

Business Valuation in the Frequency Domain

A Dynamical Systems Approach

X. A. van Ardenne

Master of Science Thesis

Business Valuation in the Frequency Domain

A Dynamical Systems Approach

MASTER OF SCIENCE THESIS

For the degree of Master of Science in Systems and Control at Delft
University of Technology

X. A. van Ardenne

December 4, 2020

Faculty of Mechanical, Maritime and Materials Engineering (3mE) · Delft University of
Technology



The work in this thesis was supported by PriceWaterhouseCoopers. Their cooperation is hereby gratefully acknowledged.



Copyright © Delft Center for Systems and Control (DCSC)
All rights reserved.



DELFT UNIVERSITY OF TECHNOLOGY
DEPARTMENT OF
DELFT CENTER FOR SYSTEMS AND CONTROL (DCSC)

The undersigned hereby certify that they have read and recommend to the Faculty of
Mechanical, Maritime and Materials Engineering (3mE) for acceptance a thesis
entitled

BUSINESS VALUATION IN THE FREQUENCY DOMAIN

by

X. A. VAN ARDENNE

in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE SYSTEMS AND CONTROL

Dated: December 4, 2020

Supervisor(s):

Dr. ir. M. B. Mendel

Reader(s):

Dr. ir. A. J. J. van den Boom

Dr. ir. K. Batselier

R. Jellema

Abstract

Business valuation is a set of procedures used by financial market participants to determine the price they are willing to pay or receive for businesses. Valuations play a crucial role in financial reporting, capital budgeting, and investment analysis. Current approaches to business valuation rely on professional expertise, causing the valuation to be as good as the analyst's assumptions.

In this thesis, the entire valuation effort is translated into a systems and control problem. Economic engineering is used to formulate signal- and energy-based analogs for valuation concepts. A bond-graph model is developed to express general business dynamics as a set of differential equations. The model's analogy with the economic theory is shown by representing the model as a complex port-Hamiltonian system, and giving all elements and signals economic interpretations.

A theory is developed to perform valuations in the frequency domain. This theory extends both the existing economic-engineering framework, and the existing valuation effort. Business valuation is performed by using this theory, by using the bond-graph model for business dynamics. The developed theory goes beyond the field of business valuations, as frequency-domain valuation can be applied to any type of financial instrument.

Contents

Preface	xvii
1 Introduction	1
1-1 Economic Engineering for Frequency-Domain Valuations	1
1-2 Part 1: Business Valuation as a Systems and Control Problem	2
1-3 Part 2: A Fundamental Model for Business Dynamics	3
1-4 Part 3: Frequency-Domain Valuation and Financial Analysis	4
2 Background on the Business Economics and Valuations	9
2-1 Introduction	9
2-2 The Value of a Company	10
2-2-1 Value is the Company's Future Performance	10
2-2-2 Value as a Bandwidth and the Role of Future Scenarios	12
2-2-3 Cyclical Behavior of the Discount Rate	13
2-3 Current Valuation Techniques and their Shortcomings	13
2-3-1 The Discounted Cash Flow Method	13
2-3-2 Multiples and Asset-Based Methods	16
2-3-3 The Field of Research on Predictive Analytics	16
2-4 Business Economics	17
2-4-1 Rational Firm Behavior	18
2-4-2 The Firm as Part of the Macroeconomic System	18
2-4-3 Economic Laws and their Mathematical expression	19
2-4-4 Competition	21
2-5 Connected Markets	21
2-5-1 Commodity Market	22
2-5-2 Labor Market	23
2-5-3 Capital Asset Market	23
2-5-4 Product Market	24
2-6 Conclusions and Thesis Objective	24

3	The Economic-Engineering Analogy in Business Economics	31
3-1	Introduction	31
3-2	Signal-Based Modeling of Business Dynamics	32
3-2-1	Analog of Newtonian Mechanics Extended to Business Economics	32
3-2-2	Bond-Graphs Modeling for Business Dynamics	33
3-3	Business Economics in the Energy Domain	36
3-3-1	The Least Action Principle for Profit Maximization	36
3-3-2	The Firm's Earnings as the Kinetic Part of the Hamiltonian	38
3-3-3	Depreciation causes Economic Systems to be Non-Conservative	40
3-3-4	Interconnection of Port-Hamiltonian Systems	40
3-4	Conclusions and Next Steps	42
4	A General Bond-Graph Model for Business Dynamics	45
4-1	Introduction	45
4-2	The Firm as a Subsystem	46
4-2-1	Bond Graph for Profit Maximization	46
4-3	Commodity Market	48
4-3-1	Bond Graph for the Production Cost	48
4-3-2	State-Space Equations for Commodity Market Dynamics	51
4-4	Labor Market	52
4-4-1	Bond Graph for the Unit Labor Cost	52
4-4-2	State-Space Equations for Labor Market Dynamics	55
4-5	Capital Asset Market	55
4-5-1	Bond Graph for the Economies of Scale	56
4-5-2	State-Space Equations for Capital Asset Market Dynamics	59
4-6	Product Market	59
4-6-1	Bond Graph for the Benefit Orders	60
4-6-2	State-Space Equations for Product Market Dynamics	63
4-7	The Total Business Model	63
4-8	Conclusions	64
5	Complex Port-Hamiltonian Mechanics for Cash flow-Based Modeling	69
5-1	Introduction	69
5-2	Hamiltonian Mechanics for Conservative Systems	70
5-3	Hamiltonian Mechanics for Non-Conservative Systems	71
5-3-1	Complexification of the Hamiltonian	71
5-3-2	The Bicomplex Hamiltonian of Damped Harmonic Oscillators	72
5-4	Complex Port-Hamiltonian Mechanics	73
5-4-1	The Influence of Bond-Graph Elements on States and Signals	73
5-4-2	Complex Port-Hamiltonian Interconnection	74

5-5	The Firm as a Complex Port-Hamiltonian System	75
5-5-1	Economic Interpretations	76
5-5-2	Comparison of the Regular System with the CpH System	77
5-5-3	Aggregation of Subsystems for Cash Flow Determination	77
5-6	Conclusions	79
6	A Frequency-Domain Theory for Business Valuation	83
6-1	Introduction	83
6-2	The Net Present Value as a Frequency-Domain Variable	84
6-3	A Complex Representation of the Internal Rate of Return	85
6-4	The Cash-Flow Equivalent as an Ex-Ante Quantity	86
6-5	Frequency-Domain Signals for Cash-Flow Determination	87
6-6	A Frequency-Domain Approach to Scenario Analysis	89
6-6-1	Transfer Functions for Liquidity, Illiquidity, or Economic Efficiency	89
6-6-2	Bode Plots for Visualization of the Frequency Response	91
6-7	Conclusions	92
7	The Systems and Control Toolbox for Financial Analysis	97
7-1	Introduction	97
7-2	Frequency-Domain Analysis for Optimal Trading Frequencies and Scenario Analysis	98
7-3	Pole-Zero Maps for Economic Factors, Cycles and Stability	100
7-4	State Trajectories for Visual Analysis of Business Cycles	102
7-5	Qualitative Time Responses for Verification and Comparison	104
7-5-1	Verification of the Bond-Graph Model	104
7-5-2	Comparison of the Commodity Market with the Labor Market	105
7-5-3	Scenario Analysis of the Capital Asset Market with External Inputs	107
7-5-4	Complex Hamiltonian for Cash Flow Dynamics	108
7-6	Conclusions	109
8	Conclusions	113
9	Discussion	115
10	Recommendations	117
10-1	Introduction	117
10-2	System Identification	117
10-2-1	Data Collection	117
10-2-2	Identification Techniques	118
10-3	Control	118
10-3-1	Cash Flow-Based Control for Anticipating Economic Shocks	119
10-3-2	Model Predictive Control for Profit and Value Maximization	119
10-4	Economic Engineering in the Frequency Domain	120

10-4-1	Frequency-Domain Analysis for Valuing other Financial Instruments . . .	121
10-4-2	Development of Complex Bond-Graph Modeling Technique	121
10-5	Model Refinement	122
10-5-1	Capital Structure	122
10-5-2	Competition	122
A	Variables for Qualitative Analysis	127
B	Time Responses of the Markets	129
B-1	Commodity Market	129
B-2	Labor Market	130
B-3	Capital Asset Market	131
B-4	Product Market	132
C	Derivations	133
C-1	State-Space Derivation from the Bond Graph Model	133
C-1-1	Commodity market	133
C-1-2	Labor market	135
C-1-3	Capital asset market	136
C-1-4	Product market	139
C-1-5	The Firm and Total Model for Busines Dynamics	140
C-2	Bicomplex Hamiltonian Derivation	143
C-2-1	Basic Formulas	143
C-2-2	DHO in Series: $\mathcal{H}_1$	144
C-2-3	Bicomplex State Equations	144
C-2-4	DHO in Parallel: $\mathcal{H}_2$	145
C-2-5	Double-Damped DHO: $\mathcal{H}_3$	147
C-2-6	Bicomplex State Equations	148
D	Development of a Complex Discounting Method for Cash Flows	151
D-1	Non-Cyclical Risk as a Complex Exponential	152
D-2	Cyclical Risk as a Complex Exponential	152
D-3	A Cyclical Discount Rate as Aggregated Complex Exponentials	154
E	Simulink Models	159
E-1	Full Model	159
E-2	Commodity Market	160
E-3	Labor Market	161
E-4	Capital Asset Market	162
E-5	Product Market	163
E-6	Firm	164
F	Matlab Code	165

List of Figures

1-1	Business dynamics are determined by five subsystems: The firm and its four connected markets.	3
1-2	A block diagram representation shows how businesses interact with the economy.	4
2-1	Three definitions for value consider different time horizons. The present value is the only fundamental definition and considers the future.	10
2-2	The company's market value (share price) oscillates around the company's present value (intrinsic value) due to the presence of noise traders on the financial market. Present value is indicated with a bandwidth, that incorporates different future scenarios to account for uncertainties of the future [2].	11
2-3	Optimistic, pessimistic and the most probable future scenarios are formulated to test the validity of the valuation and verify proneness to uncertainties of the future.	12
2-4	Risk and discount factors show cyclical behavior due to the oscillating behavior of economic markets, debt and equity issuance, financial constraints, fiscal constraints and innovation shocks.	13
2-5	The discounted cash flow method forecasts business performance with a future cash flow on the short term, and a terminal value on the long term. A constant discount rate is used to account for the decrease in worth of money over time [6].	14
2-6	The weighted average cost of capital consists of the cost of equity, which is calculated with an equity risk premium, market correlation beta and free risk rate, and the cost of debt, which calculated with the average debt yield and tax shield [5].	16
2-7	Current valuation practices allow subjective inputs in their models. The purely data-driven predictive analytics approach does not allow subjectivity, but lacks economic interpretability and abundant data. These limitations are overcome by the economic-engineering approach, as it uses economic laws to construct a causal model.	17
2-8	From a macroeconomic perspective, businesses are connected to economic markets by exchanging cash for products. Items are bought from markets for factors of production, and sold to markets for goods and services. Between buying and selling, the business adds a competitive advantage to the items to generate profits [8].	19
2-9	Perfectly competitive and monopolistic markets cause different cost curves. Perfect competition creates an economically efficient market, whereas dynamics of the monopolistic market are in the hands of a single firm [7].	21

2-10	The firm is connected to four markets: The commodity-, labor-, and capital asset markets provide factors of production, and the product market allows the firm to sell its products to costumers.	22
3-1	Schematics of second-order systems from different physical domains that show equivalent behavior. This equivalent behavior is caused by analogies between the mechanical and electrical domain [4].	32
3-2	Bond graph model representing a second-order system that is equivalent to the mechanical and electrical systems in Figure 3-4.	34
3-3	Firm use factors of production in a certain ratio to produce products. In this situation, the firm needs four wheels, two fte, and 10% of the plant to product a single car. This ratio is modeled by using gyrators.	35
3-4	The economic tetrahedra of state in two coordinate systems: Momentum-position coordinates use costs, flows, prices and goods. The action-angle coordinates use cash flow, yield, performance and assets.	36
3-5	According to Lagrange's least action principle, the motion of every particle is described by minimization of the action integral over its path. The path is determined by its two state variables: q and $\dot{q}$	37
3-6	The time evolution of a conservative Hamiltonian system in phase space has a symplectic structure. Consequently, Hamilton's equations can be used to determine its motion.	39
3-7	The time evolution of a non-conservative Hamiltonian system in phase space has no symplectic structure. For this reason, Hamilton's equations cannot be directly applied to determine its motion.	40
3-8	The conjugate input-output interconnection of economic subsystems returns an cost or benefit to a product flow (or vice versa). Multiplication of the pair yields the economic growth over that interconnection.	41
4-1	Firms are connected to four different economic markets: Product, capital asset, commodity and labor. Firms impose a product flow on the markets, which then return a certain cost or benefit.	46
4-2	The Bond-graph model of the firm consists of one state: The value-added. This element measures the value that is created by converting factors of production into a product, and weighs off the costs and benefits of the different markets.	47
4-3	Firms interact with the commodity market by buying commodities that moves from the source, via the market and its inventory, towards the product market. In return, cash flows from the firm in the direction of the source. The direction of the arrows indicates causality.	48
4-4	Bond-graph model of the commodity market. Production cost is determined by three states: Current assets, forward price, and reserves. Cash is lost through its two dissipative elements: Inventory write down and competitors. Energy-routing elements are used for capitalization and market interconnection (bill of materials).	49
4-5	Firms interact with the labor market by buying labor via the market and its human capital from the labor force. In return, cash flows from the firm in the direction of the labor force. The direction of the arrows indicates causality.	52
4-6	Bond-graph model of the labor market. Unit labor cost is determined by the three states of the labor market: intangible assets, wage, and labor force. Cash is lost through its two dissipative elements amortization, and competitors. Energy-routing elements are used for capitalization and market interconnection (labor intensity).	53
4-7	Firms interact with the capital asset market by utilizing the plant property and equipment (PPE) of the firm, and recieving the benefit of economies of scale in return. PPE is accumulated through fixed investment or capital leases. The direction of the arrows indicates causality.	56

4-8	Bond-graph model of the capital asset market. Two states, construction in progress and asset price, determine from the capital leases and fixed investments the cash flow for investing (CFFI). Another two states, fixed assets and asset performance, determine from the product flow and CFFI the economies of scale that is supplied to the firm.	57
4-9	Firms interact with the product market by receiving the order yield, and supplying the product flow. Benefit orders is determined by the quantity demanded and customer budget of the customer that end up in the backlog. The direction of the arrows indicates causality.	60
4-10	Bond-graph model of the product market. Customers interact with the market by buying products through effort and flow sources. In return, cash is supplied to the system. C-elements represent the amount of cash absorbed by competitors, and the number of orders stored in the firm's backlog. The product price is determined by weighing off the benefit orders against the opportunity cost determined by the competitive market.	61
4-11	Bond graph of the total model. Dynamics are determined by fourteen states and four inputs. Useful outputs are the costs and benefits of the markets and the product flow.	64
5-1	Time-evolution of the undamped harmonic oscillator is determined by Hamilton's equations as energy is conserved.	70
5-2	Time-evolution of the damped harmonic oscillator is not determined by Hamilton's equations due to the presence of dissipative force $-Rq$	71
5-3	Bond graph notation of the interconnection of two DHOs. Elements C_1 and I_2 are both connected to a resistive element, causing energy to be dissipated. This is accounted for by splitting operator $\Delta\mathcal{H}$ into storage part ΔS , which influences both states and signals, and resistive part ΔR , which only influences the state.	74
5-4	The bond-graph model for business dynamics as a set of DHOs. DHOs marked with blue are either damped in series or double-damped, and need a cpH representation to obtain the correct equations of motion. DHOs marked with red do not contain oscillating behavior, and use the regular Hamiltonian expression.	76
5-5	Comparison of state responses of the regular bond graph with the cpH representation shows that both representations yield the exact same dynamics.	78
5-6	Division of the bond-graph model into the subsystems endogenous to the business, and the systems that are exogenous and form the economy. Each market that is connected to the business has a similar structure: A DHO in series.	78
5-7	Block diagram resulting from aggregating endogenous subsystems into the business, and exogenous subsystems into the economy. Cash stored in the business is equal to the revenue minus the total expenses and represents the business' earnings. Inputs supplied to the firm are controllable as they represent the investments. Inputs supplied to the economy are uncontrollable, as they represent the consumer behavior.	79
6-1	The operational cash flow appears in the model for business dynamics as the element value-added. By considering the element itself, the cash flow is measured ex-post. By considering the power-bond that connects the element to the system, the cash flow is measured ex-ante.	87
7-1	The Bode plot of the commodity market shows a resonance valley that indicates an optimal trading frequency. When this frequency is attained, an exceptional low production cost is returned to the firm. The phase plot shows that the immediacy of the labor market depends on the trading frequency of the firm.	98

7-2	The Bode plot of the total model shows the effects of consumer demand on the different costs and benefits. The optimal trading frequency has high economic efficiency for benefit orders and economies of scale, and low economic efficiency for production and unit labor cost. The phase plot shows that different markets impose different time-lags.	99
7-3	Pole-zero maps of the markets connected to the firm	101
7-5	The time-domain step response of the total model shows that increased customer demand and investments results in increased costs and benefits. Costs are negative benefits, and are plotted in the negative direction. The overshoot indicates the extra costs and benefits resulting from the markets being in disequilibrium. . . .	104
7-6	The time-domain step response of the commodity market shows that an increased product flow causes an increased costs. The state-response is in agreement with economic theory: Demand increase raises the price, and depletes the inventory and reserves.	105
7-7	The time-domain step response of the labor market shows an overshoot in the output response, which is a measure for the liquidity of the market. The state-response gives the cost breakdown of the unit labor cost: It is predominantly determined by the wage.	106
7-8	Input-output response of the capital asset market with equal product flow and varying investments. High investments increase the level of capital assets, so that a higher benefit of economies of scale is supplied to the firm. Low investments cause a lack of maintenance, causing the level of capital assets to decrease and a lower economies of scale.	107
7-9	Time evolution of the complex Hamiltonian of the firm's operations consists of a storage, squeeze and contraction function. The storage function is equivalent to the regular Hamiltonian and represents the cash flow. The contraction function models the average cash losses over a business cycle, and the squeeze function models the dynamics of those losses.	109
10-1	Signal trajectories of position q of a damped harmonic oscillator [1]	120
10-2	Three different damped harmonic oscillators and their state-space representations.	121
10-3	Porter's five forces model for competition. This figure is taken from [3].	123
B-1	Time response of the commodity market with increased product flow	129
B-2	Time response of the labor market with increased product flow	130
B-3	Time response of the capital asset market with high investments	131
B-4	Time response of the capital asset market with low investments	131
B-5	Time response of the product market with increased product demand	132
B-6	Time response of the product market with increased product demand	132
C-1	Bond-graph representation of the commodity market with numbered elements and signals	133
C-2	Bond-graph representation of the labor market with numbered elements and signals	135
C-3	Bond-graph representation of the capital asset market with numbered elements and signals	137
C-4	Bond-graph representation of the product market with numbered elements and signals	139
C-5	Bond-graph representation of the firm with numbered elements and signals	141

D-1	Comparison of the regular exponential discounting function to the complex exponential for non-cyclical risk shows that the pair can be equal under the conditions of Equation D-4.	152
D-2	The constraints on a_n and b_n lie inside the region of convergence (ROC) of the complex exponential. As a result, the transform always ascribes a countable value to businesses.	153
D-3	Cyclical risk functions defined by Equation D-3 and following the constraints of Equation D-7 allow to specify the amplitude with parameter a_n and frequency with parameter b_n	154
D-4	Aggregation of cyclical and non-cyclical risk functions produces the cyclical discount function $D(t)$. The function allows for continuous and discrete discounting and helps investors decide on the length of their investment horizon.	154
E-1	Simulink model of the full system	159
E-2	Simulink model of the commodity market	160
E-3	Simulink model of the labor market	161
E-4	Simulink model of the capital asset market	162
E-5	Simulink model of the product market	163
E-6	Simulink model of the firm	164

List of Tables

3-1	Analogs in different physical domains	32
3-2	Analogs in the economic domain	33
4-1	Inputs, outputs and states of the firm	46
4-2	The interpretation of definitions in the bond graph representing the firm	47
4-3	Inputs, outputs and states of the commodity market	48
4-4	The interpretation of definitions in the commodity bond graph	50
4-5	Inputs, outputs and states of the labor market	52
4-6	The interpretation of definitions in the labor bond graph	54
4-7	Inputs, outputs and states of the capital asset market	56
4-8	The interpretation of definitions in the capital asset bond graph	58
4-9	Inputs, outputs and states of the product market	60
4-10	The interpretation of definitions in the product market bond graph	62
4-11	Economic interpretations of the elements in the bond graph for business dynamics	63
5-1	Classes of bond-graph elements and the influence they have on states and signals	73
5-2	Economic interpretations of cpH subsystems	77
5-3	Economic interpretations of complex variables	77
7-1	Applications of systems and control tools in business valuations	110
A-1	Variables used for analysis of the model for firm dynamics	127

Bibliography

- [1] M. Mendel C. Hutters. *Overcoming the dissipation obstacle with bicomplex port-Hamiltonian Mechanics*. Delft Center for Systems and Control, 2020.
- [2] T. Koller et al. *Valuations*. John Wiley and Sons Inc, 2015.
- [3] D. Fadeev. *Creating and sustaining superior performance*. New York Free Press, 1985.
- [4] L. S. Huisman. *Modelling and control of a dynamical labour market system*. Delft Center for Systems and Control, 2019.
- [5] Corporate Finance Institute. *Business valuation modelling course*. CFI, 2018.
- [6] Corporate Finance Institute. *Introduction to corporate finance*. CFI, 2018.
- [7] Triple A Learning. *Microeconomics*. Oxford study Courses, 2013.
- [8] N. G. Mankiw. *Principles of macroeconomics*. Cengage Learning, 2018.

Preface

I am very grateful to be given the opportunity to finalize the master's programme Systems and Control in economic engineering. The thesis work has pushed me to my limits while allowing me to develop myself personally every step of the way. Economic engineering is a new field of research, and could be described as atypical to other thesis tracks within the Delft Center for Systems and Control (DCSC). There are three reasons why I particularly enjoyed being part of this group:

1. The endless number of promising research topics.
2. The synergy of narrow group collaboration.
3. The exceptional opportunity to use one's creativity.

I want to thank my supervisor, dr.ir. M.B. Mendel. Besides taking the time for our extensive discussions and granting me the freedom to shape my thesis, I am foremost grateful for inspiring a genuine interest in physics and economics. This interest has pushed the boundaries of my thesis research and added great enjoyment to it. Special thanks to ir. C. Hutters for his guidance in bond-graph modeling and bicomplex Hamiltonian mechanics. Also, I want to thank the members of the economic-engineering group for their feedback and ideas. Furthermore, I want to thank R. Jellema for helping me as an intern to develop my financial knowledge and M. de Goeij for allowing me to do an internship at the PwC Valuations department. Finally, I want to thank the members of the thesis committee for showing interest in my work.

Notes to reader:

1. The goals of each chapter are stated at the start of each chapter.
2. At the end of each chapter, I specify the contributions made in that chapter.
3. The references used in each chapter are listed at the end of that chapter.

“Give me a one-handed economist. All my economists say ‘on the one hand...’,
and then ‘but on the other...’”

— *Harry Truman*

Chapter 1

Introduction

1-1 Economic Engineering for Frequency-Domain Valuations

In December 2019, the state-owned oil company Saudi Aramco was taken to the market to publicly sell its shares. An accurate valuation was essential for risk management of underwriters and investors. However, involved parties were not capable of agreeing on its value [7]. A consensus from foreign investors estimated the company at \$1.2 trillion [3], whereas J.P Morgan expected almost double by valuing the company over \$2 trillion [26].

Business valuations allow large offsets between estimates, as above-mentioned example illustrates. However, valuations play a crucial role in investment analysis, capital budgeting and financial reporting. Business value is determined by two variables: Future cash flow and a discount rate. Current techniques determine the pair by using subjective inputs, resulting in the valuation to be as good as the assumptions made by the analyst [8, 25, 27]. Cash-flow forecasts are more widely demanded now, than historically [18]. However, even the most skilled analysts cannot construct accurate predictions of the cash flow statement [15]. Estimating errors frequently remain, and continue to be substantial [15, 24, 9]. Using similar methods and more information, only lowers the forecasting errors by a small amount [16].

This thesis aims to develop a theory to perform business valuation in the frequency domain. The theory is a pioneering effort for extending the current perspective on valuations. Key insight in this theory is the recognition of business value as a frequency-domain variable, instead of a time-domain variable (See Chapter 6).

Economic engineering is used for the development of the theory. Economic-engineering models use causal and dynamical relations to describe economic systems [21]. The approach uses analogies between economic principles and physical laws. The analogies yield dynamical relations in economic systems, to forecast their behavior and allow the application of control techniques. In economic engineering, generally two approaches are taken: the use of Newtonian mechanics to describe dynamical changes in price and quantities [23, 13], and the use of

analytical mechanics to model utility maximization [14, 19]. This thesis shows that also in economics the two approaches are fundamentally the same (See Chapter 5).

Previously, the economic-engineering approach is taken to value a car-leasing business [22]. The theory developed in this thesis is general, and applies to all businesses (See Chapter 4).

The development of the frequency-domain valuation theory consists of three parts:

Development of the Frequency-Domain Valuation Theory

1) Business valuations as a S&C problem

- *Economic background (Chapter 2)*: Review the field of business economics and identify shortcomings to valuations.
- *Engineering background (Chapter 3)*: Develop physical analogs for the field of business economics to formulate thesis proposals.

2) A fundamental model for business dynamics

- (a) *Bond-graph modeling (Chapter 4)*: Create a model for business dynamics.
- (b) *Complex port-Hamiltonian mechanics (Chapter 5)*: Express business dynamics in terms of cash flows and determine which cash flow yields value.

3) Frequency-domain valuation and analysis

- (c) *A frequency-domain theory for valuation (Chapter 6)*: Use the Laplace transform to develop a fundamental theory for valuation and scenario analysis.
- (d) *S&C tools for financial analysis (Chapter 7)*: Use systems and control tools to perform financial analysis.

Each part consists of two steps, which are handled in separate chapters. In Part 1, background is provided on business valuations and economic engineering. In Part 2, a fundamental model for business dynamics is developed. In Part 3, a frequency-domain valuation theory is formulated, and S&C tools are used for financial analysis. Below, each part of the thesis is explained in more detail.

1-2 Part 1: Business Valuation as a Systems and Control Problem

In Part 1 of this thesis, background is provided on business valuations and economic engineering. Firstly, shortcomings in the current valuation approach are identified. Secondly, the economic-engineering framework is used to translate business valuations to become a systems and control problem. Part 1 concludes with proposals to perform Parts 2 and 3 of this thesis.

In Chapter 2, a review of business valuations is done. The review shows that value is determined by two variables: Future cash flow and a discount rate. It appears however, that current valuation practices rely on subjective inputs to determine the pair [25, 8]. The use

of data-driven models overcomes this limitation, but the lack of abundant available data and economic interpretability precludes the use of predictive analytics [6, 12, 4].

This thesis aims to develop a general theory for business valuations. Therefore, I use a general definition for businesses: Profit optimization [10]. Four different types of markets are connected to the firm [20, 2]. To ensure generality, models must be made for each market. Figure 1-1 shows the five systems that determine business dynamics.

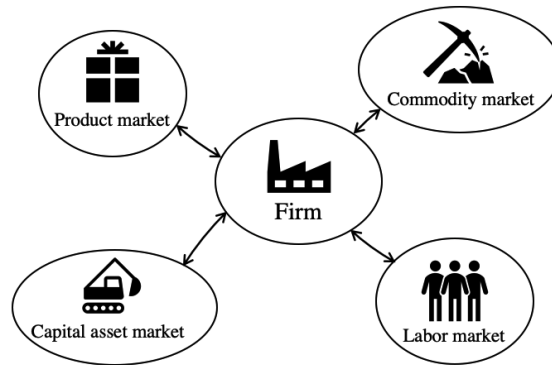


Figure 1-1: Business dynamics are determined by five subsystems: The firm and its four connected markets.

In Chapter 3, a review of economic engineering is done. Physical models use Newtonian or analytical mechanics to causally derive the equations of motion [11, 17]. Economic systems do not have equations of motion to specify their behavior. However, there are economic laws they obey [28, 25]. In the field of economic engineering, these economic laws are reformulated as mathematical definitions to specify the dynamics of economic systems [21]. A contribution of this thesis is to make the economic-engineering analogy specific to business valuations.

With these insights, business valuations are put as a S&C problem, thereby completing Part 1 of this thesis. Steps are proposed to perform Parts 2 and 3, which appear as items (a), (b), (c), and (d) in the development of the frequency-domain valuation theory.

1-3 Part 2: A Fundamental Model for Business Dynamics

In Part 2 of this thesis, a fundamental model for business dynamics is developed. This model mimics the behavior of businesses, so that cash-flow dynamics are forecasted. These cash-flow forecasts determine business value [8, 2].

In Chapter 4, a model for business dynamics is developed by using a bond-graph representation. The model is fundamental as its dynamics are determined by economic laws. The business is divided in the five subsystems that are shown in Figure 1-1. The fundamental model maximize profits, by considering the costs and benefits of its connected markets [5]. Differential equations are derived, to describe business dynamics in a causal and general way. All elements and signals are given economic interpretations, so that the model is intuitive to both economists and engineers.

In Chapter 5, it is shown that cash flows underlie the dynamics of the fundamental business model. In economics, business dynamics are not expressed in terms of costs and product flows, but in terms of cash flows [8, 25]. By giving the model a complex port-Hamiltonian (cpH) representation, it is shown that the model for business dynamics is in line with the economic approach to business valuations.

The cpH system is used to determine what cash flow yields business value. This is done by dividing the model into the business itself, and the economy. This is shown in Figure 1-2.

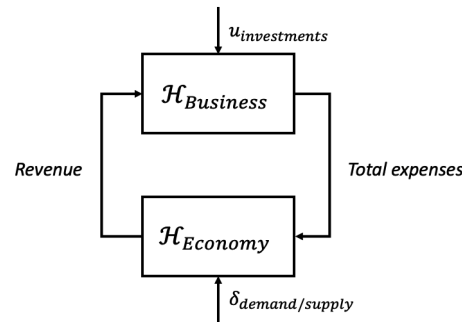


Figure 1-2: A block diagram representation shows how businesses interact with the economy.

With this, the fundamental model for business dynamics is developed, thereby completing Part 2 of the frequency-domain valuation method.

1-4 Part 3: Frequency-Domain Valuation and Financial Analysis

In Part 3 of this thesis, I develop a theory to perform business valuations in the frequency domain, and use S&C tools to perform financial analysis.

In Chapter 6, a theory is developed that uses the Laplace transform to determine the value of future cash flows. A dollar to be paid out tomorrow is worth less than a dollar paid out today, due to the uncertainty of the future. In valuation models, this value decrease is accounted for by discounting the future cash flow [8, 25]. I develop a theory that translates the entire valuation effort of businesses to the frequency domain. The use of the frequency domain for valuations is particularly effective, as the time dimension of cash flows is eliminated and the profitability of economic projects is directly compared. The existing discounting approach is extended by using a complex number for the discount rate. The imaginary part models the oscillating behavior of the economy.

In Chapter 7, S&C tools are used to perform financial analysis. S&C engineers benefit the availability of many tools for analysis. I use Bode plots, pole-zero maps, state-space trajectories, and time responses to perform analysis on economic systems. It becomes clear that in particular frequency-domain tools extend the existing financial analysis with numerous insights.

An important type of financial analysis is scenario analysis. In economics, scenario analysis is performed to review the business' proneness to future uncertainties [1]. However, scenario analysis is a time-consuming process and requires abundant professional expertise [25, 8]. I show that Bode plots and time responses are effective tools to perform this type of analysis.

The development of a frequency-domain valuation theory and the use of S&C tools for financial analysis complete Part 3 of this thesis.

Bibliography

- [1] E. R. Arzac. *Valuations for mergers, buyouts and restructuring*. John Wiley and Sons Inc., 2008.
- [2] J. W. Brewer. *Bilinear, dynamic single-ports and bond graphs of economic systems*. The Franklin Institute, 1982.
- [3] N. Martin D. N. J. Blas. *The Wall Street bankers who burst Aramco's \$2 trillion bubble*. Bloomberg, 2019.
- [4] L. Einav and J. D. Levin. *The data revolution and economic analysis*. The National Bureau for Economic Research, 2013.
- [5] D. C. Karnopp et al. *System dynamics: Modeling, simulation, and control of mechatronic systems*. John Wiley & Sons, Inc., 2012.
- [6] L. Taylor et al. *Emerging practices and perspectives on big data analysis in economics: bigger and better or more of the same?* Sage Journals, 2014.
- [7] S. Azhar et al. *Give or take a trillion: Investors still in the dark on Saudi Aramco*. Reuters, 2019.
- [8] T. Koller et al. *Valuations*. John Wiley and Sons Inc, 2015.
- [9] T. Ward et al. *A research note on the issue of non-articulation and the method used to calculate net operating cash flow*. Journal of Accounting Research, 2009.
- [10] M. Friedman. *The social responsibility of business is to increase its profits*. Times Magazine, 1970.
- [11] P. Stehle H. C. Corben. *Classical mechanics*. John Wiley & Sons, Inc., 1950.
- [12] D. F. Hendry. *Econometrics - Alchemy or science?* Economica, 1980.
- [13] L. S. Huisman. *Modelling and control of a dynamical labour market system*. Delft Center for Systems and Control, 2019.

- [14] C. Hutter. *The analytical mechanics of consumption*. Delft Center for Systems and Control, 2019.
- [15] S. Orpurt J. Hales. *A Review of Academic Research on the Reporting of Cash Flows from Operations*. Financial Accounting Standards Board, 2013.
- [16] M. Bradburry L. Austin. *The accuracy of cash flow estimation procedures*. Accounting and Finance, 2009.
- [17] G. Hrabovsky L. Susskind. *Classical mechanics: The theoretical minimum*. Penguin Books, 1978.
- [18] M. Hung M. L. Defond. *An empirical analysis of analysts' cash flow forecasts*. Journal of Accounting and Economics, 2003.
- [19] N. Manders. *The thermodynamics of economic engineering*. Delft Center for Systems and Control, 2019.
- [20] N. G. Mankiw. *Principles of macroeconomics*. Cengage Learning, 2018.
- [21] M. B. Mendel. *Principles of economic engineering*. Delft Center for Systems and Control, 2019.
- [22] J. J. D. Mulder. *The dynamical behavior and control of an enterprise*. Delft Center for Systems and Control, 2019.
- [23] N. G. Orie. *Dynamic modelling and control of the oil market*. Delft Center for Systems and Control, 2019.
- [24] D. W. COLLINS P. Hribar. *Errors in estimating accruals: Implications for empirical research*. Journal of Accounting Research, 2002.
- [25] T. Rietveld. *Handboek Investeren & Financier*. Vakmedianet Management B.V., 2017.
- [26] A. Salzman. *J. P. Morgan takes a big step in valuing Saudi Aramco at \$2 trillion*. Bloomberg, 2019.
- [27] X. A. van Ardenne. *A review of business valuations and a proposal for the application of economic engineering*. Delft Center for Systems and Control, 2020.
- [28] H. R. Varian. *Intermediate microeconomics: A modern approach*. W. W. Norton & Company, 2010.

Background on the Business Economics and Valuations

2-1 Introduction

In this chapter, background on valuations and business economics is provided. The background on valuations shows shortcomings in the valuation process. The background on business economics provides the economic laws that will be used for modeling the business.

To establish the background, research is conducted on both a theoretical and practical level. The economic literature is reviewed to obtain a theoretical basis, and practical experience is gained by completing an internship at the PwC Valuations department. I state the goals of this chapter as follows:

Chapter goals:

1. Formulate a fundamental value definition, and learn how valuations are performed.
2. Determine the fundamental role of businesses as economic agent, and review the economic laws that apply to them.
3. Identify shortcomings to the valuations process and formulate the thesis objective.

In Section 2-2, I give a fundamental definition of value. In Section 2-3, an overview of current valuation techniques and their shortcomings is given. In Sections 2-4 and 2-5, I review the field of business economics, and present the markets businesses interact with. In Section 2-6, the thesis objective is stated as the creation of an economic-engineering valuations method.

In this chapter, I use the terms firm, company, and business interchangeably to stay close to the terminology used in the literature.¹

2-2 The Value of a Company

According to Leszinski et al., value is the most overused and misused term in marketing and pricing [49]. As this is a thesis on business valuations, it is important to have a clear understanding of the definition of value.

2-2-1 Value is the Company's Future Performance

Value is created for the owners of a company when they invest cash now to generate more cash in the future. The value created is the difference between the investments made and the incoming cash flows. Money paid tomorrow is worth less than the money paid today, and future cash flows must be adjusted to reflect this time value of money. This is done by applying a discount rate. Tim Koller et al. use the term present value for this fundamental value definition in [21]:

Definition 2-2.1: Present value

The future expected cash flow adjusted for its value decrease over time.

This definition expresses value in two variables: A future cash flow and discount rate. Value is expressed in [\$], future cash flows in [\$ / t] and the discount rate in [% / t].

To prevent confusion in terminology, I compare the present value with two views on value that are commonly used in economics and accounting, but do not represent actual company value. The main distinction between these three views is the considered time horizon. The present value looks forward in time. It uses forecasts to analyze the company's future performance and uses a discount rate to project it on the present. This is visualized in green in Figure 2-1.

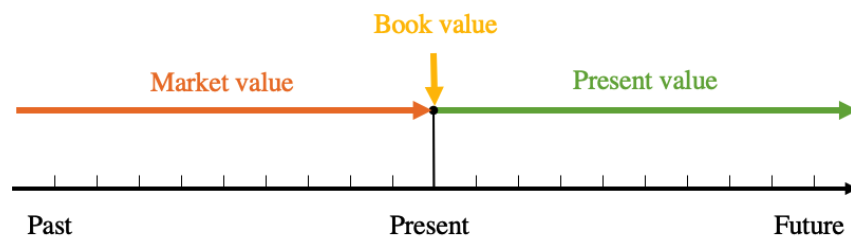


Figure 2-1: Three definitions for value consider different time horizons. The present value is the only fundamental definition and considers the future.

¹In Chapter 5, I use the terms firm and business for different parts of the bond-graph model to emphasize their difference.

Book Value

Book value is a term used in accountancy that indicates the company value according to its balance sheet² Historically, the book value was supposed to show the value of companies, but this idea has almost completely disappeared. The reason for this is that the cash for which assets are sold bears no relationship to their costs or the company's earnings [30]. Book value sees the value of a company as a snapshot of its current situation and only considers the present. This is visualized in yellow in Figure 2-1.

Market Value

Market value is the price at which a sale takes place in the open market [28]. Market value is based on transactions made in the past and therefore looks back in time [75].³ The timeline considered by the market value is visualized in orange in Figure 2-1.

Market value cannot be assumed to be equal to the company value. The reason for this lies with the behavior of traders on the stock market. Traders either base their decisions on fundamental knowledge of economics and the company, or solely on daily price changes [27]. The latter group of traders is called “noise traders” and can be responsible for significant off-sets between the market value and the company value [21]. Figure 2-2 shows this comparison. Note that in this case the market value is defined as the share price and the company value as the intrinsic value.⁴

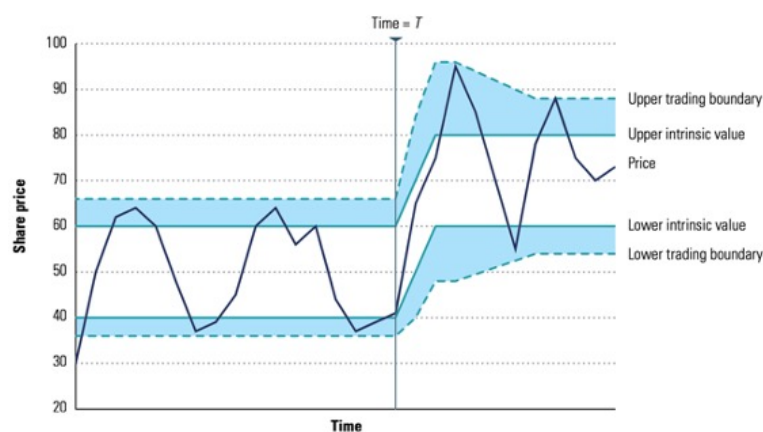


Figure 2-2: The company's market value (share price) oscillates around the company's present value (intrinsic value) due to the presence of noise traders on the financial market. Present value is indicated with a bandwidth, that incorporates different future scenarios to account for uncertainties of the future [21].

²The balance sheet is a financial statement that reports on the company's assets, liabilities, and shareholder's equity [63].

³Some New Keynesian models link the price to future performance as traders speculate about future performance. I refer the interested reader to e.g. [14, 64].

⁴The share price is the company value divided by the number of outstanding shares [63]. The intrinsic value is the term used for company value in [21].

The determination of value by aggregation of transactions in the past or future has a parallel in control engineering: the unilateral Laplace. I formulate this analog in Section 6-2.

2-2-2 Value as a Bandwidth and the Role of Future Scenarios

Present value in Figure 2-2 is not a point estimate, but approximated with a bandwidth. The bandwidth is caused by different possible future scenarios, with varying assumptions on product prices, customer metrics, costs, inflation, interest rates, and other business drivers [57].

Analysts formulate scenarios to bracket the baseline projection (investor/bank case) with optimistic (management case) and pessimistic cases (worst case). This is done to test the validity of the valuation and obtain insight into the company's proneness to uncertainties of the future [2]. Figure 2-3 shows these three scenarios by expressing the firm's performance as its free cash flow.

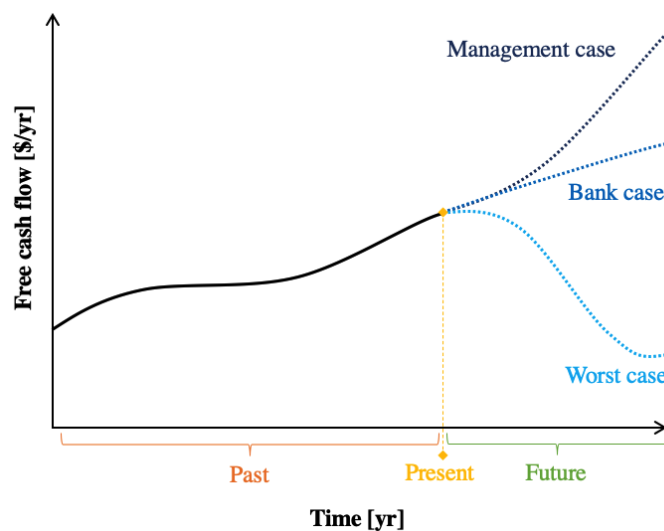


Figure 2-3: Optimistic, pessimistic and the most probable future scenarios are formulated to test the validity of the valuation and verify proneness to uncertainties of the future.

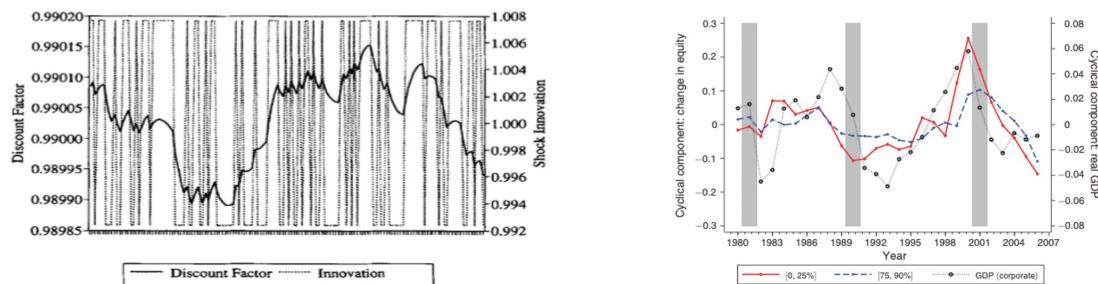
Scenario analysis is a time-consuming process that requires extensive professional expertise. It is difficult to envision all possible scenarios, and the actual outcome may be entirely unexpected [63].⁵ Nevertheless, it is critical to evaluate multiple future scenarios, as the most probable case is only a part of the process of a business valuation.

These uncertainties in variables can be adopted in the dynamical system I develop in this thesis. By varying inputs, initial conditions or parameters, scenario analysis is instinctively preformed. This is explained in Section 6-6-2.

⁵An approach to incorporate different future scenarios with their probability is using Monte Carlo simulations. I refer the interested reader to e.g. [57, 35].

2-2-3 Cyclical Behavior of the Discount Rate

Economists use a discount rate to calculate today's worth of future cash flows. The rate accounts for the time value of money (See Section 2-3-1). In practice, the rate is assumed to be constant in time, but its dynamical behavior is a field of ongoing research [71].



(a) Cyclical discount behavior caused by innovation shocks [60].

(b) Net equity is issued cyclical, causing cyclical equity risk [23].

Figure 2-4: Risk and discount factors show cyclical behavior due to the oscillating behavior of economic markets, debt and equity issuance, financial constraints, fiscal constraints and innovation shocks.

Research indicates that the discount rate shows cyclical behavior. The discount function changes over time [26, 70]. The rate shows oscillating behavior, due to cyclical financial constraints, fiscal constraints, and its link to the cyclical financial market [17, 58]. Thereby, debt and equity issuance are cyclical [23] (Figure 2-4b), there is a cyclical discount factor resulting from innovation shocks [60] (Figure 2-4a), and banks are subject to cyclical capital requirements [32].

Currently, no discounting methods exist to incorporate this cyclical behavior. In Section 6-3, I incorporate the oscillating behavior of economic markets by using complex numbers.

2-3 Current Valuation Techniques and their Shortcomings

Business valuations play a crucial role in investment analysis, capital budgeting, merger and acquisition transactions, and financial reporting [8]. Many factors influence the value of a business, and the amount of available data is limited. These factors make business valuations a complex process [21]. In this section, the field of business valuations is reviewed, and limitations are identified.

2-3-1 The Discounted Cash Flow Method

The discounted cash flow (DCF) method uses future cash flow and a discount rate to value businesses. It is the most fundamental valuation approach, as it is based on the two variables that define business value [21].

The future cash flow is commonly divided into two parts: A cash-flow projection for the

next few years, and a terminal value. This terminal value represents the company value after a certain moment in the future, because long-term predictions are seen as unreliable [63]. Equation 2-1 is used to calculate the present value.⁶

$$\text{Present value} = \sum_{n=0}^N \frac{\text{Cash flow}_n}{(1+r)^n} + \frac{\text{Terminal value}}{(1+r)^N} \quad (2-1)$$

With discount rate r , future years n , Cash flow _{n} for the cash paid out in year n , and end-year forecast period N . A visualization of the DCF method is shown in Figure 2-5.

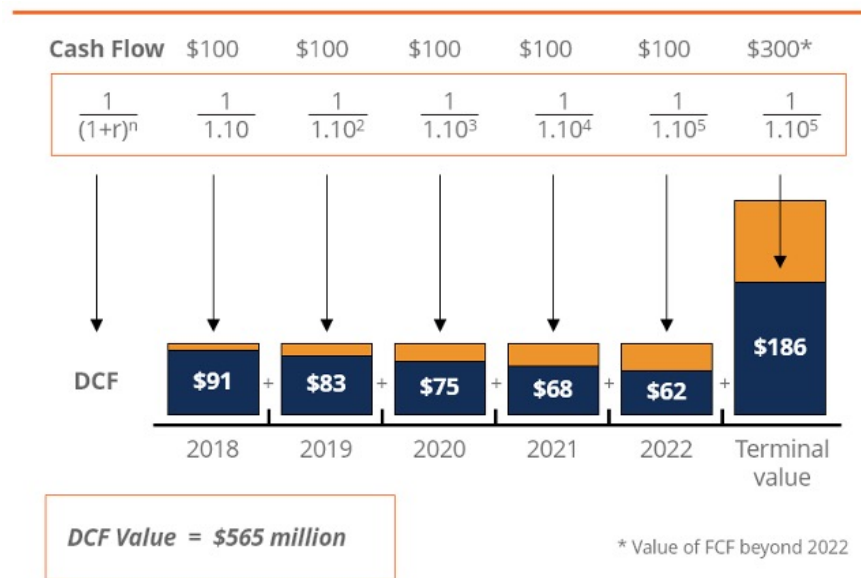


Figure 2-5: The discounted cash flow method forecasts business performance with a future cash flow on the short term, and a terminal value on the long term. A constant discount rate is used to account for the decrease in worth of money over time [40].

With this, the mathematical framework that economists use for the DCF method is set. In Chapter 6, I show how the frequency domain is used to determine this present value.

Future Cash Flow Forecast:

The cash flow used in the DCF method is the free cash flow (FCF)⁷, which is forecasted in two steps. First, the company's management forecasts its performance by stating an expected FCF, based on its knowledge of the company and the industry's expectations. Independent

⁶Many adaptations for the DCF method exist. I refer the interested reader to [21, 63, 68].

⁷The free cash flow is the cash remaining to the company's owners after subtracting operating- and investing cash flows from the income. See for more information [63].

experts then check this proposal. These experts do not have the same amount of information as the company's management, and analyze historic financials⁸ to forecast revenue, the income statement, and balance sheet. These forecasts are then used to estimate the future cash flow. Management and experts compare their forecasts to come to a definite cash flow prediction [21, 9].

There are several shortcomings to this method for cash flow forecasting. Forecasts are based on experience and professional know-how, causing the valuation to be as good as the assumptions made by the analyst [21, 63, 74]. Even the most skilled analyst is cannot construct accurate predictions of the financial statements [42]. Estimation errors frequently remain, and continue to be substantial [42, 61, 22].

Discount Rate Determination:

The discount rate is determined by using the weighted average cost of capital (WACC) model.⁹ This model sums the cost of debt and cost of equity, weighted for their capitalization.¹⁰ Equation 2-2 is used to determine the WACC.

$$\text{WACC} = \frac{E}{E + D} \times R_e + \frac{D}{E + D} \times R_d(1 - T_c) \quad (2-2)$$

E is the value of the firm's equity, D is the value of the firm's debt, R_e is the return on equity, R_d is the return on debt and T_c is the corporate tax rate.

The debt part consists of rates to account for factors as country risk, company size, and credit spread, combined with a tax shield¹¹ [56]. The cost of equity is determined by the risk-free rate, the company's correlation to the overall stock market (β), and the equity risk premium. One approximates the company's β by handpicking some comparable and public companies and averaging theirs [9, 56]. This approach is summarized in Figure 2-6.

The WACC model has two shortcomings. Again, this model is prone to subjectivity due to the manual selection of the company's peer group, and thereby not supported by empirical evidence [24]. Also, the WACC model gives a discount rate that is constant in time, contradicting the cyclical discount rate as found in the literature as explained in Section 2-2-3. In Section 6-3, I incorporate the cyclical behavior of the economy by using a complex-valued discount rate.

⁸A company's financials are the income statement, balance sheet, and cash flow statement. The documents disclose the company's financial information on a yearly or quarterly basis. The amount of data is very limited. See for more information [63].

⁹Many alternatives exist, though the valuations industry most commonly applies the WACC model. See for example [21, 63, 18].

¹⁰The capitalization of a company is the division of the company's assets into debt- and equity financed. See for more information [63].

¹¹Companies do not pay taxes over the assets financed by debt. This is called a tax shield. See for example [63, 18].

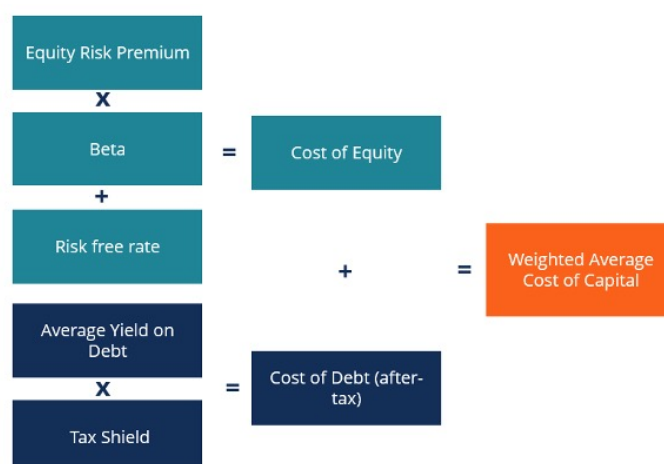


Figure 2-6: The weighted average cost of capital consists of the cost of equity, which is calculated with an equity risk premium, market correlation beta and free risk rate, and the cost of debt, which calculated with the average debt yield and tax shield [39].

2-3-2 Multiples and Asset-Based Methods

The multiples method is a relative valuation approach that looks at past transactions of comparable companies and is based on the idea that similar assets sell for a similar prices. The method is less accurate than the DCF method but provides quick insight and helps to summarize the valuation in a single number [21, 63].

Asset-based valuations count the value of individual assets to arrive at fair market value. It is often preferred when a business faces liquidity. Measuring intangible assets is hard, and the technique disregards prospective earnings. It is therefore not often used in practice [21, 9].

These two valuation methods can be projected on the timeline of Figure 2-1. The multiples method looks to the past and is comparable to the market value. The asset-based approach only considers the company in the present and is aligned with the company's book value. It can therefore be concluded that neither of these methods is based on the fundamental definition of value.

2-3-3 The Field of Research on Predictive Analytics

Predictive analytics is a broad term describing a variety of statistical and analytical techniques for developing models that predict future events or behavior.¹² Its application to business valuations limits a deal's potential downside risks and increases value creation [72]. In the field of business valuations, the data-driven modeling uses machine learning ([10, 66]), and time-series models ([50]). Either accrued earnings or cash flows are used as data type, but the literature is inconclusive about a superior method [25, 52, 16, 11].

The approach still suffers from several shortcomings in business practice, including:

¹²A description of the predictive analytics modeling effort can be found in [59].

1. **Lack of data:** Financial data on companies only consist of financial statements that are update on a yearly or quarterly basis, leading to a limited amount of data points [63].
2. **Lack of economic interpretability:** The models are based on data-fitting only, so that the models have no economic interpretation. As a consequence the models are unable to (1) separate the effects of specific economic shocks and (2) deal with changing economic circumstances [15, 33, 13].

In addition to these shortcomings, predictive analytics cannot serve as a basis for determining policy actions and other business decisions [33]. The outcomes of such actions will influence the model's predictions, further restricting its practical use. For these reasons, there has been a call for the development of structural models to support predictive analytics [15].

Figure 2-7 shows three modeling approaches to forecast the future cash flow: The current approach of cash flow projection, predictive analytics, and economic engineering.

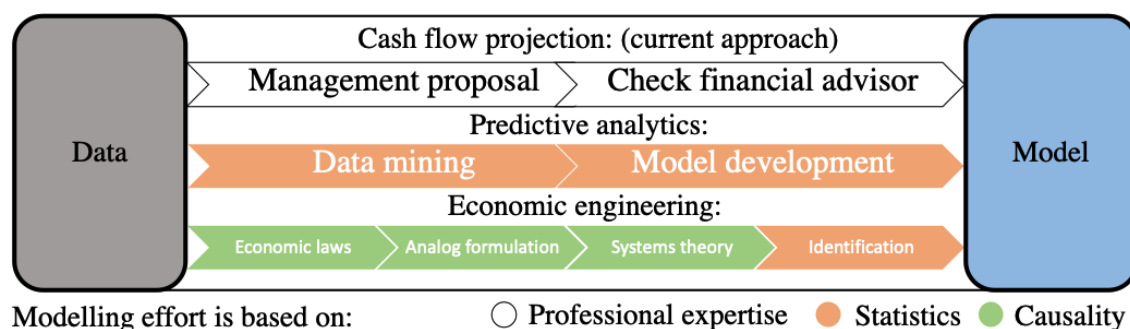


Figure 2-7: Current valuation practices allow subjective inputs in their models. The purely data-driven predictive analytics approach does not allow subjectivity, but lacks economic interpretability and abundant data. These limitations are overcome by the economic-engineering approach, as it uses economic laws to construct a causal model.

By using predictive analytics, the forecasts are no longer prone to subjectivity, therefore resolving the first shortcoming of the current approach. However, as it uses statistics only, there is a lack of sufficient data and no economic interpretability. The economic-engineering approach resolves this issue as well, as the model is based on economic principles.

2-4 Business Economics

Business economics is a field in applied economics using economic theory and quantitative methods to perform analysis. The field provides a thorough understanding of businesses and fundamental laws that are accepted as economic principles. These principles can be used to formulate a causal and economically interpretable model for business dynamics.

2-4-1 Rational Firm Behavior

The rational behavior of the firm as an economic agent was defined at the outset of modern economics by Adam Smith in [67]: *“It is not from the benevolence of the butcher, the brewer, or the baker, that we expect our dinner, but from their regard to their own interest.”* More than a century later, Edgeworth stated this point explicitly in [12]: *“The first principle of economics is that every agent is actuated only by self-interest.”* Again, almost 100 years later, Friedman makes that definition specific in [26] for businesses:

Definition 2-4.1: The business

"The social responsibility of businesses is to increase their profits."

Per definition, companies as economic agents¹³ maximize their profits. This definition is oblivious to size, industry, connected markets, and geographical location [75]. The definition can therefore be taken as starting point for the development of a general business model.

Though this point of view is adopted as an economic theory, it is challenged by some schools of thought. For example, Vriend challenges the approximation of firm behavior by the sum of individual behavior [76], and Kolstad questions what role corporate social responsibility plays [46]. Even with these debates in mind, I adopt the general economic definition of businesses in this thesis.

2-4-2 The Firm as Part of the Macroeconomic System

From a macroeconomic perspective, the firm's profit maximization is essentially the same as the individual's utility optimization [34].¹⁴

The firm converts factors of production into products. By this conversion, the factors (land, labor, capital, and entrepreneurship [67]) are moved up the value chain.¹⁵ Keynes's general theorem states that it is only possible to increase output when the input is increased [45]. The definition makes no distinction between factors (or products) acquired from (or sold to) consumers (B2C¹⁶) or other firms (B2B¹⁷) [34]. Figure 2-8 displays the firm as part of a macroeconomic system.

Profits are maximized by optimizing the costs and benefits of the connected markets. The growth of a company and its optimal product flow are determined by customer demand and scarcity of factors of production [34].

Economic markets behave according to the stability assumption. When the realized price

¹³An economic agent is an actor (decision maker) in an economic model, solving some optimization problem. See for more information for example [75].

¹⁴Background information on macroeconomics can be found in [54, 3].

¹⁵A firm does not necessarily use all factors of production. For example, software companies do not need land (or the commodities it produces).

¹⁶Business to consumer

¹⁷Business to business

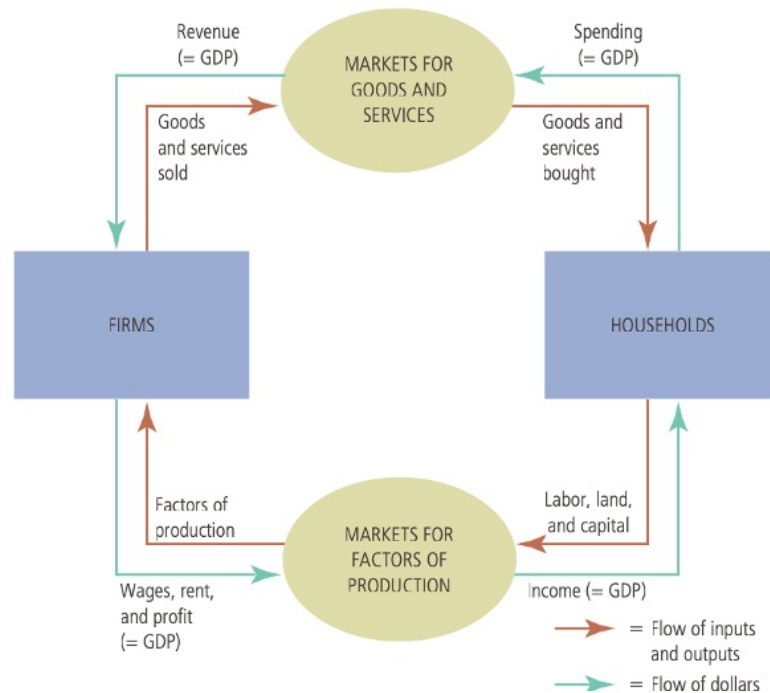


Figure 2-8: From a macroeconomic perspective, businesses are connected to economic markets by exchanging cash for products. Items are bought from markets for factors of production, and sold to markets for goods and services. Between buying and selling, the business adds a competitive advantage to the items to generate profits [54].

of the product does not meet the expected price, due to inconsistency in pricing or planning, resources are wasted, and an economic disequilibrium arises. Trading occurs until the realized and expected prices are aligned again, and the market equilibrium is restored. [34].

2-4-3 Economic Laws and their Mathematical expression

Conservation of Money

Money cannot be created from null, nor can it be destroyed. The change of net money in a system is equal to the amount transferred to that system minus the transfers out of it. This phenomenon is known as the law of conservation of money. The change in a firm's balance sheet is thus the credits minus the debits [1]. The cash flows in are equal to the cash flows out, as is stated in Equation 2-3.

$$\text{Cash in} = \text{Cash out} \quad (2-3)$$

This law has an interpretation in physics as the law of conservation of energy. By formulation of the proper analogs, economic systems satisfy the properties of a Hamiltonian system. This analog is further explained in Section 3-3.

Costs are also conserved, as imposed costs always result in a similar benefit received by

some other party. In bond graph modeling, this is denoted by a 0-junction. This analog is further explained in Section 3-2-2.

Market Clearing

Say's law states that supply always equals demand in economic systems [65]. This ruling is called market clearing, and ensures that no goods are lost in economic systems. Walras assumes that the product price changes to assure this phenomenon [77]. Equation 2-4 expresses market clearing mathematically.

$$\text{Goods in} = \text{Goods out} \quad (2-4)$$

This law is known in physics as the conservation of momentum. In bond-graph modeling, it is represented by 1-junction. This analog is further explained in Section 3-2-2.

Free Market Pricing

Walras's assumption was preceded by Smith's theory of the invisible hand. It is a metaphor for unseen forces that move the price in a free market economy. These so-called forces are costs and benefits from the supply- and demand side that move the market price towards its equilibrium [67]. Since the difference between costs and benefits is a change in sign (cost = -benefit), free market pricing is defined by Equation 2-5.

$$\text{Price} = \sum \text{costs} \quad (2-5)$$

Free market pricing is analogous to the I-element in bond graph modeling of physical systems. This analog is further explained in Section 3-2-2.

Asset accumulation

Companies accumulate assets through investment, purchases, and human capital build-up. But these same assets depreciate over time through waste, amortization, impairment, and wear tear. The company's net asset position is thus determined by the assets accumulated, minus the assets depreciated. Financial reports express assets in their monetary value [63]. By taking flows in as positive, and flows out as negative, asset accumulation is defined by Equation 3-13.

$$\text{Asset position} = \sum \text{flows} \quad (2-6)$$

Asset accumulation is the economic analog of the physical C-element. This analog is further explained in Section 3-2-2.

2-4-4 Competition

Firms compete with each other by trying to secure the business of a consumer or supplier by offering the most attractive terms. Competition is the allocation of productive resources to the highest bidder. Therefore, it encourages economic efficiency [67].

The degree of competitiveness of a market is bracketed between perfect competition and a monopoly.¹⁸ In perfectly competitive markets, profit-maximizing producers meet a price that is equal to its marginal cost, at the lowest point of the average cost curve. This results in the neoclassical supply curve and guarantees maximum economic efficiency. This is plotted on the left-hand side of Figure 2-9.

In a monopoly, there is only a single market player that fully determines the product price. In this situation, the neoclassical supply curve does not apply, and the price is not equal to the marginal cost. As a result, the monopolistic market is not guaranteed to be economically efficient [75]. This is plotted on the right-hand side of Figure 2-9.

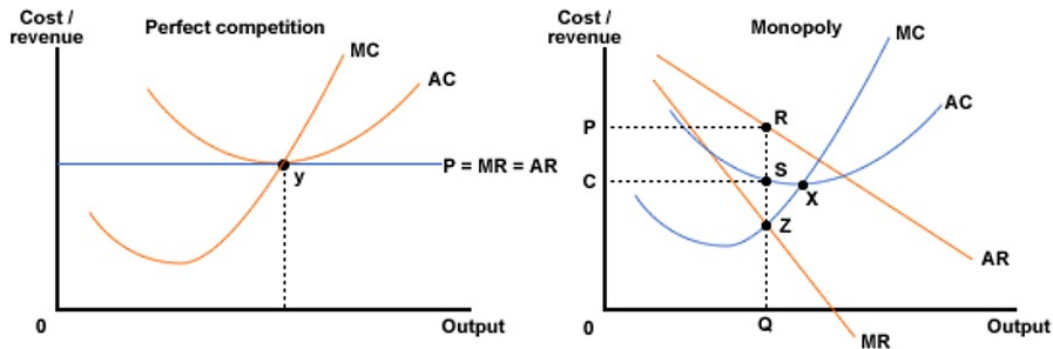


Figure 2-9: Perfectly competitive and monopolistic markets cause different cost curves. Perfect competitions creates an economically efficient market, whereas dynamics of the monopolistic market are in the hands of a single firm [48].

Pareto classifies firms in competitive markets as either price takers or price makers. Price takers are individuals or small firms whose product demand or supply is too low to have an impact on value. Price makers are governments or firms that are large enough to influence the price with their demand or supply [62].

In this thesis, I model the markets as being perfectly competitive. By using a spring-damper interconnection I allow the firm to be anything between a price taker and price maker (See Chapter 4).

2-5 Connected Markets

Firms interact with four classes of economic markets. Items are bought on the commodity-, labor-, and capital asset market, and sold on the product market [54, 3, 4, 5]. Figure 2-10 gives this interconnection schematically.

¹⁸There is also oligopoly and monopolistic competition. I refer the interested reader to [75].

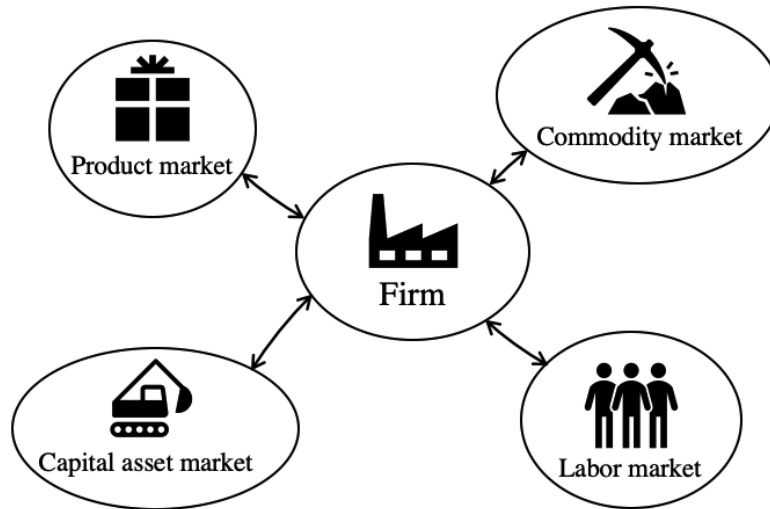


Figure 2-10: The firm is connected to four markets: The commodity-, labor-, and capital asset markets provide factors of production, and the product market allows the firm to sell its products to costumers.

This model incorporates all four factors of production, as defined by Smith and general macroeconomic theorems [67, 3]. Labor and capital are supplied by their markets, commodities originate from the factor land, and entrepreneurship is in the business itself. The markets are discussed below in more detail. In Chapter 4, a bond-graph model is developed for each market.

2-5-1 Commodity Market

The commodity market trades in useful items that can be turned into a commercial advantage [47]. The items are uniform in value, meaning that a similar product can come from different producers [55]. I use the definition as stated in [38] for commodities:

Definition 2-5.1: Commodity

Tradeable item that can usually be processed further and sold.

Originally, hard commodities are depleted, and soft commodities are grown [38]. Nowadays, virtually everything is for sale, and thus is almost everything a commodity [78, 41]. I assume a market in hard commodities in this thesis.

The commodity price is determined by the market as a whole, as commodities are traded actively. This trading takes place on the futures, spot, and derivatives markets [37]. In this thesis, I assume that commodities are traded on the futures market only, and that the firm uses one type of commodity only. This simplification can be readdressed in future research.

Firms use inventories to store their commodities, to compensate for time lags, seasonal de-

mand, and uncertainty [19]. The company's inventory appears as a current asset on the balance sheet and is expressed in units of dollars [\$], instead of goods [#] [63].

2-5-2 Labor Market

In the labor market, firms interact with employees by hiring them. The market of supply (employees) and demand (firm) influences the working hours and compensation. Firms pay the employees with salaries, wages or benefits [44]. In this thesis, I adopt the definition of labor, as stated in [47]:

Definition 2-5.2: Labor

Work performed by people.

The Centraal Plan Bureau (CPB) uses the SAFFIER II model to predict labor market dynamics [7]. The model consists of 25 behavioral equations, 270 rules of thumb, and 1455 identities. This complexity objects, among other reasons, its use in this firm model [36].

As the commodity market trades in different commodities, the labor market trades in various types of labor. The distinction made in the literature is in the level of education [69]. The educated labor force can be divided further in different skill sets [43]. In this thesis, I assume that the firm uses one labor type, and that salaries are paid per full-time equivalent (FTE). These simplifications can be readdressed in future research.

Not all labor that employees perform is directly linked to the production of products [29]. This distinction is known as direct or indirect labor [6]. Examples of indirect labor are process optimization, marketing, and R&D [51]. In this thesis, I model the firm to store this indirect labor as human capital [53]. Increased human capital yields higher productivity for the firm's labor force.

2-5-3 Capital Asset Market

The capital asset market trades in the firm's long term assets. Capital assets are needed for business operations, and are a factor of production [67]. The assets are physical in nature and not easily converted into cash. Vital capital assets are property, plant and equipment (PP&E) [73]. I adopt the following definition for capital assets [20]:

Definition 2-5.3: Capital asset

Significant piece of property that is useful for more than a year, and not intended for sale.

The capital assets contribute to profit generation and are linked to the production cost. The benefit they yield for production is known as economies of scale [21].

Large investments are expensed over their lifetime to obtain an accurate picture of the company's profitability [20]. In this thesis, I assume those expenses to be real cash flows. Two variables present the capital asset position: Their book value in [\$], and their performance in [%] [20]. I model the pair with a C-I-interconnection, expressed in action-angle variables. It is possible to lease or buy capital assets [20]. I model the pair with effort and flow sources.

2-5-4 Product Market

Firms sell their finished goods or services on the product market. The products result from combining the acquired commodities, labor and capital assets into a commercial advantage. I use the definition as stated in [47]:

Definition 2-5.4: Product

Something produced, created or grown for sale.

Product demand can either be expressed as quantity demanded [$\#/wk$] or the budget constraint that is spent on that product [$\$/\#wk$] [75, 31]. Both expressions can be imposed on the model by using flow and an effort sources.

Either the firm or competitors give answer to this demand. The ratio in which the firm supplies in comparison to its competitors, is its market position [75]. I model the market position of competitors with a C-element.

When business is secured by the firm, the demanded products are saved in the order book until the firm delivers [63]. A negative order book position indicates that the firm has items in stock.

2-6 Conclusions and Thesis Objective

Value is determined by the company's future performance. According to this definition, only the discounted cash flow (DCF) method is fundamental. The DCF method ascribes value to businesses by using two variables: The future expected cash flow, and a discount rate [21, 63]. However, both variables are determined subjectively as they are based on professional expertise [21, 9].

In the field of predictive analytics, data-driven models are developed to bypass the necessity for subjective inputs for business valuations. However, research shows that they lack the availability of abundant data, and that they are not economically interpretable [15, 33, 13]. This is resolved by using causal models for mimicking business dynamics, which are based on economic principles. The behavior of businesses as economic agent is profit maximization and interacts with four different markets. By taking this general definition of businesses as a corner stone in the modeling effort, the model applies to all businesses. In Chapter 4 of this thesis, a causal and general model for business dynamics is developed.

The discount rate is assumed to be constant in time, though research indicates that the economy shows cyclical behavior [58, 17]. In Chapter 6, a complex-valued discount rate is used to account for these oscillations.

Based on these conclusions, I state the objective of this thesis as follows:

Development of the Frequency-Domain Valuation Theory

1. Business valuations as a S&C problem and thesis proposal (Chapters 3).
2. A fundamental model for business dynamics (Chapters 4 and 5)
3. Frequency-domain valuation and financial analysis (Chapters 6 and 7)

Bibliography

- [1] M. Al-Shibli. *The fundamental principle of conservation of physical money: Its violation and the global financial system collapse*. Scientific Research, 2011.
- [2] E. R. Arzac. *Valuations for mergers, buyouts and restructuring*. John Wiley and Sons Inc., 2008.
- [3] O. Blanchard. *Macroeconomics*. Pearson, 2017.
- [4] J. W. Brewer. *Structure and cause and effect relations in social system simulations*. IEEE Transactions on Systems, Man, and Cybernetics, 1977.
- [5] J. W. Brewer. *Bilinear, dynamic single-ports and bond graphs of economic systems*. The Franklin Institute, 1982.
- [6] B. Chiang. *Indirect labor costs and implications for overhead allocation*. The College of New Jersey, 2013.
- [7] CPB. *SAFFIER II: 1 model voor de Nederlandse economie, in 2 hoedanigheden, voor 3 toepassingen*. CPB, 2010.
- [8] A. Damodaran. *Damodaran on valuations*. John Wiley & Sons Inc, 2006.
- [9] Valuations Department. *Expert opinion*. PriceWaterhouseCoopers, 2019.
- [10] Y. Ding. *Forecasting financial condition of Chinese listed companies based on support vector machine*. Expert Systems with Applications: An International Journal, 2008.
- [11] I. Ebaid. *Accruals and the prediction of future cash flows: Empirical evidence from an emerging market*. Management Research Review, 2011.
- [12] F. Y. Edgeworth. *Mathematical physics*. Routledge, 1881.
- [13] L. Einav and J. D. Levin. *The data revolution and economic analysis*. The National Bureau for Economic Research, 2013.

- [14] B. Keen et al. *What is a realistic value for price adjustments costs in New Keynesian models?* Applied Economics Letters, 2007.
- [15] L. Taylor et al. *Emerging practices and perspectives on big data analysis in economics: bigger and better or more of the same?* Sage Journals, 2014.
- [16] M. Barth et al. *Accruals and the prediction of future cash flows.* The Accounting Review, 2001.
- [17] P. Aghion et al. *Cyclical fiscal policy, credit constraints, and industry growth.* Elsevier, 2010.
- [18] R. A. Brealey et al. *Principles of corporate finance.* McGraw-Hill, 2014.
- [19] R. B. Chase et al. *Production and operations management: Manufacturing and services.* Irwin-McGrawHill U.S.A., 1998.
- [20] R. Libby et al. *Financial accounting.* McGraw-Hill Education, 2019.
- [21] T. Koller et al. *Valuations.* John Wiley and Sons Inc, 2015.
- [22] T. Ward et al. *A research note on the issue of non-articulation and the method used to calculate net operating cash flow.* Journal of Accounting Research, 2009.
- [23] W. J. Den Haan F. Covas. *The cyclical behavior of debt and equity finance.* American Economic Review, 2011.
- [24] K. R. French F. Fama. *The capital asset pricing model: Theory and evidence.* Journal of Economic Perspectives, 2004.
- [25] C. A. Finger. *The ability of earnings to predict future earnings and cash flow.* Journal of Accounting Research, 1994.
- [26] M. Friedman. *The social responsibility of business is to increase its profits.* Times Magazine, 1970.
- [27] W. N. Goetzmann and M. Massa. *Daily momentum and contrarian behavior of index fund investors.* The McGraw-Hill Companies, Inc., 2002.
- [28] E. L. Gordon. *What is fair market value.* University of Columbia, 1952.
- [29] A. Gorz. *From strategy for labor.* PlatyPus, 1964.
- [30] B. Graham and D. L. Dodd. *Security Analysis.* The McGraw-Hill Companies, Inc., 2009.
- [31] A. Hayes. *Economic rent definition.* Investopedia, 2019.
- [32] F. Heid. *Cyclical implications of minimum capital requirements.* Deutsche Bundesbank, 2005.
- [33] D. F. Hendry. *Econometrics - Alchemy or science?* Economica, 1980.
- [34] J. Hicks. *Value and capital.* Oxford University Press, 1975.

-
- [35] D. Higham. *An introduction to financial option valuation: mathematics, stochastics and computation*. Cambridge University Press, 2004.
 - [36] L. S. Huisman. *Modelling and control of a dynamical labour market system*. Delft Center for Systems and Control, 2019.
 - [37] J. C. Hull. *Options, futures, and other derivatives*. Pearson, 2015.
 - [38] iMinds. *Commodities*. iMinds Pty Ltd, 2010.
 - [39] Corporate Finance Institute. *Business valuation modelling course*. CFI, 2018.
 - [40] Corporate Finance Institute. *Introduction to corporate finance*. CFI, 2018.
 - [41] UN COMTRADE ITC Statistics. *Trade map - International trade statistics*. ITC, 2019.
 - [42] S. Orpurt J. Hales. *A Review of Academic Research on the Reporting of Cash Flows from Operations*. Financial Accounting Standards Board, 2013.
 - [43] D. B. Keesing. *International economics: Progress and transfer of technical knowledge*. The American Economic Review, 1990.
 - [44] W. Kenton. *The economic order quantity - EOQ definition*. Investopedia, 2020.
 - [45] J. M. Keynes. *The general theory of employment, interest and money*. Palgrave Macmillan, 1936.
 - [46] I. Kolstad. *Why firms should not always maximize profits*. Journal of Business Ethics, 2007.
 - [47] Oxford Languages. *Oxford dictionary of English*. Oxford University Press, 2019.
 - [48] Triple A Learning. *Microeconomics*. Oxford study Courses, 2013.
 - [49] R. Leszinski and M. v. Marn. *Setting value, not price*. McKinsey&Company, 1997.
 - [50] K. S. Lorek. *A multivariate time-series prediction model for cash-flow data*. Accounting Review, 1996.
 - [51] M. Wester M. Kilbridge. *An economic model for the division of labor*. INFORMS, 1966.
 - [52] W. Kross M. Kim. *The ability of earnings to predict future operating cash flows has been increasing -not decreasing*. Journal of Accounting Research, 2005.
 - [53] N. Manders. *The thermodynamics of economic engineering*. Delft Center for Systems and Control, 2019.
 - [54] N. G. Mankiw. *Principles of macroeconomics*. Cengage Learning, 2018.
 - [55] K. Marx. *A contribution to the critique of political economy*. Franklin Classics, 1859.
 - [56] M. B. Miller. *Quantitive financial risk management*. Wiley, 2018.
 - [57] P. Modesti. *Measures for firm values in random scenarios*. Faculty of Economics, 2010.

- [58] T. Mukoyama. *A note on dynamical discount factors and labor market volatility*. University of Virginia, 2013.
- [59] C. Nyce. *Predictive analytics white paper*. American Institute for CPCU, 2007.
- [60] J. Greenwood P. Gomme. *On the cyclical allocation of risk*. Elsevier, 1993.
- [61] D. W. COLLINS P. Hribar. *Errors in estimating accruals: Implications for empirical research*. Journal of Accounting Research, 2002.
- [62] V. Pareto. *Manual of political economy*. Oxford University Press, 1906.
- [63] T. Rietveld. *Handboek Investeren & Financieren*. Vakmedianet Management B.V., 2017.
- [64] J. J. Rotemberg. *Sticky prices in the United States*. Journal of the Political Economy, 1982.
- [65] J. Say. *A treatise of political economy*. Taylor & Francis Inc, 1803.
- [66] K. Y. Shen. *Modeling of earnings prediction by time-delay neural network*. Materials Science and Information Technology, 2012.
- [67] A. Smith. *An inquiry into the nature and causes of the wealth of nations*. W. Strahan and T. Cadell, 1776.
- [68] M. Stalla-Bourdillon. *Continuous DCF method*. HEC Paris, 2012.
- [69] G. J. Stigler. *Information in the labor market*. The University of Chicago Press Journals, 1962.
- [70] R. H. Strotz. *Myopia and inconsistency in dynamic utility maximization*. Oxford University Press, 1955.
- [71] S. E. Svedhem. *Using mechanical system dynamics to model time-discounting in behavioral economics*. Delft Center for Systems and Control, 2019.
- [72] D. Tiemann and J. Hartman. *Data analytical due diligence is driving M&A deals*. Financial Executives Research, 2013.
- [73] A. Tuovila. *What is a capital asset*. Investopedia, 2019.
- [74] X. A. van Ardenne. *A review of business valuations and a proposal for the application of economic engineering*. Delft Center for Systems and Control, 2020.
- [75] H. R. Varian. *Intermediate microeconomics: A modern approach*. W. W. Norton & Company, 2010.
- [76] N. J. Vriend. *Rational behavior and economic theory*. Elsevier, 1996.
- [77] L. Walras. *Elements d'economie politique pure*. Literary Licensing LiC, 1874.
- [78] M. Watts. *Commodities*. Introducing Human Geographies, 1999.

Chapter 3

The Economic-Engineering Analogy in Business Economics

3-1 Introduction

Systems engineering is an interdisciplinary field of engineering that focuses on causal modeling of complex physical systems. The discipline uses physical principles to determine the system's time evolution. However, as we aim to model an economic system, these physical principles cannot directly be applied.

In this chapter, the business valuation process is mapped to the physical domain, by making economic-engineering analogs specific for business economics. Consequently, systems engineering can be applied and next steps for developing the valuation method are proposed. The goals of this chapter are as follows:

Chapter goals:

1. Development of a framework to see business economics as a S&C engineer.
2. Formulation of proposals for cash-flow determination and financial analysis.

In Section 3-2, business-economics analogs and bond graphs are introduced. In Section 3-3, energy-cash flow analogs are presented. In Section 3-4, I conclude Part 1 of this thesis (putting business valuations as a S&C problem), and propose an approach to complete Parts 2 and 3.

3-2 Signal-Based Modeling of Business Dynamics

3-2-1 Analogs of Newtonian Mechanics Extended to Business Economics

Engineers use models to mimic the behavior of systems by expressing them in their physical variables. The engineering field consists of multiple domains, each having a different set of variables. Still, all domains can be modeled with a single technique, since the similarities between these domains are well-known [12, 9].

These so-called similarities are defined through the formulation of analogs. Analogs are pairs of physical variables from different domains that show identical behavior. They are often used in modeling mechatronic systems, where electrical and mechanical subsystems are regularly interconnected. In this scenario, the electrical current is analogous to the mechanical velocity, and the mechanical force is analogous to the electrical voltage. As a result, the two second-order systems depicted in Figure 3-4 show similar behavior [12, 9].

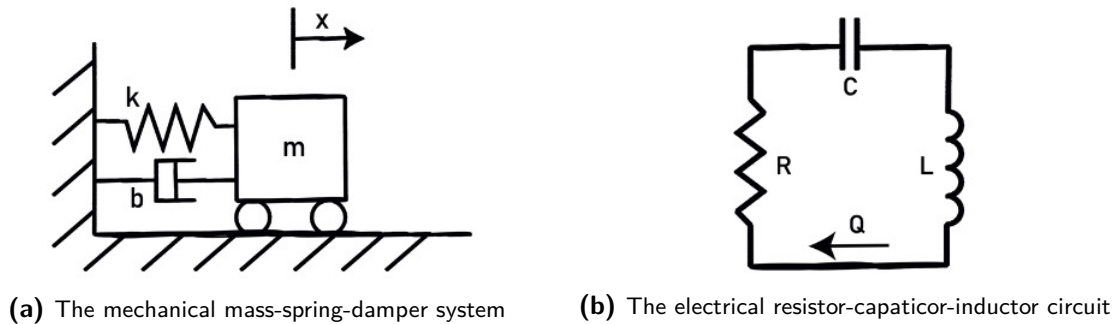


Figure 3-1: Schematics of second-order systems from different physical domains that show equivalent behavior. This equivalent behavior is caused by analogies between the mechanical and electrical domain [20].

Analogs arise from the fact that the behavior of physical systems is determined by the system's energy and power flows. In a general sense, energy is stored by either momentum or displacement, and the power (energy flow) is determined by the effort and the flow [12, 9]. Table 3-1 gives the energy and power variables in the general, mechanical and electrical domain.

Table 3-1: Analogs in different physical domains

General	Mechanics		Electronics	
	Variable	Unit	Variable	Unit
Effort (e)	Force (F)	N	Voltage (U)	V
Flow (f)	Velocity (v)	$\frac{m}{s}$	Current (I)	A
Momentum (p)	Momentum (p)	Ns	Flux linkage (λ)	Vs
Displacement (q)	Displacement (x)	m	Charge (Q)	C
Power (P)	$F(t)v(t)$	$\frac{Nm}{s}$	$U(t)I(t)$	VA = $\frac{Nm}{s}$
Energy (E)	$\int F dx, \int V dp$	Nm	$\int e dQ, \int I d\lambda$	VAs = Nm

Mendel extends this idea of analogization by transcending the engineering domain and formu-

lating physical analogs for economic variables. The idea of formulating analogs for economic principles is not new [4, 5, 6, 15, 16, 17], but Mendel's research is unique in the analogs he chooses.¹ Table 3-2 displays the economic-engineering analogs interpreted in the context of business economics [25].

Table 3-2: Analogs in the economic domain

General	Business economics	
	Variable	Unit
Effort (e)	Cost ($\dot{p}$)	$\frac{\$}{\# \text{wk}}$
Flow (f)	Flow ($\dot{q}$)	$\frac{\#}{\text{wk}}$
Momentum (p)	Price (p)	$\frac{\$}{\#}$
Displacement (q)	Asset position (q)	$\#$
Power (P)	Growth (G)	$\frac{\$}{\text{wk}^2}$
Energy (E)	Cash flow (CF)	$\frac{\$}{\text{wk.}}$

This choice of analogs directly yields two benefits. Firstly, economic variables have an intuitive interpretation.² Secondly, two major value drivers arise naturally from the framework: Growth and cash flow.

3-2-2 Bond-Graphs Modeling for Business Dynamics

Bond graphs represent dynamical systems graphically. Karnopp et al. explain how to use them for mechatronic systems in [12], and Brewer proposes their use in economics [5]. There are three reasons for using this technique for modeling firm dynamics as economic engineer.

1. **Graphical modeling:** Economic models are mathematically not as sophisticated as engineering models, as no economic equations of motion exist. A graphic representation of economic systems allows us to capture its dynamics intuitively.
2. **Domain neutrality:** Bond graphs are domain neutral. As the economic-engineering modeling approach is based on analogs, this domain-neutrality is essential. Figure 3-2 represents both physical systems displayed in Figure 3-4, and is thereby capable of representing a second-order economic system.
3. **System derivation:** The model allows for direct state-space or transfer function derivation, so that systems and control theory can be applied.

1-Ports and Multi-Ports for Economic Laws

Table 3-2 states the analogs between economic and domain-neutral variables, enabling us to map the economic variables to the bond graph:

¹Brewer and Franksen juxtapose physical force to economic price, while Mendel interprets the economic price as the physical momentum [25, 5, 15]

²There is a duality between the flow and effort, and the momentum and displacement. One can interchange the variables, but then the economic intuition is lost. See for example [9].

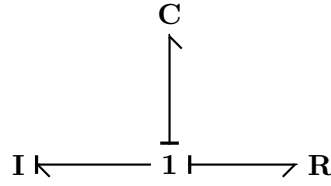


Figure 3-2: Bond graph model representing a second-order system that is equivalent to the mechanical and electrical systems in Figure 3-4.

- The I-element stores the price, and the C-element stores the asset position.
- The costs and flows are the two power variables working along the bonds.
- Multiplication of the pair gives the economic growth over that bond.
- Integration of either power variable gives the cash flow.

The 1-ports and multi-ports as described in [12] are analogous to the economic laws that are mentioned in Section 2-4-3.

The law of conservation of money (Equation 2-3) states that having some product flow entails one cost that is both paid and received. This law corresponds to the definition of the multi-port 0-junction as Equation 3-1 states.

$$\begin{aligned} e_1 &= e_2 = e_3 \\ \sum f_{in} &= \sum f_{out} \end{aligned} \quad (3-1)$$

Market clearing (Equation 2-4) stipulates that no goods are lost in economic systems, which is ensured by balancing out costs and benefits. The definition of the multi-port 1-junction states this law (Equation 3-2).

$$\begin{aligned} f_1 &= f_2 = f_3 \\ \sum e_{in} &= \sum e_{out} \end{aligned} \quad (3-2)$$

Costs and benefits acting upon a market determine the free market price (Equation 2-5). The 1-port I-element is the domain-neutral analog of the price, by doing this continuously instead of discretely.

$$p = \int e(t)dt \quad (3-3)$$

Assets are accumulated by summing the assets coming in and subtracting the assets going out (Equation 3-13). The 1-port C-element performs this action, again by doing this continuously instead of discretely.

$$q = \int f(t)dt \quad (3-4)$$

Depreciation and Consumption through R-Elements

The R-element is the only 1-port element that does not store energy and directly links the two power variables without integration. The element dissipates energy, and can be connected to both junctions [12].

When connected to a 0-junction, the element returns a flow as a result of an imposed cost. This phenomenon is known in economics as consumption [26]. When connected to a 1-junction, the element returns a cost that is caused by a flow variable. In economics, this is known as depreciation [26]. We shall see that the firms mainly dissipate energy through depreciation.

2-ports for Interconnecting Markets and Asset Capitalization

2-ports change specific attributes of the effort and flow while conserving power. They can change the energy domain, power direction, or effort-flow ratio. There are two types of 2-ports. The transformer maps flow variables to flow variables, and effort variables to effort variables. The gyrator maps flow variables to effort variables, and effort variables to flow variables [12]. The model for business dynamics needs 2-ports for two reasons.

1. **Market interconnection:** The firm connects different markets by buying factors of production (labor, commodities, and capital assets) to develop its product. The amount of labor, commodities, and capital assets that the firm needs per product differ, and so transformers are needed to relate the product to the correct amount of these factors of production. Figure 3-3 visualizes this by the example of a car producer.

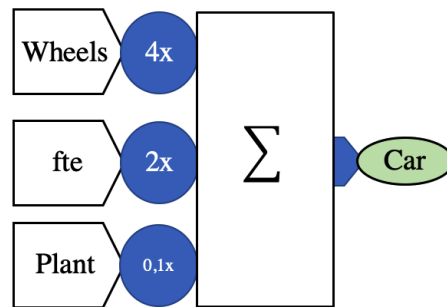


Figure 3-3: Firm use factors of production in a certain ratio to produce products. In this situation, the firm needs four wheels, two fte, and 10% of the plant to product a single car. This ratio is modeled by using gyrators.

2. **Asset capitalization:** The firm's financials display its assets in terms of dollars, instead of the number of capital assets, level of commodities, and labor force. Transformers and gyrators capitalize the assets by expressing them in terms of money.

Figure 3-4a shows the tetrahedon of state that is used for modeling firm dynamics. Note that the goods [#] could either indicate commodities [#], labor [FTE], capital assets [#], or products [#].

When modeling capitalized assets, one energy variable must express the assets in terms of money [\$]. This choice obliges the other energy variable to be dimensionless, and have the units of [%], to preserve the power-conserving properties of the bond graph. Figure 3-4b shows the tetrahedron of state in action-angle coordinates.

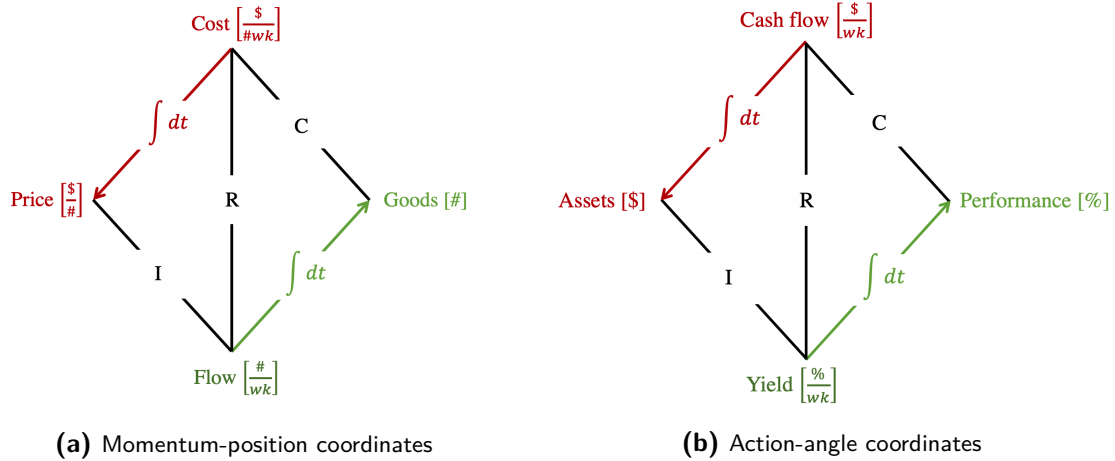


Figure 3-4: The economic tetrahedra of state in two coordinate systems: Momentum-position coordinates use costs, flows, prices and goods. The action-angle coordinates use cash flow, yield, performance and assets.

The pair has the physical analogs of action and angle, therefore expressing the capitalized assets in action-angle coordinates. In physics, C-elements conventionally store the angle variable, and I-elements store the action variable [19]. To stay close to this agreement, I assign their analogs to corresponding elements: Dollars are stored in the I-element, and percentages are stored in the C-element.

3-3 Business Economics in the Energy Domain

There is one analog of particular importance: The energy-cash flow analog. In the physical domain, energy plays a central role in Lagrangian and Hamiltonian mechanics. In business economics, financial performance is expressed in terms of cash flows, which are ultimately needed to determine the present value.

3-3-1 The Least Action Principle for Profit Maximization

The state of a particle at a given time t consists of its generalized position q and generalized velocity $\dot{q}$ [22]. This statement translates to business economics by asserting that the state of a firm at a given time t is determined by its assets q and product flow $\dot{q}$. Parallel to the utility Lagrangian for economic agents as defined in [21], I introduce the Profit Lagrangian for firms:

$$L(q, \dot{q}) \tag{3-5}$$

In physics, particles base their movement on the principle of least action. They do so by minimizing the following integral [22]:

$$S = \int_{t_1}^{t_2} L(q, \dot{q}) dt \quad (3-6)$$

Figure 3-5 shows the path of a particle in two dimensional space. $\dot{q}$ is visualized as tangents to the path of the particle, as it is the time derivative of q .

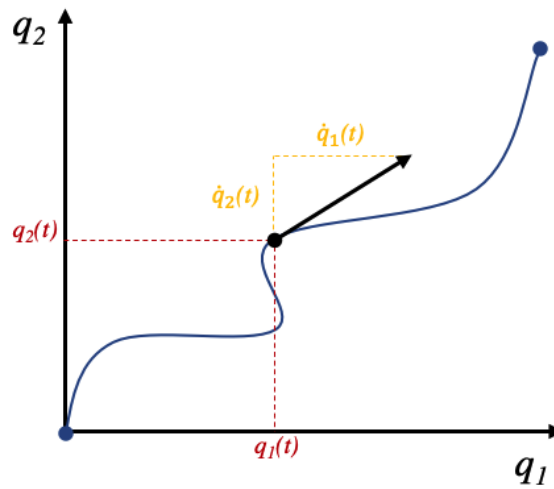


Figure 3-5: According to Lagrange's least action principle, the motion of every particle is described by minimization of the action integral over its path. The path is determined by its two state variables: q and $\dot{q}$.

Firms maximize their profit by maximizing this same integral.³ They trade off variable cost, which is a function of the product flow $\dot{q}$, and fixed cost, which is a function of their assets q [25]. By assuming a concave profit function [26], and using the calculus of variations [22, 16], the total profit is maximized if the first variation of S vanishes:

$$\partial S = 0 \quad (3-7)$$

This condition is in economic engineering considered as a fundamental principle and will be referred to as the principle of maximum profit. By evaluating this principle with the calculus of variations, we arrive at a maximum profit when the Euler-Lagrange equation is satisfied.

$$-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) + \left(\frac{\partial L}{\partial q}\right) = 0 \quad (3-8)$$

From this equation, the characterizations of the marginal profits are obtained. The marginal change of profit as a result of a change in product flow is the price that the firm is willing to pay or receive for that change in product flow, expressed in [\$/#]:

³This is explained in Section 2-4-1.

$$\frac{\partial L}{\partial \dot{q}} \equiv p \quad (3-9)$$

The marginal change of profit resulting from a change in asset position, is equal to the cost or benefit from holding that asset, expressed in [\$/#wk]:

$$\frac{\partial L}{\partial q} \equiv \dot{p} \quad (3-10)$$

The integration of the Euler-Lagrange equation shows how prices evolve dynamically in time. The price at time t is determined by adding and subtracting the costs and benefits up to time t to some initial price:

$$p(t) = p(0) + \int_0^t \dot{p} dt \quad (3-11)$$

Reconsider the integral of Equation 3-6. The time horizon $[t_1, t_2]$ can economically be seen as the investment horizon, which plays a significant role in business decisions. If the company has a short investment horizon, the company aims for high returns, without any long-term investments. If the investment horizon is long, the Profit Lagrangian allows these investments as they yield higher returns in the future. This perfectly establishes the growth-profit nexus [2, 8].

The principle of least action provides a physical interpretation for rational firm behavior. Newton's equations are derived from the Lagrangian and the principle of least action, but this principle is not axiomatically derived [3]. Coherently, it is known that firms trade off their costs and benefits to achieve maximum profit. However, it is unclear where this profit maximization comes from [18].

Apart from this theoretical explanation for profit maximization, the Lagrangian is of limited use when modeling firm dynamics. This is due to the dependency between its two variables q and $\dot{q}$. When forces are not conserved, the physical Lagrangian is not conserved in time and does not have a symplectic structure⁴. As a result, the Euler-Lagrange equation cannot be used to determine the time evolution of particles. Similarly, when the costs of the Profit Lagrangian are not conserved, the Euler-Lagrange equation cannot be used to determine the dynamics of a firm [22, 3].

3-3-2 The Firm's Earnings as the Kinetic Part of the Hamiltonian

The Hamiltonian is a function that gives the total energy of physical systems and is analogous to the surplus of an economic system [25]. The Hamiltonian can always be retrieved from the Lagrangian by application of the Legendre transform [13]:

⁴A symplectic structure possesses the geometry of a closed non-degenerative skew-symmetric bilinear form. A bilinear form maps two vector spaces linearly in a scalar: $V \times V \rightarrow K$. The closedness states that the structure matrix is finite-dimensional, the non-degenerativeness assures that its determinant is 1, the skew-symmetric part says that it is symmetric over the diagonal. A mathematical explanation of symplectic structures is given by Cannas da Silva in [11].

$$H(p, q) = p\dot{q} - L(q, \dot{q}) \quad (3-12)$$

The Hamiltonian uses two variables: The company's assets q , and the price p . According to Definition 2-2-1, only price variable p yields company value as this variable measures cash flowing in or out of the business. Contrarily, the q -variable measures the cash stored in economic systems, and does not contribute to the value of businesses. Business value is thus determined by the kinetic energy of the Hamiltonian:

$$\text{Earnings} = H(p) \quad (3-13)$$

Closed economies preserve cash (Section 2-4-3), making sure that the Hamiltonian is conserved: $\frac{dH}{dt} = 0$. Using its description in independent p - and q -coordinates, the time-evolution of systems can be visualized in phase-space, which shows its symplectic structure:

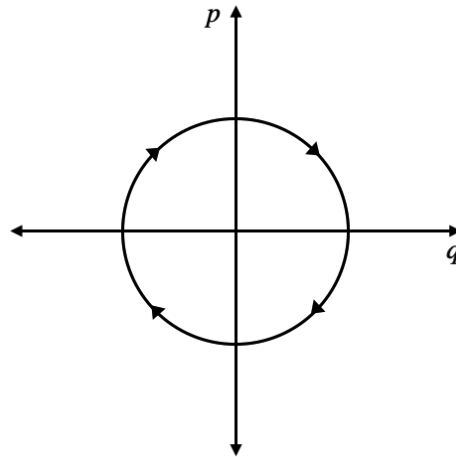


Figure 3-6: The time evolution of a conservative Hamiltonian system in phase space has a symplectic structure. Consequently, Hamilton's equations can be used to determine its motion.

This allows us to use Hamilton's equations to derive the dynamics of the firm:

$$\frac{\partial H}{\partial q} = -\dot{p} \quad (3-14)$$

$$\frac{\partial H}{\partial p} = \dot{q} \quad (3-15)$$

The two equations have an economic interpretation. Equation 3-14 says that an increase in earnings due to increased assets leads to a price decrease. This Equation is a mathematical expression of economies of scale: Larger companies can produce cheaper [14].

Equation 3-15 states that increased earnings resulting from a price increase lead to asset accumulation. This statement says that profitable companies are bound to grow (profit-growth nexus) [14, 8].

3-3-3 Depreciation causes Economic Systems to be Non-Conservative

Dissipation in physical systems converts energy into heat. It is an irreversible process, since it is impossible to reverse the process without adding additional energy. This is analogous to the economic phenomena of depreciation [10]. When depreciation occurs, it cannot be reversed [25, 24].

We shall see in Chapter 4 that depreciation makes its appearance in the model for business dynamics. Accordingly, the system is not conservative, which imposes restrictions on the use of Lagrangian and Hamiltonian mechanics. More specifically, the condition of non-degenerateness no longer holds [11]. Figure 3-7 shows that the time-evolution of a non-conservative Hamiltonian system no longer has a symplectic structure:

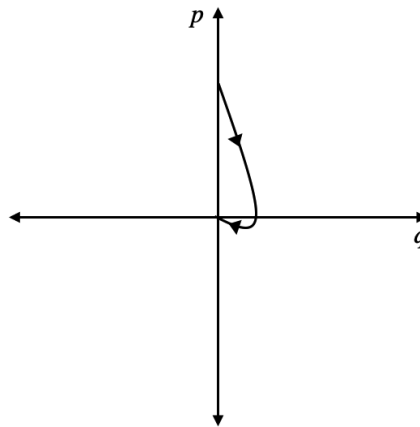


Figure 3-7: The time evolution of a non-conservative Hamiltonian system in phase space has no symplectic structure. For this reason, Hamilton's equations cannot be directly applied to determine its motion.

Since the Hamiltonian is not conserved, Hamilton's Equations can no longer be used to derive the system's dynamics. In Chapter 5, this obstacle is overcome by expressing the Hamiltonian in bicomplex coordinates, following the approach developed by Hutters in [7]. The bicomplex Hamiltonian preserves a symplectic structure in dissipative systems so that Hamilton's equations can still be used to derive the system's time evolution.

3-3-4 Interconnection of Port-Hamiltonian Systems

Port-Hamiltonian (pH) systems theory combines the properties of a Hamiltonian system with a geometric interconnection structure⁵. PH systems model the input-output behavior of the subsystem and the system it is connected with. This system could either be another subsystem, a resistive element, or the environment. The geometric structure allows interconnecting multiple subsystems so that highly complex systems can be modeled [1].

⁵This geometric structure is called a Dirac structure [1].

The pH framework is used in macroeconomic models [23].⁶ By adopting the economic-engineering analogs as listed in Section 3-2-1, the input and output of economic subsystems are the product flow and cost or benefit. Bond graphs (Section 3-2-2) are a pH modeling technique [1]. The economic pH interconnection is visualized in Figure 3-8.

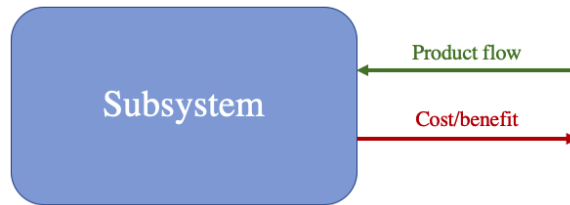


Figure 3-8: The conjugate input-output interconnection of economic subsystems returns an cost or benefit to a product flow (or vice versa). Multiplication of the pair yields the economic growth over that interconnection.

In this thesis, I model the firm and its connected markets as pH subsystems. The subsystems have their own structure, but obey the economic law of conservation of cash. By doing this, it is possible to add or remove markets straightforwardly. This is further explained in Section 5-4.

⁶Macchelli uses different economic analogs than the analogs adopted in this thesis. Macchelli's analogs are in line with those of Brewer and Franksen [23, 5, 15].

3-4 Conclusions and Next Steps

Profit maximization determines business behavior, the way the least-action principles determine the motion of all particles. Due to this general definition, both Newtonian and analytical mechanics can be used to model business dynamics.

Modeling businesses with Hamiltonian mechanics is economically fundamental: Energy is used to determine the dynamics of physical systems, the way cash flow is used in current valuation practices. However, as businesses are modeled with dissipative systems, regular Hamiltonian mechanics do not give the equations of motion or business value. In Chapter 5, I overcome this obstacle by using a complex port-Hamiltonian representation of the business model.

By formulation of the business-economics analogy, Part 1 of this thesis is completed (putting business valuations as a S&C problem). To develop the frequency-domain valuation theory, I propose to complete Parts 2 and 3 in the following four steps:

Next Steps towards the Frequency-Domain Valuation Theory

2) A fundamental model for business dynamics

- (a) *Bond-graph modeling (Chapter 4)*: Create a model for business dynamics.
- (b) *Complex port-Hamiltonian mechanics (Chapter 5)*: Express business dynamics in terms of cash flows and determine which cash flow yields value.

3) Frequency-domain discounting and financial analysis

- (c) *A frequency-domain theory for valuation (Chapter 6)*: Use the Laplace transform to develop fundamental frameworks for valuation and scenario analysis.
- (d) *S&C tools for financial analysis (Chapter 7)*: Use systems and control tools to perform financial analysis.

Bibliography

- [1] D. Jeltsema A. van der Schaft. *Port-Hamiltonian systems theory: An introductory survey*. Foundations and Trends in Systems and Control, 2014.
- [2] P. Lorange B. Chakravarthy. *Profit or growth? You don't have to choose*. Wharton School Publishing, 2007.
- [3] V. Balakrishnan. *Lecture series on classical physics*. Department of Physics, IIT Madras, 2009.
- [4] J. W. Brewer. *Structure and cause and effect relations in social system simulations*. IEEE Transactions on Systems, Man, and Cybernetics, 1977.
- [5] J. W. Brewer. *Bilinear, dynamic single-ports and bond graphs of economic systems*. The Franklin Institute, 1982.
- [6] J. W. Brewer. *Progress in the bond graph representations of economics and population dynamics*. The Franklin Institute, 1991.
- [7] M. Mendel C. Hutters. *Overcoming the dissipation obstacle with bicomplex port-Hamiltonian Mechanics*. Delft Center for Systems and Control, 2020.
- [8] M. Cowling. *The growth-profit nexus*. Kluwer Academic Publishers, 2004.
- [9] J. M. A. Scherpen D. Jeltsema. *Multidomain modeling of nonlinear networks and systems*. IEEE Control Systems Magazine, 2012.
- [10] D. Marius D. Radu. *Issues related to the accounting treatment of the tangible and intangible assets depreciation*. European Integration - New Challenges, 2011.
- [11] A. Cannas da Silva. *Lectures on Symplectic Geometry*. Springer, 2000.
- [12] D. C. Karnopp et al. *System dynamics: Modeling, simulation, and control of mechatronic systems*. John Wiley & Sons, Inc., 2012.

- [13] R. K. P. Zia et al. *Making sense of the Legendre transform*. Department of Physics, Virginia Polytechnic Institute and State University, 2009.
- [14] T. Koller et al. *Valuations*. John Wiley and Sons Inc, 2015.
- [15] O. I. Franksen. *Mathematical programming in economics by physical analogies. Part 1: The analogy between engineering and economics*. Simulation Councils, Inc., 1969.
- [16] O. I. Franksen. *Mathematical programming in economics by physical analogies. Part 2: The economic network concept*. Simulation Councils, Inc., 1969.
- [17] O. I. Franksen. *Mathematical programming in economics by physical analogies. Part 3: System equilibrium and mathematical programming*. Simulation Councils, Inc., 1969.
- [18] M. Friedman. *The social responsibility of a business is to increase its profits*. Times Magazine, 1970.
- [19] P. Stehle H. C. Corben. *Classical mechanics*. John Wiley & Sons, Inc., 1950.
- [20] L. S. Huisman. *Modelling and control of a dynamical labour market system*. Delft Center for Systems and Control, 2019.
- [21] C. Hutters. *The analytical mechanics of consumption*. Delft Center for Systems and Control, 2019.
- [22] E. M. Lipshitz L. D. Landau. *Mechanics*. Butterworth-Heinenann, 1976.
- [23] A. Macchelli. *Port-Hamiltonian formulation of simple macro-economic systems*. Conference on Decision and Control, 2013.
- [24] N. Manders. *The Thermodynamics of Economic Engineering*. Delft Center for Systems and Control, 2019.
- [25] M. B. Mendel. *Principles of economic engineering*. Delft Center for Systems and Control, 2019.
- [26] H. R. Varian. *Intermediate microeconomics: A modern approach*. W. W. Norton & Company, 2010.

A General Bond-Graph Model for Business Dynamics

4-1 Introduction

In this chapter, I develop a bond-graph model for firm dynamics by using the economic-engineering analogs established in Chapter 3. The goal of this chapter is as follows:

Chapter goals:

1. Development of a bond-graph model for business dynamics.

The firm is connected to four markets¹, which are all modeled as port-Hamiltonian subsystems. In Section 4-2 I model the firm, in Section 4-3 the commodity market, in Section 4-4 the labor market, in Section 4-5 the capital asset market, and in Section 4-6 the product market. In Section 4-7 I integrate the subsystems into the total business model, and in Section 4-8 I conclude that the model predicts dynamics for businesses in general.

The modeling effort consists of four steps: (1) Subsystems' behavior is derived from the information presented in Sections 2-4 and 2-5, (2) bond-graph representations are given by using the analogs of Section 3-2, (3) all signals and elements are given an economic interpretation so that the model is intuitive to both engineers and economists, and (4) the state-space representation is derived.

¹This is explained in Section 2-5.

4-2 The Firm as a Subsystem

The firm as a subsystem has four inputs: The costs (or benefits) of connected markets.² They determine one output, which is then returned to the markets: The product flow. One state determines the firm's dynamics: The value-added. Table 4-1 gives an overview:

Table 4-1: Inputs, outputs and states of the firm

Inputs		Outputs	States
Order yield	[\$/#wk]	Product flow	[#/wk]
Economies of scale	[\$/#wk]		Value-added
Production cost	[\$/#wk]		[\$/#]
Unit labor cost (ULC)	[\$/#wk]		

Figure 4-1 depicts the firm's interconnections to different economic markets. Note that the direction of the green and red arrows indicate causality and not the direction of positive flows.

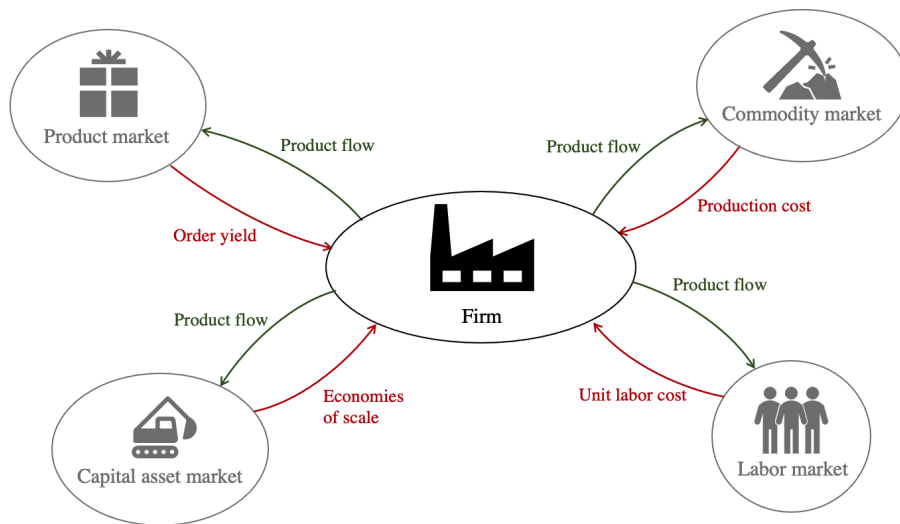


Figure 4-1: Firms are connected to four different economic markets: Product, capital asset, commodity and labor. Firms impose a product flow on the markets, which then return a certain cost or benefit.

Firms maximize profit by balancing off the costs against the benefits to determine an optimal product flow (Section 3-3-1). This optimal flow is called the economic order quantity (EOQ) [3]. I use the terms product flow and EOQ interchangeably.

4-2-1 Bond Graph for Profit Maximization

Figure 4-2 gives the bond graph of the subsystem that represents the firm. An I-element keeps track of the state value-added, for which costs and benefits determine the increase or

²The four markets are described in Section 2-5.

decrease. This state is the economic enhancement that the firm gives to its product. After deducting taxes with an R-element, the product flow is obtained from the value-added and returned to the markets.

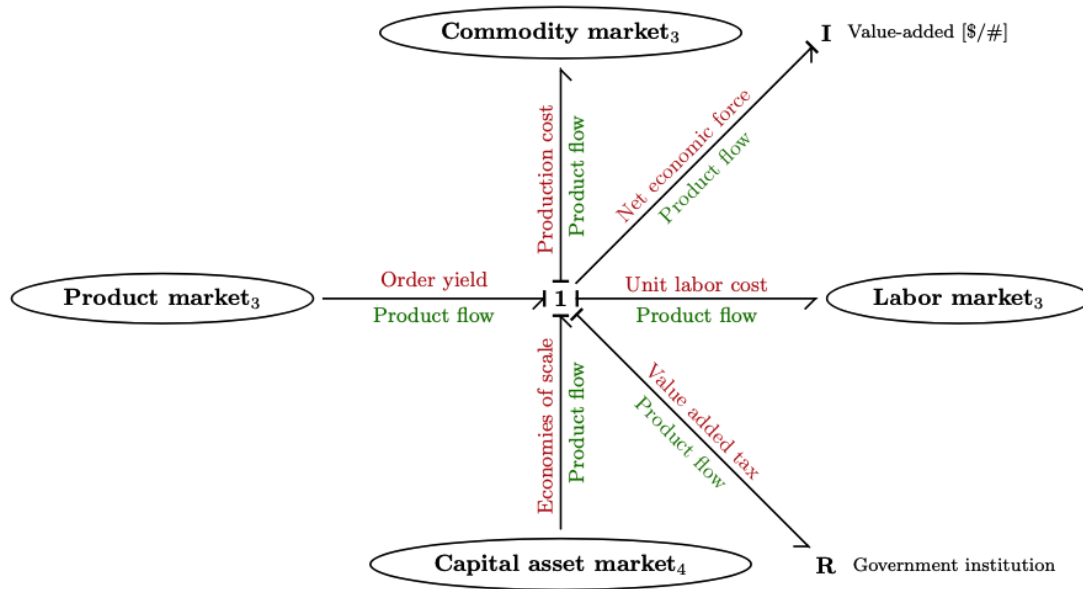


Figure 4-2: The Bond-graph model of the firm consists of one state: The value-added. This element measures the value that is created by converting factors of production into a product, and weighs off the costs and benefits of the different markets.

Table 4-2 gives the economic interpretation of the elements and signals in the subsystem:

Table 4-2: The interpretation of definitions in the bond graph representing the firm

Definition	Interpretation	Units
Elements		
Value-added	Economic enhancement that the firm gives its products [3].	[\$/#]
Government institution	System or method that controls the country [9].	
Efforts		
Order yield	Average amount customers spend on products from the firm [18].	[\$/#wk]
Production cost	Cost incurred by a business from manufacturing a product [14].	[\$/#wk]
Unit labor cost (ULC)	Average cost of labor per unit of output [18].	[\$/#wk]
Economies of scale	Cost advantages through efficient production of large scales [14].	[\$/#wk]
Value added tax (VAT)	Tax levied on each stage of production when value is added [14].	[\$/#wk]
Δ Net economic force	The change in economic enhancement of the product [3].	[\$/#wk]
Flows		
Product flow (Economic order quantity)	Ideal product flow to minimize costs [18].	[#/wk]

4-3 Commodity Market

The commodity market receives the product flow as only input from the firm and returns the production cost as only output. Three states determine the market's dynamics. Table 4-3 gives an overview.

Table 4-3: Inputs, outputs and states of the commodity market

Inputs		Outputs		States	
Product flow	[#/wk]	Production cost	[\$/#wk]	Current assets	[\$]
				Forward price	[\$/com]
				Available commodities	[com]

The market consists of three parts: The firm's inventory, the market itself, and the source. Commodities flow from the source, through the market and inventory, to the firm. Cash flows from the firm, through the inventory and market, to the source. Figure 4-3 gives an illustration. Note that the direction of the green and red arrows indicate causality and not the direction of positive flows.

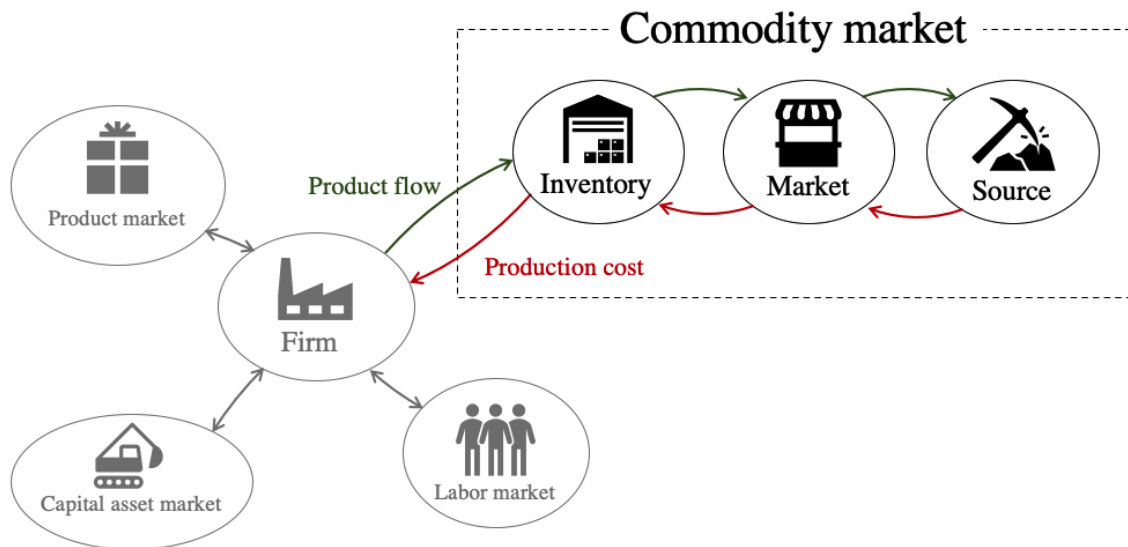


Figure 4-3: Firms interact with the commodity market by buying commodities that moves from the source, via the market and its inventory, towards the product market. In return, cash flows from the firm in the direction of the source. The direction of the arrows indicates causality.

4-3-1 Bond Graph for the Production Cost

The bond graph representing the commodity market is given in Figure 4-4. The dynamics of the inventory, market and source are explained below. Table 4-4 gives the economic interpretation of the elements and signals of the commodity market.

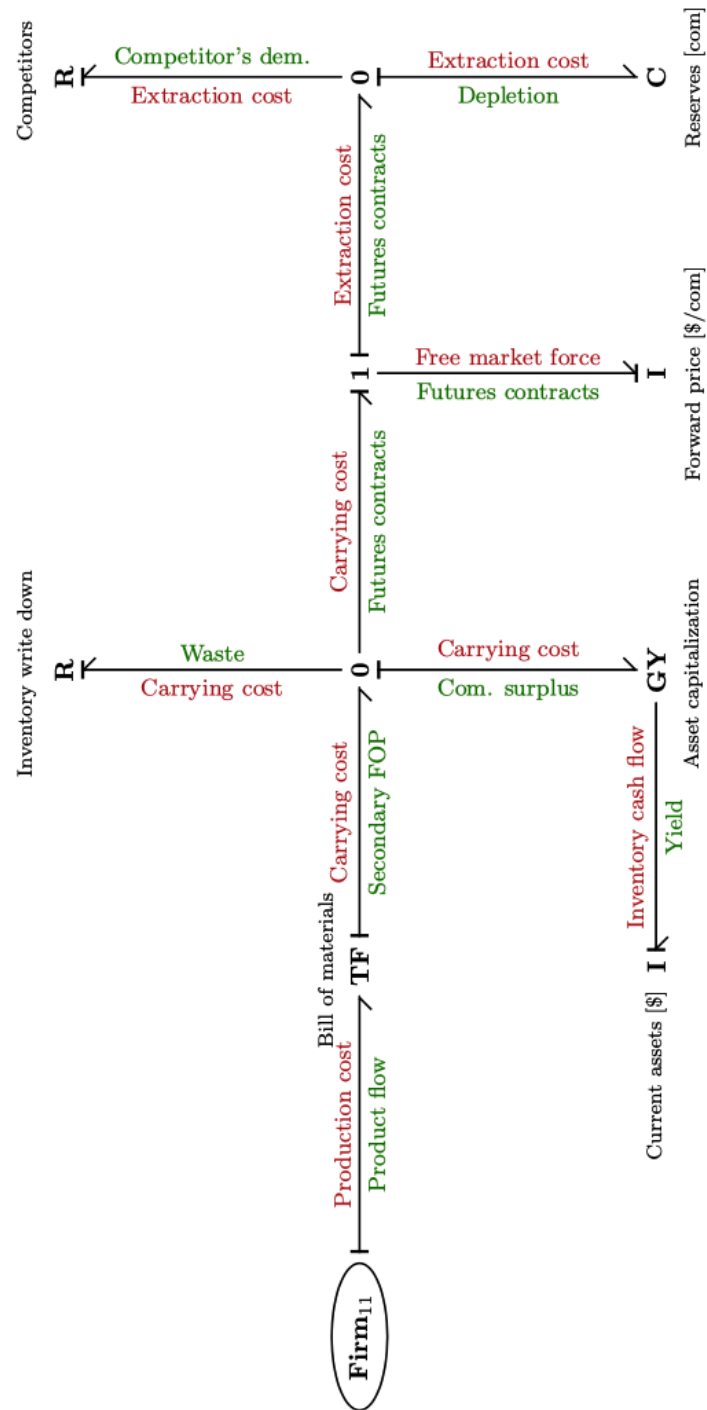


Figure 4-4: Bond-graph model of the commodity market. Production cost is determined by three states: Current assets, forward price, and reserves. Cash is lost through its two dissipative elements: Inventory write down and competitors. Energy-routing elements are used for capitalization and market interconnection (bill of materials).

Table 4-4: The interpretation of definitions in the commodity bond graph

Definition	Interpretation	Units
Elements		
Bill of materials	Raw materials and components needed to manufacture a product [15].	
Current assets	Assets that are expected to be sold or used for standard business operations over the next year [14].	[\$]
Asset capitalization	Recording of an item as an asset, rather than an expense [14].	
Inventory write down	Recognition of a portion of a company's inventory that no longer has value [14].	
Forward price	Predetermined delivery price for an underlying commodity of a forward contract [6].	[\$/com]
Competitors	Competing firms that secure the business offered by a supplier [18].	
Reserves	Level of commodities in a (natural) reserve [7].	[com]
Efforts		
Production cost	Cost incurred by a business from manufacturing a product [18].	[\$/comwk]
Carrying cost	Total cost of holding inventory [13]	[\$/ comwk]
Inventory cash flow	Cash flow spent on inventory increase [14].	[\$/wk]
Free market force	Unobservable market force that helps the demand and supply of goods in a free market to reach equilibrium [17].	[\$/comwk]
Extraction cost	Cost at which raw materials are depleted [10].	[\$/comwk]
Flows		
Product flow (Economic order quantity)	Ideal product flow to minimize costs [18].	[#/wk]
Secondary factors of production (FOP)	Inputs obtained from land, labor and capital [12].	[com/wk]
Commodity surplus	Amount of an asset or resource that exceeds the portion that's actively utilized [18].	[com/wk]
Yield	Earnings generated and realized on an investment over a particular period of time [14]	[%/wk]
Waste	Amount of raw materials lost in the production process [14].	[com/wk]
Futures contracts	Agreement to buy commodity assets, at a predetermined price and time in the future [6].	[com/wk]
Competitor's demand	Competitor's desire to purchase goods and services and willingness to pay a certain price [18].	[com/wk]
Depletion	Extraction of natural resources [6].	[com/wk]

Inventory

The inventory stores commodities and links the firm's product flow to its commodity demand on the open market. The firm's product flow is via a bill of materials linked to the amount of secondary factors of production (FOP) needed. The production cost is similarly related to a commodity's carrying cost.

The firm's inventory consists of an I-element that keeps track of the state representing the current assets, and an R-element for modeling the write down. The carrying cost depends on the current assets attained by the firm. The model determines this level by weighing off the firm's product flow against the market price of the commodity. Inventory is added to the balance sheet by using a gyrator for capitalization.

Market

The open market consists of an I-element that keeps track of the state that represents the forward price of commodities. The free market force weighs off the extraction cost against the carrying cost to determine the price increase or decrease.

Source

A C-element keeps track of the state reserves, which is the level of commodities that are available to the firm.³ An R-element represents the firm's competition in buying those commodities. The firm is a price taker if the competitor's demand is relatively high compared to the firm's demand. If this competitor's demand is relatively low, the firm is as a price maker.

4-3-2 State-Space Equations for Commodity Market Dynamics

The bond graph allows for state-space derivation, which is done in Appendix C-1. By defining state vector $x = [p_4 \quad q_5 \quad p_5]^T$ with p_4 the current assets, q_5 the forward price, and p_5 the reserves, the input u as the product flow, and output y as the production cost, the commodity market has the following state-space equation:

Dynamics commodity market

$$\begin{aligned} \dot{x} &= \underbrace{\begin{bmatrix} -\frac{GY_2^2}{R_6 I_4} & \frac{GY_2}{I_5} & 0 \\ 0 & -\frac{1}{R_7 C_5} & -\frac{1}{I_5} \\ -\frac{GY_2}{I_4} & 0 & \frac{1}{C_5} \end{bmatrix}}_A x + \underbrace{\begin{bmatrix} -GY_2 T F_2 \\ 0 \\ 0 \end{bmatrix}}_B u(t) \\ y &= \underbrace{\begin{bmatrix} \frac{T F_2 G Y_2}{I_4} & 0 & 0 \end{bmatrix}}_C x + \underbrace{\begin{bmatrix} 0 \end{bmatrix}}_D \end{aligned} \quad (4-1)$$

³I assume a market in hard commodities in this model. Adding a flow source, the market trades in soft instead of hard commodities.

Economic interpretations of the state-space variables are given in Table 4-11.

4-4 Labor Market

The labor market takes in the product flow as input from the firm and returns the unit labor cost (ULC) as output. Three states determine market dynamics. Table 4-5 gives an overview:

Table 4-5: Inputs, outputs and states of the labor market

Inputs		Outputs		States	
Product flow	[#/wk]	Unit labor cost (ULC)	[\$/#wk]	Goodwill/Intangibles	[\$]
				Salary	[\$/FTE]
				Labor force	[FTE]

The labor market consists of three parts: The firm's human capital, the market itself, and the supply side. Labor is supplied to the market and flows through the human capital to the firm. The cash flows from the firm, through the human capital and the market, to the supply. Figure 4-5 illustrates. Note that the direction of the arrows indicates causality and not the direction of positive flows.

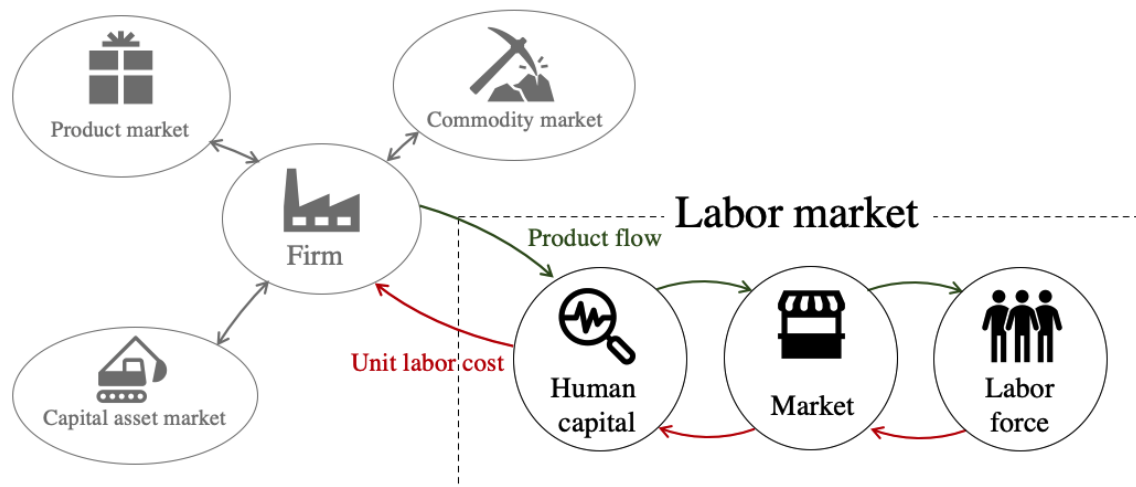


Figure 4-5: Firms interact with the labor market by buying labor via the market and its human capital from the labor force. In return, cash flows from the firm in the direction of the labor force. The direction of the arrows indicates causality.

4-4-1 Bond Graph for the Unit Labor Cost

Figure 4-6 displays the bond graph of the labor market. The dynamics of the human capital, market, and supply are explained below. Table 4-6 gives the economic interpretations of the elements and signals of the labor market.

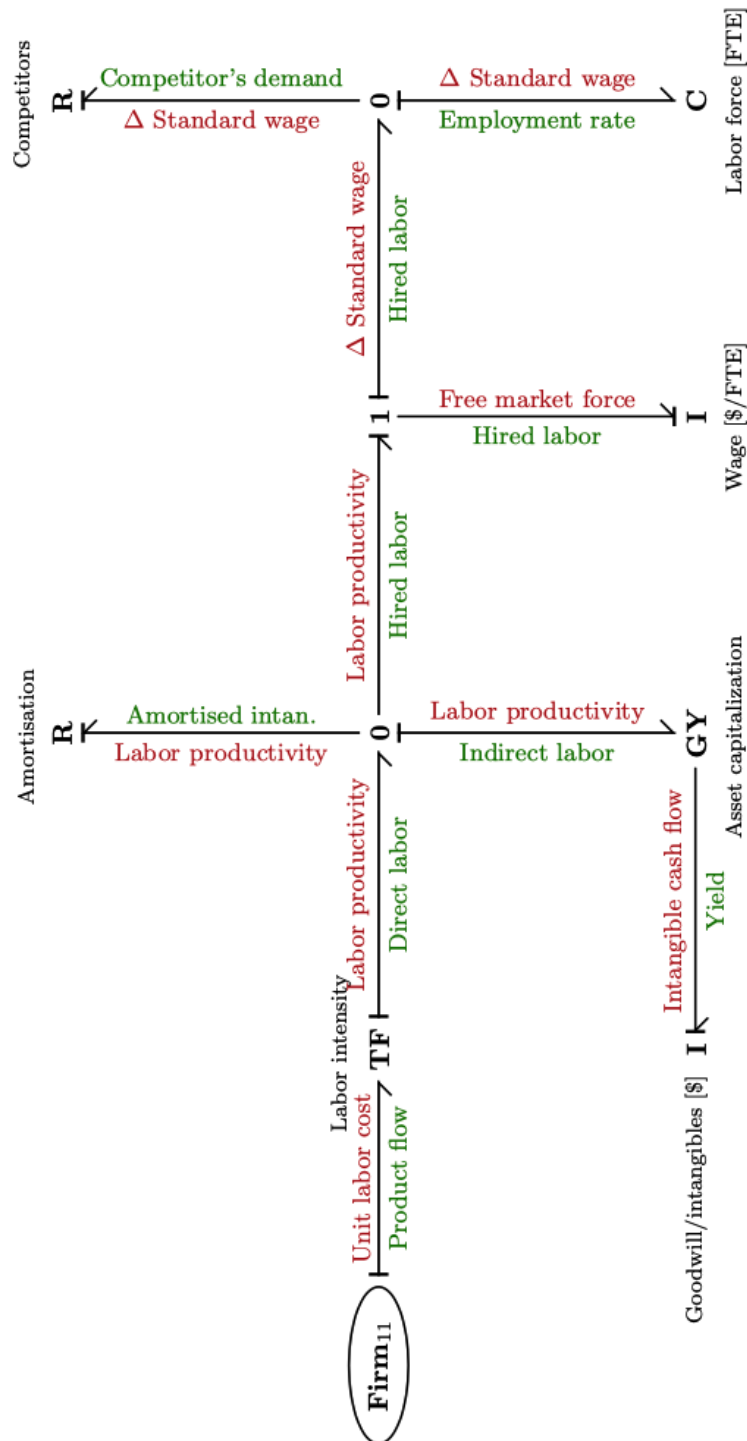


Figure 4-6: Bond-graph model of the labor market. Unit labor cost is determined by the three states of the labor market: intangible assets, wage, and labor force. Cash is lost through its two dissipative elements amortization, and competitors. Energy-routing elements are used for capitalization and market interconnection (labor intensity).

Table 4-6: The interpretation of definitions in the labor bond graph

Definition	Interpretation	Units
Elements		
Labor intensity	Amount of labor needed to produce a product [9].	
Goodwill/intangibles	Assets that are not physical in nature [14].	[\$]
Asset capitalization	Recording of indirect labor as asset, rather than expense [14].	
Amortisation	Lowering of value of intangibles over time [14].	[\$]
Wage	Amount paid for working one full time equivalent (FTE) [18].	[\$/FTE]
Competitors	Competing firms that secure labor [18].	
Labor force	All persons that are available for work, external to the firm [5].	[FTE]
Efforts		
Unit labor cost (ULC)	Average cost of labor per unit of output [18].	[\$/#wk]
Labor productivity	Value output per FTE [11].	[\$/FTEwk]
Intangible cash flow	Cash flow spent on goodwill/intangibles [14].	[\$/wk]
Free market force	Unobservable market force that helps the demand and supply of labor in a free market to reach equilibrium [17].	[\$/FTEwk]
Δ Standard wage	Difference in wage paid for laborers with a standard skill set [16].	[\$/FTEwk]
Flows		
Product flow (economic order quantity)	Ideal product flow to minimize costs [18].	[#/wk]
Direct labor	Work done directly on producing products [1]	[FTE/wk].
Indirect labor	Work done on activities other than production (Process optimization, marketing, R&D) [1]	[FTE/wk]
Amortised intangibles	Value decrease of intangibles over time [14].	[FTE/wk]
Yield	Earnings generated and realized on an investment over a particular period of time [14]	[%/wk]
Hired labor	Amount of persons moving in or out of the firm's employment [1].	[FTE/wk]
Competitor's demand	Competitor's desire to hire employees and willingness to pay a certain price [18].	[FTE/wk]
Employment rate	Measure of the extent to which available labour resources are used [5].	[FTE/wk]

Human Capital

Human capital links the firm's product flow to its demand for labor on the market and stores indirect labor. The labor intensity states how many labor hours are needed to produce a single product, linking the product flow to the amount of direct work required. Similarly, labor productivity is mapped to the unit labor cost.

An I-element keeps track of the state representing the goodwill/intangibles, and an R-element

represents the amortization. Labor productivity depends on the firm's goodwill/intangibles.⁴ Goodwill/intangibles are added to the balance sheet by capitalizing indirect labor.

Market

An I-element keeps track of the wage, which is determined by the free market force. The hired labor flows is then available to the firm.

Supply

A C-element keeps track of the labor force that is external to the firm⁵, and an R-element gives the competitor's demand for that labor. The firm is a price taker if the competitor's demand is high, compared to the firm's demand. The firm is a price maker if this competitor's demand is low.

4-4-2 State-Space Equations for Labor Market Dynamics

The bond graph allows for state-space derivation, which is done in Appendix C-1. By defining state vector $x = [p_6 \ q_6 \ p_7]^T$ with p_6 the goodwill/intangible assets, q_6 the labor force, and p_7 the wage, the input u as the product flow, and output y as the unit labor cost, the commodity market has the following state-space equation:

Dynamics labor market

$$\begin{aligned} \dot{x} &= \underbrace{\begin{bmatrix} -\frac{GY_3^2}{R_8 I_6} & \frac{GY_3}{I_7} & 0 \\ 0 & -\frac{1}{R_9 C_6} & -\frac{1}{I_7} \\ -\frac{GY_3}{I_6} & 0 & \frac{1}{C_6} \end{bmatrix}}_A x + \underbrace{\begin{bmatrix} -GY_3 T F_3 \\ 0 \\ 0 \end{bmatrix}}_B u(t) \\ y &= \underbrace{\begin{bmatrix} \frac{T F_3 G Y_3}{I_6} & 0 & 0 \end{bmatrix}}_C x + \underbrace{\begin{bmatrix} 0 \end{bmatrix}}_D \end{aligned} \quad (4-2)$$

Economic interpretations of the state-space variables are given in Table 4-11.

4-5 Capital Asset Market

The capital asset market has three inputs: The firm's product flow, and the capital leases and fixed investment controlled by management. There is one output: Economies of scale, which is a benefit supplied to the firm. Four states determine the dynamics. Table 4-7 gives an overview.

⁴For example, human capital that increases labor productivity are process optimization, R&D, and marketing. See Section 2-5.

⁵I assume a decreasing labor force in this model. By adding a flow source, the labor force will be growing.

Table 4-7: Inputs, outputs and states of the capital asset market

Inputs		Outputs		States	
Product flow	[#/wk]	Economies of scale	[\$/#wk]	Constr. in progress	[#]
Capital leases	[\$/#wk]			Asset price	[\$/#]
Fixed investment	[#/wk]			Asset performance	[%]
				Fixed assets	[\$]

The capital asset market consists of two parts: the firm's property, plant, and equipment (PPE), and the market itself. The fixed investments and capital leases are inputs to the market and determine how many assets are accumulated. The product flow determines asset utilization. Together, the three inputs decide on the amount of PPE, which yields the economies of scale as output. Figure 4-7 gives an illustration. Note that the direction of the arrows indicates causality and not the direction of positive flows.

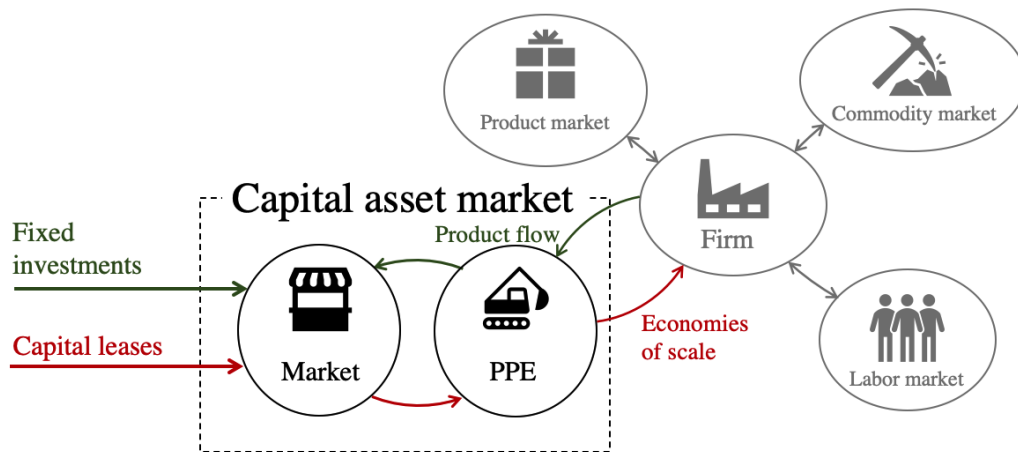


Figure 4-7: Firms interact with the capital asset market by utilizing the plant property and equipment (PPE) of the firm, and receiving the benefit of economies of scale in return. PPE is accumulated through fixed investment or capital leases. The direction of the arrows indicates causality.

4-5-1 Bond Graph for the Economies of Scale

Figure 4-8 gives the bond graph of the capital asset market. The dynamics of the market itself and the firm's PPE are explained below. Table 4-8 gives the economic interpretation of elements and signals of the capital asset market.

Market

The market links the inputs fixed investments and capital leases to the cash flow for investments (CFFI) supplied to the firm's PPE. A C-element keeps track of the assets in construction, and an I-element keeps track of the asset price by considering the construction cost and

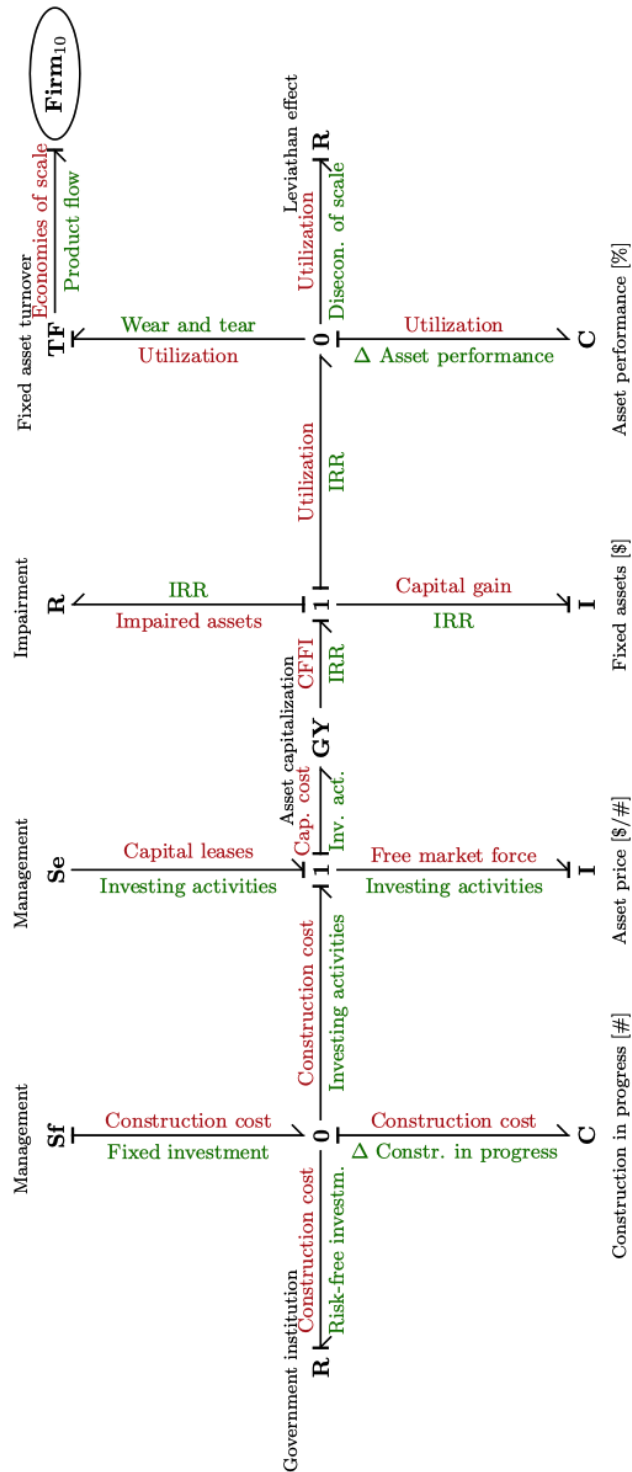


Figure 4-8: Bond-graph model of the capital asset market. Two states, construction in progress and asset price, determine from the capital leases and fixed investments the cash flow for investing (CFFI). Another two states, fixed assets and asset performance, determine from the product flow and CFFI the economies of scale that is supplied to the firm.

Table 4-8: The interpretation of definitions in the capital asset bond graph

Definition	Interpretation	Units
Elements		
Management	Party that controls the firm [14].	
Government institution	System or method that controls the country [9].	
Construction in progress	Non-current assets that are being constructed [4].	[#]
Asset price	Price of an asset determined by demand and supply [18].	[\$/#]
Asset capitalization	Recording of an item as asset rather than expense [14].	
Fixed assets	Long-term tangible pieces of property or equipment used for income generation [14].	[\$]
Impairment	Diminishing of long-term asset value [4].	
Asset performance	Profit generation of assets [4].	[%]
Leviathan effect	Firms becoming too large to run efficiently [2].	
Fixed asset turnover	Efficiency of a business using its assets for sales [4].	
Efforts		
Construction cost	Cost of constructing the fixed assets [4].	[\$/#wk]
Free market force	Unobservable market force that helps the demand and supply of fixed assets to reach equilibrium [17].	[\$/#wk]
Capital leases	Temporary use of assets in exchange for rent [14].	[\$/#wk]
Capital cost	Cost for acquiring fixed assets [4].	[\$/#wk]
Cash flow from investing (CFFI)	Cash flow used to buy long-term assets [4].	[\$/wk]
Capital gain	Increase in asset value [14].	[\$/wk]
Impaired assets	Diminished asset value [14].	[\$/wk]
Utilization	Use of fixed assets for production [4].	[\$/wk]
Economies of scale	Cost advantages through efficient production of large scales [2].	[\$/#wk]
Flows		
Fixed investment	Purchasing of fixed capital, measured as flow variable [4].	[#/wk]
Risk-free investments	Investments with no chance on default [3].	[#/wk]
Δ Construction in progress	Change in fixed assets that are being constructed [4].	[#/wk]
Investing activities	Purchase and sale of long-term assets [4].	[#/wk]
Internal rate of Return (IRR)	Expected return of the fixed assets [14].	[%/wk]
Δ Asset performance	Difference in the assets' ability to generate profitable returns [4].	[%/wk]
Diseconomies of scale (DOS)	Efficiency decrease resulting from scale rises [2].	[%/wk]
Wear and tear	Degradation of assets from usage [4].	[%/wk]
Product flow (economic order quantity)	Ideal product flow to minimize costs [18].	[#/wk]

the firm's capital cost. The R-element accounts for the risk-free return that cash could have yielded when not invested.

The traded items are some nominal amount of fixed assets, having units [#].⁶ A gyrator capitalizes the assets so that the PPE is supplied with the CFFI.

Property, Plant & Equipment

Two states determine the dynamics of PPE:⁷ Asset position and asset performance. The asset position determines the internal rate of return (IRR), which is given back to the market as capital cost. The asset performance determines the utilization, which is transformed into the output economies of scale.

Three processes decrease the value of PPE. The firm provides the product flow as input, which causes the wear and tear of assets. Also, R-elements account for impairment, and diseconomies of scale.

4-5-2 State-Space Equations for Capital Asset Market Dynamics

The bond graph allows for state-space derivation, which is done in Appendix C-1. The state vector is $x = [q_3 \ p_2 \ q_4 \ p_3]^T$ with construction in progress q_3 , Asset price p_2 , fixed assets p_3 and asset performance q_4 . The inputs are $u = [u_1 \ u_2 \ u_3]$, with fixed investments u_1 , capital leases u_2 , and product flow u_3 . Output y is the economies of scale. The capital asset market has the following state-space equation:

Dynamics capital asset market

$$\begin{aligned} \dot{x} &= \underbrace{\begin{bmatrix} -\frac{1}{R_3 C_3} & -\frac{1}{I_2} & 0 & 0 \\ \frac{1}{C_3} & 0 & -\frac{GY_1}{I_3} & 0 \\ 0 & 0 & -\frac{R_5 C_4}{1} & \frac{1}{I_3} \\ 0 & \frac{GY_1}{I_2} & -\frac{1}{C_4} & -\frac{R_4}{I_3} \end{bmatrix}}_A x + \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{TF_1} \\ 0 & 0 & 0 \end{bmatrix}}_B u(t) \\ y &= \underbrace{\begin{bmatrix} 0 & 0 & \frac{1}{TF_1 C_2} \end{bmatrix}}_C x + \underbrace{\begin{bmatrix} 0 \end{bmatrix}}_D \end{aligned} \quad (4-3)$$

Economic interpretations of the state-space variables are given in Table 4-11.

4-6 Product Market

The product market has three inputs: The firm's product flow, and the quantity demanded and force of demand of the customer. There is one output: the benefit orders. Three states

⁶These units are not the same as the firm's product, which is indicated by the same symbol.

⁷When management decides to divest, the CFFI will be negative, and assets decrease.

determine market dynamics. Table 4-9 gives an overview.

Table 4-9: Inputs, outputs and states of the product market

Inputs		Outputs		States	
Product flow	[#/wk]	Order yield	[\$/#wk]	Competitors' backlog	[#]
Quantity demanded	[#/wk]			Product price	[\$/#]
Force of demand	[\$/#wk]			Backlog	[#]

The product market consists of two parts: the firm's backlog and the market itself. Quantity demanded and budget constraint are inputs to the market, which determines the number of sales that end up in the firm's backlog. The backlog then produces the benefit orders to the firm. Figure 4-9 gives a visualization. Note that the direction of the arrows indicates causality and not the direction of positive flows.

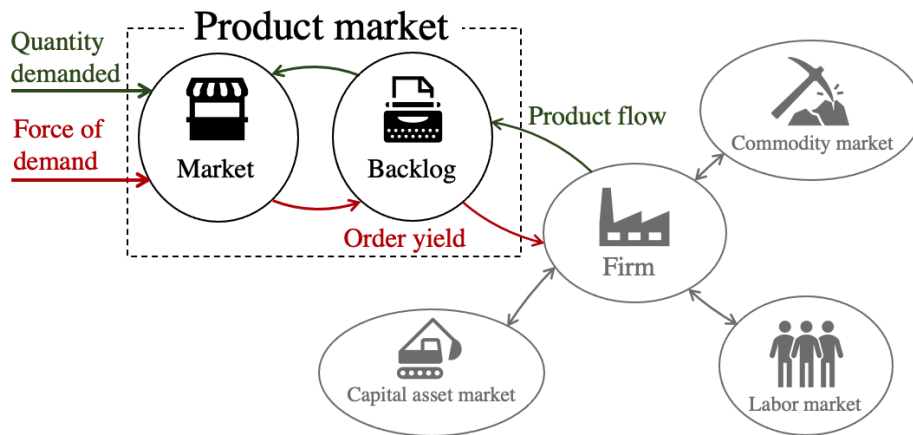


Figure 4-9: Firms interact with the product market by receiving the order yield, and supplying the product flow. Benefit orders is determined by the quantity demanded and customer budget of the customer that end up in the backlog. The direction of the arrows indicates causality.

4-6-1 Bond Graph for the Benefit Orders

Figure 4-10 presents the bond graph of the product market. The dynamics of the market and the backlog are explained below. Table 4-10 gives the economic interpretations of elements and signals.

Market

The market links inputs quantity demanded and force of demand to the sales that end up in the firm's backlog. A C-element keeps track of the market backlog of competitive firms, which returns the opportunity cost of the product. An I-element keeps track of the product price by weighing off this opportunity cost against the benefit that the firm receives for its orders.

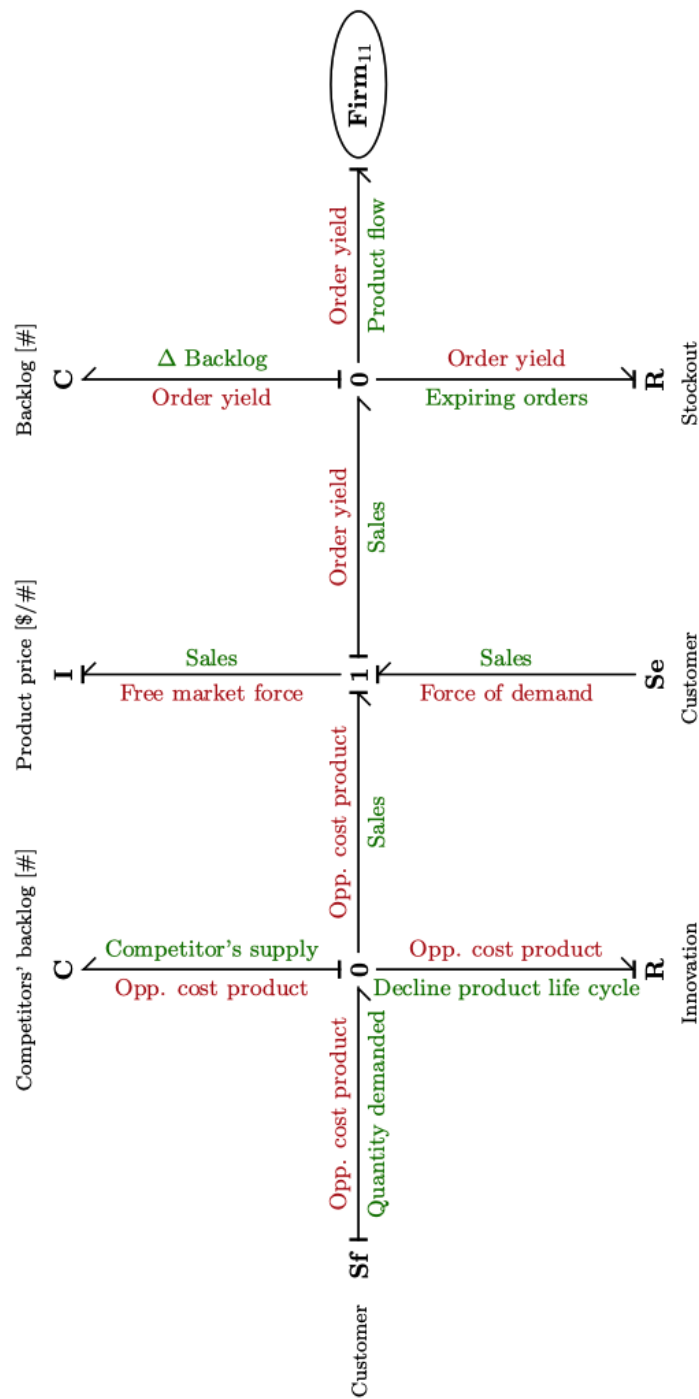


Figure 4-10: Bond-graph model of the product market. Customers interact with the market by buying products through effort and flow sources. In return, cash is supplied to the system. C-elements represent the amount of cash absorbed by competitors, and the number of orders stored in the firm's backlog. The product price is determined by weighing off the benefit orders against the opportunity cost determined by the competitive market.

Table 4-10: The interpretation of definitions in the product market bond graph

Definition	Interpretation	Units
Elements		
Customer	Person or organization that buys goods or services [9].	
Competitors' backlog	Share of the product market secured by competitors [18].	[#]
Innovation	Improvements of existing products or creation of new products [18].	
Product price	Price of a product determined by demand and supply [18].	[\$/#]
Backlog	Buildup of work that needs completing [14].	[#]
Stockout	Exhaustion of inventory [14].	
Efforts		
Opportunity cost product	Potential benefits the customer misses out on when choosing the firm's product [18].	[\$/#wk]
Free market force	Unobservable market force that helps the demand and supply of the product reach equilibrium [17].	[\$/#wk]
Force of demand	The amount customers are willing to spend on the product [18].	[\$/#wk]
Order yield	Benefits from orders met by the firm [18].	[\$/#wk]
Flows		
Quantity demanded	Customer's demand to buy the product [18].	[#/wk]
Competitor's supply	The number of products that competitors supply [18].	[#/wk]
Declining product life cycle	Decrease in product demand due to its ending life cycle [14].	[#/wk]
Sales	Exchange of products for money [14].	[#/wk]
Δ Backlog	The change in buildup of work that needs completing [14].	[#/wk]
Expiring orders	Orders not fulfilled in time by the firm [14].	[#/wk]
Product flow (Economic order quantity)	The ideal product flow to minimize costs [8].	[#/wk]

Customer demand fuels the product market and is modeled with a flow source and effort source representing quantity demanded and force of demand. Product innovations cause the market to contract, which I model with an R-element.

Backlog

The backlog connects sales to the product flow of the firm. If sales transcend the product flow, the remaining orders end up in the C-element representing the backlog. Backlogged orders either expire through the R-element representing stockout, are fulfilled by the firm, or reclaimed by competitors. Product flows that transcend the sales result in a negative backlog or product inventory.

4-6-2 State-Space Equations for Product Market Dynamics

The bond graph allows for state-space derivation, which is done in Appendix C-1. The state vector is defined as $x = [q_1 \ p_1 \ q_2]^T$, with q_1 the competitor's backlog, p_1 the product price, and q_2 the backlog. The input is $u = [u_1 \ u_2 \ u_3]$, with quantity demanded u_1 , force of demand u_2 , and product flow u_3 . Output y is the order yield. the product market has the following state-space equation:

Dynamics product market

$$\begin{aligned} \dot{x} &= \underbrace{\begin{bmatrix} -\frac{1}{R_1 C_1} & -\frac{1}{I_1} & 0 \\ \frac{1}{C_1} & 0 & -\frac{1}{C_2} \\ 0 & \frac{1}{I_1} & -\frac{1}{R_2 C_2} \end{bmatrix}}_A x + \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}}_B u(t) \\ y &= \underbrace{\begin{bmatrix} 0 & 0 & \frac{1}{C_2} \end{bmatrix}}_C x + \underbrace{\begin{bmatrix} 0 \end{bmatrix}}_D \end{aligned} \quad (4-4)$$

Economic interpretations of the state-space variables are given in Table 4-11.

4-7 The Total Business Model

The general bond-graph model for business dynamics is obtained by interconnecting the five subsystems. Figure 4-11 depicts this total bond-graph model and Table 4-11 gives the economic interpretation of the elements.

Table 4-11: Economic interpretations of the elements in the bond graph for business dynamics

Commodity market		Labor market		Capital asset market	
C ₅	Reserves	C ₆	Labor force	C ₃	Constr. in progress
I ₄	Current assets	I ₆	Goodwill/Intangibles	C ₄	Asset performance
I ₅	Forward price	I ₇	Wage	I ₂	Asset price
R ₆	Inventory write down	R ₈	Amortisation	I ₃	Fixed assets
R ₇	Competitors	R ₉	Competitors	R ₃	Government institution
TF ₂	Bill of materials	TF ₃	Labor intensity	R ₄	Impairment
GY ₂	Asset capitalization	GY ₃	Asset capitalization	R ₅	Leviathan effect
				TF ₁	Fixed asset turnover
				GY ₁	Asset capitalization
Product market		Firm			
C ₁	Comp. backlog	I ₈	Value-added		
C ₂	Backlog	R ₁₀	Government institution		
I ₁	Product price				
R ₁	Innovation				
R ₂	Stockout				

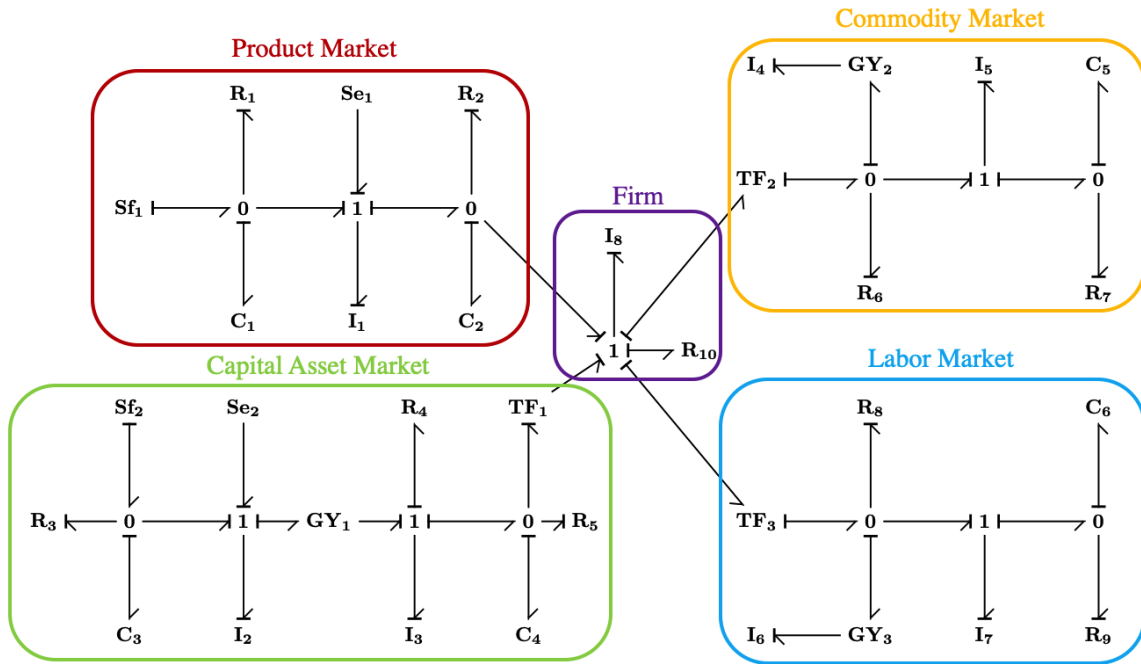


Figure 4-11: Bond graph of the total model. Dynamics are determined by fourteen states and four inputs. Useful outputs are the costs and benefits of the markets and the product flow.

Appendix C-1 provides the derivation of the state-space equations of the total model. The state-space representations allow the application of the entire systems and control toolbox. Analysis of the system's behavior is performed in Chapter 7.

The four markets are capable of mimicking every business structure by adding or removing markets. Some businesses, like tech companies, do not use commodities. When modeling these businesses, the commodity market is removed. Other businesses, like electronics companies, sell multiple products. When modeling these businesses, an additional product market is added.

4-8 Conclusions

The bond-graph for business dynamics is based on economic theory, in contrary to current business models. A set of differential equations describes business dynamics in a causal way. Moreover, all elements and signals have direct economic interpretations, making the model understandable to both engineers and economists. The state-space representation allows the application of system and control techniques (See Chapter 7).

The model generally applies to all businesses, as adding or removing markets allows to mimic different business structures.

The bond-graph model addresses the lack-of-data obstacle in current valuation models (Chapter 2). The parameters of each subsystem can be identified with data from those particular

markets. It is unnecessary to obtain large amounts of data of the business itself, as the firm is only a small part of the entire system.

The model is not directly applicable to business valuations, as dynamics are expressed in terms of product flow and costs, instead of cash flow. In Chapter 5, the model's dynamics are represented in terms of cash flow, so that the model can be applied to business valuations.

Contributions:

1. Development of fundamental models for the firm, product market, commodity market, labor market, and capital asset market.
2. Formulation of economic interpretations of all signals and elements.
3. Definition of business dynamics as a set of differential equations.

Bibliography

- [1] B. Chiang. *Indirect labor costs and implications for overhead allocation*. The College of New Jersey, 2013.
- [2] J. Drost. *Microeconomics*. Lumen Academy, 2018.
- [3] R. A. Brealey et al. *Principles of corporate finance*. McGraw-Hill, 2014.
- [4] R. Libby et al. *Financial accounting*. McGraw-Hill Education, 2019.
- [5] L. S. Huisman. *Modelling and control of a dynamical labour market system*. Delft Center for Systems and Control, 2019.
- [6] J. C. Hull. *Options, futures, and other derivatives*. Pearson, 2015.
- [7] Corporate Finance Institute. *Introduction to commodities*. CFI, 2018.
- [8] W. Kenton. *The economic order quantity - EOQ definition*. Investopedia, 2020.
- [9] Oxford Languages. *Oxford dictionary of English*. Oxford University Press, 2019.
- [10] A. Marvasti. *Resource Characteristics, Extraction Costs, and Optimal Exploitation of Mineral Resources*. Environmental and Resource Economics, 2000.
- [11] OECD. *Measuring Productivity*. OECD, 2001.
- [12] W. D. Nordhaus P. Samuelson. *Economics*. McGraw-Hill, 1948.
- [13] B. W. Bernard R. S. Russell. *Operations Management: Quality and Competitiveness in a Global Environment*. John Wiley Sons, 2006.
- [14] T. Rietveld. *Handboek Investeren & Financieren*. Vakmedianet Management B.V., 2017.
- [15] M. Rouse. *Bill of materials (BOM)*. TechTarget, 2018.
- [16] A. Smith. *An inquiry into the nature and causes of the wealth of nations*. W. Strahan and T. Cadell, 1776.

- [17] The Economic Times. *Definition of invisible hand*. Bennett, Coleman & Co. Ltd., 2020.
- [18] H. R. Varian. *Intermediate microeconomics: A modern approach*. W. W. Norton & Company, 2010.

Complex Port-Hamiltonian Mechanics for Cash flow-Based Modeling

5-1 Introduction

In practice, one does not use prices, stocks, flows, and costs to perform business valuations, but the firm's future cash flow.¹ This cash flow is used for two things: (1) Describe business dynamics, and (2) give the business value.

In this chapter, the energy-cash flow analog and a complex port-Hamiltonian (cpH) [10] system representation show that the bond-graph model agrees with the economic view on business valuations. It is explained how cash flows determine bond-graph dynamics, and which cash flow is determines present value. The cpH system thereby yields the following benefits:

- Expression of firm dynamics in business cycles ω_i and contraction β_i (Section 5-5).
- The resistive part of the complex Hamiltonian models the cash losses (Section 7-5-4).
- The possibility to perform cash flow-based control (Section 10-3).

I state the goals of this chapter as follows:

Chapter goals:

1. Representation of the bond graph for business dynamics as a cpH system.
2. Determination of the valuable cash flow for present value.

¹See Chapter 2.

In Section 5-2, I use Hamiltonian mechanics to determine the dynamics of conservative systems. In Section 5-3, I use the bicomplex Hamiltonian to model non-conservative systems. Complex Hamiltonian systems are connected in Section 5-4, using port-Hamiltonian mechanics. The bond graph for business dynamics is represented with a cpH system in Section 5-5. In Section 5-6, it is concluded that thesis objective 2 (modeling future cash flow) is completed as the bond graph's dynamics are derived from cash flows, and the valuable cash flow is determined.

5-2 Hamiltonian Mechanics for Conservative Systems

Before extending the framework of Hamiltonian mechanics to non-conservative systems, I show how regular Hamiltonian mechanics is used to derive the dynamics of conservative systems. The harmonic oscillator is used as an example, of which Figure 5-1a shows the bond graph.

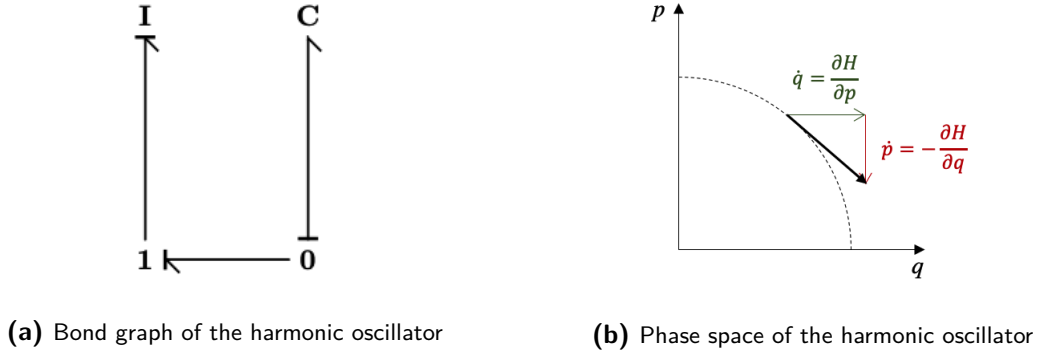


Figure 5-1: Time-evolution of the undamped harmonic oscillator is determined by Hamilton's equations as energy is conserved.

The system's energy flows back and forth between the C- and I-element. The harmonic oscillator interchanges potential energy stored in position q , for kinetic energy stored in momentum p [1, 11]. Energy is conserved, as there is no dissipation. The phase plane in Figure 5-1b shows this energy flow.

The system is described using Hamilton's equations and the Hamiltonian. The Hamiltonian is:

$$H = \frac{q^2}{2C} + \frac{p^2}{2I} \quad (5-1)$$

In port-Hamiltonian mechanics, Hamilton's equations use matrix notation [19]:

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = J \begin{pmatrix} \frac{\partial H}{\partial p} \\ \frac{\partial H}{\partial q} \end{pmatrix} \quad (5-2)$$

J is a skew-symmetric matrix ($J^2 = -I$).

5-3 Hamiltonian Mechanics for Non-Conservative Systems

Dynamics of non-conservative systems are not only determined by Hamilton's equations, but by forces imposed by resistive elements as well. I use the example of the damped harmonic oscillator (DHO) to show how this causes trouble for the Hamiltonian approach. Figure 5-2a shows a DHO in series.

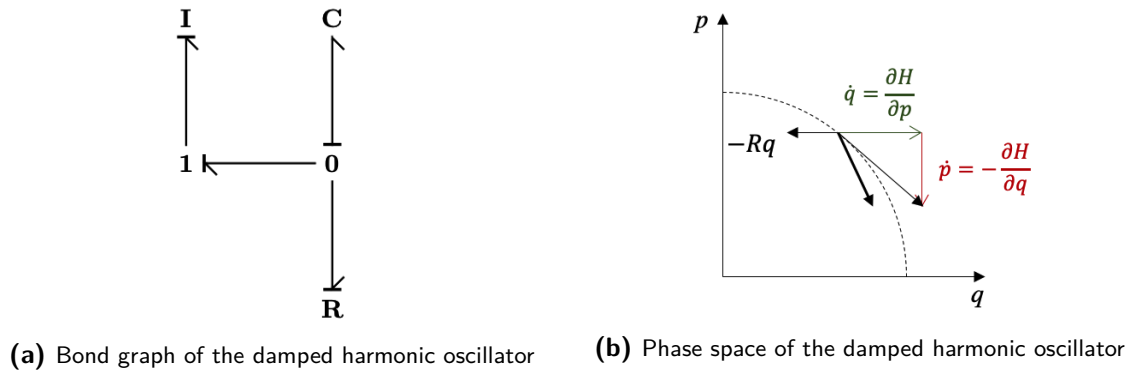


Figure 5-2: Time-evolution of the damped harmonic oscillator is not determined by Hamilton's equations due to the presence of dissipative force $-Rq$.

The phase space of Figure 5-2b shows the forces determined by Hamilton's equations and the resistive force imposed by the damper. Similar to conservative systems, the total energy is stored in the I- and C-elements and is given by Equation 5-3:

$$H = \frac{q^2}{2C} + \frac{p^2}{2I} \quad (5-3)$$

Energy is not conserved, and the symplectic structure of the Hamiltonian is lost.² Therefore, considering only Hamilton's equations does not yield the dynamics of non-conservative systems. Port-Hamiltonian mechanics extends the Hamiltonian approach to non-conservative systems by introducing a dissipation matrix R :

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = (J - R) \begin{pmatrix} \frac{\partial H}{\partial p} \\ \frac{\partial H}{\partial q} \end{pmatrix} \quad (5-4)$$

However, the R -matrix causes the dissipation obstacle in energy-based control techniques, precluding its application [18, 19, 5, 8]. In Section 10-3, it is explained why the application of energy-based control to business dynamics is particularly interesting.

5-3-1 Complexification of the Hamiltonian

I use a complex Hamiltonian to derive the dynamics of non-conservative systems by following the approach of Hutters [10, 2]. The starting point is that every point in phase space corresponds to a point in the Z -plane. Therefore, the system's time evolution can be described with a single equation of a complex-valued state variable [9]:

²The symplectic structure is lost as the system is no longer non-degenerate [3].

$$\dot{x} = -i\omega x \quad (5-5)$$

The complex-valued state variable and its conjugate are defined as follows:

$$x_j = \sqrt{\frac{I\omega_n}{2}}(q_j + \frac{i}{I\omega_n}p_j), \quad \bar{x}_j = \sqrt{\frac{I\omega_n}{2}}(q_j - \frac{i}{I\omega_n}p_j) \quad (5-6)$$

ω_n is the natural frequency of the system, here defined as $\omega_n = \sqrt{1/IC}$. The bicomplex Hamiltonian is:

$$\mathcal{H} = x\bar{x} = S - iR \quad (5-7)$$

The bicomplex Hamiltonian consists of two parts. The real-valued storage function S is the system's total energy and equal to the regular Hamiltonian. The resistive function R is imaginary and accounts for dissipation in the system.

Dynamics are obtained by using the complex-valued expression for Hamilton's equations:

$$\begin{pmatrix} \dot{x} \\ \dot{\bar{x}} \end{pmatrix} = \mathcal{J} \begin{pmatrix} \frac{\partial \mathcal{H}}{\partial \bar{x}} \\ \frac{\partial \mathcal{H}}{\partial x} \end{pmatrix} \quad (5-8)$$

With structure matrix $\mathcal{J} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$. This structure matrix is symplectic as $\det(\mathcal{J}) = -1$ [3].

Equation 5-8 determines the dynamics of non-conservative systems [10, 2].

5-3-2 The Bicomplex Hamiltonian of Damped Harmonic Oscillators

We will see in Section 5-5 that the bond graph for business dynamics is an interconnection of DHOs. In this section, I explain how the bicomplex Hamiltonian is obtained from the regular state space. I use the DHO in series as an example. Appendix C-2 shows the derivation in more detail, as well as the derivations of the double- and parallel-damped DHOs.

The regular state space of a DHO in series is as follows:

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} -1/CR & 1/I \\ -1/C & 0 \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix} \quad (5-9)$$

The bicomplex state-equations are as follows:

$$\begin{bmatrix} \dot{x} \\ \dot{\bar{x}} \end{bmatrix} = \begin{bmatrix} -(\beta + i\omega) & -\beta \\ -\beta & -(\beta - i\omega) \end{bmatrix} \begin{bmatrix} x \\ \bar{x} \end{bmatrix} \quad (5-10)$$

With $\beta_C = \frac{1}{2CR}$. Using Equation 5-8, we can equate the right-hand side to the partial derivatives of the bicomplex Hamiltonian. We find the bicomplex Hamiltonian through integration:

$$\mathcal{H} = \omega x\bar{x} - i\beta x\bar{x} - i\beta\left(\frac{x^2 - \bar{x}^2}{2}\right) \quad (5-11)$$

The bicomplex Hamiltonian can be used to split the state equations into the resistive and storage functions. We shall see in Section 5-4 that this distinction is needed for interconnecting the DHOs.

$$\underbrace{\begin{bmatrix} \dot{x} \\ \dot{\bar{x}} \end{bmatrix}}_{\dot{\chi}} = \mathcal{J} \left(\underbrace{\begin{bmatrix} 0 & \omega \\ \omega & 0 \end{bmatrix}}_{\nabla S} + \underbrace{\begin{bmatrix} i\beta & i\beta \\ -i\beta & -i\beta \end{bmatrix}}_{\nabla R} \right) \underbrace{\begin{bmatrix} x \\ \bar{x} \end{bmatrix}}_{\chi} \quad (5-12)$$

5-4 Complex Port-Hamiltonian Mechanics

In this section, I explain how bicomplex systems are interconnected using port-Hamiltonian (pH) mechanics.³ The combination of the bicomplex Hamiltonian and port-Hamiltonian systems theory is known as complex-port-Hamiltonian (cpH) theory [10].

5-4-1 The Influence of Bond-Graph Elements on States and Signals

The cpH formalism divides system elements in three categories: Energy-storing, energy-dissipating, and energy-routing.⁴ Table 5-1 summarizes:

Table 5-1: Classes of bond-graph elements and the influence they have on states and signals

cpH Formalism	Bond-Graph Elements	Influence on:
Energy-storing	C & I	States and signals
Energy-dissipating	R	States
Energy-routing	TF & GY	Signals

The column on the right-hand side states the fundamental difference between the elements: Their influence on states and signals. Energy-storing elements influence both the states and signals, as they store energy, and consequently return an effort or flow. Dissipating elements disperse energy, thereby affecting only the state. Routing elements conserve energy but alter the signal.

Structure matrix $\mathcal{J}$ is split into two parts, to dissociate the different influence that elements have on states and signals:

$$\dot{\chi} = \underbrace{(\mathcal{J}_S \nabla S + \mathcal{J}_R \nabla R)}_A \chi \quad (5-13)$$

³See Section 3-3-4.

⁴This is in contradiction to what is stated in [10].

$\mathcal{J}_s$ and $\mathcal{J}_r$ both preserve the symplectic structure, contrary to the regular port-Hamiltonian formalism. $\mathcal{J}_s$ is structured exclusively by elements that influence the state, and $\mathcal{J}_r$ contains only elements that influence signals. As a result, the cpH-formalism allows for the interconnection of subsystems.

5-4-2 Complex Port-Hamiltonian Interconnection

CpH-systems use physical inputs and outputs for their interconnection to allow measurements and actuation [10, 2]. The input-state-output of the system is [19]:

$$\begin{aligned}\dot{\chi} &= \nabla \mathcal{H} \chi + \mathcal{G} u \\ y &= \mathcal{G}^T \nabla \mathcal{H} \chi\end{aligned}\tag{5-14}$$

There is a standard approach to interconnect cpH systems, like the two DHOs displayed in Figure 5-3. The systems are connected in five steps:

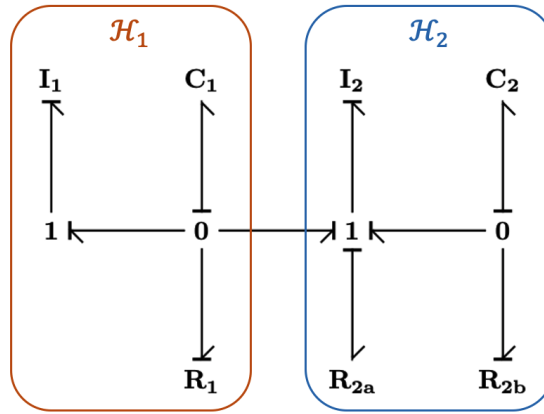


Figure 5-3: Bond graph notation of the interconnection of two DHOs. Elements C_1 and I_2 are both connected to a resistive element, causing energy to be dissipated. This is accounted for by splitting operator $\Delta \mathcal{H}$ into storage part ΔS , which influences both states and signals, and resistive part ΔR , which only influences the state.

1. Obtain the operator $\nabla \mathcal{H}_i$ from Appendix C-2 and use Equation 5-12 to split into ∇S_i and ∇R_i :

$$\nabla \mathcal{H}_i = \nabla S_i + \nabla R_i\tag{5-15}$$

2. Construct the augmented ∇S_{ij} and ∇R_{ij} matrices:

$$\nabla S_{ij} = \begin{bmatrix} \nabla S_i & 0 \\ 0 & \nabla S_j \end{bmatrix}, \quad \nabla R_{ij} = \begin{bmatrix} \nabla R_i & 0 \\ 0 & \nabla R_j \end{bmatrix}\tag{5-16}$$

3. Determine the input of the systems, and use Equation 5-6 to give the input a complex-valued state variable:

$$\mathcal{G}_i, \mathcal{G}_j\tag{5-17}$$

4. Determine the structure matrices $\mathcal{J}_S$ and $\mathcal{J}_R$. Note that the input and output matrices end up in the structure matrix $\mathcal{J}_S$, as the storage function determines the signals:

$$\mathcal{J}_S = \begin{bmatrix} \mathcal{J} & \mathcal{G}_i \mathcal{G}_j^T \\ \mathcal{G}_j^T \mathcal{G}_i & \mathcal{J} \end{bmatrix}, \quad \mathcal{J}_R = \begin{bmatrix} \mathcal{J} & 0 \\ 0 & \mathcal{J} \end{bmatrix} \quad (5-18)$$

5. The augmented A-matrix for the interconnected cpH system is defined by equation 5-13:

$$A_{ij} = \mathcal{J}_R \nabla R_{ij} + \mathcal{J}_S \nabla S_{ij} \quad (5-19)$$

To complete the cpH-interconnection framework, I add two notes of caution:

Energy-Routing Elements

Transformers and gyrators appear only in the storage structure matrix $\mathcal{J}_S$, as they exclusively influence signals. Gyrators map flows to efforts, and efforts to flows [6]. When situated between two cpH-systems, they flip the power flow of one of the subsystems. As a result, the sign of the respective storage structure matrix changes as follows:

$$\mathcal{J}_S = \begin{bmatrix} \mathcal{J} & \mathcal{G}_i \mathcal{G}_j^T \\ \mathcal{G}_j^T \mathcal{G}_i & -\mathcal{J} \end{bmatrix} \quad (5-20)$$

The sign change does not influence the system's eigenvalues. One verifies the system's correct interconnection by checking the time response.

Interconnection of Regular and Complex Systems

The cpH-interconnection framework allows the interconnection of regular to complex port-Hamiltonian systems. It is necessary to express systems in their regular states when no oscillating dynamics are present (bond graphs with only an I- or C-element).

Input matrices map physical signals to their complex representation. Regular port-Hamiltonian systems have physical inputs and outputs, causing the use of input matrices to be unnecessary.

5-5 The Firm as a Complex Port-Hamiltonian System

In this section, I give the model for business dynamics a cpH representation, and I determine which cash flow yields business value.

The bond graph for firm dynamics is divided into eight cpH-subsystems so that the system almost exclusively consists of DHOs. Figure 5-4 shows the cpH bond-graph model:

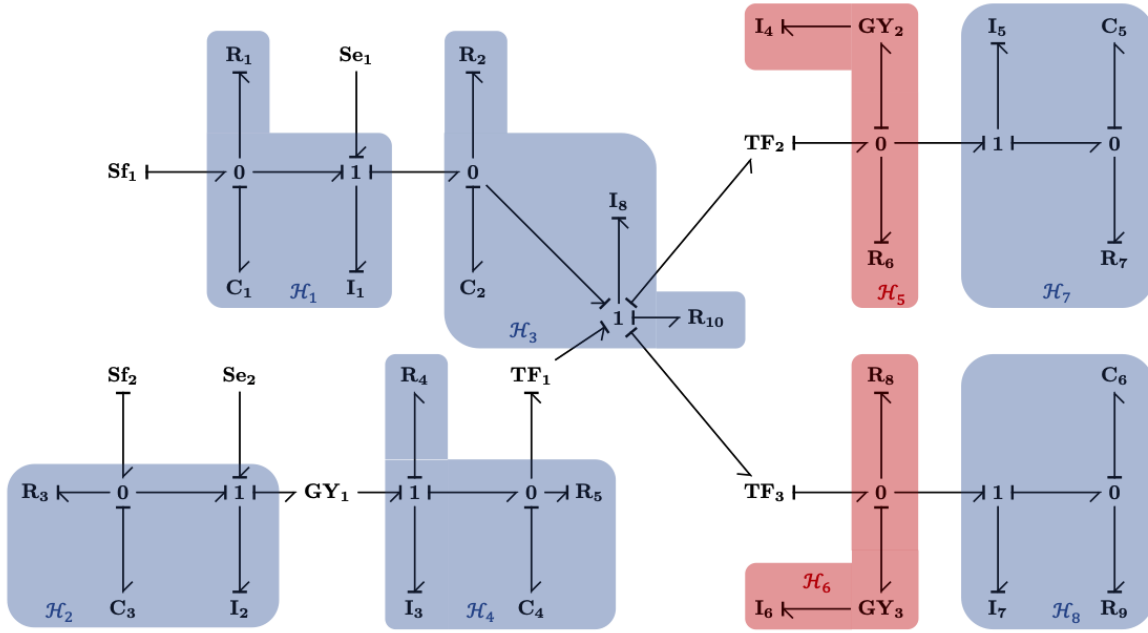


Figure 5-4: The bond-graph model for business dynamics as a set of DHOs. DHOs marked with blue are either damped in series or double-damped, and need a cpH representation to obtain the correct equations of motion. DHOs marked with red do not contain oscillating behavior, and use the regular Hamiltonian expression.

Complex subsystems are marked with blue, and regular subsystems are marked with red. Sources are external to the system and not included in the cpH system. The transformers and gyrators between the subsystems are not part of any cpH subsystem but appear in input matrices $\mathcal{G}_i$.

5-5-1 Economic Interpretations

In this section, I give economic meaning to each of the cpH subsystems, the way they appear in the bond-graph model of Figure 5-4. The subsystems are interpreted as economic systems in Table 5-2. It is thereby indicated whether the subsystems are part of the firm itself (endogenous) or influence the firm from outside (exogenous) [16].

The cpH subsystems express dynamics in natural frequency ω_n , damping coefficient β_C for dampers in series, and damping coefficient β_I for dampers in parallel. Table 5-3 gives an overview:

The cpH representation expresses the system in only 14 parameters. This amount is less than half of the number of parameters needed for the regular bond graph. The subsystems that represent markets ($\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{H}_7$, and $\mathcal{H}_8$) have similar interpretations for their variables. The literature only shows specified terminology for the capital asset market. Economically this is intuitive, as economic models aggregate the different markets into a single economy.

Table 5-2: Economic interpretations of cpH subsystems

cpH	Economic System	Endo-/Exogenous
$\mathcal{H}_1$	Product market [20]	Exogenous
$\mathcal{H}_2$	Capital asset market [15]	Exogenous
$\mathcal{H}_3$	Operations [15]	Endogenous
$\mathcal{H}_4$	PPE [15]	Endogenous
$\mathcal{H}_5$	Inventory [15]	Endogenous
$\mathcal{H}_6$	Human capital [13]	Endogenous
$\mathcal{H}_7$	Commodity market [17]	Exogenous
$\mathcal{H}_8$	Labor market [17]	Exogenous

Table 5-3: Economic interpretations of complex variables

cpH	ω_n	β_C	β_I
$\mathcal{H}_1$	Business cycle [14]	Contraction [14]	-
$\mathcal{H}_2$	Investment cycle [15]	Cost of capital [15]	-
$\mathcal{H}_3$	Inventory turnover [15]	Order expiration [15]	Corporate taxation [15]
$\mathcal{H}_4$	Fixed asset life cycle [7]	Impairment [7]	Diseconomies of scale [4]
$\mathcal{H}_5$	-	-	-
$\mathcal{H}_6$	-	-	-
$\mathcal{H}_7$	Business cycle [14]	Contraction [14]	-
$\mathcal{H}_8$	Business cycle [14]	Contraction [14]	-

5-5-2 Comparison of the Regular System with the CpH System

The parallel between the firm's regular bond-graph and cpH representations is verified by comparing the state responses to similar input. Figure 5-5 shows that both representations show the exact same system response:

Appendix F shows the code used for calculating both system responses. Note that the complex-valued states of the cpH system are transformed into their physical representations by using the inverse of Equation 5-6. In Chapter 7, I perform analysis using the phase spaces and Complex Hamiltonian.

The regular state-space model and the cpH model yield the same business dynamics. The similarity in responses shows that both Newtonian and analytical mechanics can be used to model economic systems. It is also shown that cash flow dynamics underlie the dynamics of the bond-graph model of Chapter 4. Therefore, the model is concluded to be fundamental to business dynamics.

5-5-3 Aggregation of Subsystems for Cash Flow Determination

In this section, I determine what cash flow yields business value.

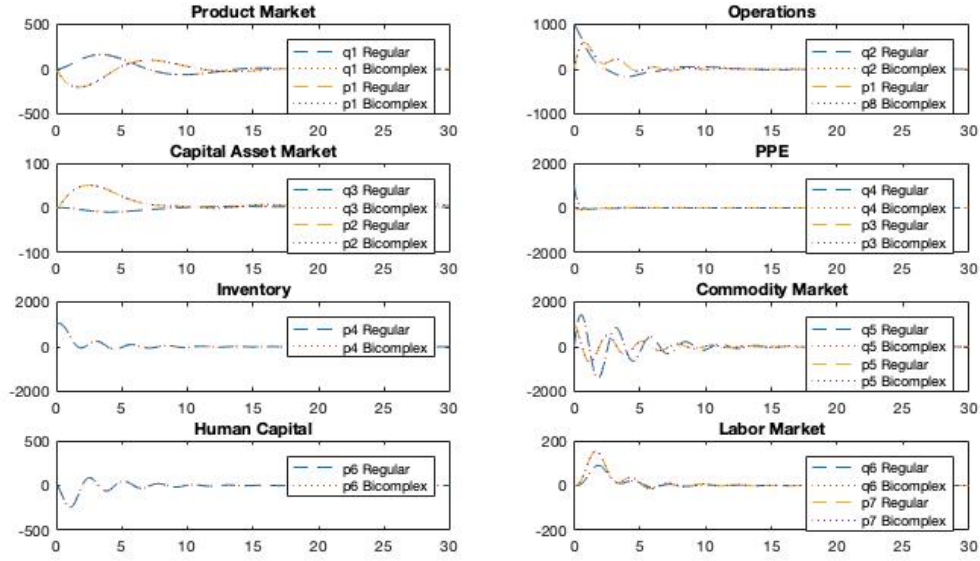


Figure 5-5: Comparison of state responses of the regular bond graph with the cpH representation shows that both representations yield the exact same dynamics.

The system is split into two parts: An endogenous part defined as the business⁵, and the exogenous part defined as the economy. Figure 5-6 shows this division:

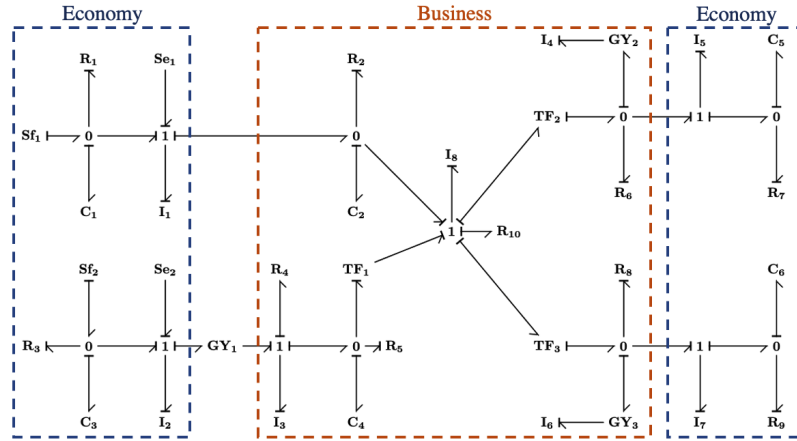


Figure 5-6: Division of the bond-graph model into the subsystems endogenous to the business, and the systems that are exogenous and form the economy. Each market that is connected to the business has a similar structure: A DHO in series.

The aggregation of markets into a single economy is intuitive for two reasons: Markets attain the same structure of DHOs in series, and interact with the business similarly by receiving an effort from the firm and returning a flow. Figure 5-7 shows the block diagram of the interconnection of the cpH systems for the business and the economy:

⁵As stated in Chapter 2, the terms firm and business are used interchangeably in economic theory. Here, I use the different terms to emphasize the distinction between the indicated parts of the bond graph.

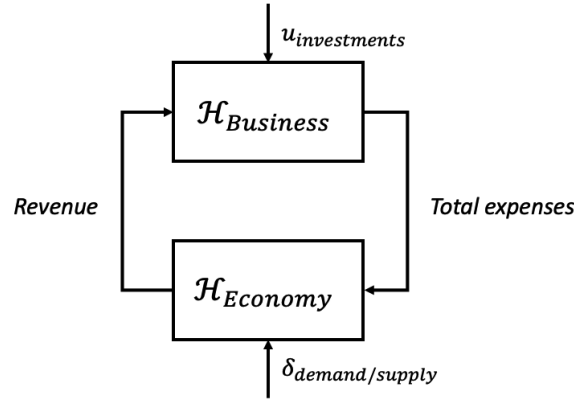


Figure 5-7: Block diagram resulting from aggregating endogenous subsystems into the business, and exogenous subsystems into the economy. Cash stored in the business is equal to the revenue minus the total expenses and represents the business' earnings. Inputs supplied to the firm are controllable as they represent the investments. Inputs supplied to the economy are uncontrollable, as they represent the consumer behavior.

This representation shows that the bond graph is a fundamental cash flow modeling technique. We know that the earnings of a firm are the total cash going in, also known as the revenue, minus the total cash going out, also known as total expenses [15]:

$$\text{Earnings} = \text{Revenue} - \text{Total expenses} \quad (5-21)$$

Not all cash flows in the business yield value: The cash flow for business value only consists of the kinetic part of the Hamiltonian (Section 5-2), causing elements C_2 , I_4 , and I_6 not to yield value.⁶ Also, the I_3 - C_4 interconnection does not produce value as it represents the PPE of the business. The cash flow that changes between kinetic and potential, represents the fixed asset life cycle, and not real cash flowing in. As a result, only the cash flow stored in I_8 contributes to the business value:

$$\text{Earnings} = \mathcal{H}(I_8) \quad (5-22)$$

5-6 Conclusions

The bond-graph model for business dynamics is fundamental to value determination. The complex port-Hamiltonian (cpH) representation of the model has shown that cash flows underlie the bond-graph model for business dynamics. This is in line with the economic view on modeling businesses. Also, all elements and signals have economic interpretations, as was shown in Chapter 4. This completes Part 2 of the frequency-domain valuation method.

⁶ I_4 and I_6 are gyrated I-elements, and have therefore the function of a C-element that stores potential energy [6].

The cash flow for business valuations is equal to the Hamiltonian of the value-added (I_8). This cash flow is a measure for profits and is derived from economic principles. The recognizing of this cash flow as valuable addresses the ongoing debate on what cash flow to use for valuations. This cash flow is comparable to the net operating profit less adjusted taxes (NOPLAT), which is used in some valuation methods [15, 14].

The cash flows that are modeled by the cpH system are not directly usable for valuation. The cash flows are what economists call an ex-post quantity, which means that the quantity is measured after the event. Because business value is determined by future cash flows, the cash flows are to be considered as an ex-ante quantity, which means that the quantity is predicted beforehand [12]. This problem is addressed in Chapter 6.

Contributions:

1. Development of a framework to express bond graphs in their bicomplex form and interconnect cpH subsystems.
2. Formulation of economic interpretations for bicomplex variables.
3. Determination of the cash flow that fundamentally yields present business value.

Bibliography

- [1] V. Balakrishnan. *Lecture series on classical physics*. Department of Physics, IIT Madras, 2009.
- [2] M. Mendel C. Hutters. *Overcoming the dissipation obstacle with bicomplex port-Hamiltonian Mechanics*. Delft Center for Systems and Control, 2020.
- [3] A. Cannas da Silva. *Lectures on Symplectic Geometry*. Springer, 2000.
- [4] J. Drost. *Microeconomics*. Lumen Academy, 2018.
- [5] A. Macchelli et al. *Control by interconnection for distributed port Hamiltonian systems*. Elsevier, 2005.
- [6] D. C. Karnopp et al. *System dynamics: Modeling, simulation, and control of mechatronic systems*. John Wiley & Sons, Inc., 2012.
- [7] R. Libby et al. *Financial accounting*. McGraw-Hill Education, 2019.
- [8] R. Ortega et al. *Energy-shaping of port-controlled Hamiltonian systems by interconnection*. Conference of decision and control, 1999.
- [9] P. Stehle H. C. Corben. *Classical mechanics*. John Wiley & Sons, Inc., 1950.
- [10] C. Hutters. *The analytical mechanics of consumption*. Delft Center for Systems and Control, 2019.
- [11] G. Hrabovsky L. Susskind. *Classical mechanics: The theoretical minimum*. Penguin Books, 1978.
- [12] Oxford Languages. *Oxford dictionary of English*. Oxford University Press, 2019.
- [13] N. Manders. *The Thermodynamics of Economic Engineering*. Delft Center for Systems and Control, 2019.
- [14] N. G. Mankiw. *Principles of macroeconomics*. Cengage Learning, 2018.

- [15] T. Rietveld. *Handboek Investeren & Financieren*. Vakmedianet Management B.V., 2017.
- [16] D. Romer. *Advanced Macroeconomics*. McGraw-Hill Irwi, 2012.
- [17] A. Smith. *An inquiry into the nature and causes of the wealth of nations*. W. Strahan and T. Cadell, 1776.
- [18] A van der Schaft. *L2-gain and passivity techniques in nonlinear control*. Springer, 2005.
- [19] A. van der Schaft. *Port-Hamiltonian systems: An introductory survey*. Proceedings of the International Congress of Mathematicians, 2006.
- [20] H. R. Varian. *Intermediate microeconomics: A modern approach*. W. W. Norton & Company, 2010.

Chapter 6

A Frequency-Domain Theory for Business Valuation

6-1 Introduction

In this chapter, a theory is developed for using frequency-domain concepts in economic systems. The chapter builds an analogy between business valuations and scenario analysis on the one hand, and the frequency domain on the other. The analogs are developed by using mathematical relations of the Laplace transform and definitions that economists use. The goals of this chapter are as follows:

Chapter goals:

- Extension of the economic-engineering theory to the frequency domain.
- The use of frequency-domain tools for business valuations and scenario analysis.

In Section 6-2, I map the economic definition of net present value to the frequency domain. In Section 6-3, I explain the analog between the complex frequency that appears in the Laplace transform, and the internal rate of return. In Section 6-4, I show that the Laplace transform of future cash flows gives the ex-ante measure for the cash-flow equivalent. In Section 6-5, I explain how this cash-flow equivalent is determined by using signals appearing in the model for business dynamics. In Section 6-6, I show how transfer functions and Bode plots are used to perform scenario analysis. In Section 6-7, I conclude that the use of frequency domain for business valuations and scenario analysis is a useful extension to current approaches.

6-2 The Net Present Value as a Frequency-Domain Variable

In this section, I express the net present value (NPV) as a frequency-domain variable. The NPV gives the value of a series of cash flows, while accounting for the time value of money: A dollar to be paid out tomorrow is worth less than a dollar paid out today. The decrease in present value of future cash flows is based on a discount rate.

Without loss of generality, the discount factor can be written in an exponential manner. This means that the NPV of a cash flow can be obtained with the Laplace transform of that cash flow [19]. With discount rate s , the NPV is the Laplace transform of cash flow $C(t)$:

$$\text{NPV}(s) = \int_0^{\infty} C(t)e^{-st} dt \quad (6-1)$$

In Section 6-3, it is shown that discount rate s is equal to the internal rate of return.

The use of the Laplace transform for calculating NPV expresses the NPV as a frequency-domain variable. Corporate and government decision makers are not so much concerned with the timing of a project's cash flows as with the profitability of the project. Therefore, analytical tools are needed to compare projects involving receipts and disbursement occurring at different times, with the goal of identifying the largest eventual profitability [7].

Using the Laplace transform for NPV calculation has several benefits. Only a handful of analytical solutions is found for the NPV problem, whereas more than a hundred solutions exist for the Laplace transform [32]. In particular, cyclical cash flows have a surprisingly simple representation [4]. Also, the continuous form of the Laplace transform uses less parameters than the currently used discrete discounting approach [44].

Economic equivalence is used by economists to compare alternatives. It is established when we are indifferent between a future payment, or a series of future payments, and a present sum of money. When cash flows are function of the discount rate, they are called cash flow equivalent [14]. I use the Laplace transform of the time derivative of the cash flow to determine the cash-flow equivalent:

$$C(s) = \int_0^{\infty} \dot{C}(t)e^{-st} dt \quad (6-2)$$

In Section 6-4, it is shown how cash-flow equivalent $C(s)$ is determined from the model for business dynamics.

Using the derivative property of the Laplace transform, we can express the NPV in terms of the cash-flow equivalent by relating Equations 6-1 and 6-2 [27]:

$$\text{NPV}(s) = \frac{C(s)}{s} \quad (6-3)$$

This equation expresses the NPV as a frequency-domain parameter. With this insight, business valuations are rephrased as frequency-domain problems. This expression for NPV is

mathematically equivalent to the time-discrete formula of the net present value as is used in the current practice [4, 32]. From the use of the Laplace transform for calculating NPV follow simplifications known from control theory. Analytical solutions to NPV problems exist, and cyclical cash flows have a simple representation [4].

6-3 A Complex Representation of the Internal Rate of Return

In this section, I explain why complex frequency parameter s is the internal rate of return (IRR) of the business that we value. I show that s satisfies the economic definition of IRR, and give economic meaning to its real and imaginary part.

The IRR on an investment or project is the rate of return that sets the net present value of all cash flows (both positive and negative) from the investment equal to zero [10]. Given a project, the net present value is a function of the rate of return. Equivalently, it is the discount rate at which the net present value of the future cash flows is equal to the initial investment. By rearranging Equation 6-3, we see that the discount rate satisfies this definition of the IRR:

$$\text{NPV}(s) - \frac{C(s)}{s} = 0 \quad (6-4)$$

where the NVP is expressed as function of the IRR. Equation 6-4 is identical to the mathematical relation between NPV and IRR as defined in the economic literature [10].

Economists need separate calculations for the NPV and IRR for their investment decisions. The NPV method focuses on project surpluses, while IRR is focused on the breakeven cash flow level of a project. Management pays close attention to the IRR, as it does not trust the forecasts of the investment opportunity it receives [10]. Equation 6-4 gives both measures in a single expression, without the need for separate calculations.

The Laplace transform is a function of complex variable s :

$$s = \sigma + i\omega \quad (6-5)$$

In engineering the real and imaginary part are interpreted as follows: The real part is responsible for describing some dissipative behavior, and the imaginary part for describing some oscillating behavior [18].

In economics, the NPV is calculated by using only the real part of the discount rate. This real-valued rate is the proportional share of the amount owed, that must be paid to compensate for payment delays [21]. The real-valued discount rate grows as the delay in payment is extended [2]. This fact is directly tied to the time value of money and its potential earning capacity [21].

The imaginary-valued discount rate describes some oscillating behavior. Imaginary discount rates do not appear in current NPV calculations, but the presence of oscillations is suggested

by the economic literature [9, 37, 15]. The cycles appear for example in the Cobweb model, which asserts that prices are subject to periodic fluctuations in certain types of markets [31]. Another example are porc cycles, which describe the phenomenon of cyclical fluctuations of supply and prices in livestock markets [11].

For economic systems, complex frequency parameter s is analogous to the IRR. Laplace transforms that discount projects with their IRR, result in an NPV of zero. This is precisely how NPV and IRR are defined in the economic literature [10]. The use of a IRR extends the current discounting approach. The imaginary part of the IRR models the oscillating behavior of the economy.

6-4 The Cash-Flow Equivalent as an Ex-Ante Quantity

In this section, I explain the need for expressing the cash-flow equivalent $C(s)$ as an ex-ante quantity. $C(s)$ is determined by taking the Laplace transform of $\dot{C}(t)$, which is the time-derivative of the cash flow. I thereby show how $\dot{C}(t)$ appears in the model for business dynamics, to derive from it the cash flow equivalent and NPV.

Cash flow forecasts are more widely demanded now, than historically [28]. It is a critical tool to forecast future performance of businesses, for equity investors valuing firms, and for assessing a firms ability to repay its loans [8, 42]. Accounting standards shape financial statements in such a way that cash flow can be predicted [17]. However, even skilled analysts cannot construct an accurate cash flow statement, because it is impossible to relate amounts to particular income statement line items [22]. Even when using potentially more accurate methods of estimating cash flows, estimating errors frequently remain, and continue to be substantial [22, 38, 13]. Using known methods and more information, only lowers the forecasting errors by a small amount [23].

Cash flows are either ex-ante, which indicates the expectation of some future event, or ex-post, which indicates the measurement of some past event [24]. In business valuations we consider forecasts of future cash flows [12, 41], which means that we consider ex-ante cash flows.

The ex-ante cash flow is not at parity with the ex-post cash flow, as it depends on the time-dimension. Myrdal proposes to reduce the actual time-dimension of ex-ante cash flows to a point of time [33].

I define the cash flow equivalent $C(s)$ as the Laplace transform of the time derivative of the cash flow $\dot{C}(t)$, as was seen in Section 6-2:

$$C(s) = \int_0^{\infty} \dot{C}(t)e^{-st}dt \quad (6-6)$$

The Laplace transform eliminates the time-dimension of the expected cash flow and expresses the cash flow equivalent in terms of the discount rate. This approach is in line with Myrdal's view on ex-ante and ex-post cash flows [33].

The dissociation between ex-ante and ex-post is seen in the bond graph for business dynamics. The state value-added keeps track of the operational profits that the business makes, as is seen in Section 5-5. Figure 6-1 shows how this element appears in the model for business dynamics:



Figure 6-1: The operational cash flow appears in the model for business dynamics as the element value-added. By considering the element itself, the cash flow is measured ex-post. By considering the power-bond that connects the element to the system, the cash flow is measured ex-ante.

I define the ex-post cash flow as the Hamiltonian of the value-added that is stored in the I-element. According to the definition of an I-element, value-added $p(t)$ is obtained from Net economic force $e(t)$ with the following operation [6]:

$$p(t) = \int_{-\infty}^0 e(t) dt \quad (6-7)$$

The operation uses a backward-looking integral, making the cash flow that is extracted from it an ex-post cash flow per definition [33].

I define the ex-ante cash flow as the Laplace transform of the power over the bond $\dot{C}(t)$, that connects I-element Value-added to the system. With product flow $f(t)$, this power is defined as follows:

$$\dot{C}(t) = e(t)f(t) \quad (6-8)$$

With this equation, we express cash-flow equivalent $C(s)$ in terms of two signals that appear in the model for business dynamics. The expression satisfies the ex-ante modeling approach for cash flows. With this insight, the NPV of businesses is determined by a discount rate and two signals (Equation 6-3): The net economic force and product flow.

6-5 Frequency-Domain Signals for Cash-Flow Determination

In this section, I show that the Laplace transforms of the net economic force $e(t)$ and product flow $f(t)$ are used for determination of the cash flow equivalent, and consequently the NPV.

There is a dissociation between calculating economic quantities ex-ante and ex-post, as there is with cash flows [33]. Net economic force and product flow are functions of time, which makes them ex-post calculations. I use the Laplace transform of both signals, to eliminate their time-dimension, and calculate the pair ex-ante.

I define the Laplace transform of the product flow $f(t)$ as the total output $Q(s)$:

$$Q(s) = \int_0^{\infty} f(t)e^{-st} dt \quad (6-9)$$

In economics, factors of production are used in production processes to produce output [43]. Equation 6-9 calculates the total output of the business by integrating the product flow over the entire future. The product flow is what the business makes available for sale, by using factors of production from the commodity, labor, and capital asset market.

A discounting factor is applied to the product flow to account for the economic depreciation. This is a measure of the decrease in market value of assets over time, caused by economic factors. The depreciation is either physical, due to obsolescence, or due to changes in demand for the good or service. In economics, this depreciation may be modeled as the present value of the flow of services or goods [5]. This is identical to how Equation 6-9 calculates the total output.

I define the Laplace transform of the net economic force $e(t)$ as the unit value $V(s)$:

$$V(s) = \int_0^{\infty} e(t)e^{-st} dt \quad (6-10)$$

The unit value is the total value of sales, divided by the sum of the quantities. Unit values may change over time as a result of shifts in either demand or supply [29]. These shifts appear in Equation 6-10 in changes in the net economic force $e(t)$.

Net economic force is discounted for unit value similarly to how cash flow is discounted for NPV. The difference is that economic force and unit value are both measurements per unit. The discount factor compensates for payment delays, and is directly tied to the time value of money and its potential earning capacity [21].

An expression for cash flow equivalent $C(s)$ in terms of total output $Q(s)$ and unit value $V(s)$ is found by using the multiplication property of the Laplace transform [27]:

$$C(s) = Q(s) * V(s) \quad (6-11)$$

This equation results from inserting Equation 6-8 into Equation 6-6 for $\dot{C}(s)$, and taking the Laplace transforms of $f(t)$ and $e(t)$.

Convolution express the amount of overlap of a function as it is shifted over another function. The observed quantity depends on the distribution of the desired quantity with the weighting function, rather than of the quantity itself [3].

In economics, convolutions are used to aggregate different types of data into a single-dimensional sufficient statistic, while preserving asymmetric distribution properties [30]. Total output and product value both have an asymmetric distribution due to the shape of the Laplace transform. This asymmetry is preserved in the cash flow equivalent, because of the convolution integral.

NPV is expressed in terms of total output $Q(s)$, product value $V(s)$, and discount rate s , by using Equations 6-3 and 6-11:

$$\text{NPV}(s) = \frac{Q(s) * V(s)}{s} \quad (6-12)$$

With this equation, business valuations are performed in the frequency domain. Equation 6-12 gives the NPV in terms of a discount rate and signals from the model for business dynamics. The use of the Laplace transform has eliminated the time-dimension of the NPV, and uses the discount rate as a parameter.

This expression of the NPV presents an analytical solution for many valuation problems. Analytical solutions to the Laplace transform of many functions are found, and the model for business dynamics provides analytical expressions for signals $e(t)$ and $f(t)$. Apart from offering faster and more reliable answers, analytical solutions show how variables relate to each other. Consequently, one can see the effects of inputs on the output with strong mathematical backing.

6-6 A Frequency-Domain Approach to Scenario Analysis

In this section, I show how scenario analysis of businesses is performed in the frequency domain. This approach uses transfer functions for measuring the liquidity between signals, and Bode plots to visualize their frequency response.

In economics, the process of scenario analysis analyzes future events by considering multiple possible outcomes, instead of showing one picture of the future. Scenario-building improves decision-making by considering the implications of these outcomes [1, 16]. Scenario analysis is no substitute for factual economic studies. Experience has shown that the use of around three scenarios is most appropriate, as more scenarios make the analysis overly complicated [41]. Especially when analyzing complex businesses, factors and assumptions do not interact in a similar way. Thereby, scenario analysis without reporting some parameters of measurement accuracy calls the entire study into question [20]. Another challenge of scenario-building is that predictors influence the forecasting process, causing them to become self-contradicting [35].

6-6-1 Transfer Functions for Liquidity, Illiquidity, or Economic Efficiency

In this section, I explain how transfer functions relate to the economic concepts of liquidity, illiquidity, and economic efficiency.

Engineers use transfer functions to analyze systems in the frequency domain. Transfer functions are mathematical functions that model the device's output for each possible input [26]. Transfer function $H(s)$ gives the linear mapping of any two signals [25]. The following transfer function gives the mapping of $P(s)$, which is the Laplace transform of input $e(t)$, to $Q(s)$, which is the Laplace transform of output $f(t)$:

$$H(s) = \frac{Q(s)}{P(s)} \quad (6-13)$$

The descriptions are given in terms of complex discount rate $s = \sigma + i\omega$, where σ gives the value-decrease of money over time, and ω gives the frequency of some economic cycle. These economic interpretations are explained in Section 6-3.

I give $Q(s)$ and $P(s)$ general economic interpretations, before considering transfer function $H(s)$. In Section 6-5, this was already done for two specific signals: Net economic force and Product flow. The interpretations that I present here for $Q(s)$ and $P(s)$ are generalizations of these signals, and follow the same arguments that were used in Section 6-5.

I interpret $Q(s)$ as output. Output is the result of an economic process that uses inputs to produce a product or service that is available for sale [39]. Output may change over time, due to changes in product flow and depreciation. Output is depreciated linearly or exponentially over time [41].

I define output $Q(s)$ as the Laplace transform of some flow $f(t)$:

$$Q(s) = \int_0^\infty f(t)e^{-st}dt \quad (6-14)$$

The Laplace transform expresses output as an ex-ante quantity, following the argument that is explained in Section 6-4. $f(t)$ is the time derivative of the goods that we consider, and keeps track of changes in those goods. Discounting factor e^{-st} performs exponential depreciation on the products. With this intuition, $Q(s)$ satisfies the economic definition for output.

I interpret $P(s)$ as economic value. Economists describe economic value as the benefit that will be derived from a good or service [43]. Multiple theories on economic value suggest that it is changed by economic forces [43, 45]. Similar to business value, economic value is discounted over time [12].

I define value $P(s)$ as the Laplace transform of some economic force $e(t)$:

$$P(s) = \int_0^\infty e(t)e^{-st}dt \quad (6-15)$$

The Laplace transform expresses value as an ex-ante quantity, following the argument that is explained in Section 6-4. $e(t)$ is the time derivative of the economic value, and keeps track of the economic forces that interact with it. Discounting factor e^{-st} performs exponential discounting. With this intuition, $P(s)$ satisfies the economic definition for economic value.

Transfer function $H(s)$ performs three sorts of mappings between economic value $P(s)$ and output $Q(s)$.

The first mapping that $H(s)$ performs, is from value $P(s)$ to output $Q(s)$. I interpret this transfer function as the liquidity of that relation. Economists classify assets as liquid if those

assets can be bought or sold without loss of value [24]. This economic definition coincides with how $H(s)$ relates output to value: High liquidity or high value for $H(s)$ causes that value $P(s)$ changes only little, when output $Q(s)$ is changed.

The second mapping that $H(s)$ performs, is from output $Q(s)$ to value $P(s)$. This is the inverse of liquidity of the relation, which I interpret as the illiquidity. Illiquid goods cannot be traded without substantial loss of value [24]. With high illiquidity $H(s)^{-1}$, or low liquidity $H(s)$, output changes result in large value offsets.

The third mapping that $H(s)$ performs, is from output $Q(s)$ to output $Q(s)$, or value $P(s)$ to value $P(s)$. I interpret this transfer function as the economic efficiency. Economists indicate economic systems as efficient, if production proceeds at the lowest possible average total cost [40]. With high efficiency $H(s)$, the input-output relation has high value transfer in the case of $P(s)$, and high output transfer in the case of $Q(s)$.

With these insights, all signals of the model for business dynamics are given economic meaning in the frequency-domain. Transfer functions relate any pair of those signals in terms of a complex discount rate. Transfer functions of economic systems are a tool to evaluate future scenarios. As the functions have the discount rate as variable, it provides the response of the business to multiple trading frequencies and time-values of money.

6-6-2 Bode Plots for Visualization of the Frequency Response

In this section, I show how Bode plots are used to visualize the response of the business to different interpretations of the economic environment.

In control theory, a Bode plot is a graph of the frequency response of a system, and is function of frequency ω . It consists of a Bode magnitude plot, and a Bode phase plot. The Bode magnitude plot gives the gain or amplification of the input signal at different frequencies. The Bode phase plot gives the phase shift of the output with respect to the input at different frequencies [46].

I interpret frequency ω as the frequency of some business cycles that is present in the system's input. In the economic theory, business cycles are often ascribed to overproduction, underconsumption, and in particular wealth inequalities [36]. By varying the frequency of the business cycles of the input, multiple interpretations of the economic environment are shown. For this reason, Bode plots are used to perform scenario analysis.

I interpret the magnitude of the frequency response of economic systems as the absolute value of the liquidity, illiquidity or efficiency. This dissociation depends on the input-output relation of the transfer function, as is explained in Section 6-6-1. The magnitude tells us how much cash or products are transferred from the input to the output of economic systems at different frequencies [25].

I interpret the phase of the frequency response as the immediacy of the input-output relation. Economists define the immediacy as the speed at which orders can be fulfilled in a

particular market, and is one of the dimensions of market liquidity [34]. The phase plot tells us the delay between the input and output of the economic system at differences. This coincides with the economic definition of immediacy.

With these economic interpretations of Bode plot variables, the frequency-domain response of economic systems can be analyzed. This is done in Section 7-2. We shall see that the use of Bode plots in economic systems is particularly useful for performing scenario analysis and identifying optimal trading frequencies.

6-7 Conclusions

The entire valuation effort of businesses is translated to the frequency domain. This is done by identifying key analogs between frequency-domain concepts, and the field of business valuations. The overall analogy that this chapter develops fits within the economic-engineering framework. For the first time however, the economic-engineering theory is extended to the frequency domain.

An important result of this chapter is the definition of the Laplace transform of some function, as the ex-ante measure of that function. Laplace transforms discount future quantities with some depreciation rate to account for the uncertainty of the future. This concept translates to the model for business dynamics by considering the signals over the bonds, instead of the energy stored in the elements. This signal-based approach is exactly how engineers do systems analysis and control.

Contributions:

1. Extension of the economic-engineering theory to the frequency domain.
2. Use of frequency-domain tools to perform business valuations and scenario analysis.

Bibliography

- [1] D. A. Aaker. *Strategic Market Management*. John Wiley & Sons, 2001.
- [2] A. C. Alpha. *Fundamental Methods of Mathematical Economics*. McGraw-Hill, 1984.
- [3] R. N. Bracewell. *The Fourier Transform and its Applications*. McGraw-Hill Higher Education, 2000.
- [4] S. A. Buser. *Laplace Transforms as Present Value Rules*. The Journal of Finance, 1986.
- [5] F. C. Wykoff C. R. Hulten. *The Measurement of Economic Depreciation*. Urban Institute, 1980.
- [6] D. C. Karnopp et al. *System dynamics: Modeling, simulation, and control of mechatronic systems*. John Wiley & Sons, Inc., 2012.
- [7] D. G. Newman et al. *Engineering Economic Analysis*. Oxford University Press, 2004.
- [8] G. Clinch et al. *The usefulness of direct and indirect cash flow disclosures*. Review of Accounting Studies, 2002.
- [9] P. Aghion et al. *Cyclical fiscal policy, credit constraints, and industry growth*. Elsevier, 2010.
- [10] R. A. Brealey et al. *Principles of corporate finance*. McGraw-Hill, 2014.
- [11] S. Rosen et al. *Cattle Cycles*. Journal of Political Economy, 1994.
- [12] T. Koller et al. *Valuations*. John Wiley and Sons Inc, 2015.
- [13] T. Ward et al. *A research note on the issue of non-articulation and the method used to calculate net operating cash flow*. Journal of Accounting Research, 2009.
- [14] W. G. Sullivan et al. *Engineering Economy*. Prentice Hall, 2012.
- [15] W. J. Den Haan F. Covas. *The cyclical behavior of debt and equity finance*. American Economic Review, 2011.

- [16] J. Haas F. X. Bea. *Strategisches Management*. Lucius & Lucius, 2005.
- [17] FASB. *Statement of Financial Accounting Standards No. 95*. Financial Accounting Standards Board, 1987.
- [18] U. Graf. *Applied Laplace Transforms and z-Transforms for Scientists and Engineers: A Computational Approach using a Mathematica Package, Graphs, and Mathematical Tables*. Birkhäuser, 2004.
- [19] R. W. Grubbström. *On the Application of the Laplace Transform to Certain Economic Problems*. Management Science, 1957.
- [20] B. Hassani. *Scenario Analysis in Risk Management*. Springer, 2016.
- [21] J. E. Goodman J. Downes. *Dictionary of Finance and Investment Terms*. Baron's Financial Guides, 2003.
- [22] S. Orpurt J. Hales. *A Review of Academic Research on the Reporting of Cash Flows from Operations*. Financial Accounting Standards Board, 2013.
- [23] M. Bradburry L. Austin. *The accuracy of cash flow estimation procedures*. Accounting and Finance, 2009.
- [24] Oxford Languages. *Oxford dictionary of English*. Oxford University Press, 2019.
- [25] W. S. Levine. *The Control Handbook*. CRC-Press, 1996.
- [26] D. F. Warne M. A. Laughton. *Electrical Engineer's Reference Book*. Newnes, 2002.
- [27] I. A. Stegun M. Abramowitz. *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*. Dover, 1972.
- [28] M. Hung M. L. Defond. *An empirical analysis of analysts' cash flow forecasts*. Journal of Accounting and Economics, 2003.
- [29] C. McKelvey. *Price, Unit Value, and Quality Demanded*. Journal of Development Economics, 2008.
- [30] E. Miravete. *Convolution and Composition of totally Positive Random variables in Economics*. Journal of Mathematical Economics, 2011.
- [31] E. Mordecai. *The Cobweb Theorem*. The Quarterly Journal of Economics, 1938.
- [32] R. Murray. *Laplace Transforms*. McGraw-Hill, 1965.
- [33] G. Myrdal. *Monetary Equilibrium*. W. Hodge, 1939.
- [34] Pricing Liquidity: The Quantity Structure of Immediacy Prices. *G. Chacko et al.* Harvard Business School, 2006.
- [35] I. Overland. *The geopolitics of renewable energy: Debunking four emerging myths*. Energy Research & Social Science, 2019.

-
- [36] R. Owen. *Mr. Owen's Report to the Committee of the Association for the Relief of the Manufacturing and Labouring Poor*. R. Watts, 1817.
 - [37] J. Greenwood P. Gomme. *On the cyclical allocation of risk*. Elsevier, 1993.
 - [38] D. W. Collins P. Hribar. *Errors in estimating accruals: Implications for empirical research*. Journal of Accounting Research, 2002.
 - [39] W. D. Nordhaus P. Samuelson. *Economics*. McGraw-Hill, 1948.
 - [40] V. Zelenyuk R. Sickles. *Measurement of Productivity and Efficiency: Theory and Practice*. Cambridge University Press, 2019.
 - [41] T. Rietveld. *Handboek Investeren & Financier*. Vakmedianet Management B.V., 2017.
 - [42] Y. Zang S. F. Orpurt. *Do direct cash flow disclosures help predict future operating cash flows and earnings?* The Accounting Review, 2009.
 - [43] A. Smith. *An inquiry into the nature and causes of the wealth of nations*. W. Strahan and T. Cadell, 1776.
 - [44] M. Stalla-Bourdillon. *Continuous DCF method*. HEC Paris, 2012.
 - [45] L. Walras. *Elements d'economie politique pure*. Literary Licensing LiC, 1874.
 - [46] R. K. R. Yarlagadda. *Analog and Digital Signals and Systems*. Springer Science, 2010.

The Systems and Control Toolbox for Financial Analysis

7-1 Introduction

The economic-engineering approach to business economics opens the opportunity to use systems and control (S&C) tools to perform financial analysis. In this chapter, I show how these tools apply to financial analysis. The goal of this chapter is as follows:

Chapter goals:

1. Find applications of systems and control tools in financial analysis.

In Section 7-2, I use Bode plots to perform scenario analysis and identify optimal trading frequencies. In Section 7-3, I use pole-zero maps to identify the economic factors and cycles that appear in economic systems. In Section 7-4, I use state-space trajectories for visual analysis of business cycles. In Section 7-5, I use time-responses for model verification, comparison, scenario analysis, and cash losses. In Section 7-6, it is concluded that the S&C toolbox shows many applications in financial analysis.

The analysis is performed on qualitative data by using the parameters as stated in Appendix A. System responses use three types of models: The Simulink model of Appendix E, the regular state-space description derived in Appendix C-1, and the bicomplex state-space description derived in Appendix C-2. All three methods yield similar responses.

7-2 Frequency-Domain Analysis for Optimal Trading Frequencies and Scenario Analysis

In this Section, I use Bode plots to identify optimal trading frequencies and perform scenario analysis. This analysis uses the general concepts that were derived in Section 6-6-2. The model for business dynamics is crucial for performing this type of analysis, as it provides the transfer functions needed for generating Bode plots.

The first Bode plot we consider, gives the frequency response of the commodity market. The input is the product flow of the firm, and the output is the unit labor cost. The transfer function that relates the pair is a measure of illiquidity, as it maps a quantity of output to a quantity of value. Figure 7-1 shows the Bode plot:

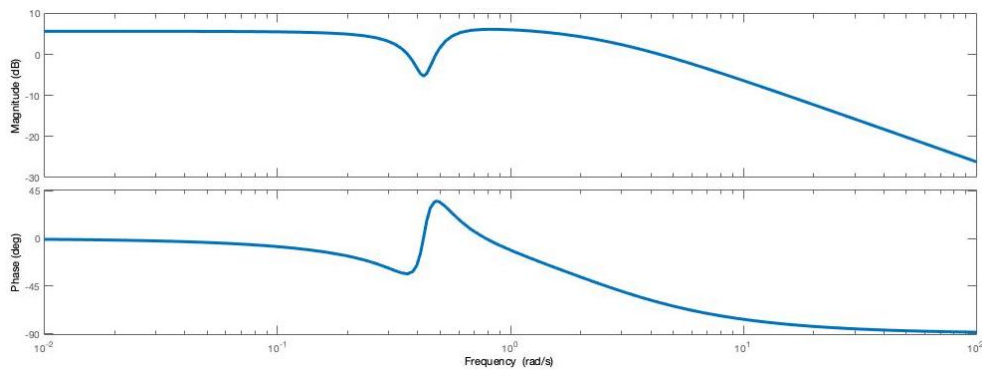


Figure 7-1: The Bode plot of the commodity market shows a resonance valley that indicates an optimal trading frequency. When this frequency is attained, an exceptional low production cost is returned to the firm. The phase plot shows that the immediacy of the labor market depends on the trading frequency of the firm.

The frequency of the response is the trading frequency of the business itself, as the business' product flow is the input to the commodity market.

The Bode magnitude plot shows with a resonance valley that the commodity market has an exceptional low illiquidity at some trading frequency. I interpret this frequency as an optimal trading frequency. The resonance valley tells us that there is an exceptional low production cost for trading at a certain frequency. As we consider the commodity market, we need valleys to show us at what frequencies low costs are made. When considering markets that supply benefits to the firm, we need peaks to show us at what frequencies exceptional high benefits are made. I give the general interpretation for resonance peaks and valleys as follows:

Resonance peak

Resonance peaks show the optimal trading frequencies.

The Bode magnitude plot shows a roll-off [9]. The roll-off shows that trading at higher frequencies results in a lower illiquidity, and therefore a lower production cost. This is in line with expectations as trading at lower frequencies generates longer business cycles, which en-

tail higher storage costs [13].

The phase plot shows the immediacy of labor that occurs when trading at different frequencies. This is useful information for the business when anticipating demand shocks [10]. The input-output relation of the commodity market shows a negligible immediacy when trading at low frequencies, and a immediacy of a quarter business cycle (90 degrees) at high frequencies [18]. Around the market's optimal trading frequency, mayor shifts in immediacy occur. The peak in the phase plot shows that when trading exactly at the optimal trading frequency, not only the absolute value of the market's liquidity is optimal, but also the immediacy.

The second Bode plot we consider, relates the customer demand to the different costs and benefits of the firm. The input is the customers demand, and the outputs are the order yield, economies of scale, production cost and labor cost. The transfer functions are a measure of illiquidity, as a quantity of output is mapped to quantities of value. The transfer functions are based on the full model for business dynamics. Figure 7-2 shows the Bode plot:

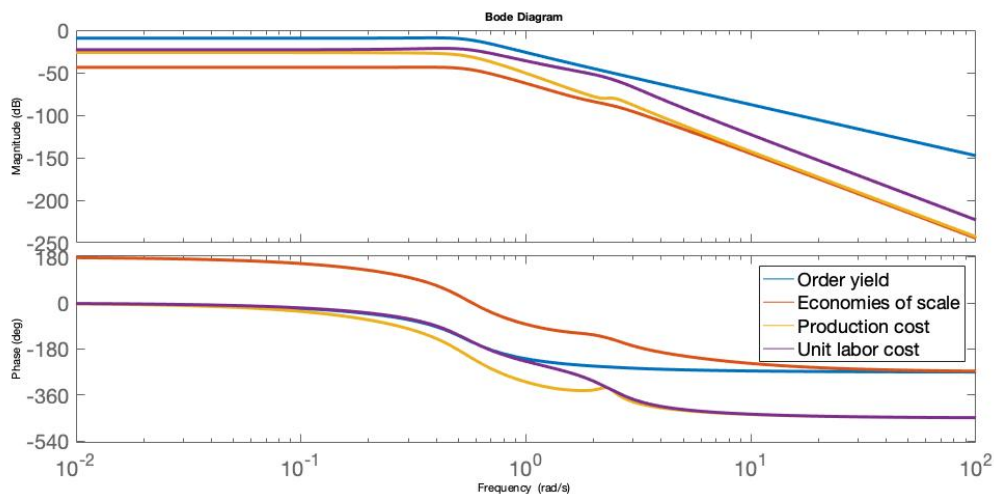


Figure 7-2: The Bode plot of the total model shows the effects of consumer demand on the different costs and benefits. The optimal trading frequency has high economic efficiency for benefit orders and economies of scale, and low economic efficiency for production and unit labor cost. The phase plot shows that different markets impose different time-lags.

I interpret this Bode plot as a method to perform scenario analysis. The frequency of the response is the frequency of the demand cycle in which customers interact with the business. The plot shows how the costs and benefits of the firm react to different customer behavior.

The magnitude plot shows how the cost structure of the business depends on the customer behavior. The plot shows how much of this customer demand is transferred to the costs and benefits at different trading frequencies. We aim to find resonance peaks in the magnitude plots that relate the customer demand to benefits, as these peaks indicate exceptionally high benefits. We aim to find resonance valleys in the Bode plots that map the consumer demand to costs, as these indicate exceptionally low costs.

The phase plot shows the immediacy between the customer demand and the different costs and benefits. The immediacy is useful information for the business, as it shows the time-lags between a change in customer demand on the one hand, and the costs and benefits on the other.

Bode plots are used to perform scenario analysis on economic systems and identify optimal trading frequencies. The scenario analysis is a factual economic study, as the outcomes are based on ex-ante measures of the model for business dynamics [24]. Also, Bode plots provide a continuous presentation of the input-output response for all different trading frequencies, instead of showing only a handful. These properties make Bode plots a useful extension to currently existing tools for scenario analysis.

The presence of peaks, valleys and roll-offs in the Bode plots is determined by the causal structure of the model for business dynamics. The parametrization of the model determines the values of the plots, and can cause the disappearance of resonance peaks through pole-zero cancellation [9]. This analysis is purely qualitative. Therefore, no conclusions can be drawn on the differences in value of the transfer functions that relate the customer demand to the different costs and benefits.

7-3 Pole-Zero Maps for Economic Factors, Cycles and Stability

In this section, I use pole-zero maps to identify economic factors and cycles that are present in economic systems, and to verify stability. Poles and zeros are the roots of the numerator and denominator of the transfer function of some input-output relation [4].

The first pole-zero maps we consider, uses the transfer functions that relates the inputs and outputs of each market that is connected to the firm. The commodity market and labor market are SISO systems. Here the transfer functions relate their only input, which is in both cases the firm's product flow, to their only output, which are the production cost and the unit labor cost respectively. The capital asset market and product market are MIMO systems. For these markets, the pole-zero maps are shown for the input-output relations that connect the two markets to the firm. The input for both markets is again the product flow, the output for the capital asset market is the economies of scale, and the output for the product market is the order yield. Figure 7-3 shows the pole-zero maps of the markets connected to the firm.

The maps provide qualitative information about the system. Each market has poles in the left-half plane, which ensures stability [4]. This stability is in line with expectations, as economists define markets to be stable [7].

I interpret the presence of real-valued poles as having some stable form of depreciation in the economic system. In engineering, real-valued poles in the right-half plane represent an exponentially decaying component [9]. In economics, depreciation is often modeled with exponential functions [13]. I give a general interpretation for real-valued poles as follows:

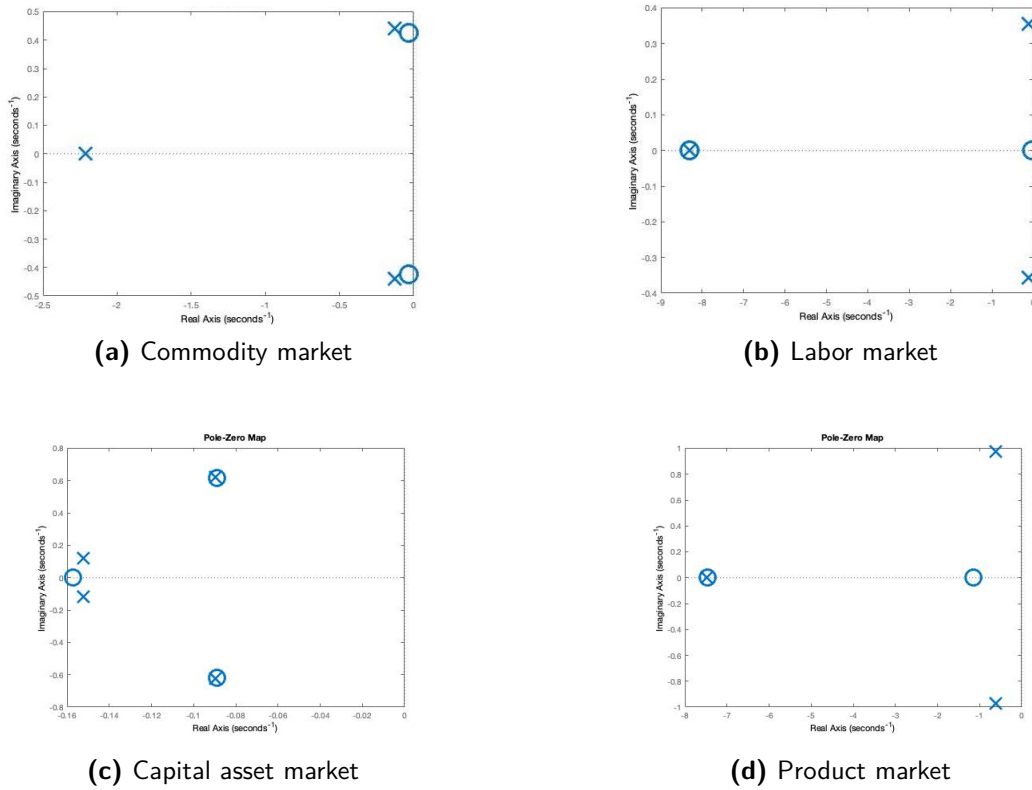


Figure 7-3: Pole-zero maps of the markets connected to the firm show what economic factors and cycles are present in that market. Complex pole pairs indicate the presence of some cycle, and real-valued poles represent the exponential decay imposed by an economic factor. As all poles are situated in the left-half plane, the markets are stable.

Real-valued poles

Real-valued poles indicate the presence of some economic factor. Poles in the left-half plane indicate depreciation, poles in the right-hand plane indicate appreciation.

I interpret the presence of complex pole pairs as having oscillating market dynamics in the economic system. In engineering, complex pairs in the left-half plane represent some oscillating decay [9]. In economics, the cyclical behavior and contraction of economic markets is well known [7].¹ I give a general interpretation for complex pole pairs as follows:

Complex pole pairs

Complex pole pairs indicate the presence of some economic cycle.

The labor market, capital asset market, and product market have pole-zero cancellations. These pole-zero cancellations eliminate dynamics that are modeled in the bond-graph model [9]. If the pole-zero cancellation was situated in the right-hand plane, it would incur stability [4].

¹Decay only occurs with poles in the left-half plane. If the poles were situated in the right-half plane, they would impose some unstable form of growth [9].

The economic interpretation of poles allows for quick insight into financial data sets. By analysis of the pole-zero map, one knows how many economic cycles and factors that are present in the system. This information can be used to decide upon what dynamics to model. In the context of the model for business dynamics, poles help decide on what markets to connect to the firm.²

7-4 State Trajectories for Visual Analysis of Business Cycles

In this section, I use the trajectories of the damped harmonic oscillators (DHOs) that appear in the model for business dynamics for visual analysis of business cycles in equilibrium and disequilibrium. State-Space trajectories display the time-evolution of two state variables [4].

The C- and I- elements that appear in the model for business dynamics are coupled to analyze the cyclical exchange between kinetic and potential energy. In engineering, the interconnection (with damper) is known as a damped harmonic oscillator [10]. In economics, linear forecasting is used to anticipate business-cycle turning points [14]. I show here how the cyclical properties of the DHOs are used for anticipating business cycles by using the state-space of the DHOs.

For this analysis, the model for business dynamics was given positive demand and investment shocks. Figure 7-4 shows the state-space responses of the six DHOs that appear in the model (See Section 5-4).

This representation of the model's state response underscores its ability to model economic cycles. I give interpretations to the state trajectories in equilibrium and disequilibrium.

Business cycles in equilibrium show what cash-flow variable predominantly generates profit. The trajectories converge to a stable cycle: This is the business cycle in equilibrium [7]. By looking at the difference in the p - and q -variables, it is seen what cash-flow variable predominantly generates value. For example, the phase plane of operations (Figure 7-4a) shows $dp = 0.15$ and $dq = 1.5$,³ indicating that the profit margin is relatively small to the amount of products that are sold. Profits increase most from increasing the price instead of selling more products [15]. In formulate a general interpretation for business cycles in equilibrium as follows:

Business cycles in equilibrium

Show which cash-flow variable predominantly generates profit.

Business cycles in disequilibrium provides estimates of the amount of assets and cash needed to take in economic shocks. The subsystems' trajectory before entering the business cycles in equilibrium is the business cycle in disequilibrium [7]. This trajectory shows the offset in price and assets that result from the economic shocks. The trajectories provide an estimate

²Firms can be connected to multiple times or not at all to a certain market. See Section 2-5.

³The analysis is performed qualitatively, so the numbers do not represent any economic meaning.

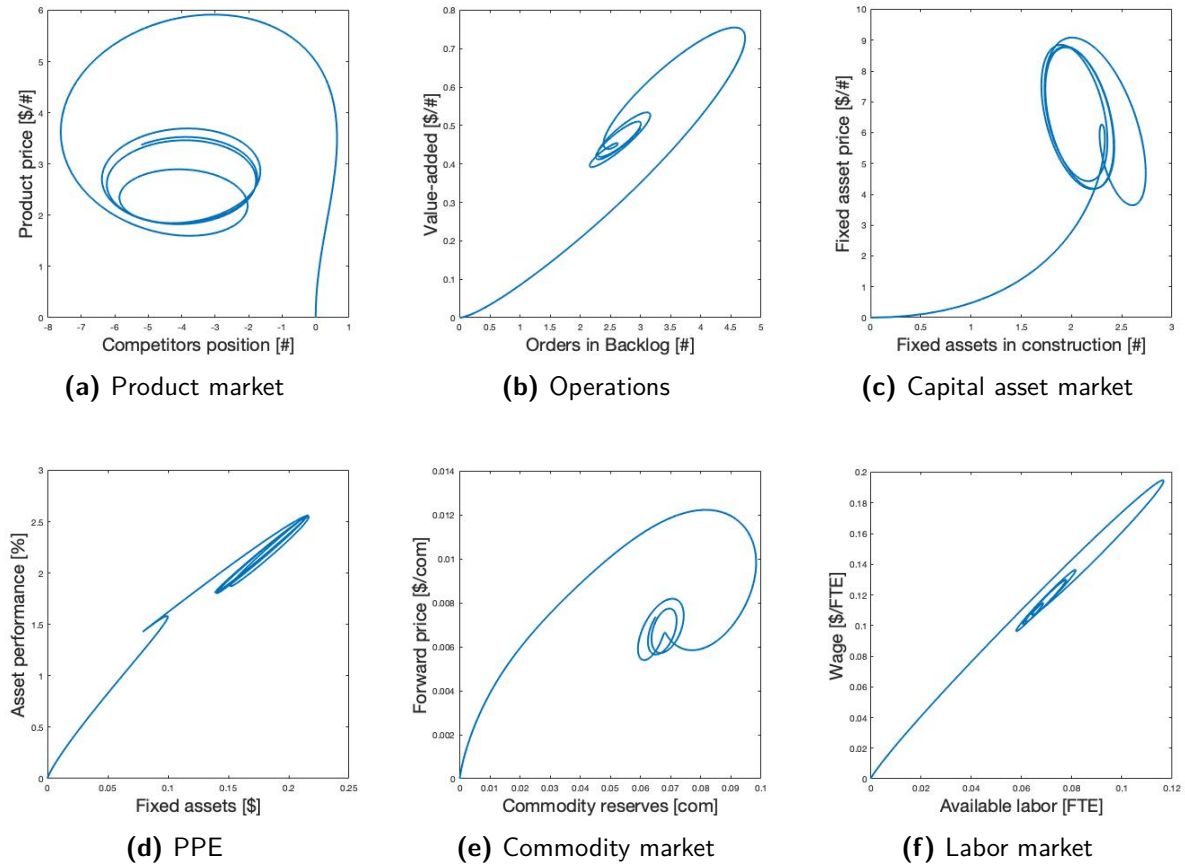


Figure 7-4: State-space trajectories of the damped harmonic oscillators in the business shows dynamics of business cycles in equilibrium and disequilibrium. In equilibrium, the state-space trajectories shows what cash-flow variable predominantly generates value. In disequilibrium, the shows the offset in price and assets needed to take in economic shocks.

of the necessary assets and cash to take in these shocks. In formulate a general interpretation for business cycles in disequilibrium as follows:

Business cycles in disequilibrium

Estimates the assets and cash needed to take in economic shocks.

State trajectories underscore the need for a dynamical system to model economic cycles. By consideration of these plots, we can analyze system's behavior in equilibrium and disequilibrium. However, the representation is not as effective as frequency-domain tools like the Bode plot, as the state trajectories give the response of the system to a single frequency, while the Bode plot shows the entire frequency response in a single plot.

7-5 Qualitative Time Responses for Verification and Comparison

In this section, I perform qualitative analysis on the time responses of the model for business dynamics.

7-5-1 Verification of the Bond-Graph Model

I use the time-domain response of the business to qualitatively verify the bond graph model. I do this by comparing the input-output and state responses to how the economic literature describes the behavior of the states and output.

Figure 7-5 shows the output response of the total system when given positive demand and investments as input. The input causes an increased product flow as the costs and benefits from the different markets change.

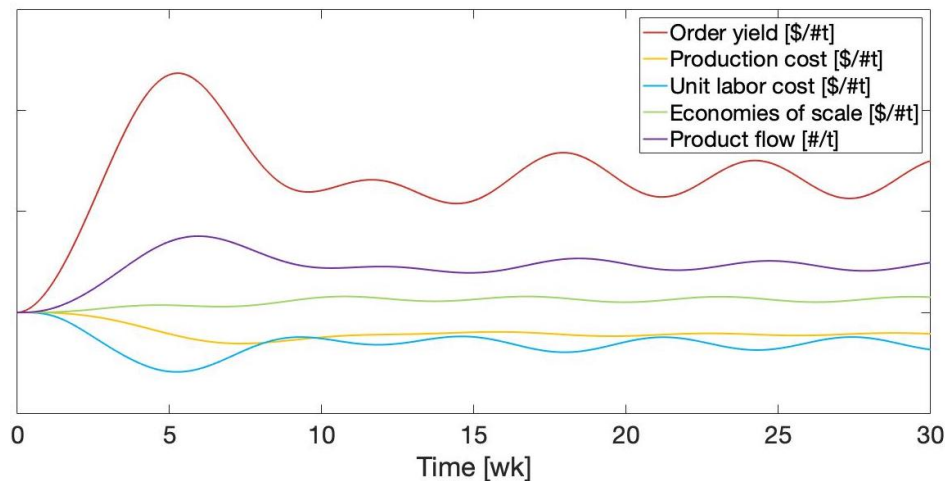


Figure 7-5: The time-domain step response of the total model shows that increased customer demand and investments results in increased costs and benefits. Costs are negative benefits, and are plotted in the negative direction. The overshoot indicates the extra costs and benefits resulting from the markets being in disequilibrium.

The input-output response of the full model is verified by comparing it to the economic literature. The order yield increases as a consequence of the increased product demand. This is in line with the economic law of supply and demand [15]. Economies of scale increase due to increased investments and increased fixed asset position. This is how economists describe the relation between fixed assets and economies of scale [11]. Costs (negative benefits) for production and labor increase due to the increased demand of the business. This is in line with how economies define scarcity [15]. There is an increased product flow as the benefits surpass the costs. The system's response shows an overshoot before converging to a new equilibrium. According to economic literature, economic systems exchange costs and benefits in disequilibrium, until a new equilibrium is found [7]. With these insights, I conclude that the outputs of the model for business dynamics behave according to the economic literature.

I use the time-domain responses of the model's subsystems to verify the model in more

detail. This verification is done for each subsystem. Here, I only analyze the time response of the commodity market, but the other subsystems are done in a similar way (See Appendix B for the time responses of each market). Figure 7-6 shows the input-output and state response of the commodity market when subjected to an increased product flow.

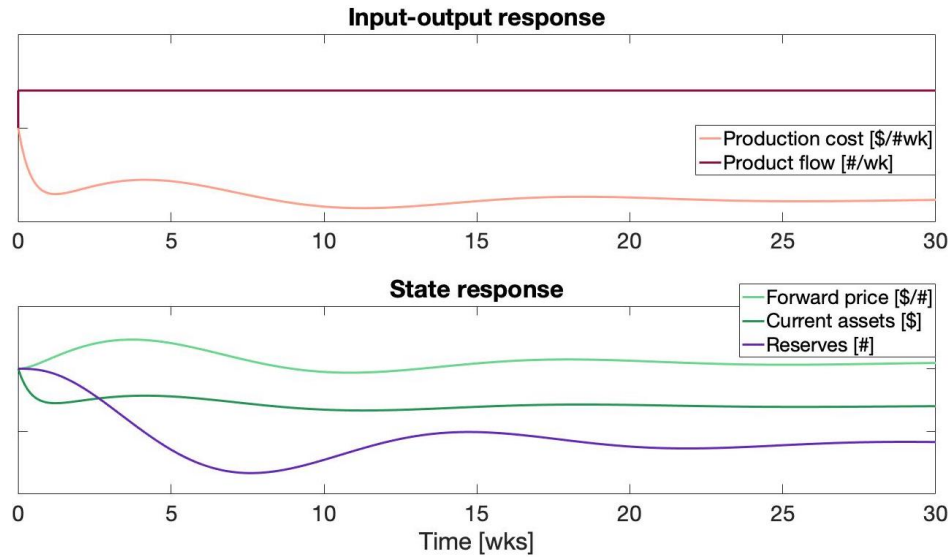


Figure 7-6: The time-domain step response of the commodity market shows that an increased product flow causes an increased costs. The state-response is in agreement with economic theory: Demand increase raises the price, and depletes the inventory and reserves.

The input-output and state response of the commodity market is verified by comparing them to the economic literature. The production cost decreases as more money is paid to the firm, resulting in higher benefits for the commodity market (cost = -benefit). This is in line with the microeconomic law for supply and demand [15]. The states behave according to economic theory as well: The increased product flow diminishes the current assets, as the number of commodities in the firm's inventory is depleted [5]. The increased demand for commodities increases the forward price, agreeing to the law of demand [15]. Consequently, the law of supply imposes that commodities are depleted from the source, causing a lower commodity level in the reserves [15]. With these insights, I conclude that the input-output and state response of the commodity market are in line with the economic literature.

The input-output response and the state response of the model for business dynamics are in line with the economic theory. I conclude this after checking the economic intuition of the state responses of each subsystem. With this insight, we have taken a huge step towards the model's verification. One last step can be taken to underscore the model's validity. This is the use of data to verify if the model is capable of mimicking the behavior of real-world markets. By doing this, the model will be practically applicable.

7-5-2 Comparison of the Commodity Market with the Labor Market

In this section, I compare the time responses of the commodity and the labor market. The two markets have the same structure, but use a different parametrization. In this comparison,

I show how different parameters affect the time response of the market. Economic interpretations for overshoot and state response are formulated.

Figure 7-7 shows the time response of the labor market to an increased product flow. This input is similar to the input of the time response of the commodity market in Figure 7-6.

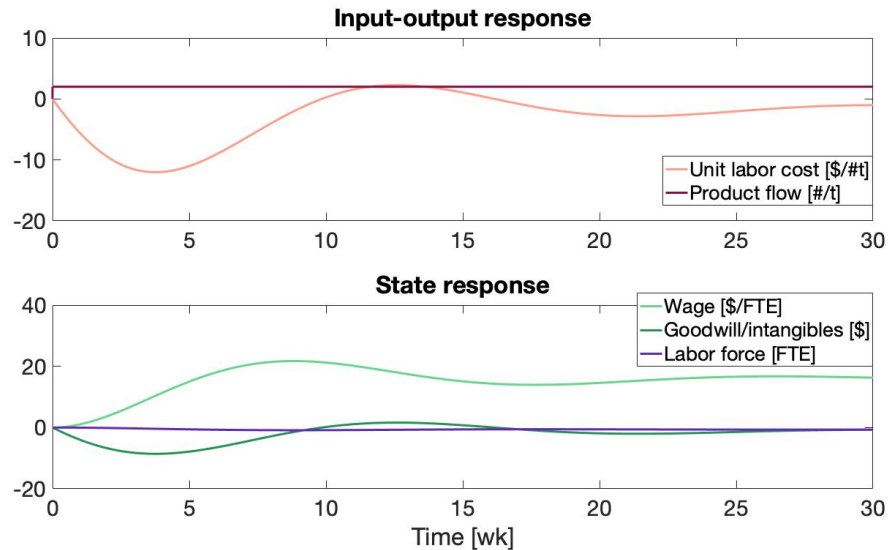


Figure 7-7: The time-domain step response of the labor market shows an overshoot in the output response, which is a measure for the liquidity of the market. The state-response gives the cost breakdown of the unit labor cost: It is predominantly determined by the wage.

The time responses of the labor and commodity markets show two differences. There is a difference in overshoot, and there is a difference in state response.

I interpret the overshoot of the output as a measure for the liquidity of the market. Overshoot in physical systems refers to the output exceeding its final steady-state value, and measures the system's ability to take in inputs [1]. In economics, market liquidity is a market's ability to absorb economic shocks [15]. The unit labor cost has a large overshoot, showing that the market is very costly in disequilibrium (temporary contracts, outsourcing), but cheap when it reaches equilibrium (fixed contracts) [16]. Contrarily, the commodity market shows close to no overshoot, which corresponds to continuous availability of the larger part of commodities [2]. In this market, the increased demand only gradually heightens the price. I formulate a general economic interpretation for overshoot as follows:

Overshoot

Overshoot of a market step response measure the liquidity of that market.

The state response is interpreted as the cost breakdown of the market. In engineering, the state response shows how the energy that is supplied to a system is divided over the different elements [9]. In economics, a cost breakdown shows where in economic systems or processes cash is lost [6]. By following the energy-cash flow analog, the two are parallel. The state response of the commodity market is equally distributed, which indicates comparable costs

from its inventory, the open market, and the supply side. The labor market's state response is predominantly showing up in the wage increase, identifying it as the main contributor to the increased unit labor cost. I formulate a general economic interpretation for the state response as follows:

State response

State responses give the cost breakdown of markets by showing where cash is stored.

With these economic interpretations for overshoot and state response, financial analysts retrieve abundant information from the time response of their models. This is a result of using an economically interpretable model where each state and each element has economic meaning. In economic modeling, line item of the financial statements is forecasted separately [12]. By taking this approach, all costs and benefits follow from a single state response.

7-5-3 Scenario Analysis of the Capital Asset Market with External Inputs

In this section, I perform scenario analysis on the capital asset market. I do this by giving the model different inputs. Figure 7-8 shows the input-output response of the capital asset market. In Figure 7-8a, the capital asset market is given high investments, and in Figure 7-8b the market is given low investments.

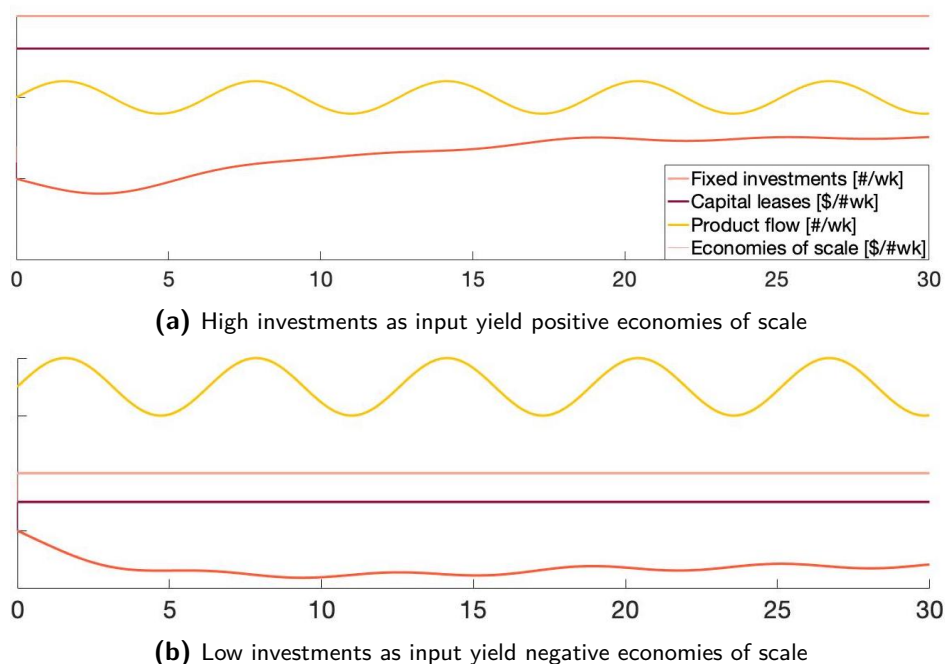


Figure 7-8: Input-output response of the capital asset market with equal product flow and varying investments. High investments increase the level of capital assets, so that a higher benefit of economies of scale is supplied to the firm. Low investments cause a lack of maintenance, causing the level of capital assets to decrease and a lower economies of scale.

This time responses show the amount of investments that are needed to maintain certain economies of scale. High investments increase the economies of scale supplied to the firm, as

more capital assets are acquired, than that are depreciated or used up [11]. Low investments cause a decrease in economies of scale, as more capital assets are depreciated or used up than there are acquired [11]. The economic interpretation is that insufficient maintenance and reinvestments cause deteriorations in the production process [17].

This type of scenario analysis can be performed on any market or on the complete system. Besides varying the inputs, it is possible to change the initial state of the business. However, it is challenging to dissociate the effect of each input on the output by using the time response. To this extend, Bode plots are more effective as they provide insight into the input-output relationships of specific signals (Section 7-2).

7-5-4 Complex Hamiltonian for Cash Flow Dynamics

In this section, I use the complex Hamiltonian to derive cash flow dynamics. The complex Hamiltonian contains more information than the regular Hamiltonian. The complex Hamiltonian consists of real-valued storage function S , and imaginary-valued resistive function R [3]:

$$\mathcal{H} = S - iR \quad (7-1)$$

The storage function represents the energy stored in the system and is equal to the regular Hamiltonian. According to the energy-cash flow analog, the real part is interpreted as follows:

Storage function

Time-evolution of the total cash flow.

The resistive function indicates where cash is lost in the system. I propose the following interpretation:

Resistive function

Time-evolution of the irreversible cash losses in the system.

The resistive function consists of two parts: The contraction and the squeeze. The contraction models the average cash losses over a cycle, and the squeeze provides the dynamics over those losses.⁴ Figure 7-9 shows the three components of the complex Hamiltonian [3].

The regular Hamiltonian and storage function of the bicomplex Hamiltonian behave in the same way, confirming the storage function's economic interpretation.

As the contraction function models the average cash losses over a cycle, it discloses the

⁴When considering the phase plane of a damped harmonic oscillator, the three components of the bicomplex Hamiltonian are projected on the trajectory as follows; a rotation represents the storage function, the contraction works perpendicular to the rotation, and the squeeze forms hyperbolae. See [3] for more information and a visualisation.

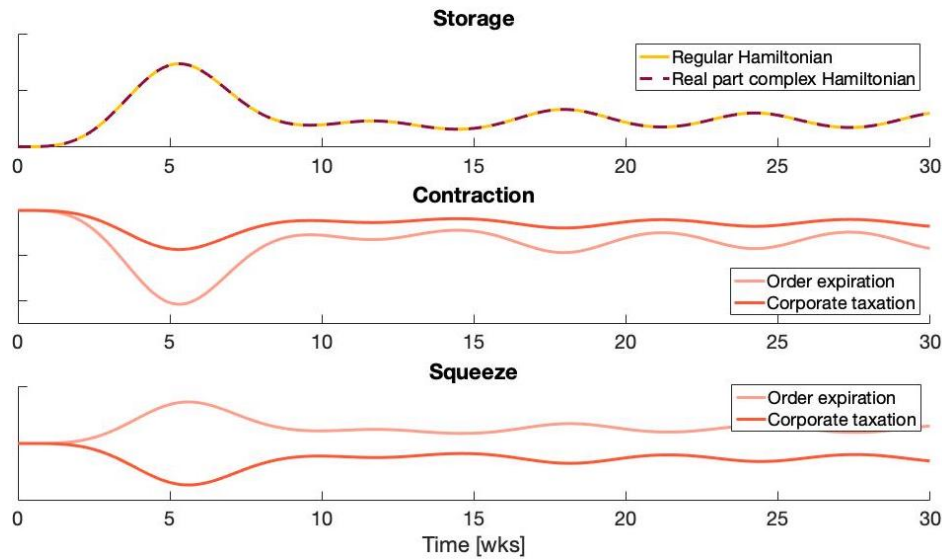


Figure 7-9: Time evolution of the complex Hamiltonian of the firm's operations consists of a storage, squeeze and contraction function. The storage function is equivalent to the regular Hamiltonian and represents the cash flow. The contraction function models the average cash losses over a business cycle, and the squeeze function models the dynamics of those losses.

proportion between the losses through order expiration and corporate taxation. The function is purely imaginary and always directed perpendicular to the business cycle's direction in the phase plane. Effects of the squeeze function cancel each other out over a business cycle, so no cash is lost through this function. The function is real-valued so that the effects of the order expiration and corporate taxation work along the p - and q -axis of the phase plane, thereby imposing dynamics on the cash losses.

The time response of the complex Hamiltonian shows that there is additionally information available about the cash losses of the system. However, this visualization is not effective as no conclusions can be drawn on the cash dynamics. An alternative approach is to be found to efficiently present this information: I recommend research on power density spectra, as they have the physical analogs of business cycle length and growth their axis. These two parameters are important quantities in business valuations [12].

7-6 Conclusions

Systems and Control (S&C) tools are particularly useful for economic analysis, as many of those tools have a direct application to (business) economics. The number of parallels with economic analysis is notable, but it is evident that the toolbox available to S&C engineers is far more extensive.

Parallels between S&C and economic analysis show that economists could enhance their existing techniques of time-domain analysis, eigenvalues, and logarithmic scaling. Economists perform all their examinations in the time domain but without the notion of overshoots or state responses. Eigenvalues make their appearance in the principal component analysis, but

the study uses the entire dataset's covariance matrix for calculating eigenvalues. Also, the eigenvalues are real-valued, which precludes the possibility to mimic economic cycles [8].

The S&C toolbox surpasses economic analysis with tools that are new to the field. Phase planes visualize business cycles in equilibrium and disequilibrium. A complex function for cash flow dynamics is provided by the complex Hamiltonian. Still, the application of many other S&C tools in economics is to be researched.

The contributions of this chapter are the propositions of economic applications of S&C tools for analysis. Table 7-1 summarizes:

Table 7-1: Applications of systems and control tools in business valuations

Systems and Control	Economics
Resonance peak	Optimal trading frequency
Overshoot	Market liquidity
State response	Cost structure
Real-valued pole	Economic factor
Complex-valued poles	Economic cycle
Phase plane	Visualization of business cycles
Storage function	Cash flow
Resistive function	Cash losses

Bibliography

- [1] M. F. Golnaraghi B. C. Kuo. *Automatic control systems*. Wiley & Sons, 2003.
- [2] A. L. Stinchcombe B. G. Carruthers. *The Social Structure of Liquidity: Flexibility, Markets, and States*. Springer, 1999.
- [3] M. Mendel C. Hutters. *Overcoming the dissipation obstacle with bicomplex port-Hamiltonian Mechanics*. Delft Center for Systems and Control, 2020.
- [4] G. F. Franklin et al. *Feedback Control of Dynamic Systems*. Prentice Hall, 2009.
- [5] R. B. Chase et al. *Production and operations management: Manufacturing and services*. Irwin-McGrawHill U.S.A., 1998.
- [6] G. Garrett. *Cost Estimating and Contract Pricing: Tools, Techniques and Best Practices*. Wolters Kluwer, 2008.
- [7] J. Hicks. *Value and capital*. Oxford University Press, 1975.
- [8] I. Jolliffe. *Principal component analysis*. Springer Berlin Heidelberg, 2011.
- [9] W. S. Levine. *The control handbook*. CRC-Press, 1996.
- [10] G. Lorenzoni. *A Theory of Demand Shocks*. MIT, 2008.
- [11] F. T. Moore. *Economies of scale: Some Statistical evidence*. Quarterly Journal of Economics, 1959.
- [12] R. Moses P. Stoica. *Spectral Analysis of Signals*. Prentice Hall, 2005.
- [13] T. Rietveld. *Handboek Investeren & Financieren*. Vakmedianet Management B.V., 2017.
- [14] S. Schreiber. *Anticipating business-cycle turning points in real time using density forecasts from a VAR*. Journal of Macroeconomics, 2016.
- [15] H. R. Varian. *Intermediate microeconomics: A modern approach*. W. W. Norton & Company, 2010.

- [16] C. von Hippel et al. *Temporary Employment: Can Organizations and Employees Both Win?* Academy of Management Perspectives, 1997.
- [17] T. Wireman. *Developing performance indicators for managing maintenance*. Industrial Press, 2005.
- [18] R. K. R. Yarlagadda. *Analog and Digital Signals and Systems*. Springer Science, 2010.

Chapter 8

Conclusions

The goal of this thesis was to develop a develop methods for business valuation in the frequency domain. To that effect, this thesis has one main contribution.

The main contribution of this thesis is the development of the theory needed to perform business valuations in the frequency domain. This theory is a pioneering effort for extending the current perspective on valuations. The current approach to business valuations makes cash-flow forecasts in the time domain. However, decision makers are concerned with the value of projects, and not so much with the timing of the cash flows. Huge efforts are made to analyse value in the time domain, while it is not a time-domain variable. The key insight of this thesis was that value is a frequency-domain variable. The frequency domain provides the analytical tools to directly compare the value of projects. The development of this theory indicates the dawn of an entire new method to perform business valuations in the frequency domain. The challenging task remains to make this method fundamental to the field of business valuations. Either economists have to revise their view on business valuations, or engineers have to extend their perspective to the economic domain.

A crucial step towards frequency-domain valuations was the development of the model for business dynamics. The model applies to all businesses, and replaces the current reliance on professional expertise with economic laws. With its causal structure, the model mimics the cyclical behavior of the economy and addresses the lack of data that prevents the use of data-driven models. The approach is a translation of an economic problem to the engineering domain, so that engineering tools are used to perform analysis, and control techniques can be applied. Steps towards its practical application are limited, as only system identification is left as a concluding task (Recommendation 10-2).

The theory is not only applicable to businesses, but to any type of financial instrument. The theory extends the existing economic-engineering framework to the frequency domain. The frequency domain can be used in conjunction with Newtonian and analytical methods to increase the scope and insight of analysis of all economic systems. Instead of using the

theory on a bond-graph for business dynamics, any micro- or macro-economic system can be analyzed in the frequency domain.

Chapter 9

Discussion

Fundamental Economics

In this thesis, I use the terms "economically fundamental" or "economic first-principles" to describe models based on economic theory. The study of the economic literature, however, suggests that there is not one single economic theory. In economics, there are different schools of thought.¹ Economic theory developed from premodern, through early modern, to modern economics. There are many schools of thought within modern economics, but the majority follow mainstream economics. Even within mainstream economics, there is a distinction between micro- and macroeconomics. Contrarily, there is only one set of laws of nature to model physical systems due to the presence of empirical evidence.

This thesis's economic models are based on the school of mainstream economics while obeying both micro- and macro-economic laws. Essential assumptions are market clearing, price as the sum of the costs, and the firm's interconnection to its four markets. Due to the model's economic interpretability, each assumption can be revisited without the necessity to build a new model.

Verification

The model and tools developed in this thesis are verified by comparison to the economic theory and by inspecting qualitative responses. As discussed above, economists do not agree on one economic theory. By comparing the model and tools to some economic theory, they are verified according to that theory and not to reality.

Qualitative analysis shows what dynamics are modeled and what dynamics are not. The qualitative system response can be compared with existing models or economic intuition so that the model is verified in more detail.

¹The presence of multiple schools of thought in economics is underscored by the quote at the start of this thesis.

Verification can be taken one step further by performing quantitative analysis. This analysis shows whether predicted business dynamics are in line with reality by using real data. If dynamics are predicted correctly, it is shown that the model is capable of mimicking the firm's behavior. Although a data fit does not prove the model's economic assumptions and relations, it could serve as a basis for economic axioms.

Chapter 10

Recommendations

10-1 Introduction

This thesis was focused on the development of an economic-engineering valuation method. I recommend to perform the last step towards its practical application: System identification. The research thereby produced many interesting research opportunities, which I recommend to pursue: Control, economic engineering in the frequency domain, and model refinements.

10-2 System Identification

Identification of the system for firm dynamics enables its practical application and underscores the verification of the model. I suggest to identify the system in two steps: Data collection and the use of identification techniques.

10-2-1 Data Collection

The field of business valuations faces a severe lack of data. In contrary to physical systems, no experiments can be performed on economic systems to excite them and generate measurement data. Still, large amounts of economic data is available. It is shown that this data can be used for the identification of the oil market [22], and macro-economic systems [18]. However, financial data on businesses is concealed, to protect the competitive advantage of their cost structures [11]. The only data businesses are obliged to publish comes in the form of financial statements, which are updated yearly or quarterly [26].

I recommend research on the use of different data sources for the separate identification of each subsystem. I suggest the use of the following data sources:

1. **Commodity market:** World bank commodity price data for data on commodity prices [14].

2. **Labor market:** World Bank data bank for labor force structure, or reports by the International Labour Organization [15, 25].
3. **Capital asset market:** Capital assets mainly consist of property, plant and equipment. I suggest the use of data on the real-estate market [3], or equipment specific to the firm. For example trucks [1].
4. **Product market:** Financial data on the market of the product that the business sells. For example smartphones [28].
5. **Firm:** Financial statements give the balance sheet, income statement and cash-flow statement [26].

When using a data set, it is important to take into account the geographic location. Also, data on financial markets is often paid for. I recommend research on finding suitable data sources for system identification.

10-2-2 Identification Techniques

Matlab's *System Identification Toolbox* provides extensive options to identify the system's parameters [20]. Orie identified the system representing the global oil market as follows [22]:

1. Set normalized parameters to an initial value.
2. Possibly bound parameters by a minimum or maximum.
3. Set search options.
4. Use *greyest()* to perform greybox estimation.

This approach is an example of how greybox identification can be performed. I recommend research on what identification technique is most effective to identify the model for firm dynamics.

10-3 Control

The goal of this thesis was the development of an economic-engineering valuations method. Although control is not a part of the research, I recognize many interesting applications for its use. I recommend research on two controllers: An energy-based controller to assist management on anticipating economic shocks, and a model predictive controller (MPC) to maximizing growth or profits for investors.

10-3-1 Cash Flow-Based Control for Anticipating Economic Shocks

Energy-based control has made its appearance in economics to control macro-economic systems [19, 16], and is by the energy-cash flow analog parallel to cash flow-based control. I see its application to business economics particularly interesting for the following reasons:

1. Cash flow, the commonly used variable in business economics, is included as quantity [16].
2. The Hamiltonian is leveraged as energy function so that system dynamics are exploited. This results in a low control gain and thus low investment costs, compared to classical control [9].
3. The ability to shape state-equilibrium [27].

An important restriction to the use of energy-based control techniques is the dissipation obstacle [10]. However, this obstacle is overcome by expressing the model for business dynamics in bicomplex coordinates [2], and the model is ready for energy-based control techniques.

The application in business economics is as follows: Management anticipates an economic shock that causes an increase (or decrease) in demand for the firm's product. As a result, management wants to fill (or deplete) its inventories, while making the minimum amount of costs. This objective is in line with benefits (2) and (3) of energy-based control.

I suggest the use of a passivity-based controller (PBC) for full-state feedback. An example is stabilization via energy balancing [10], which uses a controller with energy H_a to obtain the desired energy function H_d :

$$H_d(x) = H(x) + H_a(x) \quad (10-1)$$

Control by interconnection (CbI) controllers only use the output of the output signal of the system, and are modeled as a port-Hamiltonian system with states $\xi \in \mathbb{R}$. The closed-loop energy function is as follows [10]:

$$H_{cl}(x, \xi) = H(x) + H_c(\xi) \quad (10-2)$$

Hutters shows that the PBC or CbI controllers can assign arbitrary state-equilibria to dissipative systems by using a bicomplex system representation [2]. As the Hamiltonian is leveraged, low investment costs are needed. Figure 10-1 shows the arbitrary offset of a damped harmonic oscillator by using energy-based control:

10-3-2 Model Predictive Control for Profit and Value Maximization

I recommend the use of a model predictive controller (MPC) to assist investors on their investment decisions for profit or value maximization. The MPC's application to business economics is specifically interesting for three reasons:

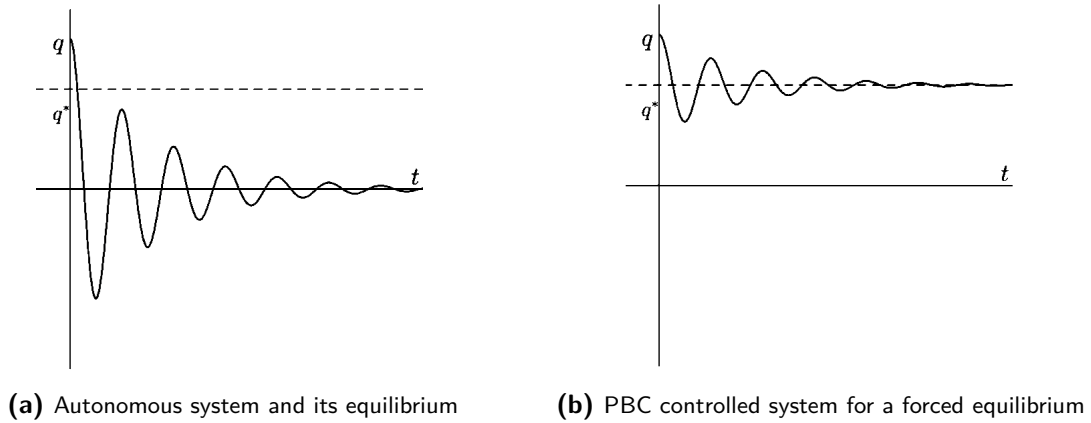


Figure 10-1: Signal trajectories of position q of a damped harmonic oscillator [2]

1. The controller's ability to optimize cost functions over a finite time horizon helps investors to formulate their strategy for a fixed investment horizon [5].
2. As the controller allows for the incorporation of constraints, it is possible to bound the available cash flow for investments [5].
3. The functionality of the MPC in economic systems is shown [21, 7].

The objective function that is used to optimize profits of businesses has the following form:

$$\text{Profits} = \sum_{t=0}^T CF_t \quad (10-3)$$

with investment horizon T and cash flow CF_t . In Chapter 5, it was determined what part of the system for business dynamics represent profits and yield company value. My suggestion is to use this function for profit maximization.

Note that Equation 10-4 shows similarities to the fundamental definition of value (Equation 2-1). If the discount function is adopted in the cost function, the MPC controller maximizes net present value:

$$\text{NPV} = \sum_{t=0}^T d(t)CF_t \quad (10-4)$$

where NPV is the net present value of the business, and $d(t)$ is the discount function.

10-4 Economic Engineering in the Frequency Domain

In this thesis, I extend the economic-engineering analogy to the frequency domain. I recommend the further development of the analogy.

10-4-1 Frequency-Domain Analysis for Valuing other Financial Instruments

In this thesis, a theory is developed that uses the frequency domain for business valuation. This theory applies to all financial instruments that are valued with ex-ante measures. Frequency-domain analysis applies to all sorts of valuations, as value is per definition dependent on the future. The frequency domain measures quantities ex-post, as should be done when considering quantities in the future. For example, the following instruments can be valued in the frequency domain:

1. **Real estate appraisal:** The real estate market is huge. To profit from it, investors must know how to value real estate and estimate how much profit each project will make, whether through property appreciation, rent, or both [6].
2. **Bonds:** The value of a bond is the discounted stream of cash flows it is expected to generate [12].
3. **Future contracts:** A futures contract is an agreement to trade something at a pre-determined price at a specified time in the future [17]. Future contracts made their appearance in the bond-graph model of the commodity market.

10-4-2 Development of Complex Bond-Graph Modeling Technique

It is shown in this thesis, that the regular bond graph of Chapter 4 is equivalent to the bicomplex representation of Chapter 5. The parameters used by the bicomplex model (business cycle and contraction) are in some economic fields (business economics, finance) a more intuitive representation. Therefore, I recommend the development of a complex bond-graph modeling technique, that allows for modeling in terms of business cycles and contraction. As a starting point, I propose to use the three damped harmonic oscillators as bond-graph elements to represent different economic markets:

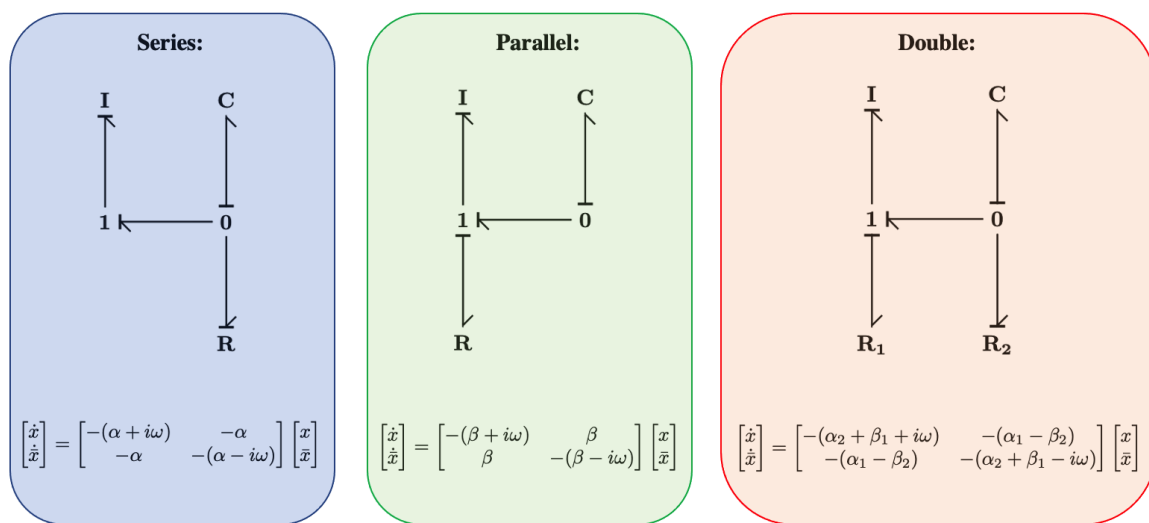


Figure 10-2: Three different damped harmonic oscillators and their state-space representations.

This modeling technique then uses only three elements, and three different parameters. Regular 2-port elements are used for their interconnection [4].

10-5 Model Refinement

The model for firm dynamics is a simplification of reality, as all models are. Only after the model's identification it becomes clear if it is sophisticated enough to capture all firm dynamics. If this is not the case, I suggest the following modeling refinements.

10-5-1 Capital Structure

The capital structure indicates what part of the business is financed by equity, and what amount by debt. The ratio is useful for determining the riskiness of lending to the firm. I chose not to model the capital structure of businesses, because the Modigliano-Miller theorem suggests in its first proposition that present value is independent from the capital structure [8]:

$$V_U = V_L \quad (10-5)$$

where V_U is the unlevered business value (without debt finance), and V_L is the levered business value (with debt finance). However, the second proposition states a difference in costs of levered and unlevered equity [8]:

$$r_E(L) = r_E(U) + \frac{D}{E}(r_E(U) - r_D) \quad (10-6)$$

where $r_E(L)$ is the levered cost of equity, $r_E(U)$ is the unlevered cost of equity, D/E is the part of the business financed by debt, and r_D is the cost of debt. It appears that the levered cost of equity is higher than the unlevered cost of equity to account for the higher risk of the equity holders [8]. I recommend research on the impact of the firm's capital structure on (1) the business value, and (2) the risk.

10-5-2 Competition

In the model for firm dynamics, I model competition I follow Pareto by modeling the firm either as a price taker or a price maker in its different markets [23]. In practice however, it is most common to use Porter's Five Forces model to analyse competition [24]. Figure 10-3 gives an illustration:

Due to the economic interpretability of the model for dynamics of the firm, the five forces can be incorporated in the subsystems they act on. I recommend research on how to do so. This recommendation is currently being pursued by the economic-engineering group.

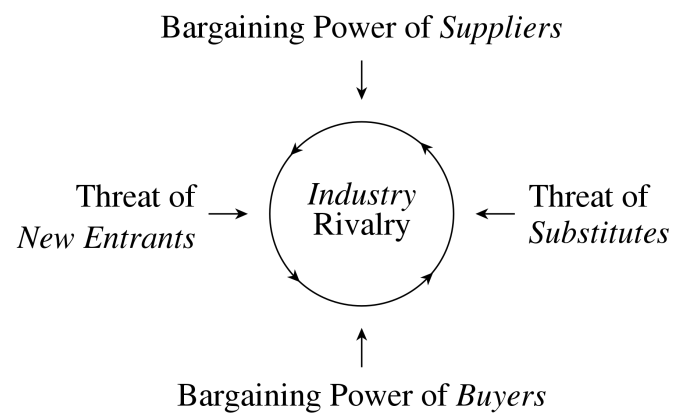


Figure 10-3: Porter's five forces model for competition. This figure is taken from [13].

Bibliography

- [1] Roland Berger. *Trends in truck & trailer market*. Roland Berger GmbH, 2020.
- [2] M. Mendel C. Hutters. *Overcoming the dissipation obstacle with bicomplex port-Hamiltonian Mechanics*. Delft Center for Systems and Control, 2020.
- [3] L. C. Delmendo. *Strong house price rises continue in The Netherlands*. Global Property Guide, 2020.
- [4] D. C. Karnopp et al. *System dynamics: Modeling, simulation, and control of mechatronic systems*. John Wiley & Sons, Inc., 2012.
- [5] D. Q. Mayne et al. *Constrained model predictive control: Stability and optimality*. Elsevier Ltd, Inc, 2000.
- [6] L. Bourne et al. *Urban Housing Markets :Recent directions in research and policy*. University of Toronto Press, 1978.
- [7] M. Coroz et al. *Automating Trades in Oil Futures using Model Predictive Control*. Delft Center for Systems and Control, 2020.
- [8] R. A. Brealey et al. *Principles of corporate finance*. McGraw-Hill, 2014.
- [9] R. Ortega et al. *Energy-shaping of port-controlled Hamiltonian systems by interconnection*. Conference of Decisions and Control, 1999.
- [10] R. Ortega et al. *Putting energy back in control*. University of Groningen, 2001.
- [11] T. Koller et al. *Valuations*. John Wiley and Sons Inc, 2015.
- [12] F. Fabozzi. *Valuation of fixed income securities and derivatives*. John Wiley, 1998.
- [13] D. Fadeev. *Creating and sustaining superior performance*. New York Free Press, 1985.
- [14] Prospects Group. *World bank commodities prices data (pink sheet)*. The World Bank Group, 2020.

- [15] World Bank Group. *Databank world development indicators, labour force structure*. The World Bank Group, 2020.
- [16] C. Hutters. *The analytical mechanics of consumption*. Delft Center for Systems and Control, 2019.
- [17] R. Keith. *Financial Derivatives: An Introduction to Futures, Forwards, Options and Swaps*. Prentice-Hall, 1997.
- [18] G. Kruimer. *First-Principles Modelling for interpretable Economic Scenario Models*. Delft Center for Systems and Control, 2020.
- [19] A. Macchelli. *Port-Hamiltonian formulation of simple macro-economic systems*. Conference on Decision and Control, 2013.
- [20] MatLab. *System Identification Toolbox*. MathWorks, Inc, 2020.
- [21] A. J. J. Meegdes. *Scheduling the supply chain with a model predictive control algorithm based on economic engineering systems theory*. Delft Center for Systems and Control, 2020.
- [22] N. G. Orie. *Dynamic modelling and control of the oil market*. Delft Center for Systems and Control, 2019.
- [23] V. Pareto. *Manual of political economy*. Oxford University Press, 1906.
- [24] M. E. Porter. *The five competitive forces that shape strategy*. Harvard Business Review, 2008.
- [25] Data Production and ILO Department of Statistics Analysis Unit. *The global labour income share and distribution*. International Labour Office, 2020.
- [26] T. Rietveld. *Handboek Investeren & Financieren*. Vakmedianet Management B.V., 2017.
- [27] A. Schaft. *L2 gain and passivity techniques in nonlinear control*. Springer, 1999.
- [28] Statista. *Mobile Phones - The Netherlands*. Statista, 2020.

Appendix A

Variables for Qualitative Analysis

Table A-1: Variables used for analysis of the model for firm dynamics

Commodity market	Labor market	Capital asset market	Product market	Firm
$C_5 = 5$	$C_6 = 1.2$	$C_3 = 2.5$	$C_1 = 4.1$	I_8
$I_4 = 3$	$I_6 = 1.6$	$C_4 = 6.1$	$C_2 = 0.5$	$R_{10} = 2$
$I_5 = 1.1$	$I_7 = 3$	$I_2 = 1.4$	$I_1 = 2.1$	
$R_6 = 2$	$R_8 = 0.2$	$I_3 = 8.3$	$R_1 = 2.1$	
$R_7 = 0.1$	$R_9 = 3$	$R_3 = 2.1$	$R_2 = 0.2$	
$TF_2 = 3.2$	$TF_3 = 2.8$	$R_4 = 1.2$		
$GY_2 = 1.2$	$GY_3 = 0.8$	$R_5 = 0.11$		
		$TF_1 = 1.3$		
		$GY_1 = 1.1$		

Time Responses of the Markets

B-1 Commodity Market

Figure B-1 shows the input-output and state response of the commodity market with input $u = 2$.

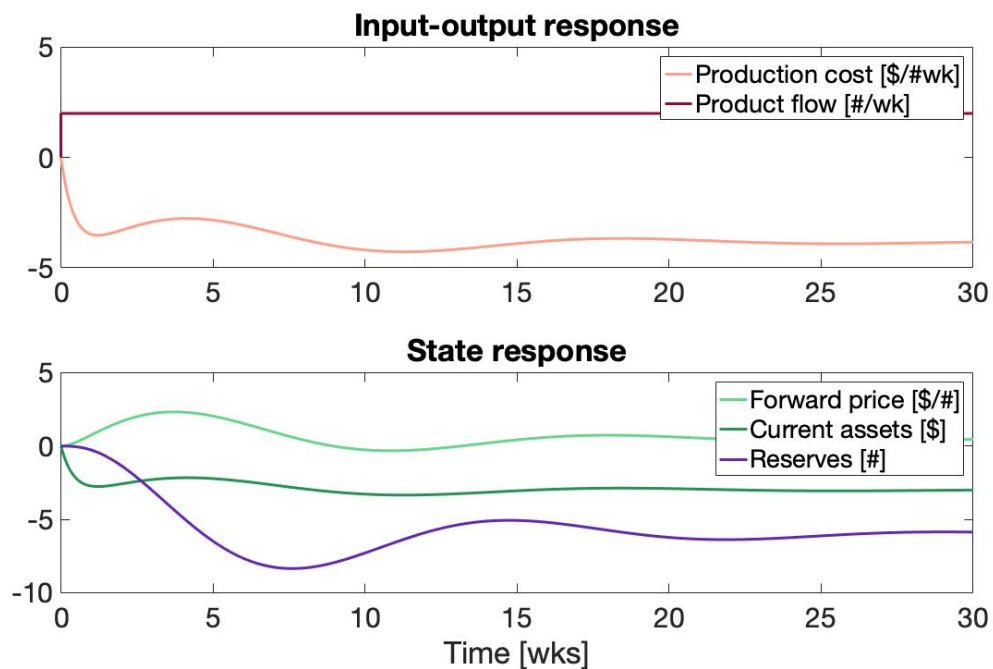


Figure B-1: Time response of the commodity market with increased product flow

B-2 Labor Market

Figure B-2 shows the input-output and state response of the labor market with input $u = 2$.

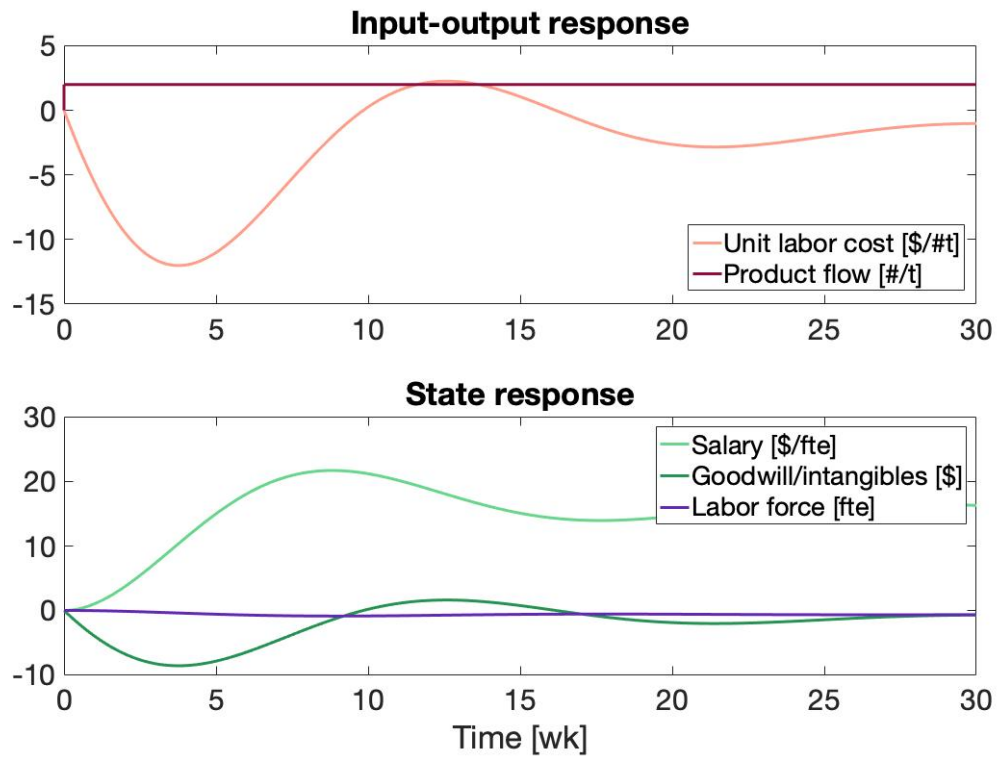


Figure B-2: Time response of the labor market with increased product flow

B-3 Capital Asset Market

Figure B-3 shows the input-output and state response of the capital asset market with high investments and input $u = [10 \ 8 \ 5 + \sin(t)]^T$. Figure B-4 has low investments and input $u = [2 \ 1 \ 5 + \sin(t)]^T$.

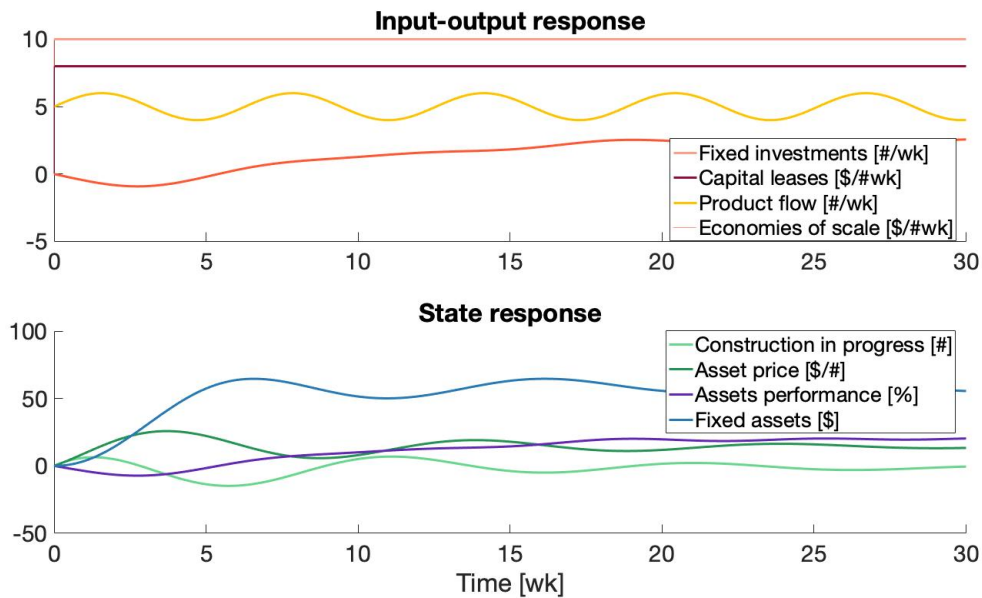


Figure B-3: Time response of the capital asset market with high investments

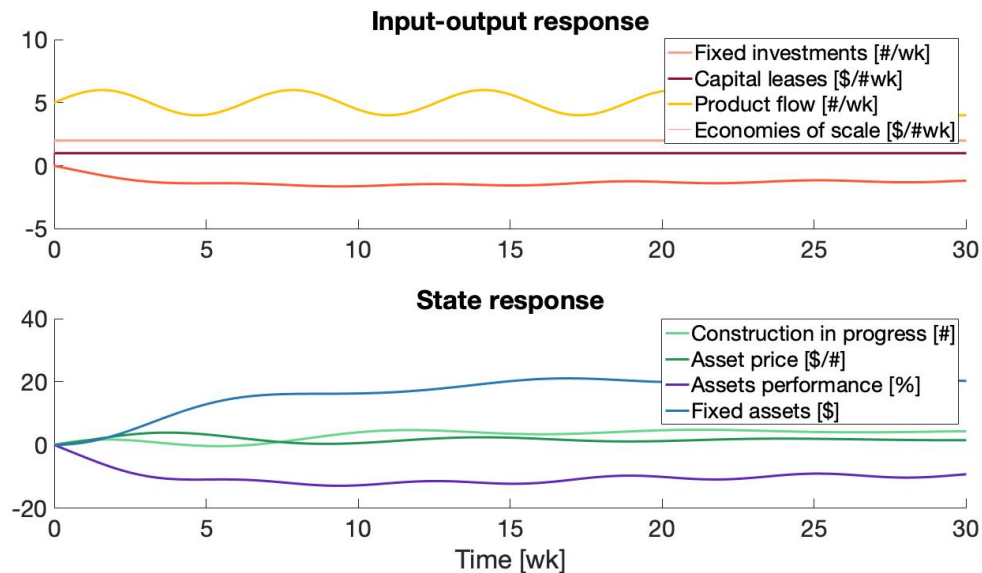


Figure B-4: Time response of the capital asset market with low investments

B-4 Product Market

Figure B-6 shows the input-output and state response of the capital asset market with increased product demand and input $u = [15 + 3\sin(t) \quad 4 \quad 5]^T$. Figure ?? has decreased product demand and input $u = [-15 - 3\sin(t) \quad -4 \quad 5]^T$.

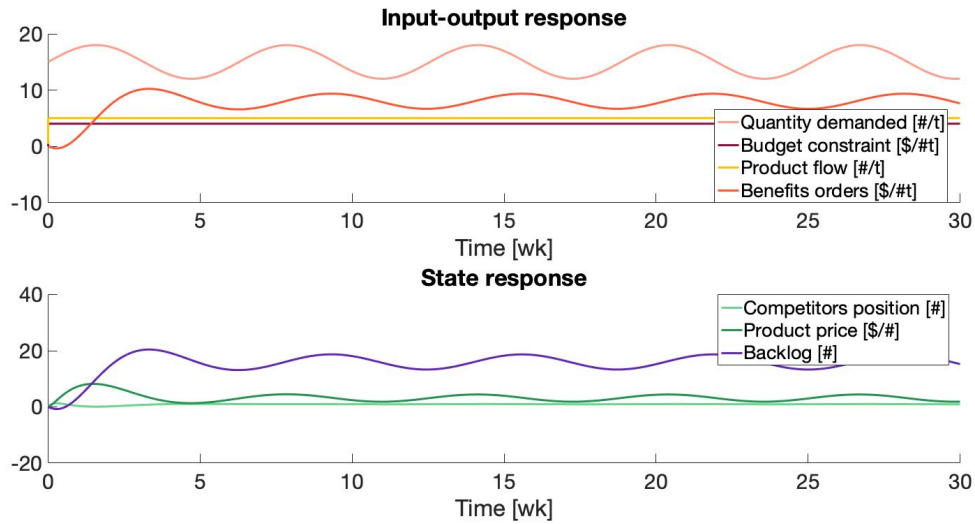


Figure B-5: Time response of the product market with increased product demand

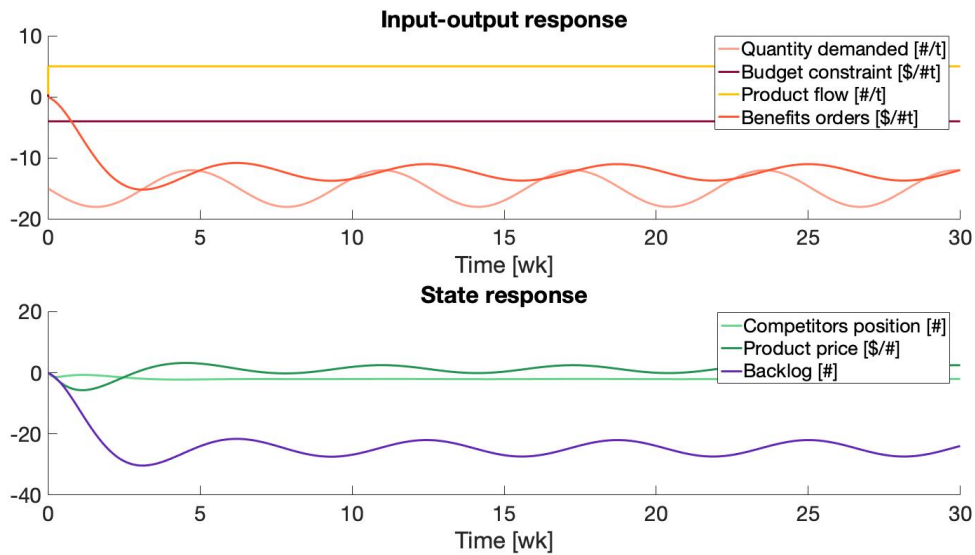


Figure B-6: Time response of the product market with increased product demand

Appendix C

Derivations

C-1 State-Space Derivation from the Bond Graph Model

C-1-1 Commodity market

In this section the equations of motion, linked to the bond graph representing the commodity market are derived. The bond graph is depicted in Figure C-1:

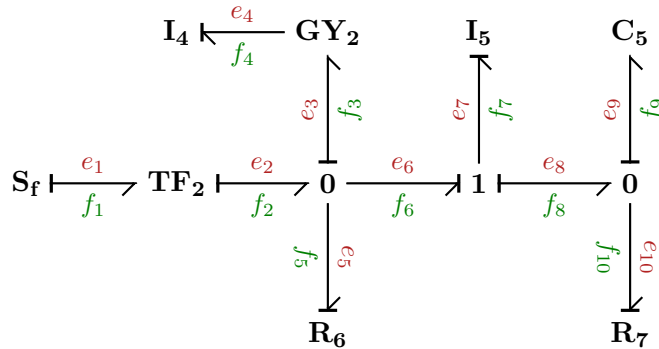


Figure C-1: Bond-graph representation of the commodity market with numbered elements and signals

Bonds and elements are numbered from left to right and top to bottom as depicted above. The states are the current assets p_4 , forward price p_5 and reserves q_5 . The equations of motion are using the method described by Karnopp et al. in Chapter 5 of [2]:

$$\dot{p}_4 = e_4 = GY_2 f_3 = GY_2 (-f_2 - f_5 + f_6) \quad (C-1)$$

$$f_2 = TF_2 f_1 = TF_2 S f(t) \quad (C-2)$$

$$f_5 = \frac{e_5}{R_6} = \frac{e_3}{R_6} = \frac{f_4 GY_2}{R_6} = \frac{GY_2}{R_6 I_4} p_4 \quad (C-3)$$

$$f_6 = f_7 = \frac{p_5}{I_5} \quad (C-4)$$

$$\begin{aligned} \dot{p}_4 &= GY_2(TF_2 S f(t) - \frac{GY_2}{R_6 I_4} p_4 - \frac{1}{I_5} p_5) \\ &= -\frac{GY_2^2}{R_6 I_4} p_4 + \frac{GY_2}{I_5} p_5 - GY_2 T F_2 u(t) \end{aligned} \quad (C-5)$$

$$\dot{p}_5 = e_7 = -e_6 + e_8 \quad (C-6)$$

$$e_6 = e_3 = GY_2 f_4 = \frac{GY_2}{I_4} p_4 \quad (C-7)$$

$$e_8 = e_9 = \frac{1}{C_5} q_5 \quad (C-8)$$

$$\dot{p}_5 = -\frac{GY_2}{I_4} p_4 + \frac{1}{C_5} q_5 \quad (C-9)$$

$$\dot{q}_5 = f_9 = -f_8 - f_{10} \quad (C-10)$$

$$f_8 = f_7 = \frac{1}{I_5} p_5 \quad (C-11)$$

$$f_{10} = \frac{1}{R_7} e_{10} = \frac{1}{R_7} e_9 = \frac{1}{R_7 C_5} q_5 \quad (C-12)$$

$$\dot{q}_5 = f_9 = -\frac{1}{I_5} p_5 - \frac{1}{R_7 C_5} q_5 \quad (C-13)$$

By defining state vector $x = [p_4 \quad q_5 \quad p_5]^T$ and the input $u = Sf(t)$, equations (C-5), (C-9) and (C-13) are summarized in matrix notation to obtain the A and B matrices.

$$\dot{x} = \underbrace{\begin{bmatrix} -\frac{GY_2^2}{R_6 I_4} & \frac{GY_2}{I_5} & 0 \\ 0 & -\frac{1}{R_7 C_5} & -\frac{1}{I_5} \\ -\frac{GY_2}{I_4} & 0 & \frac{1}{C_5} \end{bmatrix}}_A x + \underbrace{\begin{bmatrix} -GY_2 T F_2 \\ 0 \\ 0 \end{bmatrix}}_B u(t) \quad (C-14)$$

The output is $e_1 = \frac{T F_2 GY_2}{I_4} p_4$. The C and D matrices are derived as:

$$y = \underbrace{\begin{bmatrix} \frac{T F_2 GY_2}{I_4} & 0 & 0 \end{bmatrix}}_C x + \underbrace{\begin{bmatrix} 0 \end{bmatrix}}_D \quad (C-15)$$

C-1-2 Labor market

In this section the equations of motion, linked to the bond graph representing the labor market are derived. Figure C-2 shows the bond graph:

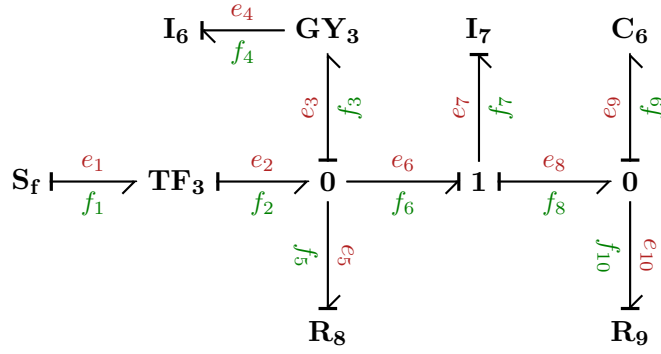


Figure C-2: Bond-graph representation of the labor market with numbered elements and signals

Bonds and elements are numbered from left to right and top to bottom as depicted above. The states are the intangible assets p_6 , wage p_7 and labor force q_6 . The equations of motion are using the method described by Karnopp et al. in Chapter 5 of [2]:

$$\dot{p}_6 = e_4 = GY_3 f_3 = GY_3(-f_2 - f_5 + f_6) \quad (\text{C-16})$$

$$f_2 = TF_3 f_1 = TF_3 S f(t) \quad (\text{C-17})$$

$$f_5 = \frac{e_5}{R_8} = \frac{e_3}{R_8} = \frac{f_4 GY_3}{R_8} = \frac{GY_3}{R_8 I_6} p_6 \quad (\text{C-18})$$

$$f_6 = f_7 = \frac{p_7}{I_7} \quad (\text{C-19})$$

$$\begin{aligned} \dot{p}_6 &= GY_3(TF_3 S f(t)) - \frac{GY_3}{R_8 I_6} p_6 - \frac{1}{I_7} p_7 \\ &= -\frac{GY_3^2}{R_8 I_6} p_6 + \frac{GY_3}{I_7} p_7 - GY_3 TF_3 u(t) \end{aligned} \quad (\text{C-20})$$

$$\dot{p}_7 = e_7 = -e_6 + e_8 \quad (\text{C-21})$$

$$e_6 = e_3 = GY_3 f_4 = \frac{GY_3}{I_6} p_6 \quad (\text{C-22})$$

$$e_8 = e_9 = \frac{1}{C_6} q_6 \quad (\text{C-23})$$

$$\dot{p}_7 = -\frac{GY_3}{I_6} p_6 + \frac{1}{C_6} q_6 \quad (\text{C-24})$$

$$\dot{q}_6 = f_9 = -f_8 - f_{10} \quad (\text{C-25})$$

$$f_8 = f_7 = \frac{1}{I_7} p_7 \quad (\text{C-26})$$

$$f_{10} = \frac{1}{R_9} e_{10} = \frac{1}{R_9} e_9 = \frac{1}{R_{10} C_6} q_6 \quad (\text{C-27})$$

$$\dot{q}_6 = f_9 = -\frac{1}{I_7} p_7 - \frac{1}{R_9 C_6} q_6 \quad (\text{C-28})$$

By defining state vector $x = [p_6 \quad q_6 \quad p_7]^T$ and the input $u = Sf(t)$, equations (C-20), (C-24) and (C-28) are summarized in matrix notation to obtain the A and B matrices.

$$\dot{x} = \underbrace{\begin{bmatrix} -\frac{GY_3^2}{R_8 I_6} & \frac{GY_3}{I_7} & 0 \\ 0 & -\frac{1}{R_9 C_6} & -\frac{1}{I_7} \\ -\frac{GY_3}{I_6} & 0 & \frac{1}{C_6} \end{bmatrix}}_A x + \underbrace{\begin{bmatrix} -GY_3 T F_3 \\ 0 \\ 0 \end{bmatrix}}_B u(t) \quad (\text{C-29})$$

The output is $e_1 = \frac{T F_3 G Y_3}{I_6} p_6$. Then the C and D matrices are derived as:

$$y = \underbrace{\begin{bmatrix} \frac{T F_3 G Y_3}{I_6} & 0 & 0 \end{bmatrix}}_C x + \underbrace{\begin{bmatrix} 0 \end{bmatrix}}_D \quad (\text{C-30})$$

C-1-3 Capital asset market

In this section the equations of motion, linked to the bond graph representing the commodity market are derived. The bond graph is depicted in Figure C-3:

Bonds and elements are numbered from left to right and top to bottom as depicted above. The states are the construction in progress q_3 , Asset price p_2 , fixed assets p_3 and asset performance q_4 .

The equations of motion are using the method described by Karnopp et al. in Chapter 5 of [2]:

$$\dot{q}_3 = f_3 = (f_2 - f_1 - f_4) \quad (\text{C-31})$$

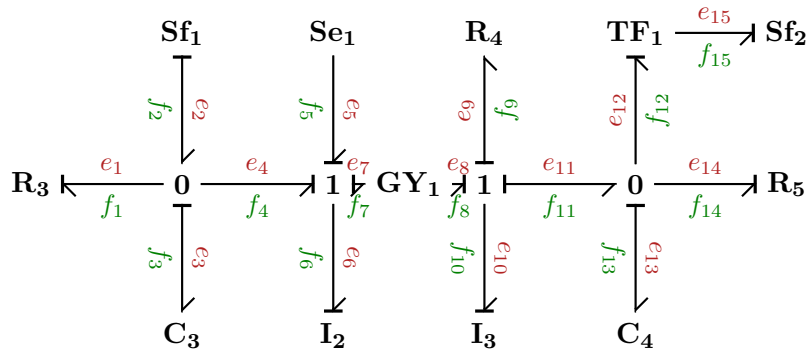


Figure C-3: Bond-graph representation of the capital asset market with numbered elements and signals

$$f_1 = \frac{e_1}{R_3} = \frac{e_3}{R_3} = \frac{1}{R_3 C_3} q_3 \quad (\text{C-32})$$

$$f_2 = S f_1(t) \quad (\text{C-33})$$

$$f_4 = f_6 = \frac{p_2}{I_2} \quad (\text{C-34})$$

$$\dot{q}_3 = -\frac{1}{R_3 C_3} q_3 - \frac{1}{I_2} p_2 + S f_1(t) \quad (\text{C-35})$$

$$\dot{p}_2 = e_6 = e_4 + e_5 - e_7 \quad (\text{C-36})$$

$$e_4 = e_3 = \frac{1}{C_3} q_3 \quad (\text{C-37})$$

$$e_5 = S e_1(t) \quad (\text{C-38})$$

$$e_7 = G Y_1 f_8 = \frac{G Y_1}{I_3} p_3 \quad (\text{C-39})$$

$$\dot{p}_2 = \frac{1}{C_3} q_3 - \frac{G Y_1}{I_3} p_3 + S e_1(t) \quad (\text{C-40})$$

$$\dot{p}_3 = e_{10} = e_8 - e_9 - e_{11} \quad (\text{C-41})$$

$$e_8 = G Y_1 f_7 = \frac{G Y_1}{I_2} p_2 \quad (\text{C-42})$$

$$e_9 = R_4 f_9 = \frac{R_4}{I_3} p_3 \quad (\text{C-43})$$

$$e_{11} = e_{13} = \frac{1}{C_4} q_4 \quad (\text{C-44})$$

$$\dot{p}_3 = \frac{GY_1}{I_2} p_2 - \frac{R_4}{I_3} p_3 - \frac{1}{C_4} q_4 \quad (\text{C-45})$$

$$\dot{q}_4 = f_{13} = f_{11} - f_{12} - f_{14} \quad (\text{C-46})$$

$$f_{11} = \frac{1}{I_3} p_3 \quad (\text{C-47})$$

$$f_{12} = \frac{1}{TF_1} f_{15} = \frac{1}{TF_1} S f_2(t) \quad (\text{C-48})$$

$$f_{14} = \frac{1}{R_5} e_{14} = \frac{1}{R_5} e_{13} = \frac{1}{R_5 C_4} q_4 \quad (\text{C-49})$$

$$\dot{q}_4 = \frac{1}{I_3} p_3 - \frac{1}{R_5 C_4} q_4 - \frac{S f_2(t)}{TF_1} \quad (\text{C-50})$$

By defining state vector $x = [q_3 \ p_2 \ q_4 \ p_3]^T$ and defining the input as $u = [S f_1 \ S e_1 \ S f_2]^T$, equations (C-35), (C-40), (C-45) and (C-50) are summarized in matrix notation to obtain the A and B matrices.

$$\dot{x} = \underbrace{\begin{bmatrix} -\frac{1}{R_3 C_3} & -\frac{1}{I_2} & 0 & 0 \\ \frac{1}{C_3} & 0 & 0 & -\frac{GY_1}{I_3} \\ 0 & 0 & -\frac{1}{R_5 C_4} & \frac{1}{I_3} \\ 0 & \frac{GY_1}{I_2} & -\frac{1}{C_4} & -\frac{R_4}{I_3} \end{bmatrix}}_A x + \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{TF_1} \\ 0 & 0 & 0 \end{bmatrix}}_B u(t) \quad (\text{C-51})$$

The output is $e_{10} = \frac{1}{C_2} q_2$. The C and D matrices are derived as:

$$y = \underbrace{\begin{bmatrix} 0 & 0 & \frac{1}{TF_1 C_2} \end{bmatrix}}_C x + \underbrace{\begin{bmatrix} 0 \end{bmatrix}}_D \quad (\text{C-52})$$

C-1-4 Product market

In this section the equations of motion, linked to the bond graph representing the commodity market are derived. The bond graph is depicted in Figure C-4:

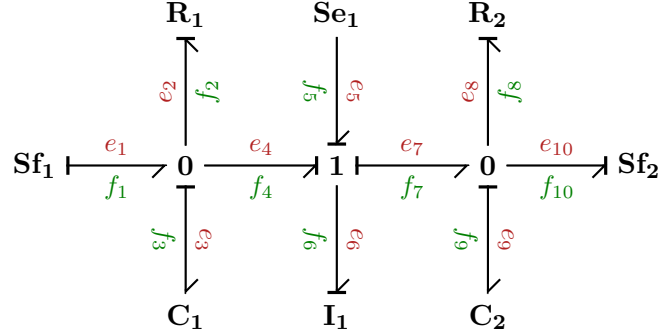


Figure C-4: Bond-graph representation of the product market with numbered elements and signals

Bonds and elements are numbered from left to right and top to bottom as depicted above. The states are the competitors' position q_1 , product price p_1 and backlog q_2 . The equations of motion are using the method described by Karnopp et al. in Chapter 5 of [2]:

$$\dot{q}_1 = f_3 = (f_1 - f_2 - f_4) \quad (\text{C-53})$$

$$f_1 = Sf_1(t) \quad (\text{C-54})$$

$$f_2 = \frac{e_2}{R_1} = \frac{e_3}{R_1} = \frac{1}{R_1 C_1} q_1 \quad (\text{C-55})$$

$$f_4 = f_6 = \frac{p_1}{I_1} \quad (\text{C-56})$$

$$\dot{q}_1 = -\frac{1}{R_1 C_1} q_1 - \frac{1}{I_1} p_1 + Sf_1(t) \quad (\text{C-57})$$

$$\dot{p}_1 = e_6 = e_4 + e_5 - e_7 \quad (\text{C-58})$$

$$e_4 = e_3 = \frac{1}{C_1} q_1 \quad (\text{C-59})$$

$$e_5 = Se_1(t) \quad (\text{C-60})$$

$$e_7 = e_9 = \frac{1}{C_2} q_2 \quad (\text{C-61})$$

$$\dot{p}_1 = \frac{1}{C_1} q_1 - \frac{1}{C_2} q_2 + S e_1(t) \quad (\text{C-62})$$

$$\dot{q}_2 = f_9 = f_7 - f_8 - f_{10} \quad (\text{C-63})$$

$$f_7 = f_6 = \frac{1}{I_1} p_1 \quad (\text{C-64})$$

$$f_8 = \frac{1}{R_2} e_8 = \frac{1}{R_2} e_9 = \frac{1}{R_2 C_2} q_2 \quad (\text{C-65})$$

$$f_{10} = S f_2(t) \quad (\text{C-66})$$

$$\dot{q}_2 = \frac{1}{I_1} p_1 - \frac{1}{R_2 C_2} q_2 - S f_2(t) \quad (\text{C-67})$$

By defining state vector $x = [q_1 \ p_1 \ q_2]^T$ and defining the input as $u = [S f_1 \ S e_1 \ S f_2]^T$, equations (C-57), (C-62) and (C-67) are summarized in matrix notation to obtain the A and B matrices.

$$\dot{x} = \underbrace{\begin{bmatrix} -\frac{1}{R_1 C_1} & -\frac{1}{I_1} & 0 \\ \frac{1}{C_1} & 0 & -\frac{1}{C_2} \\ 0 & \frac{1}{I_1} & -\frac{1}{R_2 C_2} \end{bmatrix}}_A x + \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}}_B u(t) \quad (\text{C-68})$$

The output is $e_{10} = \frac{1}{C_2} q_2$, and by assuming no initial offset, the C and D matrices are derived as:

$$y = \underbrace{\begin{bmatrix} 0 & 0 & \frac{1}{C_2} \end{bmatrix}}_C x + \underbrace{\begin{bmatrix} 0 \end{bmatrix}}_D \quad (\text{C-69})$$

C-1-5 The Firm and Total Model for Business Dynamics

In this section the equations of motion, linked to the bond graph representing the commodity market are derived. The bond graph is depicted in Figure C-5:

Bonds and elements are numbered from left to right and top to bottom as depicted above. The state is value-added p_8 . The input signals of the firm are the output signals of the market models. Using equations (C-30), (C-15), (C-69) and (C-52):

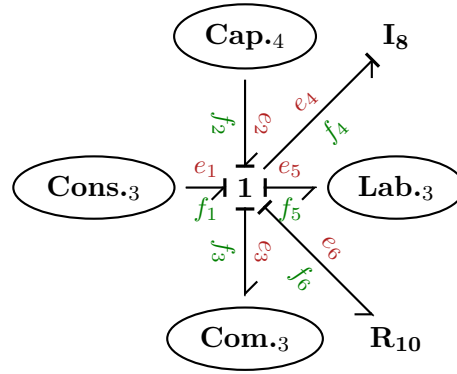


Figure C-5: Bond-graph representation of the firm with numbered elements and signals

$$e_1 = \frac{1}{C_2} q_2 \quad (\text{C-70})$$

$$e_2 = \frac{1}{TF_1 C_4} q_4 \quad (\text{C-71})$$

$$e_3 = \frac{TF_2 GY_2}{I_4} p_4 \quad (\text{C-72})$$

$$e_5 = \frac{TF_3 GY_3}{I_6} p_6 \quad (\text{C-73})$$

The equation of motion is derived by means of the method described by Karnopp et al. in Chapter 5 of [2]:

$$\dot{p}_8 = e_4 = e_1 + e_2 + e_3 + e_4 + e_6 \quad (\text{C-74})$$

$$e_6 = R_{10} f_6 = R_{10} f_4 = \frac{R_{10}}{I_8} p_8 \quad (\text{C-75})$$

$$\dot{p}_8 = \frac{1}{C_2} q_2 + \frac{1}{TF_1 C_4} q_4 + \frac{TF_2 GY_2}{I_4} p_4 + \frac{TF_3 GY_3}{I_6} p_6 - \frac{R_{10}}{I_8} p_8 \quad (\text{C-76})$$

The flow returned by element p_8 is the flow source Sf of the commodity- and labor market, and the flow source Sf_2 of the product- and capital asset market. The markets and firm can be merged in a single state-space model by replacing these flow sources with $f = \frac{1}{I_8} p_8$:

$$\dot{x} = [\dot{q}_1 \quad \dot{p}_1 \quad \dot{q}_2 \quad \dot{q}_3 \quad \dot{p}_2 \quad \dot{q}_4 \quad \dot{p}_3 \quad \dot{p}_4 \quad \dot{q}_5 \quad \dot{p}_5 \quad \dot{p}_6 \quad \dot{q}_6 \quad \dot{p}_7 \quad \dot{p}_8]^T \quad (\text{C-77})$$

The A matrix is structured as follows:

$$\begin{bmatrix}
-\frac{1}{R_1 C_1} & -\frac{1}{I_1} & 0 & 0 & 0 & 0 & 0 & \text{zeros}(6) & 0 \\
\frac{1}{C_1} & 0 & -\frac{1}{C_2} & 0 & 0 & 0 & 0 & \text{zeros}(6) & 0 \\
0 & \frac{1}{I_1} & -\frac{1}{R_2 C_2} & 0 & 0 & 0 & 0 & \text{zeros}(6) & -\frac{1}{I_8} \\
\text{zeros}(3) & -\frac{1}{R_3 C_3} & -\frac{1}{I_2} & 0 & 0 & 0 & 0 & \text{zeros}(4) & 0 \\
\text{zeros}(3) & \frac{1}{C_3} & 0 & -\frac{GY_1}{I_3} & 0 & 0 & 0 & \text{zeros}(4) & 0 \\
\text{zeros}(3) & 0 & \frac{GY_1}{I_2} & -\frac{R_4}{I_3} & -\frac{1}{C_4} & 0 & 0 & \text{zeros}(4) & 0 \\
\text{zeros}(3) & 0 & 0 & \frac{1}{I_3} & -\frac{1}{R_5 C_4} & 0 & 0 & \text{zeros}(4) & -\frac{1}{TF_1 I_8} \\
\text{zeros}(6) & -\frac{GY_2^2}{R_6 I_4} & \frac{GY_2}{I_5} & 0 & 0 & 0 & 0 & 0 & -\frac{GY_2 TF_2}{I_4} \\
\text{zeros}(6) & 0 & -\frac{1}{R_4 C_5} & -\frac{1}{I_5} & 0 & 0 & 0 & 0 & 0 \\
\text{zeros}(6) & -\frac{GY_2}{I_4} & 0 & \frac{1}{C_5} & 0 & 0 & 0 & 0 & 0 \\
\text{zeros}(6) & 0 & 0 & 0 & 0 & -\frac{GY_3^2}{R_8 I_6} & \frac{GY_3}{I_7} & 0 & -\frac{GY_3 TF_3}{I_6} \\
\text{zeros}(6) & 0 & 0 & 0 & 0 & 0 & -\frac{1}{R_9 C_6} & -\frac{1}{I_7} & 0 \\
\text{zeros}(6) & 0 & 0 & 0 & 0 & -\frac{GY_3}{I_6} & 0 & \frac{1}{C_6} & 0 \\
\text{zero}(2) & \frac{1}{C_2} & \text{zeros}(3) & \frac{1}{TF_1 C_4} & \frac{TF_2 GY_2}{I_4} & \text{zeros}(2) & \frac{TF_3 GY_3}{I_6} & \text{zeros}(2) & -\frac{R_{10}}{I_6}
\end{bmatrix}$$

A

(C-78)

The inputs are the flow- and effort input of the product market $u = [Sf_1 \ Se_1]^T$, and the flow- and effort input of the capital asset market $u = [Sf_2 \ Se_2]^T$. This gives the following input vector:

$$u = \begin{bmatrix} Sf_1 \\ Se_1 \\ Sf_2 \\ Se_2 \end{bmatrix} \tag{C-79}$$

Then the B matrix belonging to the system is as follows:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{bmatrix}}_B \tag{C-80}$$

The C matrix can be shaped to provide the desired information. For example, one could derive the costs and benefits of the markets, leading to the following output vector:

$$y = \begin{bmatrix} e_{cons} \\ e_{cap} \\ e_{comm} \\ e_{lab} \end{bmatrix} \tag{C-81}$$

The C matrix is then:

$$\underbrace{\begin{bmatrix} 0 & \frac{1}{C_2} & \text{zeros}(11) \\ \text{zeros}(5) & \frac{TF_1 C_2}{I_4} & \text{zeros}(8) \\ \text{zeros}(7) & \frac{TF_2 G Y_2}{I_4} & \text{zeros}(7) \\ \text{zero}(10) & \frac{TF_3 G Y_3}{I_4} & \text{zeros}(3) \end{bmatrix}}_C \quad (\text{C-82})$$

The D matrix is a 4×4 matrix of zeros, as no outputs are directly linked to the effort and flow sources.

C-2 Bicomplex Hamiltonian Derivation

In this section, I derive the bicomplex Hamiltonians and state-space equations of the three damped harmonic oscillators (DHOs) following Hutter [1].

C-2-1 Basic Formulas

Bicomplex Coordinates

$$x_j = \sqrt{\frac{I_i \omega_k}{2}} \left(q_j + \frac{i}{I_i \omega_k} p_j \right) \quad (\text{C-83})$$

$$\bar{x}_j = \sqrt{\frac{I_i \omega_k}{2}} \left(q_j - \frac{i}{I_i \omega_k} p_j \right) \quad (\text{C-84})$$

Regular Coordinates

$$q_j = \sqrt{\frac{1}{2I_i \omega_k}} (\bar{x} + x) \quad (\text{C-85})$$

$$p_j = i \sqrt{\frac{I_i \omega_k}{2}} (\bar{x} - x) \quad (\text{C-86})$$

Constants

$$\omega_j = \sqrt{\frac{1}{I_i C_k}} \quad (\text{C-87})$$

$$\beta_j = \frac{R_j}{2I_k} \quad (\text{C-88})$$

$$\beta_j = \frac{1}{2C_i R_j} \quad (\text{C-89})$$

$$\begin{aligned}
\dot{\bar{x}} &= \sqrt{\frac{I\omega}{2}} (\dot{q}(x, \bar{x}) - \frac{i}{I\omega} \dot{p}(x, \bar{x})) \\
&= \sqrt{\frac{I\omega}{2}} \left(-\frac{1}{CR} \sqrt{\frac{1}{2I\omega}} (\bar{x} + x) + \frac{i}{I} \sqrt{\frac{I\omega}{2}} (\bar{x} - x) + \frac{i}{IC\omega} \sqrt{\frac{1}{2I\omega}} (\bar{x} + x) \right) \\
&= \sqrt{\frac{I\omega}{2}} \left(-\frac{1}{CR} \sqrt{\frac{1}{2I\omega}} - \frac{i}{I} \sqrt{\frac{I\omega}{2}} + \frac{i}{IC\omega} \sqrt{\frac{1}{2I\omega}} \right) x \\
&\quad + \sqrt{\frac{I\omega}{2}} \left(-\frac{1}{CR} \sqrt{\frac{1}{2I\omega}} + \frac{i}{I} \sqrt{\frac{I\omega}{2}} + \frac{i}{IC\omega} \sqrt{\frac{1}{2I\omega}} \right) \bar{x} \\
\dot{\bar{x}} &= -\beta x - (\beta - i\omega) \bar{x}
\end{aligned} \tag{C-95}$$

$$\begin{bmatrix} \dot{x} \\ \dot{\bar{x}} \end{bmatrix} = \begin{bmatrix} -(\beta + i\omega) & -\beta \\ -\beta & -(\beta - i\omega) \end{bmatrix} \begin{bmatrix} x \\ \bar{x} \end{bmatrix} \tag{C-96}$$

Equating the right hand side to the general expression of the bicomplex state variables yields:

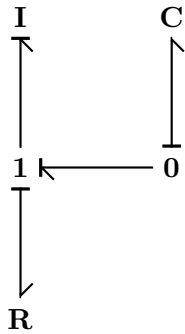
$$\mathcal{J} \begin{bmatrix} \frac{\partial \mathcal{H}}{\partial \bar{x}} \\ \frac{\partial \mathcal{H}}{\partial x} \end{bmatrix} = \begin{bmatrix} -(\beta + i\omega) & -\beta \\ -\beta & -(\beta - i\omega) \end{bmatrix} \begin{bmatrix} x \\ \bar{x} \end{bmatrix} \tag{C-97}$$

The complex Hamiltonian is obtained through integration:

$$\mathcal{H}_1 = \omega x \bar{x} - i\beta x \bar{x} - i\beta \left(\frac{x^2 - \bar{x}^2}{2} \right) \tag{C-98}$$

C-2-4 DHO in Parallel: $\mathcal{H}_2$

Bondgraph Description



Regular Hamiltonian and State-Space

$$\mathcal{H}_2 = \frac{p^2}{2I} + \frac{Cq^2}{2} \quad (\text{C-99})$$

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & 1/I \\ -1/C & -R/I \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix} \quad (\text{C-100})$$

System Dynamics in Bicomplex Coordinates

$$\dot{q}(x, \bar{x}) = \frac{i}{I} \sqrt{\frac{I\omega}{2}} (\bar{x} - x) \quad (\text{C-101})$$

$$\dot{p}(x, \bar{x}) = -\frac{1}{C} \sqrt{\frac{1}{2I\omega}} (\bar{x} + x) - \frac{iR}{I} \sqrt{\frac{I\omega}{2}} (\bar{x} - x) \quad (\text{C-102})$$

Bicomplex State Equations

$$\begin{aligned} \dot{x} &= \sqrt{\frac{I\omega}{2}} \left(\dot{q}(x, \bar{x}) + \frac{i}{I\omega} \dot{p}(x, \bar{x}) \right) \\ &= \sqrt{\frac{I\omega}{2}} \left(\frac{i}{I} \sqrt{\frac{I\omega}{2}} (\bar{x} - x) - \frac{i}{IC\omega} \sqrt{\frac{1}{2I\omega}} (\bar{x} + x) + \frac{R}{I^2\omega} \sqrt{\frac{I\omega}{2}} (\bar{x} - x) \right) \\ &= \sqrt{\frac{I\omega}{2}} \left(-\frac{i}{I} \sqrt{\frac{I\omega}{2}} - \frac{i}{IC\omega} \sqrt{\frac{1}{2I\omega}} - \frac{R}{I^2\omega} \sqrt{\frac{I\omega}{2}} \right) x \\ &\quad + \sqrt{\frac{I\omega}{2}} \left(\frac{i}{I} \sqrt{\frac{I\omega}{2}} - \frac{i}{IC\omega} \sqrt{\frac{1}{2I\omega}} + \frac{R}{I^2\omega} \sqrt{\frac{I\omega}{2}} \right) \bar{x} \\ \dot{x} &= -(i\omega + \beta)x + \beta\bar{x} \end{aligned} \quad (\text{C-103})$$

$$\begin{aligned} \dot{\bar{x}} &= \sqrt{\frac{I\omega}{2}} \left(\dot{q}(x, \bar{x}) - \frac{i}{I\omega} \dot{p}(x, \bar{x}) \right) \\ &= \sqrt{\frac{I\omega}{2}} \left(i \sqrt{\frac{\omega}{2I}} (\bar{x} - x) - \frac{i}{I\omega} \left(-\frac{1}{C} \sqrt{\frac{1}{2I\omega}} (\bar{x} + x) - \frac{Ri}{I} \sqrt{\frac{I\omega}{2}} (\bar{x} - x) \right) \right) \\ &= \sqrt{\frac{I\omega}{2}} \left(-i \sqrt{\frac{\omega}{2I}} + \frac{i}{IC\omega} \sqrt{\frac{1}{2I\omega}} + \frac{R}{I^2\omega} \sqrt{\frac{I\omega}{2}} \right) x \\ &\quad + \sqrt{\frac{I\omega}{2}} \left(i \sqrt{\frac{\omega}{2I}} + \frac{i}{IC\omega} \sqrt{\frac{1}{2I\omega}} - \frac{R}{I^2\omega} \sqrt{\frac{I\omega}{2}} \right) \bar{x} \\ \dot{\bar{x}} &= \beta x - (-i\omega + \beta)\bar{x} \end{aligned} \quad (\text{C-104})$$

$$\begin{bmatrix} \dot{x} \\ \dot{\bar{x}} \end{bmatrix} = \begin{bmatrix} -(\beta + i\omega) & \beta \\ \beta & -(\beta - i\omega) \end{bmatrix} \begin{bmatrix} x \\ \bar{x} \end{bmatrix} \quad (\text{C-105})$$

Equating the right hand side to the general expression of the bicomplex state variables yields:

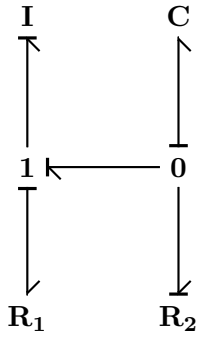
$$\mathcal{J} \begin{bmatrix} \frac{\partial \mathcal{H}}{\partial \bar{x}} \\ \frac{\partial \mathcal{H}}{\partial x} \end{bmatrix} = \begin{bmatrix} -(\beta + i\omega) & \beta \\ \beta & -(\beta - i\omega) \end{bmatrix} \begin{bmatrix} x \\ \bar{x} \end{bmatrix} \quad (\text{C-106})$$

The complex Hamiltonian is obtained through integration:

$$\mathcal{H}_2 = \omega x \bar{x} - i\beta x \bar{x} + i\beta \left(\frac{x^2 - \bar{x}^2}{2} \right) \quad (\text{C-107})$$

C-2-5 Double-Damped DHO: $\mathcal{H}_3$

Bond-Graph Description



Regular Hamiltonian and State-Space

$$\mathcal{H}_3 = \frac{p^2}{2I} + \frac{Cq^2}{2} \quad (\text{C-108})$$

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} -1/CR_2 & 1/I \\ -1/C & -R_1/I \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix} \quad (\text{C-109})$$

System Dynamics in Bicomplex Coordinates

$$\dot{q}(x, \bar{x}) = -\frac{1}{CR_2} \sqrt{\frac{1}{2I\omega}} (\bar{x} + x) + \frac{i}{I} \sqrt{\frac{I\omega}{2}} (\bar{x} - x) \quad (\text{C-110})$$

$$\dot{p}(x, \bar{x}) = -\frac{1}{C} \sqrt{\frac{1}{2I\omega}} (\bar{x} + x) - \frac{iR_1}{I} \sqrt{\frac{I\omega}{2}} (\bar{x} - x) \quad (\text{C-111})$$

C-2-6 Bicomplex State Equations

$$\begin{aligned}
\dot{x} &= \sqrt{\frac{I\omega}{2}}(\dot{q}(x, \bar{x}) + \frac{i}{I\omega}\dot{p}(x, \bar{x})) \\
&= \sqrt{\frac{I\omega}{2}} \left(-\frac{1}{CR_2}\sqrt{\frac{1}{2I\omega}}(\bar{x} + x) + \frac{i}{I}\sqrt{\frac{I\omega}{2}}(\bar{x} - x) \right. \\
&\quad \left. - \frac{i}{IC\omega}\sqrt{\frac{1}{2I\omega}}(\bar{x} + x) + \frac{R_1}{I^2\omega}\sqrt{\frac{I\omega}{2}}(\bar{x} - x) \right) \\
&= \sqrt{\frac{I\omega}{2}} \left(-\frac{1}{CR_2}\sqrt{\frac{1}{2I\omega}} - \frac{i}{I}\sqrt{\frac{I\omega}{2}} - \frac{i}{IC\omega}\sqrt{\frac{1}{2I\omega}} - \frac{R_1}{I^2\omega}\sqrt{\frac{I\omega}{2}} \right) x \\
&\quad + \sqrt{\frac{I\omega}{2}} \left(-\frac{1}{CR_2}\sqrt{\frac{1}{2I\omega}} + \frac{i}{I}\sqrt{\frac{I\omega}{2}} - \frac{i}{IC\omega}\sqrt{\frac{1}{2I\omega}} + \frac{R_1}{I^2\omega}\sqrt{\frac{I\omega}{2}} \right) \bar{x} \\
\dot{x} &= (-\beta_2 - \beta_1 - i\omega)x + (-\beta_2 + \beta_1)\bar{x}
\end{aligned} \tag{C-112}$$

$$\begin{aligned}
\dot{\bar{x}} &= \sqrt{\frac{I\omega}{2}}(\dot{q} - \frac{i}{I\omega}\dot{p}(x, \bar{x})) \\
&= \sqrt{\frac{I\omega}{2}} \left(-\frac{1}{CR_2}\sqrt{\frac{1}{2I\omega}}(\bar{x} + x) + \frac{i}{I}\sqrt{\frac{I\omega}{2}}(\bar{x} - x) \right. \\
&\quad \left. + \frac{i}{IC\omega}\sqrt{\frac{1}{2I\omega}}(\bar{x} + x) - \frac{R_1}{I^2\omega}\sqrt{\frac{I\omega}{2}}(\bar{x} - x) \right) \\
&= \sqrt{\frac{I\omega}{2}} \left(-\frac{1}{CR_2}\sqrt{\frac{1}{2I\omega}} - \frac{i}{I}\sqrt{\frac{I\omega}{2}} + \frac{i}{IC\omega}\sqrt{\frac{1}{2I\omega}} + \frac{R_1}{I^2\omega}\sqrt{\frac{I\omega}{2}} \right) x \\
&\quad + \sqrt{\frac{I\omega}{2}} \left(-\frac{1}{CR_2}\sqrt{\frac{1}{2I\omega}} + \frac{i}{I}\sqrt{\frac{I\omega}{2}} + \frac{i}{IC\omega}\sqrt{\frac{1}{2I\omega}} - \frac{R_1}{I^2\omega}\sqrt{\frac{I\omega}{2}} \right) \bar{x} \\
\dot{\bar{x}} &= (-\beta_2 + \beta_1)x + (-\beta_2 - \beta_1 + i\omega)\bar{x}
\end{aligned} \tag{C-113}$$

$$\begin{bmatrix} \dot{x} \\ \dot{\bar{x}} \end{bmatrix} = \begin{bmatrix} -(\beta_2 + \beta_1 + i\omega) & -(\beta_1 - \beta_2) \\ -(\beta_1 - \beta_2) & -(\beta_2 + \beta_1 - i\omega) \end{bmatrix} \begin{bmatrix} x \\ \bar{x} \end{bmatrix} \tag{C-114}$$

Equating the right hand side to the general expression of the bicomplex state variables yields:

$$\mathcal{J} \begin{bmatrix} \frac{\partial \mathcal{H}}{\partial \bar{x}} \\ \frac{\partial \mathcal{H}}{\partial x} \end{bmatrix} = \begin{bmatrix} -(\beta_2 + \beta_1 + i\omega) & -(\beta_1 - \beta_2) \\ -(\beta_1 - \beta_2) & -(\beta_2 + \beta_1 - i\omega) \end{bmatrix} \begin{bmatrix} x \\ \bar{x} \end{bmatrix} \tag{C-115}$$

The complex Hamiltonian is obtained through integration:

$$\mathcal{H}_3 = \omega x \bar{x} - i(\beta_2 + \beta_1)x \bar{x} - i(\beta_1 - \beta_2)\left(\frac{x^2 - \bar{x}^2}{2}\right) \tag{C-116}$$

Bibliography

- [1] M. Mendel C. Hutters. *Overcoming the dissipation obstacle with bicomplex port-Hamiltonian Mechanics*. Delft Center for Systems and Control, 2020.
- [2] D. C. Karnopp et al. *System dynamics: Modeling, simulation, and control of mechatronic systems*. John Wiley & Sons, Inc., 2012.

Appendix D

Development of a Complex Discounting Method for Cash Flows

In this section, a complex discounting method for cash flows is developed. The complex discount rate consists of a real and a complex part: The real part accounts for the exponential decay of the value of cash flows, which corresponds to the currently used discount rate. The complex part accounts for the cyclicity of economic markets, cyclical debt and equity issuance, and cyclical financial and fiscal constraints (Section 2-2-3). The discounting method has the following form:

$$X(s) = \int_0^{\infty} x(t)D(t)dt \quad (\text{D-1})$$

Where $x(t) \in \mathbb{R}$ is a continuous cash flow, and $D(t)$ is the discount function with $t \in \mathbb{R}$. The discount function aggregates cyclical and non-cyclical risk functions as follows:¹

$$D(t) = \prod_{n=1}^N d_n \quad (\text{D-2})$$

with $d_n(t)$ the n th risk function, and N the number of risk functions. Risk functions have the following form²:

$$d_n(t) = e^{a_n + b_n it} + c_n \quad (\text{D-3})$$

By setting constraints on parameters a_n , b_n , and c_n , risk functions can represent either cyclical or non-cyclical risk. With Equation D-2, different risk functions are aggregated in a complex discount function.

¹For aggregating risk functions, I use the following formula for multiplying exponential functions: $(a_1 e^{b_1 i})(a_2 e^{b_2 i}) = a_1 a_2 e^{(b_1 + b_2)i}$ [4].

²This risk function uses the complex description for the motion of a particle [4]: $M(t) = ae^{bit} + c$.

D-1 Non-Cyclical Risk as a Complex Exponential

This risk function performs exponential discounting by using the following values:³

$$\begin{aligned} a_n &= -1, \quad b_n \leq 0, \quad c_n = 0 \\ \text{with } a_n, c_n &\in \mathbb{R} \quad \text{and} \quad b_n \in \mathbb{I} \end{aligned} \quad (\text{D-4})$$

The regular exponential discounting function is compared with the complex transform with values satisfying Equation D-4 in Figure D-1:

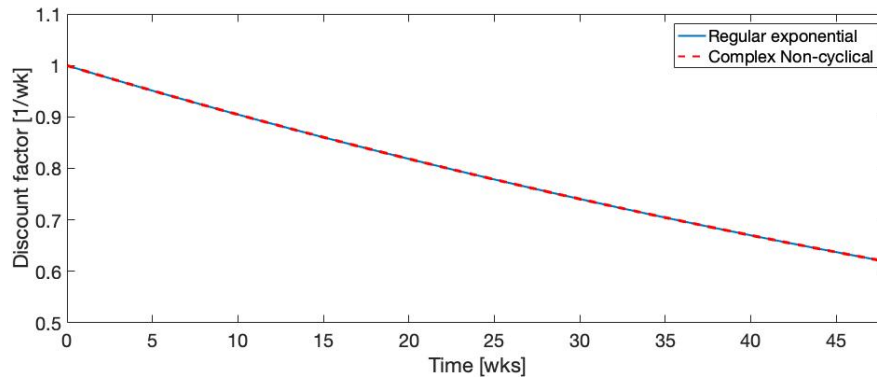


Figure D-1: Comparison of the regular exponential discounting function to the complex exponential for non-cyclical risk shows that the pair can be equal under the conditions of Equation D-4.

D-2 Cyclical Risk as a Complex Exponential

Complex exponentials for cyclical risk functions have to satisfy two conditions. Firstly, the value of money needs to decrease over time [1, 6]. Therefore, the risk function's value is constrained between zero and one:⁴

$$0 \leq d_n \leq 1 \quad (\text{D-5})$$

Secondly, the present value of a cash flow paid out now needs to be equal to the undiscounted value [1, 6]. The risk function at time zero therefore needs to satisfy:

$$d_n(0) = 1 \quad (\text{D-6})$$

Both conditions are incorporated in the discounting framework by adopting the following constraints:

³Using the code displayed in Appendix F.

⁴This equation uses the complex description for the motion of particles and a modulus smaller than one: $|a| \leq 0$ [4].

$$\begin{aligned}
a_n &= -\ln\left(\frac{1}{y_n}\right) \quad \text{with } 0 \leq y_n \leq 0.5 \\
c_n &= 1 - y_n \\
\text{with } a_n, b_n, c_n &\in \mathbb{R}
\end{aligned} \tag{D-7}$$

Parameter a_n represents the amplitude of the risk function. The constraint ensures that the function does not surpass the maximum amplitude of unity imposed by Equation D-5. b_n gives the frequency of the cyclical risk. The parameter can take any real value without violating the two above-mentioned constraints. c_n gives the risk function an initial offset. The constraint ensures that the risk function satisfies Equation D-6.

Figure D-2 shows the constraints on a_n and b_n graphically.

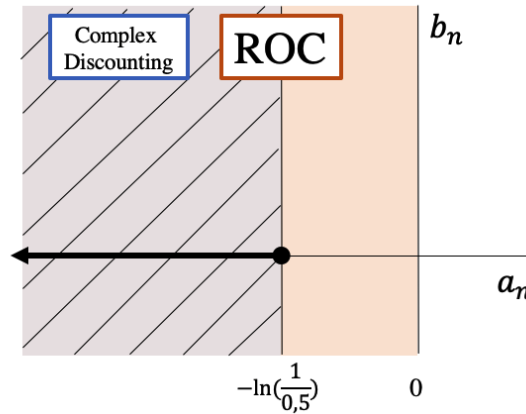


Figure D-2: The constraints on a_n and b_n lie inside the region of convergence (ROC) of the complex exponential. As a result, the transform always ascribes a countable value to businesses.

Transforms for business valuations must converge, to ascribe a countable value to the business. The plane spanned by a_n and b_n is congruent to the s -plane known from the Laplace transform, as the parameters represent the real and imaginary part of the exponential. An exponential is convergent if s lies within the region of convergence (ROC): the left-half plane. The constraints on the complex discounting function ensure that the parameters in the exponential lie inside the ROC. Therefore, the transforms always converge, and no additional constraints are needed [4].

Figure D-3 shows how Equation D-3 is used to mimic the cyclical risk functions. As an example, the risk functions represent risk imposed by fiscal policy, innovation shocks and credit risk [2, 5, 3].

The cyclical risk function allows for modeling cyclical risks, while satisfying the constraints on the regular discounting rate. As a result, the cyclical behavior that is seen in economic markets, debt and equity issuance, and financial and fiscal constraints can be adopted in the discount rate (Section 2-2-3).

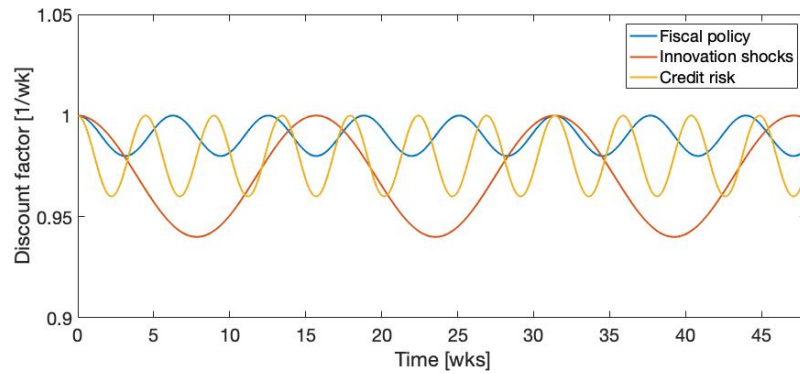


Figure D-3: Cyclical risk functions defined by Equation D-3 and following the constraints of Equation D-7 allow to specify the amplitude with parameter a_n and frequency with parameter b_n .

D-3 A Cyclical Discount Rate as Aggregated Complex Exponentials

Cyclical and non-cyclical risk functions are aggregated using the following function:⁵

$$D(t) = \prod_{n=1}^N d_n \quad (\text{D-8})$$

Figure D-4 shows the aggregated discount function that combines the non-cyclical and cyclical risk function of Figures D-1 and D-3. Note that the discount function can be used both continuously and discretely.

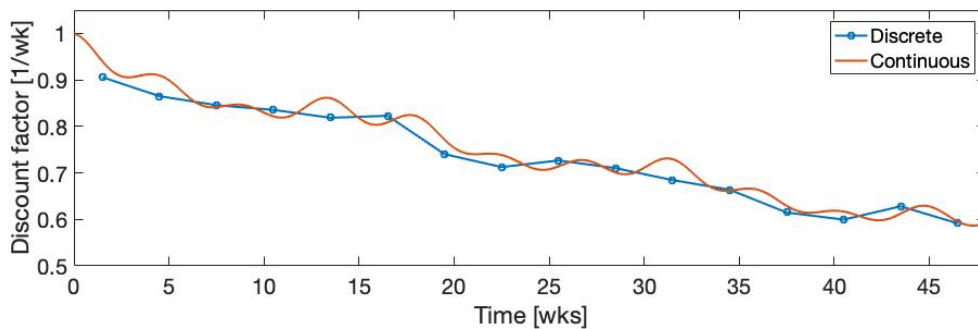


Figure D-4: Aggregation of cyclical and non-cyclical risk functions produces the cyclical discount function $D(t)$. The function allows for continuous and discrete discounting and helps investors decide on the length of their investment horizon.

The ability to perform discounting both continuously and discretely is key to comparing the economic-engineering valuation method to current practices. The model created in this thesis predicts cash flows continuously, and therefore is to be discounted continuously. However, current valuation practice forecast cash flows on a yearly or quarterly basis, thereby using a

⁵Also Equation D-2.

discrete discounting approach [1, 6].

The complex discounting transform surpasses currently existing discounting techniques, as it mimics the real-valued discount rate and adds to it the imaginary-valued rate for cyclical risk. The dynamics of this rate contain crucial information on the riskyness of investments in certain moments in time. It can help investors to decide on the moment to buy or sell their shares in businesses, while accounting for the cyclic behavior of the economy. As a result, the investor can decide on the optimal investment horizon [6].

Complex discounting of cash flows can be used without the need for the dynamical model of Chapter 4. However, this discounting method is not analogous to the Laplace transform. This precludes the expression of business value in terms of the discount rate s . In the next section, I show how this analog is preserved by performing signal-based discounting.

Summarizing, I state the complex transform for cyclical discounting:

Complex transform for cyclical discounting

The present value X of cash flow $x(t)$ is calculated as follows:

$$X = \int_0^{\infty} x(t)D(t)dt \quad (\text{D-9})$$

Using discount function:

$$D(t) = \prod_{n=1}^N d_n \quad (\text{D-10})$$

With N risk functions of the following form:

$$d_n(t) = e^{a_n + b_n it} + c_n \quad (\text{D-11})$$

Non-cyclical risk functions satisfying:

$$\begin{aligned} a_n &= -1, \quad b_n \leq 0, \quad c_n = 0 \\ \text{with } a_n, c_n &\in \mathbb{R} \quad \text{and } b_n \in \mathbb{I} \end{aligned} \quad (\text{D-12})$$

And cyclical risk functions satisfying:

$$\begin{aligned} a_n &= -\ln\left(\frac{1}{y_n}\right) \quad \text{with } 0 \leq y_n \leq 0.5 \\ c_n &= 1 - y_n \\ \text{with } a_n, b_n, c_n &\in \mathbb{R} \end{aligned} \quad (\text{D-13})$$

Bibliography

- [1] T. Koller et al. *Valuations*. John Wiley and Sons Inc, 2015.
- [2] W. J. Den Haan F. Covas. *The cyclical behavior of debt and equity finance*. American Economic Review, 2011.
- [3] F. Heid. *Cyclical implications of minimum capital requirements*. Deutsche Bundesbank, 2005.
- [4] T. Needham. *Visual Complex Analysis*. Oxford University Press, 1997.
- [5] J. Greenwood P. Gomme. *On the cyclical allocation of risk*. Elsevier, 1993.
- [6] T. Rietveld. *Handboek Investeren & Financieren*. Vakmedianet Management B.V., 2017.

Appendix E

Simulink Models

E-1 Full Model

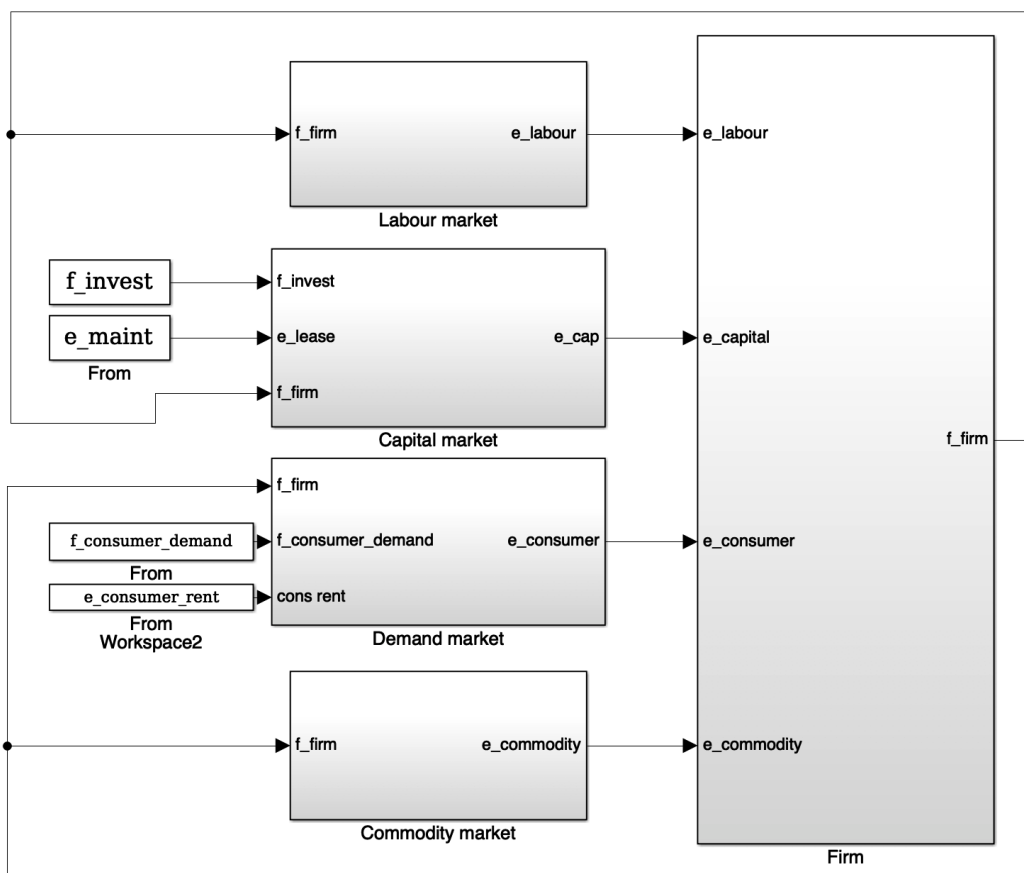


Figure E-1: Simulink model of the full system

E-2 Commodity Market

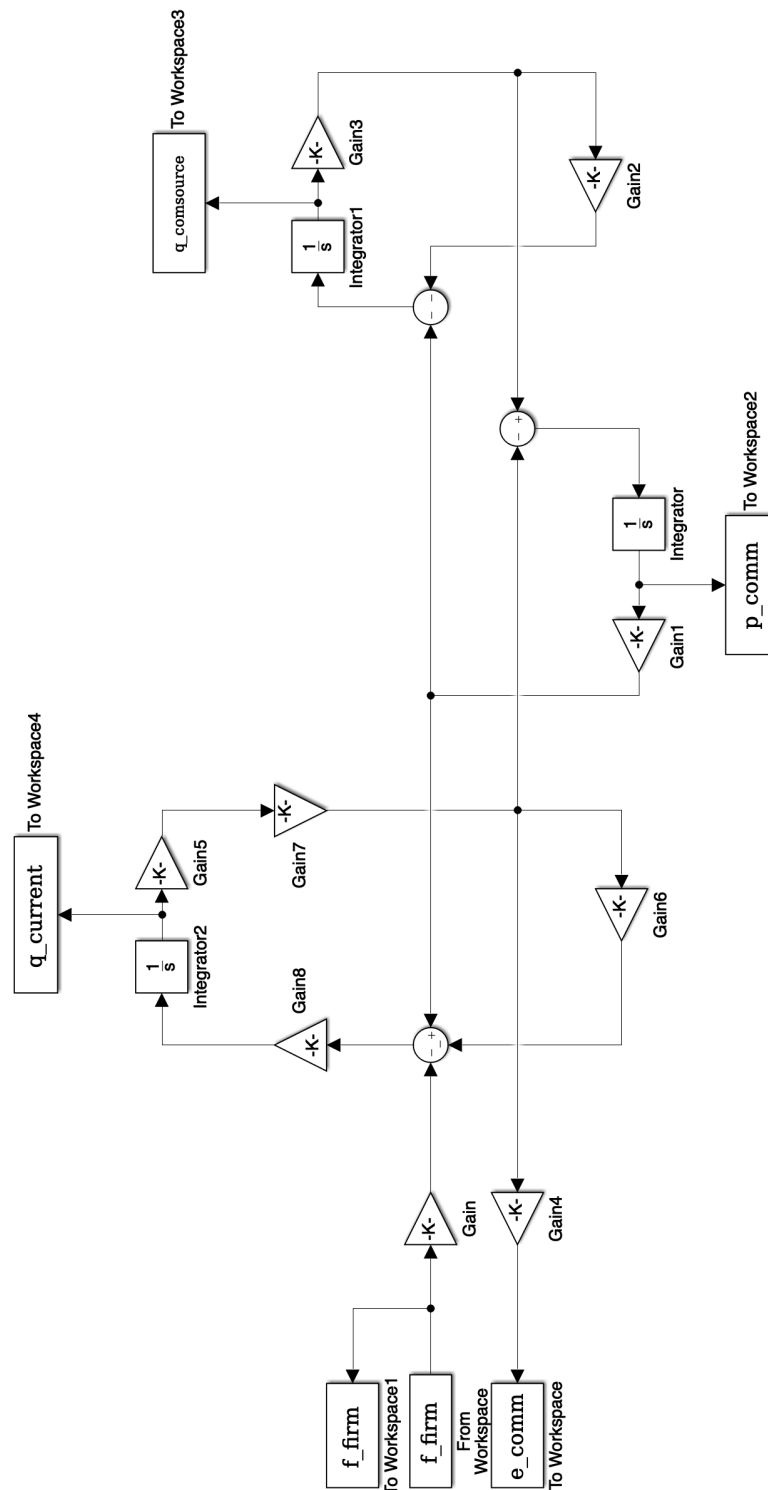


Figure E-2: Simulink model of the commodity market

E-3 Labor Market

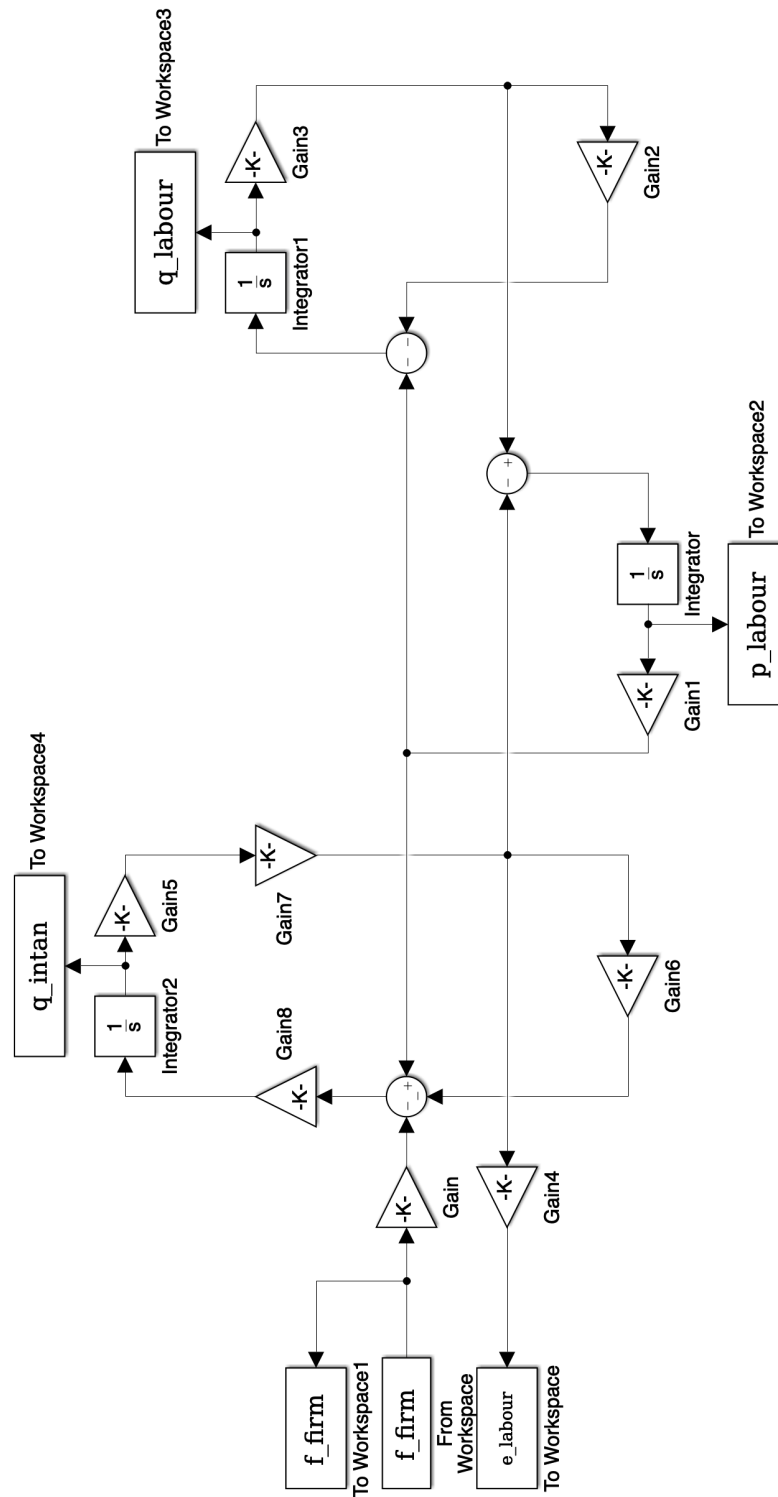


Figure E-3: Simulink model of the labor market

E-4 Capital Asset Market

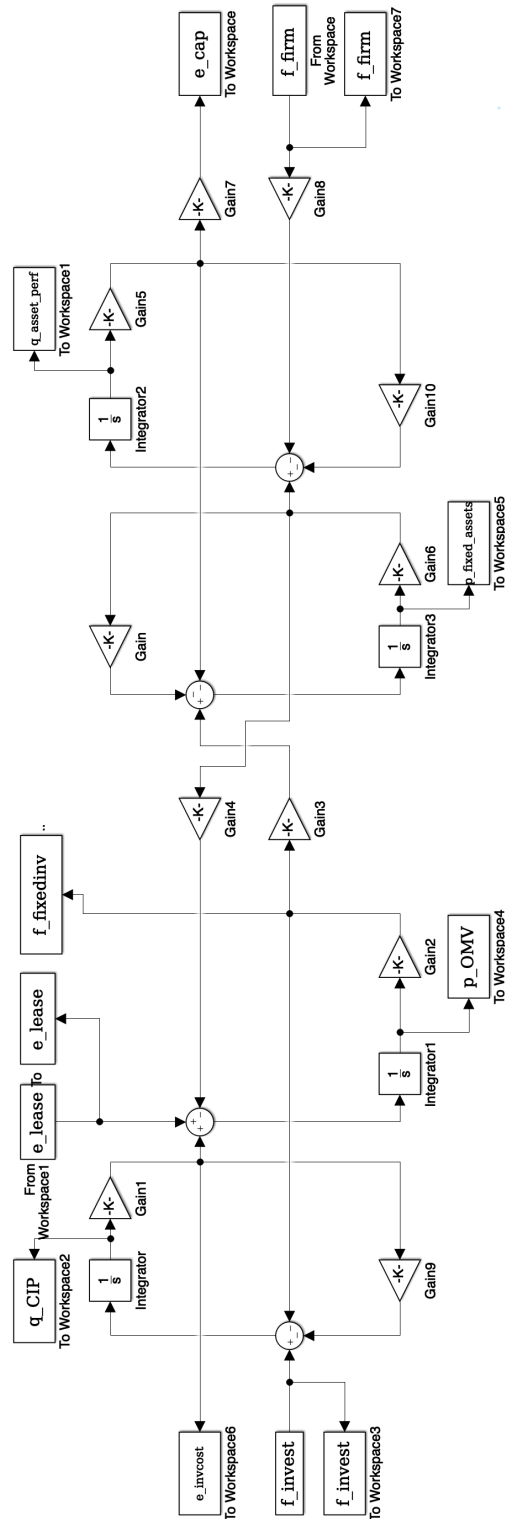


Figure E-4: Simulink model of the capital asset market

E-5 Product Market

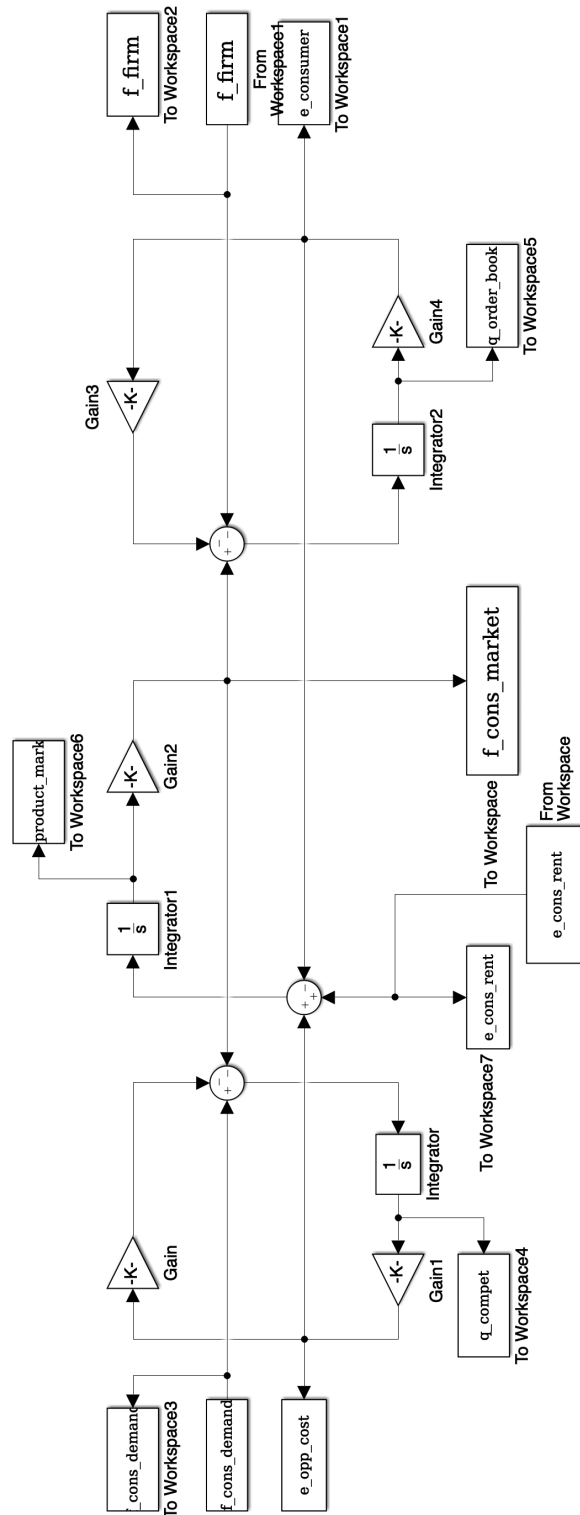


Figure E-5: Simulink model of the product market

E-6 Firm

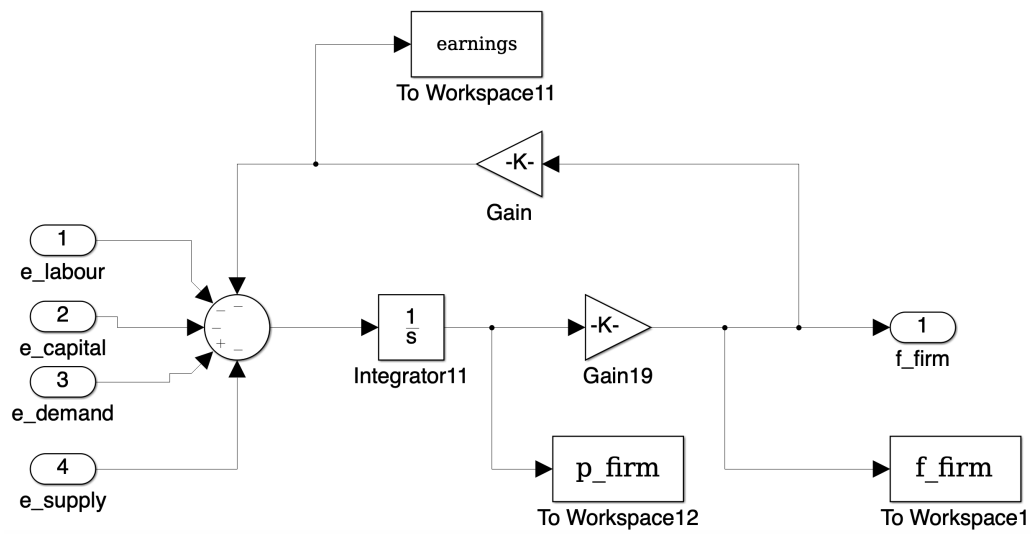


Figure E-6: Simulink model of the firm

Appendix F

Matlab Code

```
1 %MATLAB SCRIPT THESIS XAVIER VAN ARDENNE
2 %2019-2020
3 %MASTERS SYSTEMS AND CONTROL
4 %TU DELFT
5
6 clear; clc; close all;
7
8 %% SIMULINK MODEL
9
10 %Inputs
11 T_end = 30;
12 N = 10*T_end;
13 t = linspace(0,T_end,N);
14 u_step = ones(1,N);
15 u_sine = sin(t);
16 u_zeros = ones(1,N);
17
18 u_cons_dem = 2*u_sine;
19 u_cons_rent = 2*u_step;
20 u_e_inv = 2*u_sine;
21 u_f_inv = 2*u_step;
22
23 f_consumer_demand = [t; u_cons_dem]';
24 e_consumer_rent = [t; u_cons_rent]';
25 e_maint = [t; u_e_inv]';
26 f_invest = [t; u_f_inv]';
27
28 % Parameters
29
30 %Labor
31 m_labour = 3;
32 C_labourmarket = 1.2;
33 m_intangibles = 1.6;
```

```
34 R_labourmarket = 0.1;
35 R_amor = 2;
36 TF_labour_eff = 2.8;
37 GY_human_prod = 0.8;
38
39 %Commodity
40 m_commodities = 1.1;
41 C_comsource = 5;
42 I_current = 3;
43 R_depl = 3;
44 R_storage = 0.2;
45 TF_comm_product = 3.2;
46 GY_stock_eff = 1.2;
47
48 % Consumer
49 m_product_market = 2.1;
50 C_compet = 4.1;
51 C_order_book = 0.5;
52 R_consump = 2.1;
53 R_cancel = 0.2;
54
55 %Capital
56 C_CIP = 2.5;
57 C_asset_perf = 6.1;
58 m_OMV = 1.4;
59 m_fixed_assets = 8.3;
60 R_impair = 1.2;
61 R_riskfree = 2.1;
62 TF_FAT = 1.3;
63 GY_asset_act = 1.1;
64 R_levi = 0.11;
65
66 % Firm
67 m_firm = 2;
68 R_earnings = 2;
69
70 sim("full_model4.slx",T_end)
71
72 % Data extraction
73
74 %Outputs
75 e_consumer_vec = e_consumer.data;
76 e_labour_vec = e_labour.data;
77 e_commodity_vec = e_commodity.data;
78 e_capital_vec = e_capital.data;
79
80 e_net_vec = e_net.data;
81 f_firm_vec = f_firm.data;
82
83 e_opp_cost_vec = e_opp_cost.data;
84 e_invcost_vec = e_invcost.data;
85 f_fixedinv_vec = f_fixedinv.data;
86
```

```

87 %Inputs
88 f_consumer_demand_vec = f_consumer_demand.data;
89 e_consumer_rent_vec = e_consumer_rent.data;
90 e_lease_vec = e_lease.data;
91 f_invest_vec = f_invest.data;
92
93 % States
94
95 %Capital asset
96 q_CIP_vec = q_CIP.data;
97 q_asset_perf_vec = q_asset_perf.data;
98 p_invest_vec = p_OMV.data;
99 p_fixed_assets_vec = p_fixed_assets.data;
100
101 %Labor
102 p_labour_vec = p_labour.data;
103 q_labour_source_vec = q_labour.data;
104 q_intan_vec = q_intan.data;
105
106 %Commodity
107 p_comm_vec = p_comm.data;
108 q_comsource_vec = q_comsource.data;
109 q_current_vec = q_current.data;
110
111 %Product
112 p_product_market_vec = p_product_market.data;
113 q_order_book_vec = q_order_book.data;
114 q_compet_vec = q_compet.data;
115
116 p_firm_vec = p_firm.data;
117 earnings_vec = earnings.data;
118
119 %% STATE-SPACE MODEL
120
121 %Capital asset
122 C3 = C_CIP;
123 C4 = C_asset_perf;
124 I2 = m_OMV;
125 I3 = m_fixed_assets;
126 R3 = R_riskfree;
127 R4 = R_impair;
128 R5 = R_levi;
129 GY1 = GY_asset_act;
130 TF1 = TF_FAT;
131
132 A_cap = [-1/(R3*C3)    -1/I2    0    0;
133          1/C3    0    0    -GY1/(I3);
134          0    0    -1/(C4*R5)    1/I3;
135          0    GY1/(I2)    -1/C4    -R4/(I3)    ];
136 B_cap = [1 0; 0 1; 0 0; 0 0];
137 C_cap = [0 0 1/(TF1*C4) 0];
138 D_cap = zeros(1,2);
139

```

```

140 capital_sys = ss(A_cap,B_cap,C_cap,D_cap);
141
142 % Product
143 C1 = C_compet;
144 I1 = m_product_market;
145 C2 = C_order_book;
146 R1 = R_consump;
147 R2 = R_cancel;
148
149 A_cons = [-1/(R1*C1) -1/I1 0;
150           1/C1 0 -1/C2;
151           0 1/I1 -1/(R2*C2)];
152 B_cons = [1 0 ;0 1 ;0 0];
153 C_cons = [0 0 1/C2];
154 D_cons = zeros(1,2);
155
156 consumer_sys = ss(A_cons,B_cons,C_cons,D_cons);
157
158 % Commodity
159 I4 = I_current;
160 C5 = C_comsource;
161 I5 = m_commodities;
162 R6 = R_storage;
163 R7 = R_depl;
164 TF2 = TF_comm_product;
165 GY2 = GY_stock_eff;
166
167 A_comm = [-GY2^2/(I4*R6) 0 -GY2/I5 ;
168           0 -1/(R7*C5) 1/I5;
169           GY2/I4 -1/C5 0];
170 B_comm = [TF2*GY2;0;0];
171 C_comm = [(TF2*GY2)/I4 0 0];
172 D_comm = 0;
173
174 commodity_sys = ss(A_comm,B_comm,C_comm,D_comm);
175
176 % Labor
177 I6 = m_intangibles;
178 I7 = m_labour;
179 C6 = C_labourmarket;
180 R8 = R_amor;
181 R9 = R_labourmarket;
182 TF3 = TF_labour_eff;
183 GY3 = GY_human_prod;
184
185 A_lab = [-GY3^2/(I6*R8) 0 -GY3/I7 ;
186           0 -1/(R9*C6) 1/I7;
187           GY3/I6 -1/C6 0];
188 B_lab = [TF3*GY3;0;0];
189 C_lab = [(TF3*GY3)/I6 0 0];
190 D_lab = 0;
191
192 labour_sys = ss(A_lab,B_lab,C_lab,D_lab);

```

```

193
194 % Firm
195 I8 = m_firm;
196 R10 = R_earnings;
197
198 % Full model
199 A_full = [-1/(R1*C1) -1/I1 0 zeros(1,11);
200           1/C1 0 -1/C2 zeros(1,11);
201           0 1/I1 -1/(R2*C2) zeros(1,10) -1/I8;
202
203           zeros(1,3) -1/(R3*C3) -1/I2 0 0 zeros(1,7);
204           zeros(1,3) 1/C3 0 0 -GY1/(I3) zeros(1,7);
205           zeros(1,3) 0 0 -1/(C4*R5) 1/I3 zeros(1,6) -1/(TF1*I8);
206           zeros(1,3) 0 GY1/I2 -1/C4 -R4/I3 zeros(1,7) ;
207
208           zeros(1,7) -GY2^2/(R6*I4) 0 -GY2/I5 zeros(1,3) TF2*GY2/I8;
209           zeros(1,7) 0 -1/(R7*C5) 1/I5 zeros(1,4);
210           zeros(1,7) GY2/I4 -1/C5 0 zeros(1,4);
211
212           zeros(1,10) -GY3^2/(R8*I6) 0 -GY3/I7 GY3*TF3/I8;
213           zeros(1,10) 0 -1/(R9*C6) 1/I7 0;
214           zeros(1,10) GY3/I6 -1/C6 0 0;
215
216           0 0 1/C2 0 0 1/(TF1*C4) 0 -GY2*TF2/I4 0 0 -TF3*GY3/I6 0 0 -R10/I8];
217
218 B_full = [1 0 0 0;
219           0 1 0 0;
220           0 0 0 0;
221           0 0 1 0;
222           0 0 0 1;
223           zeros(9,4)];
224
225 C_full = [zeros(1,13) 1/I8;
226           zeros(1,2) 1/C2 zeros(1,11);
227           zeros(1,5) 1/(C4*TF1) zeros(1,8);
228           zeros(1,7) TF2*GY2/I4 zeros(1,6);
229           zeros(1,10) TF3*GY3/I6 zeros(1,3)];
230
231 D_full = zeros(5,4);
232
233 u_full = [u_cons_dem;
234           u_cons_rent;
235           u_f_inv;
236           u_e_inv];
237
238 x0_full = zeros(12,0);
239
240 full_sys = ss(A_full,B_full,C_full,D_full);
241 [Y_full,T,X_full] = lsim(full_sys,u_full,t,x0_full);
242
243 %% ANALYSIS
244
245 % Pole-zero

```

```

246 clc
247
248 tf_full = tf(full_sys);
249 poles_lab = pole(tf_full);
250 zeros_lab = zero(tf_full);
251
252 figure();
253 pzmap(full_sys)
254 %grid on
255
256 % Bode
257 TF_full = tf(full_sys)
258 figure(); % Consumer demand to costs and benefits
259 hold on
260 bode(TF_full(2,1))
261 bode(TF_full(3,1))
262 bode(TF_full(4,1))
263 bode(TF_full(5,1))
264 legend('Benefit orders','Economies of scale','Production cost','Unit
        labor cost')
265
266 figure(); % Fixed investments or leases
267 hold on
268
269 set(gca,'fontsize',30)
270 bode(TF_full(1,3))
271 bode(TF_full(1,4))
272 legend('Fixed investments','Capital leases')
273
274 %% BICOMPLEX HAMILTONIAN SYSTEM
275
276 J = [0 -i;
277       i 0];
278
279 %H1H3
280
281 Wn1 = sqrt(1/(C1*I1));
282 beta1 = 1/(2*C1*R1);
283
284 Wn3 = sqrt(1/(C2*I8));
285 beta3 = ((R10)/(2*(I8)));
286 beta3a = 1/(2*C2*R2);
287
288 % Input matrices
289
290 % effort in
291 g1e= [ i*sqrt(1/(2*I1*Wn1));
292        -i*sqrt(1/(2*I1*Wn1)) ];
293
294 % flow in
295 g1f = [sqrt(I1*Wn1/2);
296         sqrt(I1*Wn1/2) ];
297

```

```

298 % effort in
299 g3e= [ i*sqrt(1/(2*I8*Wn3));
300        -i*sqrt(1/(2*I8*Wn3)) ];
301
302 % flow in
303 g3f = [ sqrt((I8*Wn3)/2);
304        sqrt((I8*Wn3)/2) ];
305
306 % Interconnection matrix
307 Js13= [-J -g1e*transpose(g3f);
308        g3f*transpose(g1e) -J];
309
310 Jr13= [ J zeros(2,2);
311        zeros(2,2) J ];
312
313 % Del(H1)/ x1
314 dS1 = [0 Wn1;
315        Wn1 0];
316
317 dR1 = [+i*beta1 (+i*beta1);
318        (-i*beta1) -i*beta1];
319
320 % Del(H2)/x2
321 dS3 = [0 Wn3;
322        Wn3 0];
323
324 dR3 = [-i*beta3 (+i*beta3);(-i*beta3) i*beta3]+[i*beta3a (+i*beta3a);(-i*
        beta3a) -i*beta3a];
325
326 % Del(Htot)* xt看
327 dS13 = [dS1 zeros(2,2);
328         zeros(2,2) dS3];
329
330 dR13 = [dR1 zeros(2,2);
331         zeros(2,2) dR3];
332
333 Ac13 = (Js13)*(dS13)+Jr13*dR13;
334
335 %H2H4
336
337 beta2 = 1/(2*C3*R3);
338 Wn2 = sqrt(1/(I2*C3));
339 beta4a = 1/(2*C4*R5);
340 beta4 = R4/(2*I3);
341 Wn4 = sqrt(1/(I3*C4));
342
343 %input matrices
344 ge2= [ i*sqrt(1/(2*I2*Wn2));
345        -i*sqrt(1/(2*I2*Wn2)) ];
346 gf2 = [ sqrt(I2*Wn2/2);
347        sqrt(I2*Wn2/2) ];
348 ge4= [ i*sqrt(1/(2*I3*Wn4));
349        -i*sqrt(1/(2*I3*Wn4)) ];

```

```

350 gf4 = [sqrt(I3*Wn4/2);
351         sqrt(I3*Wn4/2)];
352
353 %del(H1)/x1
354 dS2 = [0 Wn2;
355        Wn2 0];
356
357 dR2 = [i*beta2 (i*beta2);(-i*beta2) -i*beta2];
358
359 %del(H2)/x2
360 dS4 = [0 Wn4;
361        Wn4 0];
362
363 dR4 = [-i*beta4 (i*beta4);(-i*beta4) i*beta4]+[i*beta4a (i*beta4a);(-i*
        beta4a) -i*beta4a];
364
365 % Del(Htot)* xtot
366 dS24 = [dS2 zeros(2,2);
367         zeros(2,2) dS4];
368
369 dR24 = [dR2 zeros(2,2);
370         zeros(2,2) dR4];
371
372 %Interconnection matrix
373 Js24= [-J -GY1*ge2*transpose(ge4);
374        GY1*ge4*transpose(ge2) J];
375
376 Jr24= [J zeros(2,2);
377        zeros(2,2) J];
378
379 Ac24 = (Js24)*(dS24)+Jr24*dR24;
380
381 %H5H7
382
383 beta7 = 1/(2*C5*R7);
384 Wn7 = sqrt(1/(I5*C5));
385
386 % effort in
387 ge7 = [ i*sqrt(1/(2*I5*Wn7));
388        -i*sqrt(1/(2*I5*Wn7))];
389
390 % flow in
391 gf7 = [sqrt(I5*Wn7/2);
392        sqrt(I5*Wn7/2)];
393
394 %Del(Htot)*xtot
395 dS5 = 1;
396 dS7 = [0 Wn7;
397        Wn7 0];
398
399 dS57 = [dS5 zeros(1,2);
400         zeros(2,1) dS7];
401

```

```

402 % Interconnection matrix
403 Js57 = [0 -GY2*transpose(ge7);
404         GY2*ge7/I4 J];
405
406 Jr57 = [1 zeros(1,2);
407         zeros(2,1) J];
408
409 dR5 = -GY2^2/(R6*I4);
410 dR7 = [i*beta7 (i*beta7);(-i*beta7) -i*beta7];
411 dR57 = [dR5 zeros(1,2);
412         zeros(2,1) dR7];
413
414 A_comm_bicomplex = (Js57)*(dS57)+Jr57*dR57;
415
416 %H6H8
417
418 beta8 = 1/(2*C6*R9);
419 Wn8 = sqrt(1/(I7*C6));
420
421 % effort in
422 ge8 = [ i*sqrt(1/(2*I7*Wn8));
423        -i*sqrt(1/(2*I7*Wn8)) ];
424
425 % flow in
426 gf8 = [sqrt(I7*Wn8/2);
427        sqrt(I7*Wn8/2) ];
428
429 %Del(Htot)*xtot
430 dS6 = 1;
431 dS8 = [0 Wn8;
432        Wn8 0];
433
434 dS68 = [dS6 zeros(1,2);
435         zeros(2,1) dS8];
436
437 % Interconnection matrix
438 Js68 = [0 -GY3*transpose(ge8);
439         GY3*ge8/I6 J];
440
441 Jr68 = [1 zeros(1,2);
442         zeros(2,1) J];
443
444 dR6 = -GY3^2/(R8*I6);
445 dR8 = [i*beta8 (i*beta8);(-i*beta8) -i*beta8];
446 dR68 = [dR6 zeros(1,2);
447         zeros(2,1) dR7];
448
449 A_comm_bicomplex = (Js68)*(dS68)+Jr68*dR68;
450
451 B_comm_bicomplex = [TF3*GY3;
452                     0;
453                     0];
454

```

```

455 C_comm_bicomplex = [(TF3*GY3)/I6 0 0];
456
457 D_comm_bicomplex = 0;
458
459 dSfull = [dS1 zeros(2,12);
460           zeros(2) dS3 zeros(2,10);
461           zeros(2,4) dS2 zeros(2,8);
462           zeros(2,6) dS4 zeros(2,6);
463           zeros(1,8) dS5 zeros(1,5);
464           zeros(2,9) dS7 zeros(2,3);
465           zeros(1,11) dS6 zeros(1,2);
466           zeros(2,12) dS8];
467 %
468 dRfull = [dR1 zeros(2,12);
469           zeros(2) dR3 zeros(2,10);
470           zeros(2,4) dR2 zeros(2,8);
471           zeros(2,6) dR4 zeros(2,6);
472           zeros(1,8) dR5 zeros(1,5);
473           zeros(2,9) dR7 zeros(2,3);
474           zeros(1,11) dR6 zeros(1,2);
475           zeros(2,12) dR8];
476
477 Jrfull = [J zeros(2,12);
478           zeros(2) J zeros(2,10);
479           zeros(2,4) J zeros(2,8);
480           zeros(2,6) J zeros(2,6);
481           zeros(1,8) 1 zeros(1,5);
482           zeros(2,9) J zeros(2,3);
483           zeros(1,11) 1 zeros(1,2);
484           zeros(2,12) J];
485
486 Jsfull = [-J -g1e*transpose(g3f) zeros(2,10);
487           g3f*transpose(g1e) -J zeros(2) (1/TF1)*g3e*transpose(gf4) -
488           TF2*GY2*g3e/I4 zeros(2) -TF3*GY3*g3e/I6 zeros(2);
489           zeros(2) zeros(2) -J -GY1*ge2*transpose(ge4) zeros(2,6);
490           zeros(2) -(1/TF1)*gf4*transpose(g3e) GY1*ge4*transpose(ge2) J
491           zeros(2,6);
492           zeros(1,2) TF2*GY2*transpose(g3e) zeros(1,5) -GY2*transpose(
493           ge7) zeros(1,3);
494           zeros(2,8) GY2*ge7/I4 J zeros(2,3);
495           zeros(1,2) TF3*GY3*transpose(g3e) zeros(1,8) -GY3*
496           transpose(ge8);
497           zeros(2,11) GY3*ge8/I6 J];
498
499 Acfull = (Jsfull)*(dSfull)+Jrfull*dRfull;
500
501 [eig(Acfull) eig(A_full)]
502
503 Bcfull = [gif g1e zeros(2);
504           zeros(2,4);
505           zeros(2) gf2 ge2;
506           zeros(8,4)];
507
508 Ccfull = [zeros(1,2) transpose(g3e) zeros(1,10);

```



```

504         zeros(1,2) transpose(g3f) zeros(1,10);
505         zeros(1,6) 1/TF1*transpose(gf4) zeros(1,6);
506         zeros(1,8) TF2*GY2/I4 zeros(1,5);
507         zeros(1,11) TF3*GY3/I6 zeros(1,2)];
508
509 sysc= ss(Acfull,Bcfull,[Ccfull*(dSfull)],0);
510 sysr= ss(A_full,B_full,C_full,0);
511
512 %Initial states
513
514 %regular
515 qs01 = 0; qs02 = 0; qs03 = 0; qs04 = 0; qs05 = 0; qs06 = 0;
516
517 ps01 = 0; ps02 = 0; ps03 = 0; ps04 = 0; ps05 = 0; ps06 = 0; ps07 = 0;
    ps08 = 0;
518
519 x0 = [qs01;ps01;qs02;qs03;ps02;qs04;ps03;ps04;qs05; ps05; ps06;qs06;ps07;
    ps08];
520
521 %Ladder
522 a01 = sqrt(I1*Wn1/2) *(qs01 + i/(Wn1*I1)*ps01);
523 a03 = sqrt(I8*Wn3/2) *(qs02 + i/(Wn3*I8)*ps08);
524 a02 = sqrt(I2*Wn2/2) *(qs03 + i/(Wn2*I2)*ps02);
525 a04 = sqrt(I3*Wn4/2) *(qs04 + i/(Wn4*I3)*ps03);
526 a05 = ps04;
527 a06 = ps06;
528 a07 = sqrt(I5*Wn7/2) *(qs05 + i/(Wn7*I5)*ps05);
529 a08 = sqrt(I7*Wn8/2) *(qs06 + i/(Wn8*I7)*ps07);
530 a0 = [a01; conj(a01);a03; conj(a03); a02; conj(a02);a04; conj(a04
    ); a05; a07; conj(a07); a06; a08; conj(a08)];
531
532 [ycfull,tt,xcfull] = lsim(sysc,u_full,t,a0);
533 [yrfull,tt,xrfull] = lsim(sysr,u_full,t,x0);
534
535 %Plots
536
537 % Hamiltonians for cash flow dynamics
538
539 %H1
540 H1c = (Wn1-i*beta1)*xcfull(:,1).*xcfull(:,2)+i/2*beta1*(xcfull(:,1).^2-
    xcfull(:,2).^2);
541 H1cSt = (Wn1)*xcfull(:,1).*xcfull(:,2);
542 H1cCo = -i*beta1*xcfull(:,1).*xcfull(:,2);
543 H1cSq = -i/2*beta1*(xcfull(:,1).^2-xcfull(:,2).^2);
544
545 H1r = (1/(C1*2))*xrfull(:,1).^2 + xrfull(:,2).^2/(2*I1);
546
547 %H3
548 H3c = (Wn3-i*(beta3+beta3a))*xcfull(:,3).*xcfull(:,4)-i/2*(beta3-beta3a)
    *(xcfull(:,3).^2-xcfull(:,4).^2);
549 H3cSt = (Wn3)*xcfull(:,3).*xcfull(:,4);
550 H3cCoq = -i*(beta3)*xcfull(:,3).*xcfull(:,4);
551 H3cCop = -i*(beta3a)*xcfull(:,3).*xcfull(:,4);

```

```

552 H3cSqq = -i/2*(beta3a)*(xcfull(:,3).^2-xcfull(:,4).^2);
553 H3cSqp = -i/2*(-beta3a)*(xcfull(:,3).^2-xcfull(:,4).^2);
554
555 H3r = (1/(C2*2))*xrfull(:,3).^2 + xrfull(:,14).^2/(2*I8);
556
557 %% COMPLEX DISCOUNTING
558 clear; close all; clc;
559
560 % Exponential vs hyperbolic
561 %time
562 T = 48;
563 t = 0:0.01:T;
564 Te = length(t);
565
566 %Parameters
567 r = [0.1:0.15:0.9];
568 N = length(r);
569 y = zeros(N,T);
570
571 %Discrete DCF
572 clearvars -EXCEPT t T Te
573
574 Q = 3;
575 Y = 12;
576 D = .05;
577
578 d = D/2:D:T-D/2;
579 y = Y/2:Y:T-Y/2;
580 q = Q/2:Q:T-Q/2;
581 FCF = @ (t) 10 +0.1*t+ 6*sin(0.4*t);
582
583 r = 0.01;
584 ry = r*Y;
585 rq = r*Q;
586 rd = r*D;
587
588 dcf_func_y = exp(-ry*[1:length(y)]);
589 dcf_func_q = exp(-rq*[1:length(q)]);
590 dcf_func_d = exp(-rd*[1:length(d)]);
591
592 risk1 = riskf(0.01,0.05,1:length(d));
593 risk2 = riskf(0.03,0.02,1:length(d));
594 risk3 = riskf(0.02,0.07,1:length(d));
595 risk4 = riskf(1,-rd*i,1:length(d));
596 risktotal = risk1.*risk2.*risk3.*risk4;
597
598 risk1d = riskd(0.01,0.05*Q/D,1:length(q));
599 risk2d = riskd(0.03,0.02*Q/D,1:length(q));
600 risk3d = riskd(0.02,0.07*Q,1:length(q));
601 risk4d = riskd(1,rq*i,1:length(q));
602 risktotald = risk1d.*risk2d.*risk3d.*risk4d;
603
604 for l = 1:T/Y

```

```

605     CF_an(1) = integral(FCF,Y*(1-1),Y*1) ;
606     NPV_an(1) = dcf_func_y(1)*CF_an(1);
607 end
608
609 for k = 1:T/Q
610     CF_mo(k) = integral(FCF,Q*(k-1),Q*k) ;
611     NPV_mo(k) = dcf_func_q(k)*CF_mo(k);
612     CD_mo_const(k) = risk4d(k)*CF_mo(k);
613     CD_mo_total(k) = risktotald(k)*CF_mo(k);
614 end
615
616 for n = 1:T/D
617     n1(n) = n;
618     CF_da(n) = integral(FCF,D*(n-1),D*n) ;
619     NPV_da(n) = dcf_func_d(n)*CF_da(n);
620     CD_da_const(n) = risk4(n)*CF_da(n);
621     CD_da_total(n) = risktotal(n)*CF_da(n);
622 end
623
624 [sum(CF_an) sum(CF_mo) sum(CF_mo) sum(CF_mo) sum(CF_da) sum(CF_da) sum(
    CF_da);
625 sum(NPV_an) sum(NPV_mo) sum(CD_mo_const) sum(CD_mo_total) sum(NPV_da)
    sum(CD_da_const) sum(CD_da_total)]
626
627 %Discrete complex
628 clearvars -EXCEPT t T Te
629
630 Q = 3;
631 Y = 12;
632 D = .05;
633
634 d = D/2:D:T-D/2;
635 y = Y/2:Y:T-Y/2;
636 q = Q/2:Q:T-Q/2;
637 FCF = @(t) 10 +0.1*t+ 6*sin(0.4*t);
638
639 r = 0.01;
640 ry = r*Y;
641 rq = r*Q;
642 rd = r*D;
643
644 dcf_func_y = exp(-ry*[1:length(y)]);
645 dcf_func_q = exp(-rq*[1:length(q)]);
646 dcf_func_d = exp(-rd*[1:length(d)]);
647
648
649
650 for l = 1:T/Y
651     CF_an(l) = integral(FCF,Y*(l-1),Y*l) ;
652     NPV_an(l) = dcf_func_y(l)*CF_an(l);
653 end
654
655 for k = 1:T/Q

```

```

656     CF_mo(k) = integral(FCF,Q*(k-1),Q*k) ;
657     NPV_mo(k) = dcf_func_q(k)*CF_mo(k);
658
659 end
660
661 for n = 1:T/D
662     n1(n) = n;
663     CF_da(n) = integral(FCF,D*(n-1),D*n) ;
664     NPV_da(n) = dcf_func_d(n)*CF_da(n);
665     % CD_da_const(n) = risk4(n)*CF_da(n);
666     % CD_da_total(n) = risktotal(n)*CF_da(n);
667 end
668
669 [sum(CF_an) sum(CF_mo) sum(CF_da) sum(CF_da) sum(CF_da);
670  sum(NPV_an) sum(NPV_mo) sum(NPV_da) sum(CD_da_const) sum(CD_da_total)]
671
672 %% riskf.m function
673
674 function r = riskf(ampl,freq,t)
675 off = 1-ampl;
676 a = log(1/ampl);
677 r = off +exp(-a-freq*li*t);
678 end

```