Output Regulation of Nonlinear Systems in a Koopman Operator Framework

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Abstract

This thesis considers the problem of nonlinear output regulation in a Koopman operator framework. The goal of output regulation is to asymptotically track a reference and/or simultaneously reject a disturbance signal, both generated by some external autonomous system called the exosystem. The nonlinear output regulation problem is solvable if and only if a set of partial differential equations (PDE) are satisfied. From the solution, a feedback law can be obtained that achieves output regulation. However, solving the PDE is difficult. In this thesis, we instead aim to construct a feedback law by utilizing the Koopman operator instead.

The Koopman operator associated with a state-space model of a (nonlinear) dynamical system describes the evolution of functions of the states, called observable functions, by propagating the state forward in time according to the flow of the system, and evaluating this at each possible observable function. The space of observables is an infinite-dimensional vector field. Therefore, the Koopman operator is infinite-dimensional and linear. The Koopman operator of an autonomous system associated with a nonlinear control system provides a bilinear description of the control system instead.

The use of the Koopman operator to tackle the output regulation problem has not been done before in the literature. We identify conditions under which the Koopman operator can be used to rephrase the nonlinear output regulation problem as a bilinear output regulation problem. We then show when the bilinear output regulation problem is solved using linear dynamic error feedback. In particular, a Lyapunov-based approach is used to characterize a set of initial conditions for which the output is regulated. Finally, to verify the results, a numerical example is presented.

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Chapter 1

Introduction

This thesis considers the nonlinear output regulation problem in a Koopman operator framework. The output regulation problem is a well-known known problem in control theory. The goal is to asymptotically track a reference signal and/or simultaneously reject a disturbance signal. Both signals are generated by an external autonomous dynamical system called the exosystem. It has been applied to various dynamical systems. For instance, in [18] the output regulation framework is used for disturbance rejection in formation keeping control of non-holonomic wheeled robots. In the work [16] the output regulation is considered in the context of biology and neuroscience.

A special case of the output regulation problem is the linear output regulation problem. It concerns a linear plant with linear output and a linear exosystem. The linear output regulation problem has been studied by [11, 12], and [10] provides necessary and sufficient conditions for its solvability. Solving the linear output regulation problem requires a solution to a set of linear matrix equations called the regulator equations (and internal model principle). From these equations, a linear dynamic error feedback controller can be derived that solves the linear output regulation problem. The nonlinear output regulation problem considers a more general class of systems and exosystems. Namely, one where the system, system output, and/or exosystem are nonlinear. Necessary and sufficient conditions for its solvability are given in [17]. They show that a set of partial differential equations must be satisfied. From the solution, a controller may be synthesized. However, solving the PDE is hard. In this thesis, we instead take the novel approach of utilizing the Koopman operator to tackle the nonlinear output regulation problem. The Koopman operator provides an alternative way to model (nonlinear) dynamical systems. It was originally introduced by Bernard Koopman in [24, 25] and popularized for the study of dynamical systems by [29, 30]. The Koopman operator provides a framework to better deal with the complexity of nonlinear systems. Furthermore, it is particularly well suited for data-driven methods. It has already seen numerous applications in a wide range of topics. A few are mentioned here.

In [28], a relationship between Koopman eigenfunctions and nonlinear observability is established. In [26], the Koopman operator is utilized to allow for linear model predictive control methods to be used on nonlinear systems. In [27], the well-known Extended dynamic mode

decomposition algorithm [34] is used to perform linear system identification of nonlinear systems. The work [4] uses the Koopman operator in the analysis of chaotic systems. On the more practical side, [1] employed the Koopman operator to generate a model and controller in real-time to stabilize a faulty quadcopter.
The Koopman operator, associated with a (state-space) description of a nonlinear dynamical system, is a linear and infinite-dimensional operator that acts on functions of the state of the system, often called observables or observable functions. The action of the Koopman operator on such observables amounts to evaluating the observable at the state advanced in time according to the dynamics (flow) of the system. Since projections to the states of the underlying nonlinear system are observable functions themselves, full knowledge of the Koopman operator allows one to compute trajectories of the system through linear dynamics. Its infinite-dimensional linear nature stems from the space of observables $\mathcal{F}$, which is a linear vector space consisting of all possible observable functions of the state.
The infinite-dimensional nature of the Koopman operator limits its practical use. However, for some systems, finite-dimensional subsets of $\mathcal{F}$, that include the projections to the state exist, such that the action of the Koopman operator on any observable of that set yields an observable that is also in that set, resulting in a finite-dimensional linear description. In general, given a set of observables, an observable for which the action of the Koopman operator yields an observable that is in the given set is called invariant under the action of the Koopman operator. A set of observables for which each element is invariant under the action of the Koopman operator is called a Koopman invariant subspace [5]. The dynamics on a finite-dimensional Koopman invariant subspace that includes the state projections provide an equivalent, finite-dimensional linear description of a nonlinear system in the state-space description. A trivial example of such a set is the set of state projections for an LTI system. The class of nonlinear systems that admit such a description is limited [5]. Hence, to utilize the Koopman operator description, we often have to find approximate Koopman invariant subspaces instead. We thus have to find a set of observable functions, often called a dictionary or lifting, that approximately captures the Koopman dynamics. Finding a suitable dictionary depends on the system of interest and effective ways to find it is an open problem. For control affine systems, the Koopman operator associated with the related unforced system will result in an equivalent bilinear dynamical system instead [13]. As such, the nonlinear output regulation may be tackled by considering the related bilinear output regulation problem. To solve the latter, we take inspiration from the solution of the linear output regulation problem and make use of linear dynamic error feedback. A bilinear system under such control leads to a quadratic system. This thesis presents a stability analysis of the resulting quadratic system that utilizes quadratic Lyapunov functions and linear matrix inequalities, based on the work of [21, 22]. The analysis allows for the characterization of a set of initial conditions for which output regulation is achieved. Finally, a numerical example highlighting the problem is presented to verify the results.

Thesis structure

In chapter 2, we cover the preliminaries of the thesis. This includes the Koopman operator and some relevant theoretical background, as well as the nonlinear and linear output regulation problem. It also provides a description of a Lyapunov-based technique for the analysis of quadratic systems. Chapter 3 contains the main contributions of this thesis. First, the problem is formulated, and the nonlinear output regulation problem is related to the bilinear output regulation problem. We then consider the bilinear output regulation problem for linear dynamic error feedback and use this to tackle the output regulation problem for the special case of matched input disturbance for nonlinear systems and verify the result in a numerical example. Finally, we suggest future directions for the research on the utilization of the Koopman operator in nonlinear output regulation. The thesis is concluded in chapter 4.

Chapter 2

Preliminaries

2-1 Koopman operator theory

In this section, we introduce the Koopman operator. The Koopman operator is an infinite-dimensional linear operator that describes the dynamics of functions of the state of dynamical systems. Therefore, it provides an alternative description of the dynamical system. In particular, it allows a certain class of nonlinear systems to be approximated by finite, albeit higher, dimensional linear dynamical systems. Nonlinear control systems, on the other hand, at best admit to descriptions of bilinear in the input.
The following text is based primarily on [2, 5, 6]

2-1-1 Koopman operator

Consider the autonomous continuous-time dynamical system described by the ordinary differential equation

$$\dot{x} = f(x), \tag{2-1}$$

where $x \in X \subseteq \mathbb{R}^n$, and $f : X \to X$ a smooth function assumed to be Lipschitz. Integrating (2-1) yields the flow $\boldsymbol{F}^t : X \mapsto X$ of the system, that is,

$$x(t) = \boldsymbol{F}^t(x_0) \quad \text{with} \quad \boldsymbol{F}^t(x_0) = x_0 + \int_0^t f(x(\tau))d\tau. \tag{2-2}$$

Equation (2-2) describes a trajectory starting from the initial condition $x(0) = x_0$. Note that in general, we cannot analytically solve equation (2-1), and we have to resort to numerical integration methods to determine the flow (2-2), e.g., Runge-Kutta method. The Koopman operator allows us to circumvent this by looking at how functions of the state evolve in time instead.

We now turn our attention to the definition of the Koopman operator. To this end, we consider functions of the state $\psi : X \to \mathbb{R}$, which are called observable functions. Denote the space of all such functions, with possible constraints, as $\mathcal{F}$. The family of Koopman operators $\mathcal{K}^t : \mathcal{F} \mapsto \mathcal{F}$, parameterized by t, is defined by

$$\mathcal{K}^t \psi(x) = \psi(x) \circ \boldsymbol{F}^t = \psi(\boldsymbol{F}^t(x)). \tag{2-3}$$

Since $\mathcal{K}^{t_2}(\mathcal{K}^{t_1}\psi(x)) = \psi(\boldsymbol{F}^{t_1+t_2}(x))$, we simply write $\mathcal{K}$ and refer to it as the Koopman operator. The action of the Koopman operator on ψ thus amounts to propagating the state forward according to the flow $\boldsymbol{F}$ and then evaluating ψ at the new state. Since $\mathcal{F}$ contains an infinite number of elements, the Koopman operator is an infinite-dimensional operator. Furthermore, it is a bounded linear operator. Linearity follows from the fact that $\mathcal{F}$ is assumed to be a linear vector space of functions. Indeed, let $\psi_1, \psi_2 \in \mathcal{F}$ and $a \in \mathbb{C}$, then

$$\mathcal{K}(a\psi_1(x) + \psi_2(x)) = a\psi_1(\boldsymbol{F}(x)) + \psi_2(\boldsymbol{F}(x)) = a\mathcal{K}\psi_1(x) + \mathcal{K}\psi_2(x)). \tag{2-4}$$

Next, we define the infinitesimal generator $\mathcal{L}$ of the Koopman operator. Suppose $\psi \in \mathcal{F}$, the action of $\mathcal{L}$ on ψ is defined by

$$\mathcal{L}\psi = \lim_{t\to 0} \frac{\mathcal{K}^t\psi - \psi}{t} = \lim_{t\to 0} \frac{\psi(\boldsymbol{F}^t(x)) - \psi(x)}{t}. \tag{2-5}$$

From its definition, we see that $\mathcal{L}$ corresponds to the time derivative of ψ along the trajectories of (2-1). Therefore, we write

$$\dot{\psi} = \mathcal{L}\psi. \tag{2-6}$$

This expression is reminiscent of the standard expression of LTI systems. Indeed, from the linearity of the Koopman operator, it follows that the infinitesimal generator $\mathcal{L}$ is a linear operator. If $\psi_1, \psi_2 \in \mathcal{F}$ and $a \in \mathbb{C}$, then

$$\begin{aligned}\mathcal{L}(a\psi_1 + \psi_2) &= \lim_{t\to 0} \frac{\mathcal{K}^t(a\psi_1 + \psi_2) - (a\psi_1 + \psi_2)}{t} \\ &= \lim_{t\to 0} a\frac{\mathcal{K}^t\psi_1 - \psi_1}{t} + \lim_{t\to 0} \frac{\mathcal{K}^t\psi_2 - \psi_2}{t} \\ &= a\mathcal{L}\psi_1 + \mathcal{L}\psi_2,\end{aligned} \tag{2-7}$$

where in the second line, we used the linearity of the Koopman operator and the linearity of the limit.

For each $\psi \in \mathcal{F}$ equation (2-5) holds. If we order all ψ in an infinite-dimensional column vector, we can think of $\mathcal{L}$ defining the infinite-dimensional matrix of a linear dynamical system. That is,

$$\frac{d}{dt}\begin{bmatrix}\psi_1 \\ \psi_2 \\ \vdots\end{bmatrix} = \begin{bmatrix}\mathcal{L} & 0 & \dots & \\ 0 & \mathcal{L} & \dots & \\ \vdots & \vdots & \ddots & \end{bmatrix}\begin{bmatrix}\psi_1 \\ \psi_2 \\ \vdots\end{bmatrix} = \begin{bmatrix}a_{11} & a_{21} & \dots \\ a_{21} & a_{22} & \dots \\ \vdots & \vdots & \ddots\end{bmatrix}\begin{bmatrix}\psi_1 \\ \psi_2 \\ \vdots\end{bmatrix}. \tag{2-8}$$

Koopman eigendecomposition

The infinitesimal generator of the Koopman operator can be used to define Koopman eigenfunctions. A function $\phi \in \mathcal{F}$ that satisfies

$$\mathcal{L}\phi(x) = \mu\phi, \tag{2-9}$$

for some $\mu \in \mathbb{C}$ is called a Koopman eigenfunction, and the number μ is its eigenvalue. As the name suggest, if ϕ is an eigenfunction of $\mathcal{L}$ then it is also an eigenfunction of $\mathcal{K}^t$ with eigenvalue $\lambda^t = \exp(\mu t)$. Furthermore, if ϕ_1 and ϕ_2 are Koopman eigenfunctions with eigenvalues μ_1 and μ_2, then the product $\phi_1\phi_2$ is also a Koopman eigenfunction. In particular, if the corresponding eigenvalues are μ_1 and μ_2, then

$$\begin{aligned}\mathcal{K}\phi_1(x)\phi_2(x) &= \phi_1(\boldsymbol{F}(x))\phi_2(\boldsymbol{F}(x)) \\ &= \exp((\mu_1 + \mu_2)t)\phi_1(x)\phi_2(x).\end{aligned} \tag{2-10}$$

Hence, the eigenvalue of the product is $\mu_1 + \mu_2$. The set of eigenfunctions that can be used to construct all other eigenfunctions is called a principle set of eigenfunctions.

The set of all Koopman eigenfunctions $\{\phi_j\}_{j=1}^{\infty}$ forms a basis on $\mathcal{F}$. Thus, for any $\psi_i \in \mathcal{F}$ we can write

$$\psi_i(x) = \sum_{j=1}^{\infty} v_{ij}\phi_j(x), \tag{2-11}$$

with $v_{ij} \in \mathbb{R}$. The time evolution of $\psi_i(x)$ is determined by the action of $\mathcal{K}$, and since ϕ_j are eigenfunctions with eigenvalue λ_j^t. Thus,

$$\psi_i(x(t)) = \sum_{j=1}^{\infty} \lambda_j^t v_{ij}\phi_j(x(0)) = \sum_{j=1}^{\infty} \exp(\mu_j t)v_{ij}\phi_j(x(0)). \tag{2-12}$$

Given a set of observable functions $\mathcal{D} = \{\psi_1, \psi_2, \dots, \psi_N\} \subseteq \mathcal{F}$, often called a dictionary or lifting, we form the vector-valued observable function $\Psi : X \mapsto \mathbb{R}^N$ as

$$\Psi(x) = \begin{bmatrix} \psi_1(x) \\ \vdots \\ \psi_N(x) \end{bmatrix}. \tag{2-13}$$

Each component of Ψ may be decomposed as in equation (2-11), so that we can write

$$\Psi(x) = \sum_{j=1}^{\infty} \boldsymbol{v}_j\phi_j(x), \tag{2-14}$$

with $\boldsymbol{v}_j \in \mathbb{R}^N$ a vector of coefficients given by $\boldsymbol{v}_j = [v_{1j}, \ldots, v_{N_j}]^T$. The vector $\boldsymbol{v}_j$ is called the j-th Koopman mode associated with Ψ. We refer to equation (2-14) as the Koopman eigen decomposition of Ψ. The time evolution of the set of observables $\{\psi_i\}_{i=1}^N$ is then fully characterized by the objects $\{\lambda_j, \phi_j, \boldsymbol{v}_j\}_{j=1}^{\infty}$ and is summarized by the following equation

$$\Psi(x(t)) = \sum_{j=0}^{\infty} \lambda_j^t \boldsymbol{v}_j \phi_j(x(0)). \tag{2-15}$$

Of particular interest is the vector-valued observable function $\mathcal{I} : X \mapsto X$, which contains the projections to each component x_i of the state x, i.e.,

$$\mathcal{I} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}. \tag{2-16}$$

The Koopman eigendecomposition of $\mathcal{I}$ then allows us to describe the nonlinear in (2-1), using the linear dynamics given in (2-15).

Koopman invariant subspaces

The clear advantage of the Koopman operator is that it allows one to describe a nonlinear system linearly. The downside of this description is that it is an infinite-dimensional one. In practice, one has to resort to finite-dimensional approximations. In some cases, an observable function $\Psi \in \mathcal{F}$ admits a finite-dimensional Koopman eigendecompostion, equation (2-15) then becomes

$$\Psi(x(t)) = \sum_{j=0}^{M} \lambda_j^t \boldsymbol{v}_j \phi_j(x(0)). \tag{2-17}$$

Defining $V = [\boldsymbol{v}_1, \ldots, \boldsymbol{v}_M]^T$, $\Lambda^t = \text{diag}(\lambda_1^t, \ldots, \lambda_M^t)$ and $\Phi(x) = [\phi_1(x), \ldots, \phi_M(x)]^T$, equation (2-17) may be written as

$$\psi(x(t)) = V^T \Lambda^t \Phi(x(0)). \tag{2-18}$$

A set of observable functions, or dictionary, $\mathcal{D}$, for which the time derivative (along the trajectory) of each element $\psi \in \mathcal{D}$ is a linear combination of the elements in $\mathcal{D}$, is called a Koopman invariant subspace. Mathematically, $\mathcal{D}$ is a Koopman invariant subspace if

$$\mathcal{D} = \{\psi \in \mathcal{F} \mid \psi \in \mathcal{D} \implies \dot{\psi} \in \text{span}(\mathcal{D}\}. \tag{2-19}$$

A Koopman invariant subspace of the nonlinear system (2-1) that contains the state projections $\mathcal{I}$ in its span thus provides an equivalent finite, albeit higher, dimensional linear description.

If $\mathcal{D}$ is a Koopman invariant subspace, we write $z = \Psi(x)$ with Ψ given by (2-13), consisting of all the elements of $\mathcal{D}$, so that

$$\dot{z} = Az. \tag{2-20}$$

We now give an example of a Koopman invariant subspace.

Example 1. Consider the following nonlinear dynamical system [19]

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \nu x_1 \\ \kappa(x_2 - x_1^2) \end{bmatrix}, \tag{2-21}$$

with $\mu, \kappa \in \mathbb{R}$ plant parameters. Let $\psi_1(x) = x_1$, $\psi_2(x) = x_2$ and $\psi_3(x) = x_1^2$. The time derivatives of ψ_1 and ψ_2 along trajectories of (2-21) are given by $\dot{\psi}_1(x) = \nu x_1$, $\dot{\psi}_2(x) = \kappa(x_2 - x_1^2)$ and $\dot{\psi}_3 = 2x_1\dot{x}_1 = 2\nu$. Hence, the set $\mathcal{D} = \{\psi_1, \psi_2, \psi_3\}$ is a Koopman invariant subspace for (2-21). Define the vector-valued observable function $\Psi(x) = [\psi_1(x), \psi_2(x), \psi_3(x)]^T$ and the transformation $z = \Psi(x)$. Then

$$\dot{z} = \begin{bmatrix} \nu & 0 & 0 \\ 0 & \kappa & -\kappa \\ 0 & 0 & 2\nu \end{bmatrix} z. \tag{2-22}$$

Hence, the 2-dimensional nonlinear dynamical system (2-21) is equivalently described by the 3-dimensional linear dynamical system (2-22) with $z_0 = \Psi(x_0)$. The eigenvalues of the system (2-22) (not to be confused with Koopman eigenvalues) are ν, κ and 2ν. If $\nu, \kappa < 0$ then the system is stable. If $\nu, \kappa < 0$ then the system is unstable. Even so, due to the triangular structure of the system matrix, we can redefine $z = -\Psi(x)$ and obtain a stable system.

2-1-2 Koopman operator for control systems

Consider now the control affine system

$$\dot{x} = f(x) + g(x)u. \tag{2-23}$$

with $x \in X \subseteq \mathbb{R}^n$ the state and $u \in \mathbb{R}^m$ the input. Using Koopman eigenfunctions the work [13] shows that the control affine system (2-23) can be written as

$$\dot{z} = Az + \sum_{i=1}^{m} Nzu_i. \tag{2-24}$$

Inspired by this observation, section 3-1 shows that a control affine plant, experiencing input disturbance, can be represented by a bilinear dynamical model aswell, which also includes linear terms.

We now present an example based on example 1, which shows this for a control affine system.

Example 2. Consider again the system studied in Example 1 but now with a scalar input entering the system through $g(x)$, i.e.,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \nu x_1 \\ \kappa(x_2 - x_1^2) \end{bmatrix} + \begin{bmatrix} \beta_1 x_1 \\ \beta_2 \end{bmatrix} u, \tag{2-25}$$

where $u \in \mathbb{R}$ the input and μ, κ, β_1, β_2 plant parameters. Consider the dictionary $\mathcal{D} = \{\psi_1, \psi_2, \psi_3\}$ with $\psi_1 = x_1$, $\psi_2 = x_2$ and $\psi_3 = x_1^2$, and define $\Psi(x) = [\psi_1(x), \psi_2(x), \psi_3(x)]^T$ and $z = \Psi(x)$. The dynamics of z are given by

$$\dot{z} = \begin{bmatrix} \nu & 0 & 0 \\ 0 & \kappa & -\kappa \\ 0 & 0 & 2\mu \end{bmatrix} z + \begin{bmatrix} 0 \\ \beta_2 \\ 0 \end{bmatrix} u + \begin{bmatrix} \beta_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2\beta_1 \end{bmatrix} zu. \tag{2-26}$$

The Koopman operator for autonomous systems thus allows one to describe the nonlinear system (with linear input) as a dynamical system with linear natural dynamics and bilinear in the control input. We refer to such a model as a bilinear Koopman model.

2-2 Output Regulation Problem

This section gives an overview of the output regulation problem, sometimes called a servomechanism problem. In the output regulation problem, the goal is to asymptotically track a reference on the output of a plant and/or simultaneously reject disturbance signals on the plant dynamics. The disturbance and reference signals are generated by an autonomous dynamical system, referred to as the exosystem. Output regulation of linear systems with linear output and a linear exosystem is called the linear output regulation problem. Necessary and sufficient conditions for the solvability of the linear output regulation problem are established in [10]. The internal model principle, which states that the controller must be able to reproduce the exogenous disturbance signal in order to regulate the output, is introduced and discussed in [11, 12]. It is essential in tackling the linear output regulation problem when no (full) state information is available. For a modern treatment of the linear output regulation problem, see [15]. See [32] for a concise summary.
Output regulation of systems with nonlinear dynamics is referred to as the nonlinear output regulation problem. In the nonlinear output regulation problem, the output of the system and the dynamics of the exosystem may or may not be nonlinear. Necessary and sufficient conditions were established by [17] for general nonlinear plants and nonlinear exosystems. For an extensive treatment of the general nonlinear output regulation problem, see [7] and [15].
We briefly mention that for both the linear and nonlinear output regulation, robust variants have been considered. See for example [9] for the linear case and [8, 7] for the nonlinear case.
Section 2-2-1 discusses the linear output regulation problem and section 2-2-2 describes the nonlinear output regulation problem. The summary is primarily based on [32, 15] for the linear case and [7, 17] for the nonlinear case.

2-2-1 Linear output regulation

We start by considering the special case of linear output regulation. In the linear output regulation problem, one considers systems of the form

$$\dot{x} = Ax + Bu + Pw, \tag{2-27a}$$
$$\dot{w} = Sw, \tag{2-27b}$$
$$e = Cx + Qw. \tag{2-27c}$$

Equation (2-27a) is the plant with the state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^m$ and exogenous disturbance $w \in \mathbb{R}^r$. Equation (2-27c) determines the plant output error with $e \in \mathbb{R}^l$. Equation (2-27c) is the exosystem that generates the exogenous disturbance w. The exogenous disturbance w may include both a reference signal and a disturbance signal. The matrix Q in the output error picks out the reference signal, and the matrix P in the plant dynamics the disturbance signal. The exosystem is assumed to be skew-symmetric. That is, the matrix S is skew-symmetric, i.e., $S^T = -S$. Such an exosystem can always be constructed from parallel interconnections of constant and sinusoidal exosystems[1].

The goal of linear output regulation is to find a linear dynamic error feedback controller

$$\dot{\xi} = F\xi + Ge, \tag{2-28a}$$
$$u = H\xi + Kx, \tag{2-28b}$$

with $\xi \in \mathbb{R}^\nu$, such that the matrix

$$A_c = \begin{bmatrix} F & GC \\ BH & A + BK \end{bmatrix} \tag{2-29}$$

is Hurwitz, and the closed loop system

$$\dot{\xi} = F\xi + GCx + GQw, \tag{2-30a}$$
$$\dot{x} = (A + BK)x + BH\xi + Pw, \tag{2-30b}$$
$$\dot{w} = Sw, \tag{2-30c}$$
$$e = Cx + Qw, \tag{2-30d}$$

satisfies

$$\lim_{t \to \infty} e(t) = 0, \tag{2-31}$$

for every initial condition (ξ_0, x_0, w_0). Figure 2-1 shows a block diagram of the linear output regulation problem.

[1] This follows from the fact that the transpose of a matrix S is similar to S, so that the non-zero eigenvalues of a skew-symmetric matrix always come in pairs, and because the eigenvalues of a real skew-symmetric all lie on the imaginary axis.

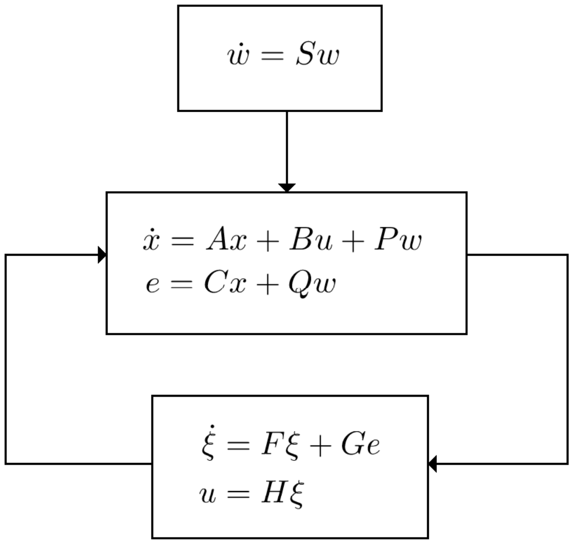

Figure 2-1: Block diagram of the linear output regulation problem.

Note that the Hurwitz requirement on the matrix A_c given by equation (2-29), corresponds to the requirement that the closed loop dynamics in the absence of the exogenous disturbance, i.e., $w = 0$ (or $P, Q = 0$), is stable. We refer to the system in the absence of the exogenous disturbance as the disconnected system. Stability of the disconnected system is achieved by means of linear state feedback Kx (see equation (2-28)). The plant dynamics of the disconnected system under this control read

$$\dot{x} = (A + BK)x, \tag{2-32}$$

which is stable if $A + BK$ is Hurwitz.

Consider now the full system. We are looking for equilibria $x = \Pi w$ and $\xi = \Sigma w$, with $\Pi \in \mathbb{R}^{n \times r}$ and $\Sigma \in \mathbb{R}^{\nu \times r}$, such that (2-31) is satisfied. These equilibria must satisfy the closed loop dynamics (2-30), this results in

$$\Pi S = A\Pi + BR + P, \tag{2-33a}$$
$$0 = C\Pi + Q, \tag{2-33b}$$

and

$$\Sigma S = F\Sigma, \tag{2-34a}$$
$$R = H\Sigma + K\Pi. \tag{2-34b}$$

Equations (2-33) are called the linear regulator equations and (2-34) is called the internal model principle. It turns out that these equations provide necessary and sufficient conditions for the solvability of the linear output regulation problem. To see this, consider the transformations $\tilde{x} = x - \Pi w$ and $\tilde{\xi} = \xi - \Sigma w$. We refer to these as the error coordinates. From equation (2-30) we find the error dynamics

$$\dot{\tilde{\xi}} = F\tilde{\xi} + GC\tilde{x} + G(C\Pi + Q)w + (F\Sigma - \Sigma S)w, \tag{2-35a}$$
$$\dot{\tilde{x}} = (A+BK)\tilde{x} + BH\tilde{\xi} + ((A+BK)\Pi + BH\Sigma + P - \Pi S)w, \tag{2-35b}$$
$$\dot{w} = Sw, \tag{2-35c}$$
$$e = C\tilde{x} + (C\Pi + Q)w, \tag{2-35d}$$

which, if equations (2-33) and (2-34) are satisfied, reduce to

$$\dot{\tilde{\xi}} = F\tilde{\xi} + GC\tilde{x}, \tag{2-36a}$$
$$\dot{\tilde{x}} = (A+BK)\tilde{x} + BH\tilde{\xi}, \tag{2-36b}$$
$$e = C\tilde{x}. \tag{2-36c}$$

Indeed, the origin is a stable equilibrium if the matrix A_c (equation (2-29)) is Hurwitz. As a result, the condition (2-31) is satisfied, and the linear output regulation problem is solved.

Conversely, suppose that the linear output regulation problem is solved. That is, condition (2-31) is satisfied for some controller (2-28) with A_c, given by equation (2-29), Hurwitz. Therefore, there exist equilibria $x = \Pi w$ and $\xi = \Sigma w$ such that

$$\lim_{t\to\infty} (C\Pi + Q)w = 0. \tag{2-37}$$

Since w is non-zero, it follows that $C\Pi + Q = 0$, which is exactly (2-33b). Furthermore, equation (2-33a), (2-34a) and (2-34b) form a Sylvester equation[2]. The matrix S is skew-symmetric and real, therefore the real part of the eigenvalues are all 0. Furthermore, $A+BK$ Hurwitz, and thus $A+BK$ and S have no eigenvalues in common. This implies that equation (2-33a) has a unique solution.

2-2-2 Nonlinear output regulation

In this section, we recall the nonlinear output regulation problem as discussed in [17, 7]. The general nonlinear output regulation problem considers nonlinear systems of the form

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i + \sum_{i=1}^{r} p_i(x)w_i, \tag{2-38a}$$
$$\dot{w} = s(w), \tag{2-38b}$$
$$e = h(x) + q(w). \tag{2-38c}$$

with $x \in X \subseteq \mathbb{R}^n$ the state, $u \in \mathbb{R}^m$ the input, $w \in W \subseteq \mathbb{R}^r$ an exogenous disturbance signal and $e \in \mathbb{R}^l$ the output error. Furthermore, $f : X \mapsto X$, $g_i : X \mapsto X$, $p_i : X \mapsto X$, $s : W \mapsto W$, $h : X \mapsto \mathbb{R}^l$, $q : W \mapsto \mathbb{R}^l$ are smooth nonlinear functions. For convenience, assume that $f(0) = 0$, $s(0) = 0$, $h(0) = 0$ and $q(0) = 0$. Equation (2-38a) is the plant, (2-38b)

[2] A Sylvester equation is a matrix equation of the form $AX + XB = C$, which admits a unique solution X if and only if A and $-B$ have no common eigenvalues [3].

the exosystem, and (2-38c) the plant output.

Consider a dynamic error feedback controller of the form

$$\dot{\xi} = \eta(\xi, e), \tag{2-39a}$$
$$u = \alpha(\xi), \tag{2-39b}$$

with $\xi \in \Xi \subseteq \mathbb{R}^{\nu}$, $\eta : \Xi \mapsto \Xi$ and $\alpha : \Xi \mapsto \mathbb{R}^m$ smooth nonlinear functions. Assume that $\eta(0,0) = 0$ and $\alpha(0) = 0$. The goal is to find $\eta(\xi, e)$ and $\alpha(\xi)$ such that $(\xi, x) = (0,0)$ is an exponentially stable equilibrium of the disconnected closed loop system, i.e., when $w = 0$, given by

$$\dot{\xi} = \eta(\xi, h(x)), \tag{2-40a}$$
$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)\alpha_i(\xi), \tag{2-40b}$$

and there exists a neighbourhood $U \subseteq X \times \Xi \times W$ of the equilibrium $(0,0,0)$ such that if $(x_0, \xi_0, w_0) \in U$, then the output is regulated. That is, the trajectories of the closed loop system

$$\dot{\xi} = \eta(\xi, h(x) + q(w)), \tag{2-41a}$$
$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)\alpha_i(\xi) + \sum_{i=1}^{r} p_i(x)w_i, \tag{2-41b}$$
$$\dot{w} = s(w), \tag{2-41c}$$

satisfy

$$\lim_{t\to\infty} e(t) = \lim_{t\to\infty} h(x(t)) + q(w(t)) = 0. \tag{2-42}$$

The following assumptions are made:

(1) the origin is a neutrally stable equilibrium point of the exosystem and there exists a neighbourhood $\hat{W} \subseteq W$ such that each initial $w_0 \in \hat{W}$ is Poisson stable[3]. This property is referred to as neutral stability of exosystem.

(2) The functions $f(x)$ and $g(x)$ are such that their linearizations at the origin form a stabilizable pair. In other words, the matrices

$$A = \left[\frac{\partial f}{\partial x}\right]_{x=0}, \qquad B = \left[\frac{\partial g}{\partial u}\right]_{x=0}, \tag{2-43}$$

are a stabilizable pair.

[3]Poisson stability of an initial conditions w_0 means that the flow $\boldsymbol{S}^t : W \mapsto W$ of $s(w)$ is defined for all $t \in \mathbb{R}$ and for each neighbourhood N of w_0 and real number $T > 0$, there exists $t_1 > T$ and $t_2 < -T$ such that $\boldsymbol{S}^{t_1}$ and $\boldsymbol{S}^{t_2}$ are in U [17]. In other words, the trajectory of a Poisson stable initial condition w_0 will eventually return to a neighbourhood of w_0.

(3) The functions $h(x)$, $q(w)$, $p(x)$ and $s(w)$ are such that the pair

$$\left(\begin{bmatrix} A & P \\ 0 & S \end{bmatrix}, \begin{bmatrix} C & Q \end{bmatrix} \right), \tag{2-44}$$

with

$$C = \left[\frac{\partial h}{\partial x}\right]_{x=0}, \quad Q = \left[\frac{\partial q}{\partial w}\right]_{w=0}, \quad P = \left[\frac{\partial p}{\partial w}\right]_{x=0}, \quad S = \left[\frac{\partial s}{\partial w}\right]_{w=0}, \tag{2-45}$$

is detectable.

Note that the first assumption implies that all the eigenvalues of the Jacobian S of $s(w)$ are on the imaginary axis. For if this is not the case, then the neutral stability and/or Poisson property of the initial condition does not hold.

Given these assumptions, the nonlinear output regulation problem is solvable if and only if there exist smooth mappings $x = \pi(w)$ and $u = c(w)$, with $\pi : W^0 \mapsto X$ and $c : W^0 \mapsto \mathbb{R}^m$, that satisfy

$$\frac{\partial \pi}{\partial w} s(w) = f(\pi(w)) + g(\pi(w))c(w) + p(\pi(w))w, \tag{2-46a}$$

$$0 = h(\pi(w)) + q(w). \tag{2-46b}$$

Equations (2-46) are called the nonlinear regulator equations.

2-3 Quadratic stabilization of bilinear control systems

For the analysis of the bilinear output regulation problem obtained by applying the Koopman operator to the nonlinear output regulation problem in question, we utilize a technique based on quadratic Lyapunov functions and linear matrix inequalities introduced in [22]. In this technique, Petersen's lemma [31], in combination with ellipsoidal constraints on the state, is used to characterize a (linear) matrix inequality that, when satisfied, guarantees that the derivative of the quadratic Lyapunov function is negative. In this section, we briefly describe the essence of this technique.

2-3-1 Petersen's lemma

The Lyapunov-based stability argument for quadratic systems presented in this section, relies on the following lemma.

Lemma 1 (Petersen's lemma [31]). *Suppose $G = G^T \in \mathbb{R}^{n\times n}$, $M \in \mathbb{R}^{n\times p}$, $N \in \mathbb{R}^{q\times n}$. The matrix inequality*

$$G + M\Delta N + N^T \Delta^T M^T \preccurlyeq 0 \tag{2-47}$$

holds for all matrices $\Delta \in \mathbb{R}^{p\times q}$ satisfying

$$||\Delta||_2 \leq 1, \tag{2-48}$$

if and only if there exists a real number $\epsilon > 0$ such that

$$\begin{bmatrix} G + \epsilon M M^T & N^T \\ N & -\epsilon I \end{bmatrix} \preccurlyeq 0. \tag{2-49}$$

For the proof, see the original work [31] and see [21] for one based on the S-procedure. Petersen's lemma has been generalized in multiple ways in [21]. Of interest here is a modification proposed in [22]. In this modification, instead of considering the matrix uncertainty Δ satisfying (2-48), we consider a vector uncertainty satisfying an ellipsoidal constraint. We now state the lemma and provide a proof, as it is not provided in the original source.

Lemma 2 ([22])**.** *Suppose $G = G^T \in \mathbb{R}^{n\times n}$, $N \in \mathbb{R}^{1\times n}$, $M \in \mathbb{R}^{n\times n}$, and $0 \preccurlyeq P = P^T \in \mathbb{R}^{n\times n}$. The matrix inequality*

$$G + M\delta N + N^T \delta^T M^T \preccurlyeq 0 \tag{2-50}$$

holds for all vectors $\delta \in \mathbb{R}^n$ satisfying

$$\delta^T P^{-1} \delta \leq 1, \tag{2-51}$$

if and only if there exists a real number $\epsilon > 0$ such that

$$\begin{bmatrix} G + \epsilon M P M^T & N^T \\ N & -\epsilon I \end{bmatrix} \preccurlyeq 0. \tag{2-52}$$

Proof. Let $\Delta = P^{-1/2}\delta$ [23] the inequality (2-50) may then be written as

$$G + MP^{1/2}\Delta N + N^T \Delta^T P^{1/2} M^T \preccurlyeq 0. \tag{2-53}$$

Furthermore, we have that

$$\begin{aligned} ||\Delta||_2 \leq 1 &\iff \sup_{x\neq 0} \frac{||P^{-1/2}\delta x||_2}{||x||_2} \leq 1, \\ &\iff \sup_{x\neq 0} \frac{x^T \delta^T P^{-1} \delta x}{x^T x} \leq 1, \\ &\iff \delta^T P^{-1} \delta \leq 1. \end{aligned} \tag{2-54}$$

The last line follows from the fact that $\delta^T P^{-1} \delta$ is a positive scalar, and the property that if $U \subseteq \mathbb{R}$ is a set and $r > 0$ then $\sup(rU) = r\sup(U)$.
Thus if (2-50) and (2-51) are satisfied, by Petersen's lemma there exists a real number $\epsilon > 0$ such that inequality (2-52) holds. □

2-3-2 Quadratic stabilization

We now present the main result of [22], on the stability of quadratic systems. The stability analysis of the bilinear output regulation problem will be based on this result. Consider the bilinear control system

$$\dot{x} = Ax + Bu + Nzu, \tag{2-55}$$

with $x \in X \subseteq \mathbb{R}^n$ the state and $u \in \mathbb{R}$ the input. Under linear state feedback $u = Kx$, with $K \in \mathbb{R}^n$, the closed loop dynamics of equation (2-55) obey

$$\dot{x} = A_c x + NxKx, \tag{2-56}$$

where we have defined the closed loop matrix $A_c = A+BK$. Next, we introduce the quadratic Lyapunov candidate function

$$V(x) = x^T P^{-1} x, \tag{2-57}$$

with P a symmetric positive semi-definite matrix. The candidate $V(x)$ is a Lyapunov function for the closed loop system (2-56) if the time derivative along trajectories of this system is non-positive. The derivative is given by

$$\begin{aligned}\dot{V}(x) &= \dot{x}^T P^{-1} x + x^T P^{-1} \dot{x} \\ &= (A_c x + NxKx)^T P^{-1} x + x^T P^{-1}(A_c x + NxKx) \\ &= x^T (A_c^T P^{-1} + P^{-1} A_c + K^T x^T N^T P^{-1} + P^{-1} NxK)x.\end{aligned} \tag{2-58}$$

From equation (2-58) we see that $V(x)$ is a quadratic Lyapunov function of the closed loop system (2-56) if

$$A_c^T P^{-1} + P^{-1} A_c + K^T x^T N^T P^{-1} + P^{-1} NxK \preccurlyeq 0. \tag{2-59}$$

Pre- and post multiplying this expression with P yields

$$A_c P + PA_c + PK^T x^T N^T + NxKP \preccurlyeq 0. \tag{2-60}$$

By imposing the ellipsoidal constraint $x \in \mathcal{E}$, with

$$\mathcal{E} = \{x \in \mathbb{R}^n \mid V(x) \leq 1\}, \tag{2-61}$$

we find that by lemma 2 the inequality (2-60) is equivalent to

$$\begin{bmatrix} A_c P + PA_c + \epsilon NPN^T & PK^T \\ KP & -\epsilon I, \end{bmatrix} \preccurlyeq 0 \tag{2-62}$$

for some $\epsilon > 0$. Therefore, any trajectory generated by the closed loop system (2-56) for which the initial state is $x_0 \in \mathcal{E}$, will tend towards zero.

We summarize the result in the following lemma

Proposition 1 ([22])**.** *Suppose* $0 \preccurlyeq P = P^T \in \mathbb{R}^{n \times n}$ *and let* $K \in \mathbb{R}^n$ *be such that* $A_c = A + BK$ *is Hurwitz. Then any trajectory of* (2-56) *for which* $x_0 \in \mathcal{E}$*, with* $\mathcal{E}$ *given by* (2-61)*, is stable if and only if there exists* $\epsilon > 0$ *such that*

$$\begin{bmatrix} A_c P + P A_c + \epsilon N P N^T & P K^T \\ K P & -\epsilon I, \end{bmatrix} \preccurlyeq 0 \tag{2-63}$$

is satisfied.

Note that there is a freedom in choosing K, P and ϵ as long as the conditions of the theorem are satisfied. The matrix inequality (2-63) is a linear matrix inequality in one of these variables if the others are fixed.

Chapter 3

Nonlinear output regulation in Koopman framework

In this chapter, we utilize the Koopman operator to tackle the output regulation problem for a certain class of nonlinear systems. To this end, we identify a class of nonlinear control systems that admit a finite-dimensional bilinear Koopman representation. That is, we can identify a finite dictionary, for which the dynamics of its elements are described by a bilinear control system. We use this to rephrase the nonlinear output regulation problem as a bilinear output regulation problem and determine necessary conditions for certain solutions to occur in the presence of linear dynamic error feedback control. Motivated by this analysis, we tackle the bilinear output regulation problem with matched input disturbance using linear dynamic error feedback. Based on the work of [22] we characterize a set of initial conditions of the system that guarantees output regulation.

3-1 Problem formulation

As discussed in section 2-2-2, the general problem of nonlinear output regulation considers system given by equations (2-38). In this section, we set $s(w) = Sw$ and $q(w) = Qw$. And assume the matrix S to be skew-symmetric. This is not a restrictive assumption. A linear exosystems with a skew-symmetric matrix is able to generate a combination of sinusoidal and constant signals. From Fourier theory, linear combinations of these signals allow, in principle, for the construction of a large class of signals. Moreover, since the exosystem is an autonomous dynamical system, the Koopman operator associated with the nonlinear exosystem (2-38b) provides an equivalent linear description. If the nonlinear exosystem admits a finite-dimensional Koopman invariant subspace, with the property that its system matrix is skew-symmetric, and the components of $q(w)$ are in this set, then the analysis of this chapter automatically generalizes to include such nonlinear exosystems.

As mentioned in section 2-1-2, the Koopman operator for autonomous dynamical systems provides a bilinear description of the control affine system associated with the said autonomous system. The same holds true for a system with an affine term with a disturbance signal, as is the case in (2-38a). Inspired by [13], we show when the system (2-38a) with output (2-38c) is described by a finite bilinear dynamical system with a linear output. We restrict our attention to the case where u and w are scalars. The general case will be considered in Lemma 3.

Define a dictionary $\mathcal{D}$ of observable functions ψ, with $N = \dim(\mathcal{D})$. Let $\Psi : X \mapsto \mathbb{R}^N$ be a vector valued observable function consisting of all observable functions $\psi \in \mathcal{D}$. Consider the transformation

$$\dot{z} = \Psi(x). \tag{3-1}$$

Taking the time derivative along the trajectories of z yields

$$\dot{z} = \frac{\partial \Psi}{\partial x}\Big(f(x) + g(x)u + p(x)w\Big). \tag{3-2}$$

If, when $u, w = 0$, $\mathcal{D}$ is a Koopman invariant subspace of (2-38a), it follows that the first term can be written as Az, with $A \in \mathbb{R}^{N\times N}$. Moreover, if the components of the vectors $\frac{\partial \Psi}{\partial x} g(x)$ and $\frac{\partial \Psi}{\partial x} p(x)$ are in the span of $\mathcal{D}$, they may be written as Nzu and Mzw, respectively, with $N, M \in \mathbb{R}^{N\times N}$. Note that if $\frac{\partial \psi_i}{\partial x_j} g_i = N_{i*}z$ is constant, then the entries of the column N_{i*} are zero everywhere except where it multiplies the constant function[1] $\psi = 1$. However, since it is constant, we can simply write $\frac{\partial \psi_i}{\partial x_j} g_i = \beta_i$, with $\beta_i \in \mathbb{R}$. Thus the second and third term in equation (3-2) can actually be written as the sum of a linear and bilinear term instead, we denote these as $Bu + Nzu$ and $Pw + Mzw$, respectively, with $B, P \in \mathbb{R}^N$. Motivated by the above observation, we redefine Ψ such that it no longer includes the constant function. Putting everything together, equation (3-2) becomes

$$\dot{z} = Az + Bu + Pw + Nzu + Mzw. \tag{3-3}$$

Finally, if we assume that the components of the vector valued function h are in $\mathcal{D}$, then the output error (2-38c) can be written as $e = Cz + Qw$, with $C \in \mathbb{R}^{l\times N}$.

The function output $z \in \mathbb{R}^N$ of the vector-valued observable function $\Psi(x)$ is interpreted as the state of the bilinear dynamical system. It is important to note that the state of the bilinear system is constrained by the transformation (3-1). Since the dynamics are directly derived from this transformation, it is sufficient to assume that $z_0 = \Psi(x_0)$, with x_0 the initial condition of the nonlinear system (2-38a).

Combining these observations, we see that under certain assumptions on $\mathcal{D}$, we can rewrite the nonlinear system (2-38a) and its output (2-38c) as a bilinear dynamical system in z, u,

[1]Without loss of generality we assume that the constant function appears only once in the dictionary $\mathcal{D}$ and is equal to 1. Any other constant function is trivially contained in the span of $\mathcal{D}$.

w. The set of nonlinear systems that admit to such a bilinear description corresponds to the set of systems that admit a dictionary $\mathcal{D}$ that satisfies these assumptions. While for many systems we can not find such dictionaries, there may exist dictionaries for which the bilinear dynamical system provides a good approximation. The more dictionary terms we include, the more accurate we expect the approximation to be[2].

We summarize the result of this section in the following lemma and provide its proof.

Lemma 3. *Consider the nonlinear dynamical system*

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i + \sum_{i=1}^{r} p_i(x)w_i, \tag{3-4a}$$

$$e = h(x) + Qw, \tag{3-4b}$$

with $x \in X \subseteq \mathbb{R}^n$, $u \in \mathbb{R}^m$, $w \in W \subseteq \mathbb{R}^r$, $e \in \mathbb{R}^l$ *and* $f : X \mapsto X$, $g_i : X \mapsto X$, $p_i : X \mapsto X$ *and* $h : X \mapsto \mathbb{R}^l$ *smooth nonlinear functions. Suppose there exists a set* $\mathcal{D}$ *of observable functions* $\psi : X \mapsto \mathbb{R}$, *with* $N = dim(\mathcal{D}) > n$, *satisfying the following properties:*

(1) If $u, w = 0$ *and* $\psi \in \mathcal{D}$ *then* $\dot{\psi} \in span(\mathcal{D})$, *i.e.,* $\mathcal{D}$ *is a Koopman invariant subspace.*

(2) If $\psi \in \mathcal{D}$ *then* $\frac{\partial \psi}{\partial x_j} g_{ij}$ *and* $\frac{\partial \psi}{\partial x_i} p_{kj}$ *are in the span of* $\mathcal{D}$ *for each* $i = 1, \dots, m$, $j = 1, \dots, n$ *and* $k = 1, \dots, r$.

(3) The components $h_i(x)$ *are in the span of* $\mathcal{D}$.

(4) The state projections $\psi(x) = x_i$ *are in the span of* $\mathcal{D}$.

The nonlinear system (3-4) *is then equivalently described by the following bilinear system*

$$\dot{z} = Az + Bu + Pw + \sum_{i=1}^{m} Nzu_i + \sum_{i=1}^{r} Mzw_i, \tag{3-5a}$$

$$e = Cz + Qw, \tag{3-5b}$$

with $z_0 = \Psi(x_0) = [\psi_1(x_0), \dots, \psi_{N-1}(x_0)]^T$ *and* $\psi_i \in \mathcal{D}$.

Proof. Choose a set $\mathcal{D}$ of functions $\psi : X \mapsto \mathbb{R}$, that satisfies the above properties. Denote the j-th element of f, g_i and p_i as f_j, g_{ij} and p_{ij}, respectively. For any $\psi_i \in \mathcal{D}$ the time derivative along the trajectories satisfying (3-4a) is given by

$$\dot{\psi}_i(x) = \sum_{j=1}^{n} \frac{\partial \psi_i}{\partial x_j} \dot{x}_j = \sum_{j=1}^{n} \frac{\partial \psi_i}{\partial x_j} \Big(f_j(x) + \sum_{q=1}^{m} g_{qj}(x)u_q + \sum_{q=1}^{r} p_{qj}(x)w_q \Big). \tag{3-6}$$

Property 1 implies that we can write

$$\sum_{j=1}^{n} \frac{\partial \psi_i}{\partial x_j} f_j(x) = \sum_{k=1}^{N} \alpha_{ik} \psi_k(x), \tag{3-7}$$

[2] Note that in practice, this may not always be the case, see [14].

with α_{ik} the expansion coefficients for $\frac{\partial \psi_i}{\partial x_j} f_j(x)$. Property 2 implies that we can write

$$\frac{\partial \psi_i}{\partial x_j} g_{qj}(x) = \sum_{k=1}^{N} \mu_{ijkq} \psi_k(x), \qquad \frac{\partial \psi_i}{\partial x_j} p_{qj}(x) = \sum_{k=1}^{N} \nu_{ijkq} \psi_k(x), \tag{3-8}$$

where μ_{ijkq} and ν_{ijkq} are the expansion coefficients of $\frac{\partial \psi_i}{\partial x_j} g_{qj}(x)$ and $\frac{\partial \psi_i}{\partial x_j} p_{qj}(x)$, respectively. We rewrite these as the sum of a linear and a bilinear terms. To this end, assume w.l.o.g. that $\psi_N = 1$ is the only constant function in $\mathcal{D}$. Using equation (3-8) the second term in equation (3-6) may then be written as

$$\begin{aligned} \sum_{q=1}^{m} \Big(\sum_{j=1}^{n} \frac{\partial \psi_i}{\partial x_j} g_{qj}(x) \Big) u_q &= \sum_{q=1}^{m} \sum_{j=1}^{n} \Big(\mu_{ijNq} \psi_N + \sum_{k=1}^{N-1} \mu_{ijkq} \psi_k(x) \Big) u_q, \\ &= \sum_{q=1}^{m} b_{iq} u_q + \sum_{q=1}^{m} \sum_{k=1}^{N-1} n_{ikq} \psi_k(x) u_q, \end{aligned} \tag{3-9}$$

with $b_{iq} = \sum_{j=1}^{n} \mu_{ijNq} \psi_N$ and $n_{ikq} = \sum_{j=1}^{n} \mu_{ijkq}$. Similarly, the third term in equation (3-6) is written as

$$\begin{aligned} \sum_{q=1}^{m} \Big(\sum_{j=1}^{n} \frac{\partial \psi_i}{\partial x_j} p_{qj}(x) \Big) w_q &= \sum_{q=1}^{r} \sum_{j=1}^{n} \Big(\nu_{ijNq} \psi_N + \sum_{k=1}^{N-1} \nu_{ijkq} \psi_k(x) \Big) u_q, \\ &= \sum_{q=1}^{r} p_{iq} w_q + \sum_{q=1}^{m} \sum_{k=1}^{N-1} m_{ikq} \psi_k(x) w_q, \end{aligned} \tag{3-10}$$

with $p_{iq} = \sum_{j=1}^{n} \nu_{ijNq} \psi_N$ and $m_{ikq} = \sum_{j=1}^{N-1} \nu_{ijkq}$. Using the equations (3-7), (3-9) and (3-10), equation (3-6) becomes

$$\dot{\psi}_i(x) = \sum_{k=1}^{N} a_{ik} \psi_k + \sum_{q=1}^{m} b_{iq} u_q + \sum_{q=1}^{r} p_{iq} w_q + \sum_{q=1}^{m} \sum_{k=1}^{N} n_{ikq} \psi_k u_q + \sum_{q=1}^{r} \sum_{k=1}^{N} m_{ikq} \psi_k w_q. \tag{3-11}$$

Next, define the vector valued observable function $\Psi : X \mapsto \mathbb{R}^n$ as

$$\Psi(x) = \begin{bmatrix} \psi_1 \\ \vdots \\ \psi_{N-1} \end{bmatrix}, \tag{3-12}$$

the transformation $z = \Psi(x)$ and matrices $A = (a_{ij})$, $B = (b_{ij})$, $P = (p_{ij})$; and $N_q = (n_{ij})_q$ and $M_q = (m_{ij})_q$ for $q = 1, \ldots, m$. From equations (3-11) and (3-12) we obtain equation (3-5a). With the dynamics of z known and the initial condition $z_0 = \Psi(x_0)$, property 4 allows the reconstruction of the state $x(t)$ from the state $z(t)$ at each time t.

Finally, property 3 implies that we can write

$$h_i(x) = \sum_{k=1}^{N} c_{ik}\psi_k(x). \tag{3-13}$$

Define $C = (c_{ij})$, then we can write $h(x) = Cz$ and the output error of the nonlinear dynamical system given by equation (3-4b) can be written as (3-5b). □

We conclude the section with two remarks:

Remark 1. *Property 4 in Lemma 3 is only required if we need to reproduce the original state x from the lifted state z. If we omit property 4, Lemma 3 remains valid if the only information we have of the system is the output error e.*

Remark 2. *The system matrices of the bilinear dynamical model* (3-5) *have a particular structure to them. As seen in the proof of Lemma 3 the function ψ_i contributes a constant control term to the i-th row in the dynamics whenever $\frac{\partial \psi_i}{\partial x_j} g_{ij}$ is constant. And a bilinear input term whenever $\frac{\partial \psi_i}{\partial x_j} g_{ij}$ is non-constant. In the case that $g(x) = B$ and $p(x) = P$ are constant columns, and the vector valued observable function $\Psi(x)$ consists of the state projections $\psi_i(x) = x_i$ in the first n components and the purely nonlinear observable functions in the remaining $N - n$ components, the bilinear system has the structure*

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u + \begin{bmatrix} P \\ 0 \end{bmatrix} w + \begin{bmatrix} 0 & 0 \\ N_{21} & N_{22} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} u + \begin{bmatrix} 0 & 0 \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} w. \tag{3-14}$$

3-2 Bilinear output regulation

In the previous section, we have seen that under the assumptions of Lemma 3, the nonlinear system described by equations (3-4) is equivalently represented by the bilinear system (3-5). Henceforth, we consider systems that admit to such an equivalent description and aim to solve the output regulation problem of these systems by tackling the output regulation problem of the bilinear system. In particular, the goal is to regulate the output using linear dynamic error feedback. We restrict our attention to the scalar input and exogenous disturbance case. The discussion below can be generalized to account for multiple input and exogenous disturbances.

From now on, we consider linear exosystems with a linear output, i.e.,

$$\dot{w} = Sw, \tag{3-15a}$$

$$v = Ew. \tag{3-15b}$$

with $w \in \mathbb{R}^r$ and $v = \in \mathbb{R}$. The output v of the exosystem now fulfills the role of disturbance and reference signal in (3-5), instead of directly w. This allows us to consider (linear combinations of) one-dimensional sinusoidal signals. Furthermore, to ease the notation we write

$N := N_1$ and $M := M$.

We assume to have full state information of x, which implies full state information of z. The goal is to find a linear dynamic error feedback controller of the form

$$\dot{\xi} = F\xi + Ge, \tag{3-16a}$$
$$u = H\xi + Kz, \tag{3-16b}$$

with $\xi \in \Xi \subseteq \mathbb{R}^{\nu}$ such the the matrix

$$A_c = \begin{bmatrix} F & GC \\ BH & A + BK \end{bmatrix} \tag{3-17}$$

is Hurwitz, and there exists a neighbourhood $U \subseteq Z \times \Xi \times W$ such that if $z_0 = \Psi(x_0)$ and $(\xi_0, z_0, w_0) \in U$ then the trajectories generated by the closed loop system

$$\dot{\xi} = F\xi + GCz + GQv, \tag{3-18a}$$
$$\dot{z} = (A + BK)z + NzKz + BH\xi + Pv + NzH\xi + Mzv, \tag{3-18b}$$
$$\dot{w} = Sw, \tag{3-18c}$$
$$v = Ew, \tag{3-18d}$$
$$e = Cz + Qv, \tag{3-18e}$$

satisfy

$$\lim_{t\to\infty} e(t) = \lim_{t\to\infty} Cz(t) + Qv(t) = 0. \tag{3-19}$$

Remark 3. *Recall that to solve the nonlinear output regulation problem (of which the bilinear output regulation problem is a special case), the point $(\xi, z) = 0$ must be an exponentially stable equilibrium of the disconnected closed loop dynamics, i.e. of* (3-18a)-(3-18b) *with $w = 0$. The disconnected closed loop dynamics has an equilibrium point at $(\xi, z) = (0, 0)$. Furthermore, the Jacobian of the disconnected closed loop dynamics at $(0, 0)$ is equal to A_c. The assumption that the matrix A_c is Hurwitz thus ensures that $(\xi, z) = (0, 0)$ is an exponentially stable equilibrium point of the closed loop dynamics [20].*

To solve the bilinear output regulation problem, we take inspiration of the linear output regulation problem and look for equilibrium points of the form $\xi = \Sigma w$ and $z = \Pi w$, with $\Sigma \in \mathbb{R}^{\nu\times r}$ and $\Pi \in \mathbb{R}^{n\times r}$. Equation (3-19) becomes

$$\lim_{t\to\infty} (C\Pi + QE)w(t) = 0. \tag{3-20}$$

Recall that the signal $w(t)$ is a (nontrivial) neutrally stable trajectory, implying that for all $T \in \mathbb{R}$ there exists $t > T$ such that $w(t) \neq 0$. Therefore, equation (3-20) holds if and only if

$$C\Pi + QE = 0. \tag{3-21}$$

Furthermore, $\xi = \Sigma w$ and $z = \Pi w$ are equilibrium points of the closed loop system (3-18a)-(3-18b) if and only if

$$\Sigma S w = F \Sigma w + G(C\Pi + QE)w, \tag{3-22a}$$
$$\Pi S w = (A + BK)\Pi w + N\Pi w K \Pi w + BH\Sigma w + PEw + N\Pi w H \Sigma w + M\Pi w E w. \tag{3-22b}$$

Clearly, if $w = 0$, this is satisfied. If $w \neq 0$, the following should be satisfied

$$\Sigma S = F\Sigma, \tag{3-23a}$$
$$\Pi S = (A + BK)\Pi + PE + BH\Sigma + N\Pi w K \Pi + N\Pi w H \Sigma + M \Pi w E, \tag{3-23b}$$

where we have used equation (3-21) to simplify the first line. Summarizing, if there exist $\Pi \in \mathbb{R}^{n\times r}$, $\Sigma \in \mathbb{R}^{\nu \times r}$ and $R \in \mathbb{R}^{m\times w}$ such that

$$\Pi S = A\Pi + BR + PE + N\Pi w R + M\Pi w E, \tag{3-24a}$$
$$0 = C\Pi + QE, \tag{3-24b}$$
$$\Sigma S = F\Sigma, \tag{3-24c}$$
$$R = H\Sigma + K\Pi, \tag{3-24d}$$

are satisfied, and if A_c, given in equation (3-17), is Hurwitz, then (3-20) holds, and thus the bilinear output regulation problem of the system (3-5) with a linear exosystem is locally solved by the linear dynamic error feedback controller (3-16). We identify equations (3-24c) and (3-24d) as the linear internal model principle. Furthermore, equations (3-24a) and (3-24b) are reminiscent of the linear regulator equations (2-33), and we appropriately call these the bilinear regulator equations[3].

Note that equation (3-24a) depends on the exogenous disturbance $w(t)$. At best, we can solve these equations with prior knowledge of w in the case that w is constant, i.e., $S = 0$. If, however, w is non-constant, in order to solve these equations, we necessarily require $N\Pi w R + M \Pi w E = 0$ for each value of $w(t)$. One way for this to hold is if $\Pi = 0$. This means that the equilibrium point of the system at which the output is regulated is $z = 0$. The case that the output is regulated for $z = 0$ coincides with the output regulation problem with no reference signal, i.e. $Q = 0$. Given these assumptions, equation (3-24b) is automatically satisfied and equations (3-24a), (3-24c) and (3-24d) become

$$0 = BR + PE, \tag{3-25a}$$
$$\Sigma S = F\Sigma, \tag{3-25b}$$
$$R = H\Sigma, \tag{3-25c}$$

which are the linear regulator equations (2-33) and linear internal model principle (2-34) for the case that $Q, \Pi = 0$.

The following proposition concerns the bilinear output regulation problem in the case that $Q = 0$. This result makes no reference to the Koopman dynamical model and thus holds for any bilinear dynamical system that satisfies the assumptions.

[3]Note that these are the bilinear regulator equations that the linear equilibrium points $x = \Pi w$ and $\xi = \Sigma w$ must satisfy in the case of linear dynamic error feedback, and not the general nonlinear regulator equations (2-46).

Proposition 2. *The bilinear output regulation problem of the system*

$$\dot{z} = Az + Bu + Pv + Nzu + Mzv, \tag{3-26a}$$
$$e = Cz, \tag{3-26b}$$

with $z \in Z \subseteq \mathbb{R}^n$*,* $u \in \mathbb{R}$ *and* $v \in \mathbb{R}$ *generated by the linear exosystem* (3-15) *and the linear dynamic error feedback controller* (3-16) *is locally solved if* A_c*, given by equation* (3-17)*, is Hurwitz and if there exist* $\Sigma \in \mathbb{R}^{n\times r}$*,* $R \in \mathbb{R}^{m\times w}$ *that satisfy equations* (3-25)*.*

Proof. Let $\tilde{\xi} = \xi - \Sigma w$. In these coordinates, the closed loop dynamics are given by

$$\dot{\tilde{\xi}} = F\tilde{\xi} + GCz + (F\Sigma - \Sigma S), \tag{3-27a}$$
$$\dot{z} = (A + BK)z + NzKz + BH\tilde{\xi} + NzH\tilde{\xi} + MzEw + (BH\Sigma + PE)w + NzH\Sigma w, \tag{3-27b}$$
$$\dot{w} = Sw. \tag{3-27c}$$

Using equations (3-25) the closed loop dynamics reduce to

$$\dot{\tilde{\xi}} = F\tilde{\xi} + GCz, \tag{3-28a}$$
$$\dot{z} = (A + BK)z + NzKz + BH\tilde{\xi} + NzH\tilde{\xi} + MzEw + NzH\Sigma w. \tag{3-28b}$$
$$\dot{w} = Sw. \tag{3-28c}$$

Assume w.l.o.g. that $w = 0$ is a neutrally stable equilibrium point of the exosystem[4]. We see that if $w = 0$, the point $(\tilde{\xi}, z) = (0, 0)$ is an equilibrium point, and the Jacobian of the dynamics at this point is equal to A_c. The matrix A_c is assumed to be Hurwitz. Hence the origin is an exponentially stable equilibrium point of the disconnected system [20].

If $w \neq 0$, we require (3-19) to hold. Since $Q = 0$, this is satisfied if $z(t) \to 0$ as $t \to \infty$. The Jacobian of (3-28) at $(\tilde{\xi}, z, w) = 0$ is given by

$$\begin{bmatrix} A_c & 0 \\ 0 & S \end{bmatrix}. \tag{3-29}$$

Since A_c is Hurwitz, it follows that there exists a set of initial conditions $(\tilde{\xi}_0, z_0, w_0)$ (close to the origin) for which $z(t) \to 0$ and $\tilde{\xi}(t) \to 0$ as $t \to \infty$. We conclude that the output is locally regulated. □

We use Proposition 2 in combination with Lemma 3 to arrive at the following corollary.

Corollary 3.1. *The nonlinear output regulation problem of the system*

$$\dot{x} = f(x) + g(x)u + p(x)w, \tag{3-30a}$$
$$e = h(x), \tag{3-30b}$$

with the linear exosystem (3-15) *and linear dynamic error feedback controller* (3-16) *is locally solved provided that the following conditions hold: (1) the assumptions of Lemma 3 and proposition 2 are satisfied; (2) the initial condition satisfy* $z_0 = \Psi(x_0)$*.*

[4]The sinusoidal exosystem automatically satisfies this, and for a constant exosystem, we can make the coordinate transformation $\tilde{w} = w - w_0$.

Proof. If $z_0 = \Psi(x_0)$, then, by Lemma 3, the nonlinear system (3-4) is equivalently described by (3-26). It follows that, by Proposition 2, the nonlinear output regulation problem is locally solved. □

In Proposition 2 we have used a linearization argument to show there exist initial conditions for which the output is regulated. However, this only tells us something about the existence of such initial conditions, not which initial conditions achieve this.

3-3 Matched input disturbance

In this section, we consider the special case of matched input disturbance. That is, the disturbance and the input enter the system dynamics identically. This is a natural problem to consider, as it models uncertainty to the input. Furthermore, in this case, we can use the Lyapunov-based stability argument for quadratic systems [22] to characterize a set of initial conditions for which output regulation is guaranteed.

The nonlinear output regulation problem with matched input disturbance considers systems given by (3-30) with $p(x) = g(x)$ and the linear exosystem (3-15). If the assumptions of Lemma 3 are satisfied and $z_0 = \Psi(x_0)$, the system is equivalently represented by a bilinear dynamical system (3-26) with $P = B$ and $M = N$. Under the control (3-16) the closed loop dynamics of the plant obeys (3-27). Since $P = B$ and $M = N$, equation (3-27b) becomes

$$\dot{z} = (A + BK)z + NzKz + BH\tilde{\xi} + NzH\tilde{\xi} + B(H\Sigma + E)w + Nz(H\Sigma + E)w. \tag{3-31}$$

From this, we see that if $H\Sigma + E = 0$, then the last two terms are canceled. In other words, the linear disturbance term is canceled if and only if the bilinear disturbance term is canceled. This is a feature of the matched input disturbance case. In the general case, the second term does not cancel and is only zero if $z = 0$. Note that this condition is equivalent to the linear regulator equation (3-25a).

If we define the quantities

$$s = \begin{bmatrix} \tilde{\xi} \\ z \end{bmatrix}, \quad \tilde{N} = \begin{bmatrix} 0 & 0 \\ 0 & N \end{bmatrix}, \quad \tilde{K} = \begin{bmatrix} 0 & K \end{bmatrix}, \quad \tilde{H} = \begin{bmatrix} H & 0 \end{bmatrix}, \tag{3-32}$$

and assume that the linear regulator equations and internal model principle given in (3-25) are satisfied, then the closed loop dynamics (3-27) for the matched input disturbance case may be written as

$$\dot{s} = A_c s + \tilde{N}s\tilde{K}s + \tilde{N}s\tilde{H}s. \tag{3-33}$$

We see that the closed loop dynamics no longer depend directly on the exogenous disturbance signal w. This allows us to apply the Lyapunov-based stability analysis proposed by [22], described in section 2-3. The result is presented in the following proposition.

Proposition 3. *The bilinear output regulation problem of the bilinear dynamical system* (3-26)*, with* $P = B$ *and* $M = N$ *and the linear exosystem* (3-15)*, is locally solved by the linear dynamic error feedback controller* (3-16)*, provided that the following conditions hold: (1) the linear regulator equations and internal model principle* (3-25) *are satisfied; (2) the matrix* A_c*, given in equation* (3-17)*, is Hurwitz; (3) if* x_0 *and* ξ_0 *are such that*

$$\begin{bmatrix} \xi_0 - w_0 \\ z_0 \end{bmatrix} = s_0 \in \mathcal{E} = \{s \in \Xi \times Z \mid s^T W^{-1} s \leq 1\}, \tag{3-34}$$

with $\epsilon > 0$ *and* $0 \preccurlyeq W = W^T \in \mathbb{R}^{(N+r)\times(N+r)}$*, satisfying*

$$\begin{bmatrix} WA_c^T + A_cW + \epsilon \tilde{N} W \tilde{N}^T & W\begin{bmatrix} H & K \end{bmatrix}^T \\ \begin{bmatrix} H & K \end{bmatrix} W & -\epsilon I \end{bmatrix} \preccurlyeq 0. \tag{3-35}$$

Proof. The closed loop dynamics of the disconnected plant obeys

$$\dot{\xi} = F\xi + GCz, \tag{3-36a}$$
$$\dot{z} = (A + BK)z + BH\xi + NzKz + BzH\xi, \tag{3-36b}$$

which has an equilibrium point at $(\xi, z) = (0, 0)$. Its Jacobian is equal to A_c, which is Hurwitz by assumption. Hence, $(\xi, z) = (0, 0)$ is an exponentially stable equilibrium point of the system, which is the first requirement to solve the nonlinear output regulation problem.

Next, define the error variable $\tilde{\xi} = \xi - \Sigma w$. In these coordinates, the closed loop dynamics are given by equations (3-15), (3-28a) and (3-31). Since the linear internal model principle is satisfied and $H\Sigma + E = 0$, the closed loop dynamics in the error coordinates reduce to

$$\dot{\tilde{\xi}} = F\tilde{\xi} + GCz, \tag{3-37a}$$
$$\dot{z} = (A + BK)z + NzKz + BH\tilde{\xi} + NzH\tilde{\xi}. \tag{3-37b}$$

We see that $(\tilde{\xi}, z) = (0, 0)$ is an equilibrium point of the system, which, since the Jacobian is again equal to A_c, is locally exponentially stable. Using the definitions (3-32), the closed loop dynamics (3-37) are equivalently described by equation (3-33). Next, define the quadratic Lyapunov function

$$V(s) = s^T W^{-1} s. \tag{3-38}$$

The time derivative of (3-38) along the closed loop dynamics (3-33) is given by

$$\dot{V}(s) = s^T(A_c^T W^{-1} + W^{-1}A_c + \tilde{K}^T s^T \tilde{N}^T W^{-1} \tag{3-39}$$
$$+ W^{-1}\tilde{N}s\tilde{K} + \tilde{H}^T s^T \tilde{N}^T W^{-1} + W^{-1}\tilde{N}s\tilde{H})s. \tag{3-40}$$

Thus, $V(s)$ is a Lyapunov function for the dynamics (3-37) if

$$A_c^T W^{-1} + W^{-1}A_c + \tilde{K}^T s^T \tilde{N}^T W^{-1} + W^{-1}\tilde{N}s\tilde{K} + \tilde{H}^T s^T \tilde{N}^T W^{-1} + W^{-1}\tilde{N}s\tilde{H} \preccurlyeq 0 \tag{3-41}$$

Pre- and post multiplying the inequality (3-41) with W yields the equivalent inequality

$$WA_c^T + A_c W + W\tilde{K}^T s^T \tilde{N}^T + \tilde{N} s \tilde{K} W + W\tilde{H}^T s^T \tilde{N}^T + \tilde{N} s \tilde{H} W \preccurlyeq 0. \tag{3-42}$$

Let $T = WA_c^T + A_c W$, $D = \tilde{N}^T$ and $V = W(\tilde{K} + \tilde{H})$, then the inequality (3-42) is written as

$$T + V^T s^T D^T + DsV \preccurlyeq 0, \tag{3-43}$$

with $T = T^T$. Because $s_0 \in \mathcal{E}$ (see (3-34)) it follows by Lemma 2 that the inequality (3-42) is satisfied if and only if there exist $\epsilon > 0$ such that

$$\begin{bmatrix} T + \epsilon DWD^T & WV^T \\ VW & -\epsilon I \end{bmatrix} \preccurlyeq 0, \tag{3-44}$$

which is exactly (3-35). The inequality (3-44) is satisfied by assumption, implying $\dot{V}(s_0) \leq 0$. As as result, any trajectory $s(t)$ for which $s_0 \in \mathcal{E}$ will tend to 0, which means $z(t)$ tends to zero and

$$\lim_{t\to\infty} e(t) = \lim_{t\to\infty} Cz(t) = 0, \tag{3-45}$$

hence the output is regulated and the bilinear output regulation is locally solved. □

The equivalence of the nonlinear dynamical system (3-30) and the bilinear dynamical system (3-26) leads to the following important corollary.

Corollary 3.2. *The nonlinear output regulation problem of the system* (3-30)*, with* $p(x) = g(x)$*, and the linear exosystem* (3-15)*, is locally solved by the linear dynamic error feedback controller* (3-16) *for the set of initial conditions* $\{\xi_0, x_0, w_0\}$*, satisfying* (3-34)*, provided that the following conditions hold: (1) the assumptions of Lemma 3 are satisfied; (2) the initial condition satisfies* $z_0 = \Psi(x_0)$*; (3) the assumptions of Proposition 3 are satisfied.*

Proof. If $z_0 = \Psi(x_0)$, then, by Lemma 3, the nonlinear system (3-30) is equivalently described by (3-26), with $P = B$ and $M = N$. The corollary then follows from Proposition 3. □

3-3-1 Numerical example

In this section, we verify the results of Proposition 3 and Corollary 3.2 by means of a numerical simulation of an example system. We design a controller that satisfies the assumptions of Corollary 3.2 and simulate the closed loop dynamics of the nonlinear system under the effect of this controller.

Inspired by [19], we consider an example system that is nonlinear system given by

$$\dot{x}_1 = \nu x_1 + x_1(u + v), \tag{3-46a}$$

$$\dot{x}_2 = \kappa(x_2 - x_1^2) + u + v, \tag{3-46b}$$

$$e = x_2 + x_1^2 x_2 + x_1 x_2^2, \tag{3-46c}$$

where u is a scalar input and $v \in \mathbb{R}$ the output of the sinusoidal exosystem

$$\dot{w}_1 = -\omega w_2, \tag{3-47a}$$

$$\dot{w}_2 = \omega w_1, \tag{3-47b}$$

$$v = w_1. \tag{3-47c}$$

Thus $v(t) = w_{10}\cos(\omega t) + w_{20}\cos(\omega t)$. The affine term in the first state highlights the use of property 2 of Lemma 3, and the nonlinear terms in the output highlights property 3. We may omit either of these terms and still apply the method.

To start, we choose a dictionary $\mathcal{D}$ such that the properties of lemma 3 are satisfied, so that the system is equivalently represented by a Koopman bilinear dynamical model. The equivalent bilinear dynamical model and the details of its construction are given in appendix A.

We use a linear dynamic error feedback controller as described in proposition 3. We choose K such that $A + BK$ is Hurwitz, and subsequently G such that A_c is Hurwitz. We fix the number $\epsilon = 0.01$ and determine W such that the matrix inequality (3-35) is satisfied. The initial conditions w_{10} and w_{20} are randomly sampled from the interval $[-1, 1]$. Next, the initial state of the plant x_0 and controller ξ_0 are randomly chosen such that $z_0 = \Phi(x_0)$ and $s_0 \in \mathcal{E}$. The trajectory of the system is then determined using Runge-Kutta integration. Figure 3-1 shows quantities of interest for a typical trajectory of the closed loop system that is initiated as described above. The top left and right plots show the value of the Lyapunov and its time derivative along the trajectory of the closed loop dynamics $z(t) = \Psi(x(t))$ as a function of t, respectively. The bottom right plot shows the output error $e(t)$ of the closed loop system as a function of time. Finally, the bottom right figure shows the residual of the disturbance signal $g(x(t))(H\xi(t) + v(t))$ (consisting of two components). as a function of time. Observe that the Lyapunov function always satisfies $V(z(t)) \leq 1$ and $\dot{V}(z(t)) \leq 0$, as it should. Furthermore, we see that the disturbances are successfully rejected, and the output error is driven to zero. Hence, the output of the nonlinear plant is successfully regulated.

Figure 3-2 shows the limit of the error[5] for 100 different trajectories satisfying the initial conditions and closed loop dynamics described above. The top figure shows the value of $s_0^T W^{-1} s_0$ and the bottom figure the corresponding limit of the error. We see that, without exception, the error is driven to machine precision, and hence we conclude that the output is regulated in the limit.

Note that in the above simulation we have made no effort to find a W for which the set of initial conditions is large (in terms of magnitude of the initial states $\tilde{\xi}_0$ and x_0 in the 2-norm.). And actually, they are quite small. Figure 3-3 shows the relevant quantities for a trajectory for which the initial condition is chosen to be much larger than those satisfying the constraint $s_0 W^{-1} s \leq 1$ (for W as chosen in the previous simulation). We see that the disturbance is still successfully rejected and the output is regulated. Furthermore, observe that $V(s)$ is still a Lyapunov function for this trajectory, even though $\max V(s) \approx 1900$. This suggest that the result may be generalized to $V(s) \leq \gamma$ with $\gamma > 1$. Indeed, [21] provides a generalization of

[5] With this we mean the average error over a fixed interval.

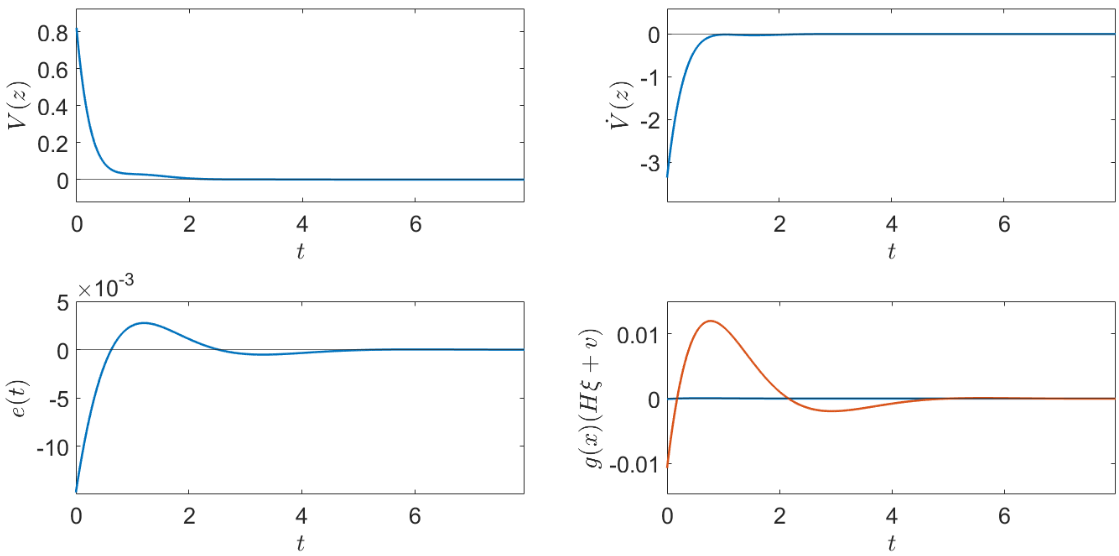

Figure 3-1: Closed loop dynamics of the example system (3-46) initiated such that $s_0 W^{-1} s_0 \leq 1$ is satisfied. *Top left:* Lyapunov function as a function of time. *Top right:* Time derivative of the Lyapunov function along the closed loop dynamics as a function of time. *Bottom left:* Output error e as a function of time. *Bottom right:* Residual of the disturbances $g(x)(H\xi + v)$ as a function of time.

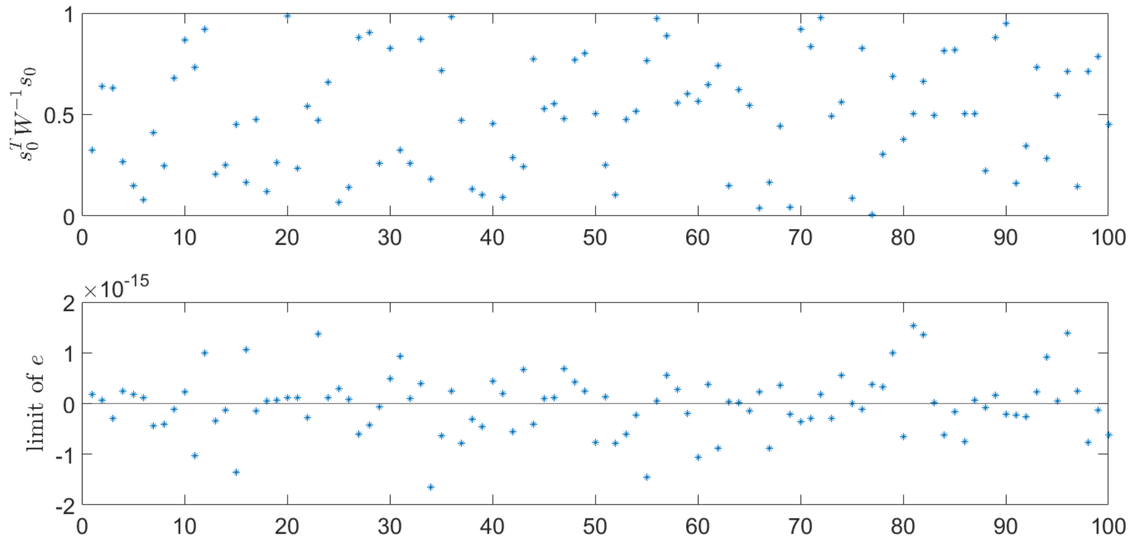

Figure 3-2: Distribution of the output error e in the limit for 100 trajectories of (3-46), satisfying the ellipsoidal constraint $s_0 W^{-1} s_0 \leq 1$.

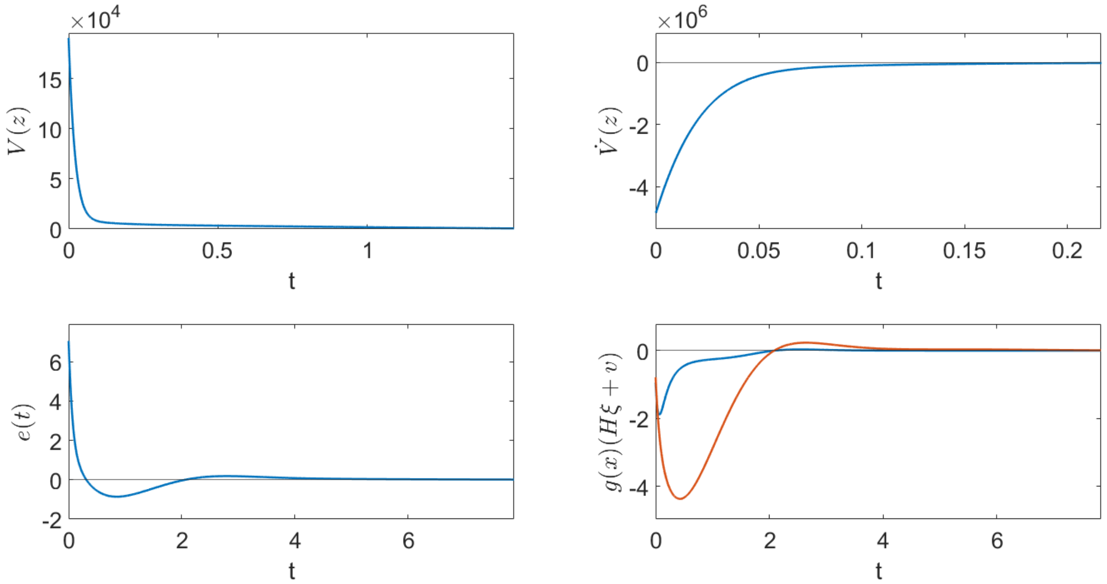

Figure 3-3: Closed loop dynamics of the example system (3-46) initiated with $s_0 W^{-1} s_0 \approx 1900$. *Top left:* Lyapunov function as a function of time. *Top right:* Time derivative of the Lyapunov function along the closed loop dynamics as a function of time. *Bottom left:* Output error e as a function of time. *Bottom right:* Residual of the disturbance $g(x)(H\xi + v)$ as a function of time.

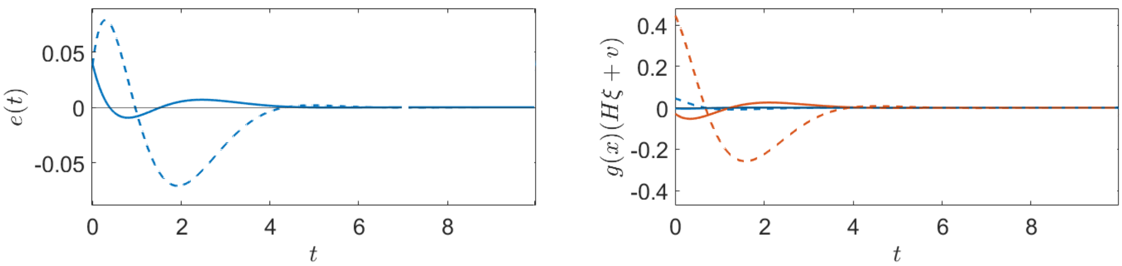

Figure 3-4: Closed loop dynamics of the example system (3-46), initiated such that $s_0 W^{-1} s_0 \leq 1$ is satisfied, using a controller based on the linearization (*dotted line*) and one based on the Koopman bilinear model (*solid line*). *Left:* Output error e as a function of time. *Right:* Residual of the disturbance $g(x)(H\xi + v)$ as a function of time.

Petersen's lemma that addresses precisely this, the result may be adapted to lemma 2 and applied to the problem at hand. More interestingly, we might be able to find a different ϵ and W (and K for that matter) for which $V(s_0) \leq 1$ is valid, but for a larger set[6] of initial conditions s_0. [22] suggests a semi-definite program to achieve this.

Finally, figure 3-4 shows a comparison of the method compared to a controller design based on the linearization of (3-46). The dynamics of the closed loop system, using the controller obtained from the linearization, are given by the dotted line, and using the controller obtained from the Koopman-based method, by the solid line. In both cases, the gain K is obtained using linear quadratic control, and the same matrix G is used. The system is initiated such that $s_0 W^{-1} s_0 \leq 1$ is satisfied. We see that the transient behavior of the output error of the controller based on the linearization is larger than the Koopman-based method. The reason for this is that the controller derived from the Koopman-based method is a nonlinear controller in the state x and utilizes nonlinear terms that appear in the dictionary used to derive the bilinear dynamical model.

3-4 Future directions

We finalize the chapter by mentioning possible paths for future research for the use of Koopman operator for the nonlinear output regulation problem. We mention a few future paths concerning the bilinear output regulation approach. We conclude with a novel suggestion for a data-driven approach of the nonlinear output regulation problem, which has not been considered so far in this thesis.

Partial state information. For most of the results, we have assumed full-state information. In practice, this is often not the case, and only output data is available. The stabilizing part of the linear dynamic error feedback controller (3-16) may be incorporated in the controller dynamics instead. For linear plants, this is achieved using the separation principle. A state-feedback controller and state-observer are designed separately and combined to achieve stabilization. However, for bilinear plants, the error dynamics of the state-observer depend on the input, thus the separation principle does not hold. It is therefore of interest to identify how a linear dynamic error feedback controller, without a state-feedback component, can be used to solve the bilinear output regulation problem, and thereby the equivalent nonlinear output regulation problem.

Quadratic stability for general disturbance signals. In the previous section, we focused on the matched input disturbance case and characterized a set of initial conditions for which output regulation is guaranteed. Essential in the argument was the fact that the closed loop dynamics (3-37) did not directly depend on the exogenous disturbance w. The reason for this was that both linear and bilinear disturbances were canceled by the same control action. In

[6] If $\mathcal{E}_i = \{s \in \mathbb{R}^{N+\nu} \mid s^T W_i^{-1} s\} \leq 1$ and $s_i^{\max} = \max_s \{||s||_2 \mid s \in \mathcal{E}_i\}$, then $\mathcal{E}_1$ is larger than $\mathcal{E}_2$ if $s_1^{\max} > s_2^{\max}$. Alternatively, one may consider a set larger than the other if the average of the two norm of its elements is larger.

the general disturbance case, considered in section 3-2, this was not the case, see equation (3-28). However, since the exogenous disturbance signal w is neutrally stable, it is bounded, i.e. $||w|| \leq \gamma$, for some $\gamma in \mathbb{R}$. We therefore expect that a similar result as in proposition 3 can be obtained for the general disturbance case. The stability argument must be modified to account for the additional bilinear terms in w.

Reference tracking. In tackling the bilinear output regulation problem we have assumed that no reference signal was present. This is an important part of the output regulation problem and cannot be ignored. In section 3-2, we have seen that linear error dynamic error feedback cannot be used to achieve output regulation of the bilinear dynamical system in the presence of a reference signal. Nonlinear techniques must therefore be explored to tackle this problem.

Extended Dynamic Mode Decomposition for nonlinear output regulation. One of the main merits of the Koopman operator framework is that it is particularly well suited for data-driven methods. The most famous example is the Extended dynamic mode decomposition (EDMD) algorithm [34]. Given a set of observable functions, the EDMD algorithm provides a best-fit linear dynamical system, in the least-squares sense, for a (nonlinear) autonomous dynamical system from data. EDMD uses as data a set of K snapshot pairs $\{x_i, x_i^+\}_{i=1}^K$, with $x_i^+ = \boldsymbol{F}(x_i)$ and $x_i, x_i^+ \in \mathcal{X} \subseteq \mathbb{R}^n$. Recall that $\boldsymbol{F}$ is the flow of the system autonomous system (2-1) given by (2-2). The EDMD algorithm is extended to actuated systems in [33].

We now sketch a proposal for the use of EDMD for actuated systems to obtain a linear dynamical model of the nonlinear system (2-38) out of data. The resulting linear system will be of the form (2-27). We subsequently solve the linear output regulation problem as described in section (2-2-1) and obtain a linear dynamic error feedback controller of the form (2-28).

The method requires one to specify a dictionary of observable functions. Looking at the system (2-38), we see that, in general, the dictionary must contain observable functions $\psi \in \mathcal{F}$, $\phi \in \mathcal{W}$, and $\theta \in \mathcal{F} \times \mathcal{W}$. Here, $\mathcal{F}$ denotes the space of functions of the plant state x and $\mathcal{W}$ the space of functions of the exogenous disturbance state w. For a finite dictionary, we define the transformations

$$z_x = \Psi(x), \tag{3-48a}$$

$$z_w = \Phi(w), \tag{3-48b}$$

$$z_{xw} = \Theta(x, w), \tag{3-48c}$$

where $\Psi(x)$, $\Phi(w)$ and $\Theta(x, w)$ are vector valued observable functions consisting of all the observable functions in $\mathcal{D}$ of the type $\psi : X \mapsto \mathbb{R}$, $\phi : W \mapsto \mathbb{R}$ and $\theta : X \times W \mapsto \mathbb{R}$, respectively.

The data consists of the snapshot pairs in the plant state $\{x_i, x_i^+\}_{i=1}^K$, exogenous disturbance $\{w_i, w_i^+\}_{i=1}^K$, output error $\{e_i, e_i^+\}_{i=1}^K$ and input $\{u_i, u_i^+\}_{i=1}^K$. The linear dynamical model is then obtained from the data by solving the following optimization problem:

$$\min_{A,B,P,S} \sum_{j=0}^{K} \left\| \begin{bmatrix} \Psi(x_{j+1}) \\ \Theta(x_{j+1}, w_{j+1}) \\ \Phi(w_{j+1}) \end{bmatrix} - \begin{bmatrix} A_{11} & A_{12} & P_1 \\ A_{21} & A_{22} & P_2 \\ 0 & 0 & S \end{bmatrix} \begin{bmatrix} \Psi(x_j) \\ \Theta(x_j, w_j) \\ \Phi(w_j) \end{bmatrix} - \begin{bmatrix} B_1 \\ B_2 \\ 0 \end{bmatrix} u_j \right\|_2^2 \tag{3-49}$$

$$\min_{C,Q} \sum_{j=0}^{K} ||e_j - C\Psi(x_j) - Q\Phi(w_j)||_2^2. \tag{3-50}$$

Often, only input-output data is available. The optimization procedure may then be modified to make use of time-delayed measurements of the input and output, as suggested by [26, 4].

Note that the resulting Koopman model we obtain from data is a linear dynamical model instead of a bilinear dynamical model. Why this is the case can be seen by looking at the closed loop dynamics of the bilinear Koopman model under linear state feedback (which is nonlinear in the original state), given by $\dot{z} = (A + BK)z + NzKz + Pw + Mzw$ and $\dot{w} = Sw$. The closed loop dynamics are autonomous in the extended state $[z^T, w^T]^T$. The dynamics of the Koopman operator can be approximated on an approximate Koopman invariant subspace. This subspace consists of functions of the extended state and thus functions of x and w. This observation provides a rationale for the choice of a linear dynamical model on a set of observable functions in x and w, instead of a bilinear dynamical model on a set of observable functions in x.

Chapter 4

Conclusion

In this thesis, we have demonstrated a novel approach to the nonlinear output regulation problem that utilizes the Koopman operator. The Koopman operator was used to identify a class of nonlinear control systems that can be equivalently described by a bilinear dynamical system. We have subsequently solved the bilinear output regulation problem using linear dynamic error feedback for the case of disturbance rejection. In particular, for the case of matched input disturbance, we have used a Lyapunov-based stability argument to characterize a set of initial conditions for which output regulation is guaranteed and validated the result by presenting a numerical example.

Many interesting problems can be framed as an output regulation problem and the Koopman operator is a promising modeling paradigm that can help our understanding of nonlinear systems. With this thesis, I hope to initiate an interest in the marriage of these frameworks to advance our theoretical understanding of numerous problems and push the boundaries for new applications.

Appendix A

Construction of the bilinear Koopman model for example system

In this appendix, we show the construction of the dictionary $\mathcal{D}$ for the example considered in section 3-3, that satisfies the properties of Lemma 3. Subsequently, we obtain the Koopman bilinear model. The plant in consideration is given by

$$\dot{x}_1 = \nu x_1 + x_1(u+v), \tag{A-1a}$$

$$\dot{x}_2 = \kappa(x_2 - x_1^2) + u + v, \tag{A-1b}$$

$$e = x_2 + x_1^2 x_2 + x_1 x_2^2. \tag{A-1c}$$

with $x \in X \subseteq \mathbb{R}^2$, $u, v \in \mathbb{R}$ and $e \in \mathbb{R}$. In the notation of equation (3-30) we have

$$f(x) = \begin{pmatrix} \nu x_1 \\ \kappa x_2 - \kappa x_1^2 \end{pmatrix}, \quad g(x) = p(x) = \begin{pmatrix} x_1 \\ 1 \end{pmatrix}, \quad h(x) = x_1 + x_2 + x_1^2 x_2 + x_1 x_2^2.$$

We show the construction of a dictionary $\mathcal{D}$ that satisfies the properties of Lemma 3 w.r.t. to the nonlinear system (A-1).

Consider the dictionary $\mathcal{D}_1 = \{\psi_1, \psi_2, \psi_3, \psi_4, \psi_5\}$ with

$$\begin{aligned} \psi_1(x) &= x_1, \\ \psi_2(x) &= x_2, \\ \psi_3(x) &= x_1^2, \\ \psi_4(x) &= x_1^2 x_2, \\ \psi_5(x) &= x_1 x_2^2. \end{aligned}$$

The functions $\psi_1(x)$ and $\psi_2(x)$ are the state projections, and therefore property 4 of Lemma 3 is satisfied. Furthermore, the output (A-1c) can be written as $e = \psi_1 + \psi_2 + \psi_4 + \psi_5$, thus property 3 of Lemma 3 is satisfied.

The time derivative of an observable function $\psi = x_1^n x_2^m$, along the trajectories of (A-1), is given by

$$\begin{aligned}\dot{\psi} &= n x_1^{n-1} x_2^m \dot{x}_1 + m x_1^n x_2^{m-1} \dot{x}_2 \\ &= n\nu x_1^n x_2^m + m\kappa(x_1^n x_2^m - x_1^{n+2} x_2^{m-1}) \\ &= (n\nu + m\kappa) x_1^n x_2^m - m\kappa x_1^{n+2} x_2^{m-1}. \end{aligned} \tag{A-2}$$

Using (A-2) we find that derivatives of along the trajectories of (A-1) of each $\psi \in \mathcal{D}_1$, in the case that $u, v = 0$, are given by

$$\begin{aligned}\dot{\psi}_1(x) &= \nu x_1, \\ \dot{\psi}_2(x) &= \kappa(x_2 - x_1^2), \\ \dot{\psi}_3(x) &= 2\nu x_1^2, \\ \dot{\psi}_4(x) &= (2\nu + \kappa) x_1^2 x_2 - \kappa x_1^4, \\ \dot{\psi}_5(x) &= (\nu + 2\kappa) x_1 x_2^2 - 2\kappa x_1^3 x_2. \end{aligned}$$

The functions $x_1^3 x_2$ and x_1^4 are not in $\mathcal{D}_1$, hence $\mathcal{D}_1$ is not a Koopman invariant subspace. Motivated by this observation, let $\mathcal{D}_2 = \{\psi_6, \psi_7, \psi_8\}$ with

$$\begin{aligned}\psi_6(x) &= x_1^4, \\ \psi_7(x) &= x_1^3 x_2, \\ \psi_8(x) &= x_1^5. \end{aligned}$$

Again, when $u, v = 0$, we find using equation (A-2) that

$$\begin{aligned}\dot{\psi}_6(x) &= 4\nu x_1^4, \\ \dot{\psi}_7(x) &= (3\nu + \kappa) x_1^3 x_2 - \kappa x_1^5, \\ \dot{\psi}_8(x) &= 5\nu x_1^5. \end{aligned}$$

We see that $\mathcal{D}_1 \cup \mathcal{D}_2$ is a Koopman invariant subspace w.r.t. the autonomous dynamics of (A-1), which is exactly property 1 of Lemma 3.

The dictionary $\mathcal{D}_1 \cup \mathcal{D}_2$ does not satisfy property 2. To see this, write down the partial derivatives of the observable functions w.r.t. x_1 and x_2 and multiply these with components $g_1(x)$ and $g_2(x)$, respectively. Using the notation $\partial_i = \frac{\partial}{\partial x_i}$, we find

$$\begin{aligned}
(\partial_1\psi_1(x))g_1(x) &= x_1, & (\partial_2\psi_1(x))g_2(x) &= 0, \\
(\partial_1\psi_2(x))g_1(x) &= 0, & (\partial_2\psi_2(x))g_2(x) &= 1, \\
(\partial_1\psi_3(x))g_1(x) &= 2x_1^2, & (\partial_2\psi_3(x))g_2(x) &= 0, \\
(\partial_1\psi_4(x))g_1(x) &= 2x_1^2 x_2, & (\partial_2\psi_4(x))g_2(x) &= x_1^2, \\
(\partial_1\psi_5(x))g_1(x) &= x_1 x_2^2, & (\partial_2\psi_5(x))g_2(x) &= 2x_1 x_2, \\
(\partial_1\psi_6(x))g_1(x) &= 4x_1^4, & (\partial_2\psi_6(x))g_2(x) &= 0, \\
(\partial_1\psi_7(x))g_1(x) &= 3x_1^3 x_2, & (\partial_2\psi_7(x))g_2(x) &= x_1^3, \\
(\partial_1\psi_8(x))g_1(x) &= 5x_1^5, & (\partial_2\psi_8(x))g_2(x) &= 0.
\end{aligned}$$

Indeed, $2x_1x_2$ and x_1^3 are not in the span of $\mathcal{D}_1 \cup \mathcal{D}_2$. To satisfy property 2[1] of Lemma 3, we require (linear combinations of)

$$\psi_9(x) = x_1x_2,$$
$$\psi_{10}(x) = x_1^3.$$

Therefore, let $\mathcal{D}_3 = \{\psi_9, \psi_{10}\}$. The time derivatives of ψ_9 and ψ_{10} along the trajectories of (A-1) are given by

$$\dot{\psi}_9(x) = (\nu + \kappa)x_1x_2 - \kappa x_1^3,$$
$$\dot{\psi}_{10}(x) = 3\nu x_1^3.$$

Furthermore, the partial derivatives w.r.t. x_1 and x_2 multiplied by $g_1(x)$ and $g_2(x)$, respectively, read

$$(\partial_1\psi_9(x))g_1(x) = x_1x_2, \qquad (\partial_2\psi_9(x))g_2(x) = x_1,$$
$$(\partial_1\psi_{10}(x))g_1(x) = 3x_1^3, \qquad (\partial_2\psi_{10}(x))g_2(x) = 0,$$

and are contained in the span of $\mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3$. Therefore, let $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3$. The dictionary $\mathcal{D}$ satisfies all four properties of Lemma 3.

A-1 Deriving the Koopman bilinear model

We now determine the bilinear Koopman model associated with the dictionary $\mathcal{D}$. For each $\psi \in \mathcal{D}$ the time derivative along the trajectories of (A-1) are given by

$$\dot{\psi} = \sum_{i=1}^{2} \frac{\partial \psi}{\partial x_i}\dot{x}_i = \sum_{i=1}^{2} \frac{\partial \psi}{\partial x_i}\Big(f_i(x) + g_i(x)(u+v)\Big). \tag{A-3}$$

Thus we find

$$\dot{\psi}_1(x) = \nu\psi_1(x) + \psi_1(x)(u+v)$$
$$\dot{\psi}_2(x) = \kappa\psi_2(x) - \kappa\psi_3(x) + u + v$$
$$\dot{\psi}_3(x) = 2\nu\psi_3(x) + 2\psi_3(x)(u+v)$$
$$\dot{\psi}_4(x) = (2\nu+\kappa)\psi_4(x) - \kappa\psi_6(x) + (\psi_3(x) + 2\psi_4(x))(u+v)$$
$$\dot{\psi}_5(x) = (\nu+2\kappa)\psi_5(x) - 2\kappa\psi_7(x) + (\psi_5(x) + 2\psi_9(x))(u+v)$$
$$\dot{\psi}_6(x) = 4\nu\psi_6(x) + 4\psi_6(x)(u+v)$$
$$\dot{\psi}_7(x) = (3\nu+\kappa)\psi_7(x) - \kappa\psi_8(x) + (3\psi_7 + \psi_{10}(x))(u+v)$$
$$\dot{\psi}_8(x) = 5\nu\psi_8(x) + 5\psi_8(x)(u+v)$$
$$\dot{\psi}_9(x) = (\nu+\kappa)\psi_9(x) - \kappa\psi_{10}(x) + (\psi_1(x) + \psi_9(x))(u+v)$$
$$\dot{\psi}_{10}(x) = 3\nu\psi_{10}(x) + 3\psi_{10}(x)(u+v)$$

[1] Actually, we only need to satisfy property 2 for the non-constant terms $\partial_i g_i$.

Next, define the transformation

$$z = \Psi(x), \tag{A-4}$$

with $\Psi(x) = [\psi_1(x), \ldots, \psi_{10}]^T$. Then we find

$$\dot{z} = Az + B(u+v) + Nz(u+v), \tag{A-5a}$$
$$e = Cz, \tag{A-5b}$$

with

$$A = \begin{bmatrix} \nu & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \kappa & -\kappa & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2\nu & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\nu+\kappa & 0 & -\kappa & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \nu+2\kappa & 0 & -2\kappa & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4\nu & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3\nu+\kappa & -\kappa & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \nu+\kappa & -\kappa \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3\nu \end{bmatrix}, \tag{A-6}$$

$$B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad N = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}, \tag{A-7}$$

$$C = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{A-8}$$

Proposition 3 requires the matrix A_c, given by equation (3-17), to be Hurwitz. To achieve this, the pair (A, B) must be stabilizable. By inspecting A and B, we see that the only controllable mode (in the linear sense, i.e., ignoring the bilinear term) is the one corresponding to the second row. Therefore, for the pair (A, B) to be stabilizable, we require all other modes to be (asymptotically) stable. The triangular structure of A allows us to easily identify the system's eigenvalues. They correspond to the elements that appear on the diagonal. We see that the stability of these modes depends on the values of ν and κ. For instance, the eigenvalue corresponding to the observable function ψ_5 is given by $\mu_5 = \nu + 2\kappa$. If μ_5 is larger than zero, the corresponding mode is unstable (in the linear sense). However, the eigenvalues of the matrix A depend on the choice of the transformation (A-4). In this example, we can redefine the transformation (A-4) such that instead of $\psi_5(x)$, it includes $-\psi_5(x)$, i.e., we set

$\Psi(x) = [\psi_1(x), \ldots, -\psi_5(x), \ldots, \psi_{10}(x)]^T$. Because of the triangular structure, all elements in the fifth row of A, B, C, and N gain a minus sign. Consequently, the corresponding mode is stable. Because of the triangular structure, we can do this for all non-zero eigenvalues of A that correspond to uncontrollable modes. Thus, if all the eigenvalues that correspond to uncontrollable modes are non-zero, we can make the pair (A, B) stabilizable. This is true when ν is unequal to κ whenever it is multiplied by the integers 1, 2 and 3.

In the simulation we use the values $\nu = -0.7$ and $\kappa = 1$. The only positive Koopman eigenvalues are those of $\psi_1(x)$, $\psi_5(x)$ and $\psi_9(x)$ (w.r.t. the autonomous system), and are given by

$$\begin{aligned}
\mu_2 &= \kappa = 1, \\
\mu_5 &= \nu + 2\kappa = 0.3 \\
\mu_9 &= \nu + \kappa = 1.3,
\end{aligned}$$

respectively. The modes corresponding to $\psi_5(x)$ and $\psi_9(x)$ are uncontrollable, thus we consider the transformation $z = \Psi(x)$, with

$$\Psi(x) = [\psi_1, \psi_2, \psi_3, \psi_4, -\psi_5, \psi_6, \psi_7, \psi_8, -\psi_9, \psi_{10}]^T. \tag{A-9}$$

The system matrices of the bilinear Koopman model are then given by

$$A = \begin{bmatrix}
\nu & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \kappa & -\kappa & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2\nu & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2\nu + \kappa & 0 & -\kappa & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\nu - 2\kappa & 0 & 2\kappa & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 4\nu & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 3\nu + \kappa & -\kappa & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 5\nu & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\nu - \kappa & \kappa \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3\nu
\end{bmatrix}, \tag{A-10}$$

$$B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad N = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -2 & 0 \\
0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 1 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3
\end{bmatrix}, \tag{A-11}$$

$$C = \begin{bmatrix} 1 & 1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0, \end{bmatrix} \tag{A-12}$$

with the pair (A, B) stabilizable.

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