Algebraic Geometry Based Design for Generalized Sidelobe Canceler

Morency, Matthew W.; Vorobyov, Sergiy A.

DOI
10.1109/IEEECONF44664.2019.9048788

Publication date
2019

Document Version
Final published version

Published in
Conference Record - 53rd Asilomar Conference on Circuits, Systems and Computers, ACSSC 2019

Algebraic Geometry Based Design for Generalized Sidelobe Canceler

Matthew W. Morency* and Sergiy A. Vorobyov†
* Department of Microelectronics, Delft University of Technology, Delft, Netherlands
† Deptment of Signal Processing and Acoustics, Aalto University, Espoo, Finland
Email: M.W.Morency@tudelft.nl and sergiy.vorobyov@aalto.fi

***Abstract*—Generalized sidelobe canceler (GSC) uses a two step procedure in order to produce a beampattern with a fixed mainlobe and suppressed sidelobes. In the first step, a beampattern with a fixed response in the look direction is produced by convolving a vector of constraints with a normalized beamforming vector with the desired mainlobe response. In the second step, the signals in the look direction are blocked out using so-called blocking matrix, while the output power is minimized. Observing that for Griffiths-Jim GSC the beamforming vector contains the coefficients of a polynomial with at least one root at 1, we find here that all rows of a blocking matrix should be the coefficients of polynomials from the polynomial ideal with a root at 1. This allows us to reveal and exploit the underlying algebraic structure for GSC blocking matrix design using methods from computational algebraic geometry. It also allows to arrive to and prove several generalized statements. For example, the necessary and sufficient condition for a signal to be blocked can be easily found. The condition to a row-space of blocking matrix for blocking multiple signals impinging upon the array from multiple directions can also be easily formulated. The linear independence of rows of blocking matrix implies that all the corresponding polynomial share a single root. In general, understanding the algebraic structure that GSC's blocking matrix has to satisfy makes the GSC's design simpler and more intuitive.**

***Index Terms*—Adaptive beamforming, Algebraic geometry, Blocking matrix design, Generalized sidelobe canceler**

I. Introduction

Adaptive beamforming is a powerful technique to significantly improve the antenna array output signal-to-interference-plus-noise ratio (SINR) as well as other performance characteristics which found a large number of applications in multiple areas [1]– [6]. In many signal processing applications, an adaptive beamforming technique known as generalized sidelobe canceler (GSC) plays a key role for faster adaptation, dimensionality reduction, and combining spatial and temporal constraints [7]– [11]. GSC uses a two step procedure to produce a beampattern with a fixed mainlobe and suppressed sidelobes. In the first step, a beampattern with a fixed response in the look direction is produced by convolving a vector of constraints with a normalized beamforming vector with the desired mainlobe response. In the second step, the signals in the look directions are blocked out, while the output power is minimized. The theory and practice of GSC is well understood, but the selection of the so-called blocking matrix used in the second step remains to be ad hoc and application dependent.

In this paper, following the recent success of algebraic geometry [12]– [14] applications in array processing [15]–[18], we build a general theory regarding the selection of the blocking matrix in GSC. The main principles of such design and the main properties are therefore explained here.

II. Adaptive beamforming and GSC

For adaptive beamforming design [4]– [6], the narrowband signal at the output of an N-antenna receive array is given by

$$\mathbf{x}(t) = \mathbf{s}(t) + \mathbf{i}(t) + \mathbf{n}(t) \tag{1}$$

where $\mathbf{s}(t)$, $\mathbf{i}(t)$, and $\mathbf{n}(t)$ are statistically independent vectors corresponding to the signal of interest (SOI), interference, and noise, respectively. In the case of point source signal, $\mathbf{s}(t)$ is expressed as $\mathbf{s}(t) = s(t)\mathbf{a}$, where $s(t)$ is the SOI waveform and $\mathbf{a}$ is its steering vector (also called as the array response or spatial signature of the SOI).

The receive beamformer output is given as

$$y(t) = \mathbf{w}^H \mathbf{x}(t) \tag{2}$$

where $\mathbf{w}$ is the $N \times 1$ vector of beamformer complex weight coefficients and $(\cdot)^H$ stands for Hermitian transpose. The beamforming problem is to find an optimal $\mathbf{w}$ maximizing the beamformer output signal-to-interference-plus-noise ratio (SINR):

$$\text{SINR} = \frac{\sigma_s^2 |\mathbf{w}^H \mathbf{a}|^2}{\mathbf{w}^H \mathbf{R}_{i+n} \mathbf{w}} \tag{3}$$

where σ_s^2 is the SOI power, $\mathbf{R}_{i+n}$ is the interference-plus-noise covariance matrix, and $|\cdot|$ denoted magnitude. In practical applications, the true covariance $\mathbf{R}_{i+n}$ is unavailable and, consequently, the data sample covariance matrix estimate:

$$\hat{\mathbf{R}} = \frac{1}{T} \sum_{t=1}^{T} \mathbf{x}(t)\mathbf{x}^H(t) \tag{4}$$

is used. Here T is the number of available snapshots and $\mathbf{x}(t)$ are the beamformer training data.

The SINR maximization problem is equivalent to the following well-known minimum variance distortionless response (MVDR) beamforming convex problem [4]:

$$\underset{\mathbf{w}}{\text{minimize}} \quad \mathbf{w}^H \hat{\mathbf{R}} \mathbf{w} \quad \text{subject to} \quad |\mathbf{w}^H \mathbf{a}| = 1. \tag{5}$$

In many practical applications, however, the source cannot be localized as a point source [8] and then multiple distortionless constraints have to be enforced. The corresponding optimization problem than can be expressed as

$$\min_{\mathbf{w}} \mathbf{w}^H \hat{\mathbf{R}} \mathbf{w} \quad \text{subject to} \quad \mathbf{C}^H \mathbf{w} = \mathbf{f} \tag{6}$$

where $\mathbf{C}$ is the matrix of constrained directions and $\mathbf{f}$ is a vector of constraints. For example in the case of point constraints, the matrix $\mathbf{C}$ consists of a multiple steering vectors for different directions of arrival (DOAs) around the presumed DOA and $\mathbf{f}$ is the vector of ones. The solution of the optimization problem (6) can be easily found as

$$\mathbf{w}_{\text{opt}} = \hat{\mathbf{R}}^{-1}\mathbf{C}\left(\mathbf{C}^H\hat{\mathbf{R}}^{-1}\mathbf{C}\right)^{-1}\mathbf{f}. \tag{7}$$

Decompose it into two components, one in the constraint subspace, and another in the subspace orthogonal to the constraint subspace, we can write that

$$\begin{aligned}\mathbf{w}_{\text{opt}} &= (\mathbf{P}_{\mathbf{C}} + \mathbf{P}_{\mathbf{C}}^{\perp})\mathbf{w}_{\text{opt}} \\ &= \mathbf{C}(\mathbf{C}^H\mathbf{C})^{-1}\mathbf{C}^H\hat{\mathbf{R}}^{-1}\mathbf{C}\left(\mathbf{C}^H\hat{\mathbf{R}}^{-1}\mathbf{C}\right)^{-1}\mathbf{f} \\ &+ \mathbf{P}_{\mathbf{C}}^{\perp}\hat{\mathbf{R}}^{-1}\mathbf{C}\left(\mathbf{C}^H\hat{\mathbf{R}}^{-1}\mathbf{C}\right)^{-1}\mathbf{f}\end{aligned} \tag{8}$$

where $\mathbf{P}_{\mathbf{C}}$ and $\mathbf{P}_{\mathbf{C}}^{\perp}$ are respectively the projection and the orthogonal projection matrices to the space spanned by the columns of $\mathbf{C}$.

Generalizing this approach, we obtain the following decomposition for $\mathbf{w}_{\text{opt}}$:

$$\mathbf{w}_{\text{opt}} = \mathbf{w}_{\text{q}} - \mathbf{B}\mathbf{w}_{\text{a}} \tag{9}$$

where $\mathbf{w}_{\text{q}} = \mathbf{C}(\mathbf{C}^H\mathbf{C})^{-1}\mathbf{f}$ is the *quiescent* non-adaptive weight vector and the vector $\mathbf{w}_{\text{a}}$ is the new adaptive weight vector. The matrix $\mathbf{B}$ here is called the blocking matrix, and it is such that the condition

$$\mathbf{B}^H\mathbf{C} = 0 \tag{10}$$

should hold.

The adaptive beamformer of the form (9) is known as GSC, and its design boils down to the design of a blocking matrix $\mathbf{B}$, since the choice of $\mathbf{B}$ is not unique. For example, $\mathbf{B} = \mathbf{P}_{\mathbf{C}}^{\perp}$ can be selected as in the example above. However, then $\mathbf{B}$ becomes not a full-rank matrix, which is an issue in many applications. A more common choise for $\mathbf{B}$ is to assume that it has to be $N \times (N - M)$ full-rank matrix. Such $\mathbf{B}$ then will also lead to the dimensionality reduction for designing the adaptive part of the beamforming vector by M elements, which is of importance in many applications suach as, for example, over-the-horizon radar as well as speech processing. Indeed, in this case, the vectors $\mathbf{z} = \mathbf{B}^H\mathbf{x}$ and $\mathbf{w}_{\text{a}}$ both have shorter length $(N - M) \times 1$ relative to the $N \times 1$ vectors $\mathbf{x}$ and $\mathbf{w}_{\text{q}}$.

Since the constrained directions are *blocked* by the matrix $\mathbf{B}$, the SOI cannot be suppressed and, therefore, the weight vector $\mathbf{w}_{\text{a}}$ can adapt freely to suppress interference by minimizing the output GSC power:

$$\begin{aligned}Q_{\text{GSC}} &= (\mathbf{w}_{\text{q}} - \mathbf{B}\mathbf{w}_{\text{a}})^H\hat{\mathbf{R}}(\mathbf{w}_{\text{q}} - \mathbf{B}\mathbf{w}_{\text{a}}) \\ &= \mathbf{w}_{\text{q}}^H\hat{\mathbf{R}}\mathbf{w}_{\text{q}} - \mathbf{w}_{\text{q}}^H\hat{\mathbf{R}}\mathbf{B}\mathbf{w}_{\text{a}} - \mathbf{w}_{\text{a}}^H\mathbf{B}^H\hat{\mathbf{R}}\mathbf{w}_{\text{q}} \\ &+ \mathbf{w}_{\text{a}}^H\mathbf{B}^H\hat{\mathbf{R}}\mathbf{B}\mathbf{w}_{\text{a}}.\end{aligned} \tag{11}$$

The solution of (11) can be found as

$$\mathbf{w}_{\text{a,opt}} = \left(\mathbf{B}^H\hat{\mathbf{R}}\mathbf{B}\right)^{-1}\mathbf{B}^H\hat{\mathbf{R}}\mathbf{w}_{\text{q}} = \hat{\mathbf{R}}_{\mathbf{z}}^{-1}\hat{\mathbf{r}}_{y\mathbf{z}} \tag{12}$$

where $y(k) = \mathbf{w}_{\text{q}}^H\mathbf{x}(k)$, $\mathbf{z}(k) = \mathbf{B}^H\mathbf{x}(k)$, and $\hat{\mathbf{R}}_{\mathbf{z}} \approx \mathbf{R}_{\mathbf{z}}$ and $\hat{\mathbf{r}}_{y\mathbf{z}} \approx \mathbf{r}_{y\mathbf{z}}$ with $\mathbf{R}_{\mathbf{z}}$ and $\mathbf{r}_{y\mathbf{z}}$ being

$$\begin{aligned}\mathbf{R}_{\mathbf{z}} &= \mathrm{E}\{\mathbf{z}(k)\mathbf{z}^H(k)\} = \mathbf{B}^H\mathrm{E}\{\mathbf{x}(k)\mathbf{x}^H(k)\}\mathbf{B} \\ &= \mathbf{B}^H\mathbf{R}\mathbf{B} \\ \mathbf{r}_{y\mathbf{z}} &= \mathrm{E}\{\mathbf{z}(k)y^*(k)\} = \mathbf{B}^H\mathrm{E}\{\mathbf{x}(k)\mathbf{x}^H(k)\}\mathbf{w}_{\text{q}} \\ &= \mathbf{B}^H\mathbf{R}\mathbf{w}_{\text{q}}.\end{aligned}$$

Here $\mathbf{R}$ is the true data covariance matrix.

Hence, the solution for the adaptive part of the GSC is given by the well known Wiener-Hopf equation (12), and the GSC design problem boils down to designing an appropriate in some sense blocking matrix $\mathbf{B}$. Obviously, a proper blocking matrix should be composed by linearly independent vectors $\mathbf{b}_i$, i.e., $\mathbf{B} = [\mathbf{b}_1, \ldots, \mathbf{b}_{N-M}]$ such that (10) also holds, that is, $\mathbf{b}_i \perp \mathbf{c}_k$, $i = 1, \ldots, N - M$; $k = 1, \ldots, M$, where $\mathbf{c}_k$ is the kth column of $\mathbf{C}$. There exists many possible choices of $\mathbf{B}$. The choice of the blocking matrix is, however, typically ad hoc and application dependent. In this paper, we attempt to build a more general theory regarding the selection of the blocking matrix $\mathbf{B}$. It will be base on algebraic geometry, the basic concepts of which are then introduced in the next section.

III. Algebraic Geometry Preliminaries

The key insight of the algebraic geometry based design of the GSC's blocking matrix relies on a basic understanding of polynomial ideals (and their associated varieties) and their relationship to linear vector spaces. As these concepts are not commonly used in the signal processing literature, we introduce them here. First we must define an algebraic group.

Definition III.1. *A group is a set of elements $\mathcal{G}$ with a binary operation $\bullet$ with the following properties:*

Property 1. $a \bullet b \in \mathcal{G}$, $\forall a, b \in \mathcal{G}$
Property 2. $\exists e \in \mathcal{G} \mid \; a \bullet e = e \bullet a = a, \; \forall a \in \mathcal{G}$
Property 3. $\forall a \in \mathcal{G}$, $\exists b \in \mathcal{G} \mid a \bullet b = b \bullet a = e$
Property 4. $(a \bullet b) \bullet c = a \bullet (b \bullet c)$.

If a group G is commutative with respect to $\bullet$ then G is said to be Abelian. As an example, consider the set of all permutations of an n-tuple. This set is a group (in fact, it is known as the symmetric group, S_n), with operation of composition of functions.[1] In this case, the identity element is the zero permutation, which takes each element of the n-tuple to itself. This is not an Abelian group as composition of functions depends, generally speaking, on the order of the composition. An example of an Abelian group are the integers under addition.

[1]Note, that although it is common to refer to the operation $\bullet$ as multiplication or addition, the operation $\bullet$ need not be the conventional notion of multiplication or addition.

Remark 1. *Every vector space is an Abelian group with respect to vector addition, which is a property which will be used later.*

Building on this definition, we next introduce the concept of a subgroup.

Definition III.2. *A subgroup $H < G$ is a subset of G for which all of the group properties hold with respect to $\bullet$.*

Thus, every subgroup is a group unto itself which is also contained in $\mathcal{G}$.

Remark 2. *Since all vector spaces are Abelian groups with respect to vector addition, it follows that every subspace of a vector space is an Abelian subgroup of that space with respect to addition.*

Ideals are algebraic substructures of rings, however, which we now define.

Definition III.3. *A ring is a set $\mathcal{R}$ with operations $\bullet$ and $\diamond$ with the following properties:*
Property 1. $(\mathcal{R}, \bullet)$ *is an Abelian group*
Property 2. $\exists 1 \in \mathcal{R} \mid \; 1 \diamond r = r \diamond 1 = r, \; \forall r \in \mathcal{R}$
Property 3. $(a \diamond b) \diamond c = a \diamond (b \diamond c), \forall a, b, c \in \mathcal{R}$
Property 4. $a \diamond (b \bullet c) = (a \diamond b) \; \bullet (a \diamond c), \forall a, b, c \in \mathcal{R}$
Property 5. $(a \bullet b) \diamond c = (a \diamond c) \bullet (b \diamond c), \forall a, b, c \in \mathcal{R}.$

Properties 2 and 3 state that a ring has a multiplicative identity, and that a ring is associative respectively. Properties 4 and 5 of Definition III.3 state that in a ring, multiplication $\diamond$ is left and right distributive over addition $\bullet$. We will only consider commutative rings which have the addition property that $a \diamond b = b \diamond a, \forall a, b \in R$.

Definition III.4. *An ideal $\mathcal{I}$ in a commutative ring R is a subgroup of R with the following properties:*
Property 1. $\mathcal{I}$ is a subgroup of R
Property 2. $\forall a \in \mathcal{I}, \; r \in R, a \diamond r \in \mathcal{I}, r \diamond a \in \mathcal{I}$

A relevant example of a ring with a non-trivial ideal, which will be used in the contribution of this paper, is the *ring of univariate polynomials over a field* $\mathbb{K}$, which we will denote as $\mathbb{K}[x]$. To show that this is a commutative ring we need to show that the relevant properties of Definitions III.1 and III.3 hold.

The fact that the polynomials are an Abelian group under addition is obvious since $P(x) + Q(x) = Q(x) + P(x) = R(x) \in \mathbb{K}[x]$, $0 \in \mathbb{K}[x]$, and the additive inverse of a polynomial $P(x)$ is trivially $-P(x) \in \mathbb{K}[x]$.

To show that it is a commutative ring, we note that $A(x)B(x) = B(x)A(x) = C(x) \in \mathbb{K}[x]$ for any polynomials $A(x), B(x), C(x)$ with coefficients in $\mathbb{K}$. The set $\mathbb{K}[x]$ is associative with respect to both multiplication and addition, and polynomial multiplication is distributive over polynomial addition since $A(x)(B(x) + C(x)) = A(x)B(x) + A(x)C(x)$ and $(A(x) + B(x))C(x) = A(x)C(x) + B(x)C(x)$. Finally, we note that the set $\mathbb{K}[x]$ has the multiplicative identity 1, thus completing the proof.

It is important to mention that polynomials in $\mathbb{K}[x]$ also form a vector space over $\mathbb{K}$. Without restriction on the degree of the polynomials, this space has infinite dimension, and as such a finite dimensional vector space of polynomials implies a restriction of the degree of the polynomials to a finite number N. This vector space has a basis of $\{x^i, 0 \leq i \leq N\}$. We denote the space of polynomials with degree strictly less than N by $\mathbb{K}_N[x]$. Fig. 1 demonstrates that the subscript N corresponds to the dimension of the vector space, where $N-1$ is the restriction on the degree of the polynomials. Thus, there should be no confusion with the definition of $\mathbb{K}_N[x]$ being the space of polynomials of degree strictly less than N.

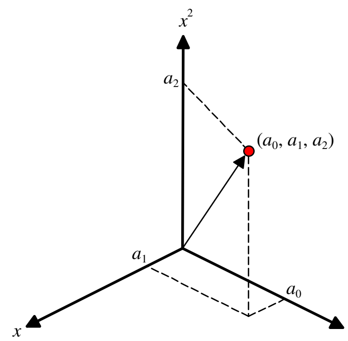

Fig. 1. Depiction of an element of $\mathbb{C}_3[x]$. Notice that a polynomial of degree 2, corresponds to a 3 dimensional vector. Hence, $\mathbb{C}_3[x]$ denotes the space of all polynomials of degree strictly less than 3.

Let $\mathcal{I}$ be the set of all univariate polynomials $\mathbb{C}[x]$ with a root at $\alpha_0 \in \mathbb{C}$. This set forms an ideal in the ring $\mathbb{C}[x]$, which we refer hereafter as a *polynomial ideal*. To see this, consider a polynomial with a single root at α_0. By Euclid's division algorithm, a univariate polynomial has a root at a point α_0 if and only if it can be written as $P(x) = Q(x)(x - \alpha_0)$. Consider Definition III.4 and let $P(x) = Q(x)(x - \alpha_0)$. Then $P(x)R(x) = Q(x)R(x)(x - \alpha_0)$ which is again in $\mathcal{I}$. The polynomials with a root at α_0 are also clearly a subgroup of $\mathbb{C}[x]$ since $P_1(x) - P_2(x) \in \mathcal{I}$, where $P_1(x) \triangleq Q_1(x)(x - \alpha_0)$, $P_2(x) \triangleq Q_2(x)(x - \alpha_0)$ for any $Q_1(x), Q_2(x) \in \mathbb{C}[x]$. This ideal is also an infinite dimensional vector space over $\mathbb{C}$. Let $A(x) = P(x)(x - \alpha_0)$ and $B(x) = Q(x)(x - \alpha_0)$. It is easy to show that

$$\begin{aligned}\alpha(A(x) + B(x)) &= \alpha A(x) + \alpha B(x), \alpha \in \mathbb{C}, A(x), B(x) \in \mathcal{I} \\ &= \alpha(P(x) + Q(x))(x - x_1) \\ &= \alpha R(x)(x - x_1), \; R(x) = P(x) + Q(x).\end{aligned}$$

Furthermore, it is easy to show that if the polynomials $A(x)$ and $B(x)$ are coprime, that is, if their greatest common divisor (gcd) is a constant, the ideal is the entire ring. To show this, we invoke Bezout's identity

$$ap + bq = gcd(p, q)$$

for some $a, b \in \mathcal{R}$ where $\mathcal{R}$ is a principal ideal domain (every ideal is generated by a single element). Since every univariate polynomial ideal is generated by a single element $f \in \mathcal{I}$ [13],

[14], the univariate polynomials are a principal ideal domain. Moreover, if $1 \in \mathcal{I}$, then $\mathcal{I}$ is the whole ring since $r \diamond 1, 1 \diamond r \in \mathcal{I}, \forall r \in R$. Thus, the only non-trivial subspaces which form an ideal in $\mathbb{C}[x]$ are the subspaces of polynomials with roots in common.

Since every ideal in $\mathbb{C}[x]$ is principal, i.e., can be generated by a single element $f \in \mathbb{C}[x]$, we denote $\mathcal{I} \subset \mathbb{C}[x]$ as$\langle f \rangle \triangleq \{f \cdot g \; \forall g \in \mathbb{C}[x]\}$, read as "the ideal generated by" f. The generator of an ideal $\mathcal{I} \subset \mathbb{C}[x]$ is given straightforwardly as $f = \prod_k (x - x_k)$. The restriction of this ideal to polynomials with degree less than N is written as $\langle f \rangle|_N$, which is a subspace of $\mathbb{C}^N$. To clarify the notations, it is important to note that $\mathcal{I}|_N \subset \mathbb{C}[x]$ is an algebraic object as a subset of the univariate polynomial ring, while $\langle f \rangle|_N$ is a vector subspace of $\mathbb{C}^N$. The key difference between the two is the notion of scalar multiplication under which both objects are closed. Scalars for the former are elements of the polynomial ring itself, while scalars for the latter are elements of $\mathbb{C}$.

Finally, polynomial ideals have an associated "variety." The variety associated with a polynomial ideal is the set of points on which the polynomials in that ideal are 0. As such, the variety "generates" an ideal. Formally, the ideal generated by a variety V is defined as

Definition III.5. $V \in \mathbb{K}^n : \mathcal{I}(V) \triangleq \{f \in \mathbb{K}[x_1, \cdots, x_n] \mid f(V) = 0\}$.

Importantly, in generating an ideal from a variety, the inclusion operation is *reversed*. To see this consider two varieties $V_1 \subset V_2$ in light of definition III.5. By definition of $\mathcal{I}(V)$, any $f \in \mathcal{I}(V_2) = 0, \; \forall p \in V_2$. But since $V_1 \subset V_2$, $f = 0, \; \forall p \in V_1$. Thus, $f \in \mathcal{I}(V_1)$. Therefore, $\mathcal{I}(V_2) \subset \mathcal{I}(V_1)$.

IV. GSC's Blocking Matrix Design via Algebraic Geometry

Assuming a uniform linear array structure, the matrix of directions $\mathbf{C}$ has a Vandermonde structure. Specifically, its L columns are the powers of a complex generator α_l. In the problem statement (6), the matrix $\mathbf{C}$ is involved to enforce a number of constraints on the beamforming vector $\mathbf{w}$. Such constraints can be used, for example, to control the location of nulls in the beampattern. The Griffiths-Jim beamformer is an example of such a constraint [11].

Obviously, in order for (10) to hold, the columns of the blocking matrix $\mathcal{C}(\mathbf{B}) \subset \mathcal{N}(\mathbf{C}^H)$, where $\mathcal{C}(\cdot)$ and $\mathcal{N}(\cdot)$ denote the column and nullspace of a matrix. Since $\mathbf{C}$ is a Vandermonde matrix, matrix multiplication becomes the evaluation of polynomials defined by the coefficients in the columns of $\mathbf{B}$ at α_l, the generators of the columns of $\mathbf{C}$. Thus, the null constraints are equivalent to designing the columns of $\mathbf{B}$ to be polynomials with roots at α_l.

We leverage the Vandermonde structure of $\mathbf{C}$ in order to develop a concise description of the nullspace of $\mathbf{C}$. Using the definition of the nullspace

$$\mathcal{N}(\mathbf{C}^H) \triangleq \{\mathbf{b} \in \mathbb{C}^N | \mathbf{C}^H \mathbf{b} = \mathbf{0}\} \tag{13}$$

it is easy to show that every vector in $\mathcal{N}(\mathbf{C}^H)$ describes the coefficients of a polynomial of degree $N-1$ with roots at $\alpha_1^*, \cdots, \alpha_l^*$, that is,

$$\mathbf{C}^H \mathbf{b} = \mathbf{0} \iff \sum_{i=0}^{N-1} (\alpha_l^*)^i \mathbf{b}_i = 0, \; \forall l \in 1, \cdots, L. \tag{14}$$

A polynomial $P(x)$ has a root at some point α if and only if $(x - \alpha)$ is a factor of $P(x)$ [13]. By induction, it can be seen that a polynomial $P(x)$ has roots at points $\alpha_1^*, \cdots, \alpha_l^*$ if and only if $P(x) = Q(x)B(x)$ where

$$Q(x) \triangleq \prod_{l=1}^{L} (x - \alpha_l^*). \tag{15}$$

From (14) and (15), $\mathcal{N}(\mathbf{C}^H)$ can be expressed as

$$\mathcal{N}(\mathbf{C}^H) = Q(x)\mathbb{C}_{N-L}[x]. \tag{16}$$

where $\mathbb{C}_{N-L}[x]$ denotes the space of all polynomials of degree strictly less than $N-L$. The degree is strictly less than $N-L$ as a constant polynomial is defined to have degree 0. $\mathbb{C}_{N-L}[x]$ has the standard polynomial basis of $\{1, x, x^2, \cdots, x^{N-L-1}\}$, and thus, a basis for $\mathcal{N}(\mathbf{C}^H)$ is $Q(x)\{1, x, x^2, \cdots, x^{N-L-1}\}$, or $\langle Q(x) \rangle | N$.

Let $\mathbf{q} \triangleq [(-1)^{L-1}s_{L-1}, (-1)^{L-2}s_{L-2}, \cdots, (-1)s_1, 1]^T$ where $s_1, \cdots, s_{L-1}$ are the elementary symmetric functions of $\alpha_1^*, \cdots, \alpha_l^*$. The k-th elementary symmetric function in L variables (in this case, $\alpha_1^*, \cdots, \alpha_l^*$) is the sum of the products of the k subsets of those L variables. For example, if $L = 3$ then

$$\begin{aligned} s_3 &= \alpha_1^* + \alpha_2^* + \alpha_3^* \\ s_2 &= (\alpha_1\alpha_2)^* + (\alpha_2\alpha_3)^* \\ s_1 &= (\alpha_1\alpha_2\alpha_3)^*. \end{aligned}$$

Let $\mathbf{q}' \triangleq [\mathbf{q}, 0, \cdots, 0]^T \; \in \mathbb{C}^N$. Then a basis of $\mathcal{N}(\mathbf{C}^H)$ is represented by the columns of the Toeplitz matrix

$$\mathbf{Q} = [\mathbf{q}', \mathbf{q}'_1, \cdots, \mathbf{q}'_{N-L-1}] \tag{17}$$

where $\mathbf{q}'_i$ is the i-th cyclical shift of $\mathbf{q}'$. For a polynomial $Q(x) = a_0 + a_1 x + \cdots + a_L x^L$, with roots $\alpha_1^*, \cdots, \alpha_l^*$, Viète's formulas yield the coefficients $a_0, \cdots, a_{L-1}$ as

$$\begin{aligned} s_1(\alpha_1^*, \cdots, \alpha_l^*) &= -\frac{a_{L-1}}{a_L} \\ s_2(\alpha_1^*, \cdots, \alpha_l^*) &= \frac{a_{L-2}}{a_L} \\ &\vdots \\ s_L(\alpha_1^*, \cdots, \alpha_l^*) &= (-1)^L \frac{a_0}{a_L}. \end{aligned}$$

Thus, the elements of the vector $\mathbf{q}$ are the coefficients of $Q(x)$, which are given as a function of the roots of $Q(x)$ by Viète's formulas, with $a_L = 1$.

Following this procedure for $\mathbf{c} = [1, 1, \ldots, 1]^T$, corresponding to a single broad-side signal, yields the Griffiths-Jim blocking matrix exactly [11], that is,

$$\mathbf{B}^H = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1 \end{bmatrix}.$$

Thus, the blocking matrix $\mathbf{B}$ corresponding to the Griffiths-Jim beamformer is actually a basis for the space of polynomials with a single root at $\alpha = 1$. However, following the procedure in this section, a general and complete description of all blocking matrices may be attained.

From this derivation we observe the direct correspondence between the number of roots of the polynomial $Q(x)$ and the size of the blocking matrix $\mathbf{B}$. From the Toeplitz structure of $\mathbf{B}$, and the fact that a polynomial with L roots has $L+1$ coefficients, the blocking matrix $\mathbf{B}$ has $N-L$ linearly independent columns. This fits exactly with the fundamental theorem of linear algebra (that rank and nullspace dimensions are complementary). However, our approach differs fundamentally from a singular value decomposition (SVD) based approach to setting nulls. Indeed, one could solve the blocking matrix problem by selecting the columns of $\mathbf{B}$ as the right singular vectors of $\mathbf{C}$. However, additional solutions to the problem also exist which are not amenable to solution via SVD.

Consider the case where we wish to set multiple roots corresponding to the same direction. If one were to attempt to construct a direction matrix $\mathbf{C}$ with multiple sources impinging from the same direction, the matrix would be rank deficient. Supposing for the moment that the matrix $\mathbf{C}$ had dimension $N \times L$, with two sources impinging from the same direction, its rank would only be $L-1$ due to the identical columns. From the fundamental theorem of linear algebra, $\mathbf{C}^T$ would have a null-space dimension of $N - L + 1$, and thus have $N - L + 1$ right singular vectors. However, by calculating $\mathbf{B}$ via Viéte's formulas, the matrix $\mathbf{B}$ would have dimension $N \times (N - L)$ with rank $N - L$, owing again to the Toeplitz structure. This solution is also fully contained in the nullspace of $\mathbf{C}$ which can be readily seen from the variety-ideal inclusion property. Specifically, if for two varieties $V_0 \subset V_1$, then $\mathcal{I}(V_1) \subset \mathcal{I}(V_0)$. The "varieties" in question here are sets of points. The variety specified by the SVD approach is the set of *unique* generators, whereas the variety in question in the proposed method would contain a double element. As the former is contained by the latter, and inclusion is reversed by generating an ideal from a variety, the solution arrived at by the proposed method is contained in the nullspace of $\mathbf{C}^T$, and thus achieves the blocking task. However, the solution is distinct from that achieved via SVD.

V. Conclusion

In this paper, a novel approch based on algebraic geometry for blocking matrix design in GSC has been developed. It provided, perhaps for the first time, a solid theoretical ground for the problem of blocking matrix design. For example, the necessary and sufficient condition for a signal to be blocked have been discusses. The condition to a row-space of blocking matrix for blocking multiple signals impinging upon the array from multiple directions have been also explained. In general, understanding the algebraic structure that the blocking matrix of GSC has to satisfy makes the design procedures simpler and more intuitive. Moreover, the GSC's blocking matrix design problem is related to many rank-constrained semidefinite programming problems with additional sum-of-squares-type constraints, which are common in such areas as downlink beamforming design for multiple-input multiple-output (MIMO) communications and transmit beamspace design in MIMO radar to name just a few. The study in this paper is also helpful to address these problems [18].

References

[1] J. Capon, R. J. Greenfield, and R. J. Kolker, "Multidimensional maximum-likelihood processing of a large aperture seismic array," *Proc. IEEE*, vol. 55, no. 2, pp. 192–211, Feb. 1967.

[2] B. Widrow and S. D. Steams, *Adaptive Signal Processing*, Prentice-Hall, 1985.

[3] H. L. Van Trees, *Optimum Array Processing. Part IV: Detection, Estimation, and Modulation Theory*, 1st Ed., New York: Wiley-Interscience, 2002.

[4] S. A. Vorobyov, "Adaptive and robust beamforming," in Academic Press Library in Signal Processing, Vol. 3, *Array and Statistical Signal Processing*, A. M. Zoubir, M. Viberg, R. Chellappa, and S. Theodoridis, Eds., Academic Press, 2014, pp. 503–552.

[5] S. A. Vorobyov, "Principles of minimum variance robust adaptive beamforming design," *Signal Processing*, vol. 93, no. 12, pp. 3264–3277, Dec. 2013.

[6] S. A. Vorobyov, A. B. Gershman, and Z.-Q. Luo, "Robust adaptive beamforming using worst-case performance optimization: A solution to the signal mismatch problem," *IEEE Trans. Signal Processing*, vol. 51, no. 2, pp. 313–324, Feb. 2003.

[7] S. P. Applebauma and D. J. Chapman, "Adaptive arrays withm ain beam constraints," *IEEE Trans. Antennas Propagat.*, vol. 24, no. 5, pp. 650–662, Sept. 1976.

[8] L. Frost, III, "An algorithm for linearly constrained adaptive array processing," *Proc. IEEE*, vol. 60, no. 8, pp. 926–935, Aug. 1972.

[9] L. J. Griffiths, "An adaptive beamformer whichi mplementsc on- straints using an auxiliary array preprocessor," in *Aspects of Signal Processing*, pt. 2, G. Tacconi, Ed., Dordrecht, Holland Reidel, 1977, pp. 517–522.

[10] L. J. Griffiths and C. W. Jim, "A generalized sidelobec ancelling structure for adaptive arrays," Signal Processing Lab., Dept. Elec. Eng., Univ. Colorado, Boulder, Tech. Rep. SPL 78-2, Nov. 1978.

[11] L. J. Griffiths and C. W. Jim, "An alternative approach to linearly constrained adaptive beamfonning," *IEEE Trans. Antennas Propagat.*, vol. 30, no. 1, pp. 27–34, Jan. 1982.

[12] A. Barvinok, *A Course in Convexity.* Graduate Studies in Mathematics, American Mathematical Society, vol. 54, 2002.

[13] E.B. Vinberg, *A Course in Algebra*. Moscow: Factorial Press, 2001.

[14] D. Cox, J. Little, D. O'Shea *Ideals, Varieties, and Algorithms*. Third Edition. Springer Science+Business Media, 2007.

[15] M. W. Morency and S. A. Vorobyov, "An algebraic approach to rank-constrained beamforming," in *Proc. 6th IEEE Int. Workshop Computational Advances in Multi-Sensor Adaptive Processing*, Cancun, Mexico, Dec. 2015, pp. 17–20.

[16] M. W. Morency, S. A. Vorobyov, and G. Leus, "An ideal-theoretic criterion for localization of an unknown number of sources," in *Proc. 50th Annual Asilomar Conf. Signals, Systems, and Computers*, Pacific Grove, California, USA, Nov. 2016, pp. 1499–1502.

[17] M. W. Morency, S. A. Vorobyov, and G. Leus, "Joint detection and localization of an unknown number of sources using algebraic structure of the noise subspace," *IEEE Trans. Signal Processing*, vol. 66, no. 17, pp. 4685–4700, Sept. 2018.

[18] M. W. Morency and S. A. Vorobyov, "An algebraic approach to a class of rank-constrained semi-definite programs with applications," `https://arxiv.org/abs/1610.02181`, Oct. 2016.