**Delft University of Technology**
**Faculty of Electrical Engineering, Mathematics and Computer Science**
**Delft Institute of Applied Mathematics**

# Explaining interest rate spreads from a debt sustainability indicator

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**MSc THESIS APPLIED MATHEMATICS**

# "Explaining interest rate spreads from a debt sustainability indicator"

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# Preface

This thesis has been submitted for the degree of Master of Science in Applied Mathematics at the Delft University of Technology, the Netherlands and was carried out by the financial consultant Ortec Finance. A company which aims to improve investment decision making by providing consistent solutions for advice and risk management through a combination of market knowledge, mathematical models and information technology. I would like to thank some people who have contributed in the process of writing this thesis. First of all, I would like to acknowledge prof.dr.ir. Kees Oosterlee and dr. Martin van der Schans for their efforts in guiding me trough the whole graduation period. Furthermore, I would like to thank prof.dr.ir. Arnold Heemink from TU Delft for being part of the examination committee. I would also like to thank the whole EFIS and Research department of Ortec Finance for the inspiring working environment. Last but certainly not the least, I would like to express my gratitude to my family and friends, and especially to my fiancé Joyce de Heer, who has supported me through the duration of my study.

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# CHAPTER 1

# Introduction

## 1.1 The debt sustainability indicator

In the face of the ongoing financial crisis in Europe, the sustainability of government debt has recently been one of the topics of debate in economics. A government's debt is sustainable when the government is expected to meet its debts obligation without an unrealistically large future correction to its fiscal policy, see Wyplosz (2007). Here, fiscal policy is the means by which a government adjusts its expenditure and revenue to monitor and influence a nation's economy. Insight into the sustainability of government debt is essential to policy-makers and financial markets for several reasons. Firstly, it gives insight towards the urgency of fiscal consolidation, i.e., the creation of strategies that are aimed at minimizing deficits while also limiting the debt accumulation. Secondly, it is of importance for the determination of the appropriate risk premium on government debt. Finally, serious debt sustainability issues can lead to a sovereign default. During the current financial crisis the government debt has rapidly accumulated for most countries. Hence, there is a need for a reliable and comprehensive measure of government debt sustainability.

As discussed in Lukkezen, Rojas-Romagosa, et al. (2012), static indicators such as the size of the government debt or the budget balance are used to assess the sustainability of government debt. While these indicators are straightforward, they provide little information on the uncertainties concerning the evolution of the government debt. Other factors are at least as important, for instance, the stability of the economic environment and the government's attitude towards debt sustainability. In Ewijk, Lukkezen, and Rojas-Romagosa (2013), a dynamic framework is proposed for assessing the sustainability of government debt. This framework takes account of economic uncertainty underlying the evolution of government debt, and incorporates a comprehensive measure of the responsiveness of fiscal policies to economic setbacks. Using stochastic simulations they evaluate an indicator that can distinguish countries with few to no debt sustainability concerns from countries with serious debt sustainability issues. This indicator will be referred to as the debt sustainability indicator.

In this thesis, we evaluate the debt sustainability indicator with a new model. The essence of this model is to produce forecasts for several economic variables, where the government debt is one of them, based on historical data. With the obtained economic forecasts, we can estimate with the indicator the risk of a significant government debt increase in the near future.

It is expected that countries with debt sustainability issues have a higher risk that the government will default on its bonds or other financial commitments. This latter type of risk is referred to as the country-specific risk. As discussed in Kwark (2002), country-specific risk influences the dynamics of the interest rates that are paid on government bonds, or more precisely the so-called interest rate spreads.

In this thesis, we consider the interest rate spread as the difference between two yields, where one of the interest rates is historically limited available. For several reasons we are interested in the dynamics of the interest rate spreads. Firstly, it is accepted that financial variables, such as stock prices and interest rates, adjust to new information much faster than other variables, see Estrella and Mishkin (1998) for more details. As a result, some variables in the financial market are used as leading indicators to predict future developments of the economy. One of these variables is the term-spread, which is the difference between the 10-years interest rate and the 3-months interest rate. The term-spread is considered as a good indicator for forecasting the future economic growth, see Estrella and Mishkin (1996), Dotsey (1998) and Wheelock, Wohar, et al. (2009) for more details. Secondly, the interest rate spreads are often seen in the financial market as a risk premium for investing in risky loans. As long as some loans are expected to be in default at the maturity date, the risky loan rate should be higher than the risk-free deposit rate in order to compensate for loan default. Hence, predicting the risk premium is for institutional investors of importance. Finally, historical data is limited available for interest rate spreads. Moreover, historical spread levels are not representative by the current changing economic circumstances, see Blinder and Baumol (1993) for more details.

Since the indicator measures whether the government is still in control of the debt evolution, it contains, just like the interest rate spreads, a part of the country-specific risk. Hence, the aim of this MSc project is to investigate whether the dynamics of the interest rate spreads can be explained by the debt sustainability indicator.

## 1.2 Organization of the thesis

We first explain in Chapter 2 the calculation of the debt sustainability indicator. We show that this calculation is easily performed by using the debt samples obtained from a stochastic simulation. The first model which performs the simulation is from Lukkezen, Rojas-Romagosa, et al. (2012). We explain the model dynamics and how this model, which will be referred to as the Benchmark model, produces a government debt prediction based on historical data. However, the Benchmark framework makes model assumptions that can be improved.

In this thesis, we propose a new model, which will be referred to as the Extended model, for evaluating the debt sustainability indicator. In Chapter 3, we explain the dynamics of the Extended model and argue why this is an extension of the Benchmark model. Furthermore, we explain the procedure for economic forecasts with the Extended model. And finally, we investigate whether the forecast accuracy of the Extended model is improved by including more information of the model's previous states.

In Chapter 4, we show that the Extended model doesn't always produce realistic economic forecasts, based on historical data. The reason for this is that the historical data is not always representative for predicting the future. In order to overcome this, we implement a new fiscal rule in the dynamics of the Extended model.

In Chapter 5, we investigate whether the debt sustainability indicator, which is calculated using the Extended model, can explain the dynamics of the interest rate spreads. In particular, we are interested in how much explanatory power the indicator contains. For this purpose, we must first specify which interest rate spreads are considered in this MSc-thesis. After this, we formulate the criteria that are used to verify the relationship between the indicator and the interest rate spreads. Finally, for several countries we discuss the results of verifying this relationship.

The main topic of Chapter 4 has let us to investigate the influence of the model parameters on the stability of the system. Therefore, in Chapter 6, we perform a stability analysis for the Extended model. In particular, we derive conditions for the existence of blow-up behaviour using general theory about discrete dynamical systems.

Finally, Chapter 7 contains the overall conclusions, further recommendations and suggestions for future research.

# CHAPTER 2

## Indicator for assessing government debt sustainability

In this chapter, we discuss an indicator for assessing government debt sustainability. The sustainability of government debt is one of the most discussed topics of the current Euro crisis. In Section 1.1, we start by introducing a model from Lukkezen, Rojas-Romagosa, et al. (2012) which describes the future evolution of the government debt. This model will be referred to as the Benchmark model. We will show that three main variables are used for describing the debt dynamics: the 10-years real interest rate, the annual return of real GDP and an error-correction component. Next, we discuss the assumptions that are made in the Benchmark model in order to produce a stochastic debt simulation. By producing samples of a future debt distribution, we can evaluate the debt sustainability indicator. In section 1.2, we discuss in more detail the calculation of the debt sustainability indicator, which is evaluated with the Benchmark model in Ewijk, Lukkezen, and Rojas-Romagosa (2013).

## 2.1 The Benchmark Model

In this section, we discuss the Benchmark model for describing the evolution of the government debt. We will show that the Benchmark model consists of the following components: the accounting equation for the government debt, an equation for the government's primary surplus and a two-dimensional linear system for the annual return of real gross domestic product (GDP) and the 10-years real interest rate. The government's primary surplus is the difference of the total revenue and total primary expenditure. Here, the primary expenditure excludes interest payments on outstanding obligations. And moreover, a financial variable is quoted as real when it is inflation-adjusted.

First, we explain the dynamics of the government debt. In Lukkezen, Rojas-Romagosa, et al. (2012), these dynamics are given by the budget accounting equation:

$$D_{t+1} = (D_t - S_t)(1 + R_t). \tag{2.1}$$

In equation (2.1), $D_t$ is the amount of government debt, $R_t$ the nominal, or not inflation-adjusted, interest rate with a maturity of 10-years and $S_t$ is the amount of government primary surplus at the beginning of year $t$. Using percentage of nominal GDP provides an economic meaningful scaling factor for analysing the amount of public debt. Therefore, the next step is to divide $D_t$

by the nominal GDP $Y_t$:

$$\frac{D_{t+1}}{Y_{t+1}} = \frac{(D_t - S_t)(1 + R_t)}{Y_t(1 + G_t)}, \tag{2.2}$$

where $G_t$ is the annual return of nominal GDP . Expression (2.2) can be simplified to:

$$d_{t+1} = (d_t - s_t)\frac{1 + r_t}{1 + y_t}, \tag{2.3}$$

where $d_t = \frac{D_t}{Y_t}$, $s_t = \frac{S_t}{Y_t}$, $r_t$ is the 10-years real interest rate and $y_t$ is the annual return of real GDP. For the derivation of (2.2), we can also use the nominal interest rate and the annual return of nominal GDP since inflation cancels out in the difference.

Equation (2.3) shows that the dynamics of the debt-to-GDP ratio, which will be referred to as the debt, is described by the 10-years real interest rate $r_t$, the return of annual real GDP $y_t$ and an error-correction component $(d_t - s_t)$. The error-correction component indicates whether the government increases its primary surplus due to changes in the debt. Hence, the dynamics of the primary surplus must be specified for investigating the evolution of the error-correction component. The primary surplus $s_t$ is given by:

$$s_t = \alpha + \rho d_t + \beta_1 \text{YVAR}_t + \beta_2 \text{GVAR}_t + \epsilon_t, \tag{2.4}$$

where $\alpha \in \mathbb{R}$ and the error term $\epsilon_t \sim \mathcal{N}(0, \sigma)$. The fiscal reaction parameter $\rho$ provides information on the long-term country-specific behaviour towards debt sustainability. In other words, the parameter provides information on the historical fiscal reaction, i.e, adjusting its policy for the revenue and expenditure, to changes in the government debt. The $\text{YVAR}_t$ is the cyclical component of the real GDP time series expressed as a percent deviation at time $t$. To be more precise, the $\text{YVAR}_t$ is calculated in the following way:

1. Denote the historical real GDP time series by $\{Z_t, t = t_0, \ldots, t_N\}$, where $t_N = t_0 + N$.
2. The trend component $\tau_t$ of $\log(Z_t)$ is extracted with a two-sided HP filter ($\lambda = 100$). In Appendix A, we discuss in more detail how this filter extracts the trend and cycle of a time series.
3. Then, the value of $\text{YVAR}_t$ is defined as:

$$\text{YVAR}_t = \frac{Z_t - \exp(\tau_t)}{Z_t}. \tag{2.5}$$

The remaining variable $\text{GVAR}_t$ in (2.4) is a measure of temporary government spending. For the United Kingdom and the United States, the temporary government spending was driven by the military spending. Therefore, the $\text{GVAR}_t$ is defined as the military spending for these two countries. For the remaining countries, the $\text{GVAR}_t$ is defined as the cyclical component of the time series for the government primary expenditure, where the trend is again extracted using a two-sided HP filter.

Substituting equation (2.4) in (2.3) yields:

$$d_{t+1} = \frac{1 + r_t}{1 + y_t}(1 - \rho)d_t - \frac{1 + r_t}{1 + y_t}(\alpha + \beta_1 \text{YVAR}_t + \beta_2 \text{GVAR}_t + \epsilon_t). \tag{2.6}$$

The next step is, after estimating the parameters $\alpha, \rho, \beta_1, \beta_2$ using ordinary least squares (OLS), see Rice (2006), to produce a stochastic debt simulation. Here, a stochastic simulation is referred to as applying a Monte Carlo simulation with start date $t_N$ and computing $M$ samples in each year $t$ of the horizon $t = t_N, \dots, T$, where $T \in [t_N, \infty)$. Hence, from the stochastic debt simulation we obtain samples $d_T^1, \dots, d_T^M$ from the unknown future debt distribution $F_{d_T}$. From this, we can approximate the quantiles of the distribution $F_{d_T}$, which will be further elaborated on in the next section. In equation (2.6), it is expected that the distribution function $F_{d_t}$ becomes less peaked when the dynamics of the 10-years real interest rate and the annual return of real GDP become more volatile. Hence, a larger fiscal response to keep the government debt under control is required. Moreover, the volatility of the variables $\text{YVAR}_t$ and $\text{GVAR}_t$ also has an important role for the shape of the distribution function $F_{d_t}$. In this regard, the Benchmark model makes important assumptions. Firstly, once the parameters $\alpha, \rho, \beta_1, \beta_2$ are estimated using historical data from the period $t = t_0, \dots, t_N$, the variables $\text{YVAR}_t$ and $\text{GVAR}_t$ are set equal to zero. As a consequence, the distribution $F_{d_t}$ doesn't depend any more on the stochastic behaviour of $\text{YVAR}_t$ and $\text{GVAR}_t$. Secondly, the Benchmark model doesn't include the error term $\epsilon_t$ from (2.4) in the stochastic debt simulation. Therefore, the stochastic behaviour of the Benchmark model only comes from the dynamics of the 10-years real interest rate $r_t$ and the annual return of real GDP $y_t$. The dynamics of $(y_t, r_t)$ are described by a VAR(2) model, see Appendix C for more details. In summary, the Benchmark model has the following dynamics:

$$
\begin{aligned}
d_{t+1} &= \frac{1+r_t}{1+y_t}(1-\rho)d_t - \frac{1+r_t}{1+y_t}\alpha, \\
\begin{pmatrix} y_t \\ r_t \end{pmatrix} &= \begin{pmatrix} \alpha^y \\ \alpha^r \end{pmatrix} + A_1 \begin{pmatrix} y_{t-1} \\ r_{t-1} \end{pmatrix} + A_2 \begin{pmatrix} y_{t-2} \\ r_{t-2} \end{pmatrix} + \begin{pmatrix} \epsilon_t^y \\ \epsilon_t^r \end{pmatrix},
\end{aligned} \tag{2.7}
$$

in order to perform a stochastic debt simulation, where $\alpha^y, \alpha^r \in \mathbb{R}$ and the error vector $\begin{pmatrix} \epsilon_t^y \\ \epsilon_t^r \end{pmatrix} \sim \mathcal{N}(0, \Sigma)$.

## 2.2 Calculation of the debt sustainability indicator

In Ewijk, Lukkezen, and Rojas-Romagosa (2013) an indicator is introduced for assessing government debt sustainability, which will be referred to as the debt sustainability indicator. In this section, we discuss the calculation of the debt sustainability indicator. We show that this calculation is easily performed by using the debt samples produced by a stochastic simulation. First, a brief overview is presented how to approximate quantiles with samples from an unknown probability distribution. Subsequently, we give the systematic procedure for calculating the debt sustainability indicator.

### 2.2.1 Approximation of the quantiles with order statistics

Suppose $X_1, \dots, X_M$ is an independent and identical distributed (i.i.d.) sample from an unknown distribution function $F$. The empirical cumulative distribution function (ECDF) is defined as

$$\hat{F}_M(x) = \frac{1}{M} \sum_{i=1}^{M} \mathbf{1}_{\{X_i \leq x\}}, \tag{2.8}$$

where

$$\mathbf{1}_{\{X_i \le x\}} = \begin{cases} 1 & \text{if } X_i \le x \\ 0 & \text{otherwise} \end{cases}.$$

The unknown distribution function $F$ can be estimated with (2.8). Note that, for a fixed point $x \in \mathbb{R}$ the quantity $M\hat{F}_M(x)$ is a sum of independent Bernoulli random variables. Hence, $M\hat{F}_M(x)$ has a binomial distribution with parameters $M$ and success probability $F(x)$. Therefore,

$$\mathbb{E}[\hat{F}_M(x)] = F(x) \quad \text{and} \quad Var[\hat{F}_M(x)] = \frac{F(x)(1 - F(x))}{M},$$

Furthermore, we find for $\epsilon > 0$

$$\mathbb{P}(|\hat{F}_M(x) - F(x)| \ge \epsilon) \le \frac{F(x)(1 - F(x))}{M\epsilon^2},$$

by Chebyshev's inequality, see Rice (2006) for more details. Hence, $\hat{F}_M(x)$ converges in probability to $F(x)$ as $M \to \infty$. With the properties of the ECDF, we can approximate the quantiles of the distribution function $F$. Let the $p$-th quantile be denoted by $q(p)$, $p \in (0, 1)$. The quantile $q(p)$ is defined in terms of the distribution function $F$ as:

$$q(p) = \inf\{x : F(x) \ge p\}. \tag{2.9}$$

From (2.9), the estimation of a quantile is given by:

$$\hat{q}(p) = \hat{F}_M^{-1}(p). \tag{2.10}$$

In order to relate (2.10) with the samples $X_1, \ldots, X_M$ we must introduce the notion of order statistics. Let $\pi : \{1, \ldots, M\} \to \{1, \ldots, M\}$ be a permutation operator such that $X_{\pi(i)} \le X_{\pi(j)}$ if $i < j$. Define the order statistics as $X_{(i)} = X_{\pi(i)}$. By this construction, we obtain the sequence $X_{(1)} \le X_{(2)} \le \ldots \le X_{(M)}$. The ECDF can be written in terms of order statistics as

$$\hat{F}_M(x) = \frac{1}{M} \sum_{i=1}^{M} \mathbf{1}_{\{X_{(i)} \le x\}}. \tag{2.11}$$

Next, by assuming that $F$ is continuous and $p \in (\frac{i-1}{M}, \frac{i}{M}]$ we have that $\hat{F}_M^{-1}(p) = X_{(i)}$. In other words, if $p \in (\frac{i-1}{M}, \frac{i}{M}]$ the estimation of the quantile $q(p)$ is given by the $i$-th order statistic $X_{(i)}$.

### 2.2.2 Evaluation of the indicator using order statistics

The debt sustainability indicator measures the degree to which governments are in control of their public finances by estimating the risk of a significant debt increase in the near future. The indicator is calculated by the following systematic procedure:

1. Estimate the model parameters using historical data from the period $t = t_0, \ldots, t_N$, where $t_N = t_0 + N$.
2. Set the length of the forecast horizon $t = t_N, \ldots, T$ equal to $k$ years, i.e., $T - t_N = k$. In Ewijk, Lukkezen, and Rojas-Romagosa (2013), the length of the forecast horizon is set to 10-years, i.e., $k = 10y$.

3. Produce the samples $d_T^1, \dots, d_T^M$ using a stochastic debt simulation with starting date $t_N$, where $M$ denote the number of samples.
4. Create the ascending sequence $d_T^{(1)} \leq \dots \leq d_T^{(M)}$, where $d_T^{(i)}$ is the $i$-th order statistic at time $T$.
5. Using the ordered sample $d_T^{(1)} \leq \dots \leq d_T^{(M)}$, the value of the debt sustainability indicator $I(t_N, k)$ is calculated as

$$\begin{aligned} I(t_N, k) &= q_{F_{t_N+k}}(0.975) - q_{F_{t_N+k}}(0.5) \\ &= q_{F_T}(0.975) - q_{F_T}(0.5) \\ &\approx \hat{q}_{F_T}(0.975) - \hat{q}_{F_T}(0.5) \\ &= \hat{F}_{M,T}^{-1}(0.975) - \hat{F}_{M,T}^{-1}(0.5) \\ &= d_T^{(0.975*M)} - d_T^{(0.5*M)}, \end{aligned} \tag{2.12}$$

where $q_{F_T}(p)$ denotes the $p$-th quantile of the distribution $F_{d_T}$, $p \in (0, 1)$, and $\hat{F}_{M,T}$ is the ECDF of the distribution function $F_{d_T}$ with $M$ samples at time $T$.

From (2.12), the indicator estimates the risk of a significant debt increase $k$-years ahead in the future. In Ewijk, Lukkezen, and Rojas-Romagosa (2013) it is shown that the indicator can distinguish countries with few to no debt sustainability concerns from countries with serious debt sustainability issues. It is expected that countries with debt sustainability issues have a higher country-specific risk. As discussed in Kwark (2002), country specific risk influences as, one of many factors, the dynamics of the interest rate spreads. Hence, the aim of this project is to investigate whether the dynamics of the interest rate spreads can be explained by the indicator. Before this topic is discussed in more detail, we first propose a new model for describing the dynamics of the government debt. This model incorporates the assumptions made in the Benchmark model in order to produce a stochastic debt simulation.

# CHAPTER 3

## The Extended model

In this chapter, we present a new model for describing the dynamics of the government debt, which we will refer to as the Extended model. As discussed in Section 2.1, the Benchmark model makes the following assumptions in order to produce a stochastic debt simulation. Firstly, once the parameters $\alpha, \rho, \beta_1$ and $\beta_2$ are estimated in (2.6), the variables $\text{YVAR}_t$ and $\text{GVAR}_t$ in (2.6) are set equal to zero in the simulation. By (2.6), the characteristics of $F_{d_T}$ depend among others on the estimated parameters $\beta_1, \beta_2$ and the stochastic behaviour of $\text{YVAR}_t$ and $\text{GVAR}_t$. Since the debt sustainability indicator is calculated by subtracting quantiles of $F_{d_T}$, it follows that this assumption influences the dynamics of the indicator. In Lukkezen, Rojas-Romagosa, et al. (2012), this assumption is justified by $\mathbb{E}[\text{YVAR}_t] = \mathbb{E}[\text{GVAR}_t] = 0$. Another reason for leaving out the $\text{YVAR}_t$ and $\text{GVAR}_t$ in the stochastic debt simulation is by the properties of the two-sided HP filter, which is used in the Benchmark model for extracting the trend of a time series. In a moment we will explain which property of the two-sided HP filter restrains to simulate the $\text{YVAR}_t$ and $\text{GVAR}_t$ in the debt forecasts. Secondly, the error component $\epsilon_t$ from (2.6) is removed after estimating the parameters of the Benchmark model. As a consequence, the stochastic behaviour of the government debt only depends on the dynamics of the annual return of real GDP $y_t$ and the real interest rate $r_t$, i.e., by a VAR(2) model. Also this assumption influences the dynamics of the indicator.

Incorporating these assumptions in a modelling framework gave us the motivation for developing an extension of the Benchmark model. In Section 3.1, we describe the dynamics of the Extended model and argue why this is an extension of the Benchmark model. Next, in Section 3.2, we discuss how to produce economic forecasts with the Extended model. In particular, this procedure is applied for all countries considered in this investigation. From this, we calculate the indicator, as discussed in Section 2.2, using the debt forecasts produced by the Extended model. Moreover, we investigate whether there exists a correspondence between the dynamics of the indicator, which is calculated with the Extended model, and the term-spread. Finally, in Section 3.3, we investigate whether the forecast accuracy of the Extended model is improved by including more information of the model's previous states.

## 3.1 The dynamics of the Extended model

In this section, we present the Extended model for describing the dynamics of the government debt. In a moment, it will be clear that the Extended model uses a filter with different properties then the two-sided HP filter for extracting the trend of a time series. For this reason, the dynamics of the Extended model include the business cycle $\text{YVAR}_t$ and the temporarily government spending $\text{GVAR}_t$.

Now, we describe the dynamics of the Extended model and give an accompanying explanation of all components. The Extended model has the following dynamics:

$$\begin{aligned}
\begin{pmatrix} y_{t+1} \\ r_{t+1} \end{pmatrix} &= \begin{pmatrix} \alpha_y \\ \alpha_r \end{pmatrix} + A_1 \begin{pmatrix} y_t \\ r_t \end{pmatrix} + A_2 \begin{pmatrix} y_{t-1} \\ r_{t-1} \end{pmatrix} + \begin{pmatrix} \epsilon^y_{t+1} \\ \epsilon^r_{t+1} \end{pmatrix}, \\
\text{rev}_{t+1} &= \alpha_{\text{rev}} + R_1 \text{rev}_t + \epsilon^{\text{rev}}_{t+1}, \\
\text{sp}_{t+1} &= \alpha_{\text{sp}} + \rho d_t + \beta YVAR_t + E_1 \text{rev}_t + E_2 \tau_t + E_3 c_t + \epsilon^{\text{sp}}_{t+1}, \\
\text{mil}_{t+1} &= \alpha_{\text{mil}} + M_1 \text{mil}_t + \epsilon^{\text{mil}}_{t+1},
\end{aligned} \tag{3.1}$$

The evolution of the government debt is described by:

$$d_{t+1} = d_t \frac{1 + r_t}{1 + y_{t+1}} - s_{t+1}, \tag{3.2}$$

where

$$s_{t+1} = \begin{cases} \text{rev}_{t+1} - (\text{sp}_{t+1} + \text{mil}_{t+1}) & \text{for UK, US,} \\ \text{rev}_{t+1} - \text{sp}_{t+1} & \text{otherwise.} \end{cases} \tag{3.3}$$

And finally, the error components have the following distribution

$$\begin{pmatrix} \epsilon^y_{t+1} \\ \epsilon^r_{t+1} \end{pmatrix} \sim \mathcal{N}(0, \Sigma_1), \quad \begin{pmatrix} \epsilon^{\text{rev}}_{t+1} \\ \epsilon^{\text{sp}}_{t+1} \end{pmatrix} \sim \mathcal{N}(0, \Sigma_2) \quad \text{and} \quad \epsilon^{\text{mil}}_{t+1} \sim \mathcal{N}(0, \sigma). \tag{3.4}$$

We can rewrite (3.1) in the following matrix-vector notation:

$$Y_{t+1} = \alpha + R X_t + \epsilon_{t+1}, \tag{3.5}$$

where

$$\begin{aligned}
Y_{t+1} &= [y_{t+1}, r_{t+1}, \text{rev}_{t+1}, \text{sp}_{t+1}, \text{mil}_{t+1}]', \\
\alpha &= [\alpha_y, \alpha_r, \alpha_{\text{rev}}, \alpha_{\text{sp}}, \alpha_{\text{mil}}]', \\
R &= \begin{pmatrix}
A_1[1,1] & A_1[1,2] & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_2[1,1] & A_2[1,2] \\
A_1[2,1] & A_1[2,2] & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_2[2,1] & A_2[2,2] \\
0 & 0 & R_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & E_1 & 0 & 0 & E_2 & E_3 & \beta & \rho & 0 & 0 \\
0 & 0 & 0 & 0 & M_1 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \\
X_t &= [y_t, r_t, \text{rev}_t, \text{sp}_t, \text{mil}_t, \tau_t, c_t, \text{YVAR}_t, d_t, y_{t-1}, r_{t-1}]', \\
\epsilon_{t+1} &= [\epsilon^y_{t+1}, \epsilon^r_{t+1}, \epsilon^{\text{rev}}_{t+1}, \epsilon^{\text{sp}}_{t+1}, \epsilon^{\text{mil}}_{t+1}]'.
\end{aligned} \tag{3.6}$$

In system (3.1), the dynamics of $(y_{t+1}, r_{t+1})$ are similar to the Benchmark model, i.e., described by a VAR(2) model. The matrix coefficients of $A_k$ are denoted by $A_k[i,j]$. Next, the dynamics of the revenue $\text{rev}_{t+1}$ are given by an AR(1) model. The dynamics of the primary spending $\text{sp}_{t+1}$ are explained by the government debt $d_t$, the business cycle $\text{YVAR}_t$, the government revenue $\text{rev}_t$, $\tau_t$ and $c_t$, where $\alpha_{\text{SP}}$, $\rho$, $\beta$, $E_1$, $E_2$ and $E_3$ are estimated using historical data. The parameter $\rho$ has the same interpretation as in the Benchmark model, i.e., the fiscal reaction parameter. The trend component

$$\tau_t = \exp(\psi_t) \tag{3.7}$$

and $\psi_t$ is the extracted trend component of $\log(\text{sp}_t)$, where the trend is extracted using a one-sided HP filter. See Appendix A for more details how a one-sided HP filter extracts the trend of a time series. The cyclical component $c_t$ is defined by

$$c_t = \text{sp}_t - \tau_t, \tag{3.8}$$

where $\tau_t$ is given by (3.7). The business cycle $\text{YVAR}_t$ is calculated by (2.5), only we use a one-sided HP filter for extracting the trend component. Here, the differences with the Benchmark model become apparent, i.e., instead of a two-sided HP filter the Extended model uses a one-sided HP filter for extracting the trend of a time series. The reason for this is that system (3.5) describes the future state $Y_{t+1}$ by using only the values at the previous state $X_t$. In other words, it is not an implicit relationship. Hence, to determine $\text{YVAR}_t, \tau_t$ and $c_t$ of the state vector $X_t$, we must use a filter that doesn't depend on future values of the time series. In other words, we want a so-called causal filter, i.e., a filter that extracts the trend component $\tau_t$ of a general time series $A_{t_0}, \ldots, A_T$, such that $\tau_t = f(A_{t_0}, \ldots, A_t)$ where $t \leq T$. Hence, we use the causal one-sided HP filter to extract the trend of a time series in the Extended model, instead of a non-causal two-sided HP filter. In Appendix A, we give the reason why the two-sided HP-filter is non-causal, and how to construct the causal one-sided HP filter.

Note that, the military spending is a form of government primary expenditure. However, for all countries the dynamics of the primary expenditure excludes the military spending. In Section 2.1, we have noticed that for the United States (US) and the United Kingdom (UK) the military spending drives the temporary government spending. Therefore, we include military spending $\text{mil}_{t+1}$ in system (3.1), which has the same model dynamics as $\text{rev}_{t+1}$, for the United States and the United Kingdom. For the remaining countries, we remove the dynamics of $\text{mil}_{t+1}$ in system (3.1). Combining the two cases, we obtain the primary surplus defined by (3.3). And finally, we note that all economic variables are observed at the end of year $t$. From this, we describe next year's debt $d_{t+1}$ with the associated interest rate to the outstanding debt $d_t$, i.e., the 10-years real interest rate $r_t$. Since the end of the year's primary surplus $s_{t+1}$ and the annual return of real GDP both influence the size of the debt $d_{t+1}$, this results in the dynamics of the government debt given by (3.2).

In system (3.1), the $\text{YVAR}_t$ is included in the modelling framework. And moreover, we can argue that the $\text{GVAR}_t$ is also included. As discussed in Section 2.1, the $\text{GVAR}_t$ is defined as the military spending for the United States and the United Kingdom. In system (3.1), we include military spending $\text{mil}_{t+1}$ for these two countries. For the remaining countries, the $\text{GVAR}_t$ is equal to $c_t$ as defined in (3.8). In other words, the $\text{GVAR}_t$ is the cyclical component of the time series for the government primary expenditure, which corresponds to the definition of $\text{GVAR}_t$ in Lukkezen, Rojas-Romagosa, et al. (2012). Therefore, both definitions of the $\text{GVAR}_t$ are included in the dynamics of the Extended model.

Including both the $\text{YVAR}_t$ and $\text{GVAR}_t$ in the dynamics of the Extended model is one of reasons why this framework is an extension of the Benchmark model. Another reason is that we don't remove components of the error vector $\tilde{\epsilon}_{t+1}$ after estimating the parameters of the Extended model. The next step is to discuss in more detail how to perform a simulation with the Extended model.

## 3.2 Forecasts of the Extended model

In this section, we give the general procedure for economic forecasts with the Extended model. The following schematically procedure is applied for all countries considered in this investigation.

1. We collect country-specific data for the variables: $y_t, r_t, \text{rev}_t, \text{exp}_t, \text{mil}_t$ and $d_t$, where the historical period is denoted by $t = t_0, \ldots, t_N$ and $t_N = t_0 + N$. In Appendix D, we describe the database which is used in this MSc-thesis. The last step of the initial phase is to specify the forecast horizon $t = t_N, \ldots, T$, $T \in [t_N, \infty)$.
2. OLS is applied to estimate the parameters of the Extended model, where we use data over the historical period $t = t_0, \ldots, t_N$. As a result, we obtain the estimates $\hat{\alpha}, \hat{R}$ and $\hat{\Sigma}$ needed in (3.5), where
$$\hat{\Sigma} = \begin{pmatrix} \hat{\Sigma}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \hat{\Sigma}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \hat{\sigma} \end{pmatrix}. \tag{3.9}$$
3. Next, we determine the coefficient of determination, denoted by $R^2$. The $R^2$ reflects how well the model fits the observed data, see Wooldridge (2012) for more details.
4. The constant $\hat{\alpha}$ is adjusted such that it contains the most recent information about the long-term behaviour of the variables: $y_t, r_t, \text{rev}_t$, and $\text{mil}_t$, see Appendix C for the technical details.
5. Determine the Cholesky decomposition $C$ of the variance-covariance matrix $\hat{\Sigma}$.
6. For each $t$ in the forecast horizon $t = t_N, \ldots, T-1$, we repeat the procedure:
    (a) Generate a sample of the random vector $\epsilon_{t+1} \sim \mathcal{N}(0, \hat{\Sigma})$ by computing the matrix-vector product $C^T Z$, where $C^T C = \hat{\Sigma}$ and $Z \sim MN(0, I)$, see Glasserman (2003) for more details. We denote this sample by $\tilde{\epsilon}_{t+1}$.
    (b) With $\hat{\alpha}, \hat{R}$, $\hat{\Sigma}$ and $\tilde{\epsilon}_{t+1}$ evaluate the recursion given by (3.5).
    (c) Using the values of $Y_{t+1}$, we obtain next year's government debt $d_{t+1}$ by calculating (3.2).

After finishing the above procedure, we obtain one Monte Carlo trajectory $\{d_{t_N}, \ldots, d_T\}$. Instead of repeating $M$ times the above procedure, the efficiency of the procedure is improved when at each time point $t$ we produce simultaneously the Monte Carlo samples $\{Y_t^1, \ldots, Y_t^M\}$, and obtain the samples $\{d_t^1, \ldots, d_t^M\}$. This is accomplished by the following steps. Firstly, we can easily produce $M$ samples of $\tilde{\epsilon}_{t+1}$ at each time point $t$. Secondly, the one-sided HP filter can extract the trend of multiple time series simultaneously, see Appendix A for more details. From this, we can create $\{X_t^1, \ldots, X_t^M\}$ at each time point $t$. The obtained Monte Carlo samples $\{Y_t^1, \ldots, Y_t^M\}$, and hence $\{d_t^1, \ldots, d_t^M\}$ for every $t = t_N, \ldots, T$ are referred to as the economic forecasts produced by the Extended model.

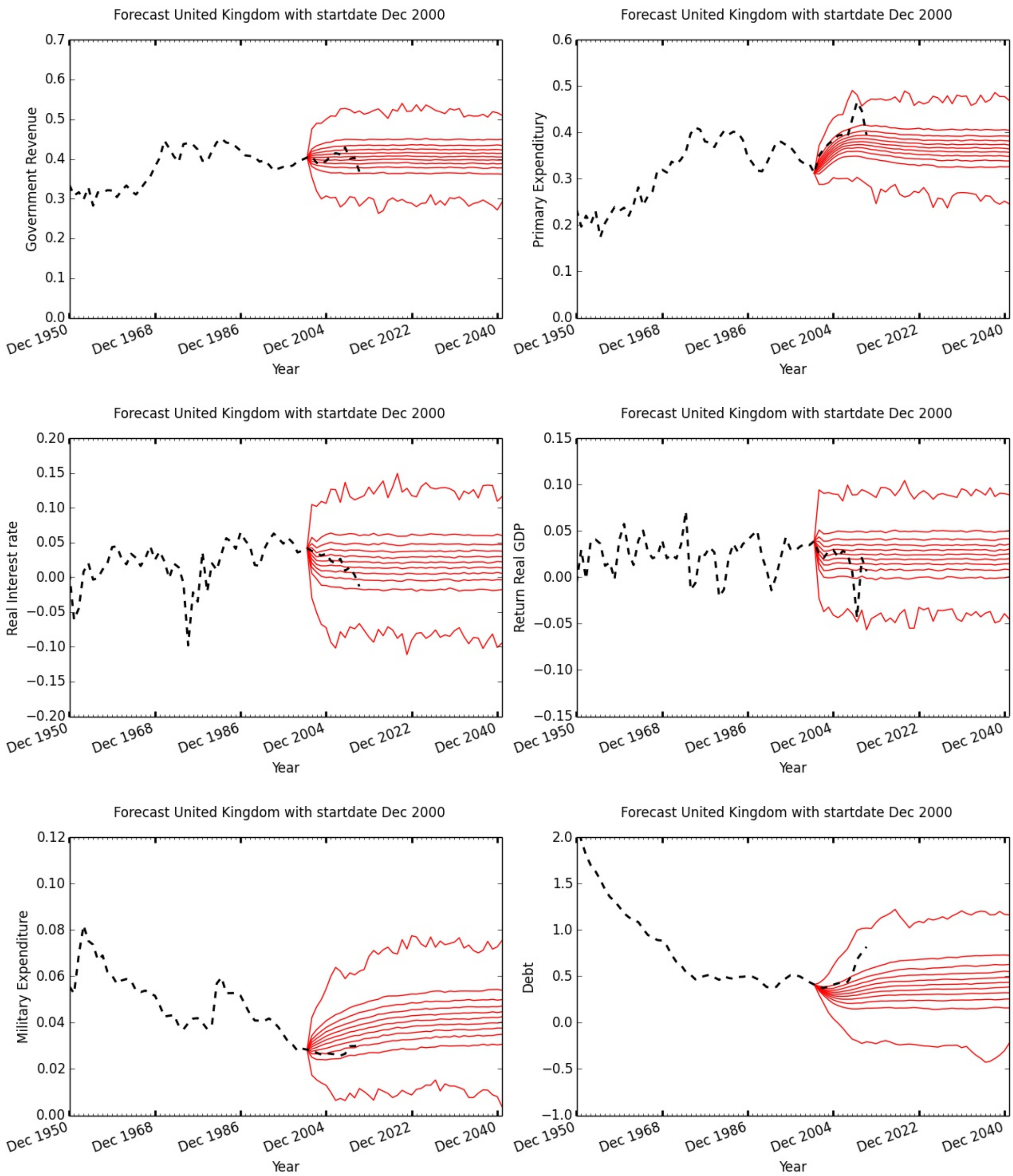


Figure 3.1: Economic variables for the United Kingdom: historical data (black-dotted) and deciles of the forecast (red). The $R^2_{\mathrm{rev}} = 0.82$, $R^2_{\mathrm{exp}} = 0.87$, $R^2_{\mathrm{r}} = 0.59$, $R^2_{\mathrm{y}} = 0.26$ and $R^2_{\mathrm{mil}} = 0.83$.

The next step is to produce forecasts for several economic variables with the Extended model. For this purpose, we consider the United Kingdom (UK), where we obtain yearly data for the economic variables: $y_t, r_t, \text{rev}_t, \exp_t, \text{mil}_t$ and $d_t$, for the period $t = 1950, \ldots, 2011$. We produce the forecasts with the Extended model until the year 2041, where the parameters of (3.5) are estimated using data over the period $t = 1950, \ldots, 2011$ and the number of samples $M = 2000$. In Figure 3.1, we show for each economic variable the historical data (black-dotted) and the deciles, which divide a distribution into ten equal parts, of the forecast (red). Historically, the government spending of the United Kingdom rapidly increases starting from the year 2000. From Figure 3.1, we observe that this behaviour is captured in the deciles of the forecast for the government expenditure. And moreover, we observe that for each fiscal variable the historical data is contained in the forecast. As a final note, we have verified numerically that the time series for $y_t, r_t, \text{rev}_t$ and $\text{mil}_t$ are stationary, Hence, the forecasts for these economic variables converge to a stationary state, see Appendix C for more details.

The calculation of the indicator, as discussed in Section 2.2, is easily accomplished with the forecasts produced by the Extended model, i.e., we use the simulated debt samples $d_T^1, \ldots, d_T^M$ to evaluate the indicator. The indicator makes a distinction between countries with and without debt sustainability concerns. Hence, the indicator contains a part of the associated country-specific risk. Because of this, we are interested how this type of risk evolves over time. For this purpose, we calculate the indicator $I(t, k)$ for several time points $t$ and obtain a country-specific indicator trajectory. For the United Kingdom, we plot in Figure 3.2 the indicator $I(t, k)$ for the period $t = 1990, \ldots, 2011$, where $k = 10y$ as in Ewijk, Lukkezen, and Rojas-Romagosa (2013). From Figure 3.2, we observe that the indicator reacts to potential debt sustainability concerns, i.e., the dynamics of the indicator change when the government debt rapidly increases after the start of the credit crisis in 2008.

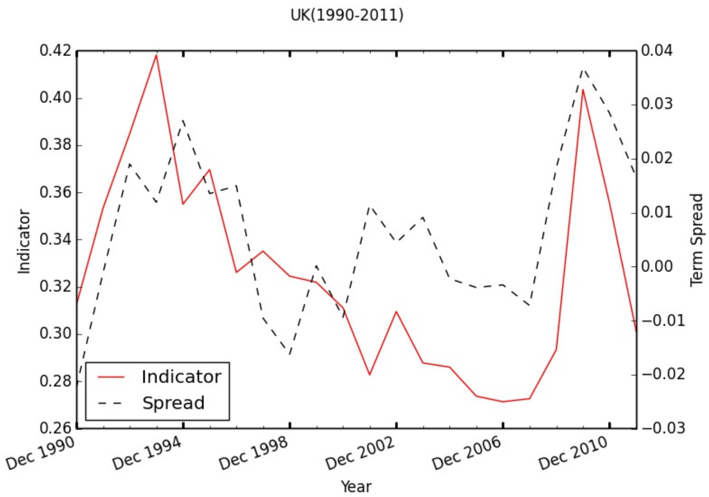


Figure 3.2: UK: Indicator (red) and term spread (black-dotted)

As discussed in Section 1.1, the country specific risk influences the dynamics of the interest rate spreads, where the term-spread is the most familiar interest rate spread. The term spread is the difference between the 10-years interest rate $r_t^{10y}$ and the 3-months interest rate $r_t^{3M}$. In Estrella and Mishkin (1996), Dotsey (1998) and Wheelock, Wohar, et al. (2009), it is argued that the term spread is an accurate indicator for forecasting a country's future economic growth. Therefore, we investigate whether the dynamics of the indicator and the term spread show similar behaviour. For this purpose, we determine the correlation between the indicator and the term spread

$$\rho(I(t,10y), S(t,10y,3M)), \quad \text{where} \quad S(t,10y,3M) = r_t^{10y} - r_t^{3M}. \tag{3.10}$$

The correlation coefficients are given for the United Kingdom (UK), the United States (US), France (FR), Italy (IT), Belgium (BEL), Germany (GER), the Netherlands (NL), Portugal (PRT) and Finland (FIN) in Table 3.1. The period where we calculate the indicator is given for each country.

| | $\rho(I^{10y}(t,10y), S(t,10y,3M))$ |
|---|---|
| UK (1990-2011) | 0.52 |
| US (1995-2011) | 0.60 |
| FR (1997-2011) | 0.36 |
| IT (1997-2011) | 0.62 |
| BEL (1995-2011) | 0.54 |
| GER (1996-2011) | 0.23 |
| NL (1996-2011) | 0.39 |
| PRT (1995-2011) | 0.81 |
| FIN (1995-2011) | -0.14 |

Table 3.1: For several countries: the correlation between $I(t,10y)$ and $S(t,10y,3M)$.

For each country, the above procedure for producing forecasts with the Extended model and evaluating the indicator is repeated, where we vary the number of samples $M$. From this, we obtain similar correlation coefficients as given in Table 3.1. Since the period where we calculate the indicator is not for an extended period, we find a correlation value above 0.50 high. Therefore, we observe from the correlation coefficients given by Table 3.1 that for most countries there exists a (linear) relationship between the indicator and the term-spread.

Historically, the interest rates with 3-months ($3M$) and 10-years ($10y$) maturity are available for an extended period. Hence, we are not interested in the explanatory power of the indicator for explaining the dynamics of the term spread. Instead, in Chapter 5, we investigate whether the indicator can explain the dynamics of the interest rate spreads where historical data is limited available.

## 3.3 Alternative model for the interest and growth rates

In Section 3.1, a stochastic VAR(2) model is used to capture the historic volatility of the real interest rate $r_t$ and the annual return of real GDP $y_t$. In this section, we compare the forecast performance of a VAR(2) model with a model which will be referred to as the alternative model. In the alternative model, we adjust the dynamics of $y_t$ in the VAR(2) model such that it contains more information of the model's previous states. Before presenting the alternative model, we first discuss how to include more information of the previous states of $y_t$ without increasing the number of lags $p$ in a VAR($p$) model. As discussed in Appendix A, the one-sided HP filter only uses the current and past state values $y_{t_0}, \ldots, y_t$ for extracting the trend component $\tau_t$ of the time series $y_{t_0}, \ldots, y_{t_N}$, where $t \leq t_N$ and $t_N = t_0 + N$. From this, another way to include more information of the model previous states is adding a trend component to the model dynamics. This leads to investigate whether including more information of the model's previous states really improves the forecast performance. For this purpose, we compare the forecasting performance of the VAR(2) model

$$\begin{pmatrix} y_{t+1} \\ r_{t+1} \end{pmatrix} = \begin{pmatrix} \alpha_y \\ \alpha_r \end{pmatrix} + A_1 \begin{pmatrix} y_t \\ r_t \end{pmatrix} + A_2 \begin{pmatrix} y_{t-1} \\ r_{t-1} \end{pmatrix} + \begin{pmatrix} \epsilon^y_{1,t+1} \\ \epsilon^r_{1,t+1} \end{pmatrix}, \tag{3.11}$$

with the alternative model

$$\begin{aligned} y_{t+1} &= \alpha_y + a_1 r(\tau_t) + b_1 \mathrm{YVAR}_t + c_1 r_t + d_1 r_{t-1} + \epsilon^y_{2,t+1}, \\ r_{t+1} &= \alpha_r + a_2 y_t + b_2 y_{t-1} + c_2 r_t + d_2 r_{t-1} + \epsilon^r_{2,t+1}, \end{aligned} \tag{3.12}$$

In (3.12), the component $r(\tau_t)$ is calculated with the following procedure:

1. Determine the time series $\{\log(Z_t), t = t_0, \ldots, t_N\}$, where $Z_t$ is the value of the real GDP at year $t$. In general, the value $Z_t$ is given by the following product:

$$Z_t = Z_{t_0} \prod_{s=t_0}^{t} (1 + y_s), \tag{3.13}$$

   where $y_t$ is the value of the annual return of real GDP at time $t$.

2. Extract the trend component $\psi_t$ from $\log(Z_t)$ by using a one-sided HP filter, see Appendix A for more details.
3. Set $\tau_t = \exp(\psi_t)$.
4. The return of the trend $\tau_t$ is given by

$$r(\tau_t) = \frac{\tau_t - \tau_{t-1}}{\tau_{t-1}}. \tag{3.14}$$

From (3.13) and the fact that $\tau_t$ depends on the values $Z_{t_0}, \ldots, Z_t$, it follows that $r(\tau_t)$ is a function of $y_1, \ldots, y_t$. Hence, the information of the model previous states $y_1, \ldots, y_t$ is contained in the component $r(\tau_t)$ and also the business cycle $\mathrm{YVAR}_t$, as discussed in Section 2.1. In Figure 3.3, we illustrate how the component $r(\tau_t)$ can be interpreted. In particular, we plot $y_t$ and $r(\tau_t)$ for the United Kingdom in the period $t = 1950, \ldots, 2011$. It is observed that $r(\tau_t)$ displays similar dynamics as $y_t$ but has smaller outliers.

Before we discuss the forecast performance of models (3.11) and (3.12), we first give some comments on the procedure how system (3.12) produces forecasts. Firstly, we apply OLS to estimate the parameters of each equation in system (3.12). After applying OLS, we collect for each equation in system (3.12) the residuals. From this, we can estimate the correlation between the components of the error vector $[\epsilon^y_{2,t+1}, \epsilon^r_{2,t+1}]'$. Secondly, we adjust the constants $\alpha_y$ and $\alpha_r$ in system (3.12) such that they contain the most recent information about the long-term behaviour of the economic variables $y_t$ and $r_t$, see Appendix C for more details. In order to do so, we must show that system (3.12) is stationary and that $\lim_{t\to\infty} \mathbb{E}[r(\tau_t)] = \tilde{\mu}_y$, where $\tilde{\mu}_y$ is specified in Appendix C. And moreover, we must show that $\lim_{t\to\infty} \mathbb{E}[\mathrm{YVAR}_t] = 0$. In Appendix B, we give the first steps of a result which can be used for proving these limits.

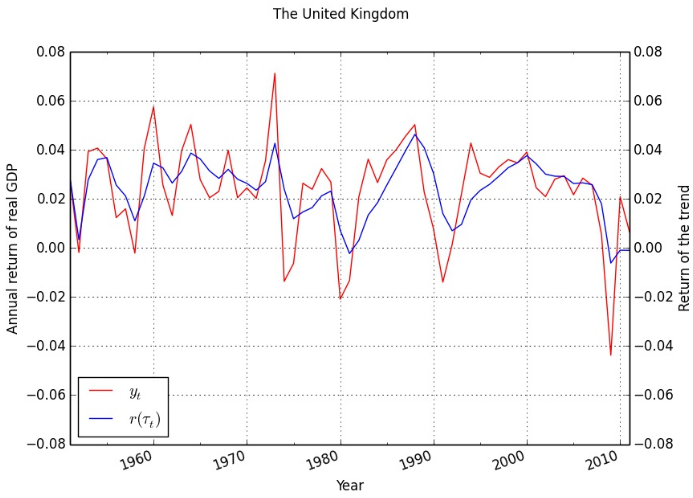


Figure 3.3: The United Kingdom: the return of annual real GDP $y_t$ versus the component $r(\tau_t)$

There are many ways to evaluate the forecasting performance of a model. We are interested in the accuracy of models (3.11) and (3.12) for predicting the $k$-years ahead annual return of real GDP $y_{t+k}$, where the forecast starts from year $t$. For this purpose, we use the mean absolute deviation (MAD) as criteria to evaluate the forecasting performance, see Tsay (2005) for the technical details how to evaluate the MAD for a $k$-years ahead forecast. From this, the model with the smallest MAD value is regarded as the best $k$-years ahead forecasting model. Next, we calculate the MAD magnitude for several $k$-years ahead forecasts produced by models (3.11) and (3.12). This is displayed in Figures 3.4 and 3.5 for the United Kingdom and the United States, respectively.

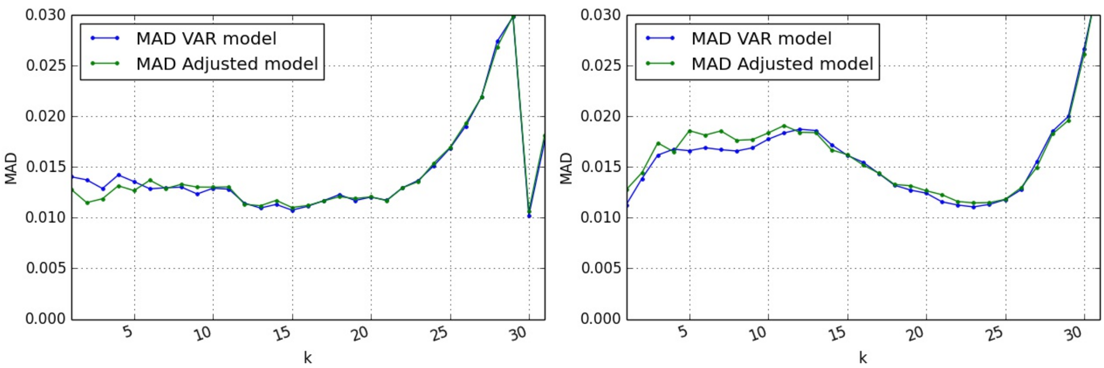


Figure 3.4: The United Kingdom: MAD value for several $k$-years ahead forecasts of $y_{t+k}$ (left) and $r_{t+k}$ (right).

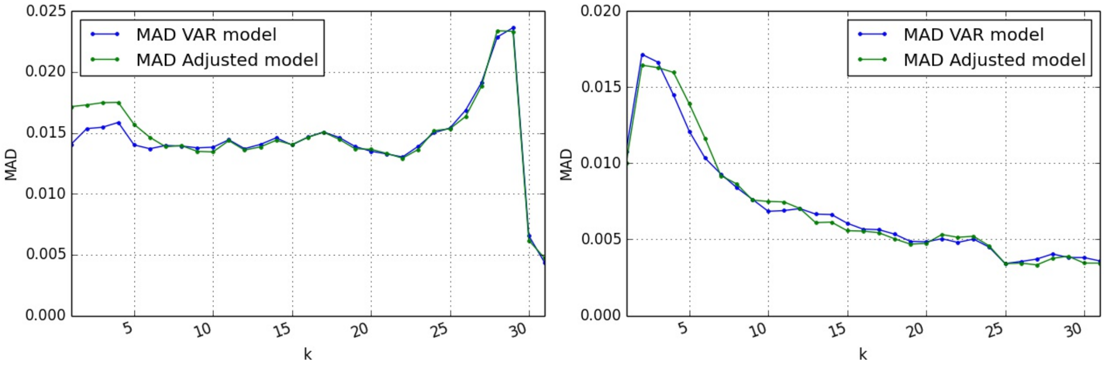


Figure 3.5: The United States: MAD value for several $k$-years ahead forecasts of $y_{t+k}$ (left) and $r_{t+k}$ (right).

From Figures 3.4 and 3.5, there is hardly a difference between the MAD values of systems (3.11) and (3.12) on the whole horizon. Only on the short-term horizon, we observe that the VAR(2) model has often a smaller MAD value then the alternative model. Similar conclusions were found for other countries. In this case, the forecast performance doesn't significantly improves by including more information of the model's previous states. From this, it follows that a VAR(2) model will be used for describing the evolution of $(y_t, r_t)$ in the Extended model.

# CHAPTER 4

## Imposing a new government expenditure rule

In this chapter, we formulate a new government expenditure rule in order to overcome a drawback of the Extended model. As discussed in Section 3.1, the Extended model describes the future evolution of the primary expenditure by:

$$\mathrm{sp}_{t+1} = \alpha_{\mathrm{sp}} + \rho d_t + \beta YVAR_t + E_1 \mathrm{rev}_t + E_2 \tau_t + E_3 c_t + \epsilon^{\mathrm{sp}}_{t+1}. \tag{4.1}$$

The parameters in (4.1) are estimated based on historical data, where the fundamental assumption is that historical data is representative for predicting the future. In Section 4.1, we show that the Extended model doesn't always produce realistic economic forecasts, which are based on historical data. Hence, we propose in this section a new government expenditure rule. This rule is based on the regulations formulated in the Stability and Growth Pact, see Beaumont and Walker (1999) for more details. In Section 4.2, we give a brief overview of the Stability and Growth Pact. In Section 4.3, after estimating the parameters of the Extended model, we discuss how this rule describes the further evolution of the primary expenditure. Next, for several countries we produce forecasts with the Extended model where the new expenditure rule is implemented. And finally, we are interested in whether the correlation between the term spread and the indicator is improved, by implementing this rule in the dynamics of the Extended model.

## 4.1 Blow-up behaviour

Before we discuss this rule in more detail, we first show that the Extended model sometimes produces unrealistic economic forecasts, which are based on historical data. One of the cases is illustrated in Figure 4.1, where we estimate the parameters of the Extended model for Germany using data over the period $t = 1970, \ldots, 1995$. We produce forecasts with the Extended model until the year 2021.

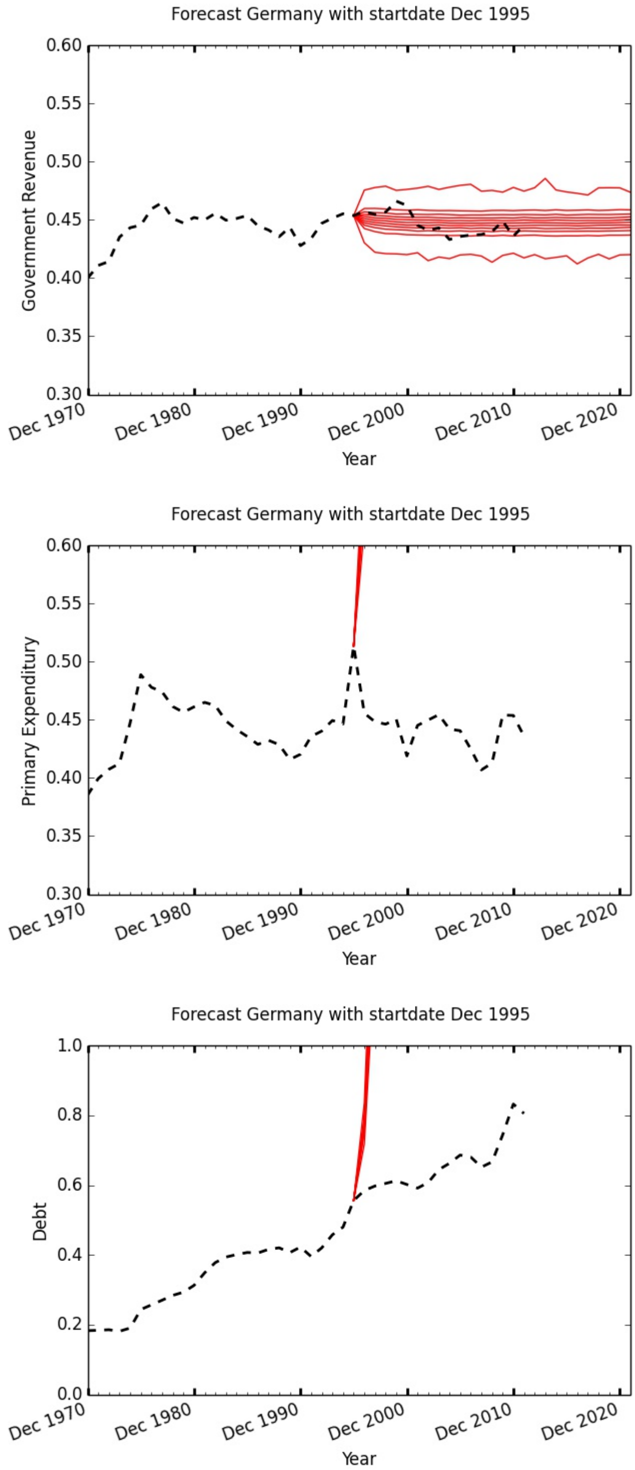


Figure 4.1: Economic variables for Germany: historical data (black-dotted) and deciles of the forecast (red). The $R^2_{\text{rev}} = 0.40$ and $R^2_{\text{exp}} = 0.69$.

We observe historically that the government has a policy that does not react to debt accumulation in the period 1970-1994. In other words, there was no intermediate change of fiscal policy, i.e, adjust its policy for the revenue and expenditure, such that the debt decreases. In particular, the historical time series of the government revenue evolves steady around the average level of 0.44, and hence the forecast converges to a stationary state, see Appendix C for more details. Both the government expenditure and debt rapidly increase starting from 1991, and with a jump in the year 1995 for the primary expenditure time series. When we estimate the parameters in (4.1) using data over the period $t = 1970, \ldots, 1995$, we want to explain the dynamics of the expenditure, where the jump is included, with historical data before the year 1995. The only way to explain this is that the government's fiscal policy, which is captured in the fiscal reaction parameter $\rho$, doesn't react correctly to the increase of the debt level. This is confirmed by Lukkezen, Rojas-Romagosa, et al. (2012), i.e, a disadvantage of using the estimated $\rho$ in (4.1) is that it doesn't represent the government response correctly for the future when severe and fully unanticipated shocks occur in the revenue and/or expenditure time series. As a consequence, the forecast of the expenditure diverges, and hence the forecast of the government debt displays also blow-up behaviour. However, these forecasts are not economically realistic since the government has taken measures to reduce its expenditure after 1995. From this, the data over the period $t = 1970, \ldots, 1995$ is not representative for predicting the government spending with (4.1).

One way of solving this is to apply another regression method for estimating the parameter $\rho$. For example, instead of using ordinary least squares, we can consider applying weighted regression, see Mendenhall, Sincich, and Boudreau (1996) for more details. In this way, we can put more weight on the data points which contain the rapid debt accumulation. Hence, we can force the government to change its fiscal policy for the increase in debt level. However, determining these weights can be challenging. And moreover, weighted regression is not easily applied in our framework; to produce government debt forecasts, for many countries, where we estimate the parameters using several data periods. Here, we propose, after estimating the parameters of the Extended model, a new government expenditure rule which is based on regulations formulated in the Stability and Growth Pact, see Beaumont and Walker (1999) for more details. This is an agreement among the 28 member states of the European Union.

## 4.2 The Stability and Growth Pact

In this section, we briefly discuss the content of the Stability and Growth Pact (SGP), see Beaumont and Walker (1999) for more details. Adopted in 1997, the SGP is a set of rules to ensure that countries in the European Union (EU) pursue sound public finances and coordinate their fiscal policies, with Germany the moving force behind the arrangement. To put it simply, the guidelines of the SGP prevent that governments spend more money than they receive. Moreover, its economic rationale was that sound public finances would be conducive to long term economic growth. This would create favourable conditions to lower interest rates. As a result, the investment, employment and eventually growth levels increase. The two major SGP criteria that member states must respect are given by:

- Annual government budget deficit no higher than 3% of Gross Domestic Product;
- Government debt lower than 60% of Gross Domestic Product,

see Ngai (2012) for more details.

The Excessive Deficit Procedure (EDP) is the EU's step by step procedure for correcting excessive deficit or debt levels, which is discussed in the corrective arm of the SGP, see OECD (2014) for more details. An EDP can be applied, for instance, whenever the debt reference value over the past three years was not reduced by at least 1/20 on average per year. If no effective measures are taken in order to reduce the debt, then the Council, the Economics and Finance Ministers of the EU member states, can impose sanctions. As outlined in the corrective arm regulation, all EU member states are each year obliged to submit a SGP compliance report that will present the country's expected fiscal development for the current and subsequent three years. This report will be evaluated by the European Commission and the Council of Ministers. However, the European Commission is charged with enforcing the SGP and has been criticized for being too lenient by not imposing penalties on countries that have not operated within the rules. In December 2011, the SGP was strengthened by a new set of regulations known as the six-pack.

## 4.3 A new government expenditure rule

In this section, we propose a new expenditure rule, which is based on the guidelines formulated in the SGP, for describing the future evolution of the government spending. This rule is obtained by assuming that all European countries have a policy such that: if there exists a time point $t$ such that $d_t > 0.60$, then

$$d_{t+1} = d_t - F_t, \quad \text{where} \quad F_t = \frac{d_t - L}{20}, \quad \text{and} \quad L = 0.60. \tag{4.2}$$

From (4.2), if the evolution of the government debt exceeds the threshold $L = 0.60$, i.e., the government does not satisfy the second criterion of the SGP. Then, the amount which exceeds the threshold must be reduced with 1/20 in the next year.

Suppose that $d_t > 0.60$ for a certain year $t$. And moreover, we assume that all European countries have a policy such that (4.2) is respected. As described in Section 3.1, the dynamics of the government debt is given by:

$$d_{t+1} = d_t \frac{1 + r_t}{1 + y_{t+1}} - \text{rev}_{t+1} + \text{sp}_{t+1}, \tag{4.3}$$

where we subtract the military spending $\text{mil}_{t+1}$ in (4.3) for the United Kingdom and the United States. In order to meet the debt requirement $d_{t+1}$ given by (4.2), it follows that next year's government primary expenditure is given by

$$\begin{aligned} \text{sp}_{t+1} &= d_{t+1} - d_t \frac{1 + r_t}{1 + y_{t+1}} + \text{rev}_{t+1} \\ &\overset{(4.2)}{=} d_t - F - d_t \frac{1 + r_t}{1 + y_{t+1}} + \text{rev}_{t+1}. \\ &= d_t - \frac{d_t - L}{20} - d_t \frac{1 + r_t}{1 + y_{t+1}} + \text{rev}_{t+1}. \end{aligned} \tag{4.4}$$

Combining (4.1) and (4.4), we obtain the following expenditure rule where the government has a policy that respects (4.2):

$$\text{sp}_{t+1} = \begin{cases} \alpha_{\text{sp}} + \rho d_t + \beta YVAR_t + E_1 \text{rev}_t + E_2 \tau_t + E_3 c_t + \epsilon^{\text{sp}}_{t+1} & \text{if } d_t \leq 0.60 \\ d_t - \frac{d_t - L}{20} - d_t \frac{1 + r_t}{1 + y_{t+1}} + \text{rev}_{t+1} & \text{if } d_t > 0.60. \end{cases} \tag{4.5}$$

From (4.5), the government expenditure is described by (4.1) if $d_t \leq 0.60$, i.e., the government satisfies the second criterion of the SGP. However, when this criterion is violated, the government adjusts its expenditure according to (4.4).

Hence, implementing (4.5) in the dynamics of the Extended model results in:

$$\begin{aligned}
\begin{pmatrix} y_{t+1} \\ r_{t+1} \end{pmatrix} &= \begin{pmatrix} \alpha_y \\ \alpha_r \end{pmatrix} + A_1 \begin{pmatrix} y_t \\ r_t \end{pmatrix} + A_2 \begin{pmatrix} y_{t-1} \\ r_{t-1} \end{pmatrix} + \begin{pmatrix} \epsilon^y_{t+1} \\ \epsilon^r_{t+1} \end{pmatrix} \\
\text{rev}_{t+1} &= \alpha_{\text{rev}} + R_1 \text{rev}_t + \epsilon^{\text{rev}}_{t+1} \\
\text{sp}_{t+1} &= \begin{cases} \alpha_{\text{sp}} + \rho d_t + \beta YVAR_t + E_1 \text{rev}_t + E_2 \tau_t + E_3 c_t + \epsilon^{\text{sp}}_{t+1} & \text{if } d_t \leq 0.60 \\ d_t - \frac{d_t - L}{20} - d_t \frac{1+r_t}{1+y_{t+1}} + \text{rev}_{t+1} & \text{if } d_t > 0.60 \end{cases} \\
d_{t+1} &= d_t \frac{1+r_t}{1+y_{t+1}} - \text{rev}_{t+1} + \text{sp}_{t+1}.
\end{aligned} \tag{4.6}$$

There is only one issue with calculating the government expenditure rule given by (4.5). Suppose that at year $t$ we observe that $d_t > 0.60$, then next year's requirement for $\text{sp}_{t+1}$ is given by (4.4). However, the values of $\text{rev}_{t+1}$ and $y_{t+1}$ are not observed. Therefore, we consider the following two possibilities for approximating $\text{rev}_{t+1}$ and $y_{t+1}$.

1. Firstly, we approximate $\text{rev}_{t+1}$ and $y_{t+1}$ by the values observed at time $t$, i.e., $\text{rev}_{t+1} \approx \text{rev}_t$ and $y_{t+1} \approx y_t$.
2. Secondly, we use the conditional expectations $\mathbb{E}[\text{rev}_{t+1}|\mathcal{F}_t]$ and $\mathbb{E}[y_{t+1}|\mathcal{F}_t]$, where $\mathcal{F}_t$ represents all historical information available up to time $t$.

The conditional expectations can be expressed by the values of the states at time $t$. As an illustration,

$$\mathbb{E}[\text{rev}_{t+1}|\mathcal{F}_t] = \mathbb{E}[\alpha_{\text{rev}} + R_1 \text{rev}_t + \epsilon^{\text{rev}}_{t+1}|\mathcal{F}_t] = \alpha_{\text{rev}} + R_1 \text{rev}_t, \tag{4.7}$$

where we assume that $\mathbb{E}[\epsilon^{\text{rev}}_{t+1}|\mathcal{F}_t] = 0$.

Similarly, we find for

$$\mathbb{E}[y_{t+1}|\mathcal{F}_t] = \alpha_y + A_1[1,1]y_t + A_1[1,2]r_t + A_2[1,1]y_{t-1} + A_2[1,2]r_{t-1}, \tag{4.8}$$

where we use the notation from Section 3.1.

Next, for several countries we produce economic forecasts by system (4.6). Hence, we can study, after estimating the parameters of system (4.6), the influence of the new government spending rule on the forecasts of the Extended model. In Figure 4.2, we plot the deciles of the forecasts produced by system (4.6) for Germany, where we estimate the parameters using data over the period $t = 1970, \ldots, 1995$. Furthermore, as discussed in Annett (2006), the SGP was a success for the Netherlands before the start of the credit crisis, especially in terms of guiding them towards debt sustainability. Therefore, we investigate whether the historical data represents the regulations of the SGP. If this is the case, then we expect that the new government expenditure rule doesn't influence the forecasts of the Extended model. In Figure 4.3, we plot the deciles of the forecasts produced by the Extended model for the Netherlands. In particular, the forecasts are produced with and without the new expenditure rule, where we estimate the parameters using data over the period $t = 1969, \ldots, 2005$.

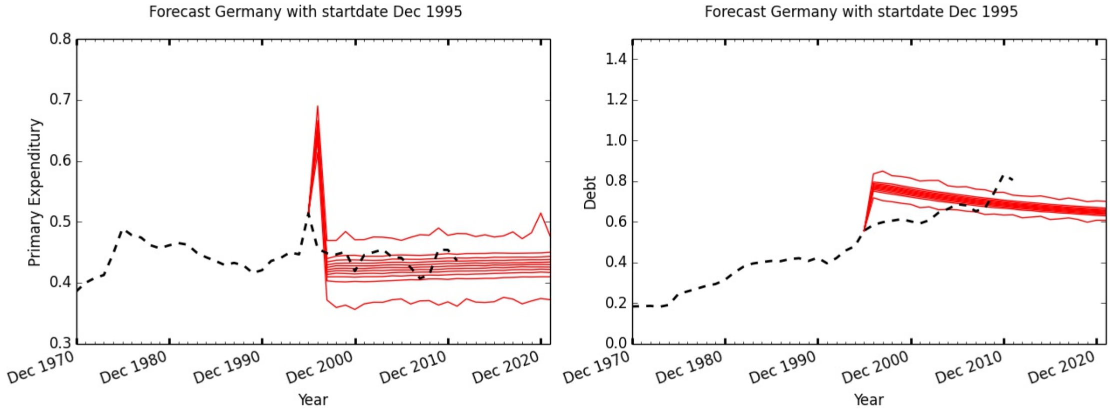


Figure 4.2: Government primary expenditure (left) and debt (right) for Germany: historical data (black-dotted) and deciles of forecast (red). System (4.6) produces the forecast, where $\text{rev}_{t+1} \approx \text{rev}_t$ and $y_{t+1} \approx y_t$.

From Figure 4.2, we observe that the blow-up behaviour of the forecasts is controlled by implementing the new government expenditure rule in system (4.6). In this case, the forecasts of system (4.6) can be interpreted as follows. Historically, the government debt is low until the year 1991, then both the government expenditure and debt suddenly increase with the peak in the year 1995. As discussed in Section 4.1, the government doesn't respond correctly to the increase in debt level due to the fiscal policy in the period 1970-1994. Hence, the new government expenditure rule changes the government attitude towards debt sustainability. In particular, the government decreased its expenditure after 1995 in the forecast, and maintained a policy such that the expenditure forecast converges to a stationary state. As a consequence, the debt forecast converges also to a stationary state, in particular, to the threshold $L = 0.60$ specified in (4.2).

For the Netherlands, the new government expenditure rule hardly affects the forecast produced by the Extended model, see Figure 4.3. The reason for this is that, after the establishment of the SGP in 1997, the fiscal policy of the Netherlands has changed such that the government meets the criteria formulated in the SGP. The parameters of the Extended model are estimated using data until 2005; the period where the Netherlands closely follows the regulations of the SGP. The government's attitude towards debt sustainability, captured by the fiscal reaction parameter $\rho$, corresponds to the guidelines formulated in the SGP. Hence, the new expenditure rule is not required for changing the government attitude towards debt sustainability.

Finally, we investigate whether the correlation between the term spread and the indicator is improved, by implementing the new government expenditure rule in the dynamics of the Extended model. In Table 4.1, we show the correlation between the indicator and the term spread. The period where we calculate the indicator is given for each country. Here, the indicator $I(t, 10y)$ is calculated as discussed in Section 3.2, and the indicator $I^i(t, 10y)$ is calculated by system (4.6) where we use approximation $i = 1, 2$ for $\text{rev}_{t+1}$ and $y_{t+1}$.

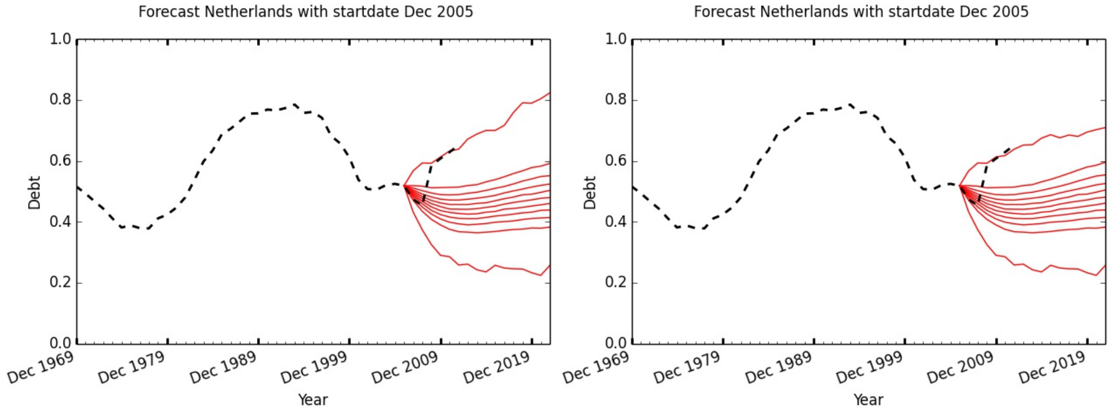


Figure 4.3: Government debt for the Netherlands without (left) and with (right) the new expenditure rule: historical data (black-dotted) and deciles of forecast (red). In the forecast of system (4.6) we use $\text{rev}_{t+1} \approx \mathbb{E}[\text{rev}_{t+1}|\mathcal{F}_t]$ and $y_{t+1} \approx \mathbb{E}[y_{t+1}|\mathcal{F}_t]$.

| | $\rho(I(t,10y), S(t,10y,3M))$ | $\rho(I^1(t,10y), S(t,10y,3M))$ | $\rho(I^2(t,10y), S(t,10y,3M))$ |
|---|---|---|---|
| BEL(1995-2011) | 0.54 | 0.32 | 0.72 |
| FIN(1995-2011) | -0.14 | 0.25 | 0.28 |
| UK (1990-2011) | 0.52 | -0.47 | -0.30 |
| BEL(1995-2011) | 0.54 | 0.23 | 0.26 |
| PRT(1995-2011) | 0.81 | 0.53 | 0.60 |

Table 4.1: For several countries: Correlation between the indicator and term spread.

From Table 4.1, the correlation between the indicator and the term spread is improved for Belgium and Denmark, where the forecasts are produced by system (4.6). In particular, the approximation $\text{rev}_{t+1} \approx \mathbb{E}[\text{rev}_{t+1}|\mathcal{F}_t]$ and $y_{t+1} \approx \mathbb{E}[y_{t+1}|\mathcal{F}_t]$ give the best improvements. A possible reason for this is that financial markets can have different expectations of debt sustainability, from what is observed from historical data. For instance, if the financial markets expect that a government has a fiscal policy which respects the regulations of the SGP, where we assume that following the regulations of the SGP by a government leads to debt sustainability, then, the financial markets expect that the government will be able to control the evolution of the debt, while this does not have to be observed from historical records. Hence, financial variables such as the term spread adjust this information from the financial market, see Estrella and Mishkin (1998) for more details. Moreover, the forecasts of system (4.6) are regulated by a spending rule which respects the guidelines formulated in the SGP. Therefore, the forecasts of the government debt are in line with the information captured by the dynamics of the term spread.

However, implementing the new government expenditure rule in system (4.6) has also negative effects. Firstly, for most countries, we observe that the correlation is not improved between the term spread and the indicator which is calculated by system (4.6). As discussed in Section 3.1, the term spread is an accurate indicator for forecasting a country's future economic growth. Secondly, system (4.6) reduces an amount of volatility in the debt forecasts. For illustration we show Figure 4.4, where we plot the deciles of the debt forecasts for Germany starting from 1996.

The Extended model produces the forecasts without and with the new government expenditure rule.

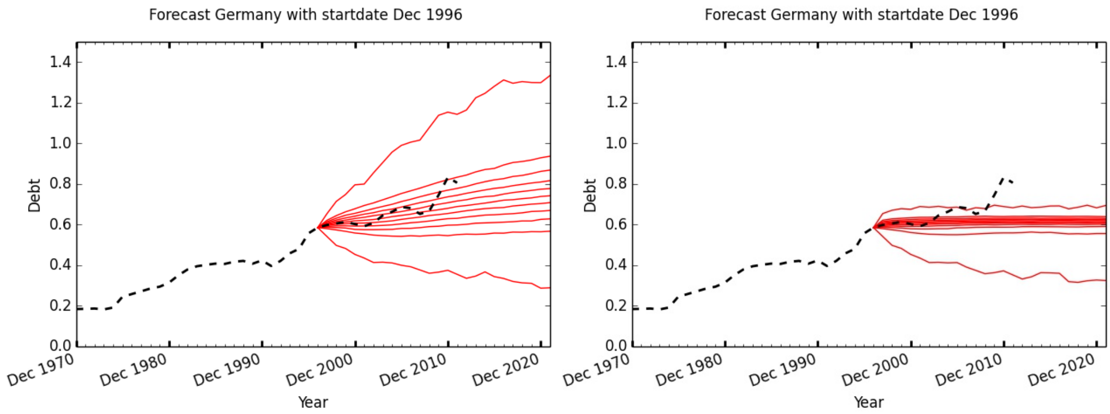


Figure 4.4: Government debt for Germany without (left) and with (right) the new expenditure rule: historical data (black-dotted) and deciles of forecast (red). In the forecast of system (4.6) we use $\text{rev}_{t+1} \approx \text{rev}_t$ and $y_{t+1} \approx y_t$.

From Figure 4.4, it is clear that the uncertainty is reduced in the debt forecasts produced by system (4.6). As a consequence, the indicator, which takes economic uncertainty underlying the debt evolution into account, can not measure the degree to which governments are in control of their public finances. This is a good reason why the correlation between the term spread and the indicator calculated by system (4.6) does not give for most countries an improvement. From both drawbacks, we conclude that the Extended model without the new government expenditure rule will be further used for calculating the debt sustainability indicator.

The main topic of this chapter has let to study more extensively the influence of the government fiscal response, captured by the parameter $\rho$, on the stability of the Extended model. Therefore, in Chapter 6, we apply dynamical systems research in order to perform a stability analysis for the Extended model.

# CHAPTER 5

## Relationship between the Indicator and the Interest rate spreads

In this chapter, we study whether the indicator as discussed in Section 2.2, can explain the dynamics of several interest rate spreads. The indicator estimates the risk of a significant debt increase in the near future. Hence, the indicator contains a part of the risk that a foreign government will default on its bonds or other financial commitments, i.e., the country-specific risk. There are many factors, such as country risk, that influence the dynamics of the interest rate spreads. Therefore, in this chapter, we investigate whether the indicator can explain the dynamics of the interest rate spreads. Firstly, in Section 5.1, we explain which interest rate spreads are considered. Secondly, in Section 5.2, we formulate criteria to verify whether there is a relationship between the indicator and the interest rate spreads. Finally, in Section 5.3, we present for several countries the numerical results of verifying this relationship. From these results, it will become clear how much explanatory power the indicator contains for describing the dynamics of the interest rate spreads.

### 5.1 3-months versus 10-years interest rate as input for the Extended model

In this section, we discuss which type of interest rate spreads are considered in this investigation. Since the yields with 3-months ($3M$ ) and 10-years ($10y$) maturity are available for an extended period, we can distinguish two types of interest rate spreads. Firstly, we consider the difference between an interest rate which is historically limited available, and the 10-years interest rate. Secondly, we consider the difference between an interest rate which is historically limited available, and the 3-months interest rate. Next, we want to investigate whether there is a relationship between the indicator and both types of interest rate spreads. In order to investigate this relationship consistently, we correspond the interest rate used as input of the Extended model with the widely available interest rate which is contained in the spread. For example, suppose that the Extended model forecasts the 10-years interest rate. Then, we investigate the relationship between the indicator which is calculated using the forecasts of the 10-years interest rate, and the spread which takes the difference with the 10-years interest rate. In this way, we establish a connection between the indicator and the interest rate spread, since both depend on the 10-years yield. The next step is to investigate how much explanatory power the indicator contains for describing the dynamics of the interest rate spreads. Note that, we can produce the same steps in the example by using a 3-months interest rate. From this, it follows that we can calculate the indicator using a 3-months or 10-years interest rate as input of the Extended model. This leads to consider the following variations of the Extended model for calculating the indicator values. Firstly, a 10-years interest rate is used as input of the Extended model:

$$\begin{aligned}
\begin{pmatrix} y_{t+1} \\ r^{10y}_{t+1} \end{pmatrix} &= \begin{pmatrix} \alpha_y \\ \alpha_r \end{pmatrix} + A_1 \begin{pmatrix} y_t \\ r^{10y}_t \end{pmatrix} + A_2 \begin{pmatrix} y_{t-1} \\ r^{10y}_{t-1} \end{pmatrix} + \begin{pmatrix} \epsilon^y_{t+1} \\ \epsilon^r_{t+1} \end{pmatrix} \\
\text{rev}_{t+1} &= \alpha_{\text{rev}} + R_1 \text{rev}_t + \epsilon^{\text{rev}}_{t+1} \\
\text{sp}_{t+1} &= \alpha_{\text{sp}} + \rho d_t + \beta YVAR_t + E_1 \text{rev}_t + E_2 \tau_t + E_3 c_t + \epsilon^{\text{sp}}_{t+1} \\
\text{mil}_{t+1} &= \alpha_{\text{mil}} + M_1 \text{mil}_t + \epsilon^{\text{mil}}_{t+1}. \\
d_{t+1} &= d_t \frac{1 + r^{10y}_t}{1 + y_{t+1}} - (\text{rev}_{t+1} - \text{sp}_{t+1} - \text{mil}_{t+1}).
\end{aligned} \tag{5.1}$$

System (5.1) corresponds to the Extended model discussed in Section 3.1. Secondly, a 3-months interest rate is used as input of the Extended model:

$$\begin{aligned}
\begin{pmatrix} y_{t+1} \\ r^{3M}_{t+1} \end{pmatrix} &= \begin{pmatrix} \alpha_y \\ \alpha_r \end{pmatrix} + A_1 \begin{pmatrix} y_t \\ r^{3M}_t \end{pmatrix} + A_2 \begin{pmatrix} y_{t-1} \\ r^{3M}_{t-1} \end{pmatrix} + \begin{pmatrix} \epsilon^y_{t+1} \\ \epsilon^r_{t+1} \end{pmatrix} \\
\text{rev}_{t+1} &= \alpha_{\text{rev}} + R_1 \text{rev}_t + \epsilon^{\text{rev}}_{t+1} \\
\text{sp}_{t+1} &= \alpha_{\text{sp}} + \rho d_t + \beta YVAR_t + E_1 \text{rev}_t + E_2 \tau_t + E_3 c_t + \epsilon^{\text{sp}}_{t+1} \\
\text{mil}_{t+1} &= \alpha_{\text{mil}} + M_1 \text{mil}_t + \epsilon^{\text{mil}}_{t+1}. \\
d_{t+1} &= d_t \frac{1 + r^{3M}_t}{1 + y_{t+1}} - (\text{rev}_{t+1} - \text{sp}_{t+1} - \text{mil}_{t+1}).
\end{aligned} \tag{5.2}$$

The economic interpretation of system (5.2) is that the government can issue only short-term debt, i.e., loans which mature in 3-months. A possible reason for this is that the government has debt sustainability issues, which can be seen as the worst case scenario for the government.

The next step is to investigate whether the 10-years interest rate differs historically from the 3-months interest rate. If this is not the case, then we can reduce systems (5.1) and (5.2) to one system since both the 10-years interest rate and the 3-months interest rate are described by a VAR(2) model. In Figure 5.1, we plot the 3-months versus the 10-years interest rate for Italy and the United Kingdom.

From Figure 5.1, it becomes clear that both the 3-months and 10-years interest rate show similar dynamics, but with different magnitudes. For all countries which we have investigated, the difference in magnitude between both yields was observed. For example, the term spread at the year 2010 is given by 337 basis points for the United Kingdom, where a basis point is equal 0.01%. Since both interest rates differ in magnitude and are described by the same model, it is expected that the forecasts of both interest rates are different. Hence, the choice of interest rate as input of the Extended model influences the dynamics of the indicator. This leads to choosing which interest rate is more appropriate as input of the Extended model.

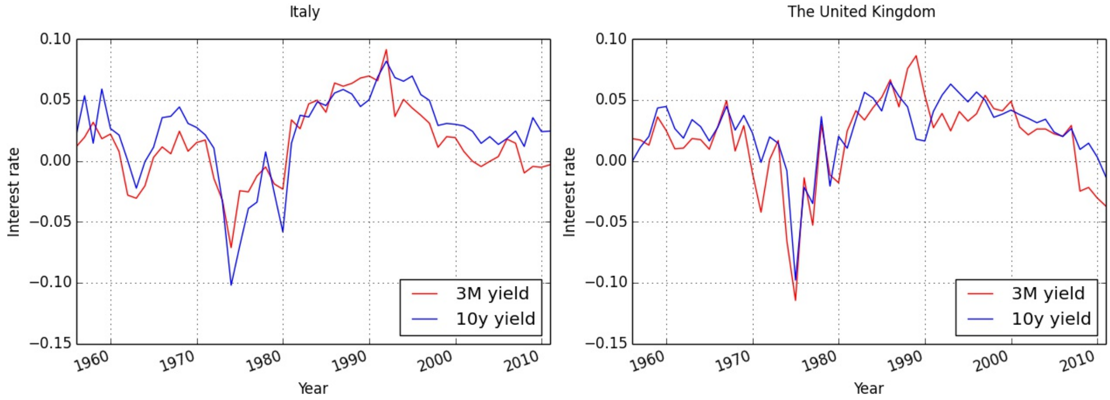


Figure 5.1: 3-month versus 10-year yields for Italy (left) and the United Kingdom(right)

Since the term spread is a leading indicator of future economic growth, it is desirable that the indicator explains well the dynamics of the term spread. For this purpose, we calculate the correlation between the indicator $I^k(t, 10y)$, calculated by an interest rate with maturity $k$, and the term spread $S(t, 10y, 3M)$, i.e.,

$$\rho(I^k(t, 10y), S(t, 10y, 3M)), \quad \text{where} \quad S(t, 10y, 3M) = r_t^{10y} - r_t^{3M}, \quad k = 3M, 10y. \tag{5.3}$$

The indicator values with the highest correlation coefficients will be used for explaining the dynamics of the interest rate spreads. The correlation coefficients are given for several countries in Table 5.1. The period where we calculate the indicator is given for each country.

| | $\rho(I^{3M}(t, 10y), S(t, 10y, 3M))$ | $\rho(I^{10y}(t, 10y), S(t, 10y, 3M))$ |
|---|---|---|
| UK (1990-2011) | 0.26 | 0.52 |
| IT (1997-2011) | 0.51 | 0.62 |
| US (1995-2011) | 0.50 | 0.60 |

Table 5.1: $\rho(I^k(t, 10y), S(t, 10y, 3M))$ for the United Kingdom, Italy and the United States

Table 5.1 shows that the dynamics of the term-spread are better explained by the indicator values produced by system (5.1). Therefore, we use a 10-years interest rate as input for the Extended model. And finally, we investigate the relationship between the indicator and the spread which takes the difference with the 10-years interest rate. In particular, we consider the following interest rate spreads:

$$S(t, 10y, k) = \begin{cases} r_t^{10y} - r_t^k & \text{if } k = 5y \\ r_t^k - r_t^{10y}. & \text{if } k = 15y, 20y, 30y \ . \end{cases} \tag{5.4}$$

In the next section, we formulate criteria for investigating this relationship.

## 5.2 Criteria for relationship between the Indicator and the Interest rate spreads

In this section, we formulate criteria to verify whether there is a relationship between the indicator and the interest rate spreads. For this purpose, we apply well known regression techniques. In particular, we are interested in how much explanatory power the indicator contains for describing the dynamics of the interest rate spreads. Below, we formulate criteria for investigating whether the indicator can describe the dynamics of the interest rate spreads given by (5.4). We explain the criteria for $S(t, 10y, k)$ where $k = 5y$. However, in the next section, we verify also the criteria for the interest rate spreads with maturities $k = 15y, 20y, 30y$.

1. First, we consider the correlation between the indicator and the interest rate spread:

$$\rho(I(t, k), S(t, 10y, k)) \quad \text{for} \quad k = 5y. \tag{5.5}$$

As discussed in Section 2.1 and 3.1, we assume that the government is financed by a 10-years interest rate. And moreover, the indicator estimates the risk of a significant debt increase $k$-years ahead in the future. By setting $k = 5y$ in (5.5), the indicator estimates the default risk of investing in a 5-years government bond. This latter type of risk is also captured in the dynamics of $S(t, 10y, 5y)$. When we observe a high correlation in (5.5), then there exists a linear relationship between the indicator and the interest rate spread.

2. And second, we investigate how well the interest rate spread can be explained using the indicator. We consider the following models for describing the evolution of the interest rate spread:

$$S(t + 1, 10y, k) = c + \alpha S(t, 10y, k) + \epsilon_{t+1} \tag{5.6a}$$
$$S(t + 1, 10y, k) = c + \alpha S(t, 10y, k) + \beta I(t + 1, k) + \epsilon_{t+1}, \tag{5.6b}$$

Then, the second criterion is decomposed in the following parts:

- Is the overall data fit improved by using model (5.6b) instead of model (5.6a)? For this, we compare the $R^2$ of both models. The $R^2$ is a statistical measure of how close the data is to the fitted regression line. Hence, the model with the highest $R^2$ has the most favourable overall goodness of fit.
- A $t$-test is applied to test the null hypothesis that the coefficient of a given predictor variable equals zero. This implies that a predictor variable can't explain the dynamics of the response variable. In this way, we can test whether the indicator does contribute for explaining the dynamics of the interest rate spread.

In (5.6a), the interest rate spread is described by an AR(1) model. Next, model (5.6b) is an extension of model (5.6a) by adding the indicator as a predictor variable. The reason for adding the indicator values at $t + 1$ is best explained by an example. In Figure 5.2, we plot the interest rate spread $S(t, 10y, 5y)$ versus the indicator for the United Kingdom. From Figure 5.2, the indicator evolves with similar dynamics as the interest rate spread. Observe that both the spread $S(t, 10y, 5y)$ and the indicator $I(t, 5y)$ rapidly increase after the start of the credit crisis. Therefore, it is possible to better explain the dynamics of the interest spread by including the future values of the indicator.

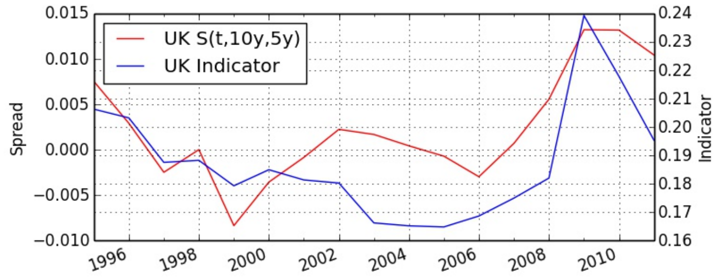


Figure 5.2: $S(t, 10y, 5y)$ versus $I(t, 5y)$ for United Kingdom

## 5.3 Results for relationship between the Indicator and the Interest rate spreads

In Section 5.2, we formulated criteria to investigate the relationship between the indicator and the interest rate spreads given by (5.4). In this section, we discuss the results of verifying this relationship for the following countries: the United States (US), France (FR) and Portugal (PRT). In Appendix E, the results of the remaining countries are presented.

For the first criterion, the results are presented in Table 5.2. In order to obtain the results, we use all available interest rate spread data, where the historical period is given for each country. Additionally, we denote the correlation defined in (5.5) with $\rho(I(k), S(k))$ where $k = 5y, 15y, 20y$ and $30y$.

| | $\rho(I(5y), S(5y))$ | $\rho(I(15y), S(15y))$ | $\rho(I(20y), S(20y))$ | $\rho(I(30y), S(30y))$ |
|---|---|---|---|---|
| US(1995-2011) | 0.68 | 0.60 | 0.57 | 0.68 |
| FR(2000-2011) | 0.62 | 0.64 | 0.66 | 0.40 |
| PRT(1995-2011) | -0.52 | -0.13 | -0.2 | -0.09 |

Table 5.2: Results of criterion 1 for US,FR and PRT

As discussed in Section 5.2, a high positive correlation indicates that there exists a linear relationship between the indicator and the interest rate spread. For the United States, we observe a high correlation between the indicator and the interest rate spreads for all maturities. For France, the correlation is high for the maturities $k = 5y, 15y, 20y$. Since the number of data points is small, we consider a correlation value above 0.50 as high. For instance, we have only eleven years of spread data available for France. From Table 5.2, the correlation between the indicator and the interest rate spreads is negatively correlated for Portugal, especially the correlation is very negative for the maturity $k = 5y$. The reason for the latter is illustrated in Figure 5.3.

From Figure 5.3, the indicator and the interest rate spreads evolve steady before the credit crisis; the period 2000-2008. Then, the indicator increases after the start of the credit crisis, especially for Portugal. And, the interest rate spread $S(t, 10y, 5y)$ rapidly decreases after the impact of the credit crisis for Portugal. Since the dynamics of $I(t, 5y)$ and $S(t, 10y, 5t)$ are strongly in the opposite direction after the credit crisis, this result is a negative correlation for Portugal. As a final point, we obtain $\rho(I(t, 5y), S(t, 10y, 5y)) = 0.42$ for PRT(1995-

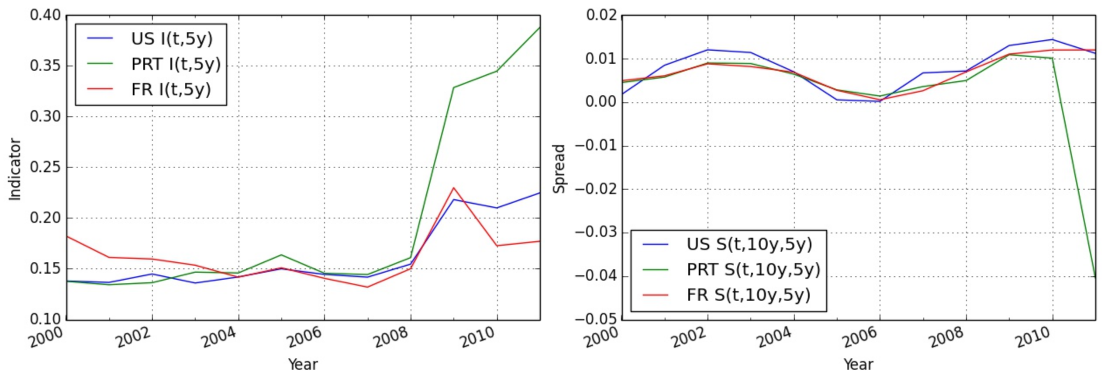


Figure 5.3: The values for $I(t, 5y)$ (left) and $S(t, 10y, 5y)$ (right) for the United States, France and Portugal

2009). Since interest rate spread data is limited available, this shows that the correlation is highly sensitive to the historical period. Therefore, it is not sufficient to use the correlation coefficient as only criterion for studying the relationship between the indicator and the interest rate spreads.

Next, we investigate how much explanatory power the indicator contains for explaining the dynamics of the interest rate spreads. Firstly, after applying ordinary least squares (OLS), we compare the estimated $R^2$ of model (5.6a) with (5.6b) for maturity $k$. From this, we can test whether model (5.6b) improves the overall goodness of fit. Secondly, we apply a $t$-test to test the null hypothesis that the coefficient of the indicator, $\beta$ in model (5.6b), is zero, see Chatterjee and Hadi (2015) for the technical details. Hence, if the null hypothesis is false, i.e., $\beta$ is significant in model (5.6b), then the indicator does contribute for explaining the dynamics of the interest rate spreads. Moreover, this test is applied for all predictor variables in models (5.6a) and (5.6b). For the United States, France and Portugal the results are presented in Tables 5.3, 5.4 and 5.5.

| $k$ | Model (5.6a)/(5.6b): $R^2$ | Is $\alpha$ significant in (5.6a) ? | Are $\alpha, \beta$ significant? in (5.6b) |
|---|---|---|---|
| 5y | 0.530/0.601 | Yes | Yes/No |
| 15y | 0.252/0.461 | Yes | No/Yes |
| 20y | 0.276/0.446 | Yes | No/Yes |
| 30y | 0.282/0.518 | Yes | No/Yes |

Table 5.3: US(1995-2011)

| $k$ | Model (5.6a)/(5.6b): $R^2$ | Is $\alpha$ significant in (5.6a) ? | Are $\alpha, \beta$ significant? in (5.6b) |
|---|---|---|---|
| 5y | 0.074/0.277 | No | No/Yes |
| 15y | 0.014/0.115 | No | No/No |
| 20y | 0.099/0.451 | No | Yes/Yes |
| 30y | 0.220/0.586 | Yes | Yes/Yes |

Table 5.4: PRT(1995-2011)

| $k$ | Model (5.6a)/(5.6b): $R^2$ | Is $\alpha$ significant in (5.6a) ? | Are $\alpha, \beta$ significant? in (5.6b) |
|---|---|---|---|
| 5y | 0.568/0.729 | Yes | Yes/Yes |
| 15y | 0.365/0.582 | Yes | No/Yes |
| 20y | 0.223/0.439 | No | No/No |
| 30y | 0.173/0.391 | No | No/No |

Table 5.5: FR(2000-2011)

From Tables 5.3, 5.4 and 5.5, it follows that the $R^2$ is improved by including the indicator. The same conclusion can be drawn for the remaining countries, see Appendix E. Furthermore, for the United states, the indicator contains explanatory power for explaining the dynamics of the interest rate spreads with maturities $k = 15y, 20y$ and $30y$. In this case, the indicator in model (5.6b) better explains the dynamics, then the interest rate spreads as predictor variable. For Portugal, the explanatory power of the indicator is clearly observable. For interest spreads with maturities $k = 5y, 20y$, model (5.6a) can't explain the dynamics of the interest rate spreads. Hence, by including the indicator, we improve the $R^2$ and the indicator does contribute to explain the evolution of the interest rate spreads. For France, the indicator contains explanatory power for explaining the dynamics of the interest rate spreads with short maturities. Note that, for all countries, when we observe a high correlation, then in most cases the indicator does contribute to explain the dynamics of the interest rate spreads. However, the reverse statement doesn't always hold. For instance, the parameter $\beta$ in model (5.6b) is for Portugal often significant, but we observe only a high correlation for $k = 5y$.

For "advanced" countries such as Germany and the Netherlands, the indicator doesn't contain much explanatory power for describing the dynamics of the interest rate spreads. A reason for this is that advanced countries have a more favourable policy for debt sustainability. As a consequence, the interest rate spreads contain a non-significant amount of country risk. Therefore, for advanced countries, the dynamics of the interest rate spreads are influenced by other factors than country risk. As a consequence, the indicator which captures a part of the country risk, doesn't contribute to describe the dynamics of the interest rate spreads.

Moreover, for all countries, if the indicator does (not) contribute for explaining the dynamics of the interest rate spread with $k = 20y$, then the same conclusion holds for the interest rate spread with $k = 30y$. We give an intuitive explanation for this observation, because it is difficult to find a conclusion which holds for all countries. In Figure 5.4, we plot the interest rate spreads for Belgium with maturities $k = 20y, 30y$. From Figure 5.4, we observe that the interest rate spreads display similar dynamics. Moreover, the indicator trajectories become smaller as $k$ increases. The reason for the latter is that the forecasts of the Extended model converge to a stationary state. Hence, if the indicator does (not) contribute to explain the dynamics of the interest rate spreads with $k = 20y$, then the same conclusion holds for interest rate spreads with $k = 30y$. This line of reasoning also holds for the countries; the United Kingdom, Italy, Germany, the Netherlands and Finland.

As a final note, we have investigated how much explanatory power the 10-years interest rate contains for explaining the dynamics of the interest rate spreads. Since the spreads given by (5.4) depend on the 10-years interest rate, it is reasonable to add the 10-years interest rate as predictor variable in model (5.6a):

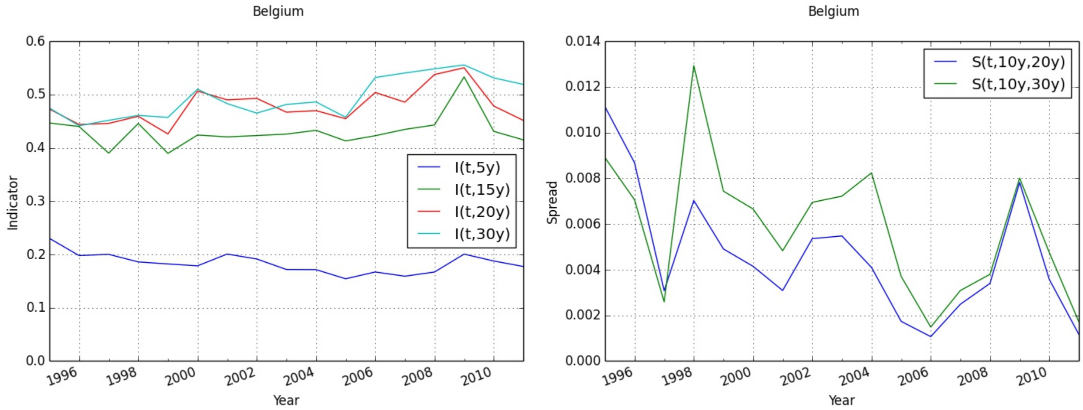


Figure 5.4: The values for $I(t,k)$ (left) and $S(t,10y,k)$ (right) for Belgium

$$S(t+1,10y,k) = c + \alpha S(t,10y,k) + \beta r_{t+1}^{10y} + \epsilon_{t+1}. \tag{5.7}$$

In Appendix F, we verified the second criterion for model (5.7). From this, we conclude that the indicator does contribute for explaining the dynamics of the interest rate spreads compared to the 10-years interest rate.

In this section, to summarize, we investigated the relationship between the indicator and the interest rate spreads. In particular, the interest rate spreads given by (5.4) and the indicator which is calculated by system (5.1). The reason for this is that $I^{10y}(t,10y)$ captures better the dynamics of the term-spread, which is a leading indicator for predicting future economic growth, than $I^{3M}(t,10y)$. From our results, we conclude that the indicator contains explanatory power for describing the dynamics of the interest rate spreads. In particular, for several interest rate spreads there exist a linear relationship with the indicator. Moreover, the overall goodness of fit is improved by model (5.6b), where we have added the indicator. And also, for several interest rate spreads the indicator in model (5.6b) explains better the dynamics, then the interest rate spreads as predictor variable. Finally, the indicator has more explanatory power for explaining the dynamics of the interest rate spreads than the 10-years interest rate.

# CHAPTER 6

## Stability analysis for the Extended model

In this chapter, we perform a stability analysis for the Extended model. In Section 4.1, we presented a case study when historical data is not representative for predicting the future evolution of the government primary expenditure, i.e., the forecasts of the primary expenditure and the government debt displayed blow-up behaviour. The reason for this is that the fiscal policy of the government does not have to react correctly to debt sustainability issues. The attitude of a government towards debt sustainability is captured by the fiscal reaction parameter $\rho$. Here, we investigate the influence of the fiscal reaction parameter $\rho$ on the stability of the Extended model. In particular, we want to derive conditions for the existence of blow-up behaviour using a discrete dynamical system approach. In Section 6.1, we discuss general theory about discrete dynamical systems. Next, in Section 6.2, we apply the theory discussed in Section 6.1 to a deterministic multi-dimensional linear system. However, the Extended model is a non-linear dynamical system. In order to perform a stability analysis, we consider two simplifications of the Extended model. First, in Section 6.3 we discuss the stability of a stochastic multi-dimensional linear system, which is a linear simplification of the Extended model. Secondly, in Section 6.4, we discuss the stability of a stochastic non-linear multidimensional system.

## 6.1 General theory about Discrete Dynamical systems

The modelling of many applications in fields, such as demography, ecology, economics, engineering, finance, and physics, can be done by discrete dynamical systems, see Lynch (2009) for some specific examples. Here, we present an overview of general theory about discrete dynamical systems, which is relevant for the stability analysis of the Extended model.

A dynamical system is a description how one state develops into another over the course of time. A discrete dynamical system is given by:

$$\begin{aligned} x_{1,t+1} &= f^1(x_{1,t}, x_{2,t}, \dots, x_{n,t}, \rho), \\ x_{2,t+1} &= f^2(x_{1,t}, x_{2,t}, \dots, x_{n,t}, \rho), \\ &\vdots \\ x_{n,t+1} &= f^n(x_{1,t}, x_{2,t}, \dots, x_{n,t}, \rho), \end{aligned} \tag{6.1}$$

where $t = 0, 1, 2, \ldots$, $x_{i,t} \in \mathbb{R}$ and $f^i : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}$, $i = 1, 2 \ldots, n$, are continuous differentiable functions. Here, the $k$-dimensional parameter $\rho$ is fixed. We denote the initial value of the vector of state variables by $x_0 = (x_{1,0}, x_{2,0}, \ldots, x_{n,0})$. System (6.1) is compactly written as:

$$x_{t+1} = f(x_t, \rho), \quad t = 0, 1, 2, \ldots, \tag{6.2}$$

where the state vector $x_t = (x_{1,t}, x_{2,t}, \ldots, x_{n,t})$ and $f : \mathbb{R}^{n+k} \to \mathbb{R}^n$. A solution of system (6.2) is a trajectory, or orbit in Kuznetsov (2013), of the vector of state variables that satisfies (6.2). A steady-state equilibrium of system (6.2) is a trajectory that is invariant under further iterations of the dynamical system.

**Definition 6.1.** *A steady-state equilibrium of the dynamical system given by* (6.2) *is a vector* $\bar{x} \in \mathbb{R}^n$ *such that*

$$\bar{x} = f(\bar{x}, \rho). \tag{6.3}$$

Suppose we start close to the steady-state equilibrium $\bar{x}$, that is, let

$$x_0 = \bar{x} + y, \tag{6.4}$$

where $y$ is a small perturbation. From this, an important question is whether system (6.2) converges to the steady-state equilibrium with initial value given by (6.4). In order to answer this question, we first consider the following types of stability for system (6.2), see Galor (2007) and Murray et al. (1994).

**Definition 6.2.** *We can distinguish the following types of stability for a dynamical system:*

1. *The steady-state equilibrium $\bar{x}$ of system* (6.2) *is locally stable (in the sense of Lyapunov) at $t = t_0$ if for any $\epsilon > 0$ there exist a $\delta(t_0, \epsilon) > 0$ such that*

$$|x_{t_0} - \bar{x}| < \delta \Rightarrow |x_t - \bar{x}| < \epsilon \quad \forall t \geq t_0. \tag{6.5}$$

2. *The steady-state equilibrium $\bar{x}$ of system* (6.2) *is asymptotically stable at $t = t_0$, if*
   - *(a) $\bar{x}$ is locally stable, and*
   - *(b) $\bar{x}$ is locally attractive; there exists a $\delta(t_0) > 0$ such that*

$$|x_{t_0} - \bar{x}| < \delta \Rightarrow \lim_{t \to \infty} x_t = \bar{x}. \tag{6.6}$$

3. *The steady-state equilibrium $\bar{x}$ of system* (6.2) *is globally stable, if*

$$\lim_{t \to \infty} x_t = \bar{x}, \quad \forall x_0 \in \mathbb{R}^n. \tag{6.7}$$

Lyapunov and asymptotic stability are local definitions; they describe the behaviour of system (6.2) nearby an equilibrium point. In general, it is challenging verifying these types of stability for general dynamical systems, especially the property of global stability. However, it is possible to determine whether a steady-state equilibrium of a non-linear system is locally stable; by examining the stability of the linear approximation near the steady-state equilibrium, see Galor (2007) for more details.

Often systems of the form (6.2) contain parameters, for instance $\rho$ in system (6.2), which are only known approximately. In particular, they are generally determined by measurements that are not exact. It can occur that a slight variation in a parameter can have a significant impact on the stability of the steady-state equilibrium, i.e., a so-called bifurcation occurs. This study leads to the area referred to as bifurcation theory. Bifurcation theory is a deep and complicated area involving a lot of current research.

**Definition 6.3.** *The appearance of a topologically nonequivalent phase portrait under variation of parameters is called a bifurcation.*

By looking at the phase portrait, we can determine the number and types of steady-state equilibria to which the system (6.2) tends as $t \to \infty$. In practice, only several key trajectories are illustrated in the diagrams that present phase portraits schematically. See Kuznetsov (2013) for the definition of whether phase portraits are topologically equivalent or not. The definition contains the option that the stability of the steady-state equilibrium of system (6.2) changes when varying parameter $\rho$, which we discuss in the next section.

## 6.2 Stability of a deterministic multi-dimensional linear system

Consider the following linear dynamical system:

$$x_{t+1} = Ax_t + B, \quad t = 0, 1, 2, \ldots, \tag{6.8}$$

where $A$ is an $n \times n$ matrix with constant coefficients and $B \in \mathbb{R}^n$. System (6.8) describes the evolution of $x_{t+1}$, whose values depend on a constant matrix $A$, a constant vector $B$ and the state at a previous time step $x_t$. System (6.8) is of the form (6.3).

As discussed in Section 6.1, the trajectory of system (6.8) is a path of state vectors; $\{x_t\}_{t=0}^{\infty}$, that satisfies (6.8). By substituting the initial value $x_0$ in system (6.8), we obtain the following recursions:

$$\begin{aligned} x_1 &= Ax_0 + B \\ x_2 &= Ax_1 + B = A^2x_0 + AB + B \\ x_3 &= Ax_2 + B = A^3x_0 + A^2B + AB + B \\ &\vdots \\ x_t &= A^tx_0 + A^{t-1}B + A^{t-2}B + \ldots AB + B. \end{aligned} \tag{6.9}$$

From (6.9), it is obvious that the state vector $x_t$ is given by:

$$x_t = A^t x_0 + \sum_{i=0}^{t-1} A^i B. \tag{6.10}$$

**Lemma 6.1.** *If the inverse of $(I - A)$ exists, then the sum of a geometric series of matrices, $\sum_{i=0}^{t-1} A^i$, whose factor is the matrix $A$, is given by*

$$\sum_{i=0}^{t-1} A^i = [I - A^t][I - A]^{-1}, \tag{6.11}$$

*where $I$ is the identity matrix.*

*Proof.*

$$\begin{aligned}\sum_{i=0}^{t-1} A^i[1-A] &= I - A + A[1-A] + A^2[1-A] + \dots A^{t-1}[1-A] \\ &= I + A + A^2 + \dots + A^{t-1} - [A + A^2 + A^3 + \dots + A^t] \\ &= I - A^t\end{aligned}$$

Hence, post-multiplication of both sides of the equation by $[1-A]^{-1}$ establishes the lemma. □

In the remaining parts of this chapter, we assume that $(I-A)^{-1}$ exists. Using the result from Lemma 6.1, we can rewrite (6.10) as follows:

$$x_t = A^t[x_0 - [I-A]^{-1}B] + [I-A]^{-1}B. \tag{6.12}$$

From (6.12), the value of the state vector $x_t$ depends on the initial state vector $x_0$, the time-independent matrix $A$ and the constant vector $B$. The next step is to characterize the evolution of (6.12) as $t \to \infty$; does the state vector converge to the steady-state equilibrium? From Definition 6.1, the steady-state equilibrium of system (6.8) is a vector $\bar{x} \in \mathbb{R}^n$ such that:

$$\bar{x} = A\bar{x} + B. \tag{6.13}$$

Thus, the steady-state equilibrium of system (6.8) is given by $\bar{x} = [I-A]^{-1}B$. Hence, we can formulate $x_t$ as a function which depends on the matrix $A$, the initial value $x_0$ and the steady-state value $\bar{x}$. By substituting the steady-state $\bar{x}$ in (6.12), the solution of system (6.8) is given by

$$x_t = A^t(x_0 - \bar{x}) + \bar{x}. \tag{6.14}$$

From (6.14), the coefficients of the matrix $A$ determine whether the state vector converges in the long run to the steady-state equilibrium. Moreover, these coefficients determine whether the system evolves monotonically or oscillatory, diverges asymptotically to plus or minus infinity, or evolves in a periodic orbit. For this, we distinguish whether the matrix $A$ has real or complex eigenvalues.

### 6.2.1 Matrix $A$ with distinct real eigenvalues

In this section, we show the influence of the eigenvalues of the matrix $A$ on the evolution of the dynamical system. For this, we formulate two examples where the matrix $A$ has a different structure. First, we consider the case that $A$ is a diagonal matrix, i.e., there is no interdependence between the variables of the state vector.

**Example 6.1.**

Consider the following two-dimensional linear system:

$$\begin{bmatrix} x_{1,t+1} \\ x_{2,t+1} \end{bmatrix} = \begin{bmatrix} a_{1,1} & 0 \\ 0 & a_{2,2} \end{bmatrix} \begin{bmatrix} x_{1,t} \\ x_{2,t} \end{bmatrix}, \tag{6.15}$$

where the initial conditions $x_0 = [x_{1,0}, x_{2,0}]'$ are given. System (6.15) is called an uncoupled system because $x_{i,t+1}$ depends only on its previous value $x_{i,t}$. The diagonal elements are the eigenvalues of the matrix $A$. From (6.14), the solution of system (6.15) is given by:

$$\begin{bmatrix} x_{1,t+1} \\ x_{2,t+1} \end{bmatrix} = \begin{bmatrix} a_{1,1}^t & 0 \\ 0 & a_{2,2}^t \end{bmatrix} \begin{bmatrix} x_{1,0} \\ x_{2,0} \end{bmatrix}. \tag{6.16}$$

The evolution of each state variable is independent of the evolution of the other state variable. Therefore, the evolution of both state variables is given by:

$$\begin{aligned} x_{1,t} &= a_{1,1}^t x_{1,0}, \\ x_{2,t} &= a_{2,2}^t x_{2,0}. \end{aligned} \tag{6.17}$$

From (6.17), the evolution of the state variables depends on the magnitude of $a_{1,1}$ and $a_{2,2}$. For instance, if $a_{1,1}, a_{2,2} \in (-1, 1)$, then the solution (6.16) converges to the steady-state equilibrium. However, if $a_{i,i} \notin (-1, 1)$ and $x_{i,0} \neq 0$, then the state variable $x_{i,t}$ diverges to plus or minus infinity. In general, the evolution of each state variable may differ qualitatively for a linear multi-dimensional system. Next, we discuss the qualitative behaviour of a coupled system.

**Example 6.2.**

Consider the following two-dimensional linear system,

$$\begin{bmatrix} x_{1,t+1} \\ x_{2,t+1} \end{bmatrix} = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} \begin{bmatrix} x_{1,t} \\ x_{2,t} \end{bmatrix}, \tag{6.18}$$

where $a_{1,2}, a_{2,1} \neq 0$ and the initial conditions $x_0 = [x_{1,0}, x_{2,0}]'$ are given. Similar to Example 6.1, the steady-state equilibrium $\bar{x} = 0$. In this case, the evolution of a state variable depends on the other state variable, i.e., $x_{1,t}$ and $x_{2,t}$ are interdependent. The idea is to use an uncoupled system for describing the evolution of a coupled system. In order to do so, we apply a well known result from linear algebra.

**Theorem 6.1.** *Let $A$ be an $n \times n$ matrix with coefficients $a_{i,j}$, $i, j = 1, \ldots, n$.*

- *If the matrix $A$ has $n$ distinct real eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$ then there exists a non-singular $n \times n$ matrix, $Q$, such that*
$$A = QDQ^{-1},$$
*where $D$ is a diagonalized matrix*
$$D = \begin{bmatrix} \lambda_1 & 0 & 0 & \ldots 0 & \\ 0 & \lambda_2 & 0 & \ldots 0 & \\ 0 & 0 & \lambda_3 & \ldots 0 & \\ \vdots & \vdots & \vdots & \ddots & \\ 0 & 0 & 0 & \ldots & \lambda_n \end{bmatrix},$$
*and $Q$ is an invertible $n \times n$ matrix whose columns are the eigenvectors of the matrix $A$, $\{f_1, f_2, \ldots, f_n\}$, i.e.*
$$Q = [f_1, f_2, \ldots, f_n].$$

*Proof.* See (Galor 2007). □

By defining the state vector $y_t = Q^{-1}x_t$, the solution of system (6.18) is described by the state vector $y_t$ as follows:

$$\begin{aligned} x_t &= Qy_t, \\ y_t &= D^t y_0. \end{aligned} \tag{6.19}$$

where $Q, D$ are known from Theorem 6.1. From (6.19), the qualitative behaviour of $x_t$ is determined by the evolution of $y_t$. As in Example 6.1, the evolution of $y_t$ depends on the diagonal elements of the matrix $D$; which are the eigenvalues of the matrix $A$. Both examples show the importance of these eigenvalues on the evolution of a multi-dimensional linear system. Therefore, we formulate conditions for the eigenvalues such that the state vector converges (monotonically or oscillatory) to the steady-state equilibrium.

**Theorem 6.2.** *Consider the system $x_{t+1} = Ax_t + B$, where $x_t \in \mathbb{R}^n$ and $x_0$ is given. Suppose that $(I - A)^{-1}$ exists and $A$ has $n$ distinct real eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$.*

1. *The steady-state equilibrium $\bar{x} = [I - A]^{-1}B$ is globally stable if and only if*

$$|\lambda_j| < 1, \quad \forall j = 1, 2, \ldots, n$$

2. *$\lim_{t\to\infty} x_t = \bar{x}$ if and only if $\forall j = 1, 2, \ldots, n$*

$$\{|\lambda_j| < 1 \quad or \quad y_{j,0} = 0\},$$

*where $y_0 = Q^{-1}(x_0 - \bar{x})$, and $Q$ is an invertible $n \times n$ matrix whose columns are the eigenvectors, $\{f_1, f_2, \ldots, f_n\}$, of the matrix $A$*

*Proof.* See (Galor 2007). □

According to Theorem 6.2, the absolute value of the eigenvalues of the matrix $A$, determines whether the steady-state equilibrium is globally stable. Next, we wish to characterize the trajectory of the state vector for a linear system. For instance, does the state vector converge monotonically or not to the steady-state equilibrium? Therefore, we consider the various types of phase portraits that characterize the evolution of the state vector, see Galor (2007) and Kuznetsov (2013) for more details. It is sufficient to look at the phase portrait of $y_t$ for describing the evolution of the state vector $x_t$. Again, the eigenvalues of the matrix $A$ give information on the evolution of each state variable. For simplicity, we discuss the various types of phase portraits for a two-dimensional linear system.

1. Suppose that $-1 < \lambda_i < 0 < \lambda_j < 1$, for $i, j = 1, 2$. Using Theorem 6.2, we conclude that the steady-state equilibrium is globally stable; which is referred to as a stable node. If the eigenvalue is positive, then the state variable converges monotonically. On the other hand, oscillatory behaviour corresponds to negative eigenvalues. Additionally, if $|\lambda_i| > |\lambda_j|$, then the convergence of the $j$-th state variable is faster towards the steady-state level.
2. The steady-state equilibrium is a saddle point in the cases: $\{-1 < \lambda_i < 1, \lambda_j < -1\}$ and $\{-1 < \lambda_i < 1, \lambda_j > 1\}$. The property of saddle points corresponds with $\lim_{t\to\infty} y_{i,t} = 0$ $\forall(y_{i,0}) \in \mathbb{R}$, whereas $\lim_{t\to\infty} y_{j,t} = 0$ if and only if $y_{j,0} = 0$. The sign of the eigenvalue

determines the qualitative behaviour of each state variable. Namely, a positive eigenvalue corresponds to monotonic behaviour, whereas a negative eigenvalue corresponds to oscillatory behaviour.

3. Suppose that both eigenvalues $\lambda_i, \lambda_j \notin (-1, 1)$. Then, the steady-state equilibrium is unstable and referred to as a source point; $\lim_{t\to} y_{1,t} = \pm\infty$ and $\lim_{t\to} y_{2,t} = \pm\infty$, $\forall (y_{1,0}, y_{2,0}) \in \mathbb{R}^2 - \{0\}$.

### 6.2.2 Matrix $A$ with distinct complex eigenvalues

In this section, we consider an $n$-dimensional linear system of the form (6.18) with $n/2$ pairs of distinct complex eigenvalues $\{\mu_1, \bar{\mu}_1, \mu_2, \bar{\mu}_2, \ldots, \mu_n, \bar{\mu}_n\}$;

$$\begin{aligned} \mu_j &= \alpha_j + \beta_j i, \\ \bar{\mu}_j &= \alpha_j - \beta_j i, \end{aligned} \tag{6.20}$$

where $\alpha_j, \beta_j \in \mathbb{R}$ and $i \equiv \sqrt{-1}$. Again, we show the influence of the eigenvalues of the matrix $A$ on the evolution of the dynamical system. First, the solution of the coupled system (6.18) is described by a dynamical system of interdependent variables. In order to do so, we state a result known from linear algebra.

**Theorem 6.3.** *If the matrix A has $n/2$ pairs of distinct complex eigenvalues, $\{\mu_1, \bar{\mu}_1, \mu_2, \bar{\mu}_2, \ldots, \mu_n, \bar{\mu}_n\}$, then there exists a nonsingular $n \times n$ matrix $Q$, such that*

$$A = QDQ^{-1}$$

*where $D$ is in block Jordan form*

$$D = \begin{bmatrix} \alpha_1 & -\beta_1 & 0 & 0 & \ldots & \ldots & 0 & 0 \\ \beta_1 & \alpha_1 & 0 & 0 & \ldots & \ldots & 0 & 0 \\ 0 & 0 & \alpha_2 & -\beta_2 & \ddots & \ddots & 0 & 0 \\ 0 & 0 & \beta_2 & \alpha_2 & \ddots & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & \ldots & \ldots & \alpha_{n/2} & -\beta_{n/2} \\ 0 & 0 & 0 & 0 & \ldots & \ldots & \beta_{n/2} & \alpha n/2 \end{bmatrix}, \tag{6.21}$$

*Proof.* See (Galor 2007). □

Using Theorem 6.3, the evolution of the state vector $x_t$ is described by $y_t$, where $y_t = Q^{-1}x_t$. To be more precise, there exists a non-singular $n \times n$ matrix $Q$, such that

$$\begin{aligned} x_t &= Qy_t \\ y_t &= D^t y_0, \end{aligned} \tag{6.22}$$

where $D$ is given by (6.21). From this, the state variables $\{y_{2j-1,t}, y_{2j,t}\}$ evolve independently of all other pairs over time, i.e.,

$$\begin{bmatrix} y_{2j-1,t+1} \\ y_{2j,t+1} \end{bmatrix} = \begin{bmatrix} \alpha_j & -\beta_j \\ \beta_j & \alpha_j \end{bmatrix}^t \begin{bmatrix} y_{2j-1,0} \\ y_{2j,0} \end{bmatrix}, \tag{6.23}$$

for $j = 1, 2, \ldots, n/2$. Note that, the evolution of the state variables $\{y_{2j-1,t}, y_{2j,t}\}$ depends on $\alpha_j$ and $\beta_j$. However, it is not clear how $\alpha_j$ and $\beta_j$ influence the qualitative behaviour of the dynamical system. For this purpose, we express system (6.23) in terms of polar coordinates. In general, a complex number $z = a + bi$ can be written in polar form as:

$$\begin{aligned} a &= r\cos(\theta) \\ b &= r\ sin(\theta), \end{aligned} \tag{6.24}$$

where $0 \leq \theta \leq 2\pi$ and the modulus of $z$ is given by:

$$r = \sqrt{a^2 + b^2}. \tag{6.25}$$

We can write the eigenvalues $\mu_j = \alpha_j + \beta_j i$ as follows:

$$\begin{aligned} \alpha_j &= r_j \cos(\theta_j) \\ \beta_j &= r_j \sin(\theta_j), \end{aligned} \tag{6.26}$$

for $j = 1, 2, \ldots, n/2$.

**Theorem 6.4.**

$$\left( r_j \begin{bmatrix} \cos\theta_j & -\sin\theta_j \\ \sin\theta_j & \cos\theta_j \end{bmatrix} \right)^t = r_j^t \begin{bmatrix} \cos t\theta_j & -\sin t\theta_j \\ \sin t\theta_j & \cos t\theta_j \end{bmatrix}.$$

*Proof.* The theorem is proven by using the trigonometric identities:

$$\begin{aligned} \cos(\theta_1 + \theta_2) &= \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2) \\ \sin(\theta_1 + \theta_2) &= \cos(\theta_1)\sin(\theta_2) - \sin(\theta_1)\cos(\theta_2) \end{aligned}$$

□

Applying Theorem 6.4, we write system (6.23) in polar form;

$$\begin{bmatrix} y_{2j-1,t+1} \\ y_{2j,t+1} \end{bmatrix} = r_j^t \begin{bmatrix} \cos t\theta_j & -\sin t\theta_j \\ \sin t\theta_j & \cos t\theta_j \end{bmatrix} \begin{bmatrix} y_{2j-1,0} \\ y_{2j,0} \end{bmatrix}. \tag{6.27}$$

Since, $0 \leq |\cos(t\theta_j)| \leq 1$ and $0 \leq |\sin(t\theta_j)| \leq 1$ for all $t$. From (6.27), we observe that the modulus $r_j$ determines whether the state variables $\{y_{2j-1,t}, y_{2j,t}\}$ converge to the steady-state level. Therefore, we state the following condition for the modulus $r_j$.

**Theorem 6.5.** *Consider the dynamical system $x_{t+1} = Ax_t + B$, where $x_t \in \mathbb{R}^n$. Suppose that $(I - A)^{-1}$ exists and suppose that $A$ has $n/2$ pairs of distinct eigenvalues $\{\mu_1, \bar{\mu}_1, \mu_2, \bar{\mu}_2, \ldots, \mu_n, \bar{\mu}_n\}$. Then the steady-state equilibrium of the dynamical system, $\bar{x}$, is globally stable if and only if the modulus of each eigenvalue of the matrix $A$ is smaller than 1, i.e. if*

$$r_j = \sqrt{(\alpha_j^2 + \beta_j^2)} < 1 \quad \forall j = 1, 2, \ldots, n/2. \tag{6.28}$$

*Proof.* See (Galor 2007). □

Next, we describe the phase portraits that characterize the evolution of the state vector $y_t$. For simplicity, we discuss the various types of phase portraits for a two-dimensional linear system. The qualitative behaviour of the system is determined by the modulus $r$, and the sign of $\alpha$ and $\beta$. The value of $r$ determines whether the system is characterized by convergence ($r < 1$), divergence ($r > 1$) or periodic orbit ($r = 1$). Here, we distinguish the behaviour of the steady-state equilibrium for $r < 1$ and $r > 1$. For a deeper discussion about the various phase portraits, see Galor (2007) and Kuznetsov (2013).

1. When $r < 1$, the dynamical system shows spiral converge towards the steady-state equilibrium. For this reason, the steady-state equilibrium is referred to as a spiral sink. If $\beta > 0$ the motion is counter-clockwise whereas if $\beta < 0$ the motion is clockwise.
2. When $r > 1$, the dynamical system exhibits spiral divergence from its steady-state equilibrium. In this case, the steady-state equilibrium is called a spiral source. If $\beta > 0$ the motion is counter-clockwise whereas if $\beta < 0$ the motion is clockwise.

### 6.2.3 Bifurcations in a multi-dimensional linear system

The conditions for global stability of a linear system are discussed in the previous sections. We have shown that the eigenvalues of the matrix $A$ determine the qualitative behaviour of the system. In this section, we investigate: how the stability of a system of the form (6.8) is affected by varying the parameters. For example, we can vary the coefficients of the matrix $A$ and determine when the system is globally stable. Before discussing the several types of bifurcations, we state a property of the steady-state equilibrium.

**Definition 6.4.** *A steady-state equilibrium is called hyperbolic, if there are no eigenvalues on the imaginary axis; all eigenvalues $\lambda_1, \dots, \lambda_n$ of the matrix $A$ have the property:*

$$|\lambda_j| \neq 1 \quad \forall j = 1, \dots, n.$$

The next step is to monitor the stability of a steady-state equilibrium, while we vary the parameters. The stability of the system is affected when the hyperbolic condition is violated. To be more precise, suppose that there exists a positive real eigenvalue $\lambda_i$ that satisfies (6.28). Then, the hyperbolic condition is violated when $\lambda_i$ approaches the unit circle and we obtain $\lambda_i = 1$. Other bifurcations are observed for negative real eigenvalues $\lambda_i$ and complex eigenvalues $\mu_i$ that satisfy (6.28). In these cases, the eigenvalues approach the unit circle and we obtain $\lambda_i = -1$ and $\mu_i = \exp^{\pm i\theta}$, $0 < \theta < \pi$. These three types of bifurcations are summarized in the following definitions.

**Definition 6.5.** *The bifurcation associated with:*

1. *the appearance of $\lambda_i = 1$ is called a fold (or tangent) bifurcation;*
2. *the appearance of $\lambda_i = -1$ is called a flip (or period-doubling) bifurcation.*
3. *the appearance of $\mu_i, \bar{\mu}_i = \exp^{\pm i\theta}$, $0 < \theta < \pi$, is called a Neimark-Sacker (or torus) bifurcation.*

For a deeper discussion of the various types of bifurcations, we refer the reader to Kuznetsov (2013). In the upcoming sections, we apply the stability analysis for a linear system to investigate the local stability of a stochastic (non)-linear system.

## 6.3 Stability of a stochastic multi-dimensional linear system

As discussed in Section 6.2, the stability of a multi-dimensional linear system; $x_{t+1} = Ax_t + B$, depends on the eigenvalues of the matrix $A$. The conditions for global stability, as described in Theorems 6.2 and 6.5, can't explain however when the Extended model produces (un)stable forecasts. This is because the Extended model is a non-linear dynamical system with complicated properties. For example, calculating the $\text{YVAR}_t$, as discussed in Section 3.1, involves the use of a one-sided HP filter in each iteration of the dynamical system. Therefore, we investigate the local stability of the following stochastic multi-dimensional linear system, which is a linear simplification of the Extended model:

$$\begin{aligned} \begin{bmatrix} \log(1+y_{t+1}) \\ \log(1+r_{t+1}) \end{bmatrix} &= \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + A \begin{bmatrix} \log(1+y_t) \\ \log(1+r_t) \end{bmatrix} + \begin{bmatrix} \epsilon^1_{t+1} \\ \epsilon^2_{t+1} \end{bmatrix}, \\ \log(d_{t+1}) &= \log(1+r_t) - \log(1+y_t) + \log(d_t - s_t), \\ \log(d_{t+1} - s_{t+1}) &= \alpha + \rho \log(d_t) + \beta \log(d_t - s_t) + \tilde{\epsilon}_{t+1}. \end{aligned} \tag{6.29}$$

Here, the error components have the following distribution:

$$\begin{pmatrix} \epsilon^1_{t+1} \\ \epsilon^2_{t+1} \end{pmatrix} \sim \mathcal{N}(0, \Sigma_1), \quad \tilde{\epsilon}_{t+1} \sim \mathcal{N}(0, \sigma). \tag{6.30}$$

In system (6.29), a VAR(1) model is used for describing the evolution of the annual return of real GDP $y_t$ and the interest rate $r_t$. Since both $y_t$ and $r_t$ are small, we use log returns, see Ruppert (2004) for more details. The non-linear Benchmark model is a simplification of the Extended model. In order to make the Benchmark model linear, we take the logarithm at both sides of equation (2.3). In system (6.29), the evolution of $\log(d_{t+1} - s_{t+1})$ depends linearly on its previous value, on the term $\rho \log(d_t)$ and on a stochastic term. We rewrite (6.29) in the following matrix-vector notation:

$$\begin{bmatrix} \log(1+y_{t+1}) \\ \log(1+r_{t+1}) \\ \log(d_{t+1}) \\ \log(d_{t+1} - s_{t+1}) \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ 0 \\ \alpha \end{bmatrix} + \begin{bmatrix} a_1 & a_2 & 0 & 0 \\ a_2 & b_2 & 0 & 0 \\ -1 & 1 & 0 & 1 \\ 0 & 0 & \rho & \beta \end{bmatrix} \begin{bmatrix} \log(1+y_t) \\ \log(1+r_t) \\ \log(d_t) \\ \log(d_t - s_t) \end{bmatrix} + \begin{bmatrix} \epsilon^1_{t+1} \\ \epsilon^2_{t+1} \\ 0 \\ \tilde{\epsilon}_{t+1} \end{bmatrix}. \tag{6.31}$$

The stability analysis discussed in Section 6.2 is for a deterministic multi-dimensional linear system. By taking the expectation at both sides of system (6.31), we obtain the following deterministic multi-dimensional linear system:

$$z_{t+1} = Rz_t + c, \tag{6.32}$$

where

$$R = \begin{bmatrix} a_1 & a_2 & 0 & 0 \\ a_2 & b_2 & 0 & 0 \\ -1 & 1 & 0 & 1 \\ 0 & 0 & \rho & \beta \end{bmatrix}, \quad z_t = \begin{bmatrix} \mathbb{E}[\log(1+y_t)] \\ \mathbb{E}[\log(1+r_t)] \\ \mathbb{E}[\log(d_t)] \\ \mathbb{E}[\log(d_t - s_t)] \end{bmatrix}, \quad c = \begin{bmatrix} c_1 \\ c_2 \\ 0 \\ \alpha \end{bmatrix}. \tag{6.33}$$

In order to reduce the number of parameters of system (6.32), we assume that both $\mathbb{E}[\log(1+y_t)]$ and $\mathbb{E}[\log(1+r_t)]$ are nearby the steady-state equilibrium. With this assumption, we approximate system (6.32) by the following two-dimensional linear system:

$$x_{t+1} = Ax_t + B, \tag{6.34}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ \rho & \beta \end{bmatrix}, \quad x_t = \begin{bmatrix} \mathbb{E}[\log(d_t)] \\ \mathbb{E}[\log(d_t - s_t)] \end{bmatrix}, \quad B = \begin{bmatrix} \mathbb{E}[\log(1+\bar{r})] - \mathbb{E}[\log(1+\bar{y})] \\ \alpha \end{bmatrix}, \tag{6.35}$$

and

$$\begin{aligned} \mathbb{E}[\log(1+\bar{y})] &= \lim_{t\to\infty} \mathbb{E}[\log(1+y_{t+1})], \\ \mathbb{E}[\log(1+\bar{r})] &= \lim_{t\to\infty} \mathbb{E}[\log(1+r_{t+1})], \end{aligned} \tag{6.36}$$

where these steady-state levels are given for each country, see Appendix C. As discussed in Section 6.2, the steady-state equilibrium of system (6.34) is given by:

$$\bar{x} = [I - A]^{-1}B = \begin{bmatrix} 1 & -1 \\ -\rho & 1-\beta \end{bmatrix}^{-1} B = \begin{bmatrix} 1 + \frac{\rho}{1-\beta-\rho} & \frac{1}{1-\beta-\rho} \\ \frac{\rho}{1-\beta-\rho} & \frac{1}{1-\beta-\rho} \end{bmatrix} B, \tag{6.37}$$

Moreover, the solution of system (6.34) is given by:

$$x_t = A^t(x_0 - \bar{x}) + \bar{x}, \tag{6.38}$$

where the steady-state equilibrium exists as long as $1-\beta-\rho \neq 0$. In order to determine whether $x_t$ converges to the steady-state equilibrium $\bar{x}$, we should verify the conditions described in Theorems 6.2 and 6.5. For this, we determine the eigenvalues of the matrix $A$, which are simply the roots of the characteristic polynomial:

$$\begin{aligned} p(\lambda) &= \det(A - \lambda I) \\ &= \det\left( \begin{bmatrix} -\lambda & 1 \\ \rho & \beta-\lambda \end{bmatrix} \right) \\ &= \lambda^2 - \beta\lambda - \rho \end{aligned} \tag{6.39}$$

Thus, the eigenvalues of the matrix $A$ are given by:

$$\lambda_1 = \frac{\beta - \sqrt{\beta^2 + 4\rho}}{2}, \quad \lambda_2 = \frac{\beta + \sqrt{\beta^2 + 4\rho}}{2}. \tag{6.40}$$

From (6.40), the stability of system (6.34) only depends on the parameters $\rho$ and $\beta$. The next step is to estimate these parameters using country specific data and to determine whether the steady-state equilibrium is globally stable. First, we consider the case of Germany, where we show how different values of $\beta$ and $\rho$ influence the stability of system (6.34).

**Example 6.3** (GER(1970-1995))**.**

In this example, the parameters of system (6.34) are estimated for Germany, with data over the period $t = 1970, \ldots, 1995$. Using ordinary least squares we obtain the estimates $\rho = 0.21$, $\beta = 0.82$. Hence, the value of the steady-state equilibrium is equal to

$$\bar{x} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} 1 + \frac{\rho}{1-\beta-\rho} & \frac{1}{1-\beta-\rho} \\ \frac{\rho}{1-\beta-\rho} & \frac{1}{1-\beta-\rho} \end{bmatrix} \begin{bmatrix} \mathbb{E}[\log(1+\bar{r})] - \mathbb{E}[\log(1+\bar{y})] \\ \alpha \end{bmatrix} = \begin{bmatrix} -2.42 \\ -2.44 \end{bmatrix}, \tag{6.41}$$

where the steady-state levels are $\mathbb{E}[\log(1+\bar{y})] = 0.013$ and $\mathbb{E}[\log(1+\bar{r})] = 0.036$. Furthermore, the eigenvalues of the matrix $A$ are given by

$$\lambda_1 = -0.20, \quad \lambda_2 = 1.03. \tag{6.42}$$

From (6.42), we conclude that the steady-state equilibrium is a saddle point. To be more precise, $\lim_{t\to\infty} x_{1,t} = \bar{x}_1, \forall x_{1,0} \in \mathbb{R}$ and $\lim_{t\to\infty} x_{2,t} = \bar{x}_2$ if and only if $\mathbb{E}[\log(d_0 - s_0)] = 0$. However, the condition $\mathbb{E}[\log(d_0 - s_0)] = 0$ never holds by the property of the logarithm. In this case, the solution of system (6.34) diverges for every initial value $x_0$. And also, the forecasts of the Extended model don't converge to a stationary state, when we estimate the parameters over the same data period, see Appendix H.

**Example 6.4** (GER(1970-1996))**.**

Similar to Example 6.3, we estimate the parameters of system (6.34) for Germany, only now with data over the period $t = 1970, \dots, 1996$. In this case, we obtain the estimates $\rho = 0.43$, $\beta = 0.52$. Hence, the steady-state equilibrium is given by:

$$\bar{x} = \begin{bmatrix} 1 + \frac{\rho}{1-\beta-\rho} & \frac{1}{1-\beta-\rho} \\ \frac{\rho}{1-\beta-\rho} & \frac{1}{1-\beta-\rho} \end{bmatrix} \begin{bmatrix} \mathbb{E}[\log(1+\bar{r})] - \mathbb{E}[\log(1+\bar{y})] \\ \alpha \end{bmatrix} = \begin{bmatrix} 0.14 \\ 0.11 \end{bmatrix}, \tag{6.43}$$

where we use the same values for $\mathbb{E}[\log(1+\bar{y})]$ and $\mathbb{E}[\log(1+\bar{r})]$ as in Example 6.3. Furthermore, the eigenvalues of the matrix $A$ are given by:

$$\lambda_1 = -0.45, \quad \lambda_2 = 0.96. \tag{6.44}$$

Combining (6.44) and Theorem 6.2, we conclude that the steady-state equilibrium is globally stable. In other words, the equilibrium is a stable node, i.e, $\lim_{t\to\infty} x_{1,t} = \bar{x}_1$ and $\lim_{t\to\infty} x_{2,t} = \bar{x}_2$ $\forall x_0 \in \mathbb{R}^2$. In Appendix H, the forecasts of all economic variables show convergence to the stationary state, where the parameters of the Extended model are estimated over the same data period.

The next step is to estimate the parameters of system (6.34) for several countries. As a result, we obtain estimates of $(\beta, \rho)$ for each country. In Figure 6.1, these estimates for $(\beta, \rho)$ are plotted; as coloured dots. For each pair of $(\beta, \rho)$, we calculate the eigenvalues $\lambda_1, \lambda_2$, as given by (6.40), and determine whether the steady-state equilibrium is globally stable. For example, the steady-state equilibrium is globally stable when we estimate the parameters of system (6.34) for Portugal over the historical period $t = 1976, \dots, 1995$ (PRT(1976-1995) in Figure 6.1).

Next, we investigate for which values of $\beta$ and $\rho$ the solution of system (6.34) converges to the steady-state equilibrium. For this purpose, we determine the following boundaries:

$$\begin{aligned} B_1 &= \{\forall(\beta,\rho) : |\lambda_1| = 1, \lambda_1 \in \mathbb{R}\}, \\ B_2 &= \{\forall(\beta,\rho) : |\lambda_2| = 1, \lambda_2 \in \mathbb{R}\}, \\ B_3 &= \{\forall(\beta,\rho) : |\lambda_1| = 1, \lambda_1 \in \mathbb{C}\}. \end{aligned} \tag{6.45}$$

The red, blue and green lines in Figure 6.1 correspond to the boundaries as given by (6.45). The purple line in Figure 6.1 is defined as

$$B_4 = \{\forall(\beta,\rho) : \beta^2 + 4\rho = 0\}. \tag{6.46}$$

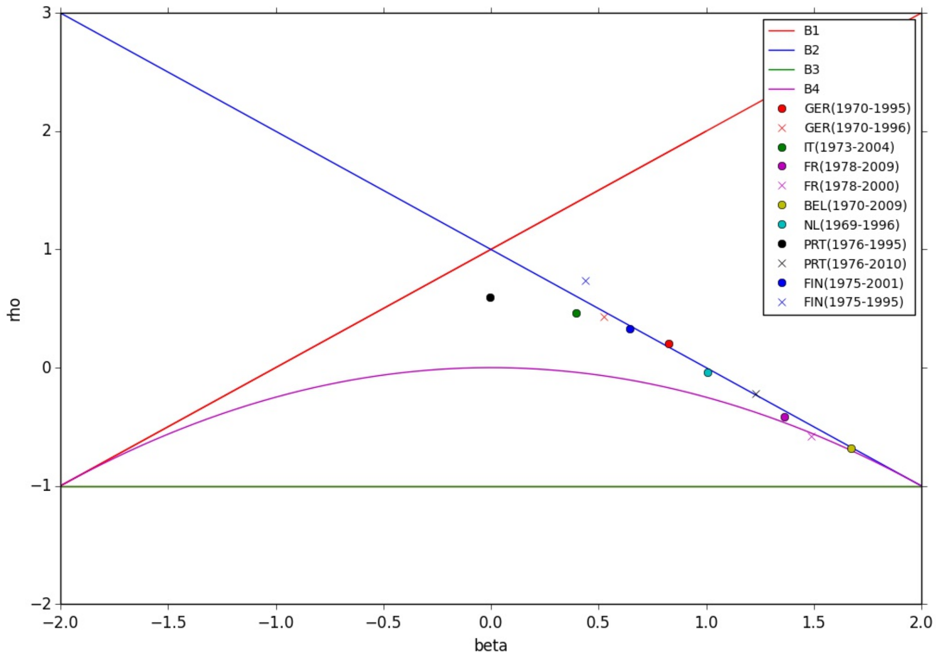


Figure 6.1: For several countries: the estimates for $(\beta, \rho)$.

In other words, for which values of $\beta$ and $\rho$ is the discriminant of the quadratic equation given by (6.39) equal to zero. As discussed in Section 6.2, system (6.34) is globally stable if and only if $(\beta, \rho)$ lies inside these boundaries given by (6.45). Hence, the steady-state equilibrium of system (6.34) is globally stable if and only if $(\beta, \rho)$ satisfies:

$$\begin{aligned} \rho + \beta &< 1, \\ \beta - \rho &< 1, \\ \rho &> -1. \end{aligned} \tag{6.47}$$

Suppose that a pair $(\rho, \beta)$ satisfies (6.47). Moreover, the pair $(\rho, \beta)$ approaches the boundaries given by (6.45), when we vary the parameters. If one of these boundaries is crossed, then the stability of system (6.34) has changed; a bifurcation has occurred. For instance, if the pair $(\rho, \beta)$ with positive real eigenvalues, crosses $B_1$ in Figure 6.1, then the bifurcation is called a fold.

From Figure 6.1, it follows that for GER(1970-1995), FIN(1975-1995) and PRT(1976-2010) the steady-state equilibrium is a saddle point; the solution of system (6.34) diverges. And also, the forecasts of the Extended model show blow-up behaviour, which is displayed in Appendix H. This suggests that if the linear system (6.34) is (not) globally stable, then the forecasts of the Extended model does (not) converge to a stationary state. This relation is again verified for Germany, as discussed in Example 6.3 and Example 6.4.

However, there exist cases where this relationship doesn't hold. For example, the steady-state equilibrium of system (6.34) for FR(1978-2009) is globally stable, but the forecasts of the Extended model don't converge to a stationary state, see Appendix H. From this, it follows that the assumptions for obtaining the linear system (6.34) are sometimes too simplistic. As a consequence, the linear system (6.34) doesn't capture fully the dynamics of the Extended model. Therefore, a more advanced approach for investigating the stability of the Extended model is required. In the next section, we discuss the local stability of a non-linear system, which is also a simplification of the Extended model.

## 6.4 Stability of a stochastic non-linear multi-dimensional system

In Section 6.3, we derived conditions for global stability of a linear simplification for the Extended model, see (6.47). It is shown that satisfying these conditions doesn't always imply that the forecasts of the Extended model converge to a stationary state. Hence, we investigate the local stability of a more advanced system. Namely, the following stochastic non-linear multi-dimensional system, which is also an approximation of the Extended model:

$$\begin{aligned}\begin{bmatrix} y_{t+1} \\ r_{t+1} \end{bmatrix} &= \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + A \begin{bmatrix} y_t \\ r_t \end{bmatrix} + \begin{bmatrix} \epsilon^1_{t+1} \\ \epsilon^2_{t+1} \end{bmatrix}, \\ s_{t+1} &= \alpha^s + \rho d_t + \beta s_t + \epsilon^s_{t+1}, \\ d_{t+1} &= (d_t - s_t)\frac{1+r_t}{1+y_t}. \end{aligned} \tag{6.48}$$

Here, the error components have the following distribution:

$$\begin{pmatrix} \epsilon^y_{t+1} \\ \epsilon^r_{t+1} \end{pmatrix} \sim \mathcal{N}(0, \Sigma_1), \quad \epsilon^s_{t+1} \sim \mathcal{N}(0, \sigma). \tag{6.49}$$

As in Section 6.3, a VAR(1) model is used for describing the evolution of the annual return of real GDP $y_t$ and the interest rate $r_t$. From (6.48), the dynamics of $d_t$ are the same as in the Benchmark model. However, system (6.48) differs from the Benchmark model by the dynamics of $s_t$; the evolution of $s_t$ depends linearly on its previous value, on the term $\rho d_t$ and on a stochastic term. Here, the parameter $\rho$ has the same meaning as in the Benchmark model.

In order to investigate the local stability of system (6.48), we assume that the expected value of the steady-state equilibrium $\mathbb{E}[\bar{\theta}] = (\mathbb{E}[r_\infty], \mathbb{E}[y_\infty], \mathbb{E}[s_\infty], \mathbb{E}[d_\infty])$ exists, where $y_t \to y_\infty$ in weak converge. For a deeper discussion of convergence of random variables we refer the reader to Vaart (2013). In Appendix G, we explain the procedure for determining $\mathbb{E}[\bar{\theta}]$. As mentioned in Section 6.1, the behaviour of the Taylor approximation nearby the equilibrium gives information over the local stability of a non-linear system.

**Theorem 6.6.** *If $Y$ is a function of several random variables,*

$$Y = g(X_1, X_2, \ldots, X_n),$$

*the Taylor series around the mean values $(\mu_{X_1}, \mu_{X_2}, \ldots, \mu_{X_n})$, yields*

$$Y = g(\mu_{X_1}, \mu_{X_2}, \ldots, \mu_{X_n}) + \sum_{i=1}^{n} (X_i - \mu_{X_i}) \frac{\partial g}{\partial X_i} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} (X_i - \mu_{X_i})(X_j - \mu_{X_j}) \frac{\partial^2 g}{\partial X_i \partial X_j} + \ldots$$

*where the derivatives are evaluated at $\mu_{X_1}, \mu_{X_2}, \ldots, \mu_{X_n}$.*

*Proof.* See Ang and Tang (2007). □

Using Theorem 6.6, we create a first-order Taylor approximation for $d_{t+1}$ around the steady-state $\mathbb{E}[\bar{\theta}]$.

$$\begin{aligned}
d_{t+1} &\approx f(\mathbb{E}[\bar{\theta}]) + \frac{\partial f}{\partial y_t}(\mathbb{E}[\bar{\theta}])(y_t - \mathbb{E}[y_\infty]) + \frac{\partial f}{\partial r_t}(\mathbb{E}[\bar{\theta}])(r_t - \mathbb{E}[r_\infty]) + \frac{\partial f}{\partial s_t}(\mathbb{E}[\bar{\theta}])(s_t - \mathbb{E}[s_\infty]) \\
&+ \frac{\partial f}{\partial d_t}(\mathbb{E}[\bar{\theta}])(d_t - \mathbb{E}[d_\infty]) \\
&= \alpha^* + \frac{\partial f}{\partial y_t}(\mathbb{E}[\bar{\theta}])y_t + \frac{\partial f}{\partial r_t}(\mathbb{E}[\bar{\theta}])r_t + \frac{\partial f}{\partial s_t}(\mathbb{E}[\bar{\theta}])s_t + \frac{\partial f}{\partial d_t}(\mathbb{E}[\bar{\theta}])d_t,
\end{aligned} \tag{6.50}$$

where

$$f(y_t, r_t, s_t, d_t) = (d_t - s_t)\frac{1 + r_t}{1 + y_t}, \tag{6.51}$$

and

$$\alpha^* = f(\mathbb{E}[\bar{\theta}]) - \frac{\partial f}{\partial y_t}(\mathbb{E}[\bar{\theta}])\mathbb{E}[y_\infty] - \frac{\partial f}{\partial r_t}(\mathbb{E}[\bar{\theta}])\mathbb{E}[r_\infty] - \frac{\partial f}{\partial s_t}\mathbb{E}[\bar{\theta}])\mathbb{E}[s_\infty] - \frac{\partial f}{\partial d_t}(\mathbb{E}[\bar{\theta}])\mathbb{E}[d_\infty]. \tag{6.52}$$

The first-order derivatives are given by:

$$\frac{\partial f}{\partial y_t} = -\frac{(d_t - s_t)(1 + r_t)}{(1 + y_t)^2}, \quad \frac{\partial f}{\partial r_t} = \frac{d_t - s_t}{1 + y_t}, \quad \frac{\partial f}{\partial s_t} = -\frac{1 + r_t}{1 + y_t}, \quad \frac{\partial f}{\partial d_t} = \frac{1 + r_t}{1 + y_t}. \tag{6.53}$$

Using (6.50), we approximate system (6.48) locally around the equilibrium $\mathbb{E}[\bar{\theta}]$ by the following linear system:

$$\begin{aligned}
\begin{bmatrix} y_{t+1} \\ r_{t+1} \end{bmatrix} &= \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + A \begin{bmatrix} y_t \\ r_t \end{bmatrix} + \begin{bmatrix} \epsilon^1_{t+1} \\ \epsilon^2_{t+1} \end{bmatrix}, \\
s_{t+1} &= \alpha^s + \rho d_t + \beta s_t + \epsilon^s_{t+1}, \\
d_{t+1} &= \alpha^* + \frac{\partial f}{\partial y_t}(\mathbb{E}[\bar{\theta}])y_t + \frac{\partial f}{\partial r_t}(\mathbb{E}[\bar{\theta}])r_t + \frac{\partial f}{\partial s_t}(\mathbb{E}[\bar{\theta}])s_t + \frac{\partial f}{\partial d_t}(\mathbb{E}[\bar{\theta}])d_t.
\end{aligned} \tag{6.54}$$

We rewrite (6.54) in the following matrix-vector notation:

$$\begin{bmatrix} y_{t+1} \\ r_{t+1} \\ s_{t+1} \\ d_{t+1} \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \alpha^s \\ \alpha^* \end{bmatrix} + \begin{bmatrix} a_1 & a_2 & 0 & 0 \\ a_2 & b_2 & 0 & 0 \\ 0 & 0 & \beta & \rho \\ \frac{\partial f}{\partial y_t}(\mathbb{E}[\bar{\theta}]) & \frac{\partial f}{\partial r_t}(\mathbb{E}[\bar{\theta}]) & \frac{\partial f}{\partial s_t}(\mathbb{E}[\bar{\theta}]) & \frac{\partial f}{\partial d_t}(\mathbb{E}[\bar{\theta}]) \end{bmatrix} \begin{bmatrix} y_t \\ r_t \\ s_t \\ d_t \end{bmatrix} + \begin{bmatrix} \epsilon^1_{t+1} \\ \epsilon^2_{t+1} \\ \epsilon^s_{t+1} \\ 0 \end{bmatrix}. \tag{6.55}$$

We proceed as in Section 6.3 by taking the expectations at both sides of system (6.55). Hence, we obtain the deterministic linear system:

$$z_{t+1} = Rz_t + c \tag{6.56}$$

where

$$R = \begin{bmatrix} a_1 & a_2 & 0 & 0 \\ a_2 & b_2 & 0 & 0 \\ 0 & 0 & \beta & \rho \\ \frac{\partial f}{\partial y_t}(\mathbb{E}[\bar{\theta}]) & \frac{\partial f}{\partial r_t}(\mathbb{E}[\bar{\theta}]) & \frac{\partial f}{\partial s_t}(\mathbb{E}[\bar{\theta}]) & \frac{\partial f}{\partial d_t}(\mathbb{E}[\bar{\theta}]) \end{bmatrix}, \quad z_t = \begin{bmatrix} \mathbb{E}[y_t] \\ \mathbb{E}[r_t] \\ \mathbb{E}[s_t] \\ \mathbb{E}[d_t] \end{bmatrix}, \quad c = \begin{bmatrix} c_1 \\ c_2 \\ \alpha^s \\ \alpha^* \end{bmatrix}. \tag{6.57}$$

As mentioned in Section 6.3, we reduce the number of parameters in system (6.57) by assuming that both $\mathbb{E}[y_t]$ and $\mathbb{E}[r_t]$ are nearby their steady-state levels. With this assumption, we approximate system (6.57) with the following two-dimensional linear system:

$$x_{t+1} = Ax_t + B, \tag{6.58}$$

where

$$x_t = \begin{bmatrix} \mathbb{E}[s_t] \\ \mathbb{E}[d_t] \end{bmatrix}, \quad A = \begin{bmatrix} \beta & \rho \\ -\phi & \phi \end{bmatrix}, \quad B = \begin{bmatrix} \alpha^s \\ \tilde{\alpha}^* \end{bmatrix}, \phi = \frac{1 + \mathbb{E}[r_\infty]}{1 + \mathbb{E}[y_\infty]}, \tag{6.59}$$

and

$$\tilde{\alpha}^* = \alpha^* + \frac{\partial f}{\partial y_t}(\bar{\theta})\mathbb{E}[y_\infty] + \frac{\partial f}{\partial r_t}(\bar{\theta})\mathbb{E}[r_\infty]. \tag{6.60}$$

As discussed in Section 6.2, the solution of system (6.58) is given by $x_t = A^t(x_0 - \bar{x}) + \bar{x}$, where

$$\begin{aligned} \bar{x} &= [I - A]^{-1}B \\ &= \begin{bmatrix} 1-\beta & -\rho \\ \phi & 1-\phi \end{bmatrix}^{-1} B \\ &= \begin{bmatrix} \frac{1-\phi}{\beta\phi+\phi\rho-\beta\phi+1} & \frac{\rho}{\beta\phi+\phi\rho-\beta\phi+1} \\ -\frac{\phi}{\beta\phi+\phi\rho-\beta\phi+1} & \frac{1-\beta}{\beta\phi+\phi\rho-\beta\phi+1} \end{bmatrix} B. \end{aligned} \tag{6.61}$$

In order to characterize the evolution of the state vector $x_t$ in system (6.58), we determine the eigenvalues of the matrix $A$, which are the roots of the characteristic polynomial:

$$\begin{aligned} p(\lambda) &= \det(A - \lambda I) \\ &= \det\left( \begin{bmatrix} \beta - \lambda & \rho \\ -\phi & \phi - \lambda \end{bmatrix} \right) \\ &= \lambda^2 - (\beta + \phi)\lambda + (\beta + \rho). \end{aligned} \tag{6.62}$$

Thus, the eigenvalues of $A$ are explicitly given by

$$\lambda_1 = \frac{\beta + \phi - \sqrt{(\beta + \phi)^2 - 4(\beta + \rho)\phi}}{2}, \quad \lambda_2 = \frac{\beta + \phi + \sqrt{(\beta + \phi)^2 - 4(\beta + \rho)\phi}}{2}. \tag{6.63}$$

The next step is to determine for which values of $\rho$ and $\beta$ the steady-state equilibrium of system (6.58) is globally stable, where the parameter $\phi$ is fixed. From (6.59), the parameter $\phi$ is known by the steady-state levels $\mathbb{E}[r_\infty]$ and $\mathbb{E}[y_\infty]$, which are given for each country, see Appendix C. We proceed as in Section 6.3 by determining the boundaries given by (6.45), where the eigenvalues $\lambda_1, \lambda_2$ are given by (6.63). The boundaries $B_1, B_2$ and $B_3$ correspond to the red, the blue, and the yellow lines in Figure 6.2 respectively. As discussed in Section 6.2, system (6.58) is globally stable if and only if the values of $(\beta, \rho)$ lie inside the region with boundaries given by (6.45). Hence, the steady-state equilibrium of system (6.58) is globally stable if $(\beta, \rho)$ satisfies:

$$\begin{aligned} \beta + \frac{\phi\rho}{\phi - 1} &> 1, \\ \beta + \frac{\phi\rho}{\phi + 1} &> -1, \\ \beta + \rho &< \frac{1}{\phi}. \end{aligned} \tag{6.64}$$

Next, we use country-specific data to estimate the parameters of system (6.58). In Figure 6.2, the estimates for $(\beta, \rho)$ are plotted, where we consider all possible data periods for Germany. As in (6.46), the purple line in Figure 6.2 represents the values of $(\beta, \rho)$ such that the discriminant $\xi$ of the quadratic equation given by (6.62) is equal to zero, where

$$\xi = (\beta + \phi)^2 - 4(\beta + \rho)\phi = 0. \tag{6.65}$$

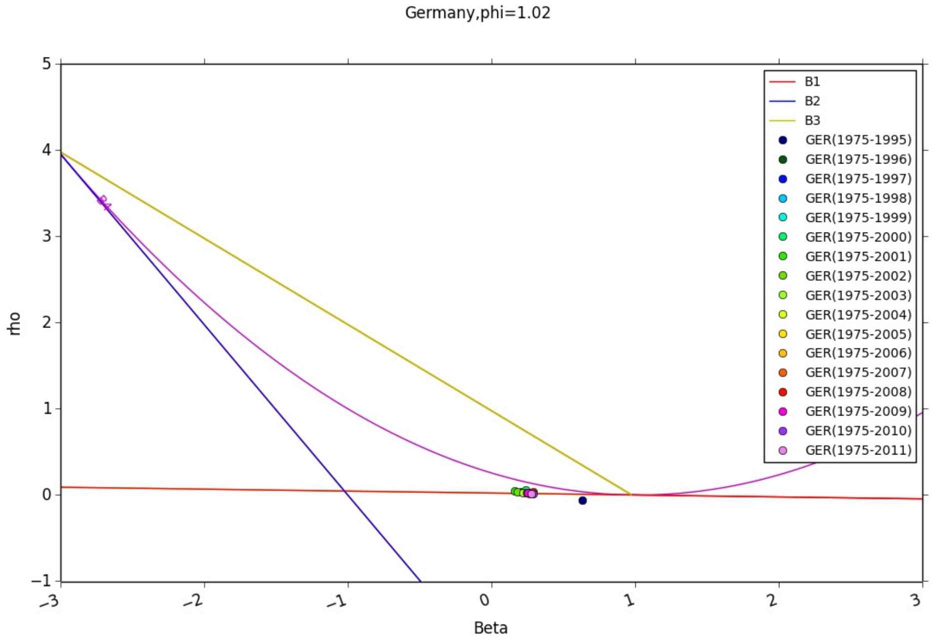


Figure 6.2: Qualitative behaviour of equilibria for Germany

Consider the period $t = 1970, \ldots, 1995$, we obtain the estimates $(\beta, \rho) = (0.635, -0.0565)$ and $\phi = 1.02$. It is easily checked that the condition $\beta + \frac{\phi\rho}{\phi-1} > 1$ doesn't hold. In Figure 6.2, the estimate of $(\beta, \rho)$ falls outside the global stability region given by (6.64). In particular, the steady-state equilibrium is a saddle point. Moreover, this corresponds to the instability of system (6.34) in Example 6.3 of Section 6.3. For the remaining data periods, we observe that the estimates for $(\beta, \rho)$ satisfy (6.64), see Figure 6.2.

As in Section 6.3, if one of the boundaries given by (6.45) is crossed, then the stability of system (6.58) has changed. In other words, starting with a pair $(\beta, \rho)$ that satisfies (6.45), a specific type of bifurcation occurs if one of the boundaries $B_1, B_2$ or $B_3$ is crossed, when we vary the parameters of system (6.58). For example, suppose a pair $(\beta, \rho)$ satisfies (6.45) and $D < 0$, i.e, the eigenvalues given by (6.63) are complex. It follows that if the boundary $B_3$ is crossed, then we observe a so-called Neimark-Sacker bifurcation.

As mentioned in Section 6.1, the local stability of the non-linear system (6.48) can be examined by the stability of the linear system (6.54), where we approximated system (6.54) by the two-dimensional system (6.58). In order to obtain the linear system (6.58), we take a Taylor expansion nearby the steady-state equilibrium of system (6.48). From this, we have derived conditions for global stability of system (6.58). Next, we investigate whether these conditions for local stability of system (6.48) also hold when the non-linear system is not nearby the steady-state equilibrium. In order to do so, we produce for several countries forecasts by system (6.48) . Then, we verify that the forecasts converge or don't converge to a stationary state, if the estimated $(\beta, \rho)$ of system (6.58) satisfies (6.64) or not.

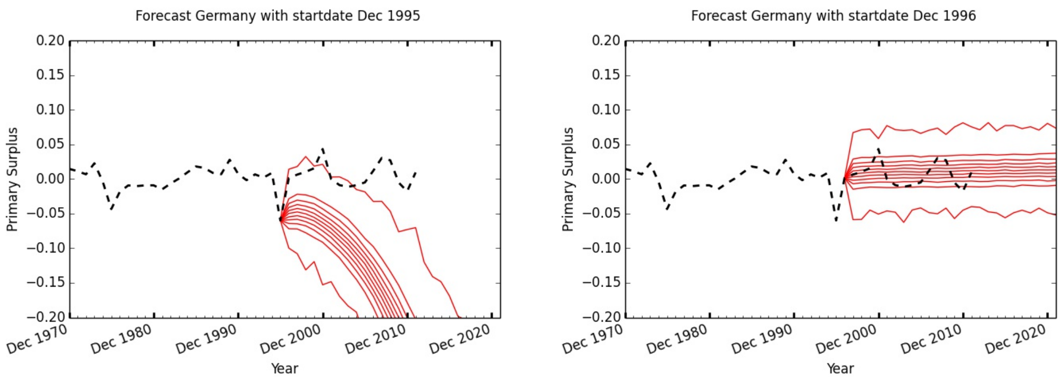


Figure 6.3: Primary surplus for Germany: historical data (black-dotted) and deciles of forecast (red). Left are parameters of system (6.48) estimated using data over period 1970-1995, where $R^2 = 0.15976$ . Right the parameters are estimated using data over period 1970-1996, where $R^2 = 0.09310$.

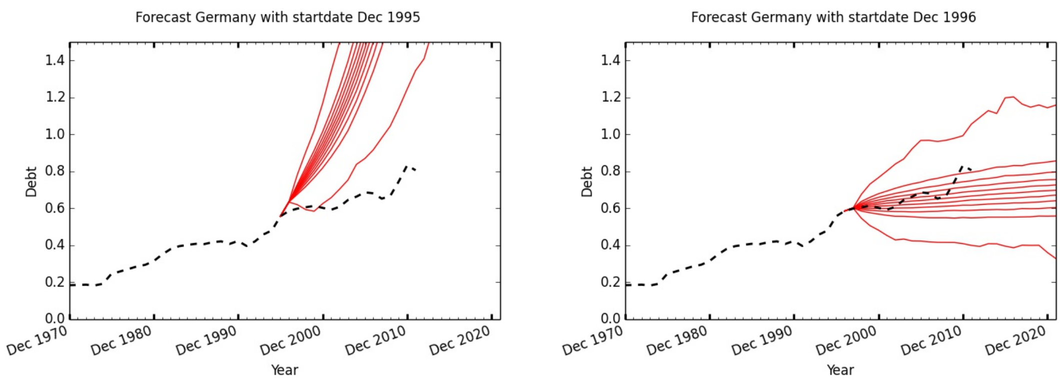


Figure 6.4: Government debt for Germany: historical data (black-dotted) and deciles of forecast (red).

In Figures (6.3) and (6.4), we plot the forecasts of system (6.48) for Germany. From these numerical results, we conclude that if the estimated $(\beta, \rho)$ of system (6.58) satisfies (6.64), then the forecasts of system (6.48) converge to a stationary state and vice versa. And moreover, this relationship is verified for the remaining countries discussed in Section 5.3. Additionally, if the forecasts of system (6.48) converge to a stationary state, then the forecasts of the Extended model converge to a stationary state as well. From this, we propose to verify the conditions given by (6.64) in order to get insight in the stability of the Extended model.

# CHAPTER 7

## Conclusion

### 7.1 Summary and conclusions

In this thesis, we have investigated whether the dynamics of the interest rate spreads can be explained by the debt sustainability indicator. We started with a general description of government debt sustainability and why this is of importance for policy-makers and financial markets. From this, it became apparent that there is a need for an accurate measure of debt sustainability. Here, we have assessed government debt sustainability using the indicator proposed in Ewijk, Lukkezen, and Rojas-Romagosa (2013). The debt sustainability indicator measures the degree to which governments are in control of their public finances by estimating the risk of a significant debt increase in the near future. In particular, we have shown that the indicator is easily calculated by using the debt samples produced by a stochastic simulation.

The first model which we have discussed, for evaluating the debt sustainability indicator, is the Benchmark model. This model consists of the following components: the accounting equation for the government debt $d_t$, an equation for the government's primary surplus $s_t$ and a VAR(2) model which describes the dynamics for the annual return of real GDP $y_t$ and the 10-years real interest rate $r_t$. It is shown that the Benchmark model makes assumptions which influence negatively the dynamics of the debt sustainability indicator.

Incorporating these model assumptions has led to the creation of the Extended model. The Extended model is an extension of the Benchmark model. The main reason for this is that the Extended model uses the causal one-sided HP filter, as opposed to the non-causal two-sided HP filter in the Benchmark model, for extracting the trend of a time series. Hence, the Extended model includes the $\text{YVAR}_t$ and $\text{GVAR}_t$ in its dynamics as opposed to the Benchmark model. Moreover, we don't remove components of the error vector after estimating the parameters of the Extended model. We have discussed the general procedure for producing economic forecasts with the Extended model. This procedure is easily applied for all countries considered in this investigation. Furthermore, we have investigated whether there exists a correspondence between the dynamics of the indicator, which is calculated with the Extended model, and the term-spread. We concluded that for most countries there exists a (linear) relationship between the indicator and the term-spread. And finally, we discussed whether an alternative model for describing the dynamics of $(y_t, r_t)$ improves the forecast accuracy. In particular, we compared the performance with the VAR(2) model of the Extended model. From the numerical results, we concluded that the VAR(2) model is better for predicting the evolution of $(y_t, r_t)$ than the alternative model. Similar conclusions were found in Lukkezen, Rojas-Romagosa, et al. (2012).

Since the parameters of the Extended model are estimated on historical data, the fundamental assumption is that historical data is representative for predicting the future. However, in Chapter 4, we have shown a case study in which historical data is not representative. It is possible that the government's fiscal policy doesn't react correctly to the debt accumulation based on historical data. As a consequence, the forecasts of the Extended model don't converge to a stationary state. In order to solve this issue, we have proposed with (4.4) a new government expenditure rule. This rule changes the fiscal policy of the government when the second criterion of the Stability and Growth Pact is not satisfied. We have verified that this rule controls the possible blow-up behaviour of the Extended model. However, we have shown with several numerical experiments that this new expenditure rule influences the quality of the debt sustainability indicator negatively. Hence, we use the original dynamics of the Extended model for investigating the relationship between the indicator and the interest rate spreads.

In order to verify this relationship we have specified which interest rate spreads can be considered in this investigation, i.e, the spread which take the difference with the 10-years interest rate. As a consequence, we have investigated whether the dynamics of the spread which take the difference with the 10-years interest rate can be explained by the debt sustainability indicator. By verifying the criteria in Section 5.2, we have made for several countries the following conclusions. Firstly, we have shown that a linear relationship exists between the debt sustainability indicator and several interest rate spreads. Secondly, the overall goodness of fit is improved by model (5.6b), where we have added the indicator as predictor variable. Thirdly, from the numerical results in Tables 5.3, 5.4 and 5.5 and Appendix E, we conclude that the indicator contains explanotory power for describing the dynamics of the interest rate spreads, especially with maturities $k = 5y, 15y$. And also, for several interest rate spreads the indicator in model (5.6b) explains better the dynamics, then the interest rate spreads as predictor variable. Therefore, we can make the overall conclusion that the dynamics of the interest rate spreads can be explained in most cases by the debt sustainability indicator. As a final note, we have shown that the indicator contains more explanatory power for describing the dynamics of the interest rate spreads than the 10-years interest rate as predictor variable.

Last but not least, we have performed a stability analysis for the Extended model. The goal was to derive conditions for the existence of blow-up behaviour using a discrete dynamical system approach. Since the Extended model is a non-linear system with complicated properties, we have derived conditions for global stability for two simplifications of the Extended model. We propose to verify the conditions given by (6.64) in order to get insight into the stability of the Extended model.

## 7.2 Outlook

From the research that has been conducted, we give some recommendations and final remarks for further research. Firstly, we give the following recommendations.

- One of the main models Ortec Finance use is a high dimensional time series model. This internally used model at Ortec Finance produces forecasts for hundreds of economic variables. The economic predictions are of crucial importance because they have large impact on model outcomes and thereby on decisions. At this time, this high dimensional time series model doesn't describe the dynamics of the government debt. Therefore, we recommend implementing the Extended model in the models of Ortec Finance as the first model for modelling the future's government debt.

- Ortec Finance has many institutional investors as client. Therefore, making economic forecasts for interest rates that are paid on government bonds is of importance. We recommend Ortec Finance to use the debt sustainability indicator as a tool for explaining the dynamics of the interest rate spreads.
- The conditions given by (6.64) can have an added value for the analysis done in (Ewijk, Lukkezen, and Rojas-Romagosa 2013), where the influence of $\rho$ on the debt evolution was investigated.

Finally, we present the following suggestions for future research.

- Changing the dynamics of the Extended model such that the instability of the model is prevented.
- Performing a stability analysis for the Extended model without making any simplifications. In this way, we can fully understand the stability of the Extended model.

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# APPENDIX A

## The Hodrick-Prescott filter

In this Appendix, we discuss how to extract the trend of a time series by using a Hodrick-Presscott filter (HP filter). In particular, we discuss the two-sided and the one-sided HP filter. From this, the properties and differences of both filters will become clear. Moreover, we give two approaches for the trend extraction by a one-sided HP filter.

## A.1 Two-Sided HP filter

A time series $\mathbf{y} = [y_1, \ldots, y_T]$ which is observed at a yearly frequency can be decomposed into a trend component $\tau_t$ and a cyclical component $c_t$. This decomposition is specified at time $t$ by:

$$y_t = \tau_t + c_t + \epsilon_t. \tag{A.1}$$

The trend component $\tau_t$ denotes the long run movement of the time series, while the cyclical component $c_t$ captures the sequence of non-periodic fluctuations. Usually, the cyclical component $c_t$ is referred to as the economic cycle. Finally, $\epsilon_t$ is a noise component.

The HP filter extracts the trend, $\tau = [\tau_1, \ldots, \tau_T]$, by minimizing the following loss function:

$$\min_{\tau} \Big( \sum_{t=1}^{T} (y_t - \tau_t)^2 + \lambda \sum_{t=2}^{T-1} [(\tau_{t+1} - \tau_t) - (\tau_t - \tau_{t-1})]^2 \Big), \tag{A.2}$$

see Leon (2012) for more details. Here, parameter $\lambda$ is a positive number which penalizes variations in the growth rate of the trend. In other words, as we increase the value of $\lambda$ the solution of (A.2) becomes smoother. The following result states that (A.2) can be solved analytically.

**Lemma A.1:** *The solution to* (A.2) *is given by:*

$$\tau = \mathbf{A}^{-1}\mathbf{y}, \tag{A.3}$$

*where*

$$\mathbf{A} = [\mathbf{I} + \lambda \mathbf{K}'\mathbf{K}], \tag{A.4}$$

$\mathbf{y} = [y_1, \ldots, y_T]'$ *are the values of the time series,* $\mathbf{I}$ *is a* $T \times T$ *identity matrix and* $\mathbf{K}$ *is a* $(T-2) \times T$ *matrix with the following structure*:

$$\mathbf{K} = \begin{pmatrix} 1 & -2 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & -2 & 1 \end{pmatrix}. \tag{A.5}$$

*Proof.* We can formulate (A.2) in the following matrix-vector notation:

$$\min_{\tau} f(\tau), \tag{A.6}$$

where

$$f(\tau) = (\mathbf{y} - \tau)'(\mathbf{y} - \tau) + \lambda(\mathbf{K}\tau)'(\mathbf{K}\tau). \tag{A.7}$$

Observe that matrix $\mathbf{A}$ is symmetric and positive definite, see (Lang 2012) for the definition of positive definite. For every nonzero vector $\mathbf{x} \in \mathrm{R}^T$,

$$\mathbf{x}'\mathbf{A}\mathbf{x} = \mathbf{x}'\mathbf{x} + \lambda(\mathbf{K}\mathbf{x})'(\mathbf{K}\mathbf{x}) \tag{A.8}$$

$$= \sum_{i=1}^{T} x_i^2 + \lambda \sum_{i=1}^{T-2} (\Delta^2 x_i)^2 \geq 0, \tag{A.9}$$

where $\Delta^2$ is the second order differences operator. Hence, the matrix $\mathbf{A}$ is invertible and $\mathbf{A}^{-1}$ is also positive definite. Next, we want to rewrite $f(\tau)$ in terms of the matrix $\mathbf{A}$.

$$\begin{aligned} f(\tau) &= \tau'\mathbf{A}\tau - \mathbf{y}'\tau - \tau'\mathbf{y} + \mathbf{y}'\mathbf{y} \\ &= \tau'\mathbf{A}(\mathbf{A}^{-1}\mathbf{A})\tau - \mathbf{y}'(\mathbf{A}^{-1}\mathbf{A})\tau - \tau'(\mathbf{A}\mathbf{A}^{-1})\mathbf{y} + \mathbf{y}'\mathbf{y} \\ &= (\tau'\mathbf{A} - \mathbf{y}')\mathbf{A}^{-1}(\mathbf{A}\tau - \mathbf{y}) + \mathbf{y}'\mathbf{y} - \mathbf{y}'\mathbf{A}^{-1}\mathbf{y} \\ &= (\mathbf{A}\tau - \mathbf{y})'\mathbf{A}^{-1}(\mathbf{A}\tau - \mathbf{y}) + \mathbf{y}'\mathbf{y} - \mathbf{y}'\mathbf{A}^{-1}\mathbf{y} \end{aligned} \tag{A.10}$$

The last two terms in (A.10) are independent of $\tau$. Therefore, minimizing $(\mathbf{A}\tau - \mathbf{y})'\mathbf{A}^{-1}(\mathbf{A}\tau - \mathbf{y})$ solves problem (A.6). Since $\mathbf{A}^{-1}$ is positive definite, it follows that

$$(\mathbf{A}\tau - \mathbf{y})'\mathbf{A}^{-1}(\mathbf{A}\tau - \mathbf{y}) \geq 0. \tag{A.11}$$

Then, the minimum value of $(\mathbf{A}\tau - \mathbf{y})'\mathbf{A}^{-1}(\mathbf{A}\tau - \mathbf{y})$ is attained when

$$(\mathbf{A}\tau - \mathbf{y})'\mathbf{A}^{-1}(\mathbf{A}\tau - \mathbf{y}) = 0 \Rightarrow \mathbf{y} = \mathbf{A}\tau \Rightarrow \tau = \mathbf{A}^{-1}\mathbf{y}. \tag{A.12}$$

□

As an example, the structure of $\mathbf{I} + \lambda\mathbf{K}'\mathbf{K}$ for $T = 7$ is given by:

$$\mathbf{I} + \lambda\mathbf{K}'\mathbf{K} = \begin{pmatrix} 1+\lambda & -2\lambda & \lambda & 0 & 0 & 0 & 0 \\ -2\lambda & 1+5\lambda & -4\lambda & \lambda & 0 & 0 & 0 \\ \lambda & -4\lambda & 1+6\lambda & -4\lambda & \lambda & 0 & 0 \\ 0 & \lambda & -4\lambda & 1+6\lambda & -4\lambda & \lambda & 0 \\ 0 & 0 & \lambda & -4\lambda & 1+6\lambda & -4\lambda & \lambda \\ 0 & 0 & 0 & \lambda & -4\lambda & 1+5\lambda & -2\lambda \\ 0 & 0 & 0 & 0 & \lambda & -2\lambda & 1+\lambda \end{pmatrix}. \tag{A.13}$$

As illustrated in (A.13), since the matrix structure of $\mathbf{I} + \lambda\mathbf{K}'\mathbf{K}$ is independent of the values of the time series $\mathbf{y}$, it follows that for constructing $\mathbf{K}$ only the size of $\mathbf{y}$ must be known. From this, it follows that extracting the trend using a two-sided HP filter can be efficiently implemented. To be more precise, the matrix $\mathbf{A}$ can immediately be determined, once the size of $\mathbf{y}$ is known. From (A.2) the constructed trend component $\tau_t$ depends on data observations $y_{t+i}$, $i > 0$. Filters with this property are called non-causal. Next, we introduce the causal one-sided HP filter.

## A.2 One-Sided HP filter

The one-sided HP filter only uses current and past state values $y_1, \ldots, y_t$ for extracting the trend component $\tau_t$. Two different approaches are discussed in (Stock and Watson 1999) and (Mehra 2004) how to construct the one-sided HP filter. Firstly, this filter is constructed using a Kalman filter estimate. Secondly, a two-sided HP filter is used iteratively.

### A.2.1 Trend extraction with a Kalman filter

An important feature in the derivation of the Kalman filter is the notion of a state space system. Here, we use the same notation as in Hamilton (1994). Let $\mathbf{y_t}$ denote an $(n \times 1)$ random vector at time $t$. Assume that $\mathbf{y_t}$ can be described in terms of an unobserved $(r \times 1)$ vector $\xi_\mathbf{t}$. Usually, $\xi_\mathbf{t}$ is referred to as the state vector. The state-space representation of the dynamics of $\mathbf{y_t}$ is given by:

$$\xi_\mathbf{t+1} = \mathbf{F}\xi_\mathbf{t} + \eta_{t+1}, \tag{A.14}$$
$$\mathbf{y_t} = \mathbf{A}'\mathbf{x_t} + \mathbf{H}'\xi_\mathbf{t} + \epsilon_\mathbf{t}, \tag{A.15}$$

where $\mathbf{F}, \mathbf{A}, \mathbf{H}$ are matrices and $\mathbf{x}_t$ is a $(k \times 1)$ vector of predetermined variables. The white noise vectors $\eta_\mathbf{t+1}$ and $\epsilon_\mathbf{t}$ have covariance matrices:

$$\mathbb{E}[\eta_\mathbf{t}\eta_\tau{}'] = \begin{cases} \mathbf{Q} & \text{for } t = \tau \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\mathbb{E}[\epsilon_\mathbf{t}\epsilon_\tau{}'] = \begin{cases} \mathbf{R} & \text{for } t = \tau \\ 0 & \text{otherwise,} \end{cases}$$

respectively. In the construction, we assume that we observe $\mathbf{y_1}, \ldots, \mathbf{y_T}, \mathbf{x_1}, \ldots, \mathbf{x_T}$ and the matrices $\mathbf{F}, \mathbf{Q}, \mathbf{A}, \mathbf{H}$, $\mathbf{R}$ are predetermined. The Kalman filter is an algorithm for calculating linear least squares forecasts of the state vector on the basis of data observed until date $t$,

$$\hat{\xi}_\mathbf{t+1|t} = \hat{E}[\xi_\mathbf{t+1}|\mathcal{F}_t], \tag{A.16}$$

where

$$\mathcal{F}_t = (\mathbf{y_1}, \ldots, \mathbf{y_t}, \mathbf{x_1}, \ldots, \mathbf{x_t})', \tag{A.17}$$

and $\hat{E}[\xi_\mathbf{t+1}|\mathcal{F}_t]$ denotes the linear projection of $\xi_\mathbf{t+1}$ on $\mathcal{F}_t$. The Kalman filter estimates these forecasts recursively by generating $\hat{\xi}_\mathbf{1|0}, \hat{\xi}_\mathbf{2|1}, \ldots, \hat{\xi}_\mathbf{T|T-1}$, see Hamilton (1994) for more details. Here, only the recursion for obtaining $\hat{\xi}_\mathbf{t+1|t}$ is presented. For $t = 1, 2, \ldots, T$, we iterate on

$$\hat{\xi}_\mathbf{t+1|t} = \mathbf{F}\hat{\xi}_\mathbf{t|t-1} + \mathbf{F}\mathbf{P}_\mathbf{t|t-1}\mathbf{H}(\mathbf{H}'\mathbf{P}_\mathbf{t|t-1}\mathbf{H} + \mathbf{R})^{-1}(\mathbf{y_t} - \mathbf{A}'\mathbf{x_t} - \mathbf{H}'\hat{\xi}_\mathbf{t|t-1}), \tag{A.18}$$

where each forecast is associated with a mean squared error (MSE) matrix,

$$\mathbf{P_{t+1|t}} = E[(\xi_{\mathbf{t+1|t}} - \hat{\xi}_{\mathbf{t+1|t}})(\xi_{\mathbf{t+1|t}} - \hat{\xi}_{\mathbf{t+1|t}})']. \tag{A.19}$$

In Stock and Watson (1999) the extracted trend by a one-sided HP filter at time $t$ is constructed as the Kalman filter estimate of $\tau_t$ in the following model:

$$\begin{aligned} y_t &= \tau_t + \epsilon_t \\ \tau_t &= 2\tau_{t-1} - \tau_{t-2} + \eta_t \end{aligned} \tag{A.20}$$

where $y_t$ is the logarithm of the time series, $\tau_t$ is the unobserved trend component and $\epsilon_t$, $\eta_t$ are mutually uncorrelated white noise sequences with relative variance $q = Var[\eta_t]/Var[\epsilon_t]$. For producing the estimate $\tau_t$, we must rewrite system (A.20) in a state-space representation. Then, the recursive procedure described in (A.18) can be used to obtain $\hat{\xi}_{\mathbf{1|0}}, \hat{\xi}_{\mathbf{2|1}}, \ldots, \hat{\xi}_{\mathbf{T|T-1}}$. This is accomplished by setting

$$\begin{aligned} &\xi_{\mathbf{t}} = [\tau_t, \tau_{t-1}]', \quad \mathbf{y_t} = y_t, \quad \mathbf{A} = \mathbf{0}, \quad \mathbf{F} = \begin{Bmatrix} 2 & -1 \\ 1 & 0 \end{Bmatrix}, \\ &\mathbf{H} = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}, \mathbf{Q} = \begin{Bmatrix} \frac{1}{\lambda} & 0 \\ 0 & 0 \end{Bmatrix} \quad \text{and} \quad \mathbf{R} = 1. \end{aligned} \tag{A.21}$$

From the construction of $\xi_{\mathbf{t}}$ in (A.21), we obtain the trend components $\tau_1, \ldots, \tau_T$ using a one-sided HP filter.

### A.2.2 Iterative two-sided HP filter

The second approach makes use of the two-sided HP filter iteratively. As discussed in (Mehra 2004), the one-sided HP filter extracts the trend component $\tau_t$ in the following way:

1. Determine the trend $\{\tilde{\tau}_s, s = 1, \ldots, t\}$ with a two-sided HP filter using all data up to time $t$;
2. Set $\tau_t = \tilde{\tau}_t$.

By repeating this procedure for all $t = 1, 2, \ldots, T$ the trend extraction by a one-sided HP filter is achieved. Note that, each trend component $\tau_t$ is determined by using only data up to time $t$. This iterative procedure is easily extended to extract simultaneously the trend components of several time series $\mathbf{y}^i$, $i = 1, \ldots, k$, where $\mathbf{y}^i = [y_1^i, \ldots, y_T^i]'$. In the second approach we rely on the use of the two-sided HP filter. The matrix $\mathbf{A}$ of (A.4) is easily determined if we assume that all time series $\mathbf{y}_i$ have the same length. Let $\mathbf{y}$ denote the matrix, where each column consists of the values of the time series $\mathbf{y}^i$, i.e.,

$$\mathbf{y} = [\mathbf{y}^1, \ldots, \mathbf{y}^k]. \tag{A.22}$$

Then, the matrix $\tau$, where each column consists of the extracted trend of the time series $\mathbf{y}^i$ by a two-sided HP filter, is determined by

$$\tau = \mathbf{A}^{-1}\mathbf{y}, \tag{A.23}$$

where $\mathbf{A}$ and $\mathbf{y}$ are given by (A.4) and (A.22). From (A.23), it is clear that we can calculate (A.23) easily for multiple time series. The recursive procedure given by (A.18) and (A.19) does not easily extend to multiple time series. For this reason, the one-sided HP filter which is constructed by using the two-sided HP filter iteratively has our preference for the trend extraction of multiple time series.

# APPENDIX B

## Convergence of the Hodrick-Presscott filter

Let $\{y_n, n \geq 1\}$ denote a one-dimensional time series with the property that $\mathbb{E}[y_n] \to \bar{y}$ as $n \to \infty$, where $\bar{y} \in \mathbb{R}$ and $\mathbb{E}[y_n] < \infty$. The trend $\{\tau_n, n \geq 1\}$ of the time series is extracted using a one-sided HP filter. In Adriana (2013), it is shown that $\tau_n = \sum_{j=1}^{n} p_{n,j} y_j$ where the weights $p_{i,j}$ are given by an exact analytical expression. In the following lemma, we prove that the trend converges in mean to the same limit as the time series.

**Lemma B.1.** *If the following holds:*

1. $\sum_{j=1}^{n} |p_{n,j}| = c_n$ *for all* $n \geq 1$*;*
2. $c_n \to c$ *as* $n \to \infty$*;*
3. *for all* $\epsilon > 0$ *there exists a* $N \in \mathbb{N}$ *such that for all* $n \geq N$*:* $\sum_{j=1}^{\frac{n}{2}} |p_{n,j}| \leq \epsilon$ *and* $|\sum_{j=\frac{n}{2}}^{n} |p_{n,j}| - c_n| \leq \epsilon$*;*
4. $\mathbb{E}[y_n] \to \bar{y}$.

*Then* $\mathbb{E}[\tau_n] \to \bar{y}$ *as* $n \to \infty$.

*Proof.* Note, for all $n \geq 2$ the following holds:

$$|\mathbb{E}[\tau_n] - \bar{y}| = \Big|\mathbb{E}\Big[\sum_{j=1}^{n} p_{n,j} y_j\Big] - \bar{y}\Big| \tag{B.1a}$$

$$= \Big|\sum_{j=1}^{n} p_{n,j} \mathbb{E}[y_j] - \bar{y}\Big| \tag{B.1b}$$

$$= \Big|\sum_{j=1}^{n} p_{n,j} (\mathbb{E}[y_j] - \bar{y})\Big| \tag{B.1c}$$

$$\leq \sum_{j=1}^{n} |p_{n,j}||\mathbb{E}[y_j] - \bar{y}| \tag{B.1d}$$

$$= \sum_{j=1}^{\frac{n}{2}} |p_{n,j}||\mathbb{E}[y_j] - \bar{y}| + \sum_{j=\frac{n}{2}}^{n} |p_{n,j}||\mathbb{E}[y_j] - \bar{y}| \tag{B.1e}$$

$$\leq \sup_{j=1,\dots,\frac{n}{2}} |\mathbb{E}[y_j] - \bar{y}| \sum_{j=1}^{\frac{n}{2}} |p_{n,j}| + \sup_{j \geq \frac{n}{2}} |\mathbb{E}[y_j] - \bar{y}| \sum_{j=\frac{n}{2}}^{n} |p_{n,j}| \tag{B.1f}$$

$$\leq \sup_{j=1,\dots,\frac{n}{2}} |\mathbb{E}[y_j] - \bar{y}| \sum_{j=1}^{\frac{n}{2}} |p_{n,j}| + c \sup_{j \geq \frac{n}{2}} |\mathbb{E}[y_j] - \bar{y}|. \tag{B.1g}$$

In (B.1c) we use the fact that $\sum_{j=1}^{n} p_{n,j} = 1$, see Leon (2012). Moreover, the inequality in (B.1g) is established by noting that $\sum_{\frac{n}{2}}^{n} |p_{n,j}| \leq c_n \leq c$. Next, we consider the case if $n \to \infty$,

$$\lim_{n\to\infty} |\mathbb{E}[\tau_n] - \bar{y}| \leq \lim_{n\to\infty} \Big( \sup_{j=1,\dots,\frac{n}{2}} |\mathbb{E}[y_j] - \bar{y}| \sum_{j=1}^{\frac{n}{2}} |p_{n,j}| + c \sup_{j \geq \frac{n}{2}} |\mathbb{E}[y_j] - \bar{y}| \Big) \tag{B.2}$$

$$= \lim_{n\to\infty} \sup_{j=1,\dots,\frac{n}{2}} |\mathbb{E}[y_j] - \bar{y}| \sum_{j=1}^{\frac{n}{2}} |p_{n,j}| + c \lim_{n\to\infty} \sup_{j \geq \frac{n}{2}} |\mathbb{E}[y_j] - \bar{y}| \tag{B.3}$$

Because $\lim_{n\to\infty} \sup_{j \geq \frac{n}{2}} |\mathbb{E}[y_j] - \bar{y}| = 0$, there exists a $K \in \mathbb{N}$ such that $\sup_{j=1,\dots,\frac{n}{2}} |\mathbb{E}[y_j] - \bar{y}| = \sup_{j=1,\dots,\frac{K}{2}} |\mathbb{E}[y_j] - \bar{y}|$ for all $n \geq K$. As a consequence, $\lim_{n\to\infty} \sup_{j=1,\dots,\frac{n}{2}} |\mathbb{E}[y_j] - \bar{y}| < \infty$. Moreover, we know that $\lim_{n\to\infty} \sum_{j=1}^{\frac{n}{2}} |p_{n,j}| = 0$. Therefore,

$$\lim_{n\to\infty} |\mathbb{E}[\tau_n] - \bar{y}| \leq \Big( \lim_{n\to\infty} \sup_{j=1,\dots,\frac{n}{2}} |\mathbb{E}[y_j] - \bar{y}| \Big) \Big( \lim_{n\to\infty} \sum_{j=1}^{\frac{n}{2}} |p_{n,j}| \Big) + c \lim_{n\to\infty} \sup_{j \geq \frac{n}{2}} |\mathbb{E}[y_j] - \bar{y}| = 0 \tag{B.4}$$

By (B.4), we may conclude that $\mathbb{E}[\tau_n] \to \bar{y}$ as $n \to \infty$.

□

# APPENDIX C

## Long-term expectation of VAR($p$) model

Let $Y_t = (y_{1,t}, \ldots, y_{n,t})'$ denote an $(n \times 1)$ vector of time series variables. The $p$-lag vector autoregressive (VAR($p$)) model has the form:

$$Y_t = c + A_1 Y_{t-1} + A_2 Y_{t-2} + \ldots + A_p Y_{t-p} + \epsilon_{t+1}, \quad t = t_0, \ldots, T. \tag{C.1}$$

Here, $Y_{t_0}$ is predetermined, $A_i$ are $(n \times n)$ coefficient matrices, $c \in \mathbb{R}^n$, $\mathbb{E}[\epsilon_{t+1}] = 0$ and

$$\mathbb{E}[\epsilon_{t+1}\epsilon_\tau] = \begin{cases} \Omega & \tau = t+1, \\ 0 & \tau \neq t+1, \end{cases}$$

where $\Omega$ is a positive definite matrix. The VAR($p$) model is a system in which each variable is regressed on a constant and $p$ of its own lags. System (C.1) is called a AR($p$) model when $n = 1$. We can use the lag operator notation to write a VAR($p$) model more compactly, i.e.,

$$A(L)Y_t = c + \epsilon_{t+1}, \tag{C.2}$$

where $A(L) = I_n - A_1 L - \ldots - A_p L^p$. Next, we investigate when a VAR($p$) model is stationary. There are several types of stationarity, see Tsay (2005) for more details.

**Definition C.1.** *The time series $Y_t$ is strictly stationary if the distribution of the vector $(Y_t, Y_{t+1}, \ldots, Y_{t+h})$ is independent of $t$, for every $h \in \mathbb{N}$.*

**Definition C.2.** *The time series $Y_t$ is stationary (or more precisely second order stationary) if $\mathbb{E}[Y_t]$ and $\mathbb{E}[Y_{t+h}Y_t]$ exist and are finite and do not depend on $t$, for every $h \in \mathbb{N}$.*

If the stationary condition is not satisfied, then any shock to the $Y_t$ series will lead to forecasts with unbounded mean and variance. On the other hand, if the process is stationary, then following such a shock, the mean of the forecasts of system (C.1) will eventually settle down and evolves unchanged. The latter case will be referred to as the forecasts of a time series converge to stationary state. From this, we want to determine conditions when a VAR($p$) model is stationary. In Tsay (2005), a VAR($p$) model is stationary when the eigenvalues of the matrix

$$\begin{pmatrix} A_1 & A_2 & \cdots & A_p \\ I_n & 0 & \cdots & 0 \\ 0 & \ddots & 0 & \vdots \\ 0 & 0 & I_n & 0 \end{pmatrix} \tag{C.3}$$

have modulus less than one. Suppose that the VAR($p$) model is stationary. Then, by taking the expectation at both sides of system (C.2) results in:

$$A(L)\mu = c, \tag{C.4}$$

where $\mu = \mathbb{E}[Y_t]$. Hence, the unconditional mean is given by:

$$\mu = (I_n - A_1 - \ldots - A_p)^{-1}c. \tag{C.5}$$

Let's consider the case of making forecasts for $Y_t$. Here, we estimate the parameters of system (C.1) using data from the historical period $t = t_0, \ldots, t_N$. Next, we produce forecasts for $Y_t$, where the forecast horizon is specified by $t = t_N, \ldots, T$. If the VAR($p$) model is stationary, it is expected that $\mathbb{E}[Y_t]$ converges to the steady-state level given by (C.5). This procedure can be repeated for several forecast horizons. As a consequence, each time we must check if the VAR($p$) model is stationary. Suppose that the estimated VAR($p$) models are stationary over all different forecast horizons. Hence, the forecasts of all estimated VAR($p$) models converge in mean towards the steady-state level given by (C.5). However, it is expected that the estimated parameters $c, A_1$ and $A_2$ are different for each considered forecast horizon, and therefore the unconditional mean given by (C.5).

In our modelling framework, we want that the forecasts of all estimated VAR($p$) models converge in mean towards a fixed unconditional mean $\tilde{\mu}$. In particular, the mean $\tilde{\mu}$ is determined such that it contains the most recent information about the long-term expectation of the economic time series. One way to obtain this is to set the regression constant $c$ for all estimated VAR($p$) models equal to:

$$\tilde{c} = (I_n - A_1 - \ldots - A_p)\tilde{\mu}. \tag{C.6}$$

Hence, the forecasts of $Y_t$, over all different forecast horizons, converge in mean to the long-term expectation $\tilde{\mu}$. The adjustment of the constant $c$ of a stationary VAR(p) given by C.6, can be applied to the constants $\alpha_y, \alpha_r$, $\alpha_{\text{rev}}$ and $\alpha_{\text{mil}}$ of system (3.1). The dynamics of $(y_{t+1}, r_{t+1})$ are described by a VAR(2) model. Furthermore, the dynamics of $\text{mil}_{t+1}$ and the marginal distribution of $\text{rev}_{t+1}$ are described by an AR(1) model. Therefore, we can adjust the constants $\alpha_y, \alpha_r$, $\alpha_{\text{rev}}$ and $\alpha_{\text{mil}}$ such that the forecasts converge in mean to the long-term expectation $\tilde{\mu} = (\tilde{\mu}_y, \tilde{\mu}_r, \tilde{\mu}_{\text{rev}}, \tilde{\mu}_{\text{mil}})$. The components of the mean vector $\tilde{\mu}$ are either observed from the World Economic Outlook (WEO) report or estimated by taking the average of all available data. Therefore, the components of $\tilde{\mu}$ give the best indication about the long-term expectation of $y_{t+1}, r_{t+1}, \text{rev}_{t+1}$ and $\text{mil}_{t+1}$.

# APPENDIX D

## Data Information

In this Appendix, we present a description of the database used in this MSc-thesis. Moreover, we discuss the choices which are made in order to use the database for estimating the parameters of the Extended model.

## D.1 Database for Fiscal Variables

The parameters of the Extended model are estimated using the database provided by (Mauro et al. 2013). The database contains a new collection of fiscal variables for a large amount of countries. In (Mauro et al. 2013), it is claimed that this database is the most comprehensive currently available. The database covers 55 countries (24 advanced economies by present day definition from the IMF's World Economic Outlook classification and 31 nonadvanced) over the period 1800-2011. The data consist of the following fiscal variables: government revenue, goverment (non-interest) expenditure, the interest paid on public debt, government primary balance and gross public debt. All these variables are expressed as percentages of nominal GDP. Moreover, the database gives access to the following economic variables: real long term interest rate and annual return of real GDP. All fiscal variables provided by the database are published at the end of each available year.

Now, we discuss the choices which are made in order to use the database from (IMF) for estimating the parameters of the Extended model. Firstly, the choice of government level is an important consideration. We consider two types of levels: general government and state government. The general government is responsible for the security of the whole country, whereas the state government looks after the developmental needs of their people and territory only. Moreover, the general government shares revenues with the state government according to a pre decided regulation. The database consists of general government level whenever available. This level is more preferable because the general government is in the end accountable for the liabilities of the whole country. It is stated in (IMF) that general government data is hardly available for all considered countries before 1960. As a result, the government level changes for most countries around the period 1960-1970. As an example, we plot the Italian government revenue-to-GDP time series over the period 1950-2011 in Figure D.1.

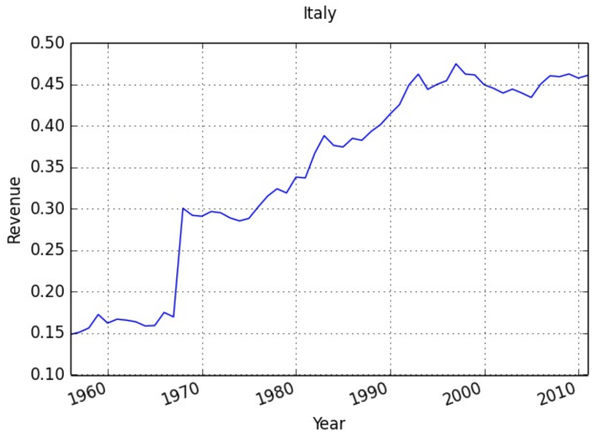


Figure D.1: Revenue-to-GDP ratio over period 1950-2011

From Figure D.1, the revenue-to-GDP time series displays the change in government level around the period 1965-1968. In particular, the general government level is available starting from the year 1968. From this, we use only data from the period when general government level is available for estimating the parameters of the Extended model. Finally, we only use data from the post Second World War (WWII) period because this event is rather unique. And moreover, data from the WWII period is not representative for analysing debt sustainability in the present economic circumstances.

# APPENDIX E

## Results for Relationship between the Indicator and the Interest rate spreads

In this Appendix, the results of verifying the relationship between the indicator and the interest rate spreads are presented for the following countries: the United States (US), France (FR), Portugal (PRT), the United Kingdom (UK), Italy (IT), Belgium (BEL), Germany (GER), the Netherlands (NL) and Finland (FIN).

| | $\rho(I(5y), S(5y))$ | $\rho(I(15y), S(15y))$ | $\rho(I(20y), S(20y))$ | $\rho(I(30y), S(30y))$ |
|---|---|---|---|---|
| US(1995-2011) | 0.68 | 0.60 | 0.57 | 0.68 |
| FR(2000-2011) | 0.62 | 0.64 | 0.66 | 0.40 |
| PRT(1995-2011) | -0.52 | -0.13 | -0.20 | -0.09 |
| UK(1995-2011) | 0.74 | 0.09 | -0.02 | -0.14 |
| IT(1997-2011) | 0.27 | -0.36 | -0.32 | 0.09 |
| BEL(1995-2011) | 0.70 | 0.51 | 0.1 | -0.37 |
| GER(1996-2011) | 0.19 | -0.04 | 0.30 | 0.55 |
| NL(1996-2011) | 0.71 | 0.32 | 0.28 | 0.08 |
| FIN(1996-2011) | 0.33 | 0.07 | 0.09 | 0.22 |

Table E.1: Results of criterion 1

| $k$ | Model (5.6a)/(5.6b): $R^2$ | Is $\alpha$ significant in (5.6a) ? | Are $\alpha, \beta$ significant? in (5.6b) |
|---|---|---|---|
| 5y | 0.539/0.686 | Yes | Yes/Yes |
| 15y | 0.301/0.334 | Yes | Yes/No |
| 20y | 0.356/0.451 | Yes | Yes/No |
| 30y | 0.404/0.405 | Yes | Yes/No |

Table E.2: UK(1995-2011)

| $k$ | Model (5.6a)/(5.6b): $R^2$ | Is $\alpha$ significant in (5.6a) ? | Are $\alpha, \beta$ significant? in (5.6b) |
|---|---|---|---|
| 5y | 0.389/0.481 | Yes | Yes/Yes |
| 15y | 0.102/0.399 | No | Yes/Yes |
| 20y | 0.107/0.184 | No | No/No |
| 30y | 0.021/0.025 | No | No/No |

Table E.3: IT(1997-2011)

| $k$ | Model (5.6a)/(5.6b): $R^2$ | Is $\alpha$ significant in (5.6a) ? | Are $\alpha, \beta$ significant? in (5.6b) |
|---|---|---|---|
| 5y | 0.337/0.544 | Yes | No/Yes |
| 15y | 0.263/0.557 | Yes | Yes/Yes |
| 20y | 0.160/0.255 | No | Yes/No |
| 30y | 0.011/0.136 | No | No/No |

Table E.4: BEL(1995-2011)

| $k$ | Model (5.6a)/(5.6b): $R^2$ | Is $\alpha$ significant in (5.6a) ? | Are $\alpha, \beta$ significant? in (5.6b) |
|---|---|---|---|
| 5y | 0.319/0.331 | Yes | Yes/No |
| 15y | 0.153/0.217 | No | No/No |
| 20y | 0.072/0.072 | No | No/No |
| 30y | 0.378/0.412 | Yes | Yes/No |

Table E.5: GER(1996-2011)

| $k$ | Model (5.6a)/(5.6b): $R^2$ | Is $\alpha$ significant in (5.6a) ? | Are $\alpha, \beta$ significant? in (5.6b) |
|---|---|---|---|
| 5y | 0.337/0.595 | No | No/Yes |
| 15y | 0.080/0.083 | No | No/No |
| 20y | 0.085/0.095 | No | No/No |
| 30y | 0.252/0.272 | No | No/No |

Table E.6: NL(1996-2011)

| $k$ | Model (5.6a)/(5.6b): $R^2$ | Is $\alpha$ significant in (5.6a) ? | Are $\alpha, \beta$ significant? in (5.6b) |
|---|---|---|---|
| 5y | 0.320/0.388 | Yes | Yes/No |
| 15y | 0.258/0.782 | Yes | Yes/Yes |
| 20y | 0.246/0.724 | Yes | Yes/Yes |
| 30y | 0.234/0.557 | Yes | Yes/Yes |

Table E.7: FIN(1996-2011)

# APPENDIX F

## Results for Relationship between the 10-years interest rate and the Interest rate spreads

In this Appendix, the explanatory power of the 10-years real interest rate for describing the dynamics of the interest rate spreads are presented for the following countries: the United Kingdom (UK), the United States (US), France (FR), Italy (IT), Belgium (BEL), Germany (GER), the Netherlands (NL) Portugal (PRT) and Finland (FIN).

| $k$ | Model (5.6a)/(5.7): $R^2$ | Is $\alpha$ significant in (5.6a) ? | Are $\alpha, \beta$ significant? in (5.7) |
|---|---|---|---|
| 5y | 0.539/0.658 | Yes | Yes/Yes |
| 15y | 0.301/ 0.487 | Yes | Yes/Yes |
| 20y | 0.356/0.596 | Yes | Yes/Yes |
| 30y | 0.404/0.618 | Yes | Yes/Yes |

Table F.1: UK(1995-2011)

| $k$ | Model (5.6a)/(5.7): $R^2$ | Is $\alpha$ significant in (5.6a) ? | Are $\alpha, \beta$ significant? in (5.7) |
|---|---|---|---|
| 5y | 0.530/0.554 | Yes | Yes/No |
| 15y | 0.252/ 0.288 | Yes | Yes/No |
| 20y | 0.276/0.310 | Yes | Yes/No |
| 30y | 0.282/0.301 | Yes | Yes/No |

Table F.2: US(1995-2011)

| $k$ | Model (5.6a)/(5.7): $R^2$ | Is $\alpha$ significant in (5.6a) ? | Are $\alpha, \beta$ significant? in (5.7) |
|---|---|---|---|
| 5y | 0.568/0.632 | Yes | Yes/No |
| 15y | 0.365/ 0.438 | Yes | Yes/No |
| 20y | 0.223/0.254 | No | No/No |
| 30y | 0.173/0.346 | No | No/No |

Table F.3: FR(2000-2011)

| $k$ | Model (5.6a)/(5.7): $R^2$ | Is $\alpha$ significant in (5.6a) ? | Are $\alpha, \beta$ significant? in (5.7) |
|---|---|---|---|
| 5y | 0.389/0.453 | Yes | Yes/No |
| 15y | 0.102/ 0.158 | No | No/No |
| 20y | 0.107/0.174 | No | No/No |
| 30y | 0.021/0.031 | No | No/No |

Table F.4: IT(1997-2011)

| $k$ | Model (5.6a)/(5.7): $R^2$ | Is $\alpha$ significant in (5.6a) ? | Are $\alpha, \beta$ significant? in (5.7) |
|---|---|---|---|
| 5y | 0.337/0.394 | Yes | Yes/No |
| 15y | 0.263/ 0.673 | Yes | No/Yes |
| 20y | 0.160/0.486 | No | No/Yes |
| 30y | 0.009/0.269 | No | No/Yes |

Table F.5: BEL(1995-2011)

| $k$ | Model (5.6a)/(5.7): $R^2$ | Is $\alpha$ significant in (5.6a) ? | Are $\alpha, \beta$ significant? in (5.7) |
|---|---|---|---|
| 5y | 0.319/0.394 | Yes | Yes/No |
| 15y | 0.153/ 0.246 | No | No/No |
| 20y | 0.072/0.102 | No | No/No |
| 30y | 0.378/0.503 | Yes | No/No |

Table F.6: GER(1996-2011)

| $k$ | Model (5.6a)/(5.7): $R^2$ | Is $\alpha$ significant in (5.6a) ? | Are $\alpha, \beta$ significant? in (5.7) |
|---|---|---|---|
| 5y | 0.337/0.430 | No | No/No |
| 15y | 0.080/ 0.111 | No | No/No |
| 20y | 0.085/0.122 | No | No/No |
| 30y | 0.252/0.273 | No | No/No |

Table F.7: NL(1996-2011)

| $k$ | Model (5.6a)/(5.7): $R^2$ | Is $\alpha$ significant in (5.6a) ? | Are $\alpha, \beta$ significant? in (5.7) |
|---|---|---|---|
| 5y | 0.074/0.227 | No | No/No |
| 15y | 0.014/ 0.157 | No | No/No |
| 20y | 0.099/0.337 | No | No/Yes |
| 30y | 0.220/0.405 | Yes | No/Yes |

Table F.8: PRT(1995-2011)

| $k$ | Model (5.6a)/(5.7): $R^2$ | Is $\alpha$ significant in (5.6a) ? | Are $\alpha, \beta$ significant? in (5.7) |
|---|---|---|---|
| 5y | 0.320/0.482 | Yes | Yes/No |
| 15y | 0.258/ 0.488 | Yes | Yes/Yes |
| 20y | 0.246/0.462 | Yes | Yes/Yes |
| 30y | 0.234/0.425 | Yes | Yes/Yes |

Table F.9: FIN(1996-2011)

# APPENDIX G

## Expected value of the steady-state equilibrium

In order to investigate the local stability of system (6.48), we assume that the expected value of the steady-state equilibrium $\mathbb{E}[\bar{\theta}] = (\mathbb{E}[r_\infty], \mathbb{E}[y_\infty], \mathbb{E}[s_\infty], \mathbb{E}[d_\infty])$ exists, where $y_t \to y_\infty$ in weak converge, see Vaart (2013) for more details. Here, we explain the procedure for determining $\mathbb{E}[\bar{\theta}]$. From system (6.48), we can rewrite the steady-state $s_\infty$ as follows:

$$s_\infty = \frac{\alpha + \rho d_\infty + \epsilon_\infty^s}{1-\beta}. \tag{G.1}$$

By substituting (G.1) in the steady-state $d_\infty$, we obtain the following expression:

$$d_\infty = \frac{-(\alpha + \epsilon_\infty^s)(1 + r_\infty)}{(1-\beta)(y_\infty - r_\infty) + \rho(1 + r_\infty)}. \tag{G.2}$$

Since system (6.48) is non-linear, it is challenging to find an expression for $\mathbb{E}[\bar{\theta}]$. Therefore, we apply a Taylor expansion for $d_\infty$ around $(\mathbb{E}[\epsilon_\infty^s], \mathbb{E}[y_\infty], \mathbb{E}[r_\infty])$, see Ang and Tang (2007) for more details. For each country, the steady-state levels $\mathbb{E}[y_\infty]$ and $\mathbb{E}[r_\infty]$ are known, see Appendix C. Moreover, we assume that $\mathbb{E}[\epsilon_\infty^s] = 0$. An approximation for $\mathbb{E}[d_\infty]$ is found by applying Theorem 6.6 and taking expectations at both sides of the Taylor expansion. For simplicity, we denote $Y = d_\infty, X_1 = \epsilon_\infty^s, X_2 = y_\infty$ and $X_3 = r_\infty$. Then, the second order Taylor expansion for $\mathbb{E}[Y]$ is given by:

$$\mathbb{E}[Y] \approx g(\mathbb{E}[X_1], \mathbb{E}[X_2], \mathbb{E}[X_3]) + \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3} \rho_{i,j}\sigma_i\sigma_j\Big(\frac{\partial^2 g}{\partial X_i \partial X_j}\Big), \tag{G.3}$$

where

$$g(X_1, X_2, X_3) = \frac{-(\alpha + X_1)(1 + X_3)}{(1-\beta)(X_2 - X_3) + \rho(1 + X_3)}. \tag{G.4}$$

By (6.49) it is given that $\rho_{1,2} = \rho_{2,1} = \rho_{1,3} = \rho_{3,1} = 0$. Therefore, many components of the summation in (G.3) have the value zero, i.e.,

$$\mathbb{E}[Y] \approx g(\mathbb{E}[X_1], \mathbb{E}[X_2], \mathbb{E}[X_3]) + \frac{1}{2}Var(X_1)\frac{\partial^2 g}{\partial X_1^2} + \frac{1}{2}Var(X_2)\frac{\partial^2 g}{\partial X_2^2} + Cov(X_2, X_3)\frac{\partial^2 g}{\partial X_2 \partial X_3} + \frac{1}{2}Var(X_3)\frac{\partial^2 g}{\partial X_3^2}, \tag{G.5}$$

where

$$\begin{aligned}
\frac{\partial g}{\partial X_1} &= \frac{-(1+X_3)}{(1-\beta)(X_2 - X_3) + \rho(1+X_3)}, \\
\frac{\partial^2 g}{\partial X_1^2} &= 0, \\
\frac{\partial g}{\partial X_2} &= \frac{(\alpha + X_1)(1+X_3)(1-\beta)}{((1-\beta)(X_2 - X_3) + \rho(1+X_3))^2}, \\
\frac{\partial^2 g}{\partial X_2^2} &= \frac{-2(\alpha + X_1)(1+X_3)(1-\beta)^2}{((1-\beta)(X_2 - X_3) + \rho(1+X_3))^3}, \\
\frac{\partial g}{\partial X_3} &= \frac{-(\alpha + X_1)}{(1-\beta)(X_2 - X_3) + \rho(1+X_3)} - \frac{(\alpha + X_1)(1+X_3)(1-\beta-\rho)}{((1-\beta)(X_2 - X_3) + \rho(1+X_3))^2}, \\
\frac{\partial^2 g}{\partial X_3^2} &= \frac{-2(\alpha + X_1)(1-\beta-\rho)}{((1-\beta)(X_2 - X_3) + \rho(1+X_3))^2} - \frac{2(\alpha + X_1)(1+X_3)(1-\beta-\rho)^2}{((1-\beta)(X_2 - X_3) + \rho(1+X_3))^3}, \\
\frac{\partial^2 g}{\partial X_2 \partial X_3} &= \frac{(\alpha + X_1)(1-\beta)}{((1-\beta)(X_2 - X_3) + \rho(1+X_3))^2} + \frac{2(\alpha + X_1)(1+X_3)(1-\beta-\rho)(1-\beta)}{((1-\beta)(X_2 - X_3) + \rho(1+X_3))^3}.
\end{aligned}$$

This results in the following final expression for $\mathbb{E}[d_\infty]$:

$$\begin{aligned}
\mathbb{E}[d_\infty] \approx\ & g(\mathbb{E}[\epsilon^s_\infty], \mathbb{E}[y_\infty], \mathbb{E}[r_\infty]) + \frac{1}{2}Var(y_\infty)\frac{\partial^2 g}{\partial y_\infty^2}(\mathbb{E}[\epsilon^s_\infty], \mathbb{E}[y_\infty], \mathbb{E}[r_\infty]) \\
& + Cov(y_\infty, r_\infty)\frac{\partial^2 g}{\partial y_\infty \partial r_\infty}(\mathbb{E}[\epsilon^s_\infty], \mathbb{E}[y_\infty], \mathbb{E}[r_\infty]) + \frac{1}{2}Var(r_\infty)\frac{\partial^2 g}{\partial r_\infty^2}(\mathbb{E}[\epsilon^s_\infty], \mathbb{E}[y_\infty], \mathbb{E}[r_\infty]).
\end{aligned} \tag{G.6}$$

By taking expectations in (G.1) at both sides and substituting (G.6), we obtain an approximation for $\mathbb{E}[s_\infty]$. Hence, we have an approximation for $\mathbb{E}[\bar{\theta}] = (\mathbb{E}[y_\infty], \mathbb{E}[r_\infty], \mathbb{E}[s_\infty], \mathbb{E}[d_\infty])$.

# APPENDIX H

## Economic forecasts produced by the Extended model

In this Appendix, we show for several countries the economic forecasts produced by the Extended model. These forecasts are supplementary to the results produced in Chapter 6. In particular, we consider the following cases:

- In H.1, we estimate the parameters of the Extended model for Germany using data over the period $t = 1970, \ldots, 1995$. We produce forecasts with the Extended model until the year 2021. The Figure shows that the forecasts of the primary expenditure and the government debt doesn't converge to a stationary state.
- In H.2, we estimate the parameters of the Extended model for Germany using data over the period $t = 1970, \ldots, 1996$. We produce forecasts with the Extended model until the year 2021. The Figure shows the convergence of all economic variables to the stationary state.
- In H.3, we estimate the parameters of the Extended model for Finland using data over the period $t = 1970, \ldots, 1995$. We produce forecasts with the Extended model until the year 2021. The Figure shows that the forecasts of the government debt doesn't converge to a stationary state.
- In H.4, we estimate the parameters of the Extended model for Portugal using data over the period $t = 1976, \ldots, 2010$. We produce forecasts with the Extended model until the year 2031. The Figure shows that the forecasts of the primary expenditure and the government debt doesn't converge to a stationary state.
- In H.5, we estimate the parameters of the Extended model for France using data over the period $t = 1978, \ldots, 2009$. We produce forecasts with the Extended model until the year 2031. The Figure shows that the forecasts of the primary expenditure and the government debt doesn't converge to a stationary state.

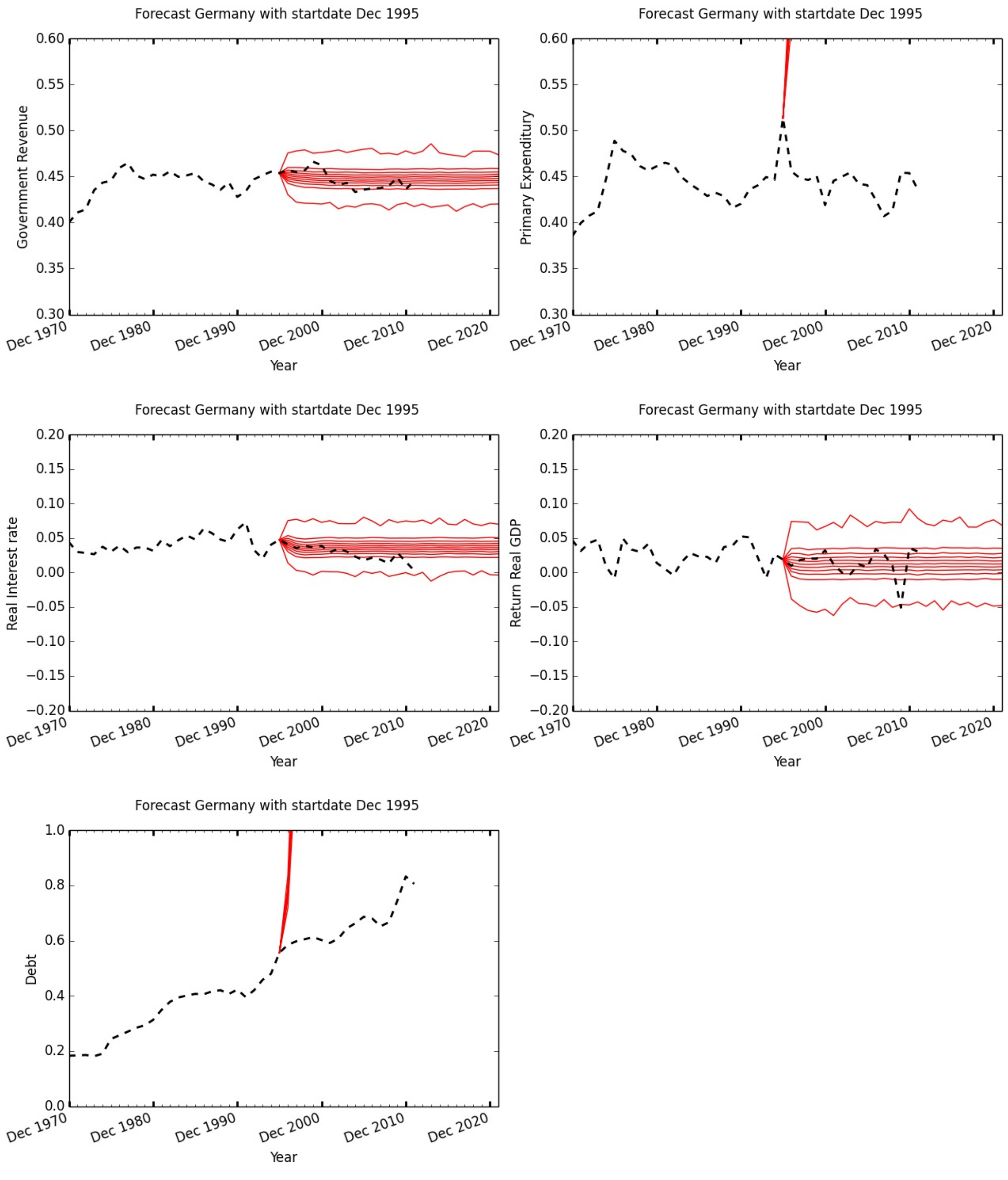


Figure H.1: Economic variables for Germany: historical data (black-dotted) and deciles of the forecast (red). The $R^2_{\text{rev}} = 0.40$, $R^2_{\text{exp}} = 0.69$, $R^2_{\text{r}} = 0.25$ and $R^2_{\text{y}} = 0.23$

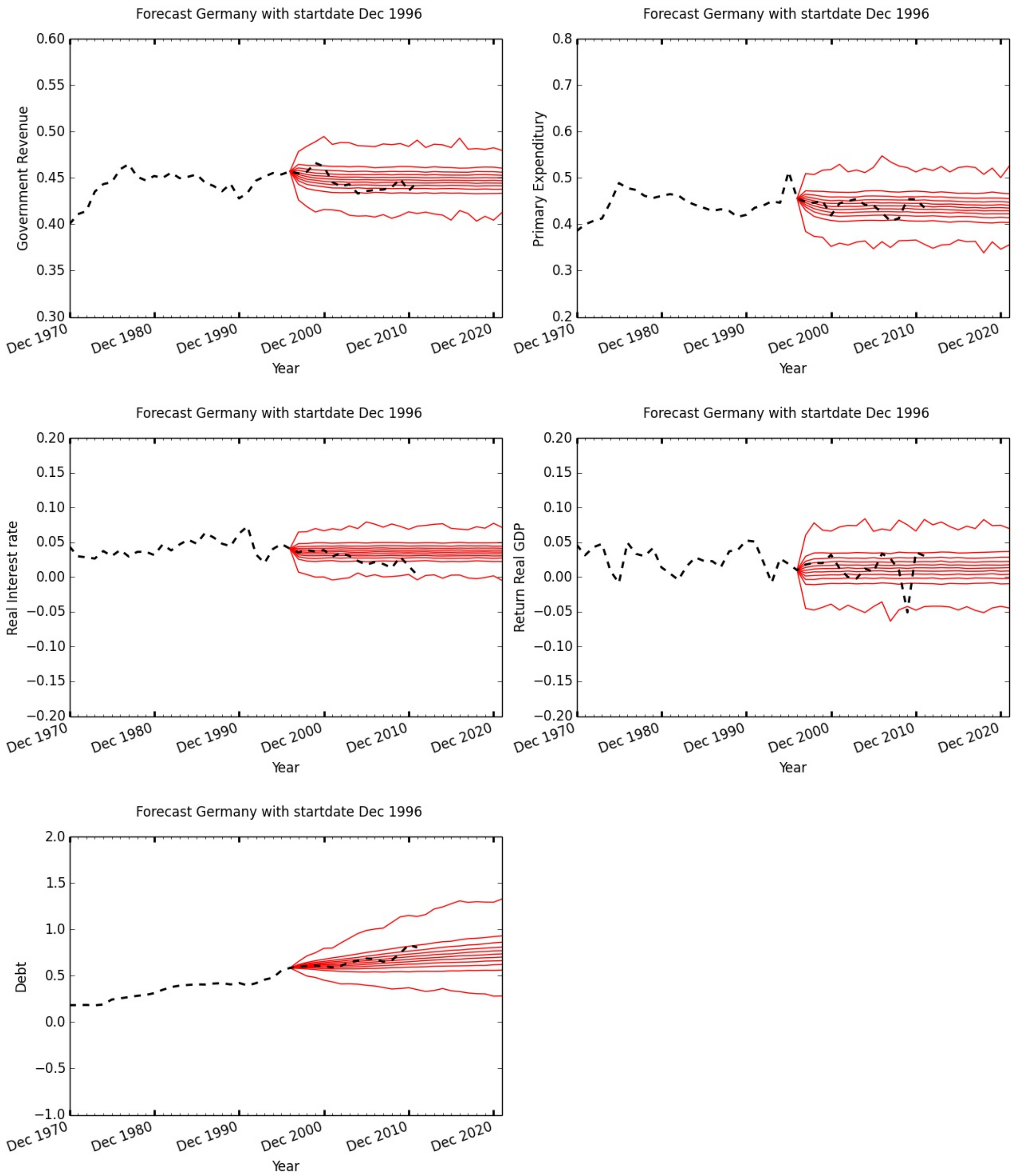


Figure H.2: Economic variables for Germany: historical data (black-dotted) and deciles of the forecast (red). The $R^2_{\text{rev}} = 0.42$, $R^2_{\text{exp}} = 0.37$, $R^2_{\text{r}} = 0.24$ and $R^2_{\text{y}} = 0.24$

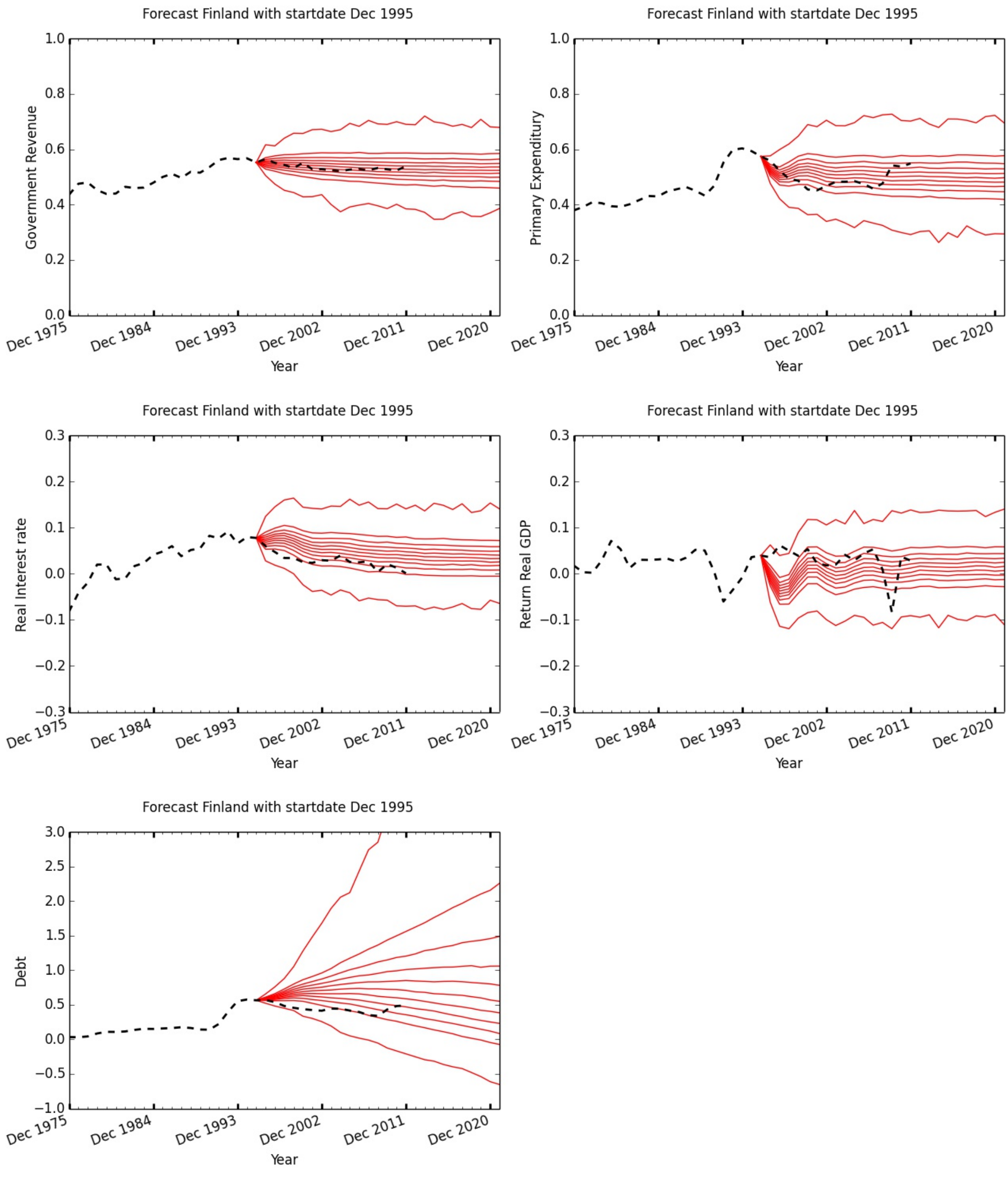


Figure H.3: Economic variables for Finland: historical data (black-dotted) and deciles of the forecast (red). The $R^2_{\text{rev}} = 0.91$, $R^2_{\text{exp}} = 0.94$, $R^2_{\text{r}} = 0.77$ and $R^2_{\text{y}} = 0.73$

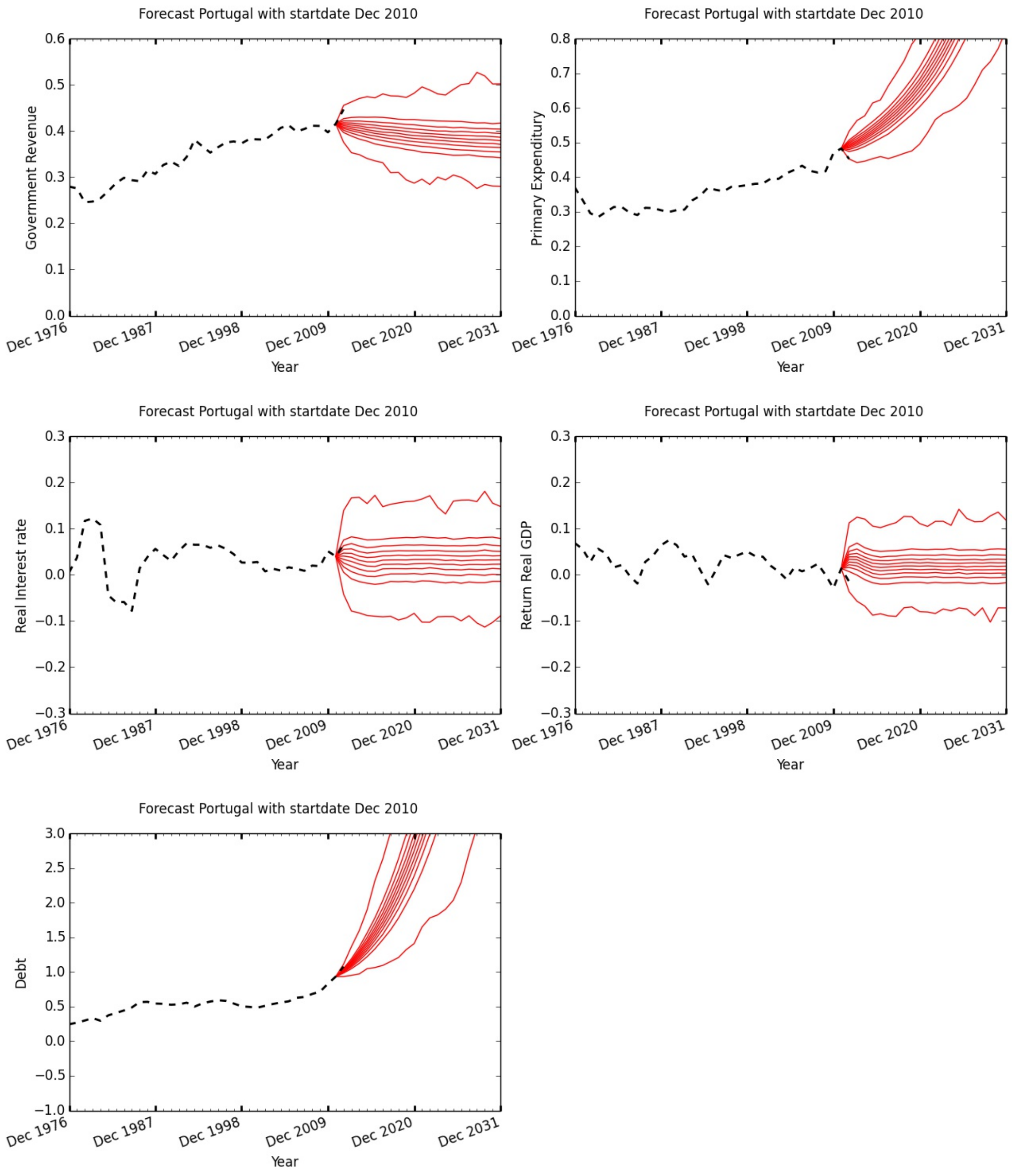


Figure H.4: Economic variables for Portugal: historical data (black-dotted) and deciles of the forecast (red). The $R^2_{\text{rev}} = 0.94$, $R^2_{\text{exp}} = 0.94$, $R^2_{\text{r}} = 0.51$ and $R^2_{\text{y}} = 0.47$

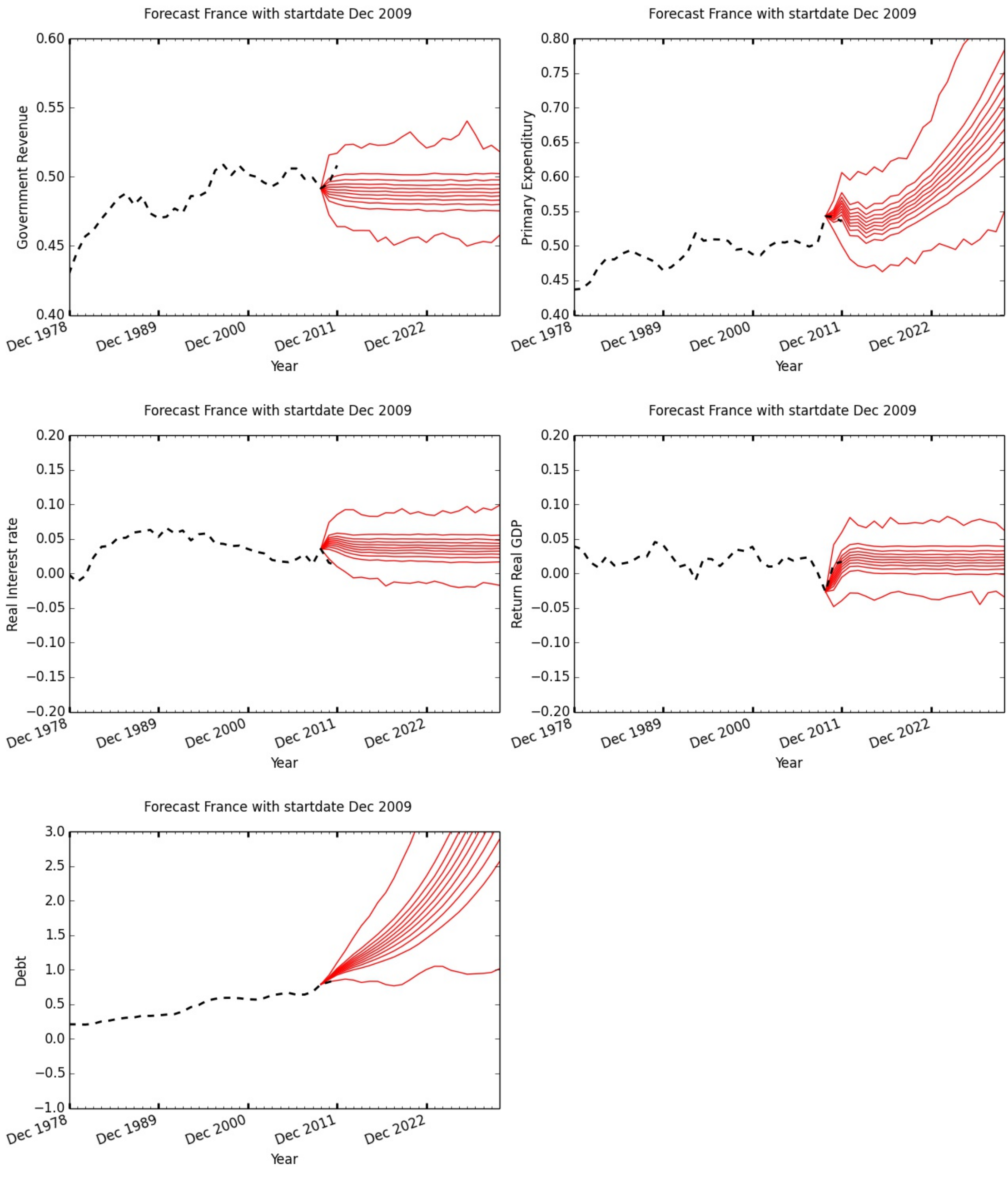


Figure H.5: Economic variables for France: historical data (black-dotted) and deciles of the forecast (red). The $R^2_{\text{rev}} = 0.75$, $R^2_{\text{exp}} = 0.81$, $R^2_{\text{r}} = 0.74$ and $R^2_{\text{y}} = 0.25$