}_2}.$$

Note that for any $Q_0 \in \mathcal{D}$, $\{h^0_{Q_0}\} \cup \{h_Q \mid Q \subseteq Q_0 \text{ dyadic cube}\}$ is an orthonormal basis of $L^2(Q_0)$. Hence we can write $b 1_{Q_0} = \langle b, h^0_{Q_0} \rangle h^0_{Q_0} + \sum_{\substack{Q \in \mathcal{D} \\ Q \subseteq Q_0}} \langle b, h_Q \rangle h_Q$. Therefore,

$$\begin{aligned}
\|b\|_{\mathrm{BMO}_2} &= \sup_{Q_0 \in \mathcal{D}} \left(\frac{1}{|Q_0|} \int_{Q_0} |b - \langle b \rangle_{Q_0}|^2 dx \right)^{1/2} \\
&= \sup_{Q_0 \in \mathcal{D}} \left(\frac{1}{|Q_0|} \int_{Q_0} |b 1_{Q_0} - \langle b, h^0_{Q_0} \rangle h^0_{Q_0}|^2 dx \right)^{1/2} \\
&= \sup_{Q_0 \in \mathcal{D}} \left(\frac{1}{|Q_0|} \int_{Q_0} | \sum_{\substack{Q \in \mathcal{D} \\ Q \subseteq Q_0}} \langle b, h_Q \rangle h_Q|^2 dx \right)^{1/2}.
\end{aligned}$$

By using the orthogonality of the Haar functions, we can pull the sum out of the integral and conclude

$$\begin{aligned}
\|b\|_{\mathrm{BMO}_2} &= \sup_{Q_0 \in \mathcal{D}} \left(\frac{1}{|Q_0|} \sum_{\substack{Q \in \mathcal{D} \\ Q \subseteq Q_0}} \int_{Q_0} |\langle b, h_Q \rangle h_Q|^2 dx \right)^{1/2} \\
&= \sup_{Q_0 \in \mathcal{D}} \left(\frac{1}{|Q_0|} \sum_{\substack{Q \in \mathcal{D} \\ Q \subseteq Q_0}} \frac{1}{|Q|} \int_{Q} |\langle b, h_Q \rangle|^2 dx \right)^{1/2} \\
&= \sup_{Q_0 \in \mathcal{D}} \left(\frac{1}{|Q_0|} \sum_{\substack{Q \in \mathcal{D} \\ Q \subseteq Q_0}} |\langle b, h_Q \rangle|^2 \right)^{1/2} \\
&= \sup_{Q_0 \in \mathcal{D}} \left(\frac{1}{|Q_0|} \sum_{\substack{Q \in \mathcal{D} \\ Q \subseteq Q_0}} |a_Q|^2 \right)^{1/2}.
\end{aligned}$$

□

Remark 6.16. In the boundedness proof of the bilinear paraproduct, we will not apply Theorem 6.14 directly; rather, we will estimate a term that arises in the proof of Theorem 6.15. We will therefore demonstrate the first step of the proof of Theorem 6.15 here, namely an application of the decoupling inequality (Theorem 6.6).

Define $f_Q := a_Q \langle f \rangle_Q h_Q$ for a dyadic cube $Q \in \mathcal{D}$. These functions fulfil the conditions in Theorem 6.6, hence we may apply the decoupling inequality and have

$$\begin{aligned}\|\pi f\|_{L^p(\mathbb{R},S_p)}^p &= \int_{\mathbb{R}} \| \sum_{Q\in\mathcal{D}} f_Q(x)\|_{S_p}^p dx \leq \beta_{p,S_p}^p \mathbb{E} \int_{\mathbb{R}} \int_{\mathcal{V}} \| \sum_{Q\in\mathcal{D}} \varepsilon_Q 1_Q(x) f_Q(y_Q)\|_{S_p}^p d\nu(y) dx \\ &= \beta_{p,S_p}^p \mathbb{E} \int_{\mathbb{R}} \int_{\mathcal{V}} \| \sum_{Q\in\mathcal{D}} \varepsilon_Q 1_Q(x) a_Q \langle f \rangle_Q h_Q(y_Q)\|_{S_p}^p d\nu(y) dx.\end{aligned}$$

Note that for fixed $y_Q \in \mathcal{V}_Q$, the families $(\varepsilon_Q h_Q(y_Q))_Q$ and $(\varepsilon_Q |h_Q(y_Q)|)_Q$ are identically distributed, hence

$$\begin{aligned}&\beta_{p,S_p}^p \mathbb{E} \int_{\mathbb{R}} \int_{\mathcal{V}} \| \sum_{Q\in\mathcal{D}} \varepsilon_Q 1_Q(x) a_Q \langle f \rangle_Q h_Q(y_Q)\|_{S_p}^p d\nu(y) dx \\ &= \beta_{p,S_p}^p \mathbb{E} \int_{\mathbb{R}} \int_{\mathcal{V}} \| \sum_{Q\in\mathcal{D}} \varepsilon_Q 1_Q(x) a_Q \langle f \rangle_Q |h_Q(y_Q)|\|_{S_p}^p d\nu(y) dx.\end{aligned}$$

By construction of the Haar functions, $|h_Q| = \frac{1}{|Q|^{1/2}} 1_Q$ is constant on $\mathcal{V}_Q$. Since $\mathcal{V} = \prod_Q \mathcal{V}_Q$ is a probability space and $\sum_{Q\in\mathcal{D}} \frac{1}{|Q|^{1/2}} \varepsilon_Q 1_Q(x) a_Q \langle f \rangle_Q |1_Q(y_Q)|$ is constant in all y_Q, $Q \in \mathcal{D}$, we now have

$$\begin{aligned}&\beta_{p,S_p}^p \mathbb{E} \int_{\mathbb{R}} \int_{\mathcal{V}} \| \sum_{Q\in\mathcal{D}} \varepsilon_Q 1_Q(x) a_Q \langle f \rangle_Q |h_Q(y_Q)|\|_{S_p}^p d\nu(y) dx \\ &= \beta_{p,S_p}^p \mathbb{E} \int_{\mathbb{R}} \int_{\mathcal{V}} \| \sum_{Q\in\mathcal{D}} \tfrac{1}{|Q|^{1/2}} \varepsilon_Q 1_Q(x) a_Q \langle f \rangle_Q |1_Q(y_Q)|\|_{S_p}^p d\nu(y) dx \\ &= \beta_{p,S_p}^p \mathbb{E} \int_{\mathbb{R}} \| \sum_{Q\in\mathcal{D}} \tfrac{1}{|Q|^{1/2}} \varepsilon_Q 1_Q(x) a_Q \langle f \rangle_Q \|_{S_p}^p dx \\ &= \beta_{p,S_p}^p \mathbb{E} \int_{\mathbb{R}} \| \sum_{Q\in\mathcal{D}} \varepsilon_Q a_Q \langle f \rangle_Q |h_Q(x)|\|_{S_p}^p dx\end{aligned}$$

This term (up to taking $1/p$-th powers) will appear in the proof of Theorem 6.17. Hence when estimating it using Theorem 6.14, we may omit a constant β_{p,S_p}, as the remaining proof of Theorem 6.15 shows

$$\left(\mathbb{E} \int_{\mathbb{R}} \| \sum_{Q\in\mathcal{D}} \varepsilon_Q a_Q \langle f \rangle_Q |h_Q(x)|\|_{S_p}^p dx \right)^{1/p} \lesssim pp' \beta_{p,S_p} \beta_{p,\mathbb{R}} C_{\mathrm{BMO}_p} \|f\|_{L^p(\mathbb{R},S_p)}.$$

We now follow [12] to prove boundedness of the paraproduct in the bilinear case.

Theorem 6.17 ([12, Theorem 5.1])**.** *Let $f_j \in L_c^\infty(\mathbb{R}, S_{p_j})$, $j = 1, 2, 3$, be such that $1/p_1 + 1/p_2 + 1/p_3 = 1$. Then*

$$|\Lambda_\pi(f_1, f_2, f_3)| \leq C_{p_1,p_2}^{\mathrm{para}} \prod_{j=1}^3 \|f_j\|_{L^{p_j}(\mathbb{R},S_{p_j})} \tag{6.13}$$

with

$$C_{p_1,p_2}^{\mathrm{para}} \lesssim \beta_{p,S_p}^- \beta_{p,S_p}^+ pp' \beta_{p_1,S_{p_1}} \beta_{p_1,\mathcal{T}} C_{\mathrm{BMO}_{p_1}}.$$

Proof. Recall from the definition of the multilinear paraproduct (see Definition 2.35) that there is one index $j_0 \in \{1, 2, 3\}$ such that the associated Haar function is cancellative. We will assume that $j_0 = 3$ (the other cases follow in the same manner) and choose $p \in (1, \infty)$ such that $1/p_1 + 1/p_2 = 1/p$.

Using the UMD property of S_p, we have

$$\|\pi(f_1,f_2)\|_{L^p(\mathbb{R},S_p)} = \|\sum_{Q\in\mathcal{D}} a_Q\langle f_1\rangle_Q\langle f_2\rangle_Q h_Q\|_{L^p(\mathbb{R},S_p)}$$
$$\leq \beta^-_{p,S_p}\left(\mathbb{E}\int_{\mathbb{R}}\|\sum_{Q\in\mathcal{D}}\varepsilon_Q a_Q\langle f_1\rangle_Q\langle f_2\rangle_Q h_Q(x)\|^p_{S_p}dx\right)^{1/p}.$$

As in Remark 6.16, we use that for fixed $x\in\mathbb{R}$, the families $(\varepsilon_Q h_Q(x))_Q$ and $(\varepsilon_Q|h_Q(x)|)_Q$ are identically distributed and apply Stein's inequality (Theorem 6.10) to show

$$\|\pi(f_1,f_2)\|_{L^p(S_p)} \leq \beta^-_{p,S_p}\left(\mathbb{E}\int_{\mathbb{R}}\|\sum_{Q\in\mathcal{D}}\varepsilon_Q a_Q\langle f_1\rangle_Q\langle f_2\rangle_Q |h_Q(x)|\|^p_{S_p}dx\right)^{1/p}$$
$$\leq \beta^-_{p,S_p}\beta^+_{p,S_p}\left(\mathbb{E}\int_{\mathbb{R}}\|\sum_{Q\in\mathcal{D}}\varepsilon_Q a_Q\langle f_1\rangle_Q f_2(x)|h_Q(x)|\|^p_{S_p}dx\right)^{1/p}.$$

Using the product property of the Schatten spaces (see Lemma 2.7) and Hölder's inequality, we hence conclude

$$\|\pi(f_1,f_2)\|_{L^p(\mathbb{R},S_p)} \leq \beta^-_{p,S_p}\beta^+_{p,S_p}\left(\mathbb{E}\int_{\mathbb{R}}\|\sum_{Q\in\mathcal{D}}\varepsilon_Q a_Q\langle f_1\rangle_Q |h_Q(x)|\|^p_{S_{p_1}}\|f_2(x)\|^p_{S_{p_2}}dx\right)^{1/p}$$
$$\leq \beta^-_{p,S_p}\beta^+_{p,S_p}\left(\mathbb{E}\int_{\mathbb{R}}\|\sum_{Q\in\mathcal{D}}\varepsilon_Q a_Q\langle f_1\rangle_Q |h_Q(x)|\|^{p_1}_{S_{p_1}}dx\right)^{1/p_1}\|f_2\|_{L^{p_2}(\mathbb{R},S_{p_2})}.$$

The remaining term is related to a linear paraproduct as discussed in Remark 6.16. We therefore obtain the estimate

$$\|\pi(f_1,f_2)\|_{L^p(\mathbb{R},S_p)} \leq C^{\text{para}}_{p_1,p_2}\|f_1\|_{L^{p_1}(\mathbb{R},S_{p_1})}\|f_2\|_{L^{p_2}(\mathbb{R},S_{p_2})}$$

with

$$C^{\text{para}}_{p_1,p_2} \lesssim \beta^-_{p,S_p}\beta^+_{p,S_p}pp'\beta_{p_1,S_{p_1}}\beta_{p_1,\mathbb{R}}C_{\text{BMO}_{p_1}}.$$

□

6.2.3. Vanishing paraproducts for T_m

It is evident that at $C^{\text{para}}_{p_1,p_2} = O(p^6)$, the paraproducts have a drastic impact on the boundedness constant of Theorem 1.4. However, as noted in Remark 2.37, the paraproducts in the decomposition of a Calderón-Zygmund operator T vanish if $\langle T(1,\dots,1),h_Q\rangle = 0$, for all considered dyadic cubes Q, as well as for adjoints of T. These adjoints are defined via $\langle T(f,g),h\rangle = \langle T^{1*}(h,g),f\rangle = \langle T^{2*}(f,h),g\rangle$ in the bilinear case.

We will show that paraproducts vanish for the operator T_m treated in Section 5.3. Note that since the kernel K (see (5.3)) has a singularity at the origin, $T_m(1,1)$ may not be well-defined. We will follow the construction in [27] and approximate T_m by operators formally defined as

$$T_{m,\varepsilon}(f,g)(x) := \int_{\max(|x-y|,|x-z|)>\varepsilon} K(x,y,z)f(y)g(z)dydz \tag{6.14}$$

and show that suitable approximations of $\langle T_{m,\varepsilon}(1,1),h_Q\rangle$, $\langle T^{1*}_{m,\varepsilon}(1,1),h_Q\rangle$, $\langle T^{2*}_{m,\varepsilon}(1,1),h_Q\rangle$ vanish for all $\varepsilon>0$ and all dyadic cubes Q.

Theorem 6.18. *Let Q be a dyadic cube and let $\varepsilon > 0$. Let $T_{m,\varepsilon}$ as in (6.14) with kernel*

$$K(x,y,z) = \frac{1}{(x-y)^2 + (x-z)^2} \sum_{\substack{k \text{ even} \\ k \neq 0}} b_k \left(\frac{(x-y) + i(x-z)}{|(x-y) + i(x-z)|} \right)^k$$

with $b_k = (-1)^{k/2}|k|(4\pi^2)^{-1}a_k$, $(a_k)_k$ scalars. Then $\langle T_{m,\varepsilon}(1,1), h_Q\rangle = 0$, $\langle T^{1*}_{m,\varepsilon}(1,1), h_Q\rangle = 0$, $\langle T^{2*}_{m,\varepsilon}(1,1), h_Q\rangle = 0$.

Proof. We will give the proof explicitly for $T_{m,\varepsilon}(1,1)$, the proof for the adjoints will be sketched.

In order to ensure the well-definedness of all terms, we approximate $\langle T_\varepsilon(1,1), h_Q\rangle$ as follows. Let

$$Q_\varepsilon(x) := \{(y,z) \in \mathbb{R}^2 \mid \max(|x-y|, |x-z|) \leq \varepsilon\}$$

and let

$$B_r(x) := \{(y,z) \in \mathbb{R}^2 \mid |x-y|^2 + |x-z|^2 \leq r^2\}$$

be the closed ball with radius $r > 0$ around the point $(x,x) \in \mathbb{R}^2$. Now choose $c_Q \in \mathbb{R}$ such that there exists $R_0 > 0$ such that for all $R > R_0$, all $x \in Q$, and all $(y,z) \notin B_R(x)$, we have

$$|x - c_Q| \leq \frac{1}{2} \max\left(|x-y|, |x-z|\right).$$

Using $\int_Q h_Q(x)dx = 0$, we can now approximate $\langle T_\varepsilon(1,1), h_Q\rangle$ as

$$\begin{aligned}
&\int_{\mathbb{R}} \int_{B_{\sqrt{2}\varepsilon}(x) \setminus Q_\varepsilon(x)} K(x,y,z) h_Q(x) dydzdx \\
&+ \lim_{R\to\infty} \left(\int_{\mathbb{R}} \int_{B_R(x) \setminus B_{\sqrt{2}\varepsilon}(x)} K(x,y,z) h_Q(x) dydzdx \right. \\
&\qquad \left. + \int_{\mathbb{R}} \int_{\mathbb{R}^2 \setminus B_R(x)} (K(x,y,z) - K(c_Q,y,z)) h_Q(x) dydzdx \right).
\end{aligned}$$

We now show that these integrals vanish separately. Letting $k(x-y, x-z) := K(x,y,z)$, translation invariance of the integral yields

$$\int_{\mathbb{R}} \int_{B_{\sqrt{2}\varepsilon}(x) \setminus Q_\varepsilon(x)} k(x-y, x-z) h_Q(x) dydzdx = \int_{\mathbb{R}} \int_{B_{\sqrt{2}\varepsilon}(0) \setminus Q_\varepsilon(0)} k(y,z) h_Q(x) dydzdx.$$

We can estimate this term using Fubini's theorem and Lemma 5.4 as

$$\left| \int_{\mathbb{R}} \int_{B_{\sqrt{2}\varepsilon}(0) \setminus Q_\varepsilon(0)} k(y,z) h_Q(x) dydzdx \right| = \left| \int_{B_{\sqrt{2}\varepsilon}(0) \setminus Q_\varepsilon(0)} k(y,z) dydz \right| \left| \int_{\mathbb{R}} h_Q(x) dx \right|,$$

where the second integral vanishes and the first integral can be uniformly bounded using Lemma 5.4 as

$$\begin{aligned}
\left| \int_{B_{\sqrt{2}\varepsilon}(0) \setminus Q_\varepsilon(0)} k(y,z) dydz \right| &\leq \int_{B_{\sqrt{2}\varepsilon}(0) \setminus Q_\varepsilon(0)} \frac{1}{(|y|+|z|)^2} dydz \\
&\leq \operatorname{vol}\left(B_{\sqrt{2}\varepsilon}(0) \setminus Q_\varepsilon(0)\right) \left(\sup_{(y,z) \in B_{\sqrt{2}\varepsilon}(0) \setminus Q_\varepsilon(0)} \frac{1}{(|y|+|z|)^2} \right) \\
&\leq \operatorname{vol}\left(B_{\sqrt{2}\varepsilon}(0) \setminus Q_\varepsilon(0)\right) \left(\sup_{\max(|y|,|z|) > \varepsilon} \frac{1}{(|y|+|z|)^2} \right) \\
&= (2\pi\varepsilon^2 - 4\varepsilon^2) \frac{1}{\varepsilon^2} = 2\pi - 4.
\end{aligned}$$

For sufficiently large R, the second integral can similarly be expressed by translation invariance as

$$\int_{\mathbb{R}} \int_{B_R(x) \setminus B_{\sqrt{2}\varepsilon}(x)} k(x-y, x-z) h_Q(x) dydzdx = \int_{\mathbb{R}} \int_{B_R(0) \setminus B_{\sqrt{2}\varepsilon}(0)} k(y,z) h_Q(x) dydzdx.$$

Transforming the inner two integrals into polar coordinates yields

$$\int_{\mathbb{R}}\int_{B_R(0)\setminus B_{\sqrt{2}\varepsilon}(0)} k(y,z)h_Q(x)dydzdx = \int_{\mathbb{R}}\int_{\sqrt{2}\varepsilon}^{R}\int_0^{2\pi}\frac{1}{r^2}\sum_k b_k e^{ik\theta} rd\theta dr\, h_Q(x)dx = 0,$$

hence the second integral vanishes for all sufficiently large R.

Finally, we estimate the remaining integral. Using Lemma 5.5, we can estimate the inner integral for $x \in Q$ as

$$\int_{\mathbb{R}^2\setminus B_R(x)} |(K(x,y,z) - K(c_Q,y,z))|\, dydz \le C_K \int_{\mathbb{R}^2\setminus B_R(x)} \frac{|x-c_Q|}{(|x-y|+|x-z|)^3}dydz.$$

By construction, we can estimate $|x - c_Q|$ by

$$|x-c_Q| \le \inf_{R>R_0}\frac{1}{2}\inf_{(y,z)\notin B_R(x)}(|x-y|,|x-z|) \le \frac{1}{2}\inf_{(y,z)\notin B_{R_0}(x)}(|x-y|,|x-z|) = \frac{R_0}{2\sqrt{2}},$$

hence by using translation invariance and transforming to polar coordinates,

$$\begin{aligned}\int_{\mathbb{R}^2\setminus B_R(x)} \frac{|x-c_Q|}{(|x-y|+|x-z|)^3}dydz &\le \frac{R_0}{2\sqrt{2}}\int_{\mathbb{R}^2\setminus B_R(x)}\frac{1}{(|x-y|+|x-z|)^3}dydz\\ &= \frac{R_0}{2\sqrt{2}}\int_{\mathbb{R}^2\setminus B_R(0)}\frac{1}{(|y|+|z|)^3}dydz\\ &\le \frac{R_0}{2\sqrt{2}}\int_{\mathbb{R}^2\setminus B_R(0)}\frac{1}{\sqrt{y^2+z^2}^3}dydz\\ &= \frac{R_0}{2\sqrt{2}}\int_R^\infty\int_0^{2\pi}\frac{1}{r^3}rd\theta dr\\ &= \frac{R_0\pi}{\sqrt{2}}\int_R^\infty \frac{1}{r^2}dr\\ &= \frac{R_0\pi}{\sqrt{2}R}.\end{aligned}$$

Hence we can estimate the full remaining integral as

$$\begin{aligned}&\left|\int_{\mathbb{R}}\int_{\mathbb{R}^2\setminus B_R(x)}(K(x,y,z)-K(c_Q,y,z))h_Q(x)dydzdx\right|\\ \le&\frac{C_K}{|Q|^{1/2}}\int_{\mathbb{R}}\int_{\mathbb{R}^2\setminus B_R(x)}\frac{|x-c_Q|}{(|x-y|+|x-z|)^3}1_Q(x)dydzdx\\ \le&\frac{C_K}{|Q|^{1/2}}\frac{R_0\pi}{\sqrt{2}R}\int_{\mathbb{R}}1_Q(x)dx\\ =&\frac{C_K R_0\pi}{\sqrt{2|Q|}R},\end{aligned}$$

which vanishes for $R \to \infty$.

For the adjoints, the statement follows the same proof idea. Formally, we have

$$\begin{aligned}\langle T_\varepsilon^{1*}(1,1), h_Q\rangle &:= \int_{\mathbb{R}}\int_{\max(|x-y|,|x-z|)>\varepsilon} k(x-y,x-z)h_Q(y)dxdzdy,\\ \langle T_\varepsilon^{2*}(1,1), h_Q\rangle &:= \int_{\mathbb{R}}\int_{\max(|x-y|,|x-z|)>\varepsilon} k(x-y,x-z)h_Q(z)dxdydz,\end{aligned}$$

where as for $\langle T_\varepsilon(1,1), h_Q\rangle$, the integral needs to be approximated to ensure well-definedness. Our construction needs some slight modifications, however, which we will demonstrate for T_ε^{1*}. Let now

$$Q'_\varepsilon(y) := \{(x,z) \in \mathbb{R}^2 \mid \max(|x-y|,|x-z|) \le \varepsilon\}$$

and let

$$B'_r(y) := \{(x,z) \in \mathbb{R}^2 \mid |x-y|^2 + |x-z|^2 \leq r^2\}, \quad r > 0.$$

This again allows us to choose $c_Q \in \mathbb{R}$ and $R_0 > 0$ such that for all $R > R_0$, all $y \in Q$, and all $(x,z) \notin B'_R(y)$, we have

$$|y - c_Q| \leq \frac{1}{2}\max(|x-y|, |x-z|).$$

The remaining key step to show is the translation invariance of the integrals in x and z; all other proof steps then proceed in the same manner as in the non-adjoint case. We demonstrate this for the integral

$$\int_{\mathbb{R}} \int_{B'_{\sqrt{2}\varepsilon}(y)\setminus Q'_\varepsilon(y)} k(x-y, x-z) h_Q(y) dxdzdy.$$

Substituting first $\xi = x - y$ we have

$$\int_{\mathbb{R}} \int_{B'_{\sqrt{2}\varepsilon}(y)\setminus Q'_\varepsilon(y)} k(x-y, x-z) h_Q(y) dxdzdy = \int_{\mathbb{R}} \int_{\Omega''_\varepsilon(y)} k(\xi, \xi + y - z) h_Q(y) d\xi dzdy,$$

where

$$\Omega''_\varepsilon(y) := \{(\xi, z) \in \mathbb{R}^2 \mid |\xi|^2 + |\xi + y - z|^2 \leq 2\varepsilon^2 \text{ and } \max(|\xi|, |\xi + y - z|) > \varepsilon\}.$$

Lemma 5.4 allows us to apply Fubini's theorem, hence we next substitute $\zeta = \xi + y - z$ to obtain

$$\int_{\mathbb{R}} \int_{\Omega''_\varepsilon(y)} k(\xi, \xi + y - z) h_Q(y) d\xi dzdy = \int_{\mathbb{R}} \int_{B_{\sqrt{2}\varepsilon}(0)\setminus Q_\varepsilon(0)} k(\xi, \zeta) h_Q(y) d\xi d\zeta dy.$$

This is (up to renaming variables) the same integral as in the non-adjoint case, hence the same proof method can be applied to the T_ε^{1*}-case. For T_ε^{2*}, the same arguments hold, allowing us to conclude

$$\langle T_\varepsilon(1,1), h_Q\rangle = \langle T_\varepsilon^{1*}(1,1), h_Q\rangle = \langle T_\varepsilon^{2*}(1,1), h_Q\rangle = 0$$

for all dyadic cubes Q and all $\varepsilon > 0$. ☐

6.3. Comparison of the resulting constants

We can now compare the constant in Theorem 1.3 resulting from the proof methods presented in Section 6.1–6.2. For this, we consider our multiplier as a map $M_{f^{[2]}} : S_p \times S_p \to S_{p/2}$ and compare the growth of the constant in p. Our results are summarised in Table 6.1.

Both proofs follow a similar structure: First, up to three triangular truncation are applied as summarised in Table 3.1. Following Theorem 2.12, this yields in total a constant of order $O(p^3)$. Next, the operator $M_{f^{[2]}}$ is decomposed into a sum of compositions of a bilinear Toeplitz form Schur multipliers with symbol $(\lambda_0, \lambda_1, \lambda_2) \mapsto \frac{\lambda_0 - \lambda_1}{\lambda_0 - \lambda_2}$ (up to permutation of variables) with a linear Schur multiplier. We now collect the growth rate of the relevant constants.

Toeplitz form Schur multiplier In Section 6.1, boundedness of the bilinear Toeplitz form Schur multiplier is shown in Theorem 6.3. The boundedness constant in our case $S_p \times S_p \to S_{p/2}$ is given by a term of the form

$$c_{p,p} \lesssim C^2_{\triangle,p} + \hbar_{p/2,S_{p/2}}(\beta_{p/2,S_{p/2}})^2 \hbar_{p,S_p}(\beta_{p,S_p})^2.$$

By Lemma 2.26, $\hbar_{p,S_p} = O(p)$, $\beta_{p,S_p} = O(p)$, and furthermore $C_{\triangle,p} = O(p)$ by Theorem 2.12. Hence in total, $c_{p,p} = O(p^6)$.

In Section 6.2, the Fourier multiplier associated with the corresponding Toeplitz form Schur multiplier is shown to be a Calderón-Zygmund operator and split into a sum over dyadic shifts and paraproducts.

- The boundedness of dyadic shifts was shown in Theorem 6.13. The boundedness constant was determined as

$$C^{\text{shift}}_{p,p} \lesssim \beta_{p/2,S_{p/2}}(\beta_{p,S_p})^2 + (\beta_{p/2,S_{p/2}})^2 \beta_{p,S_p},$$

  where we used $\beta^+_{p,S_p} \leq \beta_{p,S_p}$ from Lemma 2.25. Since $\beta_{p,S_p} = O(p)$ by Lemma 2.26, we have $C^{\text{shift}}_{p,p} = O(p^3)$.

- Bilinear paraproducts were shown to be bounded in Theorem 6.17 with constant

$$C_{p,p}^{\text{para}} \lesssim \beta^{-}_{p/2,S_{p/2}}\beta^{+}_{p/2,S_{p/2}}pp'\beta_{p,S_p}\beta_{p,\mathbb{K}}C_{\text{BMO}_p}.$$

  By [20, Corollary 4.5.15], we have $\beta_{p,\mathbb{K}} = \max(p,p') - 1 = O(p)$ for $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$. Furthermore, $C_{\text{BMO}_p} = O(p)$ by Theorem 2.39, and $\beta^{-}_{p,S_p} = O(p)$ by Lemma 2.26. Altogether, this yields $C_{p,p}^{\text{para}} = O(p^6)$.
- In Theorem 6.18, we have shown that the operator T_m that Theorem 1.4 is applied to in our proof (see Section 5) can be expressed solely as dyadic shifts without paraproducts.

Hence, the norm of the Toeplitz form Schur multiplier is bounded by $C_{p,p}^{\text{shift}} = O(p^3)$, according to the proof presented in Section 6.2.

Linear Schur multiplier We first consider the linear operators in [35]. By decomposition (6.1), we need to estimate the norms of both $M_{\phi_{1,f''}}$ and $M_{\phi_{2,f''}}$ on S_p. This was shown in Theorem 6.5. Note that we may now assume that triangular truncations have already been applied, hence the constant $C_{\triangle,p}$ is omitted. Furthermore, since we are interested in the growth of the constant for $p \to \infty$, we restrict our discussion to the constants determined for large p.

We have shown the following for $1/p + 1/r = 1/2$.

- For $m = 1$,

$$\|M_{\phi_{1,f''}}\|_{S_p\to S_p} \lesssim \max(c_p\max(c_r,1),(c_p\max(c_r,1))^{1/1-\theta})\|f''\|_\infty =: C_{p,1}.$$

  Here, $c_p \lesssim \hbar_{p,S_p}(\beta^{+}_{p,S_p})^2 = O(p^3)$ is from Theorem 6.1. For $1/p + 1/r = 1/2$, we have $r \to 2$ as $p \to \infty$, hence $c_r = O(1)$. Altogether, we conclude $C_{p,1} = O(p^3)$.
- For $m = 2$,

$$\|M_{\phi_{2,f''}}\|_{S_p\to S_p} \lesssim C_{p,2} := \max(c_p\max(c_r, 1 + c_r\|M_{\phi_{1,f''}}\|_{S_r\to S_r}), (c_p\max(c_r, 1 + c_r\|M_{\phi_{1,f''}}\|_{S_r\to S_r}))^{1/1-\theta})\|f''\|_\infty.$$

  From (6.4), we know

$$\|M_{\phi_{1,f''}}\|_{S_r\to S_r} \le \|M_{\phi_{1,f''}}\|^{\theta}_{S_p\to S_p}\|f''\|^{1-\theta}_\infty,$$

  where $\theta \in (0,1)$ is such that $\theta/p + (1-\theta)/2 = 1/r$. This allows us to estimate the growth of $\|M_{\phi_{1,f''}}\|_{S_r\to S_r}$ for $p \to \infty$ as follows.

  We know that $\|M_{\phi_{1,f''}}\|_{S_p\to S_p} = O(p^3)$, hence

$$\|M_{\phi_{1,f''}}\|^{\theta}_{S_p\to S_p} \lesssim p^{3\theta}.$$

  By elementary calculations, we have

$$\theta = \frac{\frac{1}{2} - \frac{1}{r}}{\frac{1}{2} - \frac{1}{p}} = \frac{2}{p-2}.$$

  This allows us to conclude

$$\|M_{\phi_{1,f''}}\|^{\theta}_{S_p\to S_p} \lesssim p^{3\theta} = \exp(6\tfrac{\ln p}{p-2}) \to 1, \quad p \to \infty,$$

  thus $\|M_{\phi_{1,f''}}\|_{S_r\to S_r} = O(1)$. Hence altogether, $C_{p,2} = O(p^3)$.

For the alternative proof, the linear term was shown to be bounded by Theorem 1.5. The constant in this theorem is given explicitly as $\frac{p^2}{p-1} = O(p)$.

Altogether, we conclude that the constant in the original proof in [35] is of order $O(p^{12})$, whereas our alternative proof gives a constant of order $O(p^7)$. This improvement has two sources – the use of Theorem 1.5 for the linear term, and the fact that our specific operator contains no paraproducts.

	Original proof (Section 6.1)	Alternative proof (Section 6.2)
Triangular truncations	$O(p^3)$	
Toeplitz form Schur mult.	$O(p^6)$	$O(p^3)$
Linear Schur multiplier	$O(p^3)$	$O(p)$
Total	$O(p^{12})$	$O(p^7)$

Table 6.1: Comparison of the growth rate of the constant in Theorem 1.3 in the case $S_p \times S_p \to S_{p/2}$.

In the case where paraproducts do not vanish, which is generally to be expected for Calderón-Zygmund operators with a kernel that is not of convolution type, the bound yielded by Theorem 1.4 increases. While the dyadic shifts in Section 6.2 have a bound of order $O(p^3)$, they are dominated by the paraproducts, which have a bound of order $O(p^6)$. This is mainly due to the bound of order $O(p^5)$ on the linear paraproducts in Theorem 6.14.

It is however not known if the determined constants are optimal. Improvements may furthermore be possible by avoiding the use of triangular truncations in the bilinear transference approach, this is future research.

7

Outlook: Decomposition of higher order divided differences

In this section, we give an overview over our ongoing work on proving Theorem 1.3 for any $n \in \mathbb{N}$ using multilinear transference. We first illustrate the challenges arising in higher order cases, before giving a systematic decomposition of divided differences. While a general statement is not yet available, we will finally demonstrate this approach with an example.

In Theorem 4.3, we have already shown that the linear Schur multiplier with a symbol given by an n-th order divided difference in two variables is bounded. The challenge lies in finding a suitable decomposition of the original divided difference, which not only yields multilinear Toeplitz symbols and divided differences in two variables, but does this in such a way that the associated Schur multiplier can be decomposed accordingly using Lemma 2.15. This was not problematic in the bilinear case: a linear multiplier acting on the indices (λ_0, λ_1) or (λ_1, λ_2) can always be applied before a bilinear multiplier, and a linear multiplier acting on (λ_0, λ_2) can be applied after a bilinear multiplier. We illustrate the problems that can arise in the third order case in the following example.

Example 7.1 (Unsuitable decomposition of a third-order divided difference)**.** Let $f \in C^3(\mathbb{R})$ and let $\lambda_0, \dots, \lambda_3 \in \mathbb{R}$ be such that $\lambda_0 > \lambda_3 > \lambda_1 > \lambda_2$. Permuting variables and applying Lemma 3.1 twice with $\mu = \lambda_3$ yields

$$f^{[3]}(\lambda_0, \lambda_1, \lambda_2, \lambda_3) = f^{[3]}(\lambda_0, \lambda_3, \lambda_1, \lambda_2) = \frac{\lambda_0 - \lambda_3}{\lambda_0 - \lambda_2} f^{[3]}(\lambda_0, \lambda_3, \lambda_3, \lambda_1) + \frac{\lambda_3 - \lambda_2}{\lambda_0 - \lambda_2} f^{[3]}(\lambda_3, \lambda_3, \lambda_1, \lambda_2)$$

$$= \frac{\lambda_0 - \lambda_3}{\lambda_0 - \lambda_2} \left(\frac{\lambda_0 - \lambda_3}{\lambda_0 - \lambda_1} f^{[3]}(\lambda_0, \lambda_3, \lambda_3, \lambda_3) + \frac{\lambda_3 - \lambda_1}{\lambda_0 - \lambda_1} f^{[3]}(\lambda_3, \lambda_3, \lambda_3, \lambda_1) \right) + \frac{\lambda_3 - \lambda_2}{\lambda_0 - \lambda_2} f^{[3]}(\lambda_3, \lambda_3, \lambda_1, \lambda_2).$$

Let us focus on the term

$$m_{1,3} : (\lambda_0, \dots, \lambda_3) \mapsto \frac{\lambda_0 - \lambda_3}{\lambda_0 - \lambda_2} \frac{\lambda_3 - \lambda_1}{\lambda_0 - \lambda_1} f^{[3]}(\lambda_3, \lambda_3, \lambda_3, \lambda_1).$$

This symbol is a product of Toeplitz form symbols and a divided difference in two variables. However, the corresponding Schur multiplier cannot be decomposed into Schur multipliers associated with its component functions. To see this, let x, y, z be suitable Schatten space elements and consider

$$M_{m_{1,3}}(x, y, z) = \sum_{\lambda_0, \dots, \lambda_3} \frac{\lambda_0 - \lambda_3}{\lambda_0 - \lambda_2} \frac{\lambda_3 - \lambda_1}{\lambda_0 - \lambda_1} f^{[3]}(\lambda_3, \lambda_3, \lambda_3, \lambda_1) p_{\lambda_0} x p_{\lambda_1} y p_{\lambda_2} z p_{\lambda_3}. \tag{7.1}$$

- Letting the linear multiplier with symbol $\phi_{1,3} : (\mu_1, \mu_3) \mapsto f^{[3]}(\mu_3, \mu_3, \mu_3, \mu_1)$ act first yields

$$M_{m_{1,3}}(x, y, z) = \sum_{\lambda_0, \dots, \lambda_3} \frac{\lambda_0 - \lambda_3}{\lambda_0 - \lambda_2} \frac{\lambda_3 - \lambda_1}{\lambda_0 - \lambda_1} p_{\lambda_0} x p_{\lambda_1} \left(\sum_{\mu_1, \mu_3} f^{[3]}(\mu_3, \mu_3, \mu_3, \mu_1) p_{\mu_1} y p_{\lambda_2} z p_{\mu_3} \right) p_{\lambda_3}$$

$$= \sum_{\lambda_0, \dots, \lambda_3} \frac{\lambda_0 - \lambda_3}{\lambda_0 - \lambda_2} \frac{\lambda_3 - \lambda_1}{\lambda_0 - \lambda_1} p_{\lambda_0} x p_{\lambda_1} M_{\phi_{1,3}} (y p_{\lambda_2} z) \, p_{\lambda_3}.$$

The remaining term is not of Schur multiplier form, since the projection p_{λ_2} is now inside the argument of $M_{\phi_{1,3}}$.

- Now consider the bilinear Schur multiplier with symbol

$$\psi_{0,2,3} : (\lambda_0, \lambda_2, \lambda_3) \mapsto \frac{\lambda_0 - \lambda_3}{\lambda_0 - \lambda_2}.$$

Applying it directly in (7.1) yields

$$\begin{aligned} M_{1,3}(x,y,z) &= \sum_{\lambda_0,\lambda_1,\lambda_3} \frac{\lambda_3 - \lambda_1}{\lambda_0 - \lambda_1} f^{[3]}(\lambda_3, \lambda_3, \lambda_3, \lambda_1) p_{\lambda_0} \left(\sum_{\mu_0,\mu_2\mu_3} \frac{\mu_0 - \mu_3}{\mu_0 - \mu_2} p_{\mu_0} x p_{\lambda_1} y p_{\mu_2} z p_{\mu_3} \right) p_{\lambda_3} \\ &= \sum_{\lambda_0,\lambda_1,\lambda_3} \frac{\lambda_3 - \lambda_1}{\lambda_0 - \lambda_1} f^{[3]}(\lambda_3, \lambda_3, \lambda_3, \lambda_1) p_{\lambda_0} M_{\psi_{0,2,3}}(x p_{\lambda_1} y, z) p_{\lambda_3}. \end{aligned}$$

The fact that p_{λ_1} is now in the argument of $M_{\psi_{0,2,3}}$ again means that this term is no longer of Schur multiplier form.

In this manner, one can see that symbols with arguments that are non-adjacent variables (here, $(\lambda_0, \lambda_2, \lambda_3)$ and (λ_1, λ_3)) are generally not well-suited for decomposing Schur multipliers.

This example shows that a "good" decomposition of the divided difference is one in which the component functions take adjacent variables $\lambda_j, \ldots, \lambda_{j+k}$ as arguments. In constructing such a decomposition, we follow a similar approach as in [35].

Step 1: Decomposition of $f^{[n]}$. As in Section 3.1, we restrict our symbols to a domain on which the order of $\lambda_0, \ldots, \lambda_n \in \mathbb{R}$ is fixed. Set $\lambda_{n+1} := \lambda_0$, $\lambda_{-1} := \lambda_n$. Using this cyclic notation, there now always exists a $j_* \in \{1, \ldots, n\}$ such that $\lambda_{j_*-1} \le \lambda_{j_*+1} \le \lambda_{j_*}$ or $\lambda_{j_*+1} \le \lambda_{j_*-1} \le \lambda_{j_*}$. For example, if $\lambda_0 \le \ldots \le \lambda_n$, then $j_* = n$ fulfills the second condition.

Assuming the second condition, applying Lemma 3.1 with $\mu = \lambda_{j_*-1}$ yields

$$\begin{aligned} f^{[n]}(\lambda_0, \ldots, \lambda_n) &= \frac{\lambda_{j_*+1} - \lambda_{j_*-1}}{\lambda_{j_*+1} - \lambda_{j_*}} f^{[n]}(\lambda_0, \ldots, \lambda_{j_*-1}, \lambda_{j_*-1}, \lambda_{j_*+1}, \ldots, \lambda_n) \\ &\quad + \frac{\lambda_{j_*-1} - \lambda_{j_*}}{\lambda_{j_*+1} - \lambda_{j_*}} f^{[n]}(\lambda_0, \ldots, \lambda_{j_*-1}, \lambda_{j_*-1}, \lambda_{j_*}, \lambda_{j_*+2}, \ldots, \lambda_n) \\ &=: m_g + m_b. \end{aligned}$$

Step 2: Decomposition of the "good" term m_g. First consider the term m_g. Its associated Schur multiplier, applied to suitable Schatten space elements $x_1, \ldots, x_n$, can be decomposed as

$$\begin{aligned} &\sum_{\lambda_0,\ldots,\lambda_n} \frac{\lambda_{j_*+1} - \lambda_{j_*-1}}{\lambda_{j_*+1} - \lambda_{j_*}} f^{[n]}(\lambda_0, \ldots, \lambda_{j_*-1}, \lambda_{j_*-1}, \lambda_{j_*+1}, \ldots, \lambda_n) p_{\lambda_0} x_1 p_{\lambda_1} \cdots p_{\lambda_{n-1}} x_n p_{\lambda_n} \\ &= \sum_{\lambda_0,\ldots,\lambda_n} f^{[n]}(\lambda_0, \ldots, \lambda_{j_*-1}, \lambda_{j_*-1}, \lambda_{j_*+1}, \ldots, \lambda_n) p_{\lambda_0} x_1 p_{\lambda_1} \cdots p_{\lambda_*-1} \\ &\quad \times \left(\sum_{\mu_{j_*+1} \le \mu_{j_*-1} \le \mu_{j_*}} \frac{\mu_{j_*+1} - \mu_{j_*-1}}{\mu_{j_*+1} - \mu_{j_*}} p_{\mu_{j_*-1}} x_{j_*} p_{\mu_{j_*}} x_{j_*+1} p_{\mu_{j_*+1}} \right) p_{\lambda_{j_*+1}} \cdots p_{\lambda_{n-1}} x_n p_{\lambda_n}. \end{aligned}$$

We have chosen j_* in such a way that on our chosen domain (with fixed order of $\lambda_0, \ldots, \lambda_n$), the fraction is always bounded. In fact, we recognise the Schur multiplier M_{ψ_0} from Section **??** and write

$$\begin{aligned} &\sum_{\lambda_0,\ldots,\lambda_n} \frac{\lambda_{j_*+1} - \lambda_{j_*-1}}{\lambda_{j_*+1} - \lambda_{j_*}} f^{[n]}(\lambda_0, \ldots, \lambda_{j_*-1}, \lambda_{j_*-1}, \lambda_{j_*+1}, \ldots, \lambda_n) p_{\lambda_0} x_1 p_{\lambda_1} \cdots p_{\lambda_{n-1}} x_n p_{\lambda_n} \\ &= \sum_{\lambda_0,\ldots,\lambda_n} f^{[n]}(\lambda_0, \ldots, \lambda_{j_*-1}, \lambda_{j_*-1}, \lambda_{j_*+1}, \ldots, \lambda_n) p_{\lambda_0} x_1 p_{\lambda_1} \cdots \\ &\qquad \times p_{\lambda_*-1} M_{\psi_0}(x_{j_*}, x_{j_*+1}) p_{\lambda_{j_*+1}} \cdots p_{\lambda_{n-1}} x_n p_{\lambda_n}. \\ &= M_{\phi_{j_*,n}}(x_1, \ldots, x_{j-1}, M_{\psi_0}(x_{j_*}, x_{j_*+1}), x_{j_*+2}, \ldots, x_n), \end{aligned}$$

where we used notation from Section 2.2 and

$$\phi_{j_*,n} : (\lambda_0, \ldots, \lambda_{n-1}) \mapsto f^{[n]}(\lambda_0, \ldots, \lambda^{(2)}_{j_*-1}, \lambda_{j_*+1}, \ldots, \lambda_{n-1}).$$

Since we have shown the boundedness of M_{ψ_0} in Section 5, it remains to show the boundedness of the $n-1$-linear multiplier $M_{\phi_{j_*,n}}$. Note that the map $\phi_{j_*,n}$ is again a divided difference and can be further decomposed as in Step 1.

Step 3: Decomposition of the "bad" term m_b. Since the Toeplitz form symbol in m_b takes as variables $(\lambda_{j_*-1}, \lambda_{j_*}, \lambda_{j_*+1})$, while the divided difference takes $(\lambda_0, \ldots, \lambda_{j_*-1}, \lambda_{j_*}, \lambda_{j_*+2}, \ldots, \lambda_n)$ as variables, these symbols are not suitable in the sense of Lemma 2.15 for the decomposition of Schur multipliers.

However, if one is willing to use linear transference as in [35], then one can decompose this function as

$$\begin{aligned}&\frac{\lambda_{j_*-1} - \lambda_{j_*}}{\lambda_{j_*+1} - \lambda_{j_*}} f^{[n]}(\lambda_0, \ldots, \lambda_{j_*-1}, \lambda_{j_*-1}, \lambda_{j_*}, \lambda_{j_*+2}, \ldots, \lambda_n)\\ &= \int_{\mathbb{R}} g(s)(\lambda_{j_*-1} - \lambda_{j_*})^{is}(\lambda_{j_*+1} - \lambda_{j_*})^{-is} f^{[n]}(\lambda_0, \ldots, \lambda_{j_*-1}, \lambda_{j_*-1}, \lambda_{j_*}, \lambda_{j_*+2}, \ldots, \lambda_n) ds\end{aligned}$$

by Lemma 6.2. By setting

$$\phi'_{j_*,n}(\lambda_0, \ldots, \lambda_{j_*-1}, \lambda_{j_*}, \lambda_{j_*+2}, \ldots, \lambda_n) = f^{[n]}(\lambda_0, \ldots, \lambda^{(2)}_{j_*-1}, \lambda_{j_*}, \lambda_{j_*+2}, \ldots, \lambda_n)$$

we decompose the corresponding Schur multiplier as

$$\begin{aligned}&M_{M_2}(x_1, \ldots, x_n)\\ &= \int_{\mathbb{R}} g(s) \sum_{\lambda_0, \ldots, \lambda_n} \phi'_{j_*,n}(\lambda_0, \ldots, \lambda_n) p_{\lambda_0} x_1 p_{\lambda_1} \ldots p_{\lambda_{j_*-1}} (M_{\varphi_s} x_{j_*}) p_{\lambda_{j_*}} (M_{\varphi_{-s}} x_{j_*+1}) p_{\lambda_{j_*+1}} x_{j_*+2} \ldots p_{\lambda_n} ds,\end{aligned}$$

where φ_s is as in the proof of Theorem 6.3. Note that since $\phi'_{j_*,n}$ is does not depend on λ_{j_*+1}, we have

$$\begin{aligned}&M_{M_2}(x_1, \ldots, x_n)\\ &= \int_{\mathbb{R}} g(s) \sum_{\lambda_0, \ldots, \lambda_n} \phi'_{j_*,n}(\lambda_0, \ldots, \lambda_n) p_{\lambda_0} x_1 p_{\lambda_1} \ldots p_{\lambda_{j_*-1}} (M_{\varphi_s} x_{j_*}) p_{\lambda_{j_*}} (M_{\varphi_{-s}} x_{j_*+1}) x_{j_*+2} p_{j_*+2} \ldots p_{\lambda_n} ds\\ &= \int_{\mathbb{R}} g(s) M_{\phi'_{j_*,n}}(x_1, \ldots, x_{j_*-1}, M_{\varphi_s} x_{j_*}, (M_{\varphi_{-s}} x_{j_*+1}) x_{j_*+2}, \ldots, x_n) ds.\end{aligned}$$

It is hence sufficient to show the boundedness of $M_{\phi'_{j_*,n}}$.

Step 4: Boundedness of Schur multipliers of divided differences in $k < n$ variables In the steps above, we have reduced the proof to inductively showing that Schur multipliers M_{ϕ_l}, where ϕ_l is given by a divided differences function of order n with $k+1$ pairwise different variables and $k < n$, i.e.

$$\phi_l(\lambda_0, \ldots, \lambda_k) := f^{[n]}(\lambda_0^{(l_0)}, \ldots, \lambda_k^{(l_k)})$$

such that $l_0 + \ldots + l_k = n + 1$. For $k = 1$, we have shown this statement in Theorem 4.3. Repeated application of the steps outlined above eventually leads to a reduction of the number of variables in the divided difference functions, such that the proof can further be reduced to the $k = 1$ case.

The approach outlined above is still unsatisfying, as it relies on linear transference. Let us anecdotally demonstrate that this approach can in fact yield a suitable decomposition of $f^{[3]}$ on the domain considered in Example 7.1, which does not require linear transference as in Step 3 above.

Example 7.2. Let $f \in C^3(\mathbb{R})$ and let $\lambda_0, \ldots, \lambda_3 \in \mathbb{R}$ be such that $\lambda_0 > \lambda_3 > \lambda_1 > \lambda_2$. Choose $j_* = 0$, then $\lambda_{j_*+1} \leq \lambda_{j_*-1} \leq \lambda_{j_*}$. Applying Lemma 3.1 with $\mu = \lambda_{j_*-1} = \lambda_3$ yields

$$f^{[3]}(\lambda_0, \lambda_1, \lambda_2, \lambda_3) = f^{[3]}(\lambda_0, \lambda_3, \lambda_1, \lambda_2) = \frac{\lambda_0 - \lambda_3}{\lambda_0 - \lambda_1} f^{[3]}(\lambda_0, \lambda_3, \lambda_3, \lambda_2) + \frac{\lambda_3 - \lambda_1}{\lambda_0 - \lambda_1} f^{[3]}(\lambda_3, \lambda_3, \lambda_1, \lambda_2).$$

We apply Lemma 3.1 again, with $\mu = \lambda_3$ in the first summand and $\mu = \lambda_1$ in the second summand to get

$$\begin{aligned}
&\frac{\lambda_0-\lambda_3}{\lambda_0-\lambda_1} f^{[3]}(\lambda_0,\lambda_3,\lambda_3,\lambda_2) + \frac{\lambda_3-\lambda_1}{\lambda_0-\lambda_1} f^{[3]}(\lambda_3,\lambda_3,\lambda_1,\lambda_2)\\
=&\frac{\lambda_0-\lambda_3}{\lambda_0-\lambda_1}\left(\frac{\lambda_0-\lambda_3}{\lambda_0-\lambda_2} f^{[3]}(\lambda_0,\lambda_3,\lambda_3,\lambda_3) + \frac{\lambda_3-\lambda_2}{\lambda_0-\lambda_2} f^{[3]}(\lambda_3,\lambda_3,\lambda_3,\lambda_2)\right)\\
&+\frac{\lambda_3-\lambda_1}{\lambda_0-\lambda_1}\left(\frac{\lambda_3-\lambda_1}{\lambda_3-\lambda_2} f^{[3]}(\lambda_3,\lambda_3,\lambda_1,\lambda_1) + \frac{\lambda_1-\lambda_2}{\lambda_3-\lambda_2} f^{[3]}(\lambda_3,\lambda_1,\lambda_1,\lambda_2)\right).
\end{aligned}$$

One final application of Lemma 3.1 in the last summand with $\mu = \lambda_1$ yields

$$\begin{aligned}
f^{[3]}(\lambda_0,\lambda_1,\lambda_2,\lambda_3) = &\frac{\lambda_0-\lambda_3}{\lambda_0-\lambda_1}\left(\frac{\lambda_0-\lambda_3}{\lambda_0-\lambda_2} f^{[3]}(\lambda_0,\lambda_3,\lambda_3,\lambda_3) + \frac{\lambda_3-\lambda_2}{\lambda_0-\lambda_2} f^{[3]}(\lambda_3,\lambda_3,\lambda_3,\lambda_2)\right)\\
&+\frac{\lambda_3-\lambda_1}{\lambda_0-\lambda_1}\frac{\lambda_3-\lambda_1}{\lambda_3-\lambda_2} f^{[3]}(\lambda_3,\lambda_3,\lambda_1,\lambda_1)\\
&+\frac{\lambda_3-\lambda_1}{\lambda_0-\lambda_1}\frac{\lambda_1-\lambda_2}{\lambda_3-\lambda_2}\left(\frac{\lambda_3-\lambda_1}{\lambda_3-\lambda_2} f^{[3]}(\lambda_3,\lambda_1,\lambda_1,\lambda_1) + \frac{\lambda_1-\lambda_2}{\lambda_3-\lambda_2} f^{[3]}(\lambda_1,\lambda_1,\lambda_1,\lambda_2)\right).
\end{aligned}$$

The first, second, and last divided differences in the formula above take adjacent indices as values (with $\lambda_4 := \lambda_0$), hence their associated linear Schur multipliers can immediately be applied, whereas the linear multipliers in the third and fourth summand need to be composed with a bilinear Schur multiplier. Applied to Schatten space elements x_1, x_2, x_3, each summand can be expressed as the following composition of Schur multipliers. Here, the multilinear symbols are to be understood as being restricted to the domain where $\lambda_0 > \lambda_3 > \lambda_1 > \lambda_2$. On this domain, we have the decomposition

$$\begin{aligned}
M_{f^{[3]}}(x_1,x_2,x_3) = &M_{f_1}(M_{m_1}(x_1,x_2,x_3)) + M_{m_2}(x_1,x_2,M_{f_2}x_3)\\
&+M_{m_{3,1}}(x_1, M_{f_3}(M_{m_{3,2}}(x_2,x_3)))\\
&+M_{m_{4,1}}(x_1, M_{f_4}(M_{m_{4,2}}(x_2,x_3))) + M_{m_{5,1}}(x_1, M_{m_{5,2}}(M_{f_5}x_2,x_3)),
\end{aligned}$$

where the symbols are defined as follows.

$$f_1 : (\lambda_0,\lambda_3) \mapsto f^{[3]}(\lambda_0,\lambda_3,\lambda_3,\lambda_3), \quad m_1 : (\lambda_0,\lambda_1,\lambda_2,\lambda_3) \mapsto \frac{\lambda_0-\lambda_3}{\lambda_0-\lambda_1}\frac{\lambda_0-\lambda_3}{\lambda_0-\lambda_2},$$

$$f_2 : (\lambda_2,\lambda_3) \mapsto f^{[3]}(\lambda_3,\lambda_3,\lambda_3,\lambda_2), \quad m_2 : (\lambda_0,\lambda_1,\lambda_2,\lambda_3) \mapsto \frac{\lambda_0-\lambda_3}{\lambda_0-\lambda_1}\frac{\lambda_3-\lambda_2}{\lambda_0-\lambda_2},$$

$$\begin{gathered}
f_3 : (\lambda_1,\lambda_3) \mapsto f^{[3]}(\lambda_3,\lambda_3,\lambda_1,\lambda_1),\\
m_{3,1} : (\lambda_0,\lambda_1,\lambda_3) \mapsto \frac{\lambda_3-\lambda_1}{\lambda_0-\lambda_1}, \quad m_{3,2} : (\lambda_1,\lambda_2,\lambda_3) \mapsto \frac{\lambda_3-\lambda_1}{\lambda_3-\lambda_2},
\end{gathered}$$

$$\begin{gathered}
f_4 : (\lambda_1,\lambda_3) \mapsto f^{[3]}(\lambda_3,\lambda_1,\lambda_1,\lambda_1),\\
m_{4,1} : (\lambda_0,\lambda_1,\lambda_3) \mapsto \frac{\lambda_3-\lambda_1}{\lambda_0-\lambda_1}, \quad m_{4,2} : (\lambda_1,\lambda_2,\lambda_3) \mapsto \frac{\lambda_1-\lambda_2}{\lambda_3-\lambda_2}\frac{\lambda_3-\lambda_1}{\lambda_3-\lambda_2},
\end{gathered}$$

$$\begin{gathered}
f_5 : (\lambda_1,\lambda_2) \mapsto f^{[3]}(\lambda_1,\lambda_1,\lambda_1,\lambda_2),\\
m_{5,1} : (\lambda_0,\lambda_1,\lambda_3) \mapsto \frac{\lambda_3-\lambda_1}{\lambda_0-\lambda_1}, \quad m_{5,2} : (\lambda_1,\lambda_2,\lambda_3) \mapsto \left(\frac{\lambda_1-\lambda_2}{\lambda_3-\lambda_2}\right)^2.
\end{gathered}$$

We can already see in the example above some indication on how "bad" terms can become unproblematic with further applications of Lemma 3.1. For example, the symbols in the term

$$\frac{\lambda_0-\lambda_3}{\lambda_0-\lambda_1} f^{[3]}(\lambda_0,\lambda_3,\lambda_3,\lambda_2)$$

are incompatible, yet the second application of Lemma 3.1 yields that this term is equal to

$$\frac{\lambda_0 - \lambda_3}{\lambda_0 - \lambda_1}\left(\frac{\lambda_0 - \lambda_3}{\lambda_0 - \lambda_2} f^{[3]}(\lambda_0, \lambda_3, \lambda_3, \lambda_3) + \frac{\lambda_3 - \lambda_2}{\lambda_0 - \lambda_2} f^{[3]}(\lambda_3, \lambda_3, \lambda_3, \lambda_2)\right).$$

Here, both divided difference terms take adjacent variables as input, hence by Lemma 2.15 they are compatible with any multivariable symbol for the decomposition of Schur multipliers. It remains to show that such a decomposition for which the boundedness proof does not require linear transference can be found for all $f^{[n]}$ and all orderings of variables $\lambda_0, \dots, \lambda_n$, this is future research.

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