

# Modelling a virtual photon stream from a black body source

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by

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Bachelor project

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# ABSTRACT

When measuring the light emitted by an object that is far away from earth, it is important that the detector used for the measurement is as accurate as possible. One of the detectors that is used to detect this light is the Microwave Kinetic Inductance Detector (MKID). To improve the efficiency of the MKID we would like a model that, before producing the MKID, can tell how well it will work. The input of this model would be the characteristics of the MKID, and also the light that is emitted from the object in space.

The aim of this project is to model this light, as a virtual stream of photons. Because this light can be approximated by radiation coming from a black body source, a photon stream from such a source will be modeled. The simulation of this photon stream has been done in two steps. First the energy of each photon was simulated, after this the arrival time at the detector of the photons has been generated.

In the simulation of the energies of the photons two different situations have been discussed; the case where we are looking at a three dimensional space, and the case in which we are interested in just one dimension. In the three dimensional case the power density in frequency domain can be described by Planck's law, in the one dimensional case this can be described by the Johnson-Nyquist formula. In both situations an analytic procedure has been used to generate photon energies, and in the one dimensional case also a numerical method has been used.

The arrival times of the photons have been modeled using an in-homogeneous Poisson process. For this we need an intensity function in time that describes the fluctuating possibility of detecting a photon in time. The probability of detecting a photon is proportional to the power of the detected light. So, we can use a power signal as intensity function. To simulate this intensity function, the power spectral density (PSD) has been used. This PSD describes how much each frequency component contributes to the power signal. Using the theorem of Wiener-Khinchin, we have simulated a power stream, our intensity function, with the desired PSD.

This generated intensity function takes on negative values and can therefore not be used. But, if the intensity function is smoothed using a moving average filter with a span of 50, it can be used in the in-homogeneous Poisson process. Using this smoothed intensity function, the arrival times of the photons have been simulated.

The analytic simulations of the photon energies resulted in a large stream of photon energies, that are distributed according to theory. For the further simulation the photon energies from the one dimensional case have been used. These energy values were combined with the result of the photon time simulation to get a virtual stream of photons.

The simulated photon stream has the right properties for its energy distribution and also the average power in the photon stream has the right value. When we look at the power signal that we would measure from this stream of photons and calculate its PSD, we see that it matches the theoretical curve up to  $10^{7.5}$  Hz. For higher frequencies it drops below the theoretical expectation. This is due to the use of the smooth intensity function, but because in practice a detector is too slow to measure these high-frequency components (usually they can measure up to 10 kHz), this is not a problem for the future use in modeling these detectors.

This result can not only be used to built a model for the MKIDs, it can also be used by astronomers that look at other wavelengths than those in the submillimeter range, or in the field of optics.



# CONTENTS

<b>1</b>	<b>Introduction</b>	<b>3</b>
1.1	Black body radiation . . . . .	3
1.2	MKIDs . . . . .	3
1.3	Goal of the project . . . . .	4
<b>2</b>	<b>Energy distribution of photons</b>	<b>5</b>
2.1	Planck law . . . . .	5
2.1.1	Normalizing the Photon Density . . . . .	5
2.1.2	Getting the cdf of the Planck Photon Density . . . . .	7
2.1.3	Drawing random numbers from the Planck Curve . . . . .	8
2.2	Johnson-Nyquist Noise . . . . .	11
2.2.1	Numerical Approximation . . . . .	11
2.3	Analytic Approach Johnson-Nyquist . . . . .	15
2.3.1	Determining a dominating density . . . . .	15
2.3.2	Getting the CDF of dominating density . . . . .	18
2.3.3	Generation numbers . . . . .	19
<b>3</b>	<b>Time distribution of photons</b>	<b>23</b>
3.1	Time correlation . . . . .	23
3.1.1	Classical light wave . . . . .	23
3.1.2	Photons . . . . .	24
3.1.3	Wave vs particle . . . . .	25
3.1.4	Photon Bunching . . . . .	25
3.2	Time generation . . . . .	28
3.2.1	Average amount of photons in time T . . . . .	28
3.2.2	PSD and time domain . . . . .	29
3.2.3	Generating the envelope . . . . .	30
3.2.4	Generating the Photon Times . . . . .	31
3.2.5	Coherence Time . . . . .	31
<b>4</b>	<b>Results</b>	<b>33</b>
4.1	Simulation of the photon stream . . . . .	33
4.1.1	Determining the intensity function . . . . .	33
4.1.2	Getting a smoothed intensity function . . . . .	34
4.1.3	Simulating the photon stream . . . . .	35
4.2	Discussion . . . . .	37
4.2.1	Relevance of detector sampling . . . . .	37
4.2.2	Visibility of photon bunching . . . . .	37
4.2.3	Improving the simulation . . . . .	37
<b>5</b>	<b>Conclusion</b>	<b>39</b>
	<b>Bibliography</b>	<b>41</b>
<b>A</b>	<b>Appendix - Different approach</b>	<b>43</b>
A.1	Replacing the negative part of the intensity function . . . . .	43
A.1.1	Mismatch of the PSD. . . . .	45
A.2	Distribution of amount of photons . . . . .	46

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<b>B</b>	<b>Appendix - Matlab scripts</b>	<b>47</b>
B.1	Matlab script for generating frequencies from Planck's formula . . . . .	47
B.2	Matlab script for numerical generation from Johnson-Nyquist formula . . . . .	48
B.3	Matlab script for analytical generation from Johnson-Nyquist formula . . . . .	49
B.4	Matlab script for simulating the final photon stream . . . . .	50
<b>C</b>	<b>Appendix - Figures</b>	<b>51</b>
C.1	Numerical generation from Johnson Nyquist . . . . .	51
C.2	Generation of photons from the Planck Curve . . . . .	53

# 1

## INTRODUCTION

In astronomy, scientists often want to make images of distant stars or other objects. To do so, light emitted from the object is being detected. These objects in space emitting light can be approximated by a black body source emitting light. This light is not necessarily in the visible spectrum.

### 1.1. BLACK BODY RADIATION

Radiation of a black body source has been investigated in great detail in the early 1900's by great names in physics like Max Planck and Stefan Boltzmann. A black body is a physical object that emits electromagnetic radiation. All radiation that gets to the black body is absorbed. The spectrum of the emitted light from the black body is dependent of the temperature of the black body. If the temperature of the black body increases, it will emit more light and also the highest intensity peak will shift to a higher frequency.

Light consist of photons, light particles. These photons cause so called "photons noise" when the light is measured. The noise arises from the randomness in the arrival time of photons at the detector. This photon noise was first measured in 1928 by John Bertrand Johnson<sup>[1]</sup>, while discussing his results with Harry Nyquist<sup>[2]</sup>. When we look at an object in space from earth, the light will travel through the earths atmosphere. The signal is therefore a mixture between light from the object, combined with photon noise due to the atmosphere. The effective black body temperature of the object, due to this interaction, is approximately 40 K, even though the actual temperature of the object in space can be very different.

### 1.2. MKIDS

There are several examples of telescopes that are used by astronomers to look at the skies. Some of them are on earth, others are in space while they make their images. Perhaps the most well know example of the latter is the Hubble Space telescope. It was launched into an orbit around the earth in 1990 and is still in operation.

An example of a telescope observing from the surface of the earth is Atacama Submillimeter Telescope Experiment (ASTE) in Chile.

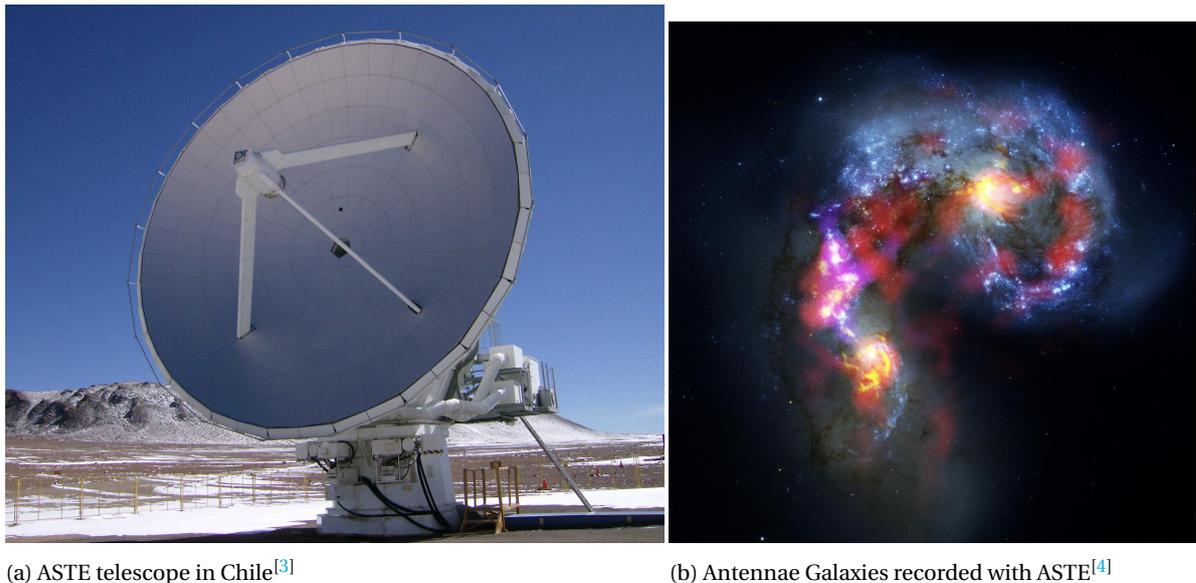
(a) ASTE telescope in Chile<sup>[3]</sup>(b) Antennae Galaxies recorded with ASTE<sup>[4]</sup>

Figure 1.1: ASTE telescope at 5100 meter altitude in Chile

As can be seen in figure 1.1a, the telescope has a parabolic shape, so the light will be focused at the center. From the focal point it is reflected to the "hole" in the middle of the telescope. Here, the light is detected with small chips called Microwave Kinetic Inductance Detectors (MKIDs). These chips can measure the power of the light. Advantages to this chip are that it is applicable in a large frequency domain<sup>[5]</sup>, it is very sensitive<sup>[6]</sup> and also it is easy to fabricate.

### 1.3. GOAL OF THE PROJECT

The amount of light from the object in space that actually reaches the telescope is small. Therefore we want the MKIDs in the telescope to be as accurate as possible. To improve the accuracy of the MKIDs a model can be made in which the outcome of a certain MKID design is modeled. The input of this model would be the parameters of the MKID like its size in multiple directions and its material, and the light that is emitted from an object in space. Then the model would indicate the efficiency of the chip design.

The first step in making this model would be to simulate the incoming light that is emitted from the object in space, as we would measure it in a detector. Because these objects are approximately black body sources, the light emitted from a black body source with a temperature of 40 K will be simulated. The goal of this bachelor project is to simulate this light such that it has the characteristics of black body radiation. The MKIDs used in the ASTE telescope measure light with a wavelength that is in the submillimeter (0.1 mm - 1 mm) range. However, the outcome of this project can be used for other wavelengths as well, such as visible light. The program can be used within the field of astronomy, but also in other fields in which optics is involved.

To include the photon noise in the simulation, the light will be modeled as a stream of photons. The result of this project can later be used to build a model of the MKID to find the optimal structure of this chip. Simulating the photons in the virtual stream of photons has been done in two steps. First, virtual energy values have been created for the photons. After this, the arrival time of the photons at the detector has been simulated. Together this gives us a virtual stream of photons that is a representation of the light from a distant object as it arrives at a detector.

# 2

## ENERGY DISTRIBUTION OF PHOTONS

### 2.1. PLANCK LAW

Our goal here is to simulate photons radiated from a black body source. In literature it can be found that the total power emitted by a black body has a certain power density<sup>[7]</sup>. The total power that is emitted by the black body source is the sum of the energy of all the photons that are emitted by the black body in one second. The power emitted at each frequency has a continuous spectrum, a power density function. For a black body source, this density is given by Planck's law:

$$B_T(f) = \frac{2hf^3/c^2}{e^{hf/kT} - 1}, \quad f > 0 \quad (2.1)$$

Here  $h$  is Planck's constant,  $k$  is the Boltzmann constant,  $c$  the speed of light,  $T$  the temperature of the black body and  $f$  the frequency of the light. We are interested in the amount of photons that are emitted by the black body at a certain frequency  $f$ . It is known that the energy of a photon depends on the frequency of the photon in the following way:

$$E = hf \quad (2.2)$$

If we divide the total energy by the energy of each individual photon, we get an expression for the amount of photons that are emitted. So if we want to obtain a density for the amount of photons emitted at each frequency, we must divide the Planck curve by the energy of each photon,  $hf$ . Then we get a density for the amount of photons, derived from the Planck curve:

$$D_T(f) = \frac{2f^2/c^2}{e^{hf/kT} - 1}, \quad f > 0 \quad (2.3)$$

#### 2.1.1. NORMALIZING THE PHOTON DENSITY

This formula is not yet a probability density function, because its integral is not equal to one. First I will find the factor that normalizes equation 2.1, making it a probability density function (PDF), which are commonly used in statistical mathematics. For a PDF holds:

$$\int_{-\infty}^{\infty} g(x) dx = 1 \quad (2.4)$$

Here  $g(x)$  is a PDF of  $x$ . In our case, since frequencies cannot be smaller than zero, we must find a factor  $A$  such that:

$$\int_0^{\infty} \frac{1}{A} D_T(f) df = 1 \quad (2.5)$$

So basically we have:

$$A = \int_0^{\infty} D_T(f) df \quad (2.6)$$

This integral can be evaluated analytically. We can rewrite equation 2.6 as<sup>[8]</sup>:

$$\begin{aligned} A &= \frac{2}{c^2} \int_0^{\infty} \frac{f^2}{e^{hf/kT} - 1} df \\ &= \frac{2}{c^2} \int_0^{\infty} \frac{e^{-hf/kT} f^2}{1 - e^{-hf/kT}} df \\ &= \frac{2}{c^2} \int_0^{\infty} e^{-hf/kT} f^2 \left( \frac{1}{1 - e^{-hf/kT}} \right) df \end{aligned}$$

Now, using the geometric series and the fact that  $-hf/kT < 0$ , we see this equals:

$$\begin{aligned} &= \frac{2}{c^2} \int_0^{\infty} e^{-hf/kT} f^2 \left( \sum_{n=0}^{\infty} e^{-nhf/kT} \right) df \\ &= \frac{2}{c^2} \int_0^{\infty} f^2 \left( \sum_{n=1}^{\infty} e^{-nhf/kT} \right) df \end{aligned}$$

Because this series is uniformly and absolutely convergent, we can interchange the integral and the summation. This way we get the expression:

$$= \frac{2}{c^2} \sum_{n=1}^{\infty} \int_0^{\infty} f^2 e^{-nhf/kT} df$$

We now choose  $u = \frac{nh}{kT} f$ , then  $df = \frac{kT}{nh} du$ . By substituting this we get:

$$\begin{aligned} &= \frac{2}{c^2} \sum_{n=1}^{\infty} \int_0^{\infty} \left( \frac{nh}{kT} \right)^{-2} u^2 e^{-u} \frac{kT}{nh} du \\ &= \frac{2}{c^2} \sum_{n=1}^{\infty} \left( \frac{kT}{nh} \right)^3 \int_0^{\infty} u^2 e^{-u} du \end{aligned}$$

Let's introduce the gamma function, defined as:

$$\Gamma(s) = \int_0^{\infty} u^{s-1} e^{-u} du \quad (2.7)$$

This function has some nice properties, for example if  $n$  is an integer:

$$\Gamma(n) = (n-1)! \quad (2.8)$$

We see:

$$\begin{aligned} &= \frac{2}{c^2} \sum_{n=1}^{\infty} \left( \frac{kT}{nh} \right)^3 \Gamma(3) \\ &= \frac{2}{c^2} \left( \frac{kT}{h} \right)^3 \Gamma(3) \sum_{n=1}^{\infty} \frac{1}{n^3} \end{aligned}$$

Here we can write the infinity sum as the so called Riemann-zeta function:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad (2.9)$$

Matlab has a standard function to calculate the Riemann-zeta function. Using 2.8 and 2.9 we get:

$$A = \frac{4}{c^2} \left( \frac{kT}{h} \right)^3 \zeta(3) \quad (2.10)$$

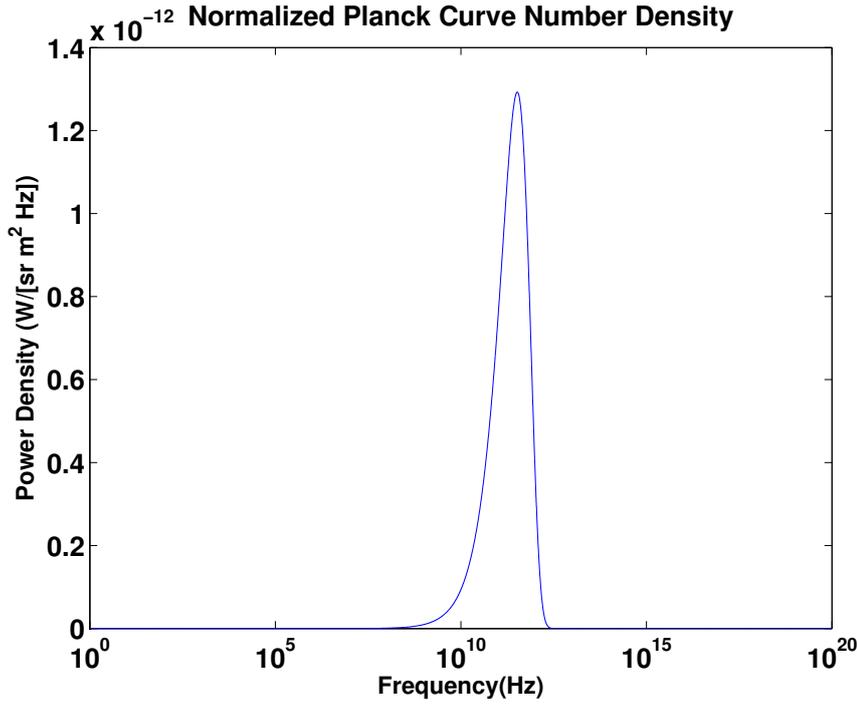


Figure 2.1: Normalized Photon Density, according to the Planck Law

$$A \approx 4.8398 \cdot 10^{14} \cdot T^3 \quad (2.11)$$

Now we have found our PDF, the Planck Photon Number Density, that describes the probability of detecting photons with frequency  $f$ :

$$D'_T(f) = \frac{2f^2/c^2}{A(e^{hf/kT} - 1)}, \quad f > 0 \quad (2.12)$$

The normalized Planck Photon Number Density is displayed in figure 2.1

### 2.1.2. GETTING THE CDF OF THE PLANCK PHOTON DENSITY

The cumulative distribution function CDF  $G(x)$  is defined as:

If  $g(x)$  a PDF, then:

$$G(x) = \int_{-\infty}^x g(\lambda) d\lambda, \quad x \geq 0 \quad (2.13)$$

The inverse of this function is often used to generate random numbers from a PDF  $g(x)$ . I will use it to show that getting numbers from the Planck Photon Number Density is actually like generating numbers from a mixture of other well known densities. Substituting the normalization constant 2.10 in our equation for the PDF 2.12, we can get the CDF using 2.13:

$$\begin{aligned} G(f') &= \int_0^{f'} \frac{1}{\frac{4}{c^2} \left(\frac{kT}{h}\right)^3 \zeta(3)} \frac{2f^2/c^2}{e^{hf/kT} - 1} df \\ &= \frac{1}{2 \left(\frac{kT}{h}\right)^3 \zeta(3)} \int_0^{f'} \frac{f^2}{e^{hf/kT} - 1} df \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2\left(\frac{kT}{h}\right)^3 \zeta(3)} \int_0^{f'} f^2 e^{-hf/kT} \frac{1}{1 - e^{-hf/kT}} df \\
&= \frac{1}{2\left(\frac{kT}{h}\right)^3 \zeta(3)} \int_0^{f'} f^2 e^{-hf/kT} \left( \sum_{n=0}^{\infty} e^{-nhf/kT} \right) df \\
&= \frac{1}{2\left(\frac{kT}{h}\right)^3 \zeta(3)} \int_0^{f'} f^2 \left( \sum_{n=1}^{\infty} e^{-nhf/kT} \right) df
\end{aligned}$$

Interchanging integral and summation results in:

$$= \frac{1}{2\left(\frac{kT}{h}\right)^3 \zeta(3)} \sum_{n=1}^{\infty} \int_0^{f'} f^2 e^{-nhf/kT} df$$

We now change variables again,  $u = \frac{nh}{kT} f$  so  $df = \frac{kT}{nh} du$ . This gives:

$$\begin{aligned}
&= \frac{1}{2\left(\frac{kT}{h}\right)^3 \zeta(3)} \sum_{n=1}^{\infty} \int_0^{\frac{kT}{nh} u'} \left( \frac{nh}{kT} \right)^{-2} u^2 e^{-u} \frac{kT}{nh} du \\
&= \sum_{n=1}^{\infty} \frac{1}{2n^3 \zeta(3)} \int_0^{\frac{kT}{nh} u'} u^2 e^{-u} du \\
&= \sum_{n=1}^{\infty} \frac{1}{n^3 \zeta(3)} \frac{1}{\Gamma(3)} \int_0^{\frac{kT}{nh} u'} u^2 e^{-u} du
\end{aligned}$$

The gamma probability distribution is defined as:

$$\frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, \quad x \geq 0 \quad (2.14)$$

It's CDF is called the incomplete gamma function:

$$\gamma(\alpha, \beta x) = \frac{1}{\Gamma(\alpha)} \int_0^{\beta x} u^{\alpha-1} e^{-u} du \quad (2.15)$$

Using this expression, now see that our CDF reduces to:

$$G = \sum_{n=1}^{\infty} \frac{1}{n^3 \zeta(3)} \gamma\left(3, \frac{kT}{nh} u'\right) \quad (2.16)$$

### 2.1.3. DRAWING RANDOM NUMBERS FROM THE PLANCK CURVE

We can look at expression 2.15 as a mixture of an infinite amount of Gamma distributions. Each function has its own weight. The mixture takes the form:

$$\sum_{n=1}^{\infty} p_n F_n(x) \quad (2.17)$$

Here  $p_n = \frac{1}{n^3 \zeta(3)}$  and  $F_n = \gamma\left(3, \frac{kT}{nh} u'\right)$ . In order for this to be a cumulative distribution function, the sum of all the weights should be one. Indeed, we see:

$$\begin{aligned}
\sum_{n=1}^{\infty} p_n &= \frac{1}{\zeta(3)} \sum_{n=1}^{\infty} \frac{1}{n^3} \\
&= \frac{1}{\zeta(3)} * \zeta(3) = 1
\end{aligned}$$

Now, if we want to generate variables from our mixture 2.15, we can apply a two-stage procedure. First, we must generate a value of  $n$ . The  $P_n$  in our mixture is the probability of getting a random number from  $F_n$ . So, first we must generate  $N$  from a discrete distribution on  $\{1, 2, 3, \dots\}$  with probabilities  $\{p_1, p_2, p_3, \dots\}$ . For this  $N$ , we then generate a random number from the corresponding gamma distribution  $F_N$ . By doing this, we got random variables from our mixture.

*Proof.*

$$\begin{aligned}
 P(X \leq x) &= \sum_{n=1}^{\infty} P(X \leq x | N = n) P(N = n) \\
 &= \sum_{n=1}^{\infty} P(Y_n \leq x) P(N = n) \\
 &= \sum_{n=1}^{\infty} p_n F_n(x) \\
 &= G(x)
 \end{aligned}$$

□

The Matlab script for this generation can be found in Appendix B.1. In figure 2.2 the frequencies that are generated are shown in a histogram.

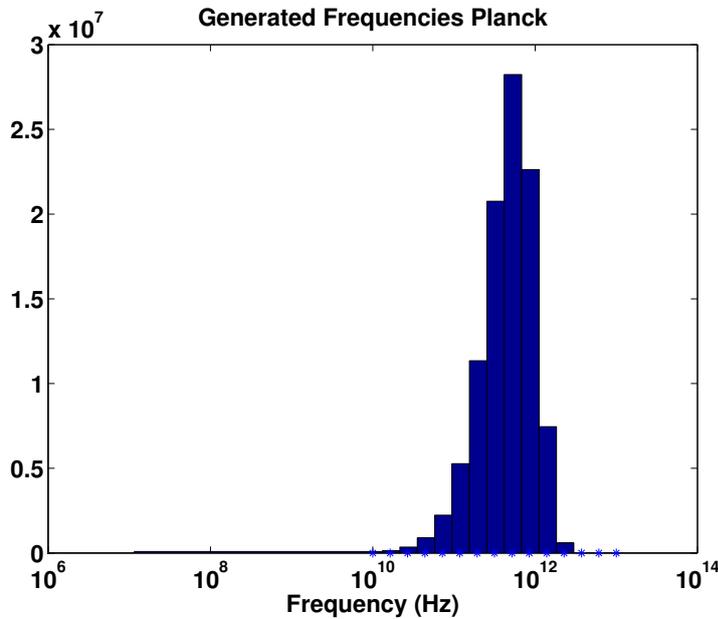


Figure 2.2: Histogram of frequencies generated using the Planck Curve

To check these results, it would be nice to generate the original density of the number of photons, according to the Planck Curve. To do this, we used the ordered frequencies that have been created. I choose a small frequency spacing  $df$  and divide the amount of photons that are generated in this frequency spacing by the total photons that have been generated. This way, we should get back the original curve for the Photon density, gotten from the Planck Curve.

For a large amount of photons simulated (here 100.000.000) we get back the normalized number density. This shows that the frequencies that are generated indeed represent the frequencies, so also the energy, of the photons that come from a blackbody source that radiates according to the Planck Law.

The same curve is plotted for different amount of simulations in Appendix C.2, but on a logarithmic scale to see the accuracy for high and low frequencies.

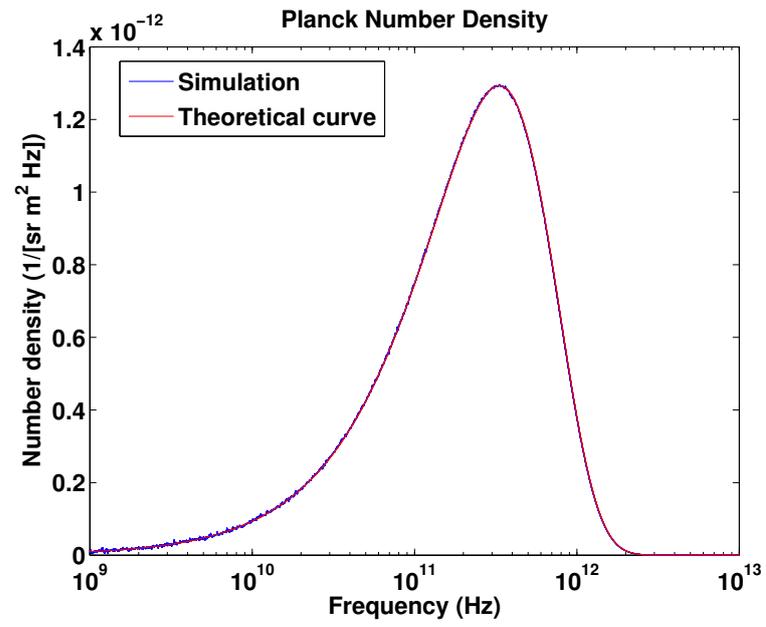


Figure 2.3: Photon density retrieved from the generated frequencies and the theoretical density.

## 2.2. JOHNSON-NYQUIST NOISE

The Planck Curve is used in a three dimensional space. For a one dimensional space, the Johnson-Nyquist formula for the power density of photons coming from a black body is used<sup>[9]</sup>:

$$P_T(f) = \frac{hf}{e^{hf/kT} - 1} \quad (2.18)$$

To derive an expression for the density of the amount of photons at frequency  $f$ , equation 2.17 must be divided by the energy of each photon,  $hf$ . This gives our photon density, from Johnson-Nyquist formula<sup>[10]</sup>:

$$D(f)_{J-N} = \frac{1}{e^{hf/kT} - 1} \quad (2.19)$$

This formula is also known as the Bose-Einstein distribution. Unfortunately, this function cannot be normalized like the density we got from the Planck Curve. The integral of  $D(f)_{J-N}$  from zero to infinity goes to infinity. This can easily be shown.

*Proof.*

$$\begin{aligned} \int_0^\infty D(f)_{J-N} df &= \int_0^\infty \frac{1}{e^{hf/kT} - 1} df \\ &= \int_0^\infty \frac{1}{-1 + 1 + x + \frac{x^2}{2!} + \dots} df \\ &= \int_0^\infty \frac{1}{x + \frac{x^2}{2!} + \dots} df \\ x > 0 &\Rightarrow \geq \int_0^\infty \frac{1}{x} df \rightarrow \infty \end{aligned} \quad (2.20)$$

□

### 2.2.1. NUMERICAL APPROXIMATION

So in fact, equation 2.18 cannot be normalized to become an actual PDF. We can however numerically approximate the area under this function, if we start at a frequency  $f_0 > 0$  and integrate from there. I picked  $f_0 = 1$  Hz. Matlab can get the area under this curve using the trapezium method. Obviously Matlab cannot integrate from  $f_0$  to infinity, therefore I choose to integrate from  $f_0$  to  $10^{15}$ . Because, for the temperatures we are interested in,  $D(10^{15})_{J-N} \approx 0$  (= 0 according to Matlab), this is a good approximation.

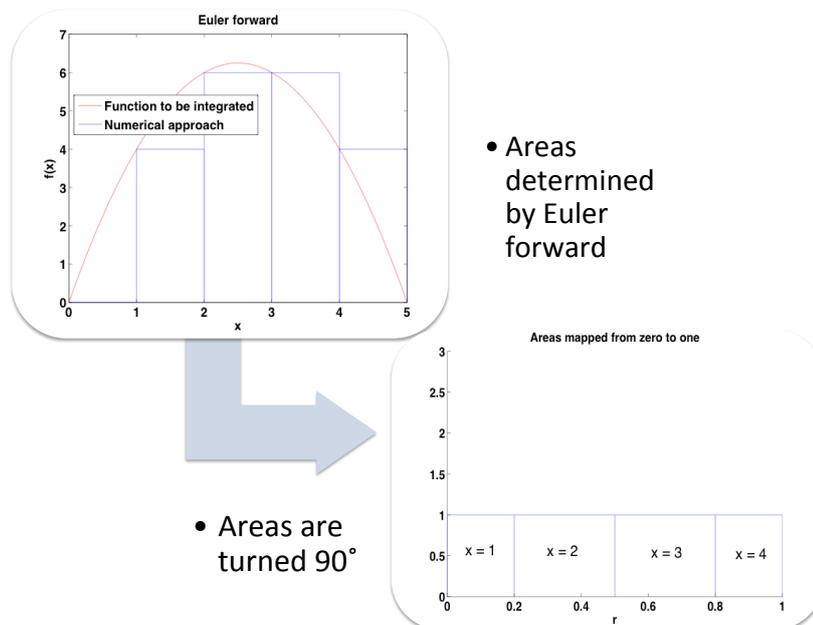


Figure 2.4: Numerical approach

The numerical method is using the ability of matlab to generate random numbers uniformly on (0,1). We take a frequency spacing vector  $df$  that contains the frequency ranges. Then, using the Euler Forward Method, the area of above these frequency ranges can easily be calculated by multiplying  $df$  by the value of  $D(f)_{J-N}$  at its lower boundary. By dividing these area's by the total area and cumulatively summing them, we have mapped them from zero to one. Now all is left is to generate random numbers from zero to one and retrieving the frequency corresponding to the area that has been selected. This process has been shown for a simple function in figure 2.4. First, the Euler-Forward method is used to integrate the function. Then, the areas are turned 90° and divided by the total area. This way, each number from zero to one corresponds with one of these areas and this way also to a value of  $x$ , the one on the lower boundary of the corresponding area. In this example, when Matlab generates the number 0.6, it will correspond with  $x = 3$ . Note that the result in this simple example is rather poor, because a spacing number of only five has been used. This means  $dx = 5$ , this spacing is too large to get a nice result. I generated 10.000 frequencies this way and they are plotted in figure 2.5.

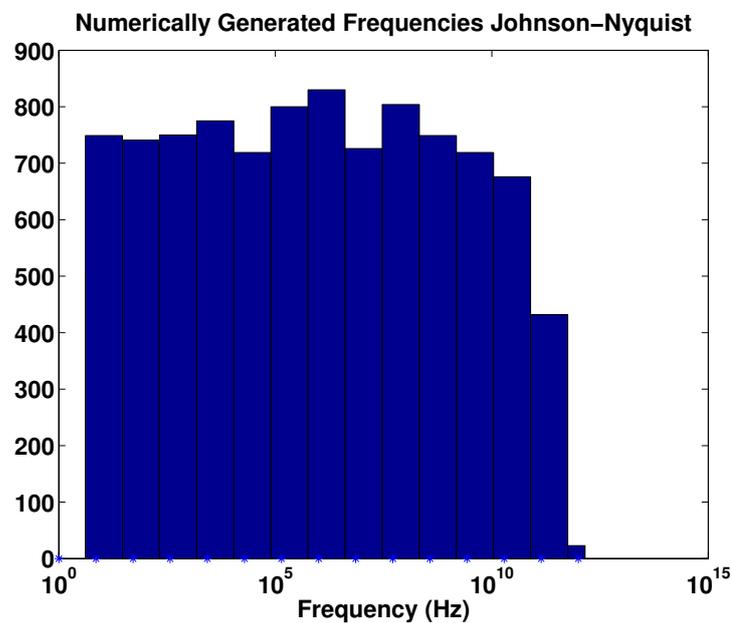


Figure 2.5: Histogram of frequencies generated by using the Johnson Nyquist number density

The accuracy of this method depends on the frequency spacing  $df$ . If  $df$  is infinitely small, the integration method would be exact and so the number generation would be too. So, by increasing the amount of frequency bins  $M$  in the used frequency range, this method will more accurate. The frequencies are separated in a logarithmic way. In Appendix C.1 the density curves retrieved from the generated frequencies are plotted for increasing values of  $M$ , on a logarithmic scale to show the behavior at small and large frequencies. In figure 2.6 I have plotted  $D(f)_{J-N}$  with the number density retrieved from the generated frequencies by the numerical method, on a linear scale. Here I took  $M = 100.000$ , and 10.000 frequencies have been generated. To create figure 2.5 I also used  $M = 100.000$ .

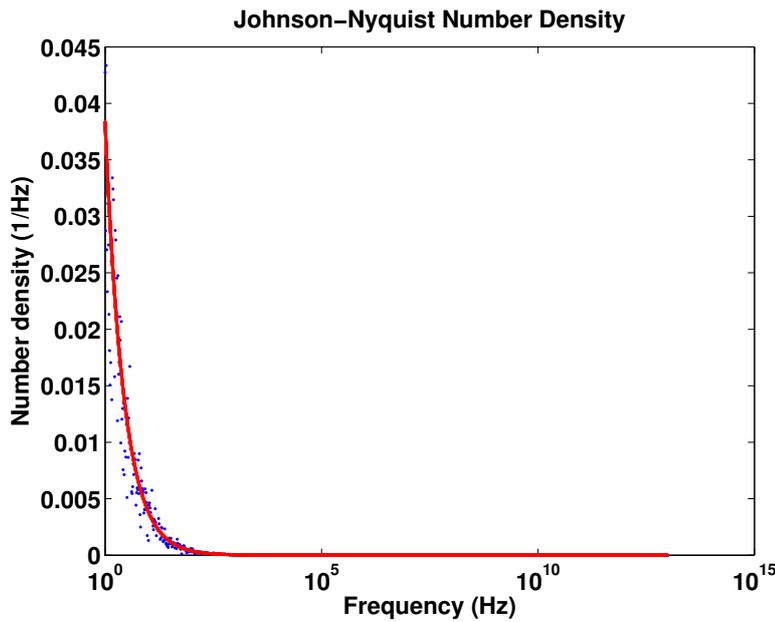
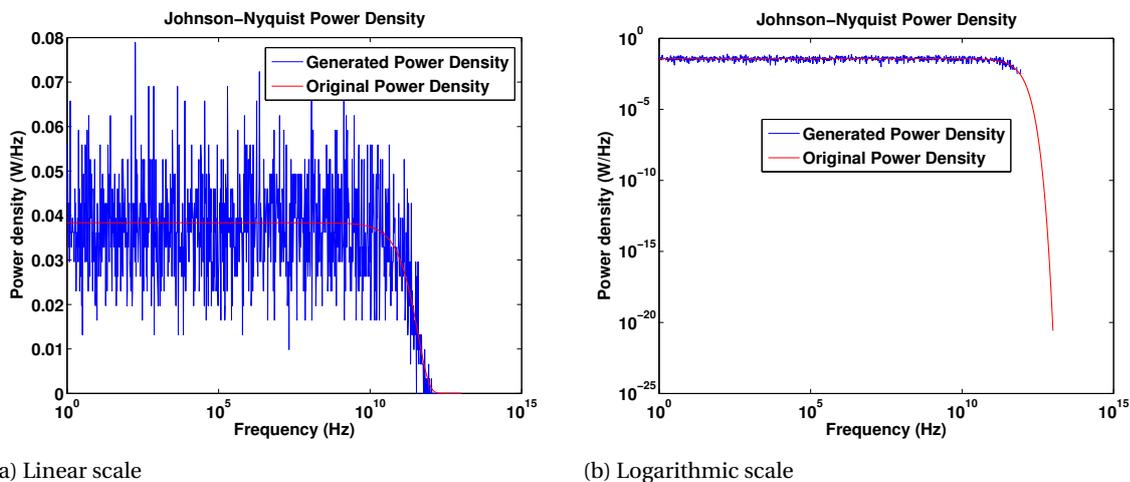


Figure 2.6: Johnson Number Density as retrieved from generated frequencies with numerical approach and theoretical line.  $M = 100.000$

Now we can recreate the original Johnson-Nyquist formula for the power density 2.17 by simply multiplying this curve by the energy, which is  $hf$ . The result is show in figure 2.7.

From this, we can determine how big the contribution of each frequency is to the total power by multiplying this curve by the frequency. This is plotted in figure 2.8. Note that there is an error in this curve that is not negligible. An explanation for this error could be the result of numerically generating the frequencies. But, as can be seen in figure 2.7a, the curve fluctuates around the value we would expect. This suggests the error is probably due to the limited amount of generated frequencies. 10.000 frequencies might seem like a lot, but on a range from 1 to  $10^{13}$  this is a not very impressive. Increasing the amount of generations is of course possible, but this increases the computation time in such a way it is no longer acceptable. The matlab code for this generation can be found in Appendix B.2

We see this numerical approach gives the result we wished for, but it is slow and it is itself an approximation. Therefore, it would be nice if more of the generation process could be done analytically.



(a) Linear scale

(b) Logarithmic scale

Figure 2.7: Johnson-Nyquist Power Density from the generated frequencies and theoretical line

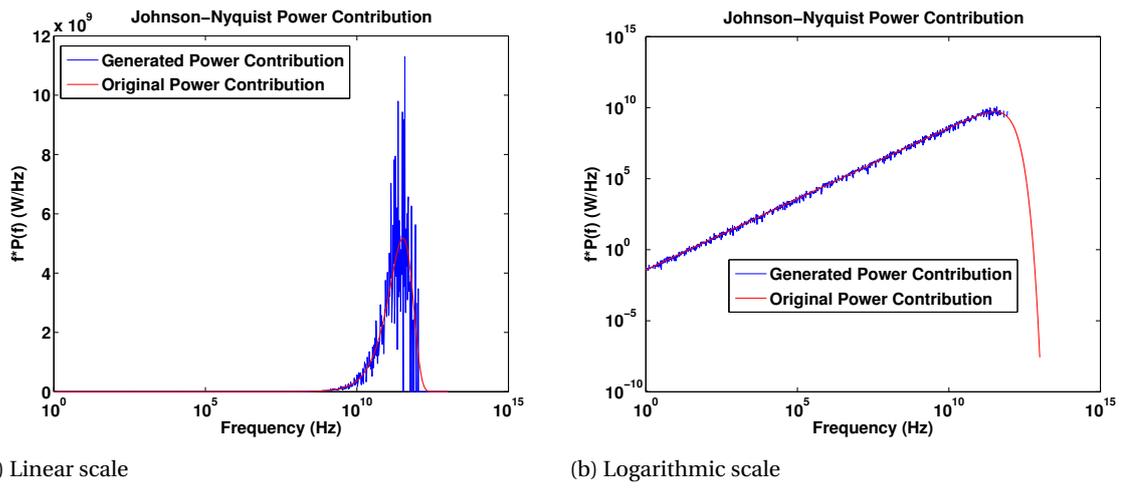


Figure 2.8: Contribution of each frequency to the total power for the generated frequencies and theoretical line

## 2.3. ANALYTIC APPROACH JOHNSON-NYQUIST

One method that is commonly used to draw random numbers from a PDF is based on getting numbers  $u \in [0, 1]$  uniformly. Now, if random numbers from a PDF  $g$  are wanted, the corresponding cumulative distribution function (cdf)  $G$  must be determined. The required random numbers from the distribution then are:

$$G^{-1}(u) \quad (2.21)$$

Here  $G^{-1}$  is the inverse function of cdf  $G$ . In our case, unfortunately, the integral 2.13 cannot be determined analytically. Therefore, we also do not have a analytic function for  $G^{-1}$ . We can see this from identity 2.19. Now we again use a starting frequency  $f_0$ , so we could calculate the CDF. Unfortunately, this does not give us a nice expression for the inverse function  $G^{-1}$ . Therefore, the method I just explained cannot be used to get numbers from the Johnson-Nyquist Number Density. So, what can we do to generate numbers from this distribution?

I will use another method that is called the rejection method<sup>[11]</sup>. The basic idea is very simple. We want to use the method introduced in the previous section, using a PDF  $m(f)$  which has a known cdf  $M(f)$  that can easily be inverted ( $M^{-1}(f)$ ). This  $m(f)$  is called the dominating distribution. We choose  $m(f)$  such that  $c * m(f)$  is greater than the density we want to get numbers from, for all frequencies. Here  $c$  is a constant with  $c \geq 1$ . To use this method, the density of interest does not have to be normalized. An example of this situation is shown in figure 2.9. Because we have  $M^{-1}$  analytically, we can get  $n$  random frequency's from  $m(f)$  by using the `rand(n)` function in Matlab. For each of these frequency's  $f_i$  we can calculate  $c * m(f_i)$ . Next, we take a random number  $x$  from 0 to  $c * m(f_i)$ . If this number satisfies  $x \leq D_{J-N}(f_i)$ , this  $f_i$  will be accepted. If  $x > D_{J-N}(f_i)$ , this frequency will be rejected. This way, we get a set of frequencies, that are generated randomly from the Johnson-Nyquist Number Density.

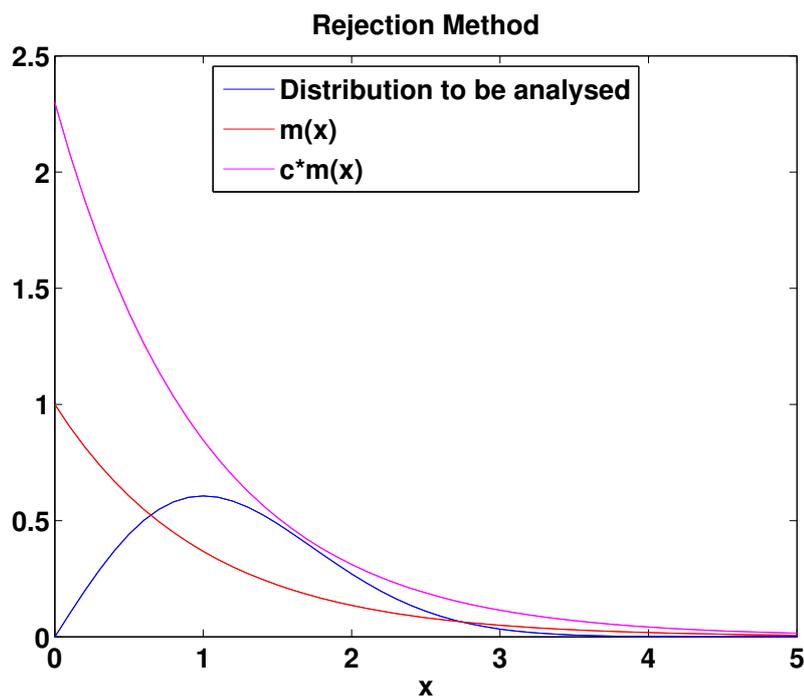


Figure 2.9: Example of curves using the Rejection Method

### 2.3.1. DETERMINING A DOMINATING DENSITY

The tricky thing about  $D_{J-N}(f)$  is that it grows very large for small frequencies, going to infinity at  $f = 0$ . It also has an infinity long 'tail', so we need a distribution that has this property as well. An exponential distribution seems appropriate, since it has support  $[0, \infty]$  and drops exponentially in the low frequency region. The exponential distribution is defined as:

$$h(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases} \quad (2.22)$$

Here usually  $x \geq 0$ . It's CDF can easily be calculated:

$$H(x) = (1 - e^{-\lambda x}) \quad (2.23)$$

We will use the exponential distribution with parameter  $\lambda = 1$ . Note that in our case, we only look at frequencies larger than  $f_0$ , so the CDF will be slightly different. To use this distribution, we apply the transformation:

$$x = \frac{hf}{kT} \quad (2.24)$$

The Johnson-Nyquist Number Density then becomes:

$$D(x)_{J-N} = \frac{1}{e^x - 1}, x > x_0 = \frac{hf_0}{kT} \quad (2.25)$$

I choose  $f_0$  to be  $f_0 = 1$  Hz. This results in:

$$x_0 \approx 4.7992 \cdot 10^{-11} \quad (2.26)$$

We now want to see for what value of constant  $K$ ,  $K * h(x)$  is larger or equal to  $D(x)_{J-N}$  for all  $x \geq x_0$ . We see:

$$D(x)_{J-N} \leq K * h(x) \quad (2.27)$$

$$\frac{1}{e^x - 1} \leq K e^{-x}$$

$$\frac{e^x}{e^x - 1} \leq K$$

$$K \geq \frac{1}{1 - e^{-x}}$$

This must hold for any  $x \geq x_0$ . So we see:

$$K \geq \frac{1}{1 - e^{-x_0}} \quad (2.28)$$

I used  $K = \frac{1}{1 - e^{-x_0}}$ , the smallest possible value of  $K$ . Both the exponential distribution and the Johnson-Nyquist Number Density are plotted in figure 2.10.

The value of  $K$  influences the efficiency of this method. Note that  $D(x)_{J-N} \leq K * h(x)$ . If we would want to calculate the probability of accepting a value that is generated by the dominating exponential distribution, the acceptance probability:

$$\begin{aligned} P &\leq \int_{x_0}^{\infty} h(x) \frac{D(x)_{J-N}}{K * h(x)} dx \\ &\leq \frac{1}{K} \int_{x_0}^{\infty} D(x)_{J-N} dx \\ &\leq \frac{1}{K} \end{aligned} \quad (2.29)$$

So if  $K = 1$  the probability of accepting is the largest, but this value of  $K$  would not make the exponential distribution dominating. The numerical value derived in equation 2.27 is  $K \approx 2.084 \cdot 10^{10}$ , so the probability of accepting  $P \leq \frac{1}{K} \approx 4.8 \cdot 10^{11}$ . We see that the probability of accepting is very small.

In figure 2.10 we see the exponential density is, for low frequencies, not even close to the density we want to get numbers from. So indeed, using the rejection method, most of the generations will be rejected. For the first part of the density we will need another distribution as domination density, in order to make the program generate numbers in an acceptable period of time. For  $x$  larger than a boundary  $d$  we can use the exponential density. We see:

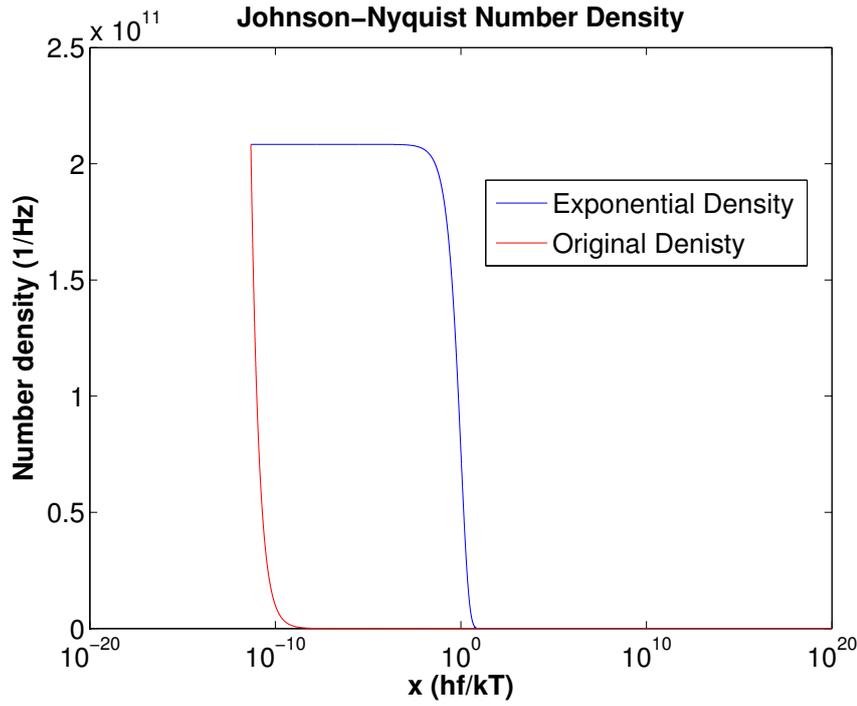


Figure 2.10: Dominating Exponential Density with original Johnson-Nyquist Density

$$\frac{1}{e^x - 1} = \frac{1}{-1 + 1 + x + \frac{x^2}{2!} + \dots}$$

$$\frac{1}{e^x - 1} = \frac{1}{x + \frac{x^2}{2!} + \dots}$$

Since  $x > x_0 > 0$  we get:

$$\frac{1}{e^x - 1} < \frac{1}{x} \quad (2.30)$$

On the interval  $[x_0, d]$  we will use  $1/x$  as domination distribution. The total distribution, not normalized, now is:

$$f(x) = \begin{cases} \frac{1}{x} & x \in [x_0, d] \\ \frac{e^{-x}}{1 - e^{-d}} & x > d \end{cases} \quad (2.31)$$

To make this an actual probability density function, 2.30 has to be normalized.

$$\begin{aligned} \int_{x_0}^{\infty} f(x) dx &= \int_{x_0}^d \frac{1}{x} dx + \int_d^{\infty} \frac{e^{-x}}{1 - e^{-d}} dx \\ &= \ln(x) \Big|_{x_0}^d - \frac{1}{1 - e^{-d}} e^{-x} \Big|_d^{\infty} \\ &= \ln\left(\frac{d}{x_0}\right) - \frac{1}{1 - e^{-d}} (0 - e^{-d}) \\ \frac{1}{Q} &= \ln\left(\frac{d}{x_0}\right) + \frac{1}{e^d - 1} \end{aligned} \quad (2.32)$$

Now we get the normalized density for the dominating distribution:

$$f_d(x) = \begin{cases} \frac{Q}{x} & x \in [x_0, d] \\ Q \frac{e^{-x}}{1 - e^{-d}} & x > d \end{cases} \quad (2.33)$$

This is a nice expression, but it depends on the boundary  $d$ . From this expression, we can determine an optimal value of  $d$ . We want this function to be as close to the Johnson-Nyquist Number Density as possible, without getting under this density. We ensured that 2.30 is always greater than Johnson-Nyquist Number Density. So, if we want to get it as close as possible, we must minimize its integral. Taking the derivative with respect to  $d$  equal to zero gives:

$$\begin{aligned} \frac{\partial \frac{1}{Q}}{\partial d} &= 0 & (2.34) \\ \frac{1}{d} + \frac{-e^d}{(e^d - 1)^2} &= 0 \\ \frac{1}{d} &= \frac{e^d}{e^{2d} - 2e^d + 1} \\ \frac{1}{d} &= \frac{1}{e^d - 2 + e^{-d}} \\ d + 2 &= e^d + e^{-d} \\ \cosh(d) &= \frac{d + 2}{2} & (2.35) \\ \Rightarrow d &= 0.93 & (2.36) \end{aligned}$$

For my generation I used  $d = 1$ , just for convenience. Choosing a larger  $d$  can be done without any problems regarding the validity of this method (since the dominating function will only be higher above the observed density if  $d$  is larger than it's minimum value), but choosing  $d$  too high will affect the efficiency of this method negatively.

### 2.3.2. GETTING THE CDF OF DOMINATING DENSITY

To be able to generate numbers from 2.32, we must determine its CDF. Using equation 2.13 we find in the case  $x \in [x_0, d]$ :

$$\begin{aligned} F_1(x) &= Q \int_{x_0}^x \frac{1}{\lambda} d\lambda \\ &= Q \ln(\lambda) \Big|_{x_0}^x \\ &= Q \ln\left(\frac{x}{x_0}\right) \end{aligned} \quad (2.37)$$

If  $x > d$  we get:

$$\begin{aligned} F_2(x) &= Q \int_{x_0}^d \frac{1}{\lambda} d\lambda + Q \int_d^x \frac{e^{-\lambda}}{1 - e^{-d}} d\lambda \\ &= Q \ln\left(\frac{d}{x_0}\right) - Q \frac{1}{1 - e^{-d}} e^{-\lambda} \Big|_d^x \\ &= Q \ln\left(\frac{d}{x_0}\right) + Q \frac{e^{-d} - e^{-x}}{1 - e^{-d}} \end{aligned} \quad (2.38)$$

So in total we have:

$$F_d(x) = \begin{cases} Q \ln\left(\frac{x}{x_0}\right) & x \in [x_0, d] \\ Q \ln\left(\frac{d}{x_0}\right) + Q \frac{e^{-d} - e^{-x}}{1 - e^{-d}} & x > d \end{cases} \quad (2.39)$$

This function has been plotted in figure 2.11

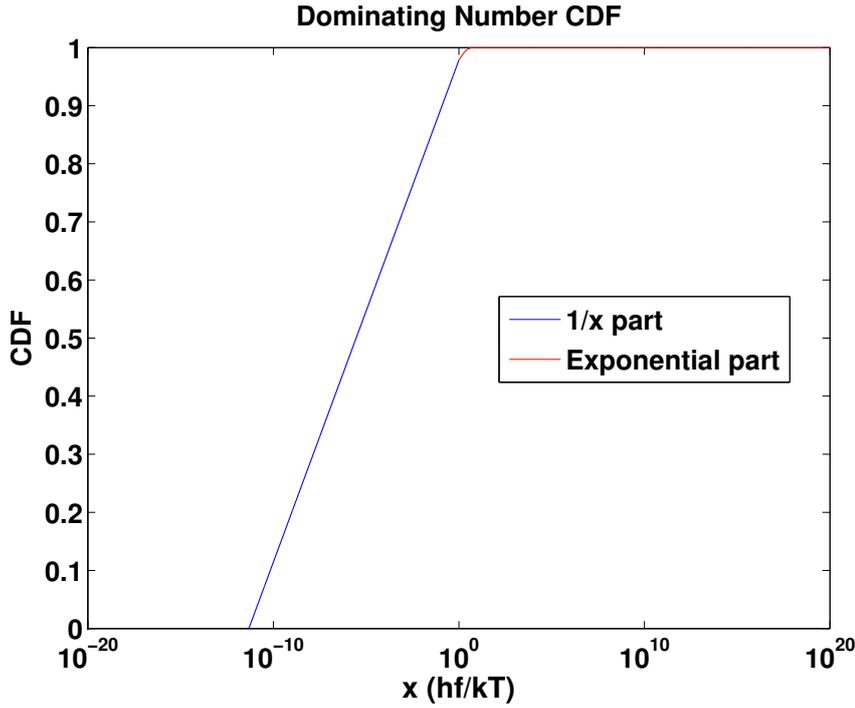


Figure 2.11: CDF of the dominating distribution function

It is not hard to find the inverse function of equation 2.38. We get:

$$F_d^{-1}(x) = \begin{cases} x_0 e^{x/Q} & x \in [F_d(x_0), F_d(d)] \\ -\ln \left[ e^{-d} + (1 - e^{-d}) \left( \ln \left( \frac{d}{x_0} \right) - \frac{x}{Q} \right) \right] & x \in (F_d(d), 1] \end{cases} \quad (2.40)$$

### 2.3.3. GENERATION NUMBERS

With these tools available, we can start generating random numbers from the Johnson-Nyquist Number Density. The matlab script to do this is attached in Appendix B.3. Using the method described in section 2.3, we will first generate random numbers of the dominating density  $f_d(x)$  by using  $F_d^{-1}(x)$  and the ability of matlab to generate uniform random number from zero to one. After implementing the rejection method and transforming back to the frequency domain, we get frequencies generated from the Johnson-Nyquist Number Density. These frequencies are shown in figure 2.12

For this simulation the script produced  $n = 10,000,000$  frequencies that are tested using the rejection method. After rejecting,  $9.8 * 10^6$  frequencies are accepted. So approximately 98% of the frequencies were accepted. We see most of the frequencies are not rejected, so the dominating function used has been a good choice.

I would like to remark that there are no frequencies generated lower than 1 Hz. This is the value of  $f_0$  we have chosen. This might seem like a high value to start with, but because the contribution in terms of the total power of these low frequency photons is negligible, we can neglect them in this generation is well. This can be seen in figure 2.15. To verify the generated frequencies, the original number density has been retrieved from these frequencies. This is shown in figure 2.13 and 2.14

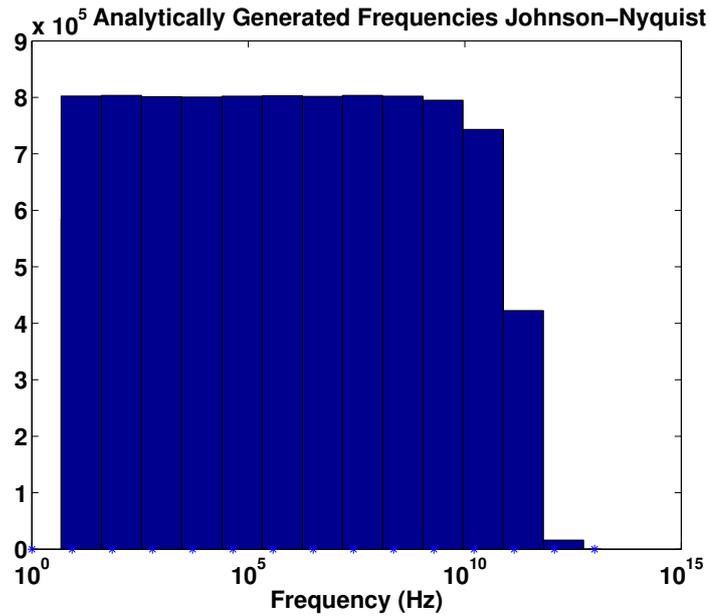


Figure 2.12: Histogram of frequencies analytically generated from the Johnson-Nyquist Number Density

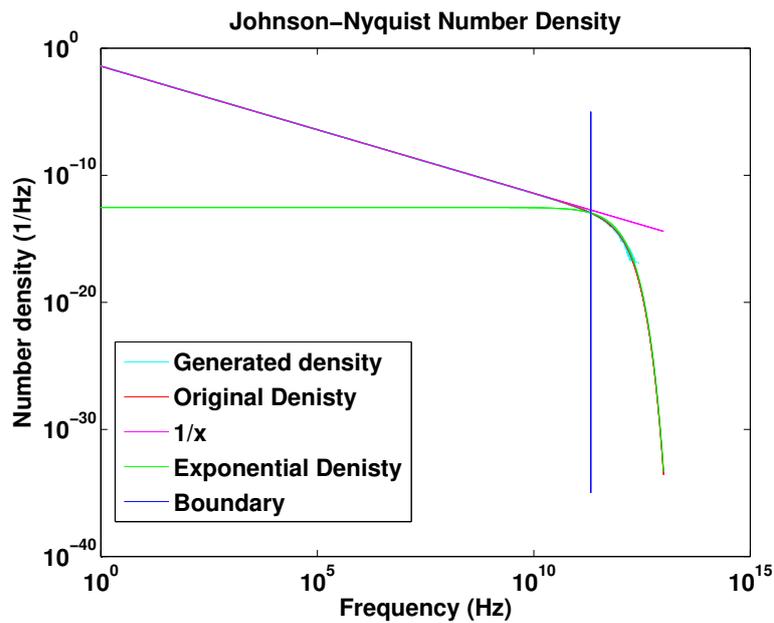


Figure 2.13: Johnson-Nyquist Number Density retrieved from the generated frequencies, with boundary  $d$ ,  $\frac{1}{x}$ , the exponential distribution and the original density

Using this, we can recreate the original formula for the Johnson-Nyquist noise, equation 2.17. This curve shows the power density per frequency. If we multiply this by the frequency we get an insight on the contribution of each frequency to the total power. This is shown in figure 2.15. Note that neglecting frequencies below 1 Hz is appropriate, because the contribution to the total power is very low in this region.

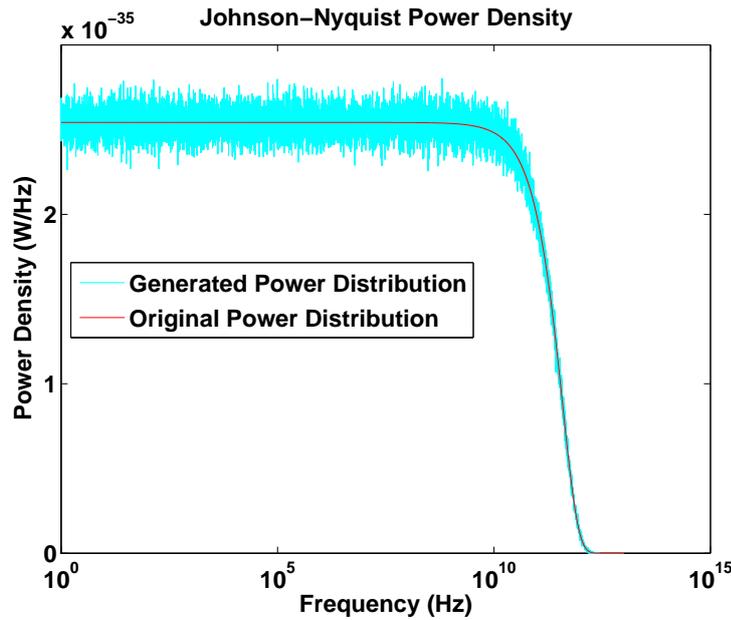
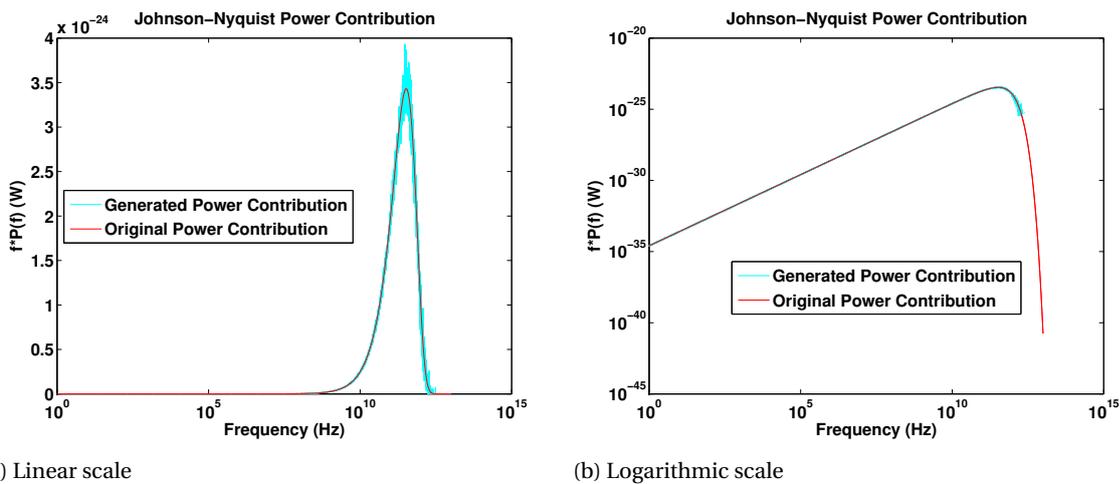


Figure 2.14: Power Density as calculated from the generated frequencies and theoretical line by Johnson-Nyquist noise

From figure 2.15 can be concluded that this method generates the right amount of photons regarding their frequencies. Also, the frequency range in which these photons are generated is appropriate.



(a) Linear scale

(b) Logarithmic scale

Figure 2.15: Contribution of each frequency to the total power for the generated frequencies and theoretical line

Using this method, up to 100.000.000 frequency values can be simulated within a minute. This is 10.000 times more photons than we got in the numerical approach in section 2.2.1. So, this analytic method is not a numerical approximation and it is also approximately 10.000 times faster than the numerical approach. Therefore we will use this method to generate the frequency values of the photons in our photon stream.



# 3

## TIME DISTRIBUTION OF PHOTONS

### 3.1. TIME CORRELATION

#### 3.1.1. CLASSICAL LIGHT WAVE

In classical point of view, light can be described as a wave traveling with speed  $c$ , an amplitude  $A$  and a frequency  $f$ . Light is an electromagnetic wave, which can be completely described by the Maxwell equations<sup>[12]</sup>. In these equations we find a precise description of the propagation of an electric field  $\mathbf{E}$  with complementary magnetic field  $\mathbf{B}$ . These equations form the foundation of electromagnetism and classical optics.

$$\Delta \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \quad (3.1)$$

$$\Delta \cdot \mathbf{B} = 0 \quad (3.2)$$

$$\Delta \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (3.3)$$

$$\Delta \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \quad (3.4)$$

These equations describe a wave consisting of a magnetic and an electric field. This might not be trivial at first sight. Let's assume we are dealing with vacuum and no currents are involved, so  $\rho = 0$  and  $\mathbf{J} = 0$ . The Maxwell equations are now reduced to:

$$\Delta \cdot \mathbf{E} = 0 \quad (3.5)$$

$$\Delta \cdot \mathbf{B} = 0 \quad (3.6)$$

$$\Delta \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (3.7)$$

$$\Delta \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \quad (3.8)$$

We now take the curl of equation 3.7 and 3.8, and simplify this using the curl of the curl identity:

$$\Delta \times (\Delta \times X) = \Delta(\Delta \cdot X) - \Delta^2 X \quad (3.9)$$

This gives us:

$$\Delta(\Delta \cdot \mathbf{E}) - \Delta^2 \mathbf{E} = \Delta \times -\frac{\partial \mathbf{B}}{\partial t}$$

$$\Delta(\Delta \cdot \mathbf{E}) - \Delta^2 \mathbf{E} = -\frac{\partial}{\partial t} (\Delta \times \mathbf{B})$$

Using equation 3.5 and 3.8  $\Rightarrow$

$$-\Delta^2 \mathbf{E} = -\frac{\partial}{\partial t} \left( \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)$$

$$\mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} - \Delta^2 \mathbf{E} = 0 \quad (3.10)$$

Doing the same but using equation 3.6 and 3.7 gives:

$$\mu_0\epsilon_0 \frac{\partial \mathbf{B}^2}{\partial t^2} - \Delta^2 \mathbf{B} = 0 \quad (3.11)$$

Equation 3.10 and 3.11 are both three dimensional wave equations with by definition a speed  $c = \frac{1}{\sqrt{\mu_0\epsilon_0}}$ . Here  $c$  is the speed of light in vacuum. When the light wave is not traveling through vacuum, this will result in a lower speed  $v$  for the traveling wave. The intensity of the electromagnetic field in the wave front is determined by the strength of the electric field.

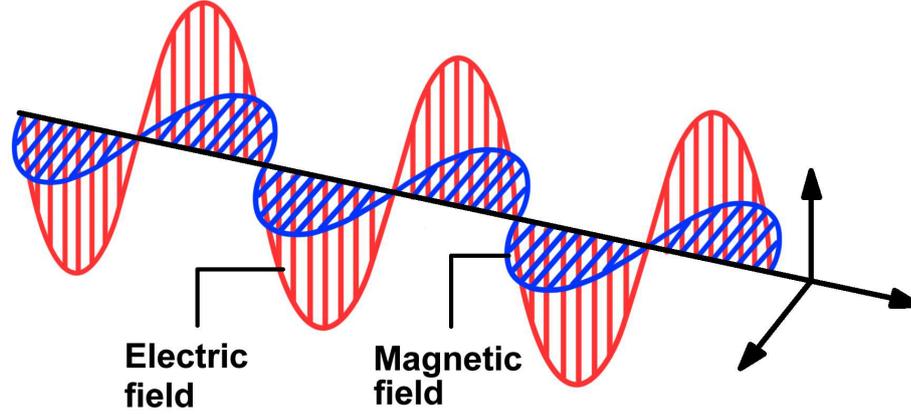


Figure 3.1: Schematic image of an electromagnetic wave<sup>[13]</sup>

The period of the wave is:

$$T = \frac{1}{f} \quad (3.12)$$

Usually, a light source will only emit photons with frequencies in a certain band of frequencies, its bandwidth. We will call this bandwidth  $\Delta f$ . When measuring the light coming from such a source, the measured amplitude will be a result of light at all the frequencies within the band. Changes in the amplitude are therefore limited in time by the bandwidth. The time that is needed to see a significant change in amplitude is called the temporal coherence time  $\tau_{coh}$  and is approximately given by:

$$\tau_{coh} \approx \frac{1}{\Delta f} \quad (3.13)$$

Because the amplitude does not change significantly within a time interval of  $\tau_{coh}$ , it is little use to design a detector that can measure for time periods smaller than  $\tau_{coh}$ , if the amplitude of the wave is what we are interested in.

### 3.1.2. PHOTONS

When simulating photons, we do not use the classical interpretation of light, but we use its quantum properties. These properties are described using photons. A photon does not have mass, but it does have an energy and, when measuring, an arrival time. We will now look at the time distribution of photons emitted from a black body source.

To understand the statistical behavior of photons in time, it is useful to gain an understanding of what a photon exactly is. The photons that come from a black-body source are emitted by the individual atoms in this black body. If we assume the Bohr model of atoms, these atoms consist of a core with electrons surrounding it. These electrons can be in different shells. By adding energy to the atom, an electron can be excited, which makes it move to a higher energy shell. When this electron falls back into its original shell, it emits the energy that was needed to excite it. This produces a photon with a frequency corresponding with the energy that is emitted by the atom. The energy of the photon is:

$$E = hf \quad (3.14)$$

Here  $h$  is Planck's constant and  $f$  is the frequency of the photon.

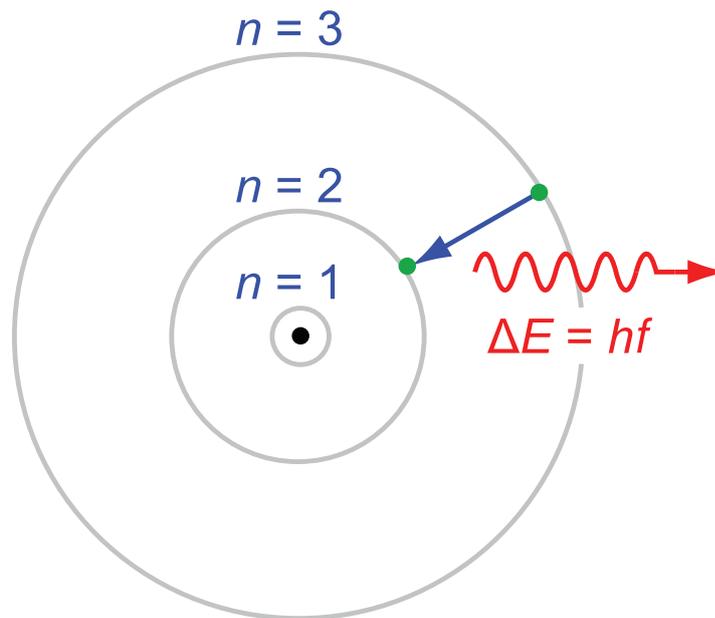


Figure 3.2: The Bohr Atom model, with an electron falling back from the third to the second shell<sup>[14]</sup>

The excited atoms do not necessarily fall directly back to their non-excited state. It might take a while for an electron to fall back. This implies that there is no coherence in the arrival of each of these photons; they are all individually emitted by an atom, so the photons do not influence one another. But, we will see that these photons are in fact *not* independent from each other.

### 3.1.3. WAVE VS PARTICLE

The wave characterization and the particle characterization of light are two different ways to describe the phenomenon of light. We will see that these two are not totally separable.

The intensity of the electromagnetic field in the wave front of a light wave is determined by the strength of the electric field. According to the Huygens-Fresnel principle, the strength of the electric field is a superposition of many elementary waves. This means all atoms in the black-body emitting the light contribute to this wave front. The photons emitted by these atoms therefore must have some kind of interaction, so they can form this wave front together. This implies that the emissions of the photons, so the transition of an electron into a lower energy level, are dependent on each other. Single photons seem to "know" about each others existence.

Photons are not classical particles. The photons traveling in a light wave obey a probability distribution, so we know the probability of observing a photon at a certain time. You are not sure there is a photon, until you have measured it. One could look at a light wave as a probability wave moving in space and time, where the probability is of a photon being measured at a certain time.

### 3.1.4. PHOTON BUNCHING

How does the particle description of light match this classical wave approach? A light wave is built up with photons. These are emitted independently, in our case by atoms in the black body. The photons can however interact with each other; together they form the light wave that we know.

In time, the power of the measured light wave fluctuates. When the power is high, the probability of finding photons is also high. If we detect a power of 0 W, we will not find any photons. The fluctuation of the power therefore describes the probability of detecting photons in time. Because the amplitude of the wave does not change significantly within time periods of  $\tau_{coh}$  as in 3.13, the power (the amplitude squared) also stays approximately the same on a time scale of  $\tau_{coh}$ . The power fluctuations and the corresponding photons are schematically shown in figure 3.3.

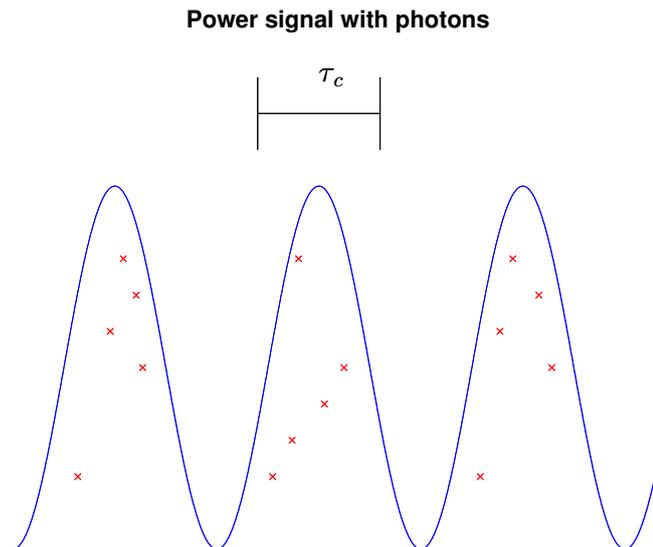


Figure 3.3: Photons traveling in a wave

Photons emitted from any source have a certain statistical behavior that describes how these photons are detected in time, that is often not just random. In the case of photons being emitted from a black body source, photon bunching occurs: the photons tend to arrive in "bunches". This means that there are periods in which there do not arrive any photons, followed by periods of time in which several photons arrive. This is schematically illustrated in figure 3.3. Note that the bunching occurs on a time scale of  $\tau_{coh}$ . If we look at larger time scales than  $\tau_{coh}$ , the power signal in figure 3.3 will be in more than just one "peak-region", so the bunching within each "peak-region" is no longer visible.

When a photon detector is measuring a wave, it has a certain detection time  $\tau_{det}$ . The detector will detect all photons in this time interval  $\tau_{det}$ . So, if  $\tau_{det}$  is larger than  $\tau_{coh}$ , the detector will not measure the photon bunching. In practice,  $\tau_{det} \gg \tau_{coh}$ , so the photon bunching will not be detected. Then, why bother about this photon bunching in our photon simulation?

Even though the photon bunching is usually not measured directly, the bunching of the photons causes noise in the measured signal. This noise is called photon-noise. If we want a model that represents the signal as we would measure it in a photon detector, the photon-noise has to be included. Therefore, the bunching of the photons must be included in the model.

The probability  $p_n$  of finding exactly  $n$  photons in a time interval  $dt$  that is approximately  $\tau_{coh}$  is given by<sup>[15]</sup>:

$$p_n = \frac{\bar{n}^n}{(\bar{n} + 1)^{n+1}} \quad (3.15)$$

Here  $\bar{n}$  is the average amount of photons emitted in time interval  $T$ . Note that the maximum value of this discrete probability distribution is at  $n = 0$ .

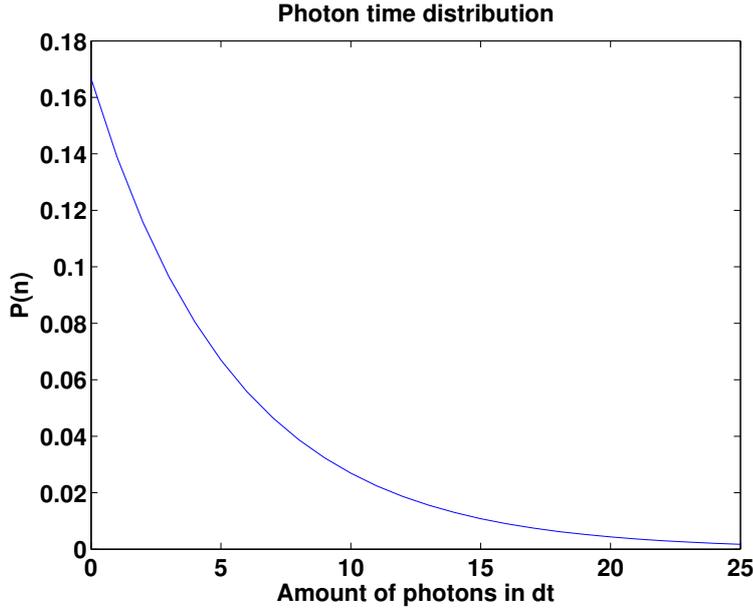


Figure 3.4: Probability time distribution of measuring a photon in time  $dt$ , with  $\bar{n} = 5$

Why does identity 3.15 not hold for time intervals that are not approximately equal to  $\tau_{coh}$ ? Lets assume we look at a  $dt$  smaller than  $\tau_{coh}$ , say  $dt = 0.1\tau_{coh}$ . In this case, the time intervals  $dt$  are not independent of each other. So, in each time interval  $dt_i$  information about the previous time interval  $dt_{i-1}$  is needed to determine the probability of finding a photon in  $dt_i$ . In this example, a measurement is dependent of nine time intervals  $dt$  preceding the measurement in  $dt_i$ . This information is not enclosed if for each  $dt$  the number of photons is independently generated using formula 3.15. Now, if we take  $dt > \tau_{coh}$ , the bunching effect is no longer visible. Therefore, identity 3.15 can no longer be used. We see  $dt$  should be approximately equal to  $\tau_{coh}$ , if we want to use identity 3.15.

That identity 3.15 is indeed a probability distribution can easily be shown:

$$\begin{aligned}
 \sum_{n=0}^{\infty} p_n &= \sum_{n=0}^{\infty} \frac{\bar{n}^n}{(\bar{n}+1)^{n+1}} \\
 &= \frac{1}{\bar{n}+1} \sum_{n=0}^{\infty} \left( \frac{\bar{n}}{\bar{n}+1} \right)^n \\
 &= \frac{1}{\bar{n}+1} \frac{1}{1 - \frac{\bar{n}}{\bar{n}+1}} \\
 &= \frac{1}{\bar{n}+1} \frac{\bar{n}+1}{\bar{n}+1 - \bar{n}} \\
 &= 1
 \end{aligned} \tag{3.16}$$

## 3.2. TIME GENERATION

### 3.2.1. AVERAGE AMOUNT OF PHOTONS IN TIME T

In the previous section we looked at the frequency of photons emitted from a black body source. With these, the energy of these photons can be calculated. This has resulted in a large amount of numbers that represent the energy of the photons emitted by a black-body source. This data does not tell us when or in which order these photons are emitted. We can however use this data to say something about the average amount of photons that will be emitted per unit of time, in a certain frequency range. This result is necessary for simulating the arrival times of the photons at the detector.

From theory we can calculate the total power that will be emitted in a certain frequency range. With this information and the total energy of the photons that have been generated in the previous section, we can calculate the average amount of photons emitted per unit time.

The power emitted in a certain frequency range  $[f_1, f_2]$  can be calculated by integrating the power density over this frequency interval. For the Johnson-Nyquist noise this gives:

$$\begin{aligned} \int_{f_1}^{f_2} D(f)_{J-N} df &= \int_{f_1}^{f_2} \frac{hf}{e^{hf/kT} - 1} df \\ &= \frac{(kT)^2}{h} \int_{f_1}^{f_2} \frac{hf/kT}{e^{hf/kT} - 1} d\frac{hf}{kT} \end{aligned}$$

Now changing variables to  $x = \frac{hf}{kT}$  gives:

$$= \frac{(kT)^2}{h} \int_{\frac{hf_1}{kT}}^{\frac{hf_2}{kT}} \frac{x}{e^x - 1} dx$$

Using the geometric series reduces this to:

$$\begin{aligned} &= \frac{(kT)^2}{h} \int_{\frac{hf_1}{kT}}^{\frac{hf_2}{kT}} x e^{-x} \sum_{n=0}^{\infty} e^{-nx} dx \\ &= \frac{(kT)^2}{h} \int_{\frac{hf_1}{kT}}^{\frac{hf_2}{kT}} x \sum_{n=1}^{\infty} e^{-nx} dx \end{aligned}$$

Say  $x_1 = \frac{hf_1}{kT}$  and  $x_2 = \frac{hf_2}{kT}$  and using integration by parts we get:

$$\begin{aligned} &= \frac{(kT)^2}{h} \left( -x \sum_{n=1}^{\infty} \frac{1}{n} e^{-nx} \Big|_{x_1}^{x_2} + \int_{x_1}^{x_2} \sum_{n=1}^{\infty} \frac{1}{n} e^{-nx} dx \right) \\ &= \frac{(kT)^2}{h} \left( \sum_{n=1}^{\infty} \frac{1}{n} (x_1 e^{-nx_1} - x_2 e^{-nx_2}) - \sum_{n=1}^{\infty} \frac{1}{n^2} e^{-nx} \Big|_{x_1}^{x_2} \right) \\ &= \frac{(kT)^2}{h} \left( \sum_{n=1}^{\infty} \frac{1}{n} \left[ e^{-nx_1} \left( x_1 + \frac{1}{n} \right) - e^{-nx_2} \left( x_2 + \frac{1}{n} \right) \right] \right) \end{aligned} \quad (3.17)$$

This series converges. This can easily be shown:

$$\int_{f_1}^{f_2} D(f)_{J-N} df = \frac{(kT)^2}{h} \left( \sum_{n=1}^{\infty} \frac{1}{n^2} e^{-nx_1} + \sum_{n=1}^{\infty} \frac{x_1}{n} e^{-nx_1} - \sum_{n=1}^{\infty} \frac{1}{n^2} e^{-nx_2} - \sum_{n=1}^{\infty} \frac{x_1}{n} e^{-nx_2} \right)$$

This converges if each of these individuals sums converges. We see:

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n^2} e^{-nx_1} &\leq \sum_{n=1}^{\infty} e^{-nx_1} \\ &= \sum_{n=1}^{\infty} (e^{-x_1})^n \\ &\Rightarrow \text{Converges if } |e^{-x_1}| < 1 \\ &\Rightarrow \text{Converges if } -x_1 < 0 \end{aligned}$$

We see that, using the comparison test, that the series  $\sum_{n=1}^{\infty} \frac{1}{n^2} e^{-nx_1}$  converges if  $x_1 > 0$  (which is always true, since  $f_1 > 0$ ). The same argument holds for  $\sum_{n=1}^{\infty} \frac{1}{n^2} e^{-nx_2}$

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{x_1}{n} e^{-nx_1} &\leq x_1 \sum_{n=1}^{\infty} e^{-nx_1} \\ &= x_1 \sum_{n=1}^{\infty} (e^{-x_1})^n \\ &\Rightarrow \text{Converges if } |e^{-x_1}| < 1 \\ &\Rightarrow \text{Converges if } -x_1 < 0 \end{aligned}$$

We see that, using the comparison test, that the series  $\sum_{n=1}^{\infty} \frac{x_1}{n} e^{-nx_1}$  converges if  $x_1 > 0$  (which is always true, since  $f_1 > 0$ ). The same argument holds for  $\sum_{n=1}^{\infty} \frac{x_2}{n} e^{-nx_2}$ . This proves identity 3.17 converges.

We have found an expression for the average power  $P$  of our photon stream:

$$P = \frac{(kT)^2}{h} \left( \sum_{n=1}^{\infty} \frac{1}{n} \left[ e^{-nx_1} \left( x_1 + \frac{1}{n} \right) - e^{-nx_2} \left( x_2 + \frac{1}{n} \right) \right] \right) \quad (3.18)$$

Now we know how much energy per second is in the photon stream. If we simulate a photon stream with a length of  $T$  seconds, we need to simulate photons that have in total approximately an energy of  $P \cdot T$ . The total energy we expect to be in the photon stream is:

$$E_{exp} = PT \quad (3.19)$$

Then, how many photons will approximately be in this photon stream? The photon energy is  $hf$ . If we take a high approximation, we look at the photon that has the least energy, that would be  $hf_{min}$ . Then, if all the photons generated would have this low energy, we would need a total number of photons of:

$$n_{gen} = \frac{PT}{hf_{min}} \quad (3.20)$$

If we let our program generate a total of  $n_{gen}$  photons, we know that this should be enough photons to fill our photon stream of  $T$  seconds. We are interested in the actual amount of photons that will be in the photon stream. For this we look at the average energy of the  $n_{gen}$  photons we have just simulated. This simply is:

$$\bar{E} = h\bar{f} \quad (3.21)$$

Here  $\bar{E}$  is the average energy and  $\bar{f}$  is the average frequency of the photons. If we multiply the average energy per photon with the amount of photons  $\bar{n}$ , we get the total energy of the photon stream in time  $T$ :

$$E_{tot} = \bar{n}\bar{E} \quad (3.22)$$

This total energy should be the same as the expected energy of 3.19. Combining 3.19 and 3.22 gives us the average amount of photons in our photon stream during of length  $T$ :

$$\bar{n} = \frac{PT}{\bar{E}} \quad (3.23)$$

### 3.2.2. PSD AND TIME DOMAIN

In signal analysis, time signals are often analyzed by calculating the so called power spectral density (PSD) of the time signal. This curve describes the power that each frequency component in the time signal contributes to the time signal.

The probability of detecting a photon is proportional to the power of the signal that arrives at the detector, as is shown in figure 3.3. This envelope can be simulated by using the PSD for light from a blackbody source, which is<sup>[6]</sup>:

$$P(f) = 2hFP_{rad}(1 + \eta_{opt}B) \frac{(dA/dP_{rad})^2}{1 + (2\pi f\tau_{qp})^2} \quad (3.24)$$

Here  $F$  is the frequency of the photons,  $P_{rad}$  is the average power of the photon stream,  $\eta_{opt}$  is the quantum efficiency of the detector,  $\frac{dA}{dP_{rad}}$  is the detector response,  $\tau_{qp}$  is the coherence time and  $B$  is the mode occupation of the photons. Because we are looking at the intrinsic behavior of the light, we can neglect the detector properties. Therefore in this situation we use:

$$\frac{dA}{dP_{rad}} = 1 \quad (3.25)$$

$$\eta_{opt} = 1 \quad (3.26)$$

Note that the frequency of the photons  $F$  is not the same as the frequency parameter of the PSD  $f$ . The  $f$  is the frequency that build up a certain time domain signal and is obtained using a Fourier transform. We can choose the range of  $F$  any way we like. For the simulation we took  $F$  to have a constant value that is the average of the bandwidth of photon frequencies that we are looking at. For this bandwidth we have already calculated the average power  $P_{rad}$ , according to formula 3.18.

For the mode occupation  $B$  holds:

$$B = \frac{1}{e^{hF/kT} - 1} \quad (3.27)$$

Here  $h$  is Planck's constant,  $k$  is the Boltzmann constant and  $T$  the temperature.

From the theoretical PSD 3.24, that is in the frequency domain, it is possible to generate an appropriate signal in the time domain. This signal represents the power of the light. The power is directly proportional to the amount of photons that is in the light. So, when such a signal has been retrieved, it can be used as an intensity function for the photons in time.

### 3.2.3. GENERATING THE ENVELOPE

How can we generate this intensity function? For this it is necessary to gain an understanding of the construction of a PSD. The PSD of a time signal can be generated in two ways, both leading to the same result. These are shown in figure 3.5.

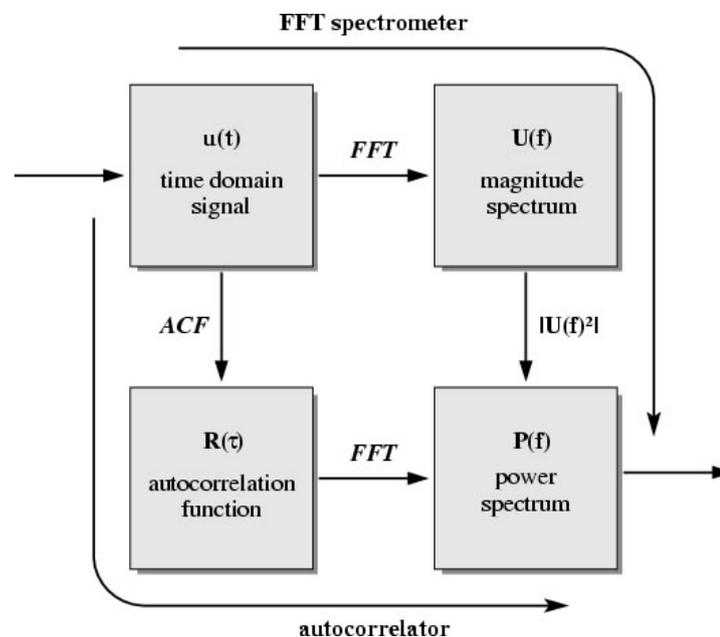


Figure 3.5: Theorem of Wiener-Khinchin<sup>[16]</sup>. There  $P(f)$  is the PSD of  $u(t)$

The PSD of a time signal  $u(t)$  can be calculated by taking the auto-correlation-function (ACF) and then taking the Fourier transform, or first taking the Fourier transform and squaring the amplitude of this Fourier

transform of  $u(t)$  [17] [18]. We will use this last method, but in reversed order, to generate a time signal  $u(t)$  that has the desired PSD.

Note that by taking the square of the amplitude of the Fourier transform, all information about the phase of the original signal is lost. This information can not be retrieved and has to be simulated. How do we do this?

The noise in light emitted from a blackbody source has a white spectrum. So, the phase is completely random. The lost phases can be simulated by taking a random phase  $\phi$  with  $0 \leq \phi < 2\pi$ . With these phases, a white noise spectrum can be created. This spectrum has to be multiplied by the square root of the PSD (which is the magnitude of the Fourier transform of  $u(t)$ ). Then, when applying the Inverse Fourier transform, a signal  $u(t)$  is created that has the desired PSD. This time signal  $u(t)$  is not unique, because of the randomness in the phase; so in every simulation  $u(t)$  will be different.

### 3.2.4. GENERATING THE PHOTON TIMES

We now have an intensity function for the photons in time:

$$\mu(t) = u(t) \quad (3.28)$$

This intensity function can be used to simulate the photon times. To do this we use an inhomogeneous Poisson Process with intensity  $\mu(t)$ . Before these times can be simulated, the parameter  $\lambda$  of the underlying Poisson Process has to be determined.  $\lambda$  is the amount of photons expected to be detected at time  $t = T$ . This is just the expected number of photons in time  $T$ ,  $\bar{n}$  as given by identity 3.23.

Generating random numbers from an inhomogeneous Poisson Process during a time  $T$  is done in two steps:

- Generate the amount of photons in time  $T$  by generating a random number from a Poisson Distribution with parameter  $\bar{n}$ . This is the amount of photons  $N$ .
- Generate  $N$  random numbers from the intensity function  $\mu(t)$ . We do this by using the method of rejection, this time with a constant dominating function that has the value of  $\max(\mu(t))$ . The time values generated in this step are the photon times we are looking for.

### 3.2.5. COHERENCE TIME

One important characteristic of the light emitted from a blackbody we want to simulate here is the bunching of the photons. As described earlier, this effect is only visible when we look at a time scale of  $\tau_{coh}$ . This means that we should take our spacing in time  $dt$  as  $dt = \tau_{coh}$ . If we choose  $dt$  this way, the photons *within* each time interval are dependent, but photons different time intervals are independent. We now have:

$$dt = \tau_{coh} \quad (3.29)$$

Here  $\tau_{coh}$  can be calculated with the bandwidth according to 3.13. Now say that we would like to simulate a photon stream with a length in time of  $T$  seconds. We know that in this case we get a time domain signal starting at time 0, with time steps of  $dt$  according to identity 3.29, going up to time  $T$ . What does this imply for the signal in the frequency domain?

In the frequency domain, the smallest frequency we can encounter is  $\frac{1}{T}$ . This is because the longest possible time we are looking at is  $T$ . Now waves with a larger period than  $T$  are not detected, so frequencies smaller than  $\frac{1}{T}$  will not be measured. So<sup>[19]</sup>:

$$f_{min} = \frac{1}{T} \quad (3.30)$$

How about the highest frequency that we will find in our signal?

The smallest time scale at which we will detect photons is  $dt = \tau_{coh}$ . So the smallest wave we can detect has a period of  $\tau_{coh}$ . This gives us a maximum frequency of:

$$f_{max} = \frac{1}{\tau_{coh}} \quad (3.31)$$

From our result for  $dt$  we get as frequency spacing:

$$df = \frac{1}{T} \tag{3.32}$$

Note that the coherence time is crucial for the simulation of the photons. At each different time  $\tau_{coh}$ , the probability of detecting a photon is not dependent on the photons detected at another time. With this choice of our spacing in time  $dt = \tau_{coh}$  we have made sure that we implement the photon bunching, because at each  $dt$  we will independently get a value for the intensity function that does not depend on another time interval  $dt$ . This way the photons are bunched within each time interval  $dt$ .

# 4

## RESULTS

### 4.1. SIMULATION OF THE PHOTON STREAM

To simulate the stream of photons we are looking for, the photon energies from the first section must be combined with the arrival times of the photons. For the photon energies, the values simulated for the one dimensional case have been used. How to simulate the arrival time can be found in the previous section. The Matlab script for the simulation of the photon stream can be found in Appendix B.4.

#### 4.1.1. DETERMINING THE INTENSITY FUNCTION

The first step in the simulation is to generate the intensity function using the power spectral density  $P(f)$  3.24. The simulation takes the square root of the power spectrum  $P(f)$  and applies the inverse Fourier transform to it. This procedure is shown schematically in figure 3.5. The result of this is the intensity function as shown in figure 4.1.

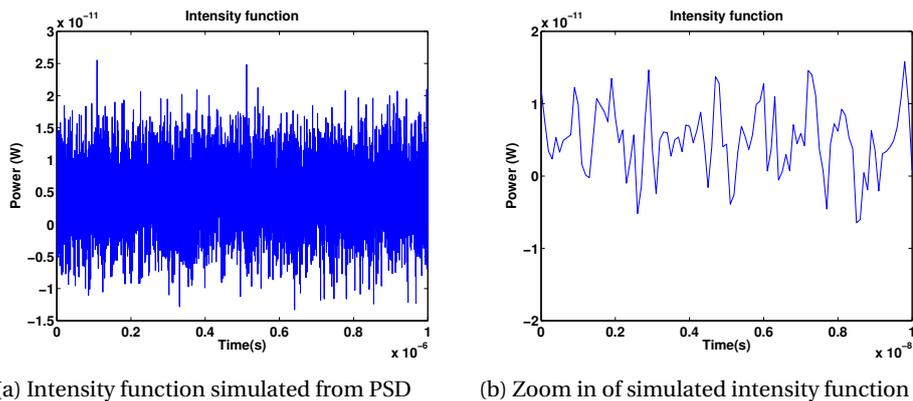


Figure 4.1: Intensity function simulated from theoretical PSD for a time  $T = 10^{-6}$  seconds

To check if this intensity function has the desired properties, the power spectral density of the simulated intensity function has been calculated. The PSD should be equal to the theoretical curve 3.24. This procedure has been repeated for 100 times. The average of the resulting power spectral densities has been plot along with the theoretical PSD in figure 4.2.

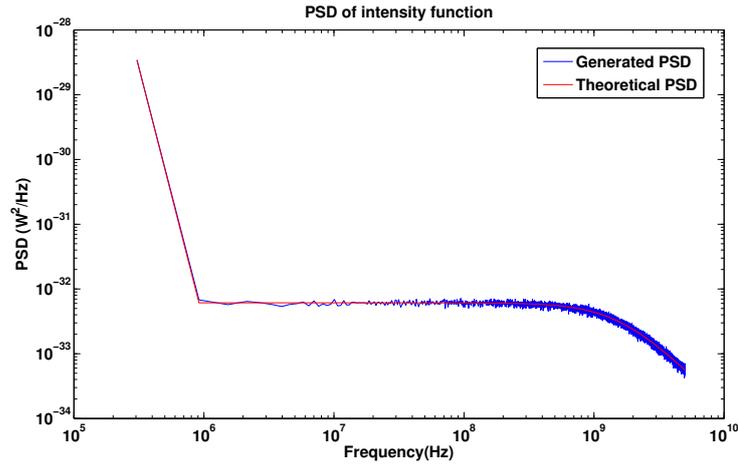


Figure 4.2: Average of 100 PSD's calculated from generated intensity functions

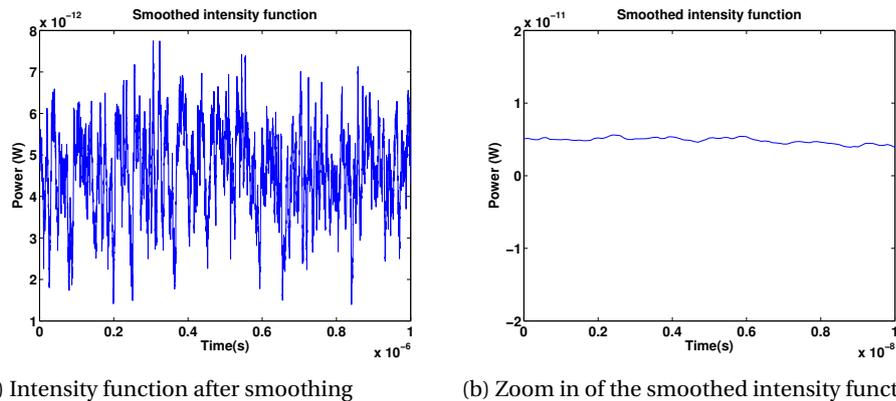
We see that the power spectral density of the generated time series represents the theoretical spectrum 3.24.

#### 4.1.2. GETTING A SMOOTHED INTENSITY FUNCTION

If we look at the intensity function that has been produced in time domain, we see that it is not positive at all times. The negative power does not have a physical meaning and is the result of fluctuations in the signal. Because these fluctuations are larger than the average of the intensity function as given by 3.18, the intensity function will not always be larger than zero. In order to use this intensity function in an in-homogeneous Poisson process, it has to be positive everywhere.

To solve this problem two methods have been tried. The first approach is to remove the negative part of the intensity function completely, and make it have a value of zero at all times the intensity function drops below zero. The result of this approach is not as good as desired. This method is discussed in Appendix A.

Another method to make sure the intensity function does not have any negative values is smoothing the intensity function. If the intensity function is smoothed enough the high frequency fluctuations will be removed and the function will be positive everywhere. To smooth the intensity function a moving average with a span of 50 data points has been used. The resulting smoothed intensity function is shown in figure 4.3.



(a) Intensity function after smoothing

(b) Zoom in of the smoothed intensity function

Figure 4.3: Smoothed intensity function for a total time of  $T = 10^{-6}$  seconds

A total of 100 of these smoothed intensity functions have been created. From each, also the PSD has been determined. The average of these PSD's is plotted in figure 4.4

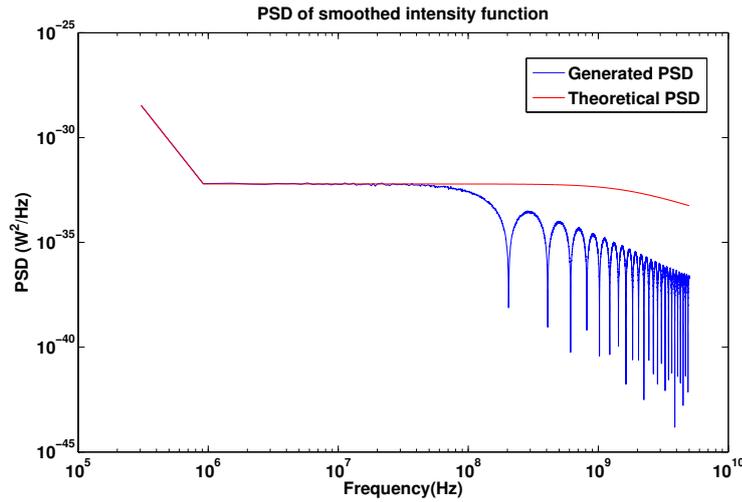


Figure 4.4: Average of 100 PSD's of smoothed intensity functions

#### 4.1.3. SIMULATING THE PHOTON STREAM

We take this smoothed intensity function as our  $\mu(t)$  of identity 3.28. Now we have all the tools to generate the arrival times of the photons. After assigning each photon randomly with an energy that we generated earlier we get the following result:

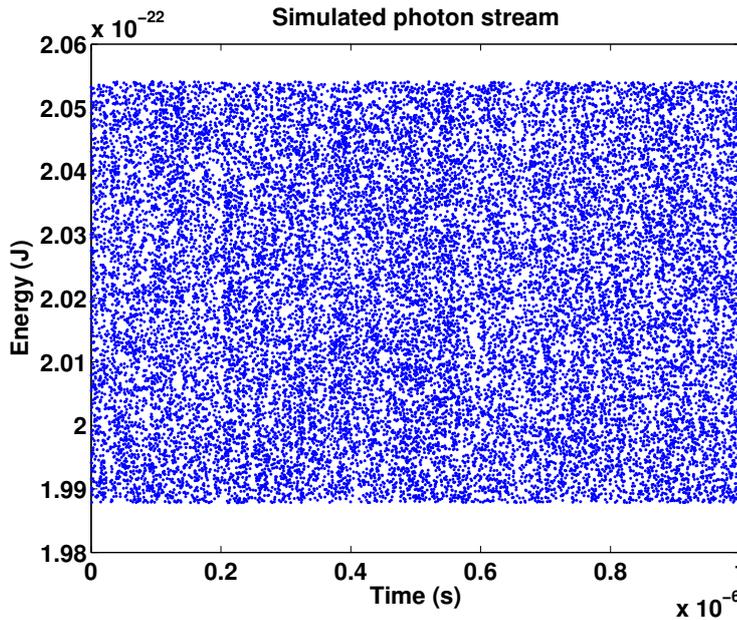


Figure 4.5: Simulated stream of photons. Each dot represents one photon with an energy and an arrival time.

From this it is possible to calculate the power of the signal that we would measure in the detector. Because the intensity function has been smoothed, the resolution has decreased. In the program, the spacing in time is still  $dt = \tau_{coh}$ , but because we used a moving average filter with a span of 50, the actual time sampling now is  $50 \tau_{coh}$ . By averaging the intensity function over a span of 50 data points, all photons within this 50 data points should be in the same time sample. The resulting stream of photons has to be compensated for this. If we want to calculate the power of the photon stream that we would measure with a detector, it is necessary to smooth this power signal with the same factor, in this case 50. After smoothing the result should be comparable to the signal in figure 4.3.

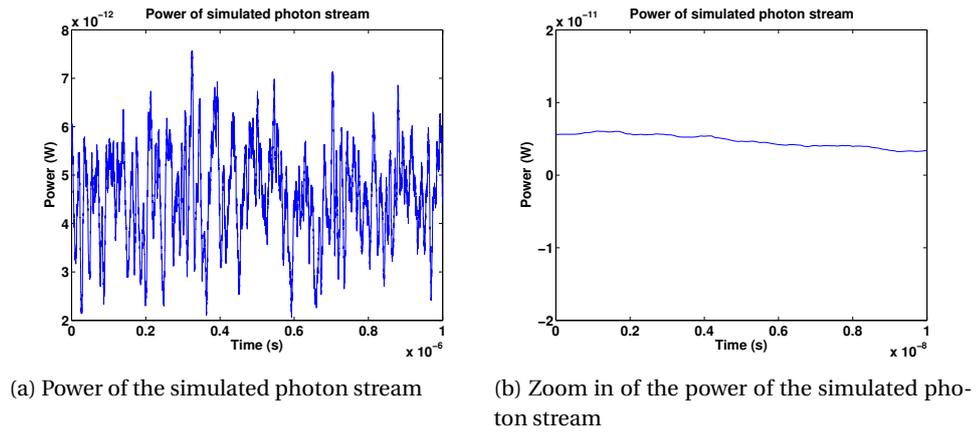


Figure 4.6: Power of the simulated stream of photons with a length of  $T = 10^{-6}$  seconds

Repeating this 100 times and taking the average of the 100 corresponding PSD's results in figure 4.7.

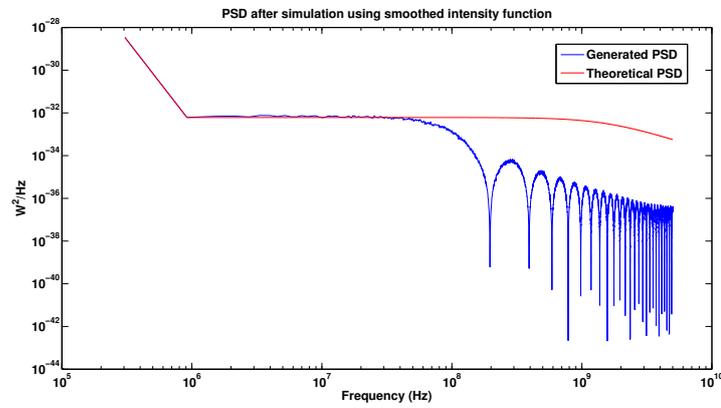


Figure 4.7: Average of 100 PSD's of the power of the simulated stream of photons

As expected is this PSD similar to the PSD of the smoothed intensity function that was used for the simulation.

## 4.2. DISCUSSION

Now we have simulated a virtual stream of photons that can be used as input for a model for detectors such as the MKID. Even though the result does not completely match the theoretical expectations, we will see that it is sufficient for the use as input for a model of light detectors.

### 4.2.1. RELEVANCE OF DETECTOR SAMPLING

In the simulation, a smoothed version of the intensity function has been used to simulate the photon stream. Because of this, high frequency components in the power signal will be filtered out. This can be seen in the PSD of the smoothed intensity function, figure 4.4. In the PSD of the resulting power signal after the simulation, the same thing can be seen. Starting from approximately  $10^{7.5}$  Hz, the contribution to the power does not match the theoretical expectation. For the further use of the photon stream this is not a problem, because the detection rate at which the detector detects the signal is at most 10 kHz, in practice even smaller. Therefore, it can not measure signals faster than 10 kHz. This is much smaller than  $10^{7.5}$  Hz. So, in practice the frequency components at which the model is inaccurate are not measured.

### 4.2.2. VISIBILITY OF PHOTON BUNCHING

By smoothing the intensity function, the sample time of the generated photon stream has increased by a factor 50. The sample time now is  $50\tau_{coh}$ , instead of  $\tau_{coh}$ . This means that photon bunching is no longer visible. So, formula 3.15 for the probability of the amount of photons per time  $\tau_{coh}$  is no longer applicable. If we, without taking this in mind, look at the probability  $P(n)$  of detecting  $n$  photons in time interval  $\tau_{coh}$ , we can see that this indeed does not match the theoretical curve of 3.15. This is shown in figure 4.8

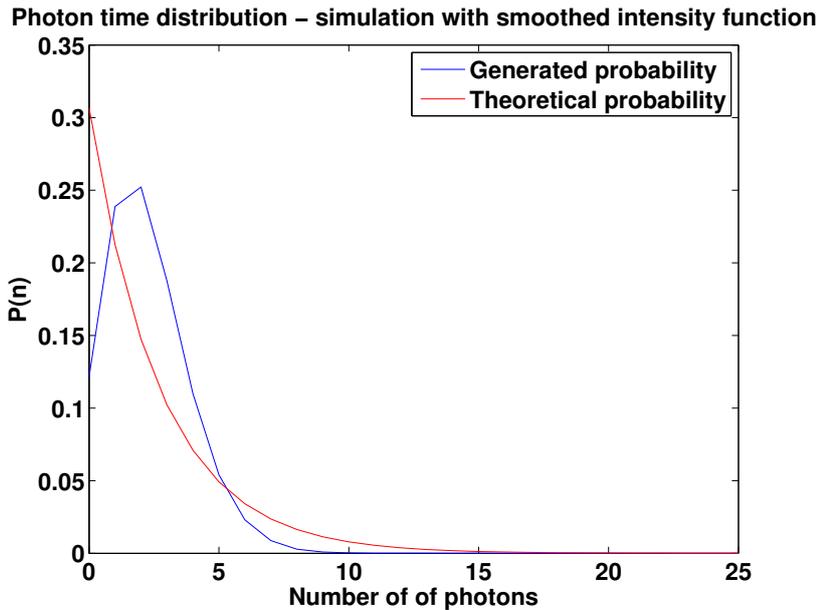


Figure 4.8: Photon distribution in time after simulation using the smoothed intensity function

This is a disadvantage of the use of the smoothed intensity function. Next we will discuss another method that could solve this problem.

### 4.2.3. IMPROVING THE SIMULATION

The photon stream simulated by this program is very spot on when it comes to the energy of the photons and the time signal in the low frequency range, but is not perfect for the large frequency components. This is the result of smoothing the intensity function to make sure it is positive at all times. By doing so, the high frequency fluctuations are neglected, and the time sampling rate has decreased. Not smoothing the intensity function would make sure the sample times stays  $\tau_{coh}$ . To do this, the negative power values have to be used in the simulation.

This could be done by introducing photons with "negative energy". It is as if the photons with a negative energy are emitted from the detector, rather than arriving at the detector. Implementing this in the simulation can be done by not only simulating photons with a positive energy when the intensity function is positive, but also simulating photons with a negative value whenever the intensity function drops below zero.

To do this we can use the same procedure, but we use the absolute value of the intensity function to simulate photon times. Whenever a photon time is simulated that is in the originally negative part of the intensity function, the energy assigned to the photon will be multiplied by  $-1$ . To make sure the average energy in the photon stream stays the same, more photons have to be simulated. However, we can still use the inhomogeneous Poisson process as described in the previous chapter, but with a little variation.

Say we want to simulate a photon stream of length in time of  $T$  seconds. After generating a Poisson value  $\bar{n}$  for the amount of photons in the photon stream we can calculate the approximate total energy in the photon stream using a rewritten version of formula 3.23:

$$\bar{E} = \frac{PT}{\bar{n}} \quad (4.1)$$

Here  $P$  is the power in the frequency spectrum we are interested in, according to identity 3.18.

Now we have to make the program simulate photons (negative and positive), until the sum of the energies is equal to or larger than  $\bar{E}$ . When the power of the resulting photon stream is calculated, it should represent the intensity function as plotted in figure 4.1.

Note that with this approach, the simulated photon stream will also contain negative photons. This is not physical, but it is a way to include the noise that is caused by photon bunching.

# 5

## CONCLUSION

In this bachelor project a virtual stream of photons radiated from a black body source has been simulated. This photon stream can be used as a start to modeling several detectors, such as the MKID. The temperature of the black body can be adjusted in the program, just like the bandwidth of the photons in the stream and the length in time of the photon stream. The simulated stream of photons has the following properties:

1. The distribution of the frequency of the photons is according to Bose-Einstein statistics.
2. The average power in the photon stream is according to the Johnson-Nyquist power density.
3. The power of the simulated stream of photons has a power spectral density that matches the theoretical expectation for frequency components that are below  $10^{7.5}$  Hz. Higher frequency components are not well represented in the power signal. Because the detector speed is limited, this result is sufficient.

This result matches the theoretical requirements up to an extend that it can be used as input for the model of light detectors.

The result could however be improved by introducing the concept of "negative photons". By doing so the photon bunching in the photon stream will be visible.



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# A

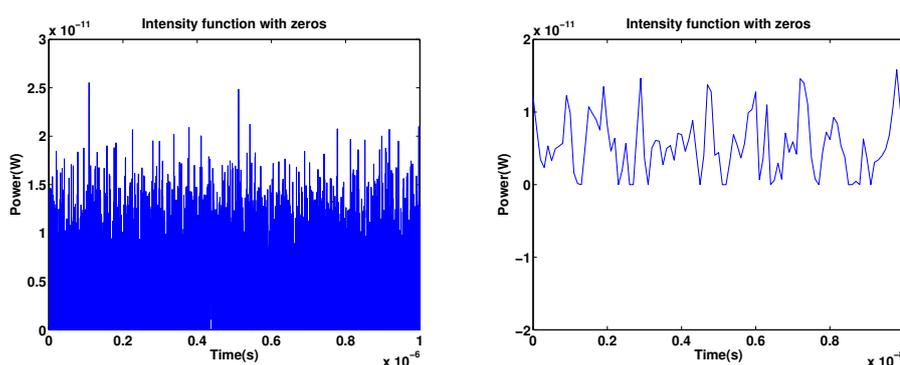
## APPENDIX - DIFFERENT APPROACH

### ADDING ZEROS TO GET A USABLE INTENSITY FUNCTION

The problem of the intensity function in figure 4.1 not being positive at all times can be solved in multiple ways. In the simulation, the intensity function has been smoothed enough so it would be larger than zero at all times. In this section another method to make sure the intensity function is positive will be discussed.

#### A.1. REPLACING THE NEGATIVE PART OF THE INTENSITY FUNCTION

When the intensity function has a negative power, one could reason that it would be as if photons are emitted by the detector instead of being detected. So, when the power is negative in the intensity function, we can be sure that at least no photons will be detected. An approach to the negative intensity function problem is therefore to remove the negative part of the intensity function completely, and make it have a value of zero at all times the intensity function drops below zero. This results in a usable intensity function shown in figure A.1.



(a) Intensity function with zeros simulated from PSD (b) Zoom in of simulated intensity function with zeros

Figure A.1: Intensity function that is always positive, simulated from theoretical PSD for a time  $T = 10^{-6}$  seconds

Repeating this and taking the average of 100 resulting PSD's results in figure A.2

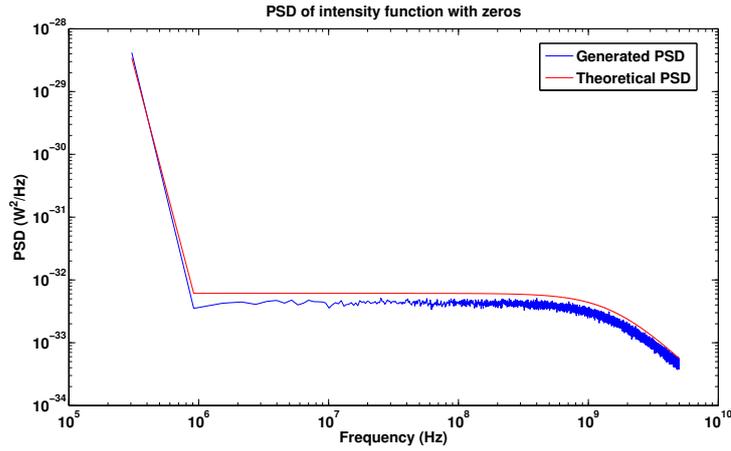
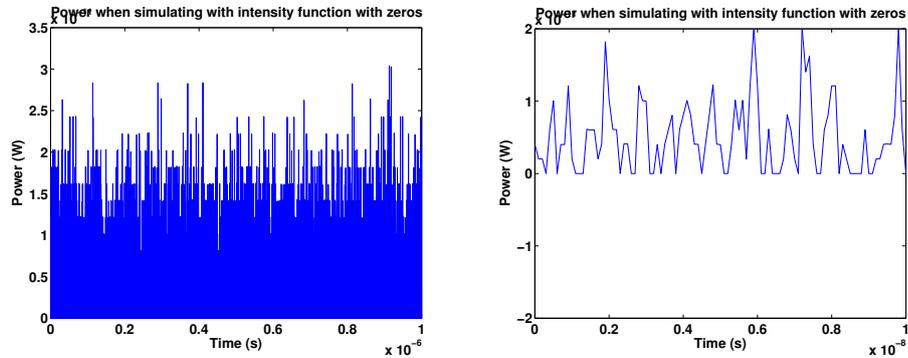


Figure A.2: Average of 100 PSD's calculated from usable generated intensity functions

We can see that the PSD of this usable intensity function has almost the same shape as the theoretical PSD, but it is lower than the theoretical curve. Except from the DC component, the DC component has the right value. This is because the fluctuations of the negative part of the PSD have been ignored. Because a part of the fluctuations have been ignored, the power in the fluctuations of the simulated signal will be lower. Therefore only the DC component has the right value.



(a) Power of the simulated photon stream

(b) Zoom in of the power of the simulated photon stream

Figure A.3: Power of the simulated stream of photons a time of  $T = 10^{-6}$  seconds

From this result we now want to find the power spectral density to look into the characterizations of the created stream of photons. The resulting plot showing the PSD is shown in figure A.4. This is again an average over 100 simulations.

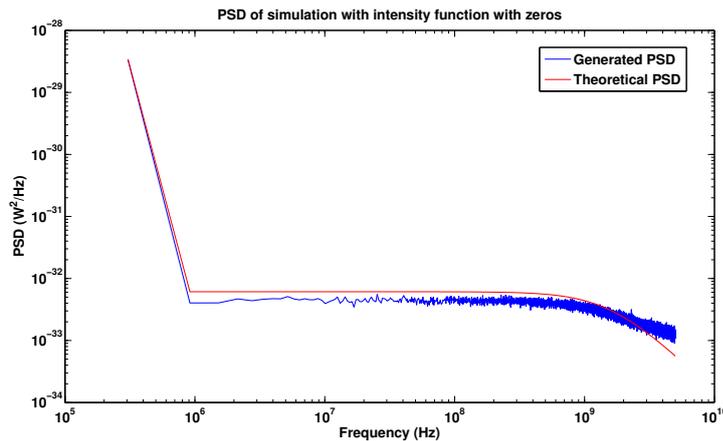
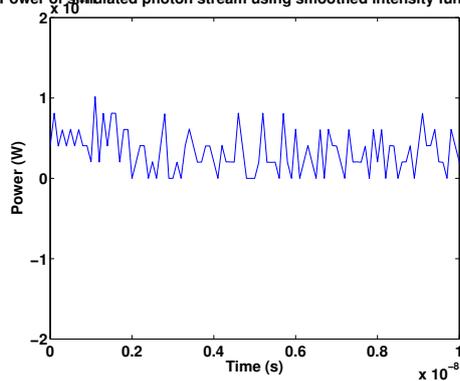


Figure A.4: Average of 100 PSD's from simulated photon streams

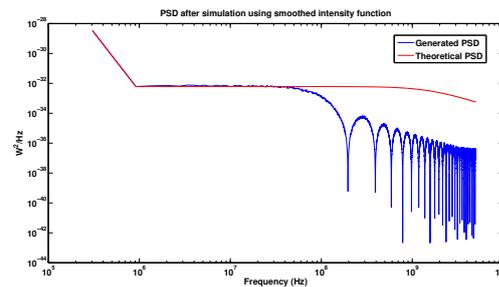
### A.1.1. MISMATCH OF THE PSD

We have seen that the PSD of the power of the simulated photon stream when using the intensity function with zero's does not completely match the theoretical curve 3.24. This is the result of neglecting the negative values in the intensity function. We can compare the result of the simulation with the intensity function with zero's with the method at which the intensity function is smoothed. Both results are plotted in figure A.5

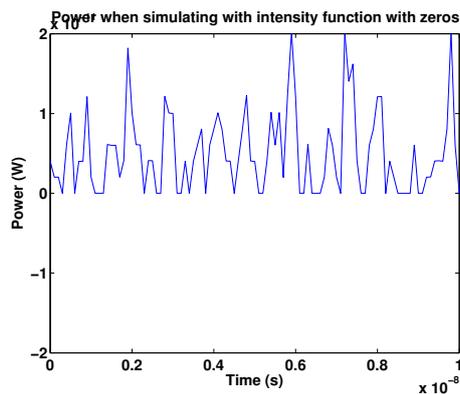
Power of simulated photon stream using smoothed intensity function



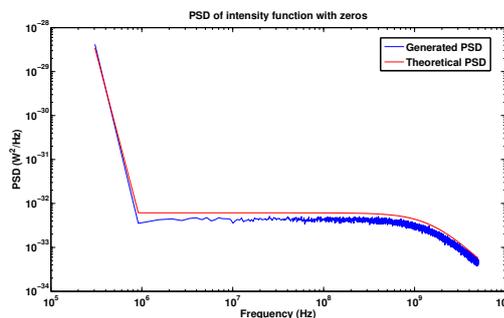
(a) Power of the simulated photon stream using the smoothed intensity function



(b) PSD of the simulated photon stream using the smoothed intensity function



(c) Power of the simulated photon stream using the intensity function with zero's



(d) PSD of simulated photon stream using the intensity function with zero's

Figure A.5: Two possible modifications of the intensity function.

We can see that if we are interested in frequency components larger than approximately  $10^{7.5}$ , the method that implements zeros gives a better result. But, as we are planning to do, we are looking only at frequency components smaller than  $10^{7.5}$ , it is better to use the smoothed intensity function.

## A.2. DISTRIBUTION OF AMOUNT OF PHOTONS

An advantage of the method of implementing zeros at negative values of the intensity function is that the sampling in time is still  $\tau_{coh}$ . So, we should be able to observe the photon bunching effect. To check whether this is the case, we use equation 3.15. The average amount of photons in this time stream  $\bar{n}$  has been calculated with identity 3.23. Also we know the amount of photons in each time slot  $dt = \tau_{coh}$ . Now we can see if the probability for the amount of photons in  $\tau_{coh}$  is what we would expect from equation 3.15. Plotting them both results in figure A.6

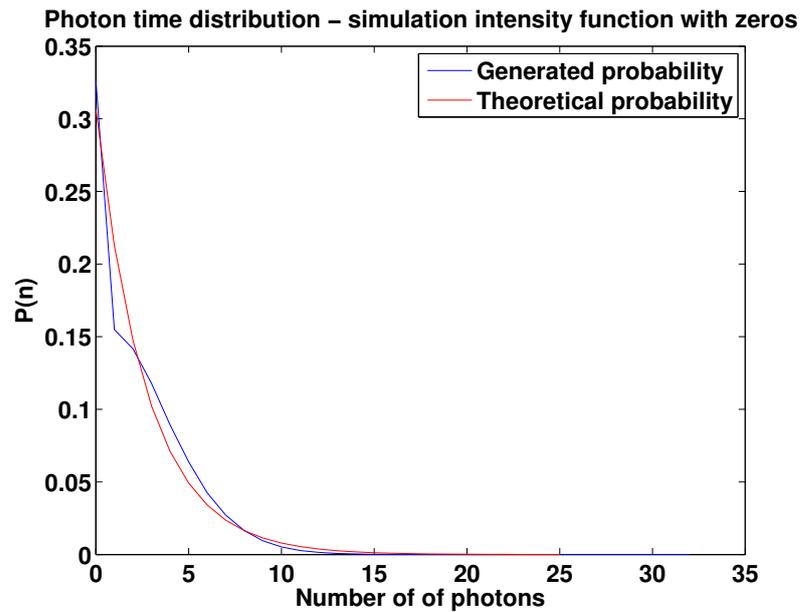


Figure A.6: Time distribution of the amount of photons, average of 100 simulations and the theoretical curve

The two curves do not match completely. Especially at  $n = 2$ , here the value of the simulation is lower than the theoretical probability. This is most likely because the fluctuations in the negative part of the intensity function are ignored. A possible solution to this problem has been presented in section 4.2.3.

# B

## APPENDIX - MATLAB SCRIPTS

### **B.1.** MATLAB SCRIPT FOR GENERATING FREQUENCIES FROM PLANCK'S FORMULA

## **B.2. MATLAB SCRIPT FOR NUMERICAL GENERATION FROM JOHNSON-NYQUIST FORMULA**

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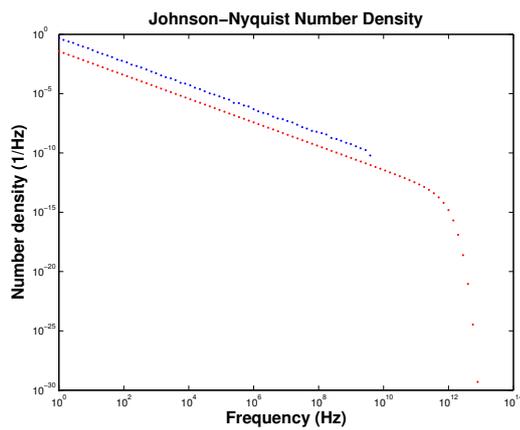
### **B.3. MATLAB SCRIPT FOR ANALYTICAL GENERATION FROM JOHNSON-NYQUIST FORMULA**

#### **B.4. MATLAB SCRIPT FOR SIMULATING THE FINAL PHOTON STREAM**

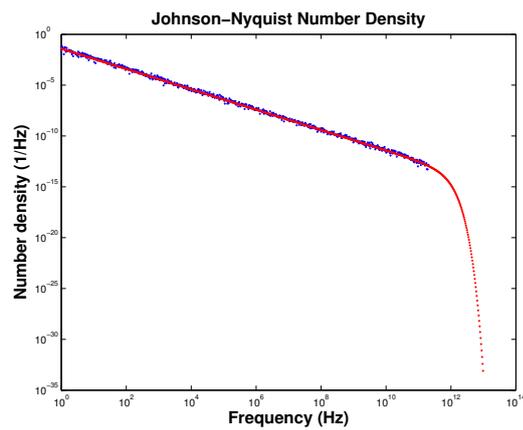
# C

## APPENDIX - FIGURES

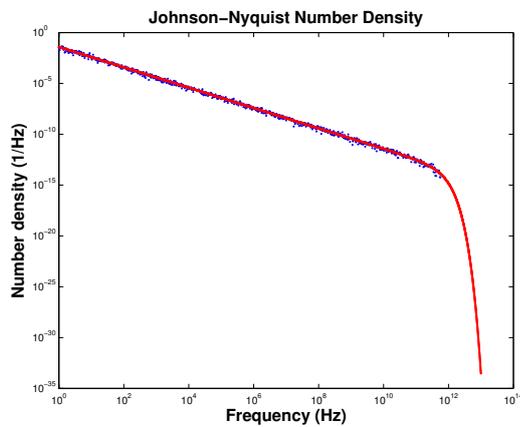
### C.1. NUMERICAL GENERATION FROM JOHNSON NYQUIST



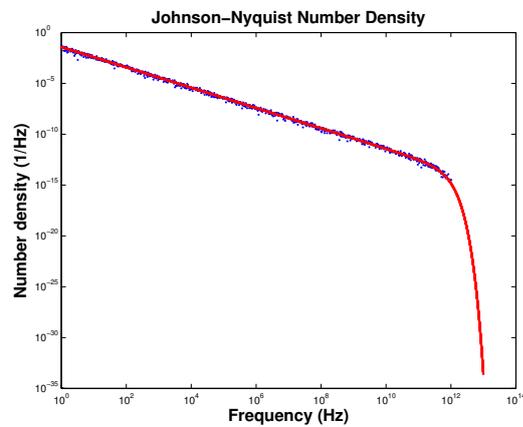
(a)  $M = 100$



(b)  $M = 1000$



(c)  $M = 10000$



(d)  $M = 100000$

Figure C.1: Numerical generation of the Johnson-Nyquist number density, with different amount of frequency spacing  $M$



## C.2. GENERATION OF PHOTONS FROM THE PLANCK CURVE

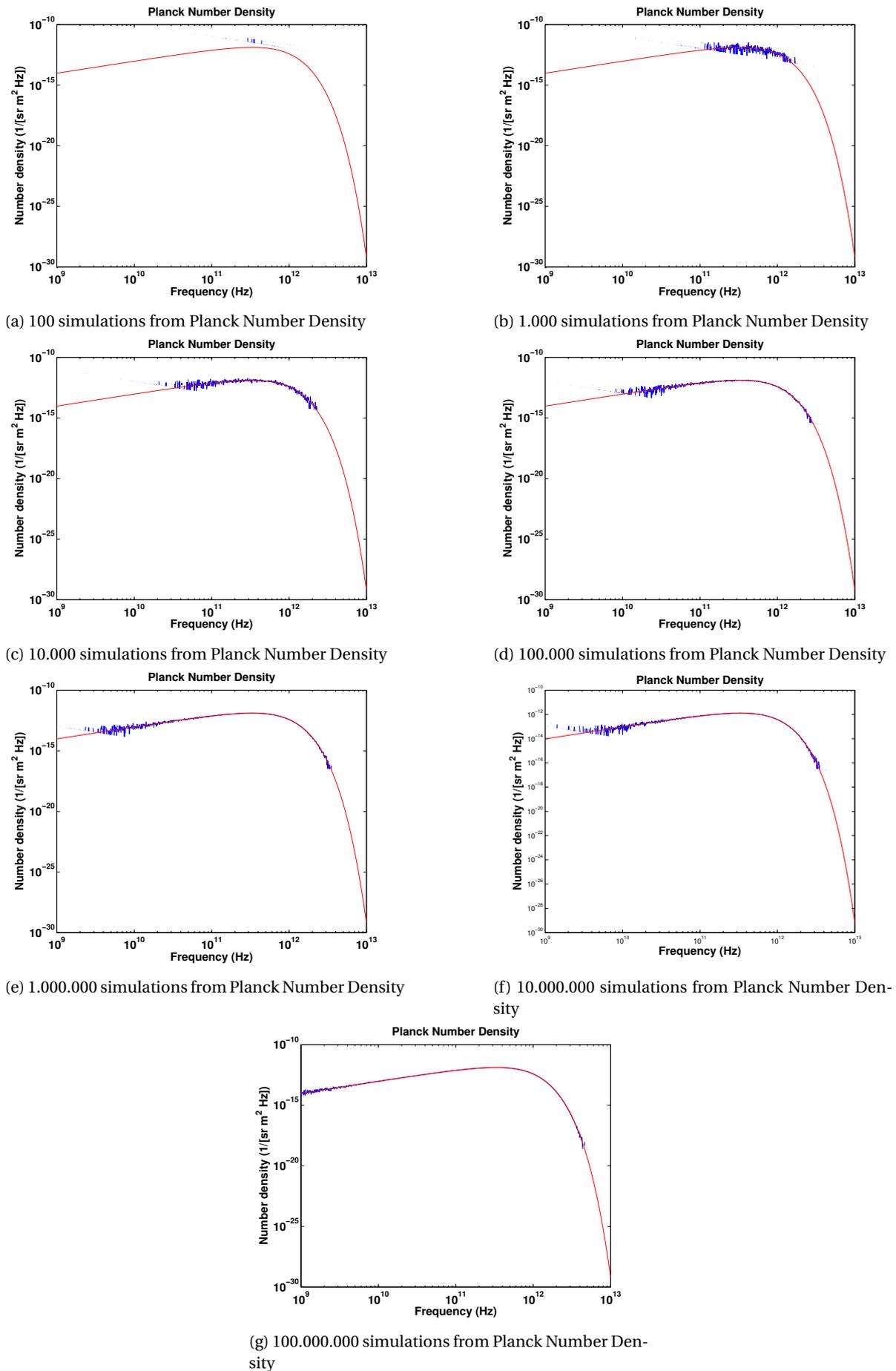


Figure C.2: Planck number density retrieved from simulations, for different amount of simulations