GEOMETRICAL CONSIDERATIONS ON SPACE KINEMATICS IN CONNECTION WITH BENNETT'S MECHANISM

PROEFSCHRIFT

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INTRODUCTION

In Engineering **4**, 777 (1903) G.T.Bennett published a paper on a mechanism, which he called the skew isogram. It is a quadrilateral ABA'B' of which the sides are rods, hinged in its vertices. He proved that this skew quadrilateral is movable if the opposite sides are equal. Further, he described several technical applications. In 1914 he published in J.London Math.Soc. **13**, 151 (1914) a series of theorems associated with the isogram, considered as a pure mathematical subject. Bennett's papers contain various theorems of which no proofs were provided. One of the aims of this thesis is to give a summary of these theorems and to supply the missing proofs.

If one of the rods, AB say, of the isogram with its two hinge-lines is fixed, it is possible to determine a moving space in which the rod opposite to AB, called the connecting rod, together with its two hinge-lines are fixed lines. Another aim is to examine the motion of this moving space. The general theory of the moving space given in chapter I as far as needed in the following chapters, is taken from Schoenflies, Geometrie der Bewegung. To make chapter I selfcontained, it was necessary to give proofs of several theorems, which are different from those of Schoenflies, especially the one of theorem IV, which is given by means of analytical geometry.

Investigating the moving space we have made use of the method of the axial reflection. The theorems which we need are given in chapter II. They are taken from two papers of J.Krames: Zur Geometrie des Bennett'schen Mechanismus (Wiener Sitz. Ber. IIa, **146**, 159 (1937); Symmetrische Schrotungen I (Monatsh. Math. Phys. **45**, 394 (1937). For the same reason as in chapter I the proofs are not the same as those given by Krames.

Chapter III contains a selection of theorems of the isogram given by Bennett. It appears possible to introduce various quadrics which are connected with the sides, hinge-lines and angle-bisectors of the quadrilateral. Further, this chapter gives necessary and sufficient conditions for the quadrilateral to be movable. In general a skew quadrilateral hinged in its vertices is triply stiff, but if the opposite sides are equal it is movable. After giving the definition of the twist of a link, we determine a relation between the twists and the lengths of the sides.

In chapter IV we consider the moving space in which the rod A'B' together with its hinge-lines are fixed lines. Every point of the moving space describes, in general, a rational space curve of the fourth degree. The parametric equations of these curves are deduced. One of the most important quantities of the moving isogram appears to be the quantity denoted by m, which only depends on the ratio of the unequal sides and on the twist of the fixed rod. This quantity m is the constant ratio of the sine of half the sum of an angle and the supplement of the adjacent angle to the sine of half the difference of these angles of the isogram.

In § 6 we prove that the space curves have four isotropic points and it is further shown that no spherical curves occur among them. In § 8 is deduced the equation of the surface of the third degree which is the locus of the points with an osculating plane with a fourth-order contact (these points are called the points of inflection or the stationary points). In § 9 are deduced the equations of the locus of the points which have a tangent with a second-order contact. This locus is, for any position of the quadrilateral, a twisted cubic.

In chapter V the theorems of chapters I and II are applied. It is shown that the hinge-lines are two by two conjugated lines, which means that the planes through the points of one of the hinge-lines normal to the tangent at these points go through another hinge-line, called its conjugated line. Therefore many of the theorems of chapter I are immediately applicable to the moving space. As the isogram has an axis of symmetry, namely the line connecting the midpoints of the diagonals AA' and BB', and as this axis describes a ruled surface during the motion of the isogram, the moving space can be considered as the reflected fixed space with regard to the generators of the ruled surface. Several loci in the moving space are given in their reflected position. In this way the locus of the points with a tangent through a given point is found. Furthermore, the equations of the instantaneous screw-axis are given, following from the theory of chapter I.

In chapter VI we consider the surface generated by the connecting rod A'B'. This surface of the fourth degree has in general two double-lines, which intersect the line of the fixed link at a right angle. Conditions are given that the double-lines be real. Further, we deduce the effect of the values m_1 and m_2 of m on the quadrilateral being crossed or not-crossed, where crossed means that the rotations of the links around the fixed hinge-lines are in opposite directions.

Chapter I

THE MOTION OF A RIGID SPACE *)

§ 1. Displacement of a line

1. We consider the motion of a rigid space S in a fixed space Σ. If the position of three points of S, which do not lie on one straight line is given, the position of each point of S is determined. Let two positions of a point P of S be denoted by P_o and P_1. The line P_oP_1 is called the chord P_oP_1 or *the chord of P.* The midpoint of P_oP_1 is denoted by P_m and the plane through P_m and normal to the chord P_oP_1 by Π^P. This plane Π^P is called the *bisecting plane* of P_oP_1 or the bisecting plane belonging to P. Analogously we get: The points A, B, etc. give the chords A_oA_1, B_oB_1, etc., the midpoints A_m, B_m, etc. and the bisecting planes α^P, β^P, etc.

2. Two positions of a line l of S are denoted by l_o and l_1 (fig. 1). A series of points on l_o corresponds to a congruent series of points on l_1. Let P and A be two points of l. P_o and A_o lie on l_o and P_1 and A_1 on l_1. The midpoints of the chords P_oP_1 and A_oA_1 are P_m and A_m respectively.

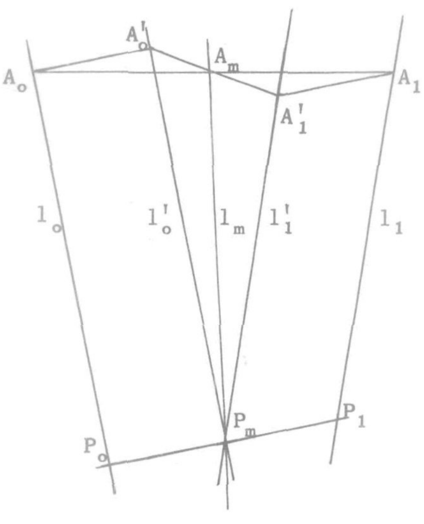

Figure 1

We draw the lines l_o' and l_1' through P_m parallel to l_o and l_1 respectively and through the points A_o and A_1 the lines A_oA_o' and A_1A_1' parallel to P_oP_1. As A_oA_o' and A_1A_1' are equal and parallel, the quadrilateral $A_oA_o'A_1A_1'$ is a parallelogram and consequently its diagonals A_oA_1 and $A_o'A_1'$ meet each other in the midpoint A_m of the chord A_oA_1. As $P_mA_o' = P_oA_o = P_1A_1 = P_mA_1'$ the triangle $A_o'P_mA_1'$ is isosceles and as $A_o'A_m = A_1'A_m$ the line P_mA_m is the angle bisector of the angle between l_o' and l_1'. The posi-

*) Schoenflies [3].
Note: The number between the signs [ ] denotes the number of the paper given in the list of literature.

tion of this bisector is independent of the position of A_o and A_1 on l_o and l_1 and now we obtain

Theorem I: *The locus of the midpoints of the chords of the corresponding points of l_o and l_1 is a line.*

This line is called the *middle-line* of l_o and l_1 and is denoted by l_m.

3. We obtain a special case if l_m is perpendicular to P_oP_1. As P_oP_1 is parallel to A_oA_1' we get that l_m is perpendicular to A_oA_1' and as l_m is perpendicular to $A_o'A_1'$, l_m is perpendicular to the plane through $A_oA_o'A_1A_1'$ and consequently l_m is perpendicular to A_oA_1. So we obtain

Theorem II: *If one of the chords connecting the corresponding points of the lines l_o and l_1 is perpendicular to the middle-line l_m, then all chords are perpendicular to l_m.*

4. If the lines l_o and l_1 approach each other we get in the limit that the line A_oA_1 through the two positions A_o and A_1 of any point A of l becomes the *tangent* at the point A to the curve described by A. The bisecting plane α^p of the chord A_oA_1 becomes the plane through A normal to the curve described by A. We obtain

Theorem III: *If the tangent at any point of a line l to the curve described by this point is perpendicular to l, the tangent at each point of l is perpendicular to the curve described by this point.*

We notice that the bisecting planes of the points of l in their limiting position go in this case through l.

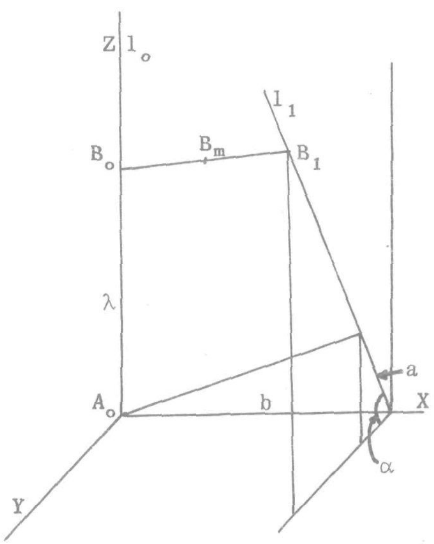

Figure 2

5. **Theorem IV**: The bisecting planes of the chords of the corresponding points of l_o and l_1 go through one line.

This line, denoted by l^p, is called the line conjugated to l_o and l_1, or briefly *conjugated to l.*

The proof of this theorem will be given by means of analytical geometry (fig. 2).

We let, without loss of generality, l_o coincide with the Z-axis and we take as the equations of l_1:

$x = b$ and $z = y \tan \alpha$.

If $A_o(0;0;0)$ and $A_1(b_1;\ a \cos \alpha;\ a \sin \alpha)$ are two corresponding points, the points B_o and

B_1 on l_o and l_1 respectively are also corresponding points if $A_oB_o = A_1B_1$. If this distance A_oB_o is denoted by λ, the coordinates of B_o and B_1 are:

$$B_o(o; o; \lambda)$$

$$B_o\{b;\ (a+\lambda)\cos\alpha;\ (a+\lambda)\sin\alpha\}$$

The coordinates of the midpoint B_m of B_oB_1 are:

$$B_m\{\tfrac{1}{2}b;\ \frac{a+\lambda}{2}\cos\alpha;\ \frac{\lambda+(a+\lambda)\sin\alpha}{2}\}.$$

The equation of the bisecting plane β^p of the chord B_oB_1 is:

$$bx+(a+\lambda)\cos\alpha.\, y+\{(a+\lambda)\sin\alpha-\lambda\}z-\tfrac{1}{2}b^2-\tfrac{1}{2}a^2-a\lambda = 0.$$

This equation represents a pencil of planes and consequently these planes go through one line l^p, the line conjugated to l.

6. If the chords are perpendicular to the middle-line l_m, the bisecting planes of the chords go through l_m and therefore the lines l_m and l^p coincide in this case. If l_1 approaches l_o we get in the limit that l_m coincides with l.

If l_m coincides with l^p we get in the limit that l coincides with l^p. If l is identically equal to its conjugated line l^p, l is called a *self-conjugated line*.

From theorem III we draw the conclusion that if the tangent at any point of a line l to the curve described by this point is perpendicular to l, the line l is a self-conjugated line.

7. The equation of the pencil of planes can be written as:

$$(bx+ay\cos\alpha + az\sin\alpha - \tfrac{1}{2}a^2-\tfrac{1}{2}b^2) + \lambda\{y\cos\alpha + z(\sin\alpha-1)-a\} = 0.$$

The planes given by $\lambda=0$ and by $\lambda=\infty$ are parallel if:

$$b:0 = a\cos\alpha : \cos\alpha = a\sin\alpha : (\sin\alpha-1).$$

These conditions are fulfilled in the following cases:

1^0	$a = 0$;	$b = 0$;	$\alpha \neq \pi/2$
2^0	$a = 0$;	$b \neq 0$;	$\alpha = \pi/2$
3^0	$a \neq 0$;	$b \neq 0$;	$\alpha = \pi/2$.

In the first case the equation of the pencil becomes:

$$\lambda\{y\cos\alpha + z(\sin\alpha-1)\} = 0,$$

that is, the pencil is degenerated into one plane and consequently l^p is not determined.

The second case gives:

$$bx - \tfrac{1}{2}b^2 = 0,$$

that is, again l^p is not determined.

The third case gives:

$$bx + az - \tfrac{1}{2}a^2 - \tfrac{1}{2}b^2 - \lambda a = 0$$

that is, the pencil is degenerated into a series of parallel planes and consequently l^p is a line at infinity.

We obtain:

1^0 If l_o and l_1 intersect each other and their common point is a self-corresponding point of l_o and l_1, the line l^p conjugated to l_o and l_1 is not determined.

2° If l_o and l_1 are parallel and the chords are perpendicular to l_o (or l_1) the line l^p is not determined.

3° *If l_o and l_1 are parallel and the chords are not perpendicular to l_o (or l_1) l^p is a line at infinity.*

§ 2. Displacement of a plane

1. We consider two positions ε_o and ε_1 of the plane ε of the moving space S. Each point of ε_o corresponds to one point of ε_1. Let A_o be a point in ε_o and A_1 its corresponding point in ε_1. The midpoint of the chord A_oA_1 is A_m. Each line l_o in ε_o through A_o corresponds to a line l_1 in ε_1 through A_1 and one line l_m through A_m corresponds to l_o and l_1. Any line m_o of ε_o not through A_o intersects all lines l_o. Hence the middle-line m_m of m_o and m_1 intersects all lines l_m through A_m. Consequently all lines l_m lie in one plane called the *middle-plane* ε_m of the planes ε_o and ε_1. We obtain

Theorem V: *The locus of the midpoints of the chords connecting the corresponding points of ε_o and ε_1 is a plane ε_m* called the middle-plane belonging to the two positions ε_o and ε_1 of a plane ε.

2. Let ε_o and ε_1 be two positions of ε and ε_m its middle-plane. If A_oA_1 and B_oB_1 are two chords connecting two pairs of corresponding points of ε_o and ε_1, their midpoints A_m and B_m lie in ε_m. We draw the line a_m through A_m in the plane ε_m perpendicular to the chord A_oA_1 and the line b_m through B_m in ε_m perpendicular to the chord B_oB_1. The common point of a_m and b_m is denoted by E_m. As a_m and b_m can be considered as middle-lines of two pairs of lines a_o, a_1 and b_o, b_1 of the planes ε_o and ε_1, the point E_m is the midpoint of the chord E_oE_1 where E_o is the common point of a_o and b_o and E_1 that of a_1 and b_1. The chord A_oA_1 is perpendicular to a_m and therefore E_oE_1 is also perpendicular to a_m (theorem II). Similarly E_oE_1 is perpendicular to b_m and hence E_oE_1 is perpendicular to ε_m. The point E_m of the middle-plane ε_m with the property that its corresponding chord E_oE_1 is normal to ε_m is called the *nullpoint of* ε_m.

3. If ε_1 approaches ε_o we get in the limit that the lines which go through corresponding points of ε_o and ε_1 become tangents to the curves described by these points. If t is the tangent at P to the curve described by P we write briefly: *t is the tangent at P.*

Let ε be the limiting position of ε_1, A and B two points in ε,

t_1 and t_2 the tangents at A and B respectively (fig. 3). We draw the line a in ε through A perpendicular to t_1 and the line b in ε through B perpendicular to t_2. As a and b are self-conjugated lines (§ 1.6), the tangents in each point of a and b are perpendicular to these lines. If, in general, a and b meet each other in E, the tangent t_3 at E is perpendicular to a and to b and hence t_3 is normal to ε.

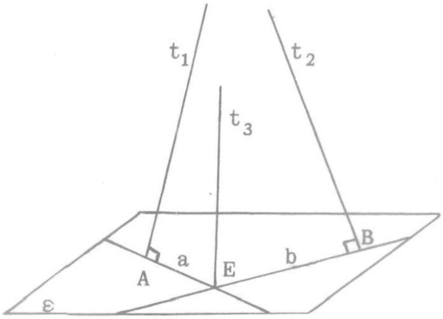

Figure 3

If a and b are parallel there does not exist a point in ε such that its tangent is normal to ε for if T is such a point, TB is a selfconjugated line, that is, TB is perpendicular to t_2, that is, T lies on b and similarly T lies on a which is impossible as a and b have no common point.

If a and b coincide (fig. 4), the point E lies on a. If we give a tangent t_4 at a point C that does not lie on AB and the line c in ε through C perpendicular to t_4, the point E is the common point of c and a.

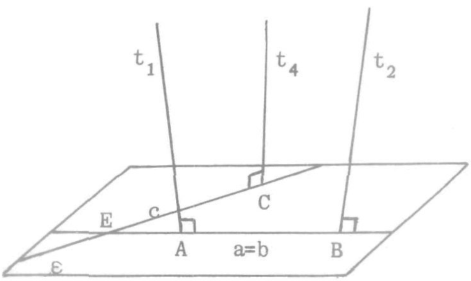

Figure 4

If a and c are parallel we have the foregoing case. So we obtain

Theorem VIa: *Each plane ε generally contains a point in which the tangent is normal to ε.*

If the tangents t_1 and t_2 at the points A and B of the plane ε are both normal to ε, we might draw through any point P of ε the lines PA and PB. As these lines are self-conjugated lines, the tangent at P is perpendicular to PA and to PB and consequently to ε. Hence we notice that if two points of ε have a tangent perpendicular to ε, each point of ε has a tangent perpendicular to ε.

Theorem VIb: *In each plane ε there is in general one point E, and only one with a tangent normal to ε.*

This point E is called the *nullpoint* of ε. It may occur that each point of a plane ε has a tangent normal to ε or that no point of ε has such a tangent.

4. The line of intersection of ε_o and ε_m is denoted by e_o and the one of ε_1 and ε_m by e_1 (fig. 5). A point P_o of e_o gives a

chord P_oP_1 with its midpoint P_m which lies in ε_m. As P_o and P_m lie in ε_m, P_1 also lies in ε_m and as P_1 is a point of e_1, P_1 is a point of the common line e_1 of ε_1 and ε_m, that is, e_o and e_1 are corresponding lines. The nullpoint E_m of ε_m is the midpoint of the chord E_oE_1 which is perpendicular to ε_m. We draw E_oF_o perpendicular to e_o and E_1F_1 perpendicular to e_1. As the figure composed of the line e_o and the point E_o of the plane ε_o corresponds to the figure composed of e_1 and E_1 in ε_1, these figures are congruent and consequently the points F_o and F_1 are corresponding points. From the congruence of the triangles $E_oF_oE_m$ and $E_1F_1E_m$ it follows that $F_oE_m = F_1E_m$ and further that $F_oS = F_1S$ where S is the common point of e_o and e_1.

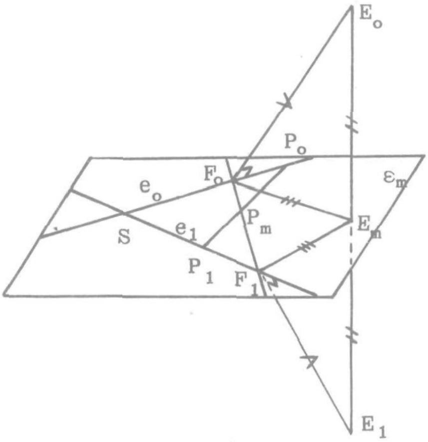

Figure 5

Let P_o be a point of e_o and P_1 a point on e_1 such that $F_oP_o = F_1P_1$. The theorem of Menelaos gives in the triangle F_oSF_1 with regard to the line $P_oP_mP_1$ that $P_oP_m = P_mP_1$ where P_m is the common point of the lines P_oP_1 and F_oF_1. Consequently P_m is the midpoint of P_oP_1 and therefore *the line F_oF_1 is the middle-line of e_o and e_1*; it is denoted by e_m. The line e_m is called the *characteristic* of the plane ε_m.

For the limiting position we obtain

Theorem VII: *The locus of the points of a plane ε in which the tangent lies in ε is a line, called the characteristic e of the plane ε.*

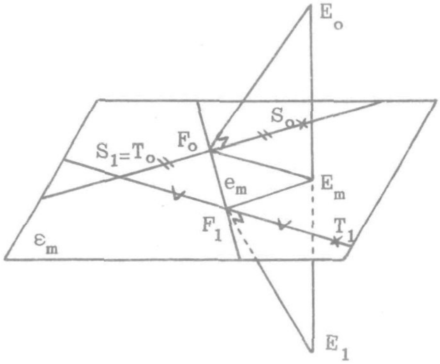

Figure 6

5. If (fig. 6) S is considered as a point of e_1, it is denoted by S_1, and as a point of e_o by T_o. The point S_o corresponding to S_1 lies such that F_o is the midpoint of the chord S_oS_1, and the point T_1 corresponding to T_o lies such that F_1 is the midpoint of the chord T_oT_1. As the plane $F_oE_oE_m$ goes through F_o and is normal to the chord S_oS_1 it is the

bisecting plane of S_oS_1. Similarly the plane $F_1E_1E_m$ is the bisecting plane of the chord T_oT_1. The line E_oE_1 is the line of intersection of these bisecting planes, and as follows from the definition, the line E_oE_1 is the line conjugated to the line $F_oF_1(e_m)$. In the limiting position we obtain: *The line e^p conjugated to the characteristic e of a plane ε is the line normal to ε and going through the nullpoint E of ε .*

6. Let a_m be a line in the middle-plane ε_m such that its conjugated line a^p is perpendicular to ε_m, and let A_oA_1 be the chord belonging to a point A_m of a_m. As the bisecting plane of A_oA_1 goes through a^p and as a^p is normal to ε_m, the chord A_oA_1 lies in ε_m, that is, A_o lies on e_o and A_1 on e_1. It follows that A_m lies on e_m and hence the lines a_m and e_m coincide.

For the limiting position we obtain

Theorem VIII: *If the line l^p conjugated to the line l is perpendicular to l, the line l is the characteristic e of the plane ε through l normal to l^p. The point of intersection of l^p and ε is the nullpoint E of the plane ε.*

7. The direction of the tangent at P to the curve described by P is called *the direction of velocity of the point P.* Let P be a point of the characteristic e of a plane ε and let E be the nullpoint of ε (fig. 7). The line PE is a self-conjugated line since the direction of velocity V_E in E is perpendicular to PE. Hence the direction of velocity V_P in P is perpendicular to PE, and as V_P lies in ε it follows from a known theorem that *the tangents at the points of the characteristic e of a plane ε are tangents to a parabola with the nullpoint E of ε as its focus.*

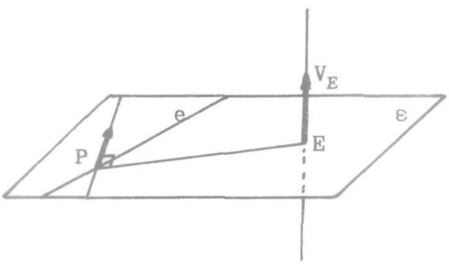

Figure 7

8. Let l_o and l_1 be two positions of a line l, l_m its middle-line and l^p its conjugated line (fig.8). A plane ε_m through l^p intersects l_m in a point, A_m say, such that A_m is the mid-point of the chord A_oA_1 which is normal to ε_m. Hence A_m is the nullpoint of ε_m. In the limiting position we obtain

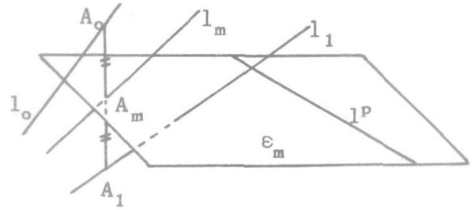

Figure 8

Theorem IXa: *If l^p is the line conjugated to the line l,* *this line l is the locus of the nullpoints of the planes through l^p.*

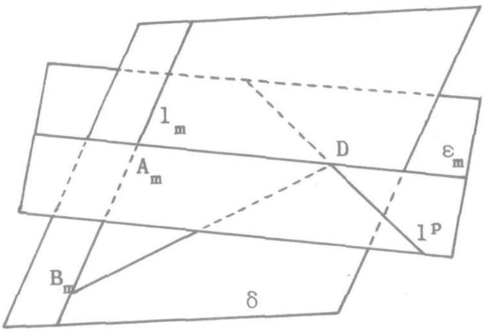

Figure 9

Let δ be a plane through l_m (fig. 9), D its point of intersection with l^p, A_m a point of l_m and ε_m the plane through l^p and A_m. As A_m is the nullpoint of ε_m and A_mD lies in ε_m, the line A_mD is a self-conjugated line. Let B_m be another point of l_m. Then also the line B_mD is a self-conjugated line. The chord of which D is the midpoint is therefore perpendicular to A_mD and to B_mD and hence normal to the plane δ. It then follows that D is the nullpoint of δ and we get in the limit:

Theorem IXb: *If l^p is the line conjugated to the line l, l^p is the locus of the nullpoints of the planes through l.*

From the theorems IXa and IXb follows:

Theorem IXc: *If l^p is the line conjugated to the line l, l is the line conjugated to l^p.*

9. Let A be a point of a line l and B a point of the line l^p conjugated to l. As A is the nullpoint of the plane through l^p and A, and, as AB lies in this plane, the tangent at A is perpendicular to AB, that is, AB is a self-conjugated line.

So we obtain

Theorem X: *The bisecants of two conjugated lines are self-conjugated lines.*

§ 3. The instantaneous screw-axis

1. Parallel planes can be considered as planes through a line l_∞ at infinity. The locus of the nullpoints of these parallel planes is a line l conjugated to l_∞. Conversely (theorem IXc) l_∞ is conjugated to l. *A line l is called an axis if the line l^p conjugated to l is a line at infinity.*

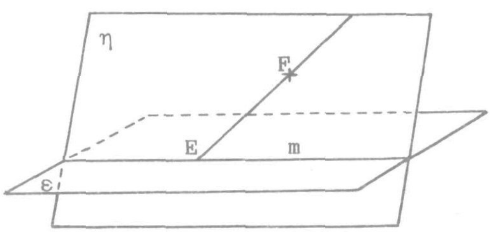

Figure 10

2. Let ε be a plane with E as its nullpoint (fig. 10). The plane η through E intersects ε in the line m. If F is the nullpoint of η, the line EF is a self-conjugated line, as the tangent at F is perpendicular to EF. As E is the nullpoint of ε, m is also a self-conjugated line. The tangent at E is perpendicular to m and also to EF. This is only possible if F lies on m. So we obtain: *The nullpoint F of a plane η through the nullpoint E of a plane ε lies on the line of intersection of the planes η and ε.*

3. Let ε' and ε'' be two parallel planes (fig. 11), E' and E'' their nullpoints, η' and η'' two other parallel planes through E' and E'' respectively, m' and m'' the lines of intersection of the planes ε' and η' and of ε'' and η'' respectively, and F' and F'' the nullpoints of η' and η''. The point F' lies on m' and F'' lies on m''. We draw the lines $E'E''$ and $F'F''$ which lie in one plane, as m' and m'' are parallel. If P be the point of intersection of the lines lines $E'E''$ and $F'F''$, we could lay a plane ε parallel to ε' through P and a plane η parallel to η' through P. The tangent at P would then be normal to ε and to η but this is impossible. Hence the lines $E'E''$ and $F'F''$ are parallel.

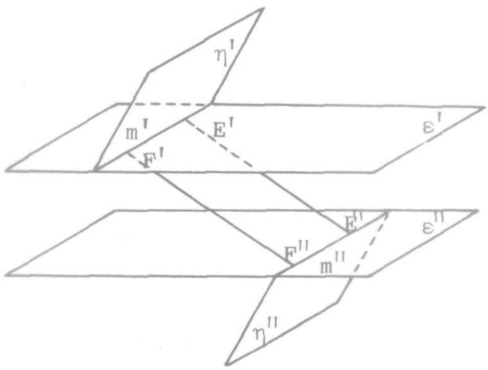

Figure 11

We obtain

Theorem XI: *The loci of the nullpoints of several series of parallel planes are parallel lines called axes.*

4. If we take a series of parallel planes normal to the axes, we obtain as the locus of the nullpoints an axis with the property that in every point the direction of velocity coincides with the direction of the axis. This axis is called the instantaneous screw-axis denoted by x-axis. So we get the following definition: *The x-axis is the line with the property that every point has a tangent in the direction of this line.*

The point at infinity of the x-axis is *the nullpoint of the plane at infinity.*

5. We consider two positions S_1 and S_2 of the moving space S. The midpoints of the chords connecting corresponding points of S_1 and S_2 give the space S_m. Let ε' and ε'' be two parallel planes of S; ε'_o and ε''_o their positions in s_o; ε'_1 and ε''_1 in S_1 and ε'_m and ε''_m in S_m. The nullpoints of ε'_m and ε''_m are denoted by E'_m and E''_m. We notice that ε'_o and ε''_o are parallel, ε'_1 and ε''_1 are parallel and ε'_m

and ε''_m are parallel. The line through E'_m and E''_m is an axis namely the locus of the nullpoints of the planes parallel to ε'_m.

Let ε_m be a plane normal to the axis $E'_mE''_m$ and E_m the nullpoint of ε_m. E_m can be considered as the midpoint of the chord E_oE_1 which is normal to ε_m. As the locus, denoted by x_m-axis, of the nullpoints of the planes parallel to ε_m is parallel to the axis through $E'_mE''_m$, the chords belonging to the common points of these planes and the x_m-axis lie on this axis. Hence the corresponding positions x_o and x_1 of the x_m-axis coincide with the x_m-axis briefly written as *x-axis* or *the axis*. The x_m-axis is called the screw-axis with regard to the positions S_o and S_1 of the space S. By a translation of S_o in the direction of the x-axis x_o coincides with x_1. If this translation is followed by a rotation around this axis, S_o coincides with S_1. We showed that *every displacement of a space S can be obtained by a screw-displacement with the x-axis as its axis.* If S_1 approaches S_o we get in the limit: *At any moment the motion of S is an infinitesimal screw-displacement.*

§ 4. Constructions of the x-axis

1. Let l and l^p be two conjugated lines and α and β two parallel planes through l and l^p respectively (fig. 12). The nullpoint

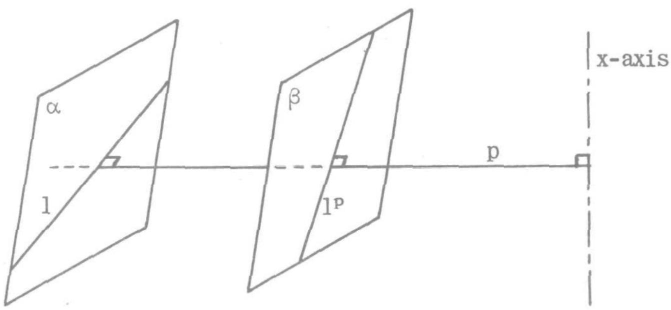

Figure 12

of α is its point of intersection with l^p. As l^p is parallel to α, the nullpoint of α is the point at infinity of l^p. Similarly the nullpoint of the plane β is the point at infinity of l. The axis y belonging to the parallel planes α and β is the line which connects the nullpoints of α and β and hence this axis y is the line at infinity of α and β.

As the x-axis is parallel to the axis y, the point at infinity of the x-axis must lie on the axis y and consequently the x-axis is parallel to the planes α and β. So we obtain

Theorem XII: *The x-axis is parallel to a plane which is parallel to two conjugated lines l and l^p.*

2. Let p be the common normal of the conjugated lines l and l^p and let α and β be two parallel planes through l and l^p respectively. As the x-axis is parallel to α and β, the angle between this axis and the common normal p is a right angle. It is possible to lay a plane γ through p normal to the x-axis. As p is a self-conjugated line, it goes through the nullpoint E of γ. Because of the definition of the x-axis, this axis goes through the nullpoint E and we obtain

Theorem XIII: *The common perpendicular p of two conjugated lines l and l^p intersects the x-axis, and the angle between p and the x-axis is a right angle.*

3. Let l and l^p be two conjugated lines. A plane δ through a point P of the x-axis normal to this axis meets l in A and l^p in B. The line AB is a self-conjugated line for it connects two points of conjugated lines. As AB lies in γ, AB goes through the nullpoint of γ. This nullpoint is the point P and we obtain: *The points of intersection of a plane normal to the x-axis with two conjugated lines lie on a line which intersects the x-axis.*

4. If two pairs of conjugated lines l, l^p and m, m^p are given, the construction of the x-axis is as follows (fig. 13).

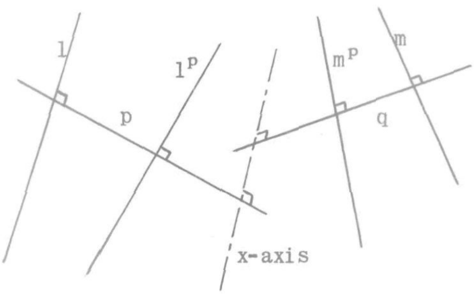

Figure 13

Draw the common perpendicular p of the lines l and l^p and the common perpendicular q of m and m^p. As the x-axis intersects p and q at a right angle, *the x-axis is the common perpendicular of p and q.*

5. If again two pairs of conjugated lines l, l^p and m, m^p are given (fig. 14), a construction of the nullpoint E of a given plane α is as follows: *E is the point of intersection of the lines LL^p and MM^p*, if L, L^p, M, M^p are the points of intersection of the lines l, l^p, m, m^p with the plane α. The lines LL^p and MM^p are self-conjugated lines and therefore they go through the nullpoint E of the plane α.

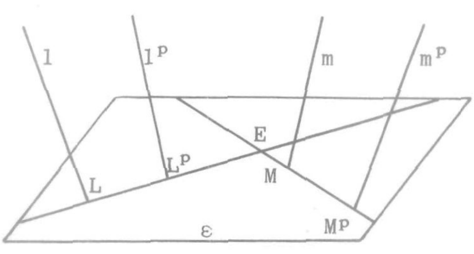

Figure 14

§ 5. Constructions of the characteristics of a plane

1. Let l and l^p be two conjugated lines such that l is perpendicular to l^p (fig. 15). The plane δ through l normal to l^p intersects l^p in the point A and the plane ε through l^p normal to l intersects l in B. A is the nullpoint of δ and B of ε. Hence the tangent at A coincides with l^p and the one at B coincides with l. So we obtain

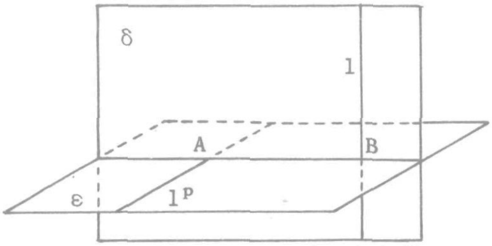

Figure 15

Theorem XIV: *If l is a line perpendicular to its conjugated line l^p, these lines are both tangents at the points in which the common perpendicular intersects l and l^p.*

2. Each line p which intersects the x-axis perpendicularly is a self-conjugated line for the tangent at the point of intersection coincides with the x-axis and is therefore perpendicular to the line p. A plane ε through p (fig. 16) contains p as a self-conjugated line and consequently p goes through the nullpoint E of ε and we obtain

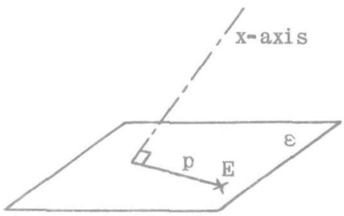

Figure 16

Theorem XV: *If ε is a plane not normal to the x-axis, the line p in ε which intersects the x-axis at a right angle goes through the nullpoint E of the plane ε.*

3. Let l and l^p be two conjugated lines (fig. 17) and ε a given plane. The line p, perpendicular to α, which intersects the lines l and l^p, cuts α in the point P. The tangent at P is normal to p for p is a self-conjugated line. Consequently the tangent at P lies in α, that is, P is a point of the characteristic e of α.

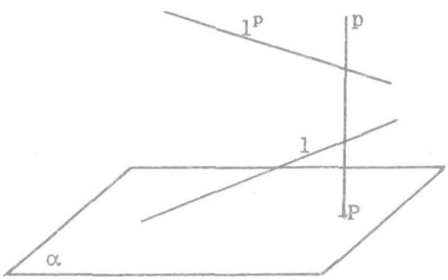

Figure 17

If in a special case l^p is normal to α (fig. 18),

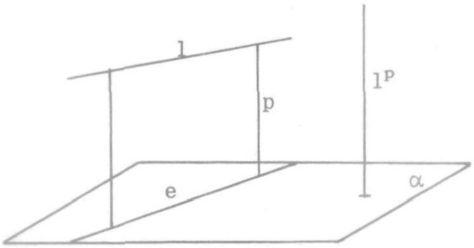

Figure 18

each line p normal to α which intersects l is parallel to l^p, that is, intersects l^p in its point at infinity. Consequently p is a self-conjugated line. The locus of the points of intersection of the lines p with the plane α is the characteristic e of α. We obtain

Theorem XVI: *The projection of a line l on a plane α is the characteristic e of α if the line l^p conjugated to l is normal to α.*

4. Another special case occurs if l^p is the line at infinity of a plane β (fig. 19). Then the line l conjugated to l^p is an axis. Let α be a plane normal to β. The lines p which intersect l and are normal to α are parallel to β and therefore they intersect l^p at infinity. The locus of the points of intersection of these lines p with α is the characteristic e of α. So we obtain: *If l is an axis and α a plane perpendicular to the planes through the line l^p at infinity conjugated to l, the projection of l on α is the characteristic e of α.*

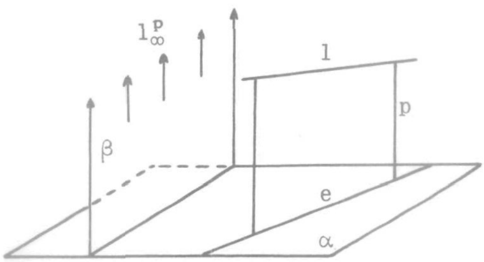

Figure 19

5. Let γ be a plane perpendicular to the plane α and not parallel to the plane β of figure 19. If the line at infinity of γ is denoted by m^p, we notice that the projection of the axis m conjugated to m^p on the plane α is also the characteristic e of α. Consequently we have:

Theorem XVII: *The locus of the axes which are the lines conjugated to the lines at infinity of the planes perpendicular to a given plane α is the plane perpendicular to α through the characteristic e of α.*

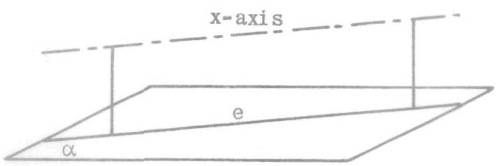

Figure 20

6. If in a special case *the x-axis is parallel to a plane* ε (fig. 20), *the projection of the x-axis on* ε *is the characteristic e of*

ε for the lines through the x-axis normal to ε are self-conjugated lines.

§ 6. Tangents

1. If we return to figure 1, we notice that all chords are parallel to the plane which is parallel to the lines P_oP_1 and $A'_oA'_1$. As all chords intersect l_o and l_1 they are generators of a paraboloid. In the limit we obtain

Theorem XVIII: *The tangents at the points of a line generate a paraboloid.*

2. Let l be a given line and η a plane normal to l (fig. 21). Each plane ε through l is normal to η. The axis conjugated to the line at infinity of the plane η is denoted by u. From the second special case of theorem XVI (see § 5.4) follows that the projection of u upon ε is the characteristic e of ε. As the tangents at the points of u are normal to the plane η, they are parallel to the line l. We obtain

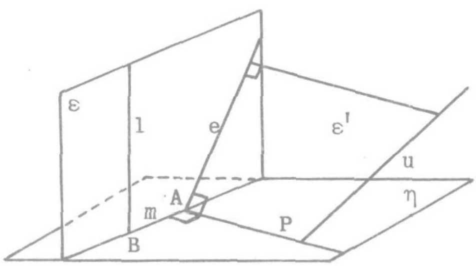

Figure 21

Theorem XIX: *The characteristic e of a plane ε through a line l is the projection upon ε of the locus of the points with a tangent parallel to l.*

3. Let the plane through u perpendicular to ε be denoted by ε' (fig. 21). The characteristics e of all planes ε through l are defined as the lines of intersection of the planes ε with the planes ε' through u perpendicular to ε. The locus of the characteristics e of the planes ε through l is the locus of the lines of intersection of the planes of two pencils through the skew lines l and u such that a plane of the first pencil is normal to a plane of the other one. It is known that this locus is an orthogonal hyperboloid and we obtain: *The locus of the characteristics of the planes through a line is an orthogonal hyperboloid H.*

Let P be the point of intersection of u and η and m the line of intersection of ε and η (fig. 21). If PA is normal to m, PA is normal to ε and hence A is a point of the characteristic e of ε. The point of intersection of l and η is denoted by B. In the several positions of ε through l the point A describes a circle with BP as its diameter and consequently *the curve of intersec-*

tion of the hyperboloid H with a plane normal to l is a circle. Similarly the curves of intersection of H with the planes perpendicular to u (that is, to the x-axis for u is parallel to this axis) are circles.

4. As the points of the characteristic of a plane ε have the property that their tangents lie in ε, these tangents intersect any line l in ε. Therefore we have

Theorem XX: *The locus of the points with a tangent which intersects a given line l is the locus of the characteristics of the planes through l.* From 3 follows that *this locus is an orthogonal hyperboloid H generated by the lines of intersection of the orthogonal planes of the pencils through l and through the line u* if u is the axis conjugated to the line at infinity of a plane normal to l.

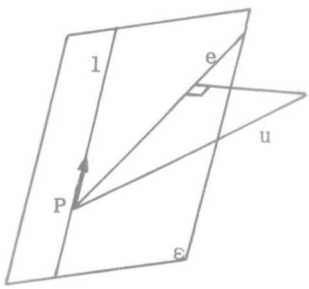

Figure 22

In the special case that l is a tangent (fig. 22), the point of contact P is a point of the characteristic of each plane through l and hence the characteristics generate a cone with P as vertex. This cone is a quadric and the axis u through P is one of its generators for at each point of u the tangent is parallel to l, that is, intersects l in its point at infinity.

5. Let l be a tangent (fig. 23), A its point of contact and B the foot of the perpendicular from A to the x-axis. As AB is perpendicular to the x-axis, AB is a self-conjugated line and hence l is perpendicular to AB. We obtain

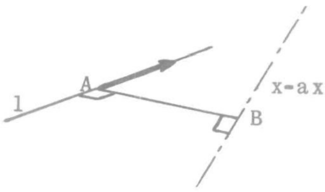

Figure 23

Theorem XXI: *If a line l is a tangent, its point of contact lies on the common normal of l and the x-axis.*

The point of intersection of a line l with its common normal with regard to the x-axis is called the *central point of the line l*.

Let P be a given point. If we lay a plane ε through P parallel to the x-axis and a plane η through the x-axis normal to ε, we state that the line of intersection of ε and η is the locus of the central points of the lines through P in ε. Let a be the line through P parallel to the x-axis. The locus of the central points of all lines through P is the locus of the lines of intersection of the planes through a with the planes through the x-axis normal to them. We obtain

Theorem XXII: *The locus of the central points of all lines through a point P is a circular cylinder through P and the x-axis such that the plane through P and the x-axis is a plane of symmetry of the cylinder.*

6. What is the locus of the points which have *a tangent through a given point P* ?

Let l be a line through P and m the line through P parallel to the x-axis (fig. 24). This line m is also the axis belonging to the planes normal to the tangent at P. The axis belonging to the planes normal to l is denoted by u.

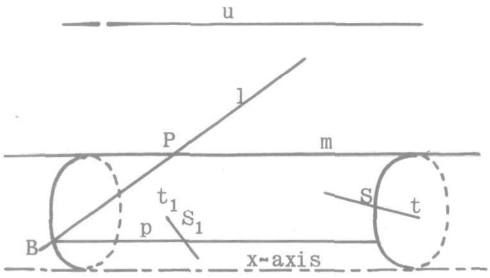

Figure 24

Each tangent through P is a tangent which intersects l. From theorem XX follows that the locus of the points with a tangent which intersects l is a hyperboloid H generated by the lines of intersection of the orthogonal planes of the pencils through l and u. The required locus is therefore a curve on H.

From theorem XXII follows that the locus of the central points of the lines through P is a circular cylinder C through m and the x-axis such that the plane through m and the x-axis is a plane of symmetry of C.

Let S be a point of the curve of intersection of the hyperboloid H and the cylinder C and let t be the tangent at S. As S is a point of H the tangent t intersects l, and as S is a point of C the tangent t intersects m. If S is not a point of the plane through l and m, t goes through P.

Let S_1 be a point of the generator p of C which intersects l in B where B is the second point of intersection of l with C (fig. 24). B is the central point of l. The tangent t_1 at S_1 intersects m for S_1 is a point of C and consequently t_1 also intersects l and hence S_1 is a point of H.

Each point of the generator p is a common point of H and C and therefore *p is a common generator of H and C*. The generator p does not belong to the locus of the points with a tangent through P. The curve of intersection of C and H is in general a twisted curve of the fourth degree. This curve is degenerated in the line p and a twisted cubic. We obtain

Theorem XXIII: *The locus of the points which have a tangent through a given point P is a twisted cubic.*

7. The curve of intersection of a plane δ perpendicular to the x-axis with the hyperboloid H is a circle and the curve of intersection of δ with the cylinder C is also a circle. The common points of these circles are 1^o a point of the common generator p, 2^o a real point of the twisted cubic and 3^o the two circular points at infinity of the plane δ. As the twisted cubic goes through these points, the cubic is called *a cubical circle*.

As the tangents at the points of the x-axis coincide with the x-axis, there is no point on this axis with a tangent through P. The generator of the cylinder C which has no common point with the cubical circle on C is the asymptotic line of the cubic. We obtain: *The x-axis is the asymptotic line of the locus of the points which have a tangent through a given point P.*

8. If l is the tangent at P the locus of the points with a tangent through P is the curve of intersection of the circular cylinder C and the cone generated by the lines of intersection of the orthogonal planes of the pencils through l and m respectively. The common generator of the cone and the cylinder is the line m namely the line through P parallel to the x-axis. In this special case the lines u, p and m coincide.

§ 7. Tangents with a second-order contact

1. Let ε_o and ε_1 be two positions of the plane ε (fig. 25). The middle-plane is ε_m and the nullpoint E_m. Let A be a point of ε and α^P the bisecting plane through the midpoint A_m of the chord A_oA_1. We draw the line E_mA_m. As the chord E_oE_1 belonging to E_m is perpendicular to the plane ε_m and as the line E_mA_m lies in ε_m, E_oE_1 is normal to E_mA_m and hence the chord A_oA_1 is perpendicular to A_mE_m (theorem II). Consequently the bisecting plane α^P of the chord A_oA_1 goes through E_m. We obtain

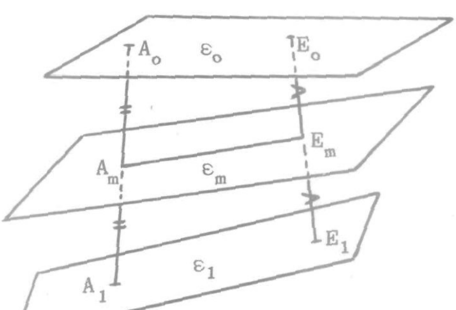

Figure 25

Theorem XXIV: *The bisecting planes of the chords of the corresponding points of a plane ε in its two positions ε_o and ε_1 go through the nullpoint E_m of the middle-plane ε_m.*

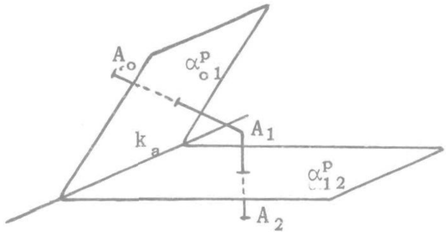

Figure 26

2. We consider three positions A_o, A_1 and A_2 of a point A (fig.26). We use the following notations: α^P_{o1} and α^P_{12} are the bisecting planes of the chords A_oA_1 and A_1A_2 respectively. The line of intersection of these two planes is denoted by k_a and is called the *axis of curvature* belonging to the three positions A_o, A_1, A_2 of the point A.

If A_o, A_1 and A_2 approach each other, the limiting position of k_a is called the axis of curvature belonging to the point A with regard to the curve described by A.

3. Let ε_o, ε_1, ε_2 be three positions of a plane ε. The middle-plane of ε_o and ε_1 is denoted by ε_{o1} and its nullpoint by E_{o1}. Similarly we get ε_{12} and E_{12}.

A point A of the plane ε gives a bisecting plane α^P_{o1} through E_{o1} and a bisecting plane α^P_{12} through E_{12}. All points of ε give a series of planes through E_{o1} and a series of planes through E_{12}. These series are projective and therefore the lines of intersection of corresponding planes of the two series are the bisecants of a twisted cubic *). These lines of intersection are axes of curvature.

This cubic has three points of intersection with a given plane δ and hence in δ lie three bisecants of the cubic. Consequently there are three points in ε which have an axis of curvature in a given plane δ. As each plane contains three points with an axis of curvature in δ, the locus of these points is a twisted cubic.

If δ is the plane at infinity, an axis of curvature lies in δ if the bisecting planes α^P_{o1} and α^P_{12} of the points A_o, A_1, A_2 are parallel. Hence A_o, A_1 and A_2 lie in one line. In the limiting position the chord $A_oA_1A_2$ becomes a tangent with a second-order contact at A to the curve described by A. We obtain

Theorem XXV: *The locus of the points in the moving space S which have a tangent with a second-order contact is a twisted cubic.*

§ 8. Osculating planes with a four-point contact

1. Let A_o, A_1, A_2 and A_3 be four positions of a point A (fig.

*) Reye, Geometrie der Lage II, p. 231 (Hannover 1880)

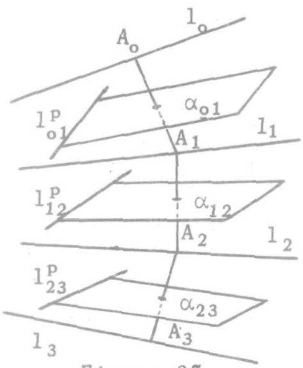

Figure 27

27). The bisecting planes α^p_{o1}, α^p_{12}, α^p_{23} go in general through one point denoted by A'. To every point A corresponds one point A'. This point A' can be considered as the centre of the sphere through A_o, A_1, A_2, A_3.

Any point A of a line l gives three bisecting planes through A'. The bisecting plane α^p_{o1} goes through the line l^p_{o1} conjugated to l_o and l_1; the plane α^p_{12} goes through l^p_{12} and the plane α^p_{23} through l^p_{23} where l_o, l_1, l_2, l_3 are four positions of l. The locus of the points A', if A is any point of l, is the locus of the points of intersection of the corresponding planes of the pencils through l^p_{o1}, l^p_{12} and l^p_{23} respectively. As these pencils are projective, the locus of A' is a twisted cubic *).

2. If ε is a given plane, this twisted cubic intersects ε in three points. Consequently on l lie three points with their corresponding point in ε. This holds for every line l. Hence *the locus of the points P in the moving space S of which the centre P' of the sphere through P_o, P_1, P_2 and P_3* (which are four positions of P) *lies in a given plane ε is a surface of the third degree.* If ε is a plane at infinity, P' is a point at infinity and hence the points P_o, P_1, P_2 and P_3 lie in one plane. If these four points approach each other the plane through them becomes the osculating plane with four points of contact in P to the curve described by P. We obtain

Theorem XXVI: *The locus of the points which have an osculating plane with four points of contact with the curves described by these points is a surface of the third degree.*

A point with an osculating plane with four points of contact is called 1° a *point of inflection,* 2° a *stationary point* or 3° a point with an *osculating plane with a third-order contact.*

4. The surface of the third degree of theorem XXVI contains the twisted cubic mentioned in theorem XXV for if A_o, A_1 and A_2 lie in one line, the points A_o, A_1, A_2 and A_3 lie in one plane.

*) Reye, Geometrie der Lage II, p. 197 (Hannover 1880)

Chapter II

THE AXIAL REFLECTION *)

§ 1. Definitions and theorems

1. The point P^r (fig. 28) is the axially reflected point of the point P with regard to the line l if the line l intersects the line PP^r in its midpoint P' and l is perpendicular to PP^r. The line l is called *the axis of reflection*. We say that P is reflected with regard to l. Any figure F is reflected with regard to a line l if each point of F is reflected to l. The locus of the reflected points gives the figure F^r.

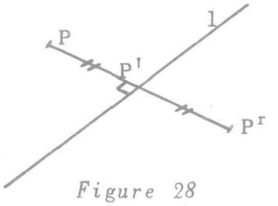

Figure 28

A space Σ is reflected if each point of Σ is reflected. The locus of the reflected points gives the space Σ^r, the reflected space of Σ. The spaces Σ and Σ^r are congruent. If Σ makes half a turn around l it coincides with Σ^r. The spaces Σ and Σ^r have the line l in common.

2. We can reflect a point P with regard to the generators of a ruled surface Γ. The locus of the reflected points P^r is a curve called the *path of P^r*. P is called the *pole of the path*. As the midpoints P' of the lines PP^r are points of the generators, that is, points of Γ, and as these midpoints P' are the feet of the perpendiculars of P on the generators, it follows that the locus of P' is a curve on Γ such that, if we multiply this curve by two with regard to P, the path of P^r is obtained. The curve described by P' is called *the pedal of P* with regard to the generators of Γ, and the ruled surface Γ is called *the basic surface of the axial reflection.*

3. We can reflect a fixed space Σ_f with regard to several generators of the basic surface Γ. Then we obtain several spaces Σ^r. As all these spaces are congruent we can consider them as the several positions of a space moving as a rigid body.

Now we take the generators g_o and g_1 of Γ (fig. 29) and we reflect any point P of Σ_f with regard to g_o and g_1 respectively. The reflected points are P^r_o and P^r_1.

*) Krames [4], [5], [6].

Let x be the common perpendicular of g_o and g_1; Q, R and S the projections of P, P^r_o, P^r_1 on x; and G_o and G_1 the points of intersection of g_o and g_1 with the line x.

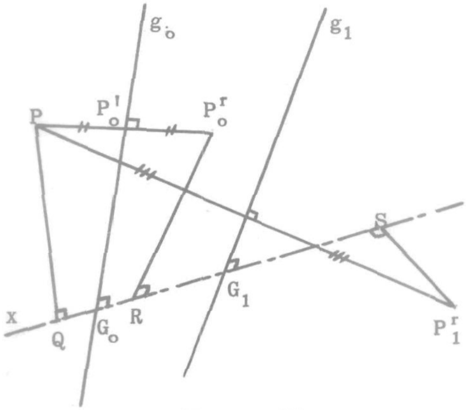

Figure 29

From $PP'_o = P'_oP^r_o$, PP^r_o and x perpendicular to g_o, PQ and P^r_oR perpendicular to x follows $PQ = P^r_oR$. Similarly we get $PQ = P^r_1S$ and hence $P^r_oR = P^r_1S$, that is, the distances of the reflected points of P to the common perpendicular of g_o and g_1 are equal.

As P is an arbitrary point of Σ_f, we obtain

Theorem Ia: *If we reflect a space Σ_f with regard to two generators g_o and g_1 we get the spaces Σ^r_o and Σ^r_1. The distances of corresponding points of these spaces to the common perpendicular x of g_o and g_1 are equal.*

Further, we deduce that $G_oR = G_oQ$ and $G_1S = G_1Q$. The points R and S are formed from the points G_o and G_1 if we multiply R and S by two with regard to Q. Hence $RS = 2\,G_oG_1$; thus we have

Theorem Ib: *The distance of the projections on the line x of two corresponding points of Σ^r_o and Σ^r_1 is twice the distance of g_o and g_1.*

As the lines PQ, g_o, P^r_oR, g_1 and P^r_1S are parallel to any plane α perpendicular to the line x, the angles between these lines are equal to the angles between the projection of these lines on α. Hence the angle between P^r_oR and P^r_1S is twice the angle between g_o and g_1. So we obtain

Theorem Ic: *The angle between the perpendiculars of two corresponding points of Σ^r_o and Σ^r_1 on the line x is twice the angle between g_o and g_1.*

4. From the theorems Iabc it follows that it is possible to determine a displacement which is composed of a rotation around the line x and a translation in the direction of this line x such that Σ^r_o is displaced to Σ^r_1. The angle of rotation is twice the angle between g_o and g_1 and the size of the translation is twice the distance of g_o and g_1.

This displacement can be obtained by a continuous motion such that the angle of rotation and the size of translation with regard to the line x are proportional. This motion is a screw-motion and *the line x is its screw-axis*. The *parameter of the* *screw-motion is the ratio of the translation and the rotation,* that is, $RS : \angle (P^r_o R, P^r_1 S) = G_o G_1 : \angle (g_o, g_1)$.

At any moment the velocity distribution of the screw-motion is such that the velocity in any point P of the moving space is composed of a constant component parallel to the screw-axis and a component perpendicular to the plane through the point P and the screw-axis. This last component is proportional to the distance of P to the axis.

5. It is also possible to determine a displacement which replaces the line g_o to the line g_1 by means of a translation $G_o G_1$ in the direction of the common perpendicular x and a rotation around this line x. Every point of g_o has in this case a translation $G_o G_1$ and the rotation around the line x is the angle between g_o and g_1.

Again we can obtain this movement by a continuous motion, namely a screw-motion with x as its axis. At any moment the points of the moving line have a velocity distribution equal to that of a screw-motion. The parameter of this motion is $G_o G_1 : \angle (g_o, g_1)$. Now we obtain

Theorem IIa: *The axial reflection of the fixed space Σ_f with regard to two generators g_o and g_1 gives the spaces Σ^r_o and Σ^r_1. It is possible to determine two screw-motions with the common perpendicular of g_o and g_1 as their common axis and with the same parameter, namely $G_o G_1 : \angle (g_o, g_1)$. The one displaces Σ^r_o to Σ^r_1 and the other g_o to g_1.*

6. If g_1 approaches g_o we get in the limit

Theorem IIb: *The velocity distributions of the points of Σ^r and of the points of g at any moment are those of screw-motions. The* common axis of these screw-motions is the limiting position of the common perpendicular of g_o and g_1. The parameters of the screw-motions are equal.

7. **Theorem III**: *If the basic surface is a ruled surface of the n^{th} degree the path of the reflected points P^r of any point P in the fixed space Σ_f is in general a twisted curve of the $2n^{th}$ degree.*

This theorem will have been proved if we show that the pedal of P with regard to the generators of Γ is a curve of the $2n^{th}$ degree.

If p is the polarline (fig.30) of the point at infinity L of any line l

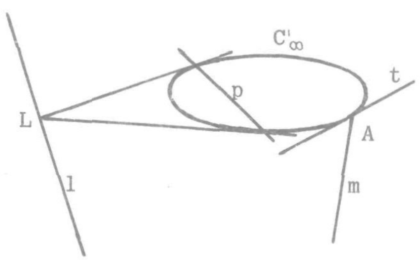

Figure 30

with regard to the isotropic conic C_∞, it is known that each plane through p is perpendicular to each line through L, that is, each line that intersects p is perpendicular to each line through L.

If m is an isotropic line, its point at infinity A lies on C_∞ and the polarline of A is the tangent t in A at C_∞. Each line which intersects m at a right angle is a line in the plane through t and m. This plane is called the *isotropic plane of m*. If P is not a point of this isotropic plane, the line PA is the line through P which intersects the isotropic line m in A at a right angle. The foot of the perpendicular is the point A.

As the degree of a twisted curve is the number of points of intersection with any plane, we can consider the number of points of intersection with the plane at infinity. The foot of the perpendicular to a generator of Γ is a point at infinity if:

1° the generator lies in the plane at infinity, or

2° the generator is an isotropic one.

We suppose that Γ has no generators at infinity.

An isotropic generator is a generator which intersects the isotropic conic C_∞. Because the curve of intersection K of Γ with the plane at infinity is a curve of the n^{th} degree, there are 2n points of intersection of K and C_∞, that is, Γ has 2n isotropic generators. Hence the pedal of P has 2n points at infinity, that is, the pedal of P is a curve of the $2n^{th}$ degree.

If Γ has one or more generators in the plane at infinity the degree of the pedal is diminished.

8. **Theorem IV**: *If there exists a one-to-one correspondence between the points of two rational twisted curves C_1 and C_2 of degrees d_1 and d_2, and the curves have p self-corresponding points of intersection the surface generated by the lines joining corresponding points is a ruled surface of the degree $d_1 + d_2 - p$.*

Proof: Let the parametric equations of C_1 and C_2 be (without restriction the parameters can be taken such that points with the same parameter value correspond)

$$C_1 \begin{cases} x = x_1(t) \\ y = y_1(t) \\ z = z_1(t) \\ w = w_1(t) \end{cases} \qquad C_2 \begin{cases} x = x_2(t) \\ y = y_2(t) \\ z = z_2(t) \\ w = w_2(t) \end{cases}$$

in which x_1, y_1, z_1 and w_1 are functions of t of the d_1^{th} degree and x_2, y_2, z_2 and w_2 are functions of t of the d_2^{th} degree.

The points of intersection of C_1 and C_2 are given by $t = t_1, t_2, \dots, t_p$.

The equations of a line l connecting two corresponding points of C_1 and C_2 are:

$$x = x_1(t) + \lambda x_2(t)$$
$$y = y_1(t) + \lambda y_2(t)$$
$$z = z_1(t) + \lambda z_2(t)$$
$$w = w_1(t) + \lambda w_2(t)$$

In these equations t differs from t_k $(k = 1, 2, \ldots, p)$ because a line through two corresponding points which coincide is undetermined.

The equations of l can be considered as the two-parametric equations of the surface S generated by the lines which connect corresponding points of C_1 and C_2. The degree of a surface is the number of points of intersection with any line m. Let the equation of m be:

$$y = \alpha x + \beta w \qquad \text{and} \qquad z = \gamma x + \delta w.$$

The points of intersection of S and m are given by:

$$\begin{cases} y_1 + \lambda y_2 = \alpha(x_1 + \lambda x_2) + \beta(w_1 + \lambda w_2) \\ z_1 + \lambda z_2 = \gamma(x_1 + \lambda x_2) + \delta(w_1 + \lambda w_2) \end{cases}$$

or:

$$\begin{cases} y_1 - \alpha x_1 - \beta w_1 + \lambda(y_2 - \alpha x_2 - \beta w_2) = 0 \\ z_1 - \gamma x_1 - \delta w_1 + \lambda(z_2 - \gamma x_2 - \delta w_2) = 0. \end{cases}$$

Elimination of λ gives:

$$(y_1 - \alpha x_1 - \beta w_1)(z_2 - \gamma x_2 - \delta w_2) - (z_1 - \gamma x_1 - \delta w_1)(y_2 - \alpha x_2 - \beta w_2) = 0$$

The degree of this equation in t is $d_1 + d_2$.

If $t = t_k$ we get a self-corresponding point of C_1 and C_2 and thus:

$$x_1(t_k) : x_2(t_k) = y_1(t_k) : y_2(t_k) = z_1(t_k) : z_2(t_k) = w_1(t_k) : w_2(t_k)$$

Substitution of these values into the last equation gives an identity, that is, $t = t_k$ is a root of the equation and the left-hand member can be divided by $t - t_k$. This holds for every value of t corresponding to self-corresponding points of C_1 and C_2. Therefore the left member of the equation can be divided by $(t-t_1)(t-t_2)\ldots(t-t_p)$ and the degree in t is diminished by p. As the degree was $d_1 + d_2$, it becomes $d_1 + d_2 - p$ and thus the number of points of intersection with the line m is $d_1 + d_2 - p$.

9. **Theorem V**: *The reflected lines p^r of any line p of the fixed space Σ_f with regard to the generators of a ruled surface Γ of the n^{th} degree generate a ruled surface of the $2n^{th}$ degree.*

Proof: If we reflect two points A and B of the line p we get the paths of A^r and B^r, a^r and b^r say. By means of the reflection of p a one-to-one correspondence is determined between the points of a^r and b^r. The curves a^r and b^r have 2n points of intersection, namely the points at infinity. These points are self-corresponding points. The degree of a^r and b^r is 2n. From theorem

IV it follows (as $d_1=2n$, $d_2=2n$ and $p=2n$) that the degree of the surface generated by the reflected lines p^r is $2n+2n-2n = 2n$.

§ 2. A hyperboloid as basic surface

1. As a hyperboloid H with one sheet has two series of generators, it is necessary to consider only one of these series if we use this quadric as a basic surface. The hyperboloid H has four points of intersection with the isotropic conic, for its curve at infinity K is a conic. The four generators (belonging to the considered series) which go through these points are the isotropic generators γ_i (i = 1,2,3,4) (fig.31). The plane τ_i through γ_i and through the tangent t_i to the isotropic conic C_∞ in the point at infinity A_i of γ_i is the isotropic plane belonging to γ_i. The hyperboloid H has four isotropic planes.

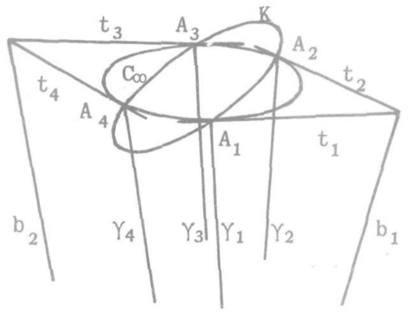

Figure 31

As the points A_i are conjugate complex two by two, the corresponding isotropic planes are also conjugate complex. Therefore two of the lines of intersection of these planes are real; they are called *the focal axes of the hyperboloid H.*

The focal axes are denoted by b_1 and b_2; b_1 is the line of intersection of the isotropic planes τ_1 and τ_2 and b_2 the one of τ_3 and τ_4. The points at infinity A_1 and A_2 are conjugate complex and so are A_3 and A_4.

2. If we reflect any point P with regard to the considered series of generators of H, we deduce from theorem III (as n=2) that the path of P^r and thus the pedal of P are twisted curves of the fourth degree. If P is a point of the focal axis b_1, P lies in the two isotropic planes τ_1 and τ_2. Each line in an isotropic plane τ is perpendicular to the isotropic generator γ of τ (see § 1.7). It follows that each line through P that intersects γ_1, γ_2 respectively is perpendicular to γ_1, γ_2 respectively, that is, *γ_1 and γ_2 are lines which belong to the pedal of P.* (The line PA_1 makes an undetermined angle with γ_1, that is, A_1 belongs to the pedal of P). Hence the twisted curve generated by P^r degenerates into two lines and into a curve of the second degree.

If P is reflected with regard to the two other isotropic generators γ_3 and γ_4 we get the points A_3 and A_4, namely the points at infinity of these generators. As these points belong to the curve of the second degree, we conclude that this conic is a circle. *Each point of* b_1 *gives a path which is a circle.* All these circles have the same isotropic points A_3 and A_4 and therefore *their planes are parallel.*

3. Two points R and S of b_1 give on reflection two circles. Between the reflected points R^r and S^r there exists a one-to-one correspondence. The lines connecting two corresponding points are reflected lines of b_1. From theorem IV it follows that *the ruled surface generated by the reflected lines of* b_1 *is a surface of the second degree* for the curves C_2 and C_2 are now circles, that is, $d_1 = d_2 = 2$ and these circles have two self-corresponding points of intersection (A_3 and A_4), that is, $p = 2$. The degree becomes $d_1 + d_2 - p = 2 + 2 - 2 = 2$. This quadric is denoted by Δ_2. As Δ_2 is a surface generated by skew lines and has no generators in the plane at infinity the quadric Δ_2 is a hyperboloid.

4. Let the distance of the two points R and S of b_1 be denoted by d (fig. 32). Hence the distance of the reflected points R^r and S^r is also d. The two circles C_1 and C_2 described by R^r and S^r respectively lie in two parallel planes, ε_1 and ε_2 say. Let p be the distance of ε_1 and ε_2. The angle α between R^rS^r and the plane ε_2 is given by $\sin \alpha = p : d$. As p and d are constant α is also constant.

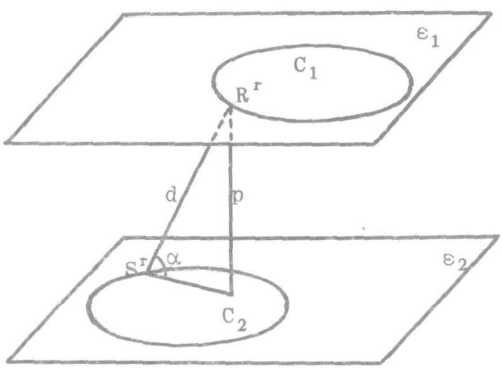

Figure 32

Let M be the centre of Δ_2. The cone generated by the lines through M parallel to the generators of Δ_2 is the asymptotic cone of Δ_2. As α is constant, the cone is a surface of revolution and therefore Δ_2 *is a hyperboloid of revolution.*

Consequently the conic at infinity K of Δ_2 has only two points of contact, namely A_3 and A_4, with C_∞.

5. **Theorem VI**: *The focal axis* b_2 *of the hyperboloid H is the axis of revolution of the hyperboloid* Δ_2 *generated by the reflected lines of the other focal axis* b_1 *with regard to the generators of H.*

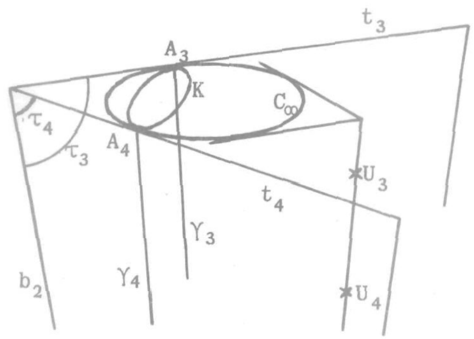

Figure 33

Proof: Let the points of intersection of b_1 with the isotropic planes τ_3 and τ_4 be U_3 and U_4 (fig. 33). Each line through U_3 and a point of γ_3 is perpendicular to γ_3, that is, γ_3 belongs to the pedal of U_3. If U_3 is reflected with regard to γ_3 we get another isotropic line γ_3' through A_3 and lying in the plane τ_3. As γ_3' is an isotropic generator of Δ_2 the plane τ_3 is a tangent plane of Δ_2. Now the conic at infinity K of Δ_2 has two points of contact with C_∞ and therefore the tangent t_3 on C_∞ in A_3 is a tangent of K. As τ_3 goes through the generator γ_3' and through t_3 the point of contact with Δ_2 is A_3. Similarly A_4 is the point of contact of τ_4 with Δ_2. Now b_2 is the line of intersection of the tangent planes τ_3 and τ_4 and hence the lines b_2 and A_3A_4 are conjugated polarlines. The conjugated polarline of the line at infinity of the planes through the circular sections of Δ_2 is the axis of revolution of Δ_2. Consequently b_2 is the axis of Δ_2.

It is evident that b_1 is the axis of revolution of the hyperboloid Δ_1 generated by the reflected lines of b_2 with regard to the generators of the hyperboloid H.

§ 3. Focal axes of a hyperboloid with one sheet

1. Let the equation of the hyperboloid be:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \qquad \text{with } a > b.$$

The equations of one of the series of generators are, if we use homogeneous coordinates:

$$\left.\begin{aligned} \frac{x}{a} + \frac{z}{c} &= \lambda\left(w - \frac{y}{b}\right) \\ \frac{x}{a} - \frac{z}{c} &= \frac{1}{\lambda}\left(w + \frac{y}{b}\right) \end{aligned}\right\} \qquad (1)$$

and the equations of the isotropic conic C_∞:

$$\left.\begin{aligned} x^2 + y^2 + z^2 &= o \\ w &= o \end{aligned}\right\} \qquad (2)$$

Elimination of x, y, z and w out of (1) and (2) gives the condition that a generator intersects C_∞:

Substraction and addition of the equations (1), after taking w = o, gives:

$$z = -\frac{c}{2b}\,y\left(\frac{1}{\lambda} + \lambda\right) \qquad \text{and} \qquad x = \frac{a}{2b}\,y\left(\frac{1}{\lambda} - \lambda\right) \qquad (3)$$

Substitution of the expressions (3) in (2) gives the condition:

$$a^2\left(\frac{1}{\lambda} - \lambda\right)^2 + 4b^2 + c^2\left(\frac{1}{\lambda} + \lambda\right)^2 = o$$

or: $$\lambda^4(a^2 + c^2) + 2\lambda^2(-a^2 + 2b^2 + c^2) + (a^2 + c^2) = o$$

or: $$\lambda_{1234} = \pm\frac{\sqrt{a^2 - b^2} \pm i\sqrt{b^2 + c^2}}{\sqrt{a^2 + c^2}}$$

These four values of λ, successively substituted in (1) *give the equations of the four isotropic generators* (belonging to the considered series) of the hyperboloid.

2. If we take

$$\lambda_1 = \frac{\sqrt{a^2 - b^2} + i\sqrt{b^2 + c^2}}{\sqrt{a^2 + c^2}} \qquad \text{and} \qquad \lambda_2 = \frac{\sqrt{a^2 - b^2} - i\sqrt{b^2 + c^2}}{\sqrt{a^2 + c^2}}$$

which are two conjugate-complex values, we obtain two conjugate isotropic generators, λ_1 and λ_2 say. Obviously we have:

$$\lambda_1 \cdot \lambda_2 = 1 .$$

The isotropic plane τ_1 through λ_1 goes through the tangent t_1 to C_∞ at the point at infinity A_1 of γ_1.

The coordinates of A_1 follow from (3) if we take $\lambda = \lambda_1$ and for instance $y = -2\lambda_1 b$. This gives:

$$A_1\ \{a(\lambda_1^2 - 1)\ ;\ -2\lambda_1 b\ ;\ c(\lambda_1^2 + 1)\ ;\ o\}$$

The equations of the tangent t_1 at A_1 to C_∞ are:

$$\left.\begin{aligned} a(\lambda_1^2 - 1)\,x - 2\lambda_1 by + c(\lambda_1^2 + 1)\,z &= o \\ w &= o \end{aligned}\right\} \qquad (4)$$

The plane τ_1 is the plane through t_1 and an arbitrary point of γ_1. Therefore we take in (1) for instance y = o and w = 1 which gives the following point on γ_1:

$$\{\tfrac{1}{2}a\left(\frac{1}{\lambda_1} + \lambda_1\right)\ ;\ o\ ;\ -\tfrac{1}{2}c\left(\frac{1}{\lambda_1} - \lambda_1\right)\ ;\ 1\} .$$

The equation of τ_1 is, using non-homogeneous coordinates:

$$a(\lambda_1^2-1)\ \{x-\tfrac{1}{2}a(\frac{1}{\lambda_1}+\lambda_1)\} -2\lambda_1 by + c(\lambda_1^2+1)\ \{z+\tfrac{1}{2}c(\frac{1}{\lambda_1}-\lambda_1) = o$$

or:

$$a(\lambda_1^2 - 1)x - 2\lambda_1 by + c(\lambda_1^2 + 1)z = \frac{a^2 + c^2}{2\lambda_1}(\lambda_1^4 - 1) \qquad (5)$$

Similarly the equation of the isotropic plane τ_2 is:

$$a(\lambda_2^2 - 1)x - 2\lambda_2 by + c(\lambda_2^2 + 1)z = \frac{a^2 + c^2}{2\lambda_2}(\lambda_2^4 - 1) \qquad (6)$$

3. The focal axis b_1 is the line of intersection of the isotropic planes τ_1 and τ_2. After dividing (5) by λ_1 and (6) by λ_2 and after addition and substraction we have for the equations of b_1:

$$\begin{cases} x = \frac{1}{a}\sqrt{(a^2 + c^2)(a^2 - b^2)} \\ z = \frac{b}{c} y \sqrt{\frac{a^2 + c^2}{a^2 - b^2}} \end{cases}$$

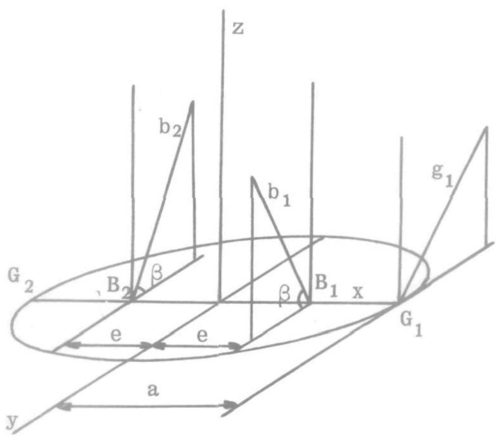

Figure 34

Similarly we get for b_2:

$$\begin{cases} x = -\frac{1}{a}\sqrt{(a^2 + c^2)(a^2 - b^2)} \\ z = -\frac{b}{c} y \sqrt{\frac{a^2 + c^2}{a^2 - b^2}} \end{cases}$$

So we obtain: *The focal axes b_1 and b_2 of the hyperboloid*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

(where $a > b$) *intersect the x-axis at a right angle in the points $B_1(e;o;o)$ and $B_2(-e;o;o)$,* if we write briefly $e = \frac{1}{a}\sqrt{(a^2 + c^2)(a^2 - b^2)}$ (fig. 34).

§ 4. A movable quadrilateral

1. Let g_1 be the generator of the considered series which goes through one of the vertices G_1 on the major axis (x-axis) of the hyperboloid H. As g_1 and b_1 are perpendicular to the x-axis the reflected line b_1^{r1} of b_1 with regard to g_1 intersects the x-axis at a right angle. Therefore the x-axis is the common perpendicular of b_1^{r1} and b_2. As b_1^{r1} is a generator of the hyperboloid of revolution Δ_2 with b_2 as its axis and the common perpendicular of b_1^{r1} and b_2 intersects b_2 in B_2, we conclude that *B_2 is the centre of Δ_2*.

The reflected point B_1^{r1} of B_1 with regard to g_1 lies on the x-axis. As $B_1^{r1}B_2$ is the shortest distance of the lines b_1^{r1} and b_2, the point B_1^{r1} is a point of the minimum circle of Δ_2. If B_1 is reflected to all generators of H the path of the reflected point B_1^r is this minimum circle.

Similarly we have that *the path of B_2^r is the minimum circle of the hyperboloid of revolution* Δ_1 with b_1 as its axis and B_1 as its centre. From the symmetry follows that the hyperboloids Δ_1 and Δ_2 are congruent.

2. If we reflect (fig. 35) the figure formed by b_1, b_2 and its common normal B_1B_2 with regard to any generator g of the hyperboloid H, we obtain the reflected figure formed by b_1^r, b_2^r and its common normal $B_1^rB_2^r$.

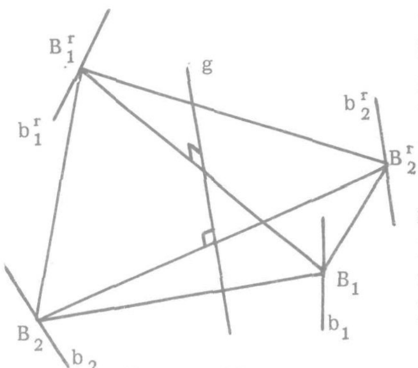

Figure 35

As B_2^r is a point of the minimum circle of Δ_1, the line $B_2^rB_1$ is the common perpendicular of b_2^r and b_1. Similarly $B_1^rB_2$ is the common perpendicular of b_1^r and b_2. The lines $B_2^rB_1$ and $B_1^rB_2$ are equal for Δ_1 and Δ_2 are congruent.

If g is a moving generator of H, the quadrilateral $B_1B_2B_1^rB_2^r$ moves, except the fixed lines b_1, b_2 and B_1B_2.

During the motion the quadrilateral has the following properties:

1. its opposite sides are equal and constant,
2. the lines b_1, b_2, b_1^r, b_2^r are in each position perpendicular to the adjacent sides,
3. the angle between b_1 and b_2 is equal to the angle between b_1^r and b_2^r and therefore this last angle is constant,
4. the angle between b_1^r and b_2 is constant, because b_1^r is a generator of the hyperboloid of revolution Δ_2 with b_2 as its axis and this angle is equal to that between b_2^r and b_1, for Δ_1 and Δ_2 are congruent.

From these properties we draw the conclusion that *if the four sides of the quadrilateral $B_1B_2B_1^rB_2^r$ are material rods and if they are joined in the vertices by hinges*, such that the lines b_1, b_2, b_1^r, b_2^r are hinge-axes, *the quadrilateral is movable.*

Chapter III

THEOREMS ON THE SKEW ISOGRAM MECHANISM *)

§ 1. Degrees of freedom of a skew n-gon

1. We consider (fig. 36) a skew n-gon $A_o A_1 \dots A_{n-1}$ of which the sides are rods, and these rods are hinged in the vertices. The hinge-axes are supposed to be in rigid connection with the rods. In each vertex A_p the hinge-line (or hinge-axis) h_p is perpendicular to the rods through that vertex. Each rod with its two hinge-axes is called a *link of the n-gon.* The angle φ_p ($p = 0, 1, \dots, n-1$) is the angle between the two rods through the vertex A_p.

2. To find the number of degrees of internal freedom of the n-gon we construct the figure in the following way:

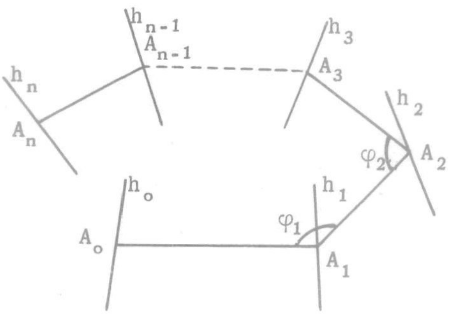

Figure 36

Let A_oA_1 with its hinge-lines h_o and h_1 be a fixed link.
If the link A_1A_2 is hinged to this fixed link A_oA_1 we obtain a mechanism with one degree of freedom, for the angle φ_1 between the two links gives the position of the mechanism. If we continue this procedure untill the n^{th} rod $A_{n-1} A_n$ is hinged to the $(n-1)^{th}$ rod $A_{n-2} A_{n-1}$, we obtain a mechanism with $(n - 1)$ degrees of freedom.

The carthesian coordinates (x, y, z) (with regard to the fixed space) of the points of the rod $A_{n-1} A_n$ and of the hinge-line h_n are functions of the angles $\varphi_1, \varphi_2, \dots \varphi_{n-1}$.

The following conditions have to be satisfied:

1. A_o and A_n must coincide,
2. h_o and h_n must coincide

or:

1. A_o and A_n must coincide,
2. Any point P_n of h_n that does not coincide with A_n must lie on h_o.

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*) Bennett [1], [2].

The first condition gives three relations between the angles $\varphi_1, \varphi_2 \ldots, \varphi_{n-1}$ and the second one gives two relations between these angles. Hence there exist 5 relations between the (n – 1) angles, that is, (n – 6) angles are independent. The position of the n-gon is given by (n – 6) data, that is, *the mechanism has in general (n – 6) degrees of internal freedom.* This is a special case of the so-called Grübler-formula.

3. If n = 4 the skew n-gon is a quadrilateral. From the formula follows that *this mechanism is triply stiff.* We obtain a quadrilateral which is not rigid only if special conditions are fulfilled.

A sufficient condition is that the four hinge-axes are parallel. The quadrilateral is then a plane one and it is well-known that it is movable with one degree of freedom.

We get another case if the opposite sides are equal. This quadrilateral has been considered in a paper of G. T. Bennett in 1903, who called it *a skew isogram mechanism.*

§ 2. Sufficient conditions for a quadrilateral to be movable

1. Let the quadrilateral be denoted by ABA'B' and the hinge-lines in its vertices by h, k, h', k' respectively (fig. 37). We take the rod AB with its hinge-lines h and k as a fixed link. If AB = A'B' = b and A'B = = AB' = a we obtain a quadrilateral with equal opposite sides. To prove that this quadrilateral is movable, we suppose first that the hinge-joints in A' and B' are replaced by ball-joints. It is evident that in this case the quadrilateral is movable (Grübler-formula).

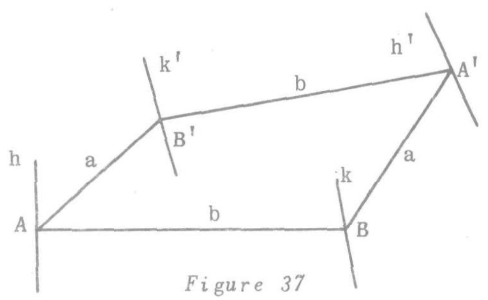

Figure 37

2. Let (fig. 38) N be the midpoint of AA' and M that of BB'. The triangle AMA' is isosceles, as AM and A'M are corresponding medians in the congruent triangles ABB' and A'B'B. Likewise the triangle BNB' is isosceles. Hence MN is perpendicular to AA' and to BB', that is, MN is the common normal of the diagonals AA' and

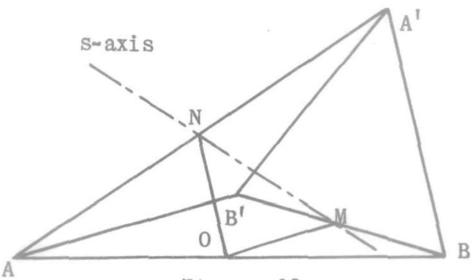

Figure 38

BB'. As AN = NA' and BM = MB' the axial reflection of AB with regard to the line MN gives A'B'. *The line MN is an axis of symmetry of the quadrilateral and is called the s-axis.*

The hinge-lines h and k of the vertices A and B give by reflecting with regard to the S-axis the lines h' and k' through the vertices A' and B'. The line h' is perpendicular to the two rods through A' and the line k' to the rods through B'. As the angle between h' and k' is equal to the angle between h and k and this last angle is constant during the motion of the quadrilateral, it follows that *the angle between h' and k' is constant.* However, we have still to prove that the angles between h' and k and between h and k' are constant.

3. Let O be the midpoint of AB. We have: OM is parallel to AB', ON is parallel to BA' and OM = ON = ½a. As the planes ABB' and ABA' are fixed planes the points M and N describe circles if the quadrilateral moves. Between the points M and N of these circles there exists a one-to-one correspondence and the points of intersection of these circles are self-corresponding points. Hence *the surface generated by MN is a quadric, namely a hyperboloid H* (Chapter II, theorem IV). As the line AB is the common diameter of two circles on H it follows that AB is an axis of H and from the symmetry it follows that O is the centre of H.

The path of the reflected point A^r of A with regard to the generators MN of H is the circle described by A'. Therefore A is a point of a focal axis of H (Chapter II, § 2.2). As h is perpendicular to the plane of the circle on H described by N and as h goes through A, *h is a focal-axis of H.* Similarly we get that k is a focal-axis of H.

4. Now we have the same problem as given in Chapter II, § 4. Hence the angle between h' and k is constant and so is the angle between h and k'. The *ball-joints in A' and B' can be replaced by hinge-joints* with h' and k' as their axes. We have proved that the conditions AB = A'B' and A'B = AB' are *sufficient conditions* for the quadrilateral to be movable.

§ 3. Necessary conditions for a quadrilateral to be movable

1. Let the sides of the quadrilateral $ABA'B'$ be: $AB = a$; $BA' = b$; $A'B' = c$; $B'A = d$ (fig. 39). A is the origin of a system of coordinates, B is a point of the X-axis and the hinge-axis h in A coincides with the Z-axis. The angle between the hinge-axis k in B and the plane XOZ is α, where $o < \alpha < \pi$. Consequently the point A' moves in the fixed plane ABA' which is the plane through AB such that the angle between this plane and the plane XOY is α.

Let the angle BAB' be denoted by φ, the angle $B'A'B$ by ξ, the angle ABA' by $\pi - \psi$ and the angle $A'B'A$ by $\pi - \varepsilon$. The projection of A' upon the plane XOY is denoted by A'', the projection of B' upon the plane ABA' by B'', the projections of A' upon AB and AB' by P and Q respectively and the projections of B' upon AB and $A'B$ by R and S respectively.

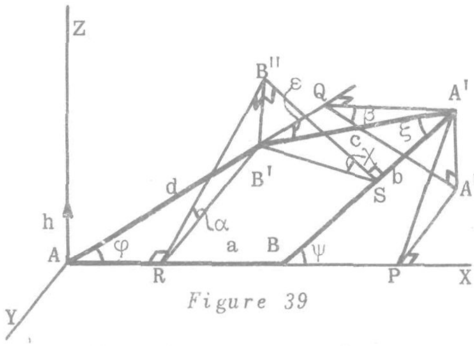

Figure 39

2. *We suppose that the quadrilateral, which is hinged in its vertices is movable.* The angles $A'QA'' = \beta$ and $B'S'B'' = \chi$ are constant during the motion, for β is the angle between the hinge-axes of A and B' and χ is the one between the hinge-axes of B and A'. As $A'A'' = A'P \sin\alpha = b \sin\psi \sin\alpha$ and $A'A'' = A'Q \sin\beta = c \sin\varepsilon \sin\beta$ we obtain

$$b \sin\psi \sin\alpha = c \sin\varepsilon \sin\beta$$

or:

$$\frac{\sin\varepsilon}{\sin\psi} = \frac{b \sin\alpha}{c \sin\beta}$$

As b, c, α and β are constant, $\frac{\sin\varepsilon}{\sin\psi}$ is also constant during the motion. We write:

$$\frac{\sin\varepsilon}{\sin\psi} = p \tag{1}$$

The cosine-law in the triangles ABA' and $A'B'A$ gives:

$$a^2 + b^2 + 2ab\cos\psi = c^2 + d^2 + 2cd\cos\varepsilon$$

or, briefly written,

$$\cos\varepsilon = q\cos\psi + r \tag{2}$$

where $q = \frac{ab}{cd}$ and $r = \frac{a^2 + b^2 - c^2 - d^2}{2cd}$

Elimination of ε from (1) and (2) gives:

$$1 = p^2 \sin^2\psi + q^2 \cos^2\psi + 2qr \cos \psi + r^2$$

or: $$(q^2 - p^2) \cos^2\psi + 2qr \cos \psi + r^2 + p^2 - 1 = o$$

If the quadrilateral is movable this equation has to be an identity with regard to ψ, *that is,*

$$q^2 - p^2 = o \; ; \qquad 2qr = o \; ; \qquad r^2 + p^2 - 1 = o$$

As q is unequal to zero the second condition gives:

$$r = o$$

We get: $$p^2 = q^2 = 1 \; ; \qquad r = o \, .$$

As q is positive we obtain:

$$q = 1 \; ; \qquad r = o \, .$$

or: $$ab = cd \; ; \qquad a^2 + b^2 = c^2 + d^2 \qquad (3)$$

Similarly we get if we use the cosine-law in the triangles B'AB and BA'B' and the relations: $B'B'' = d \sin\varphi \sin\alpha = c \sin\xi \sin\chi$:

$$ad = bc \qquad \text{and} \qquad a^2 + d^2 = b^2 + c^2 \qquad (4)$$

From the four relations (3) and (4) follows immediately:

$$a^2 = c^2 \qquad \text{and} \qquad b^2 = d^2$$

or: $$a = c \qquad \text{and} \qquad b = d$$

We obtain: *Necessary and sufficient conditions that a skew quadrilateral hinged in its vertices be movable are: the opposite sides are equal.*

§ 4. A relation between the twists and the sides

1. Let (fig. 40) A_pA_{p+1} be a link with its two hinge-lines h_p and h_{p+1}. We shall give these hinge-lines a direction. *The twist of the link* A_pA_{p+1} *is the angle of rotation of a right screw-motion* with A_pA_{p+1} as its axis *which replaces* h_p *to* h_{p+1}. This angle is equal to that of the right screw-motion with $A_{p+1}A_p$ as its axis which replaces h_{p+1} to h_p.

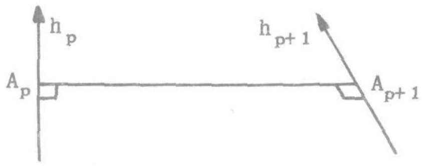

Figure 40

2. Let (fig. 41) ABA'B' be a movable quadrilateral hinged in its vertices, briefly called *a quadrilateral of Bennett or an isogram.* The twists of AB' and AB are denoted by $(2\pi - \alpha)$ and β respectively and the opposite sides AB = A'B' by b and AB' = A'B by a. The points A, B, A', B' can be considered as the vertices of a tetrahedron. Two planes go through each of its edges. As h is normal to the plane B'AB (plane ε_h) and k is normal to the plane ABA' (plane ε_k), the angle between h and k, that is, the twist β of the link AB, is equal to the angle between ε_h and ε_k. Similarly the angle between ε_h and $\varepsilon_{k'}$ (plane A'B'A) is equal to the twist $(2\pi - \alpha)$ of the link AB'.

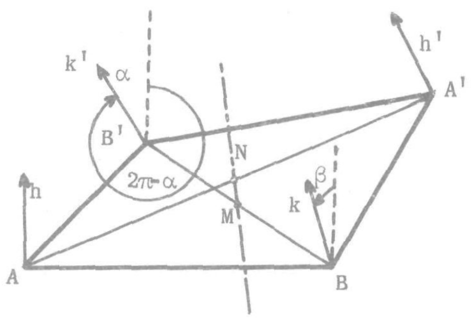

Figure 41

In § 2 we proved that the line MN connecting the midpoints M and N of the diagonals BB' and AA' is an axis of symmetry (s-axis) of the figure. From this symmetry follows the theorem:

The twists of the opposite links are equal.

If the quadrilateral moves with AB as its fixed link the twists are constant during the motion and so are the angles between the two planes through each of the four sides. For this reason the quadrilateral is called an isogram.

3. If (fig. 42) E is the projection of A' on the plane ε_h, F and G the projections of E on AB and AB' respectively we state that the angle A'GE is the angle α between the planes ε_h and $\varepsilon_{k'}$. Similarly the angle A'FE is β. From the congruence of the tri-

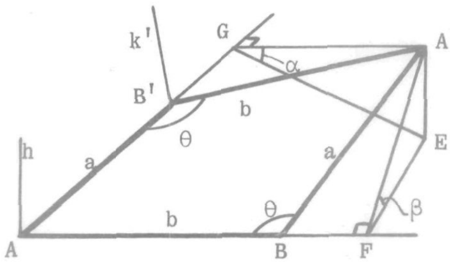

Figure 42

angles ABA' and $A'B'A$ we conclude that the angles ABA' and $A'B'A$ are equal, θ say.

In triangle BFA' we have $A'F = a \sin\theta$ and in triangle EFA' we have: $A'E = A'F \sin\beta$. Hence $A'E = a \sin\theta \sin\beta$.

Similarly we obtain in the triangles $B'GA'$ and EGA':

$A'G = b \sin\theta$; $A'E = A'G \sin\alpha$ and hence $A'E = b \sin\theta \sin\alpha$. The two expressions for $A'E$ give:

$$a \sin\theta \sin\alpha = b \sin\theta \sin\beta$$

or:

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} \quad \text{or} \quad -\frac{a}{\sin(-\alpha)} = \frac{b}{\sin\beta} \; {}^{*)}$$

We obtained the theorem:

In an isogram the ratio of the length of a link and the sine of its twist has the same value or the opposite value for each link.

§ 5. A relation between the angles of the isogram *)

1. In figure 43 (which is the same as figure 42) we denote the angles BAB' and $B'A'B$ by φ. Projection of the line AGEF upon the line AB gives:

$$AF = AG\cos\varphi + GE\sin\varphi$$

or:

$$b + a\cos(\pi - \theta) = \{a + b\cos(\pi - \theta)\}\cos\varphi + b\sin\theta\cos\alpha\sin\varphi$$

or: $$b - a\cos\theta = (a - b\cos\theta)\cos\varphi + b\sin\theta\cos\alpha\sin\varphi \quad (4)$$

The projection of the line AFEG upon the line AB' gives:

$$AG = AF\cos\varphi + FE\sin\varphi$$

or: $$(a - b\cos\theta) = (b - a\cos\theta)\cos\varphi + a\sin\theta\cos\beta\sin\varphi \quad (5)$$

*) Macmillan [13].

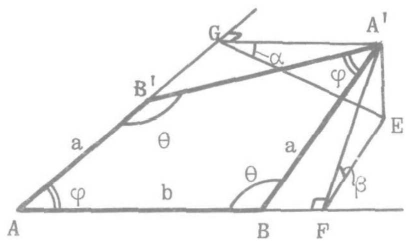

Figure 43

Addition of the equations (4) and (5) gives:

$$(a+b)-(a+b)\cos\theta =$$
$$= (a+b)\cos\varphi - (a+b)\cos\theta\cos\varphi + \sin\theta\sin\varphi\,(b\cos\alpha + a\cos\beta)$$

or:

$$(a+b)(1-\cos\theta)(1-\cos\varphi) = \sin\theta\sin\varphi\,(b\cos\alpha + a\cos\beta)$$

or:

$$(a+b)\,.\,2\sin^2\tfrac{1}{2}\theta\,.\,2\sin^2\tfrac{1}{2}\varphi =$$
$$= 2\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta\,.\,2\sin\tfrac{1}{2}\varphi\cos\tfrac{1}{2}\varphi\,(b\cos\alpha + a\cos\beta)$$

or: $$\tan\tfrac{1}{2}\theta\,.\,\tan\tfrac{1}{2}\varphi = (b\cos\alpha + a\cos\beta) : (a+b) \qquad (6)$$

As a, b, α and β are constant during the motion we obtain the theorem: *The product of the tangents of the adjacent semi-angles of a moving isogram is constant.*

2. Since $a : \sin\alpha = b : \sin\beta$ the equation (6) becomes:

$$\begin{aligned}\tan\tfrac{1}{2}\theta\tan\tfrac{1}{2}\varphi &= (\sin\beta\cos\alpha + \sin\alpha\cos\beta) : (\sin\alpha + \sin\beta)\\ &= \sin(\alpha+\beta) : 2\sin\tfrac{1}{2}(\alpha+\beta)\cos\tfrac{1}{2}(\alpha-\beta)\\ &= \cos\tfrac{1}{2}(\alpha+\beta) : \cos\tfrac{1}{2}(\alpha-\beta)\\ &= (\cos\tfrac{1}{2}\alpha\cos\tfrac{1}{2}\beta - \sin\tfrac{1}{2}\alpha\sin\tfrac{1}{2}\beta) :\\ &\quad\ (\cos\tfrac{1}{2}\alpha\cos\tfrac{1}{2}\beta + \sin\tfrac{1}{2}\alpha\sin\tfrac{1}{2}\beta)\\ &= \frac{1-\tan\tfrac{1}{2}\alpha\tan\tfrac{1}{2}\beta}{1+\tan\tfrac{1}{2}\alpha\tan\tfrac{1}{2}\beta}\end{aligned}$$

By aid of this expression we shall develop a planimetric construction to find corresponding values of θ and φ if the twists α and β are given.

3. Let $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ be the equation of an ellipse such that the excentricity $e = \frac{1}{a}\sqrt{a^2 - b^2}$ is equal to $\tan\frac{1}{2}\alpha\,.\,\tan\frac{1}{2}\beta$, where $2\pi - \alpha$ and β are the twists of a given isogram (fig. 44).

Any point P of the ellipse is connected with the foci F_1 and F_2. The angles PF_1F_2 and PF_2F_1 in triangle PF_1F_2 are denoted by θ' and φ'. We write $PF_1 = r_1$ and $PF_2 = r_2$. We have the following relations:

$r_1 + r_2 = 2a$; $F_1F_2 = 2\sqrt{a^2 - b^2} = 2c$ and $e = c : a$.

The sine-law in triangle PF_1F_2 gives:

$$r_1 : \sin\varphi' = r_2 : \sin\theta' = 2c : \sin(\varphi' + \theta')$$

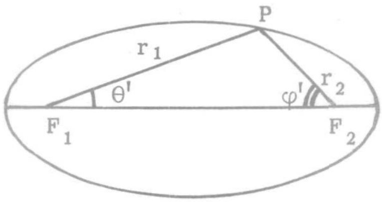

Figure 44

From $r_1 + r_2 = 2a$ follows:

$$r_1 + r_1 (\sin\theta' : \sin\varphi') = 2a$$

or

$$r_1 = 2a \sin\varphi' : (\sin\varphi' + \sin\theta')$$

Substitution in $r_1 : \sin\varphi' = 2c : \sin(\varphi' + \theta')$ gives:

$$2a \sin\varphi' : \sin\varphi' (\sin\varphi' + \sin\theta') = 2c : \sin(\varphi' + \theta')$$

or
$$a : 2\sin\tfrac{1}{2}(\varphi' + \theta') \cos\tfrac{1}{2}(\varphi' - \theta') = c : 2\sin\tfrac{1}{2}(\varphi' + \theta') \cos\tfrac{1}{2}(\varphi' + \theta')$$

or
$$a(\cos\tfrac{1}{2}\varphi' \cos\tfrac{1}{2}\theta' - \sin\tfrac{1}{2}\varphi' \sin\tfrac{1}{2}\theta') = c(\cos\tfrac{1}{2}\varphi' \cos\tfrac{1}{2}\theta' + \sin\tfrac{1}{2}\varphi' \sin\tfrac{1}{2}\theta')$$

or
$$\tan\tfrac{1}{2}\theta' \cdot \tan\tfrac{1}{2}\varphi' = (a - c) : (a + c) = (1 - e) : (1 + e) \doteq (1 - \tan\tfrac{1}{2}\alpha \tan\tfrac{1}{2}\beta) : (1 + \tan\tfrac{1}{2}\alpha \tan\tfrac{1}{2}\beta) .$$

In 2 we found that this last expression was equal to $\tan\tfrac{1}{2}\theta \tan\tfrac{1}{2}\varphi$. If we take $\theta' = \theta$, we obtain $\varphi' = \varphi$, that is, θ' and φ' *are the corresponding values of the adjacent angles of an isogram.*

§ 6. Quadrics associated with the isogram

1. The internal angle bisectors of the angles of the isogram $ABA'B'$ (fig. 45) are denoted by h_1, k_1, h'_1 and k'_1 and the external angle bisectors by h_2, k_2, h'_2 and k'_2. The plane through an angle bisector and the normal through the vertex upon the plane of the angle is called a *bisecting plane*. The internal bisecting plane of the angle A is the plane through h and h_1. It is denoted by ε_{h1}. The plane ε_{h2} is the external bisecting plane of A and it goes through h and h_2. Likewise are defined the planes ε_{k1}, ε_{k2}, $\varepsilon_{h'_1}$, $\varepsilon_{h'_2}$, $\varepsilon_{k'_1}$ and $\varepsilon_{k'_2}$.

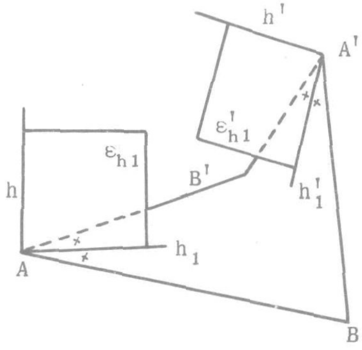

Figure 45

From the symmetry of the figure with regard to the s-axis follows that the planes ε_{h1} and ε'_{h1} intersect each other by a line, r_1 say, such that r_1 intersects the s-axis in the point C_1 at a right angle.

As C_1 is a point of ε_{h1} the distances to AB and A'B' are equal and as C_1 is a point of ε'_{h1} the distances to A'B and to A'B' are equal. As C_1 is a point of the s-axis the distances to AB and to A'B' are also equal, that is, the distances of C_1 to the four sides (or their extensions) are equal. *C_1 can be considered as the centre of a sphere touching the sides of the isogram.* Similarly the internal bisecting planes ε_{k1} and ε'_{k1} intersect the s-axis in the point C_2 which can also be considered as the centre of a sphere touching the sides. The line of intersection of ε_{k1} and ε'_{k1} is the line r_2 through C_2 perpendicular to the s-axis.

2. If a skew quadrilateral ABCD has a sphere which touches the sides in points between the vertices, the relation AB + CD = = BC + AD must hold. As an isogram is a quadrilateral with equal opposite sides it has not such a sphere, unless the four sides are equal.

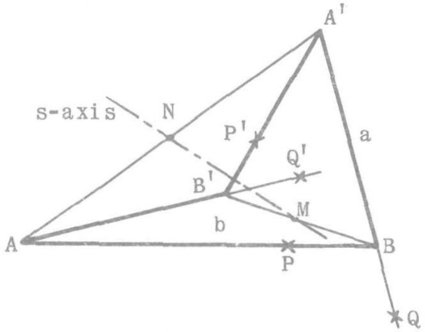

Figure 46

Let the points of contact of the sphere around C_1 with the sides be P, Q, P', Q' (fig. 46). From the symmetry follows: AP = A'P' and A'Q = AQ'. Consequently we get: PB = P'B' and QB = Q'B'.

As AP = AQ' (two tangents through B to the sphere) we obtain AP = A'Q.

If we write AP = A'Q = x we get:

$$BP = |x - b| \qquad \text{and} \qquad BQ = |x - a|$$

As BP = BQ (two tangents) we have:

$$|x - b| = |x - a|$$

or:

$$x = \tfrac{1}{2}(a + b)$$

If $b > a$, P lies on AB and Q lies on the extension of A'B and if $b < a$, P lies on the extension of AB and Q on A'B. Hence the sphere with C_1 as its centre touches either AB and the extension of A'B or A'B and the extension of AB. Therefore *the plane* ε_{k2} *goes through* C_1. So does the plane ε_{k2}'.

The line PQ goes, as follows from Menelaos theorem in triangle ABA', through the midpoint N of AA'. As P'Q' also goes through N the lines PQ and P'Q' lie in one plane. The same theorem used in triangle ABB' gives that PQ' goes through M. P'Q also goes through M.

As PQ' is parallel to h_2 we get that PQ' is normal to ε_{h1} and as r_1 lies in ε_{h1} we obtain: r_1 is normal to PQ'. As P'Q is parallel to h_2' we get likewise: r_1 is normal to P'Q. Hence r_1 is normal to the plane PQP'Q', that is, r_1 is perpendicular to PQ. PQ is normal to the external bisecting plane ε_{k2} and consequently r_1 is parallel to ε_{k2}. As r_1 and ε_{k2} go through the same point C_1, we conclude that ε_{k2} goes through r_1. Similarly ε_{k2}' goes through r_1.

Analogously we obtain that the planes ε_{h2}, ε_{h2}', ε_{k1}, ε_{k1}' go through the line r_2 which intersects the s-axis at a right angle in the point C_2. We derived the following

Theorem: *The internal bisecting planes of two opposite angles of the isogram ABA'B' and the external bisecting planes of the opposite angles go through one line r* which intersects the axis of symmetry in the point C at a right angle. This point C can be considered as the centre of a sphere which touches the sides (or their extensions) of the isogram. *The four points of contact lie in one plane which goes through the axis of symmetry and is perpendicular to the line r.* We get two points C namely C_1 and C_2 with their corresponding lines r_1 and r_2.

As h lies in ε_{h1}, k in ε_{k2}, h' in ε_{h1}' and k' in ε_{k2}' and as these planes go through the line r_1, we obtain the

Theorem: *The four hinge-lines of an isogram intersect the lines* r_1 *and* r_2.

Further we conclude:

The locus of the centres of the spheres which touch the sides (or their extensions) of an isogram consists of the lines r_1 *and* r_2 *).

3. If the line AB rotates around the line r_1 a hyperboloid of revolution R_1 is generated. As the plane ε_{h1} goes through r_1 and the lines AB and AB' are symmetrical with regard to this plane ε_{h1}, AB' generates the same hyperboloid R_1 when rotating around r_1. As ε_{k2} goes through r_1 and the lines AB and A'B (with their

*) This theorem is a completion of the theorems of Bennett.

extensions) are symmetrical with regard to this plane, A'B lies on R_1. From this follows that A'B' also lies on R_1. As ε_{h1}' goes through r_1 and A'B and A'B' are symmetrical with regard to ε_{h1}', AB and A'B' belong to one of the series of generators of R_1 and A'B and AB' belong to the other series.

Similarly we have an hyperboloid of revolution R_2 with r_2 as its axis on which the four sides of the isogram lie. So we get the

Theorem: *There exist two hyperboloids of revolution R_1 and R_2 with the four sides of an isogram as generators.* Their axes are the lines of intersection of the internal bisecting planes of opposite angles of the isogram.

4. The sides AB and AB' are generators of R_1 and R_2. As these generators intersect each other the plane ε_h through AB and AB' is a tangent plane of R_1 and of R_2. The point of contact is A. The angle bisectors h_1 and h_2 of the angle BAB' lie in the plane ε_h and go through the point of contact. Consequently they are tangents to R_1 and R_2. Generally we have that each plane through two adjacent sides of the isogram touches R_1 and R_2 in their common vertex. So we have: *The internal and external bisectors of the angles of an isogram are tangents to the hyperboloids of revolution R_1 and R_2.*

5. In 3 we showed that r_1 (the line of intersection of ε_{h1} and ε_{h1}') is perpendicular to the plane PQP'Q', where P, Q, P' and Q' are the points of contact of the sphere with C_1 (the common point of r_1 and the s-axis) as its centre touching the sides of the isogram.

As (fig. 47) PB = BQ, the line PQ is parallel to k_1. Similarly

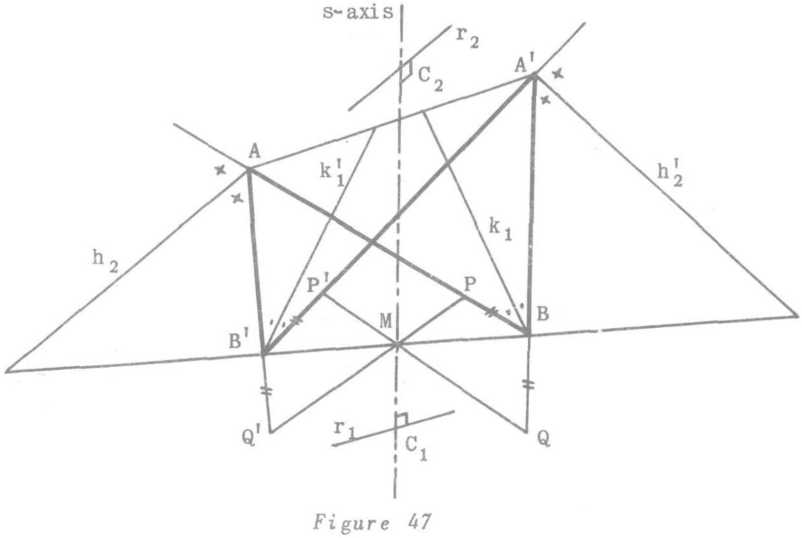

Figure 47

$P'Q'$ is parallel to k_1'. As $AP = AQ'$ and $A'P' = A'Q$ we get: PQ' is parallel to h_2 and $P'Q$ is parallel to h_2'. Hence r_1 is perpendicular to k_1, k_1', h_2 and h_2', that is, these lines are parallel to a plane normal to r_1, for instance the plane $PQP'Q'$. As these four lines intersect the lines AA' and BB', they can be considered as generators of a paraboloid. The lines lie in the planes ε_{k1}, ε_{k1}', ε_{h2} and ε_{h2}' respectively. These planes go through the line r_2 and therefore the lines k_1, k_1', h_2 and h_2' intersect r_2. Hence r_2 is a generator of the paraboloid denoted by Π_2. The s-axis intersects the generators AA', BB' and r_2 and consequently this axis is also a generator of Π_2. The lines AA', BB' and r_2 are perpendicular to the s-axis. Therefore a plane normal to the s-axis is a direction-plane of the paraboloid. As the plane $PQP'Q'$ is also a direction-plane and the two direction-planes are perpendicular to each other, the paraboloid Π_2 is a rectangular one.

Similarly the lines h_1, h_1', k_2, k_2' and the s-axis can be considered as generators of a rectangular paraboloid Π_1. The line r_1 lies on Π_1. The paraboloids Π_1 and Π_2 have the s-axis and the diagonals AA' and BB' in common. We obtain the theorem: *The internal angle bisectors of two opposite angles, the external angle bisectors of the other opposite angles and the axis of symmetry* (s-axis) *of an isogram belong to one of the series of generators of a rectangular paraboloid* Π. The diagonals of the isogram belong to the other series of generators.

6. If we take a pencil of planes through r_1 and a pencil through r_2, we can consider the lines of intersection of the planes of the first pencil with the planes of the second one which are perpendicular to the first planes. We observed in Chapter I, § 6.3 that the locus of these lines of intersection is a hyperboloid Ω. We quote the following well-known theorems of this quadric:

1. The common normal of r_1 and r_2 is an axis of Ω;
2. The points of intersection of this common normal with r_1 and with r_2 are two vertices of Ω;
3. The curves of intersection of Ω with planes normal to r_1 or to r_2 are circles;
4. The generators through the vertices on r_1 and r_2 are normal to the planes of these circles. A hyperboloid with this quality is called an orthogonal hyperboloid.

The internal and external bisecting planes of an angle of an isogram have the hinge-line through the vertex of that angle in common. These planes are perpendicular to each other. One of them goes through r_1 and the other through r_2. Consequently the hinge-lines lie on the orthogonal hyperboloid Ω and we obtain:

The four hinge-lines of an isogram are generators of an orthogonal hyperboloid Ω. One of its axes is the part of the s-axis limited by the points C_1 and C_2 which are the points in which the internal bisecting planes of the isogram intersect the s-axis.

7. Let (fig. 48) T be the projection of C_1 on the plane BAB' (or ε_h) and U the projection of C_2 on this plane. As the line C_1C_2 (s-axis) is perpendicular to BB', its projection TU on ε_k is also perpendicular to BB'. As C_1 is a point of r_1 and r_1 lies in the plane ε_{h1} which is perpendicular to ε_h, T lies on h_1. As C_2 lies in the plane ε_{h2} and this plane is perpendicular to ε_h, U lies on h_2. In the plane figure ABB' the internal bisector and the external bisector of the angle A of triangle ABB' intersect

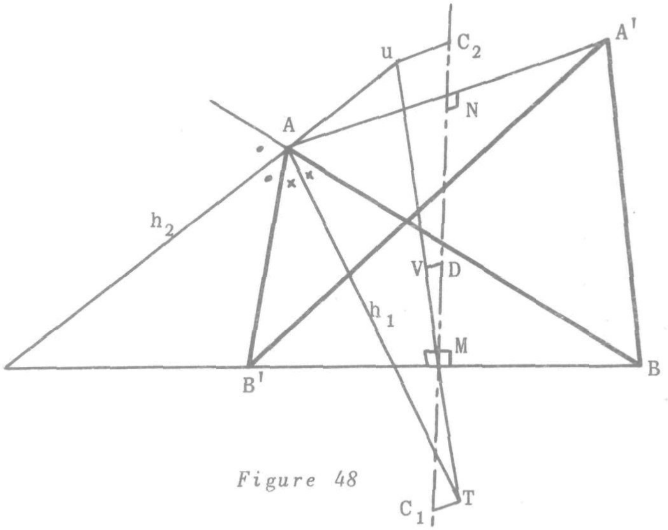

Figure 48

the line which intersects BB' in its midpoint M at a right angle in the points T and U such that TU is a diameter of the circumscribed circle of triangle ABB'. Its centre is the midpoint V of TU. The normal through V upon the plane ABB' intersects the line C_1C_2 in its midpoint D. The distances of D to the points A, B and B' are equal, and as D is a point of the s-axis, its distances to the four vertices of the quadrilateral are equal. Hence D is the centre of the circumscribed sphere of the isogram. We obtain the theorem: *The centre of the sphere through the vertices of an isogram is the midpoint of the line limited by the points of intersection* C_1 *and* C_2 *of the internal bisecting planes of the angles of the isogram with the s-axis.*

From 6 follows:

The centre D of the sphere through the vertices of an isogram coincides with the centre of the orthogonal hyperboloid Ω *which contains the hinge-lines.*

§ 7. The motion of the side $A'B'$ to one of its nullpositions

If we reflect (fig. 49) $A'B'$ with regard to the internal bisector h_1' of the angle $B'A'B$ of the isogram $ABA'B'$ we get the line $A'B_1'$. We state that $A'B_1' = A'B' = b$. If we reflect $A'B_1'$ with

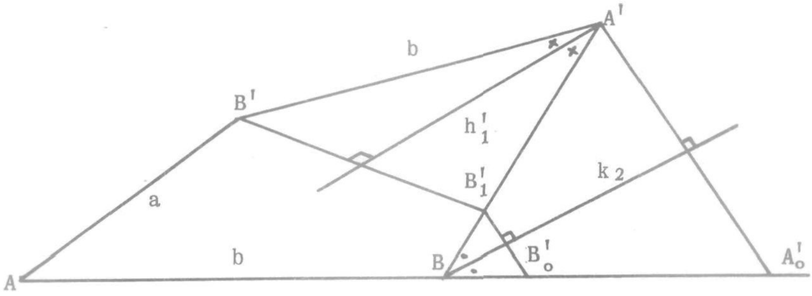

Figure 49

regard to the external bisector k_2 of the angle ABA' of the isogram we obtain the line $A_o'B_o'$ lying on the extension of AB. We notice that $A'B_1' = A_o'B_o' = b$. As $BB_1' = a - b$ and $BB_1' = BB_o'$ we get $AB_o' = AB + BB_o' = b + (a - b) = a$. The four points A, B, A_o', B_o' lie such that $AB = A_o'B_o' = b$ and $A_o'B = AB_o' = a$. The degenerated quadrilateral $ABA_o'B_o'$ can be considered as a special position of the given isogram $ABA'B'$. The positions in which one of the angles of an isogram is zero are called the *nullpositions*. An isogram has two nullpositions namely if $\angle B'AB = o$ or if $\angle ABA' = o$.

Now we consider the problem in the following way:

Take the point B_1' on $A'B$ such that $A'B_1' = AB = b$. Reflection of the line $A'B_1'$ with regard to the skew lines h_1' and k_2 gives the lines $A'B'$ and $A_o'B_o'$. We showed in chapter II, § 1 that it is possible to determine a screw motion with the common normal of h_1' and k_2 as its axis, which replaces $A'B'$ to $A_o'B_o'$. As the skew lines h_1' and k_2 are perpendicular to the line r_2 (§ 6.5), the screw axis is parallel to r_2. We obtain:

A screw motion with the common normal of the internal angle bisector of A' and the external bisector of B as its axis can be determined such that $A'B'$ can be replaced to $A'B'$ if the figure $ABA_o'B_o'$ is a nullposition of the isogram.

§ 8. The screw-axis of a small displacement of A'B'

As usual we take AB as the fixed link of the isogram ABA'B'. If the figure moves the axis of symmetry (s-axis) generates, as we showed in § 2 a hyperboloid H. We consider two positions of A'B', $(A'B')_1$ and $(A'B')_2$ say. Let the corresponding positions of the s-axis be s_1 and s_2.

Axial reflection of AB with regard to s_1 gives $(A'B')_1$ and with regard to s_2 it gives $(A'B')_2$. Therefore the positions $(A'B')_1$ and $(A'B')_2$ can be obtained by a screw-motion with the common normal n_{12} of s_1 and s_2 as its screw-axis.

If $(A'B')_2$ approaches $(A'B')_1$ and therefore s_2 approaches s_1 and we denote the limiting positions by A'B' and s respectively and the limiting position of n_{12} by n, we get in the limit: The instantaneous screw-axis of the motion of A'B' is the line n. This line n can be determined if we give the following remarks:

1. The locus of the limiting points of intersection of the common normal of two near-by generators of a ruled surface is its *line of striction*. The point of intersection of the line of striction with a generator is called the *central point* of this generator. If we have a ruled surface of the second degree there are two series of generators and hence there are two lines of striction.

2. The plane through a generator g_1 and parallel to its near-by generator g_2 is perpendicular to the common normal n_{12} of these generators. In the limiting position this plane is perpendicular to the limiting position n of n_{12}; it is the *asymptotic tangent-plane* through g.

3. The asymptotic plane is the tangent-plane through g that touches the ruled surface in the point at infinity of g.

Now we obtain the theorem: *The instantaneous screw-axis of the motion of A'B' is the line n through the central point and perpendicular to the asymptotic plane of the s-axis if this axis is considered as a generator of the hyperboloid H.*

Chapter IV

THE MOTION OF THE SPACE CONNECTED WITH THE CONNECTING-ROD A'B'

§ 1. The ruled surface generated by the s-axis

1. Let be given the isogram ABA'B' (fig. 50). The plane through AB and normal to the hinge-line h in A is denoted by ε_h and the one through AB and normal to the hinge-line k in B by ε_k. The position of the isogram with regard to a fixed rectangular system of coordinates will be chosen such that:

1. the midpoint of AB coincides with the origin O
2. AB lies in the x-axis
3. a bisecting plane of the planes ε_h and ε_k coincides with the plane XOY.

We denote: a) the angle between ε_h and ε_k by 2α
b) the length of AB = A'B' by 2a
c) the length of AB' = A'B by r
d) the angle between A'B and the positive X-axis by φ
e) the angle between AB' and the positive X-axis by ψ
f) the projections of A' and B' on the plane YOZ by A'_1 and B'_1.

If the isogram moves A' moves in the fixed plane ε_k and B' in the fixed plane ε_h. A'_1 moves along the fixed line of intersection of ε_k with the plane YOZ and B'_1 along the one of ε_h with the plane YOZ. The line MN where M and N are the midpoints of BB' and AA' is the line of symmetry or the s-axis of the isogram.

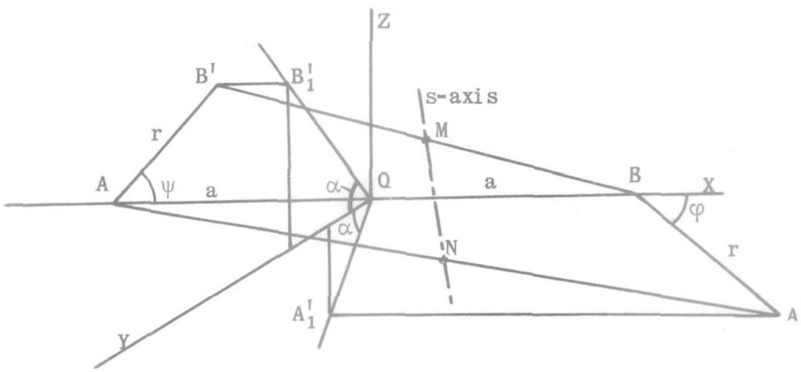

Figure 50

2. The coordinates of the points A, B, A', B', M and N are:

	x	y	z
A	$-a$	o	o
B	a	o	o
A'	$a + r\cos\varphi$	$r\sin\varphi\cos\alpha$	$-r\sin\varphi\sin\alpha$
B'	$-a + r\cos\psi$	$r\sin\psi\cos\alpha$	$r\sin\psi\sin\alpha$
M	$\frac{1}{2} r\cos\psi$	$\frac{1}{2} r\sin\psi\cos\alpha$	$\frac{1}{2} r\sin\psi\sin\alpha$
N	$\frac{1}{2} r\cos\varphi$	$\frac{1}{2} r\sin\varphi\cos\alpha$	$-\frac{1}{2} r\sin\varphi\sin\alpha$

(1)

The equations of the line MN (s-axis) are:

$$\frac{x - \frac{1}{2} r\cos\psi}{\cos\varphi - \cos\psi} = \frac{y - \frac{1}{2} r\sin\psi\cos\alpha}{\cos\alpha(\sin\varphi - \sin\psi)} = \frac{z - \frac{1}{2} r\sin\psi\sin\alpha}{-\sin\alpha(\sin\varphi + \sin\psi)} \quad (2)$$

The first and second member of (2) may be written:

$$x\cos\alpha(\sin\varphi - \sin\psi) - y(\cos\varphi - \cos\psi) =$$
$$= \tfrac{1}{2} r\cos\psi\cos\alpha(\sin\varphi - \sin\psi) - \tfrac{1}{2} r\sin\psi\cos\alpha(\cos\varphi - \cos\psi)$$
$$= \tfrac{1}{2} r\cos\alpha\sin(\varphi - \psi)$$

or, after dividing by $\sin\frac{1}{2}(\varphi - \psi)$:

$$2x\cos\alpha\cos\tfrac{1}{2}(\varphi + \psi) + 2y\sin\tfrac{1}{2}(\varphi + \psi) = r\cos\alpha\cos\tfrac{1}{2}(\varphi - \psi) \quad (3)$$

The first and third member of (2) give:

$$-2x\sin\alpha\cos\tfrac{1}{2}(\varphi - \psi) + 2z\sin\tfrac{1}{2}(\varphi - \psi) = -r\sin\alpha\cos\tfrac{1}{2}(\varphi + \psi) \quad (4)$$

If we denote $\frac{1}{2}(\varphi + \psi)$ and $\frac{1}{2}(\varphi - \psi)$ by λ and μ respectively, the equations (3) and (4) become:

$$2x\cos\alpha\cos\lambda + 2y\sin\lambda = r\cos\alpha\cos\mu \quad (5)$$
$$2x\sin\alpha\cos\mu - 2z\sin\mu = r\sin\alpha\cos\lambda \quad (6)$$

3. As the distance between A' and B' is 2a, we obtain:

$$4a^2 = \{2a + r(\cos\varphi - \cos\psi)\}^2 + r^2\cos^2\alpha(\sin\varphi - \sin\psi)^2 + r^2\sin^2\alpha(\sin\varphi + \sin\psi)^2$$

or: $$4a^2 = 4a^2 - 8ar\sin\lambda\sin\mu + 4r^2\sin^2\lambda\sin^2\mu + 4r^2\cos^2\alpha\sin^2\mu\cos^2\lambda + 4r^2\sin^2\alpha\sin^2\lambda\cos^2\mu$$

or: $$2a\sin\lambda\sin\mu = r\{\sin^2\lambda\sin^2\mu + \cos^2\alpha\sin^2\mu(1-\sin^2\lambda) + \sin^2\alpha\sin^2\lambda(1 - \sin^2\mu)\}$$

or: $$\frac{2a}{r} \sin \mu \sin \lambda = \cos^2 \alpha \sin^2 \mu + \sin^2 \alpha \sin^2 \lambda \qquad (7)$$

We write: $$\sin \lambda = m \sin \mu \qquad (8)$$

Substitution of (8) in (7) gives the relation:

$$\frac{2a}{r} m = \cos^2 \alpha + m^2 \sin^2 \alpha \qquad (7a)$$

From this relation follows that m is a function of α, a and r only. Hence *m is constant during the motion of the isogram.* The discussion about the reality of m is given in chapter VI, § 3.1

4. If in fig. 50 the values of α, r, a and ψ are given, the point A' can be considered as the point of intersection of:

1. a sphere around B' with radius 2a and
2. a circle in the plane XOA_1' around B with radius r.

In general a sphere and a circle have two points of intersection. The two positions of A' give two values of φ. Hence two values of φ correspond to a given series of values of α, r, a and ψ. As m is given by the relation (8) and λ and μ by:

$$\lambda = \tfrac{1}{2}(\varphi + \psi) \qquad \text{and} \qquad \mu = \tfrac{1}{2}(\varphi - \psi)$$

we obtain if we substitute the given value of ψ and the two corresponding values of φ in (8), two values m_1 and m_2 of m. These two values of m are the roots of the equation (7a).

Consequently *it is*, in general, *possible to construct two quadrilaterals when α, r and a are given. These two quadrilaterals are characterized by the values m_1 and m_2 of m* following from the equation (7a).

5. The equation of the ruled surface generated by the s-axis can be obtained by eliminating λ and μ from the equations (5), (6) and (8).

We write (6) as:

$$\cos \lambda = (2x \sin \alpha \cos \mu - 2z \sin \mu) : r \sin \alpha .$$

Substitution of cos λ in (5) gives:

$$2x \cos \alpha\, (2x \sin \alpha \cos \mu - 2z \sin \mu) : r \sin \alpha + 2y \sin \lambda = \\ = r \cos \alpha \cos \mu$$

and as $\sin \lambda = m \sin \mu$, we get:

$$4x^2 \sin \alpha \cos \alpha \cos \mu - r^2 \sin \alpha \cos \alpha \cos \mu = \\ = 4xz \cos \alpha \sin \mu - 2rym \sin \alpha \sin \mu$$

or: $\tan\mu = (4x^2 - r^2)\sin\alpha\cos\alpha : 2(2xz\cos\alpha - rym\sin\alpha)$ (9)

Analogously we substitute $\cos\mu$ from (5) in (6):

$$2x\sin\alpha\,(2x\cos\alpha\cos\lambda + 2y\sin\lambda) : r\cos\alpha - 2z\sin\mu = \\ = r\sin\alpha\cos\lambda$$

and as $\sin\mu = \frac{1}{m}\sin\lambda$ we obtain:

$$4x^2m\sin\alpha\cos\alpha\cos\lambda - r^2m\sin\alpha\cos\alpha\cos\lambda = \\ = 2rz\cos\alpha\sin\lambda - 4\,xym\sin\alpha\sin\lambda$$

or:

$\tan\lambda = (4x^2 - r^2)\,m\sin\alpha\cos\alpha : 2(rz\cos\alpha - 2xym\sin\alpha)$ (10)

The relation $\sin\lambda = m\sin\mu$ is reducible to:

$$\tan\lambda : \sqrt{1 + \tan^2\lambda} = m\tan\mu : \sqrt{1 + \tan^2\mu}$$

or: $$(1 - m^2)\tan^2\lambda\tan^2\mu = m^2\tan^2\mu - \tan^2\lambda$$

Substitution of $\tan\mu$ and $\tan\lambda$ from (9) and (10) gives:

$$\frac{(4x^2 - r^2)^2 m^2\sin^2\alpha\cos^2\alpha\,(4x^2 - r^2)^2\sin^2\alpha\cos^2\alpha\,(1 - m^2)}{4(2xz\cos\alpha - rym\sin\alpha)^2 \,.\, 4(rz\cos\alpha - 2xym\sin\alpha)^2} =$$

$$= \frac{m^2(4x^2 - r^2)^2\sin^2\alpha\cos^2\alpha}{4(2xz\cos\alpha - rym\sin\alpha)^2} - \frac{m^2(4x^2 - r^2)^2\sin^2\alpha\cos^2\alpha}{4(rz\cos\alpha - 2xym\sin\alpha)^2}$$

After dividing by $m^2(r^2 - 4x^2)^2\sin^2\alpha\cos^2\alpha$ and multiplying by the denominators we obtain:

$$(1 - m^2)\sin^2\alpha\cos^2\alpha\,(4x^2 - r^2)^2 = \\ = 4(rz\cos\alpha - 2xym\sin\alpha)^2 - 4(2xz\cos\alpha - rym\sin\alpha)^2$$

The right-hand member of this equation is reducible to:

$$4(rz\cos\alpha - 2xym\sin\alpha + 2xz\cos\alpha - rym\sin\alpha)(rz\cos\alpha - \\ - 2xym\sin\alpha - 2xz\cos\alpha + rym\sin\alpha) = \\ = 4(2x + r)(z\cos\alpha - ym\sin\alpha)(r - 2x)(z\cos\alpha + ym\sin\alpha) \\ = 4(r^2 - 4x^2)(z^2\cos^2\alpha - y^2m^2\sin^2\alpha)$$

The equation becomes, after dividing by $r^2 - 4x^2$:

$$(m^2 - 1)\sin^2\alpha\cos^2\alpha\,(4x^2 - r^2) = 4z^2\cos^2\alpha - 4y^2m^2\sin^2\alpha$$

or: $$(1 - m^2)\,x^2 + \frac{z^2}{\sin^2\alpha} - \frac{m^2y^2}{\cos^2\alpha} = \frac{(1 - m^2)\,r^2}{4} \qquad (11)$$

where m is given by the equation (7a).

6. From equation (11) follows that *the ruled surface generated by the s-axis is a hyperboloid* Γ. Its axes coincide with the X-, Y- and Z-axes respectively and its centre is the midpoint O of AB.

If the isogram moves the points M and N describe congruent circles in the planes ε_h and ε_k. O is their common centre and the X-axis is their common diameter. The two circles are the curves of intersection of the planes ε_h and ε_k with the hyperboloid. As $\alpha \neq 0$ these planes do not coincide and consequently *the hyperboloid* Γ *is never a hyperboloid of revolution.*

§ 2. The moving space S

1. The space connected with the X-, Y- and Z-axes is called the fixed space Σ. The line AB and the hinge-lines h and k are fixed lines in Σ. *The moving space S is the space connected with the line A'B' and the hinge-lines h' and k'.* These lines are fixed lines in S. If the quadrilateral moves, the space S also moves. Every point of S describes a curve in the fixed space Σ.

2. As the line MN(s-axis) is an axis of symmetry of the isogram this line is also an axis of symmetry for the spaces Σ and S. If any point P of Σ is reflected with regard to the s-axis, we get the point P'. The position of P with regard to the lines AB, h and k is identically equal to the position of P' with regard to the lines A'B', h' and k'. The point P' in S *corresponds* to the point P in Σ. *Axial reflection of P with regard to the s-axis in its several positions gives several positions of P' which can be considered as points of the path which the point P' of S describes in* Σ.

3. The twisted curves described by the points of S can be obtained by reflection of corresponding points of Σ with regard to one of the series of generators of the ruled surface Γ described by the s-axis (chapter II, § 1). Γ is the basic surface of the reflection. As Γ is a quadric the curves are generally space curves of the fourth degree. If we multiply Γ by two with regard to any point P in Σ we obtain a quadric Γ'. The curve described by P' lies on this quadric Γ'. In general we have:

The twisted curves described by the points of the moving space S lie on congruent quadrics Γ'. Γ' *is a quadric which is generated from the hyperboloid* Γ *by multiplying by two with regard to a point.*

§ 3. Reflection of a point P with regard to a line l

Let the coordinates of a point P be (p_1, p_2, p_3) and the equations of a line l:

$$\frac{x_1 - b_1}{a_1} = \frac{x_2 - b_2}{a_2} = \frac{x_3 - b_3}{a_3} \qquad (12)$$

or, more briefly:

$$\frac{x_k - b_k}{a_k} = t \quad (k = 1, 2, 3)$$

where x_1, x_2, x_3 are the current coordinates.

The point of reflection of P with regard to l is denoted by $S(s_1, s_2, s_3)$. The equation of the plane α through P perpendicular to l is:

$$\Sigma\, a_k(x_k - p_k) = o$$

The point of intersection F of this plane α with the line l is given by:

$$\Sigma\, a_k(a_k t + b_k - p_k) = o$$

or:

$$t\, \Sigma\, a_k^2 + \Sigma\, a_k b_k - \Sigma\, a_k p_k = o$$

or:

$$t = -\{\Sigma\, a_k b_k - \Sigma\, a_k p_k\} : \Sigma\, a_k^2$$

We obtain the coordinates of F, (f_1, f_2, f_3) say, by substitution of this value of t into the equations of l:

$$f_i = -a_i\, \{\Sigma\, a_k b_k - \Sigma\, a_k p_k\} : \Sigma\, a_k^2 + b_i \qquad (i = 1, 2, 3)$$

As F is the midpoint of PS' we have the relations:

$$s_i + p_i = 2f_i$$

and hence *the coordinates of the reflected point S of P are:*

$$s_i = 2a_i\, \{\Sigma\, a_k p_k - \Sigma\, a_k b_k\} : \Sigma\, a_k^2 + 2b_i - p_i \qquad (i = 1, 2, 3) \quad (13)$$

§ 4. Parametric equations of the curves described by the points of the moving space

1. Let $P(x_o, y_o, z_o)$ be a point of the fixed space Σ. Its corresponding point $\overset{o}{P}{}'$ in the moving space S is obtained by reflection of P with regard to the s-axis given by the equations:

$$2x \cos\alpha \cos\lambda + 2y \sin\lambda = r \cos\alpha \cos\mu \qquad (5)$$
$$2x \sin\alpha \cos\mu - 2z \sin\mu = r \sin\alpha \cos\lambda \qquad (6)$$
$$\sin\lambda = m \sin\mu \qquad (8)$$

The equations (5) and (6) are reducible to:

$$\frac{x}{\sin\lambda \sin\mu} = \frac{y - \dfrac{r\cos\alpha\cos\mu}{2\sin\lambda}}{-\cos\alpha\cos\lambda\sin\mu} = \frac{z + \dfrac{r\sin\alpha\cos\lambda}{2\sin\mu}}{\sin\alpha\cos\mu\sin\lambda}$$

These equations are written in a form analogous to the equations (12) of the line 1, mentioned in § 3, namely

$$\frac{x - b_1}{a_1} = \frac{y - b_2}{a_2} = \frac{z - b_3}{a_3}$$

where we write x, y and z instead of x_1, x_2 and x_3.

If we compare these equations we get:

$a_1 = \sin\lambda \sin\mu$ $\qquad b_1 = 0$

$a_2 = -\cos\alpha \cos\lambda \sin\mu$ $\qquad b_2 = r\cos\alpha\cos\mu : 2\sin\lambda$

$a_3 = \sin\alpha \cos\mu \sin\lambda$ $\qquad b_3 = -r\sin\alpha\cos\lambda : 2\sin\mu$

2. The formula (13) which gives the coordinates of the reflected point P' contains the expression:

$$(\Sigma\ a_k p_k - \Sigma\ a_k b_k) : \Sigma\ a_k^2$$

If this form is denoted by A, (13) becomes:

$$s_i = 2a_i A + 2b_i - p_i$$

or:
$$\frac{s_i + p_i}{2} = a_i A + b_i \qquad (i = 1,\ 2,\ 3)$$

or: $\frac{1}{2}(x + x_o) = \sin\lambda \sin\mu \,.\, A$ (14)

$\frac{1}{2}(y + y_o) = -\cos\alpha\cos\lambda\sin\mu \,.\, A + r\cos\alpha\cos\mu : 2\sin\lambda$ (15)

$\frac{1}{2}(z + z_o) = \sin\alpha\cos\mu\sin\lambda \,.\, A - r\sin\alpha\cos\lambda : 2\sin\mu$ (16)

where x, y and z are the coordinates of P' and x_o, y_o, z_o the coordinates of the points P corresponding to P'. Elimination of A from (14) and (15) and from (14) and (16) respectively gives:

$$(x + x_o)\cos\alpha\cos\lambda + (y + y_o)\sin\lambda = r\cos\alpha\cos\mu$$

and $$(x + x_o)\sin\alpha\cos\mu - (z + z_o)\sin\mu = r\sin\alpha\cos\lambda$$

These equations are (cf the equations (5) and (6) of the s-axis) *the parametric equations of the hyperboloid* Γ' which can be reduced from the hyperboloid Γ generated by the s-axis by multiplying by two with regard to the point $P(x_o, y_o, z_o)$. The space curve described by P' lies on Γ' which is in accordance with § 2.3.

3. To write down the equations (14), (15) and (16), combined with the relation $\sin\lambda = m \sin\mu$ ... (8) as parametric equations we make use of the following reducements and abreviations:

From $\sin\lambda = m\sin\mu$ follows:

$$(\sin\lambda + \sin\mu) : (\sin\lambda - \sin\mu) = (m+1) : (m-1)$$

or: $$\tan\tfrac{1}{2}(\lambda+\mu) : \tan\tfrac{1}{2}(\lambda-\mu) = (m+1) : (m-1)$$

Since (§ 1.2): $\tfrac{1}{2}(\varphi+\psi) = \lambda$ and $\tfrac{1}{2}(\varphi-\psi) = \mu$

we obtain: $$\tan\tfrac{1}{2}\varphi = \frac{m+1}{m-1}\tan\tfrac{1}{2}\psi$$

We consider $\tan\tfrac{1}{2}\psi = t$ as a parameter and we write down briefly:

$$(m+1) : (m-1) = n \qquad \text{and} \qquad (1+n^2t^2)(1+t^2) = N$$

Hence we have: $$\tan\tfrac{1}{2}\psi = n.t$$

Now we get the following reducements:

$$\sin\psi = 2\tan\tfrac{1}{2}\psi : (1+\tan^2\tfrac{1}{2}\psi) = 2t : (1+t^2)$$
$$\cos\psi = (1-\tan^2\tfrac{1}{2}\psi) : (1+\tan^2\tfrac{1}{2}\psi) = (1-t^2)(1+t^2)$$

and
$$\sin\varphi = 2nt : (1+n^2t^2)$$
$$\cos\varphi = (1-n^2t^2) : (1+n^2t^2)$$

Further we have:

$$\left.\begin{aligned}\sin\lambda\cos\mu &= \tfrac{1}{2}(\sin\varphi+\sin\psi) = t(1+n)(1+nt^2) : N\\ \sin\mu\cos\lambda &= \tfrac{1}{2}(\sin\varphi-\sin\psi) = t(n-1)(1-nt^2) : N\\ \sin\lambda\sin\mu &= \tfrac{1}{2}(\cos\psi-\cos\varphi) = t^2(n^2-1) : N\\ \cos\lambda\cos\mu &= \tfrac{1}{2}(\cos\psi+\cos\varphi) = (1-n^2t^4) : N\end{aligned}\right\} \quad (17)$$

4. In 2 we denoted

$$A = (\Sigma a_kp_k - \Sigma a_kb_k) : \Sigma a_k^2$$

We have:

$$\begin{aligned}\Sigma a_k^2 &= \sin^2\lambda\sin^2\mu + \cos^2\alpha\cos^2\lambda\sin^2\mu + \sin^2\alpha\cos^2\mu\sin^2\lambda\\ &= \sin^2\lambda\sin^2\mu + \cos^2\alpha\sin^2\mu - \cos^2\alpha\sin^2\lambda\sin^2\mu +\\ &\qquad + \sin^2\alpha\cos^2\mu\sin^2\lambda\\ &= \sin^2\lambda\sin^2\mu\sin^2\alpha + \cos^2\alpha\sin^2\mu + \sin^2\alpha\cos^2\mu\sin^2\lambda\\ &= \sin^2\lambda\sin^2\alpha + \sin^2\mu\cos^2\alpha\\ &= \frac{2a}{r}\sin\mu\sin\lambda, \text{ which follows from (7a).}\end{aligned}$$

If we denote $\frac{2a}{r} = k$ where *k is the ratio of the unequal sides of the isogram,* we get:

$$\Sigma a_k^2 = k \sin\mu \sin\lambda$$

Furthermore we have:

$$\Sigma a_k b_k = -\tfrac{1}{2}r \left[\frac{\cos^2\alpha \cos\lambda \cos\mu \sin\mu}{\sin\lambda} + \frac{\sin^2\alpha \cos\mu \cos\lambda \sin\lambda}{\sin\mu}\right]$$

$$= -\tfrac{1}{2}r \cos\lambda \cos\mu \,.\, \frac{\cos^2\alpha \sin^2\mu + \sin^2\alpha \cos^2\mu}{\sin\lambda \sin\mu}$$

$$= -\tfrac{1}{2}r \cos\lambda \cos\mu \,.\, \frac{2a}{r}$$

$$= -a \cos\lambda \cos\mu$$

And finally we get:

$$\Sigma a_k p_k = x_o \sin\lambda \sin\mu - y_o \cos\alpha \cos\lambda \sin\mu + {}$$
$$+ z_o \sin\alpha \cos\mu \sin\lambda$$

A becomes:

$$\{x_o \sin\lambda \sin\mu - y \cos\alpha \cos\lambda \sin\mu + {}$$
$$+ z_o \sin\alpha \cos\mu \sin\lambda + a \cos\lambda \cos\mu\} : k \sin\mu \sin\lambda$$

or, written with the parameter t:

$$A = \{x_o t^2(n^2 - 1) - y_o \cos\alpha \,.\, t(n-1)(1 - nt^2) + {}$$
$$+ z_o \sin\alpha \,.\, t(n+1)(1+nt^2) + a(1 - n^2t^4)\} : k\, N \sin\mu \sin\lambda \qquad (18)$$

5. Substitution of (18) in (14) gives:

$$\tfrac{1}{2} N k(x + x_o) = x_o t^2(n^2 - 1) - y_o \cos\alpha \,.\, t(n-1)(1 - nt^2) + {}$$
$$+ z_o \sin\alpha \,.\, t(n+1)(1 + nt^2) + a(1 - n^2t^4)$$

and in (15):

$$\tfrac{1}{2} N k \,.\, \frac{y + y_o}{\cos\alpha} = -\frac{\cos\lambda}{\sin\lambda} \cdot \{x_o t^2(n^2 - 1) - {}$$

$$- y_o \cos\alpha \,.\, t(n-1)(1 - nt^2) + z_o \sin\alpha \,.\, t(n+1)(1 + nt^2) + {}$$
$$+ a(1 - n^2t^4)\} + \tfrac{1}{2} N k\, r \frac{\cos\mu}{\sin\lambda}$$

According to (17) we have:

$$\frac{\cos\lambda}{\sin\lambda} = \frac{1 - nt^2}{t(n+1)}$$

and

$$\tfrac{1}{2}\, N\, k\, r\, \frac{\cos\mu}{\sin\lambda} = \frac{a\,N}{m} \cdot \frac{\cos\mu}{\sin\mu} = \frac{a}{m} \cdot \frac{(1 + t^2)(1 + nt^2)(1 + n^2t^2)}{t(n - 1)}$$

And we obtain:

$$\tfrac{1}{2}\, k\, N \cdot \frac{y + y_o}{\cos\alpha} = - x_o t(n - 1)(1 - nt^2) +$$

$$+ y_o \cos\alpha\, \frac{(1 - nt^2)^2(n - 1)}{n + 1} - z_o \sin\alpha\, (1 - n^2t^4) +$$

$$+ a \left\{\frac{-(1 - nt^2)^2(1 + nt^2)}{t(n + 1)} + \frac{1}{m} \cdot \frac{(1 + t^2)(1 + nt^2)(1 + n^2t^2)}{t(n - 1)}\right\}$$

In this expression we reduce the coefficient of $y_o \cos\alpha$ and the one of a respectively to:

$$-t^2(n^2 - 1) + \frac{N}{m} \qquad \text{and} \qquad t(1 + nt^2)(1 + n)$$

Therefore the equation (15) becomes:

$$\tfrac{1}{2}\, k\, N \cdot \frac{y + y_o}{\cos\alpha} = - x_o t(n - 1)(1 - nt^2) - y_o \cos\alpha\, t^2(n^2 - 1) +$$
$$+ y_o \cos\alpha \cdot \frac{N}{m} - z_o \sin\alpha\, (1 - n^2t^4) + at(1 + nt^2)$$

In the same way we get for (16).

$$\tfrac{1}{2}\, k\, N\, \frac{z + z_o}{\sin\alpha} = x_o t(n + 1)(1 + nt^2) - y_o \cos\alpha\, (1 - n^2t^4) -$$
$$- z_o \sin\alpha \,.\, t^2(n^2 - 1) + m\, z_o \sin\alpha \,.\, N - at(n - 1)(1 - nt^2)$$

6. Recapitulation: *The parametric equations of the twisted curves described by a point P' of the moving space S are:*

$$\left.\begin{aligned}
&\tfrac{1}{2}\, k\, N\, (x + x_o) = x_o t^2(n^2 - 1) - y_o \cos\alpha\, t(n - 1)(1 - nt^2) + \\
&\qquad + z_o \sin\alpha\, t(n + 1)(1 + nt^2) + a(1 - n^2t^4) \\
&\tfrac{1}{2}\, k\, N\, \frac{y + y_o}{\cos\alpha} = - y_o \cos\alpha\, t^2(n^2 - 1) - x_o t(n - 1)(1 - nt^2) + \\
&\qquad + at(n + 1)(1 + nt^2) - z_o \sin\alpha\, (1 - n^2t^4) + \frac{y_o \cos\alpha\, N}{m} \\
&\tfrac{1}{2}\, k\, N\, \frac{z + z_o}{\sin\alpha} = - z_o \sin\alpha\, t^2(n^2 - 1) - at(n - 1)(1 - nt^2) + \\
&\qquad + x_o t(n + 1)(1 + nt^2) - y_o \cos\alpha\, (1 - n^2t^4) + m\, z_o \sin\alpha\, N
\end{aligned}\right\} (19)$$

with the following relations:

$$m^2 \sin^2 \alpha + \cos^2 \alpha = k \, . \, m \; ; \qquad k = 2a : r$$

$$n = (m + 1) : (m - 1) \; ; \qquad N = (1 + t^2)(1 + n^2 t^2)$$

and $$t = \tan \tfrac{1}{2} \psi$$

From the equations (19) follows that *the curves described by the points of the moving space S are rational twisted curves of the fourth degree.*

§ 5. Points at infinity of the curves given by (19)

The points at infinity can be found if we use homogeneous coordinates $\bar{x}$, $\bar{y}$, $\bar{z}$, $\bar{w}$. In the parametric equations (19) we substitute: $x = \frac{\bar{x}}{\bar{w}}$; $y = \frac{\bar{y}}{\bar{w}}$; $z = \frac{\bar{z}}{\bar{w}}$. If we take $\bar{w} = N = (1 + t^2)(1 + n^2 t^2)$ we obtain the points at infinity for N = o, or:

$$(1 + t^2)(1 + n^2 t^2) = o$$

that is, $t_1 = i$; $\quad t_2 = - i$; $\quad t_3 = \frac{i}{n}$; $\quad t_4 = -\frac{i}{n}$

Substitution of $t = t_1 = i$ in the equations (19) gives:

$$\begin{aligned} \tfrac{1}{2} k \, \bar{x} &= - x_o (n^2 - 1) - i \, y_o \cos \alpha \, (n - 1)(n + 1) + \\ &\quad + i \, z_o \sin \alpha \, (n + 1)(1 - n) + a(1 - n^2) = \\ &= (1 - n^2) \, (x_o + i \, y_o \cos \alpha + i \, z_o \sin \alpha + a) \end{aligned}$$

$$\begin{aligned} \tfrac{1}{2} k \, \bar{y} : \cos \alpha &= y_o \cos \alpha \, (n^2 - 1) - i \, x_o (n - 1)(n + 1) + \\ &\quad + ia(n + 1)(1 - n) - z_o \sin \alpha \, (1 - n^2) = \\ &= i \, (1 - n^2) \, (x_o + i \, y_o \cos \alpha + i \, z_o \sin \alpha + a) \end{aligned}$$

$$\begin{aligned} \tfrac{1}{2} k \, \bar{z} : \sin \alpha &= z_o \sin \alpha \, (n^2 - 1) - ia(n - 1) \, (n + 1) + \\ &\quad + i \, x_o (n + 1)(1 - n) - y_o \cos \alpha \, (1 - n^2) = \\ &= i \, (1 - n^2) \, (x_o + i \, y_o \cos \alpha + i \, z_o \sin \alpha + a) \end{aligned}$$

Consequently we get:

$$\bar{x} : \bar{y} : \bar{z} = 1 : i \cos \alpha : i \sin \alpha$$

on the understanding that

$$x_o + i \, y_o \cos \alpha + i \, z_o \sin \alpha + a \neq o$$

If t = -i, we obtain:

$$\bar{x} : \bar{y} : \bar{z} = 1 : -i \cos \alpha : -i \sin \alpha$$

on the understanding that

$$x_o - i\,y_o \cos\alpha - i\,z_o \sin\alpha + a \neq o$$

Substitution of $t_{34} = \pm i : n$ in the equations (19) gives:

$$\tfrac{1}{2} k\,\bar{x} = \frac{n^2 - 1}{n^2}(- x_o \mp i\,y_o \cos\alpha \pm i\,z_o \sin\alpha + a)$$

$$\tfrac{1}{2} k\,\bar{y} = \frac{n^2 - 1}{n^2}(y_o \cos\alpha \mp i\,x_o \pm i\,a - z_o \sin\alpha) \cos\alpha$$

$$\tfrac{1}{2} k\,\bar{z} = \frac{n^2 - 1}{n^2}(z_o \sin\alpha \mp i\,a \pm i\,x_o - y_o \cos\alpha) \sin\alpha$$

that is, $\bar{x} : \bar{y} : \bar{z} = 1 : \pm i \cos\alpha ; \mp i \sin\alpha$

on the understanding that $- x_o \mp i\,y_o \cos\alpha \pm i\,z_o \sin\alpha + a \neq o$.

It follows that the four points at infinity of all curves are the isotropic points given by:

$$\bar{x} : \bar{y} : \bar{z} = 1 : \pm i \cos\alpha : \pm i \sin\alpha$$

and $\bar{x} : \bar{y} : \bar{z} = 1 : \pm i \cos\alpha : \mp i \sin\alpha$

These points are the isotropic points of the planes $y \sin\alpha \pm z \cos\alpha = o$ which are the equations of the planes ABA' and ABB' (fig. 50, p. 55).

If $x_o \pm i\,y_o \cos\alpha \pm i\,z_o \sin\alpha + a = o$,

that is, $x_o = - a$ and $y_o = - z_o \tan\alpha$

the point $(x_o;\ y_o;\ z_o)$ is a point of the hinge-line through A and its reflected point is a point of the moving hinge-line through A'. As the points of the hinge-line through A' describe circles which lie in planes parallel to the fixed plane ABA', the points at infinity of these circles are the isotropic points of the plane ABA'.

If $- x_o \mp i\,y_o \cos\alpha \pm z_o \sin\alpha + a = o$, we get the circles described by the points of the moving hinge-line through B'.

So we obtained the theorem:

The four points at infinity of all curves of the fourth degree described by the points of the moving space S are the isotropic points of the planes ABA' and ABB'.

§ 6. Plane curves and spherical curves

1. The curves given by the equations (19) are plane curves if *the right-hand sections are interdependent functions of t.* To fix this condition the functions might be written arranged in order of size of powers of the parameter t.

Furthermore we write briefly:

$$\left.\begin{array}{l} y\cos\alpha = \underline{y}\ ;\ z\sin\alpha = \underline{z}\ ;\ y_o\cos\alpha = \underline{y}_o\ ;\ z_o\sin\alpha = \underline{z}_o \\ \text{and } x = \underline{x}\ ;\ x_o = \underline{x}_o \end{array}\right\} \quad (20)$$

The equations (19) become:

$$\tfrac{1}{2} k N (\underline{x} + \underline{x}_o) = a + t\{(1-n)\underline{y}_o + (1+n)\underline{z}_o\} + t^2\underline{x}_o(n^2-1) + t^3 n\{(n-1)\underline{y}_o + (n+1)\underline{z}_o\} - an^2t^4$$

$$N\left\{\frac{k(\underline{y} + \underline{y}_o)}{2\cos^2\alpha} - \frac{\underline{y}_o}{m}\right\} = -\underline{z}_o + t\{(1-n)\underline{x}_o + (1+n)a\} - t^2\underline{y}_o(n^2-1) + t^3 n\{(n-1)\underline{x}_o + (n+1)a\} + \underline{z}_o n^2t^4$$

$$N\left\{\frac{k(\underline{z} + \underline{z}_o)}{2\sin^2\alpha} - m\,\underline{z}_o\right\} = -\underline{y}_o + t\{(1-n)a + (1+n)\underline{x}_o\} - t^2\underline{z}_o(n^2-1) + t^3 n\{(n-1)a + (n+1)\underline{x}_o\} + \underline{y}_o n^2t^4$$

If we denote the right-hand sections of these equations by p q and r respectively we have to examine if there exist values of the constants A, B, C and D such that:

$$A \cdot p + B \cdot q + C \cdot r = D$$

is an identity with regard to t.

We obtain the following relations:

(I) $Aa - B\underline{z}_o - C\underline{y}_o = D$

(II) $A\{(1-n)\underline{y}_o + (1+n)\underline{z}_o\} + B\{(1-n)\underline{x}_o + (1+n)a\} + C\{(1-n)a + (1+n)\underline{x}_o\}$

(III) $A\underline{x}_o - B\underline{y}_o - C\underline{z}_o = o$

(IV) $A\{(n-1)\underline{y}_o + (n-1)\underline{z}_o\} + B\{(n-1)\underline{x}_o + (n+1)a\} + C\{(n-1)a + (n+1)\underline{x}_o\} = o$

(V) $-Aa + B\underline{z}_o + C\underline{y}_o = o$

The relations (I) and (V) give: D = o.
Adding and subtracting (II) and (IV) gives, after dividing by n + 1 and by n − 1 respectively:

$$A\underline{z}_o + Ba + C\underline{x}_o = o$$
$$A\underline{y}_o + B\underline{x}_o + Ca = o$$

So we obtain with (I) and (III) the four conditions:

$$(I') \quad Aa - B\underline{z}_o - C\underline{y}_o = o$$
$$(II') \quad A\underline{z}_o + Ba + C\underline{x}_o = o$$
$$(III') \quad A\underline{z}_o - B\underline{y}_o - C\underline{z}_o = o$$
$$(IV') \quad A\underline{y}_o + B\underline{x}_o + Ca = o$$

These four equations in A, B and C have a non-zero solution if the determinants of the coefficients of A, B and C of the equations (I', II', III') and of the equations (I', II', IV') are zero, that is,

$$\begin{vmatrix} a & -\underline{z}_o & -\underline{y}_o \\ \underline{z}_o & a & \underline{x}_o \\ \underline{x}_o & -\underline{y}_o & -\underline{z}_o \end{vmatrix} = o \quad \text{and} \quad \begin{vmatrix} a & -\underline{z}_o & -\underline{y}_o \\ \underline{x}_o & -\underline{y}_o & -\underline{z}_o \\ \underline{y}_o & \underline{x}_o & a \end{vmatrix} = o$$

or:
$$\underline{y}_o^3 + \underline{y}_o(\underline{x}_o^2 - \underline{z}_o^2 + a^2) - 2a\underline{x}_o\underline{z}_o = o$$

and
$$\underline{z}_o^3 + \underline{z}_o(\underline{x}_o^2 - \underline{y}_o^2 + a^2) - 2a\underline{x}_o y_o = o$$

Subtraction of these relations after multiplying by $\underline{z}_o$ and $\underline{y}_o$ respectively gives:

$$\underline{y}_o^4 - \underline{z}_o^4 + (\underline{x}_o^2 + a^2)(\underline{y}_o^2 - \underline{z}_o^2) = o$$

or:
$$(\underline{y}_o^2 - \underline{z}_o^2)(\underline{x}_o^2 + \underline{y}_o^2 + \underline{z}_o^2 + a^2) = o$$

or, restricting ourselves to real points of the moving space,

$$\underline{y}_o^2 - \underline{z}_o^2 = o$$

Substitution of $\underline{y}_o = \underline{z}_o$ gives:

$$\underline{z}_o^3 + \underline{z}_o(\underline{x}_o^2 - \underline{z}_o^2 + a^2) - 2a\underline{x}_o\underline{z}_o = o$$

or:
$$\underline{z}_o(\underline{x}_o^2 - 2a\underline{z}_o + a^2) = o$$

or:
$$\underline{x}_o = a \qquad (\underline{z}_o \neq o)$$

Substitution of $\underline{y}_o = -\underline{z}_o$ gives: $\underline{x}_o = -a$

If we mention the abreviations (20) we get:

$$\begin{cases} x_o = a \\ y_o = z_o \tan\alpha \end{cases} \quad \text{and} \quad \begin{cases} x_o = -a \\ y_o = -z_o \tan\alpha \end{cases}$$

2. The equations of the hinge-line h through the vertex A and of the hinge-line k through B are:

$$\left\{\begin{array}{l} x = -a \\ y = -z \tan \alpha \end{array}\right. \quad \text{and} \quad \left\{\begin{array}{l} x = a \\ y = z \tan \alpha \end{array}\right.$$

It follows that $P(x_o, y_o, z_o)$ must lie on one of these hinge-lines. As in this case the reflected point P' of P with regard to the s-axis is a point of one of the moving hinge-lines, it is evident that the curve described by P' is a circle, and thus a plane curve.

We obtain the theorem:

The only points of the moving space S which describe plane curves are the points of the moving hinge-lines. These curves are circles.

3. The curve described by a point of the space S is a *spherical* curve, if it lies on a sphere. As every curve lies on a hyperboloid (§ 4.2) a spherical curve can be considered as a curve of intersection of a hyperboloid and a sphere. As the curve of intersection of two quadrics is in general not a rational curve, it follows that *the space S contains no points which describe spherical curves* except the circles mentioned in 2.

§ 7. Points of inflection or stationary points

1. A point of a twisted curve is a point of inflection or a stationary point if *the osculating plane at that point has a third-order contact* to the curve (chapter I § 8.2).

Let the equations of any curve be given in the parametric form:

$$x = x(t) \; ; \qquad y = y(t) \; ; \qquad z = z(t)$$

The differential geometry *) gives that a point is a stationary point if:

$$\begin{vmatrix} x' & y' & z' \\ x'' & y'' & z'' \\ x''' & y''' & z''' \end{vmatrix} = o \qquad (22)$$

where the accents indicate differentiations with respect to the parameter t.

2. Briefly we denoted the right-hand sections of the parametric equations (21) by p, q and r respectively. These letters represent functions of t of the fourth degree.

Further we use in (21) the abreviations:

*) Eisenhart, A treatise on the differential geometry of curves and surfaces, p. 18 (Boston 1909).

$$\tfrac{1}{2}k(\underline{x} + \underline{x}_o) = X$$
$$\tfrac{1}{2}k(\underline{y} + \underline{y}_o) : \cos^2\alpha - \underline{y}_o : m = Y$$
$$\tfrac{1}{2}h(\underline{z} + \underline{z}_o) : \sin^2\alpha - m\underline{z}_o = Z$$

Consequently we get:

$$X' = \tfrac{1}{2}k\underline{x}' ; \qquad X'' = \tfrac{1}{2}k\underline{x}'' ; \qquad X''' = \tfrac{1}{2}k\underline{x}'''$$
$$Y' = \tfrac{1}{2}k\underline{y}' : \cos^2\alpha ; \qquad \text{etc.}$$

If we multiply the first column of (22) by $\tfrac{1}{2}k$, the second one by $\tfrac{1}{2}k : \cos^2\alpha$ and the third one by $\tfrac{1}{2}k : \sin^2\alpha$, we obtain the same determinant as (22) but now written in X, Y and Z. From the abreviations follows that the parametric equations (21) become:

$$X = p : N$$
$$Y = q : N$$
$$Z = r : N$$

Substitution in (22) gives:

$$\begin{vmatrix} p'N - pN' & q'N - qN' & r'N - rN' \\ p''N - pN'' & \dots & \dots \\ p'''N - pN''' & \dots & \dots \end{vmatrix} = o$$

or, if we increase the rank of this determinant:

$$\begin{vmatrix} N & p & q & r \\ N' & p' & q' & r' \\ N'' & p'' & q'' & r'' \\ N''' & p''' & q''' & r''' \end{vmatrix} = o \qquad (23)$$

3. Let $f_k = \alpha_k + \beta_k t + \gamma_k t^2 + \delta_k t^3 + \varepsilon_k t^4$ $(k = 1, 2, 3, 4)$ be four functions of the fourth degree in t, such that

$$f_1 \equiv p ; \qquad f_2 \equiv q ; \qquad f_3 \equiv r ; \qquad f_4 \equiv N$$

The values of the coefficients α_k etc. may be tabulated as follows (see the equations (21))

k	α_k	β_k	γ_k	δ_k	ε_k
1	a	$(1-n)\underline{y}_o + (1+n)\underline{z}_o$	$\underline{x}_o(n^2-1)$	$n\{(n-1)\underline{y}_o + (n+1)\underline{z}_o\}$	$-an^2$
2	$-\underline{z}_o$	$(1-n)\underline{x}_o + (1+n)a$	$-\underline{y}_o(n^2-1)$	$n\{(n-1)\underline{x}_o + (n+1)a\}$	$-\underline{z}_o n^2$
3	$-\underline{y}_o$	$(1-n)a + (1+n)\underline{x}_o$	$-\underline{z}_o(n^2-1)$	$n\{(n-1)a + (n+1)\underline{x}_o\}$	$\underline{y}_o n^2$
4	1	o	n^2+1	o	n^2

Substitution in the determinant (23) gives, after increasing the rank again:

$$\begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & t^4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 & -4t^3 \\ \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & 6t^2 \\ \delta_1 & \delta_2 & \delta_3 & \delta_4 & -4t \\ \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4 & 1 \end{vmatrix} = o \qquad (24)$$

If we mention the abreviations (20), this determinant (24) is reducible to:

$$\begin{vmatrix} z_o \sin\alpha & a & x_o & 4n(1-n)\ t(nt^2+1) \\ x_o & -y_o\cos\alpha & -z_o\sin\alpha & 12n^2t^2-(1+n^2)(1+n^2t^4) \\ y_o\cos\alpha & x_o & a & 4n\ (n+1)\ t(nt^2-1) \\ -a & z_o\sin\alpha & y_o\cos\alpha & (1-n^2)(n^2t^4-1) \end{vmatrix} = o \qquad (25)$$

4. If we give t any value $t = t_1$, (25) is the equation of a surface of the third degree with x_o, y_o, z_o as its current coordinates. *The locus of the points of inflection of the curves described by the points of the moving space S is at any moment* (given by $t = t_1$) *found by reflection of the surface given by* (25) *with regard to the s-axis* in the position that corresponds to $t = t_1$. This is in accordance with chapter I § 8.2 (theorem XXVI).

Let $P(x_o, y_o, z_o)$ be a given point. Its reflected point P' describes a rational curve of the fourth degree. Substitution of the given values of x_o, y_o, z_o in (25) gives an equation of the fourth degree in t. Hence the curves described by the points of the moving space S have four points of inflection.

We obtain a *special* case if the terms of the second and the third column are proportional, say:

$$a = Cx_o\ ; \qquad -y\cos\alpha = -Cz_o \sin\alpha\ ; \qquad x_o = C\ .\ a$$
$$z_o \sin\alpha = Cy_o \cos\alpha$$

and hence we get: $C = \pm 1$.

If $C = -1$ we obtain: $\begin{cases} x_o = -a \\ y_o = -z_o \tan\alpha \end{cases}$

and if $C = +1$: $\begin{cases} x_o = a \\ y_o = z_o \tan\alpha \end{cases}$

From this follows that every point of one of the hinge-lines in A and B gives a corresponding curve which has only points of inflection. This is evident for these curves are circles and the osculating plane at a point of a circle coincides with the plane of the circle.

§ 8. Tangents with a second-order contact

1. The equation (11) of the hyperboloid Γ generated by the s-axis is:

$$\frac{4x^2}{r^2} + \frac{4m^2y^2}{(m^2-1)\,r^2\cos^2\alpha} - \frac{4z^2}{(m^2-1)\,r^2\sin^2\alpha} = 1$$

Briefly we write:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

where $a^2 = \frac{1}{4}r^2$; $b^2 = \frac{1}{4m^2}\cdot(m^2-1)\,r^2\cos^2\alpha$;

$$c^2 = \tfrac{1}{4}(m^2-1)\,r^2\sin^2\alpha$$

The equations of the two series of generators, called a-lines and b-lines respectively are:

$$\left.\begin{aligned}\frac{x}{a}+\frac{z}{c} &= \frac{1}{\lambda}(1+\frac{y}{b})\\ \frac{x}{a}-\frac{z}{c} &= \lambda(1-\frac{y}{b})\end{aligned}\right\}\text{ a-lines}\qquad\left.\begin{aligned}\frac{x}{a}+\frac{z}{c} &= \frac{1}{\mu}(1-\frac{y}{b})\\ \frac{x}{a}-\frac{z}{c} &= \mu(1+\frac{y}{b})\end{aligned}\right\}\text{ b-lines}$$

The b-lines coincide with the several positions of the s-axis of the moving isogram.

If we solve x, y and z out of the equations:

$$\frac{x}{a}-\frac{z}{c} = \mu(1+\frac{y}{b})$$

$$\frac{x}{a}-\frac{z}{c} = \lambda(1-\frac{y}{b})$$

$$\frac{x}{a}+\frac{z}{c} = \frac{1}{\lambda}(1+\frac{y}{b})$$

we obtain:

$$x = \frac{a\lambda}{\mu+\lambda}(\frac{1}{\lambda}+\mu)\ ;\qquad y = \frac{b(\lambda-\mu)}{\lambda+\mu}\ ;\qquad z = \frac{c\lambda}{\lambda+\mu}(\frac{1}{\lambda}-\mu)$$

which are *parametric equations of the hyperboloid* Γ.

2. If $T(x_o, y_o, z_o)$ is a point of Γ, there exist values of λ and μ such that:

$$x_o = \frac{a\lambda}{\mu+\lambda}(\frac{1}{\lambda}+\mu)\ ;\qquad y_o = \frac{b(\lambda-\mu)}{\lambda+\mu}\ ;\qquad z_o = \frac{c\lambda}{\lambda+\mu}(\frac{1}{\lambda}-\mu)$$

The direction numbers of the b-line through T written in the form

$$\left.\begin{aligned} \frac{x}{a} + \frac{y}{\mu b} + \frac{z}{c} &= \frac{1}{\mu} \\ \frac{x}{a} - \frac{\mu y}{b} - \frac{z}{c} &= \mu \end{aligned}\right\}$$

are the minors of:
$$\left\| \begin{matrix} \frac{1}{a} & \frac{1}{\mu b} & \frac{1}{c} \\ \frac{1}{a} & \frac{-\mu}{b} & -\frac{1}{c} \end{matrix} \right\|$$

or: $$-\frac{1}{\mu bc} + \frac{\mu}{bc}\,; \qquad \frac{1}{ac} + \frac{1}{ac}\,; \qquad -\frac{\mu}{ab} - \frac{1}{\mu ab}$$

or: $$a(\mu^2 - 1)\,; \qquad 2\mu b\,; \qquad -c(\mu^2 + 1)$$

The equation of the plane α through T perpendicular to the b-line through T is:

$$a(\mu^2 - 1)(x - x_o) + 2\mu b(y - y_o) - c(\mu^2 + 1)(z - z_o) = o$$

or: $$a(\mu^2 - 1)\, x + 2\mu by - c(\mu^2 + 1)\, z =$$

$$= \frac{1}{\lambda + \mu}\{a^2(1 + \lambda\mu)(\mu^2 - 1) + 2\mu b^2(\lambda - \mu) - c^2(1 - \lambda\mu)(\mu^2 + 1)\}$$

or, written in order of size of the powers of μ:

$$\mu^3(ax - cz - a^2\lambda - c^2\lambda) + \mu^2(a\lambda x + 2by - \lambda cz - a^2 + 2b^2 + c^2) +$$
$$+ \mu(-ax + 2b\lambda\mu - cz + a^2\lambda - 2b^2\lambda - c^2\lambda) +$$
$$+ (-\lambda ax - \lambda cz + a^2 + c^2) = o \qquad (26)$$

3. The coefficients of x, y and z in the equation (26) are functions of the parameters λ and μ which belong to the a- and b-line through the point T on the hyperboloid Γ. To find the number of planes through a given point P if T moves along a given a-line we have to substitute in (26) the coordinates of P and the value of λ belonging to the given a-line. As (26) is an equation of the third degree in μ, we conclude that we obtain three values of μ, that is, there are three b-lines with the property that the planes normal to these b-lines and through their point of intersection with a given a-line go through P. That is, through any point P go three perpendiculars on b-lines which intersect a given a-line. Hence, *the a-lines are trisecants of the pedal of P with regard to the b-lines*; the pedal is a (3,1)-curve on Γ. If we multiply the hyperboloid Γ and the pedal by two with regard to P, we get the hyperboloid Γ' and a curve C which is the locus of the reflected points P' of P with regard to the b-lines of Γ

(§ 2). C is a (3,1)-curve on Γ' and the a'-lines of Γ' are trisecants of C. On the other hand we have that the trisecants of C are the a'-lines of Γ'. A well-known theorem *) says that the trisecants of a rational twisted curve of the fourth degree generate a quadric. In our case this quadric is Γ'.

4. The equation (26), briefly written as

$$a_o\mu^3 + a_1\mu^2 + a_2\mu + a_3 = o$$

has *three equal roots* if the left-hand member of the equation is identically equal to:

$$a_o(\mu - p)^3 = o$$

or: $a_1 = -3pa_o$; $a_2 = 3p^2a_o$; $a_3 = -p^3a_o$

or: $3pa_o + a_1 = o$; $a_1p + a_2 = o$; $a_2p + 3a_3 = o$

The values of a_o, a_1, a_2, a_3 follow from (26). Therefore these conditions become if we write

$$q^2 = a^2 - 2b^2 - c^2$$

and

$$r^2 = a^2 + c^2$$

$$\left.\begin{aligned} 3(ax - cz - \lambda r^2) \,.\, p + (a\lambda x + 2by - \lambda cz - q^2) &= o \\ (a\lambda x + 2by - \lambda cz - q^2) \,.\, p + (-ax + 2b\lambda y - cz + \lambda q^2) &= o \\ (-ax + 2b\lambda y - cz + \lambda q^2) \,.\, p + 3 \,.\, (-\lambda ax - \lambda cz + r^2) &= o \end{aligned}\right\} \quad (27)$$

Let μ_1 be a given value of p. The b-line which corresponds to this value μ_1 is denoted by b_1. If we let coincide the s-axis with b_1 we get the position of the isogram that corresponds to $\mu = \mu_1$.

The equations (27) represent three planes. Their point of intersection P has coordinates which are functions of λ. Every value of λ gives a point P such that reflection of P with regard to the s-axis (b_1-line) gives the point P' which has a tangent to the curve belonging to P (the path of P') with a *second-order contact* (fig. 51). This tangent is the line obtained from the a-line corresponding to the considered value of λ by multiplying by two with regard to P.

5. If λ is considered as a parameter, the locus of the points

*) Schrek, D.J.E. Rationale ruimtekrommen van den vierden graad, 25 Diss. Utrecht (1915).

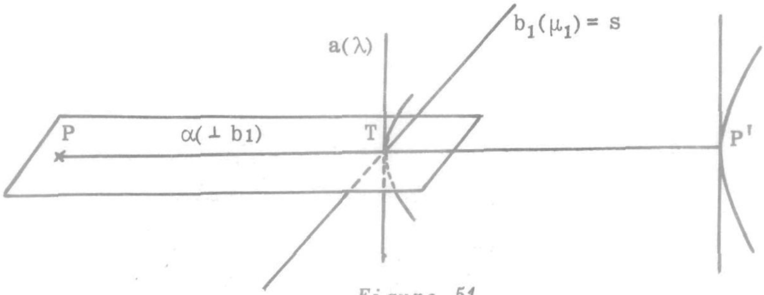

Figure 51

P in the fixed space Σ is given by the equations (27) which can be written as (p is replaced by μ_1):

$$\left.\begin{aligned}\lambda(ax - cz - 3\mu_1 r^2) &= -3a\mu_1 x - 2by + 3c\mu_1 z + q^2\\ \lambda(a\mu_1 x + 2by - c\mu_1 z + q^2) &= ax - 2b\mu_1 y + cz + q^2\mu_1\\ \lambda(-3ax + 2b\mu_1 y - 3cz + q^2\mu_1) &= a\mu_1 x + c\mu_1 z - 3r^2\end{aligned}\right\} \quad (27a)$$

or, briefly

$$\left.\begin{aligned}\lambda A &= B\\ \lambda C &= D\\ \lambda E &= F\end{aligned}\right\} \quad (27b)$$

where A, B, C, D, E and F are linear functions of x, y and z.

Elimination of λ gives:

$$\left.\begin{aligned}A.D &= B.C\\ C.F &= D.E\end{aligned}\right\} \quad (28)$$

which are the equations of a twisted cubic being the curve of intersection of two quadrics which have the line C = D = o in common.

6. Theorem XXV of chapter I, § 7 gives that the locus of the points P' of the moving space S with a tangent with a second-order contact is at any moment a twisted cubic. In our case we have:

The locus of the points P' with a tangent with a second-order contact is obtained by reflection of the curve given by the equations (28) with regard to the s-axis of the isogram.

Chapter V

THE TANGENTS TO THE CURVES

§ 1. Conjugated lines

1. The hinge-lines h' and k' are lines of the moving space S (fig. 52). The curves described by the points of these lines are circles around the hinge-lines k and h respectively. The plane through any point P of h' normal to the tangent in P goes through k. Hence *h' and k are conjugated lines* (chapter I, § 1.5). Similarly *k' and h are conjugated lines.*

2. As AB' is the common perpendicular of the conjugated lines h and k' and A'B that of the conjugated lines h' and k, *the common perpendicular of AB' and A'B is the instantaneous screw-axis* (denoted by x-axis) of the motion of the space S (chapter I, § 4.4).

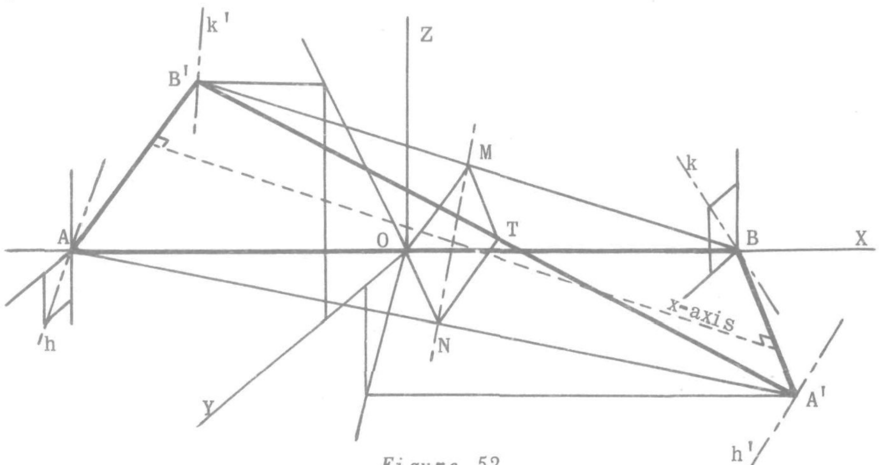

Figure 52

The line MN joining the midpoints M and N of BB' and AA' respectively is the axis of symmetry of the quadrilateral ABA'B' and consequently *the x-axis intersects MN (s-axis).*

§ 2. Points with a tangent parallel to A'B'

1. Let T be the midpoint of A'B' and δ the plane through T normal to A'B'. The tangent in the nullpoint K of δ is normal to

δ and hence parallel to $A'B'$. The locus of the nullpoints of the planes normal to $A'B'$ is the locus of the points with a tangent parallel to $A'B'$. As the locus of the nullpoints of a series of parallel planes is an axis and hence a line parallel to the x-axis, the required locus is *the line u through the nullpoint K* of the plane δ parallel to the x-axis (chapter I, § 3.3).

The nullpoint K of δ is the common point of the lines through the points of intersection of δ with two pairs of conjugated lines. Let A_1, B_1, A'_1, B'_1 be the points of intersection of h, k, h', k' respectively with δ. The point K is the common point of $A_1B'_1$ and A'_1B_1 (chapter I, § 4.5).

2. If we reflect the moving space S with regard to the line MN we obtain the space Σ. The reflected figure in Σ of a figure F in S is denoted by F^r. The figures F and F^r are congruent.

We have the following transformations:

$A'B'$ passes into AB, denoted by $(A'B')^r$, because A and A' are symmetrical with regard to MN,

h' passes into h , denoted by h'^r

h passes into h', denoted by h^r

k' passes into k , denoted by k'^r

k passes into k', denoted by k^r

δ passes into the plane YOZ, denoted by δ^r.

If F is a fixed figure in S, F^r is a fixed figure in Σ. *The consideration of the figures in S will be carried out in the fixed space.* The true position of these considered figures is obtained by reflection with regard to the axis MN.

3. The nullpoint K of the plane δ mentioned in 1 passes by reflection with regard to MN into the point K^r of the plane YOZ. The line $A_1B'_1$ passes into the line $(A_1B'_1)^r$ which is the line through the points of intersection of $h^r \equiv h'$ and $h'^r \equiv h$ with the plane YOZ. Similarly A'_1B_1 passes into the line through the points of intersection of h and k' with the plane X = o. If the points of intersection of the hinge-lines h, k, h', k' with the plane X = o are denoted by A_2, B_2, A'_2, B'_2 respectively, the point K^r is the common point of the lines $A_2B'_2$ and A'_2B_2.

As h is parallel to the plane YOZ, the point of intersection A_2 of h and this plane is the point at infinity of h. Therefore the line $A_2B'_2$ is the line through B'_2 parallel to h. Similarly the line A'_2B_2 is the line through the point of intersection A'_2 of h' and the plane YOZ parallel to k.

The plane through B' normal to AB' contains k'. As this plane is parallel to h, it contains also the point at infinity A_2 of h. Its line of intersection with the plane YOZ is therefore the line $A_2B'_2$. Similarly *A'_2B_2 is the line of intersection of the plane through A' normal to BA' with the plane YOZ.*

4. The direction numbers of AB' are (chapter IV, § 1.2):

$$r \cos\psi; \qquad r \sin\psi \cos\alpha; \qquad r \sin\psi \sin\alpha$$

The equation of the plane through B' normal to AB' is:

$$\cos\psi\,(x + a - r\cos\psi) + \sin\psi\cos\alpha\,(y - r\sin\psi\cos\alpha) + \\ + \sin\psi\sin\alpha\,(z - r\sin\psi\sin\alpha) = 0$$

The equations of the line of intersection of this plane with the plane YOZ are:

$$\left.\begin{array}{c} x = 0 \\ y\cos\alpha + z\sin\alpha = -a\cot\psi + r\,\mathrm{cosec}\,\psi \end{array}\right\} \quad (1)$$

The direction numbers of A'B are:

$$r\cos\varphi; \qquad r\sin\varphi\cos\alpha; \qquad -r\sin\varphi\sin\alpha$$

The equation of the plane through A' normal to A'B is:

$$\cos\varphi\,(x - a - r\cos\varphi) + \sin\varphi\cos\alpha\,(y - r\sin\varphi\cos\alpha) - \\ - \sin\varphi\sin\alpha\,(z + r\sin\varphi\sin\alpha) = 0$$

and the equations of the line of intersection with the plane YOZ are:

$$\left.\begin{array}{c} x = 0 \\ y\cos\alpha - z\sin\alpha = a\cot\varphi + r\,\mathrm{cosec}\,\varphi \end{array}\right\} \quad (2)$$

The two pairs of equations (1) and (2) give the point K^r in the plane YOZ. *Reflection of this point K^r* with regard to the line MN *gives the nullpoint of the plane* δ through the midpoint T of A'B' and normal to A'B'.

5. Let δ_1 be a plane parallel to the plane δ and let the equation of the plane δ_1^r, obtained by reflection of δ_1 with regard to MN, be $x = p$ where p is the distance between δ and δ_1.

The equations analogous to (1) and (2) are:

$$\left.\begin{array}{c} x = p \\ y\cos\alpha + z\sin\alpha = -(a + p)\cot\psi + r\,\mathrm{cosec}\,\psi \end{array}\right\} \quad (1a)$$

and

$$\left.\begin{array}{c} x = p \\ y\cos\alpha - z\sin\alpha = (a - p)\cot\varphi + r\,\mathrm{cosec}\,\varphi \end{array}\right\} \quad (2a)$$

These equations give the reflected nullpoint of the plane δ_1. Elimination of p gives the locus u^r of the points with a tangent parallel to $(A'B')^r$. We obtain:

$$\left.\begin{array}{l} a \cot \psi + y \cos \alpha + z \sin \alpha = - a \cot \psi + r \operatorname{cosec} \psi \\ x \cot \varphi + y \cos \alpha - z \sin \alpha = a \cot \varphi + r \operatorname{cosec} \psi \end{array}\right\} \quad (3)$$

Reflection of the line given by these equations (3) gives the locus u of the points with a tangent parallel to A'B'.

As the points of u have the same direction of velocity, the conjugated line of u is a line at infinity namely the line at infinity of the plane δ and hence, u is an axis (chapter I, § 3.1). Consequently u is parallel to the screw-axis (x-axis). As the x-axis intersects the axis MN at a right angle, the angle between u and MN is also a right angle. Hence the lines u and u^r are parallel. As the x-axis is normal to the plane OMTN we obtain: *The locus of the points which have a tangent parallel to the line A'B' is a line u normal to the midplane OMTN of the quadrilateral.*

6. A line m which intersects the x-axis at a right angle is a self-conjugated line (chapter I, § 1.6). If m is drawn perpendicular to A'B', it is possible to bring a plane through m normal to A'B'. The line m goes through the nullpoint of this plane. As this nullpoint is a point of the locus u, m intersects u. We obtain: *The lines intersecting the x-axis at a right angle and drawn perpendicular to A'B' intersect u.*

§ 3. The point of A'B' in which the tangent coincides with A'B'

If A'B' is a tangent, its point of contact is denoted by C. This point C is the nullpoint of the plane through C normal to A'B'. As the locus of the nullpoints of the planes normal to A'B' is the line u given by the equations (3), C is a point of u. Hence, C is the common point of the line u and A'B'. These lines have a common point if the lines u^r and $(A'B')^r$ intersect each other.

The equations of u^r are:

$$\left.\begin{array}{l} x \cot \psi + y \cos \alpha + z \sin \alpha = - a \cot \psi + r \operatorname{cosec} \psi \\ x \cot \varphi + y \cos \alpha - z \sin \alpha = a \cot \varphi + r \operatorname{cosec} \varphi \end{array}\right\} \quad (3)$$

and the equations of $(A'B')^r$ are:

$$\left.\begin{array}{l} y = o \\ z = o \end{array}\right\}$$

Elimination of y and z out of these four equations gives:

$$x \cot \psi = - a \cot \psi + r \operatorname{cosec} \psi$$

and $$x \cot \varphi = a \cot \varphi + r \operatorname{cosec} \varphi$$

or: $$x = - a + r \sec \psi$$

and $$x = a + r \sec \varphi$$

These two equations have a solution if:

$$- a + r \sec \psi = a + r \sec \varphi$$

or: $$2a : 2r = (\cos \varphi - \cos \psi) : 2 \cos \varphi \cos \psi$$

If we use the following substitutions and relations:

$$\tfrac{1}{2}(\varphi - \psi) = \mu \, ; \qquad \tfrac{1}{2}(\varphi + \psi) = \lambda \, ; \qquad \sin \lambda = m \sin \mu$$

(chapter IV, § 1.2 and 1.3), we obtain:

$$\frac{a}{r} = \frac{- 2 \sin \frac{1}{2}(\varphi - \psi) \sin \frac{1}{2}(\varphi + \psi)}{\cos (\varphi + \psi) + \cos (\varphi - \psi)} = \frac{- 2 \sin \mu \sin \lambda}{\cos 2\lambda + \cos 2\mu} =$$

$$= - \frac{m \sin^2 \mu}{1 - (m^2 + 1) \sin^2 \mu}$$

or: $$\sin^2 \mu = \frac{a}{a + am^2 - mr}$$

If the denominator $a + am^2 - mr$ is denoted by N we get:

$$\sin^2 \mu = \frac{a}{N}$$

The point of contact C^r of $(A'B')^r$ can be found by substitution of this value into the equation:

$$x = a + r \sec \varphi$$

We get: $$x_c = a + \frac{r}{\cos \varphi} = a + \frac{r}{\cos (\lambda + \mu)} =$$

$$= a + \frac{r}{\cos \lambda \cos \mu - \sin \lambda \sin \mu} =$$

$$= a + \frac{r}{\pm\sqrt{(1 - \sin^2\lambda)(1 - \sin^2\mu)} - m \sin^2\mu} =$$

$$= a + \frac{r}{\pm\sqrt{(1 - \frac{m^2 a}{N})(1 - \frac{a}{N})} - m\frac{a}{N}} = a + \frac{Nr}{\pm\sqrt{(a - mr)(am^2 - mr)} - am} =$$

$$= \frac{\pm a\sqrt{m(a - mr)(am - r)} - a^2 m + ar + am^2 r - mr^2}{\sqrt{m}\{\pm\sqrt{(a - mr)(am - r)} - a\sqrt{m}\}} =$$

$$= \frac{\pm a\sqrt{m(a - mr)(am - r)} - (am - r)(a - mr)}{\sqrt{m}\{\pm\sqrt{(a - mr)(am - r)} - a\sqrt{m}\}} =$$

$$= \frac{\pm\sqrt{(a - mr)(am - r)}}{\sqrt{m}}$$

or:
$$x_c = \pm\sqrt{\frac{(a - mr)(am - r)}{m}}$$

We obtain:

The line $A'B'$ is tangent in one of its points C if the position of the quadrilateral is given by: $\sin^2\mu = a : N$, where N only depends on the data a, r and α of the isogram. The point of contact C is given in its reflected position C^r as a point of the X-axis with

$$x_c = \pm\sqrt{\frac{(a - mr)(am - r)}{m}}$$

which also depends only on a, r and α.

In chapter VI, § 5 we shall prove that there exists no real position of the isogram such that A'B' is a tangent.

§ 4. Characteristics of the planes through A'B'

1. If the pencil of planes through A'B' is reflected with regard to the axis MN, we obtain the pencil of planes through the X-axis. *The locus of the characteristics of the planes through $A'B'$ is generated by the projections upon these planes of the line u* which is the line conjugated to the line at infinity of the planes normal to A'B' (chapter I, § 6.3). In the reflected position we obtain that the locus of the characteristics of the planes through the X-axis is generated by the projections of u^r upon these planes.

2. The equations (3) of u^r can be reduced by means of addition and substraction in the following way:

$$\begin{cases} x(\cot\psi + \cot\varphi) + 2y\cos\alpha = a(\cot\varphi - \cot\psi) + \\ \qquad\qquad + r(\operatorname{cosec}\varphi + \operatorname{cosec}\psi) \\ x(\cot\psi - \cot\varphi) + 2z\sin\alpha = -a(\cot\varphi + \cot\psi) + \\ \qquad\qquad + r(\operatorname{cosec}\psi - \operatorname{cosec}\varphi) \end{cases}$$

or:
$$\begin{cases} mx\cos\lambda + y\cos\alpha\sin\mu\,(m^2 - 1) = -\cos\mu\,(a - mr) \\ x\cos\mu + z\sin\alpha\sin\mu\,(m^2 - 1) = -\cos\lambda\,(am - r) \end{cases}$$

The equation of the pencil of planes through the line u^r is:

$$x(m\cos\lambda + P\cos\mu) + y\sin\mu\cos\alpha\,(m^2 - 1) + \\ + zP\sin\mu\sin\alpha\,(m^2 - 1) + \ldots\ldots = o \qquad (5)$$

where P is the parameter.

The equation of the pencil of planes through the X-axis is:

$$y + Qz = o \qquad (6)$$

where Q is the parameter.

A plane of the first pencil is normal to a plane of the second pencil if:

$$\sin\mu\cos\alpha\,(m^2 - 1) + P\,.\,Q\,.\,\sin\mu\sin\alpha\,(m^2 - 1) = o$$

or:
$$P\,.\,Q = -\cot\alpha \qquad (7)$$

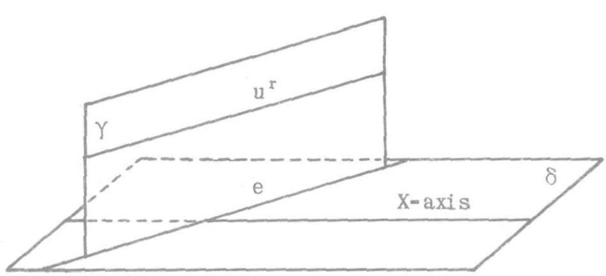

Figure 53

If (fig. 53) γ is a plane of the first pencil and δ a plane of the second pencil perpendicular to α, the line of intersection e of γ and δ is the projection of u^r upon δ and hence e is the characteristic of δ. Elimination of P and Q out of (5), (6) and (7) gives the locus of the characteristics of the planes through the X-axis.

We obtain:

$$\frac{m\,x\cos\lambda + y\sin\mu\cos\alpha\,(m^2-1) + \cos\mu\,(a-mr)}{x\cos\mu + z\sin\mu\sin\alpha\,(m^2-1) + \cos\lambda\,(am-r)} \cdot \frac{y}{z} = -\cot\alpha$$

or: $$m\,x\,y\cos\lambda\sin\alpha + y^2\sin\mu\sin\alpha\cos\alpha\,(m^2-1) + y\cos\mu\sin\alpha\,(a-mr) = -x\,z\cos\alpha\cos\mu - z^2\sin\mu\cos\alpha\sin\alpha\,(m^2-1) - z\cos\lambda\cos\alpha\,(am-r)$$

or: $$(y^2+z^2)\sin\alpha\cos\alpha\sin\mu\,(m^2-1) + m\,x\,y\cos\lambda\sin\alpha + x\,z\cos\mu\cos\alpha + y\sin\alpha\cos\mu\,(a-mr) + z\cos\alpha\cos\lambda\,(am-r) = 0 \qquad (8)$$

The quadric given by this equation is a hyperboloid. *If this hyperboloid is reflected with regard to the axis MN, we obtain the locus of the points with a tangent which intersects the line A'B'.*

§ 5. Equations of the instantaneous screw-axis

1. The instantaneous screw-axis (x-axis) is the common normal of two pairs of conjugated lines. As noticed in § 1 the x-axis is the common normal of the links AB' and A'B and intersects the axis MN at a right angle. From this follows that the *x-axis is invariable with regard to the reflection* upon the axis MN.

2. Let l_1 and l_2 be two lines given by the equations:

$$\frac{x-x_k}{a_k} = \frac{y-y_k}{b_k} = \frac{z-z_k}{c_k} \qquad (k = 1,2)$$

The equations of the *common normal* of l_1 and l_2 are:

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ a_1 & b_1 & c_1 \\ \Delta_1 & \Delta_2 & \Delta_3 \end{vmatrix} = 0 \quad \text{and} \quad \begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ a_2 & b_2 & c_2 \\ \Delta_1 & \Delta_2 & \Delta_3 \end{vmatrix} = 0$$

where:

$$\Delta_1 = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}; \qquad \Delta_2 = \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}; \qquad \Delta_3 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

3. The equations of AB' and A'B are:

$$\frac{x+a}{-a + r\cos\psi + a} = \frac{y}{r\sin\psi\cos\alpha} = \frac{z}{r\sin\psi\sin\alpha}$$

and $$\frac{x-a}{a + r\cos\varphi - a} = \frac{y}{r\sin\varphi\cos\alpha} = \frac{z}{-r\sin\varphi\sin\alpha}$$

If AB' and $A'B$ are considered as the lines l_1 and l_2 we obtain:

$$\Delta_1 = -2 \sin\alpha \cos\alpha \sin\varphi \sin\psi$$
$$\Delta_2 = \sin\alpha \sin(\varphi + \psi)$$
$$\Delta_3 = \cos\alpha \sin(\varphi - \psi)$$

Hence, the *equations of the screw-axis* are:

$$\begin{vmatrix} x + a & y & z \\ \cos\psi & \sin\psi\cos\alpha & \sin\psi\sin\alpha \\ -2\sin\alpha\cos\alpha\sin\varphi\sin\psi & \sin\alpha\sin(\varphi+\psi) & \cos\alpha\sin(\varphi-\psi) \end{vmatrix} = 0$$

and

$$\begin{vmatrix} x - a & y & z \\ \cos\psi & \sin\varphi\cos\alpha & -\sin\varphi\sin\alpha \\ -2\sin\alpha\cos\alpha\sin\varphi\sin\psi & \sin\alpha\sin(\varphi+\psi) & \cos\alpha\sin(\varphi-\psi) \end{vmatrix} = 0$$

Using the relations:

$$\varphi + \psi = 2\lambda; \qquad \varphi - \psi = 2\mu; \qquad \sin\lambda = m \sin\mu;$$
$$2a : r = k \qquad \text{and} \qquad \cos^2\alpha + m^2 \sin^2\alpha = k\,m$$

these equations are reducible to:

$$\begin{cases} x \cos\mu + z \sin\alpha \sin\mu\,(m^2 - 1) = \\ \qquad \dfrac{a \cos\lambda}{k\,(1 - k\,m \sin^2\mu)}\,(k^2 m^2 \sin^2\mu - 2\,k\,m \sin^2\mu + 1) \\ m\,x \cos\lambda + y \cos\alpha \sin\mu\,(m^2 - 1) = \\ \qquad \dfrac{a\,m \cos\mu}{k\,(1 - k\,m \sin^2\mu)}\,(k^2 \sin^2\mu - 2\,k\,m \sin^2\mu + 1) \end{cases}$$

§ 6. Points with a tangent through a given point of $A'B'$

1. The locus of the points with a tangent which intersects the line through a given point P parallel to the x-axis is the circular cylinder C through P and the x-axis such that the plane through P and the x-axis passes through the axis of the cylinder C (chapter I, § 6.5). The locus of the points with a tangent which intersects the line $A'B'$ is the locus of the characteristics of the planes through $A'B'$. This locus is the hyperboloid H given by the equation (8) of § 4. In chapter I, § 6.6 has been shown that *the locus of the points which have a tangent through the given point P is the curve of intersection of the cylinder C and the hyperboloid H.*

2. The equation of the cylinder C can be obtained if we reflect the point P of A'B' with regard to the axis MN. If the distance of P to the midpoint T of A'B' is denoted by p, the coordinates of P^r are (p ; o ; o). *The x-axis is invariant with regard to the axial reflection.*

The cylinder C can be defined as the locus of the lines of intersection of the planes of the pencil through the x-axis with the planes of the pencil through the line u_p drawn through P parallel to the x-axis which planes are normal to the planes of the first pencil.

In the reflected position we get the pencil through the line u_p^r through P^r and parallel to the x-axis and the pencil through the x-axis (this axis is invariant with regard to the reflection).

3. The equations of the x-axis are (see § 5).

$$A x + B y = C_1$$

$$P x + Q z = R_1$$

where: $A = m \cos \lambda$; $B = \sin \mu \cos \alpha \, (m^2 - 1)$

$P = \cos \mu$; $Q = \sin \mu \sin \alpha \, (m^2 - 1)$

$$C_1 = \frac{am \cos \mu}{k\,(1 - km \sin^2 \mu)} \cdot (k^2 \sin^2 \mu - 2\, km \sin^2 \mu + 1)$$

$$R_1 = \frac{a \cos \lambda}{k\,(1 - km \sin^2 \mu)} \cdot (k^2 m^2 \sin^2 \mu - 2\, km \sin^2 \mu + 1)$$

The equations of the line u_p^r which is the line through (p ; o ; o) parallel to the x-axis are:

$$Ax + By = C_2$$

$$Px + Qz = R_2$$

where $C_2 = mp \cos \lambda$ and $R_2 = p \cos \mu$

Let the equation of the pencil of planes through the x-axis be:

$$(A + \lambda' P)x + By + Q\lambda' z - C_1 - \lambda' R_1 = 0$$

and the equation of the pencil of planes through the line u_p^r:

$$(A + \mu' P)x + By + Q\mu' z - C_2 - \mu' R_2 = 0$$

A plane of the first pencil is perpendicular to a plane of the second one if:

$$(A + \lambda' P)(A + \mu' P) + B^2 + \lambda' \mu' Q^2 = 0$$

or:
$$\lambda' \mu' (P^2 + Q^2) + (\lambda' + \mu') PQ + A^2 + B^2 = 0$$

Elimination of the parameters λ' and μ' out of this last equation and the equations of the two pencils gives the following *equation of the cylinder C:*

$$\frac{Ax + By - C_1}{Px + Qz - R_1} \cdot \frac{Ax + By - C_2}{Px + Qz - R_2} \cdot (P^2 + Q^2) - A \cdot P \cdot \left(\frac{Ax + By - C_1}{Px + Qz - R_1} + \right.$$
$$\left. + \frac{Ax + By - C_2}{Px + Qz - R_2}\right) + (A^2 + B^2) = 0$$

4. This equation can be reduced in the following form:

$$\begin{aligned} &\{km + \sin^2\mu\,(m^2 - km - km^3)\}\,(x^2 - px) + \\ &+ \{1 + \sin^2\mu\,(-m^2 - km + km^3)\}\,y^2 \cos^2\alpha + \\ &+ \{m^2 + \sin^2\mu\,(-m^2 + km - km^3)\}\,z^2 \sin^2\alpha + \\ &+ \sin\lambda \cos\lambda\,(1 - km)(-2xy \cos\alpha + py \cos\alpha + az \sin\alpha) + \\ &+ m\,(k - m) \sin\mu \cos\mu\,(-2xz \sin\alpha + ay \cos\alpha + pz \sin\alpha) + \\ &+ m \cos\mu \cos\lambda\,(-2yz \sin\alpha \cos\alpha - ax + ap) = 0 \qquad (9) \end{aligned}$$

5. *The locus of the points with a tangent through a given point $P^r(p\,;\,0\,;\,0)$ is the curve of intersection of the hyperboloid H and the cylinder C given by the equations (8) and (9) respectively*. Reflection of this curve with regard to the axis MN gives the locus of the points with a tangent through the point P on the line A'B' such that p is the distance of P to the midpoint T of the link A'B'.

6. As we showed in chapter I, § 6.7 the quadrics (8) and (9) have one generator in common. As the generators of the cylinder C are parallel to the x-axis, the common generator is also parallel to this axis. The hyperboloid H given by the equation (8) is generated by the projections of the line u (which is the locus of the nullpoints of the planes normal to A'B') upon the planes through A'B'. Let γ be the plane through A'B' parallel to the x-axis, that is, to the line u. The projection u' of u upon this plane γ is a generator of H parallel to the x-axis. The point of

intersection of u' and $A'B'$ is denoted by R. If we prove that R is a point of the cylinder C, *the line u' is the common generator of C and H.*

The direction numbers of the line u^r are given by (§ 4)

$$\left\| \begin{matrix} m \cos \lambda & \sin \mu \cos \alpha \, (m^2 - 1) & o \\ \cos \mu & o & \sin \mu \sin \alpha \, (m^2 - 1) \end{matrix} \right\|$$

or:

$$\sin \mu \sin \alpha \cos \alpha \, (m^2 - 1) \; ; \quad - m \cos \lambda \sin \alpha \; ; \quad - \cos \mu \cos \alpha$$

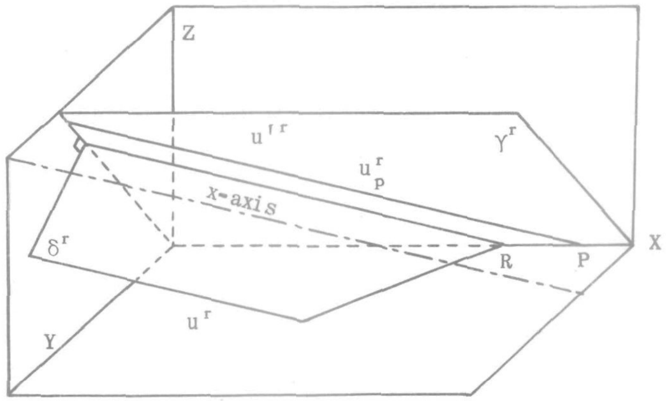

Figure 54

The equation of the plane γ^r through the X-axis parallel to u^r (fig. 54) is:

$$y \cos \alpha \cos \mu - mz \cos \lambda \sin \alpha = o$$

The pencil of planes through u^r is given by the equation (5) namely:

$$mx \cos \lambda + y \cos \alpha \sin \mu \, (m^2 - 1) + \cos \mu \, (a - mr) +$$
$$+ P\{x \cos \mu + z \sin \alpha \sin \mu \, (m^2 - 1) + \cos \lambda \, (am - r)\} = o$$

The plane δ^r of this pencil normal to the plane γ^r is given by:

$$\cos \alpha \sin \mu \, (m^2 - 1) \cos \mu \cos \alpha -$$
$$- P \sin \mu \sin \alpha \, (m^2 - 1) \, . \, m \cos \lambda \sin \alpha = o$$

or: $$P = \cos^2 \alpha \cos \mu : m \sin^2 \alpha \cos \lambda$$

If we substitute this value of P into the equation of the pencil through u^r and if we take $y = z = o$, we obtain *the equation which gives the position of R* which is the point of intersection of the X-axis with the generator of the hyperboloid H parallel to the line u^r.

We get:

$$mx \cos\lambda + \cos\mu \; (a - mr) +$$
$$+ (\cos^2\alpha \cos\mu : m \sin^2\alpha \cos\lambda) \{x \cos\mu + \cos\lambda \, (am - r)\} = o$$

or: $$x = \frac{(am - m^2 r)\sin^2\alpha + (am - r)\cos^2\alpha}{- m \sin^2\alpha} \cdot$$

$$\cdot \frac{m \cos\lambda \cos\mu \sin^2\alpha}{m^2 (1 - \sin^2\lambda) \sin^2\alpha + (1 - \sin^2\mu)\cos^2\alpha}$$

This expression is reducible to:

$$x = \frac{am \cos\lambda \cos\mu}{km + \sin^2\mu \, (- km^3 + m^2 - km)} \qquad (10)$$

7. If we substitute the values $y = z = o$ into the equation (9) of the cylinder C, we obtain an equation which gives the points of intersection of C with the X-axis. One of these points is the point P given by $x = p$.

We get:

$$\{km + \sin^2\mu \, (m^2 - km - km^3)\}(x^2 - xp) - ma \cos\mu \cos\lambda \, (x - p) = o$$

From this equation follows $x = p$ and the value of x corresponding to (10). As the generators of the cylinder C are parallel to the x-axis, the line through R parallel to the x-axis is a generator of C denoted by u'^r. This line u'^r is the common generator of the quadrics (8) and (9) and consequently *the curve of intersection is a twisted cubic.*

§ 7. Points with a tangent through a point anywhere in the space S (chapter I, § 6.6)

1. Let the coordinates of the reflected point P^r be (p ; q ; r). We draw the line l through P parallel to A'B'. The line u conjugated to the line at infinity of the planes normal to l is identical with the line conjugated to the line at infinity of the planes normal to A'B'.

The equation of the hyperboloid generated by the characteristics of the planes through l^r is equal to the equation of the locus generated by the lines of intersection of the normal planes of the pencils through u^r and l^r respectively.

The equation of the pencil of planes through u^r is:

$$mx \cos \lambda + y \sin \mu \cos \alpha (m^2 - 1) + \cos \mu (a - mr) + \\ + P\{x \cos \mu + z \sin \mu \sin \alpha (m^2 - 1) + \cos \lambda (am - r)\} = o \qquad (5)$$

and of the pencil through l^r:

$$y - q + Q(z - r) = o$$

The condition of the normal position of any plane of the first pencil to a plane of the second one is:

$$P . Q = - \cot \alpha$$

Elimination of the parameters P and Q out of this equation and the equations of the two pencils gives the following *equation of a hyperboloid:*

$$\frac{mx \cos \lambda + y \sin \mu \cos \alpha (m^2 - 1) + \cos \mu (a - mr)}{x \cos \mu + z \sin \mu \sin \alpha (m^2 - 1) + \cos \lambda (am - r)} \times \frac{y - q}{z - r} = - \cot \alpha$$

or: $$(y^2 + z^2 - qy - rz) \sin \alpha \cos \alpha \sin \mu (m^2 - 1) + \\ x(y - q) m \sin \alpha \cos \lambda + x(z - r) \cos \alpha \cos \mu + \\ (y - q) \sin \alpha \cos \mu (a - mr) + (z - r) \cos \alpha \cos \lambda (am - r) = o \qquad (11)$$

2. The locus of the points with a tangent which intersects the line u_p drawn through P parallel to the x-axis is a circular cylinder C' through the x-axis and through u_p such that the plane through P and the x-axis contains the axis of the cylinder.

The equations of the line u_p^r are: (cf. § 6.3)

$$Ax + By = mp \cos \lambda + Bq$$
$$Px + Qz = p \cos \mu + Qr$$

The equation of the cylinder C' follows from the equation (9) of the cylinder C if C_2 is replaced by $C_2 + Bq$ and R_2 by $R_2 + Qr$.

Therefore *the equation of the cylinder C' is reducible to:*

$$\{km + \sin^2\mu (m^2 - km - km^3)\} x(x - p) + \\ + \{1 + \sin^2\mu (- m^2 - km + km^3)\} y(y - q) \cos^2\alpha + \\ + \{m^2 + \sin^2\mu (- m^2 + km - km^3)\} z(z - r) \sin^2\alpha + \\ + \sin \lambda \cos \lambda (1 - km) \{- x(y - q) \cos \alpha - y(x - p) \cos \alpha + \\ + a(z - r) \sin \alpha\} + \sin \mu \cos \mu\, m(k - m) \{- x(z - r) \sin \alpha - \\ - z(x - p) \sin \alpha + a(y - q) \cos \alpha\} + m \cos \mu \cos \lambda [\{- y(z - r) - \\ - z(y - q)\} \sin \alpha \cos \alpha - a(x - p)] = o \qquad (12)$$

3. The locus of the points with a tangent through a given point P in the moving space S is obtained by reflection of the curve of intersection of the quadrics (11) and (12) with regard to the axis MN. As these quadrics have one generator in common, the curve is a twisted cubic. The point P has been chosen such that the coordinates of the reflected point P' are (p ; q ; r).

Chapter VI

THE SURFACE GENERATED BY THE CONNECTING-ROD A'B'

§ 1. Double-lines of the surface Π generated by A'B'

1. In chapter IV, § 1 we stated that the coordinates of A' and B' are:

$$A'(\ a + r\cos\varphi\ ;\ r\sin\varphi\cos\alpha\ ;\ - r\sin\varphi\sin\alpha)$$
$$B'(- a + r\cos\psi\ ;\ r\sin\psi\cos\alpha\ ;\ \ r\sin\psi\sin\alpha)$$

The *equations of the line A'B'* are:

$$\frac{x + a - r\cos\psi}{2a + r(\cos\varphi - \cos\psi)} = \frac{y - r\sin\psi\cos\alpha}{r\cos\alpha\,(\sin\varphi - \sin\psi)} =$$
$$= \frac{z - r\sin\psi\sin\alpha}{- r\sin\alpha\,(\sin\varphi + \sin\psi)}$$

or, if we write $\frac{1}{2}(\varphi + \psi) = \lambda$ and $\frac{1}{2}(\varphi - \psi) = \mu$,

$$2xr\cos\alpha\sin\mu\cos\lambda + 2ar\cos\alpha\sin\lambda\cos\mu =$$
$$= 2ay - 2yr\sin\mu\sin\lambda + r^2\cos\alpha\sin 2\mu$$

and: $$- 2xr\sin\alpha\sin\lambda\cos\mu - 2ar\sin\alpha\sin\mu\cos\lambda =$$
$$= 2az - 2zr\sin\mu\sin\lambda - r^2\sin\alpha\sin 2\lambda$$

or, writing y and z as functions of x:

$$2y(a - r\sin\mu\sin\lambda) =$$
$$= r\cos\alpha\,(2x\sin\mu\cos\lambda + 2a\sin\lambda\cos\mu - r\sin 2\mu)$$

and: $$2z(a - r\sin\mu\sin\lambda) =$$
$$= - r\cos\alpha\,(2x\sin\lambda\cos\mu + 2a\sin\mu\cos\lambda - r\sin 2\lambda)$$

From these equations we reduce:

$$\frac{y}{z} = - \cot\alpha\frac{\frac{2x}{m}\sin\lambda\cos\lambda + 2am\sin\mu\cos\mu - r\sin 2\mu}{2xm\sin\mu\cos\mu + \frac{2a}{m}\sin\lambda\cos\lambda - r\sin 2\lambda}$$

$$= - \cot\alpha\frac{\frac{x}{m}\sin 2\lambda + (am - r)\sin 2\mu}{(\frac{a}{m} - r)\sin 2\lambda + m\,x\sin 2\mu}$$

2. This expression is *independent of* λ *and* μ if:

$$\frac{x}{m} : \left(\frac{a}{m} - r\right) = (am - r) : xm$$

or:

$$x^2 = (am - r)\left(\frac{a}{m} - r\right)$$

or:

$$x = \pm\sqrt{(am - r)\left(\frac{a}{m} - r\right)}$$

The values of $\frac{y}{z}$ corresponding to these values are:

$$\frac{y}{z} = \mp\frac{\cot\alpha}{a - mr}\sqrt{(am - r)\left(\frac{a}{m} - r\right)}$$

Briefly we write:

$$\sqrt{(am - r)\left(\frac{a}{m} - r\right)} = p \qquad \text{and} \qquad \frac{a - mr}{p\cot\alpha} = q$$

where p and q are functions of a, r and α only because m only depends on a, r and α.

Consequently we get:

$$x = \pm p \, ; \qquad \frac{y}{z} = \mp\frac{1}{q}$$

The lines:

$$\left.\begin{array}{r} x = p \\ qy + z = o \end{array}\right\} \qquad \text{and} \qquad \left.\begin{array}{r} x = -p \\ qy - z = o \end{array}\right\}$$

denoted by d_1 and d_2 respectively are lines of the surface Π generated by A'B'. They are the lines of intersection of the planes $x = \pm p$ with the surface Π.

3. Between the points of the circles described by A' and B' there exists a *one-to-one correspondence*. As these circles have no self-corresponding points in common, the surface Π described by A'B' is a surface of the fourth degree. A surface of the fourth degree is not a doubly ruled surface for the quadrics are the only doubly ruled surfaces *). As A' and B' move in different planes through the X-axis, the line A'B' has a point in common with this axis only if A'B' coincides with it. The lines d_1 and d_2 intersect the X-axis at a right angle and hence, d_1 and d_2 are no generators of Π.

Π can be considered as a ruled surface with d_1 and d_2 and the circle described by A' as its directrices. Consequently *d_1 and d_2 are double-lines of the surface* Π **).

*) Eisenhart, Differential Geometry, 224 (Boston 1909)
**) H. J. van Veen, Beknopt Leerboek der Beschrijvende Meetkunde, 228 (Groningen 1946).

§ 2. The equation of the surface Π

1. The surface Π can be considered as the surface generated by the lines which intersect the lines d_1 and d_2 and the circle described by A' (fig. 55).

The equations of d_1 and d_2 are:

$$\left.\begin{aligned} x &= \pm p \\ qy \pm z &= o \end{aligned}\right\} \qquad (1)$$

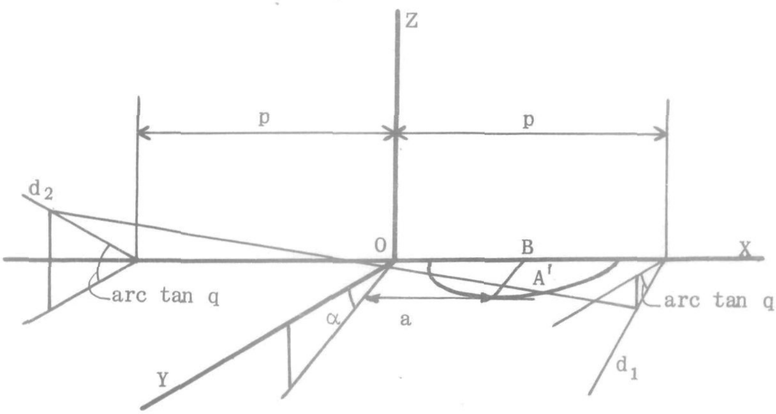

Figure 55

and the equations of the circle described by A' are:

$$\left.\begin{aligned} z &= -y \tan \alpha \\ (x - a)^2 + y^2 + z^2 &= r^2 \end{aligned}\right\} \qquad (2)$$

The lines which intersect d_1 and d_2 are the lines of intersection of the planes of the pencils:

$$(z - qy) + \lambda(x + p) = o \qquad (3)$$

and

$$(z + qy) + \mu(x - p) = o \qquad (4)$$

These lines intersect the circle given by the equations (2) if the equations (2) ; (3) ; (4) have a solution. Elimination of x, y and z out of these equations gives the condition for the common solution.

Elimination of z gives:

$$(x - a)^2 + y^2 \sec^2 \alpha = r^2 \qquad (5a)$$

$$\lambda x + \lambda p - y(\tan\alpha - q) = o \qquad (5b)$$
$$\mu x - \mu p - y(\tan\alpha + q) = o \qquad (5c)$$

If we write: $q + \tan\alpha = f$ and $q - \tan\alpha = g$ the equations (5b) and (5c) become:

$$\lambda x - fy = -\lambda p \qquad \text{and} \qquad \mu x + gy = \mu p$$

or: $$x = \frac{-\lambda gp + \mu fp}{g\lambda + \mu f} \qquad \text{and} \qquad y = \frac{2\mu\lambda p}{g\lambda + \mu f}$$

The elimination of x and y is obtained by substitution of these expressions into (5a). We get:

$$\left(p\,\frac{\mu f - g\lambda}{uf + g\lambda} - a\right)^2 + \frac{4\,\mu^2\lambda^2p^2}{\cos^2\alpha\,(\mu f + g\lambda)^2} = r^2$$

The equation of the surface Π follows by substitution of

$$\lambda = \frac{qy - z}{p + x} \qquad \text{and} \qquad \mu = \frac{qy + z}{p - x}$$

(these expressions follow from (3) and (4)) into this relation between λ and μ.

We obtain:

$$\begin{aligned} &p^2\{f\,(p + x)\,(qy + z) - g\,(p - x)\,(qy - z)\}^2 - \\ &- 2ap\,\{(p + x)^2\,(qy + z)^2\,f^2 - (p - x)^2\,(qy - z)^2\,g^2\} + \\ &+ (a^2 - r^2)\,\{f\,(p + x)\,(qy + z) + g\,(p - x)\,(qy - z)\}^2 + \\ &+ 4\,\{q^2y^2 - z^2\}^2\,p^2\sec^2\alpha = o \end{aligned} \qquad (6)$$

2. The equation (6) is reducible in the following way:
As $f = q + \tan\alpha$ and $g = q - \tan\alpha$ we have:

$$\begin{aligned} f\cdot(p + x)(qy + z) = {} & (pq^2y + qxz + pz\tan\alpha + qxy\tan\alpha) + \\ & + (pqz + q^2xy + pqy\tan\alpha + xz\tan\alpha) \end{aligned}$$

If we write this expression as A + B, we find

$$g(p - x)(qy - z) = A - B$$

Substitution into (6) gives:

$$p^2B^2 - 2apAB + (a^2 - r^2)\,A^2 + (q^2y^2 - z^2)^2\,p^2\sec^2\alpha = a$$

or: $$(pB - aA)^2 + (q^2y^2 - z^2)^2\,p^2\sec^2\alpha = r^2A^2 \qquad (7)$$

From the equation (7a) (chapter IV) namely:

$$\frac{2am}{r} = \cos^2 \alpha + m^2 \sin^2 \alpha$$

follows: $2am \cos^2 \alpha + 2am \sin^2 \alpha = r \cos^2 \alpha + m^2 r \sin^2 \alpha$

or: $$\tan^2 \alpha = (r - 2am) : m(2a - rm)$$

and $$\sec^2 \alpha = 1 + \tan^2 \alpha = r(m^2 - 1) : m(rm - 2a)$$

Furthermore we have: $p.q = \tan \alpha \, (a - mr)$ (§ 1.2)

The expression (pB − aA) in (7) becomes:

$$p^2qz + pq^2xy + p^2qy \tan \alpha + pxz \tan \alpha - apq^2y - aqxz -$$
$$- apz \tan \alpha - aqxy \tan \alpha = (pz + qxy)\ (pq - a \tan \alpha) +$$
$$+ (pqy + xz)\ (p \tan \alpha - aq)$$

where $pq - a \tan \alpha = \tan \alpha \, (a - mr) - a \tan \alpha = - mr \tan \alpha$

and $$p \tan \alpha - aq = \frac{\tan \alpha}{p} \left(p^2 - \frac{apq}{\tan \alpha}\right) =$$

$$= \frac{\tan \alpha}{p} \left\{(am - r)\ \left(\frac{a}{m} - r\right) - a(a - mr)\right\} =$$

$$= \frac{\tan \alpha}{mp} (a - mr)\ \{(am - r) - am\} = - rq : m$$

This gives:

$$pB - aA = - mr \tan \alpha \, (pz + qxy) - \frac{rq}{m} (pqy + xz)$$
$$= - \frac{r}{m} \{m^2pz \tan \alpha + m^2qxy \tan \alpha + pq^2y + qxz\}$$

If we write:

$$F = rpz \tan \alpha + rqxy \tan \alpha \quad \text{and} \quad G = rpq^2y + rqxz$$

we get: $$pB - aA = - mF - G : m$$

The expression rA from (7) becomes:

$$rA = G + F$$

Substitution of (pB − aA) and rA into (7) gives:

$$(-mF - G : m)^2 + (q^2y^2 - z^2)^2 p^2 \frac{r}{m}\cdot\frac{m^2-1}{rm-2a} = (G+F)^2$$

or:

$$(m^2 - 1)F^2 + (\frac{1}{m^2} - 1)G^2 + (q^2y^2 - z^2)^2 p^2 \frac{r}{m}\cdot\frac{m^2-1}{rm-2a} = o$$

or, after dividing by $\frac{m^2-1}{m^2}$:

$$m^2F^2 - G^2 + (q^2y^2 - z^2)^2 \frac{p^2rm}{rm-2a} = o$$

The equation of the surface Π generated by A'B' becomes:

$$m^2r^2p^2z^2\tan^2\alpha + m^2r^2q^2x^2y^2\tan^2\alpha - p^2q^4r^2y^2 - 2q^2r^2x^2z^2 +$$

$$+ 2xyzr^2pq(m^2\tan^2\alpha - q^2) + (q^2y^2 - z^2)^2 \frac{p^2rm}{rm-2a} = o$$

where: 1. r, a and α are data of the quadrilateral,
2. m is given by the equation $2am : r = \cos^2\alpha + m^2\sin^2\alpha$,
3. $p = \sqrt{(am - r)(\frac{a}{m} - r)}$ and $q = \frac{(a - mr)}{p}\tan\alpha$

§ 3. Reality of the double-lines

1. The equation (7a) from chapter IV:

$$2am : r = \cos^2\alpha + m^2\sin^2\alpha$$

or, if we write 2a : r = k,

$$k = \frac{\cos^2\alpha}{m} + m\sin^2\alpha$$

gives k as a function of α and m. If k and m are taken as current coordinates and α as a parameter, the equation gives a series of curves (fig. 56). The limiting values of α are o and π/2 in which cases the equation becomes k = 1/m and k = m respectively.

Each value of $o < \alpha < \pi/2$ gives a curve (hyperbola) through the point S(1 ; 1). The k-axis is one of the asymptotic lines of the curves. All curves lie in the shaded area of the graph.

2. The equation $k = \frac{\cos^2\alpha}{m} + m\sin^2\alpha$ is reducible to:

$$k = \{\sqrt{m}\sin\alpha - \frac{\cos\alpha}{\sqrt{m}}\}^2 + 2\sin\alpha\cos\alpha$$

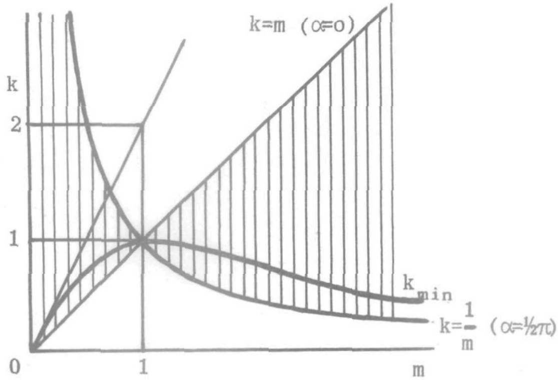

Figure 56

Hence, $k_{min} = 2 \sin \alpha \cos \alpha = \sin 2\alpha$ if $m = \cot \alpha$

or $$k_{min} = \frac{2 \sin \alpha \cos \alpha}{\cos^2 \alpha + \sin^2 \alpha} = \frac{2 \cot \alpha}{\cot^2 \alpha + 1} = \frac{2m}{m^2 + 1}$$

The curve corresponding to $k_{min} = \frac{2m}{m^2 + 1}$ goes through the origin and through S. The greatest value of k_{min} is 1 namely if $m = 1$.

3. The surface Π has two real double-lines d_1 and d_2 if $p = \sqrt{(am - r)(\frac{a}{m} - r)}$ is real. This gives the condition:

$$(am - r)(\frac{a}{m} - r) > o$$

or, as $m > o$, $$(2am - 2r)(2a - 2mr) > o$$

or: $$(km - 2)(k - 2m) > o$$

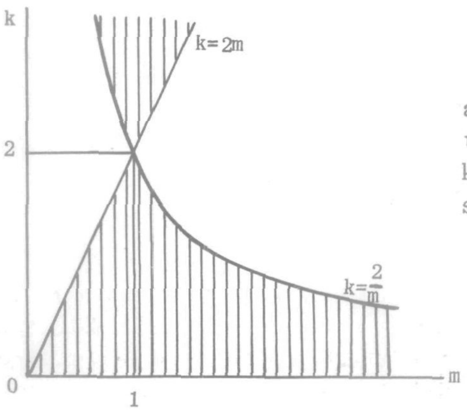

Figure 57

The points of the shaded area only (fig. 57) limited by the line $k = 2m$ and the curve $k = 2/m$ have coordinates which satisfy the inequation:

$$(km - 2)(k - 2m) > o$$

4a. Combination of the conditions given in 1 and 3 gives that the values of k and m are only the values of the coordinates of the points in the areas S_1OS and $A_\infty SS_2B_\infty$ (fig. 58), where A_∞ and B_∞ are the points at infinity of the curves km = 1 and km = 2 respectively which coincide with the point at infinity of the m-axis.

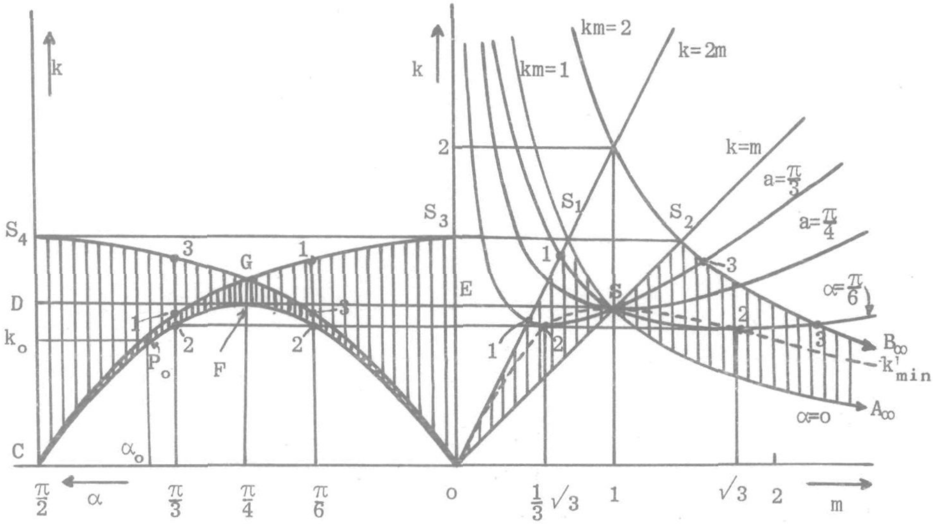

Figure 58

4b. The left-hand figure gives k as function of α. The limiting curve corresponding to the line OS_1 (k = 2m) is given by:

$$k = \frac{k}{2}\sin^2\alpha + \frac{2\cos^2\alpha}{k}$$

or:
$$k^2 = \frac{4\cos^2\alpha}{1+\cos^2\alpha}$$

or:
$$k = \frac{2\cos\alpha}{\sqrt{1+\cos^2\alpha}}$$

which gives the line CS_3.

The line corresponding to OS (k = m or $\alpha = \pi/2$) is the line CD and the line corresponding to SS_1 (α = o) is the line ES_3. The point S in which k = 1, m = 1 and α is undetermined corresponds to the line DE. Hence, the "triangle" S_1OS corresponds to the figure S_3CDES_3.

4c. The line corresponding to S_2B_∞ (km = 2) is given by:

$$k = \frac{2}{k} \sin^2 \alpha + \frac{k}{2} \cos^2 \alpha$$

or: $$k = \frac{2 \sin \alpha}{\sqrt{1 + \sin^2 \alpha}}$$

which gives the line S_4O ($o < \alpha < \pi/2$).

The curves S_4O and S_3C are symmetrical with regard to the line $\alpha = \pi/4$. The line SS_2 ($\alpha = \pi/2$) corresponds to DS_4 and the curve SA_∞ (km = 1; $\alpha = o$) corresponds to the line EO. Hence, the figure $A_\infty SS_2B_\infty$ corresponds to the figure $OEDS_4O$. The line CFO in the left part of the graph is the curve given by the equation:

$$k_{min} = \sin 2\alpha$$

As $\sin 2\alpha < \frac{2 \cos \alpha}{\sqrt{1 + \cos^2 \alpha}}$ and $\sin 2\alpha < \frac{2 \sin \alpha}{\sqrt{1 + \sin^2 \alpha}}$ the curve CFO is drawn beneath the curves CS_3 and OS_4.

If we follow the curve $\alpha = \pi/3$ in the right-hand graph, we meet the special points 1, 2 and 3. These points correspond to the points 1, 2 and 3 in the left-hand graph on the line $\alpha = \pi/3$. Similarly we get the points 1, 2 and 3 on the line $\alpha = \pi/6$. Consequently each point of the area in the left figure between the curve CFO and the curves CG and GO corresponds *to two points* of the shaded areas of the right figure.

5. The surface Π generated by A'B' has two real double-lines if:

I $\quad \sin 2\alpha < k < \frac{2 \cos \alpha}{\sqrt{1 + \cos^2 \alpha}}$ if $o < \alpha < \pi/4$.

II $\quad \sin 2\alpha < k < \frac{2 \sin \alpha}{\sqrt{1 + \sin^2 \alpha}}$ if $\pi/4 < \alpha < \pi/2$.

Let P_o be a point of the double-area of the left-hand graph and let its coordinates be d_o and k_o. There exist two values of m, m_1 and m_2 say, corresponding to $\alpha = \alpha_o$ and $k = k_o$. So we obtain: If we have an isogram with $\alpha = \alpha_o$ and $k = k_o$, there exist two surfaces Π_1 and Π_2 which can be generated by A'B' and each of them has two real double-lines (chapter IV, § 1.4).

§ 4. Special cases given by the limiting values of k

1. If $k = \sin 2\alpha$ the equation

$$\cos^2 \alpha + m^2 \sin^2 \alpha = km$$

becomes:
$$\cos^2 \alpha + m^2 \sin^2 \alpha = m \sin 2\alpha$$

or:
$$(m \sin \alpha - \cos \alpha)^2 = o$$

or:
$$m = \cot \alpha$$

The value $p = \sqrt{(am - r)(\frac{a}{m} - r)}$ given in § 1.2 becomes:

$$p = \tfrac{1}{2}\, r\sqrt{\frac{(km - 2)(k - 2m)}{m}} = \tfrac{1}{2}\, r \sqrt{\frac{(\sin 2\alpha \cot \alpha - 2)(\sin 2\alpha - 2 \cot \alpha)}{\cot \alpha}}$$

$$= r \sin \alpha \cos \alpha$$

and the value of $\frac{y}{z}$ given in § 1.2 becomes:

$$\frac{y}{z} = - \cot \alpha \frac{p}{a - mr} = - \cot \alpha \frac{r \sin \alpha \cos \alpha}{a - r \cot \alpha} =$$

$$= - \cot \alpha \frac{2 \sin \alpha \cos \alpha}{\sin 2\alpha - 2 \cot\alpha} = \tan \alpha$$

The equations of the double-lines d_1 and d_2 are in this case:

$$\begin{cases} x = \pm r \sin \alpha \cos \alpha \\ y = \pm z \tan \alpha \end{cases}$$

2. If $k = \frac{2 \sin \alpha}{\sqrt{1 + \sin^2 \alpha}}$, the equation

$$\cos^2 \alpha + m^2 \sin^2 \alpha = km$$

becomes: $\cos^2 \alpha + m^2 \sin^2 \alpha = 2m \sin \alpha : \sqrt{1 + \sin^2 \alpha}$

or: $m^2 \sin^2 \alpha - 2m \sin \alpha : \sqrt{1 + \sin^2 \alpha} + \cos^2 \alpha = o$

or:
$$m_{12} = \frac{\frac{2 \sin \alpha}{\sqrt{1 + \sin^2 \alpha}} \pm \sqrt{\frac{4 \sin^2 \alpha}{1 + \sin^2 \alpha} - 4 \sin^2 \alpha \cos^2 \alpha}}{2 \sin^2 \alpha}$$

$$= \frac{1 \pm \sqrt{1 - \cos^2 \alpha - \sin^2 \alpha \cos^2 \alpha}}{\sin \alpha \sqrt{1 + \sin^2 \alpha}}$$

$$= \frac{1 \pm \sin^2 \alpha}{\sin \alpha \sqrt{1 + \sin^2 \alpha}}$$

or:

$$m_1 = \frac{\sqrt{1 + \sin^2 \alpha}}{\sin \alpha} = \frac{2}{k} \qquad \text{and} \qquad m_2 = \frac{\cos^2 \alpha}{\sin \alpha \sqrt{1 + \sin^2 \alpha}}$$

If we take $m = m_1 = \frac{2}{k}$, we notice that $p = o$ and $\frac{y}{z} = o$. *The two double-lines coincide with the z-axis.* In every position the line A'B' intersects the z-axis. Hence, in every position the projections of A' and B' upon the plane XOY lie on a line through the origin.

3. If $k = \frac{2 \cos \alpha}{\sqrt{1 + \cos^2 \alpha}}$, the equation

$$\cos^2 \alpha + m^2 \sin^2 \alpha = km$$

becomes: $m^2 \cdot \sin^2 \alpha - 2m \cos \alpha : \sqrt{1 + \cos^2 \alpha} + \cos^2 \alpha = o$

From this equation follows:

$$m_1 = \frac{\cos \alpha}{\sin^2 \alpha} \sqrt{1 + \cos^2 \alpha} \qquad \text{and} \qquad m_2 = \frac{\cos \alpha}{\sqrt{1 + \cos^2 \alpha}} = \tfrac{1}{2} k$$

If $m = m_2 = \frac{1}{2}k$ we obtain $p = o$ and $\frac{y}{z} = o$ and again *the double-lines coincide with the z-axis.*

§ 5. The value x_c of chapter V § 3

In chapter V we found that the point of the line A'B' with a tangent which coincides with A'B' is given in its reflected position with regard to the axis MN of the isogram by:

$$x = \pm \sqrt{(am - r)(\frac{a}{m} - r)} \; ; \qquad y = o \; ; \qquad z = o$$

if the position of the quadrilateral follows from:

$$|\sin \mu| = \sqrt{\frac{a}{N}} \qquad \text{and} \qquad |\cos \lambda| = \sqrt{\frac{a - mr}{N}}$$

where $N = a + am^2 - mr$.

The considered point is a real point if:

$$(am - r)(\frac{a}{m} - r) > o$$

or, as $m > o$

$$(km - 2)(k - 2m) > o$$

From $|\sin \mu| = \sqrt{\frac{a}{N}}$ follows: $N > o$ and from $|\cos \lambda| = \sqrt{\frac{a - mr}{N}}$ follows:

$$a - mr > o \qquad \text{or:} \qquad k - 2m > o$$

Hence, the condition $(km - 2)(k - 2m) > o$ is only fulfilled if:

$$km - 2 > o \qquad \text{and} \qquad k - 2m > o$$

Fig. 56 shows that if we draw the curve $km = 2$ and the line $k = 2m$ there are no values of k and m which hold to the inequations

$$km - 2 > o \qquad \text{and} \qquad k - 2m > o$$

and to the equation: $km = \cos^2 \alpha + m^2 \sin^2 \alpha$

Consequently we obtain the theorem: *There exist no position of the isogram ABA'B' such that a real point of A'B' has a tangent that coincides with A'B'.*

§ 6. Distance of the point of intersection of the double-line d_1 with A'B' to the X-axis

1. Let FG (fig. 59) be the double-line d_1 through (p ; o ; o) and GH its projection on the plane XOY. F is the point of intersection of d_1 and the line A'B', that is, the common point of A'B' and the plane $x = p$.

In § 1.1 we found as one of the equations of A'B':

$$2y(a - r \sin \mu \sin \lambda) =$$
$$= r \cos \alpha \,(2x \sin \mu \cos \lambda + 2a \sin \lambda \cos \mu - r \sin 2\mu)$$

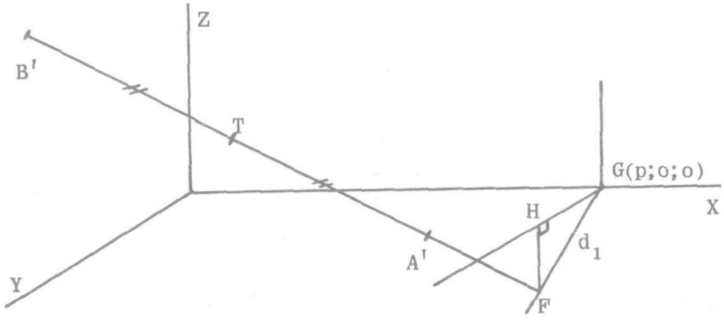

Figure 59

Substitution of x = p gives:

$$y_F = HG = r\cos\alpha\,\frac{2p\sin\mu\cos\lambda + 2a\sin\lambda\cos\mu - 2r\sin\mu\cos\mu}{2(a - r\sin\mu\sin\lambda)} =$$

$$= r\cos\alpha\,\frac{\sin\mu\,\{p\cos\lambda + (am - r)\cos\mu\}}{a - rm\sin^2\mu}$$

As cos λ is a function of μ, y_F is also a function of μ.

We write $y_F = r\cos\alpha\, f(\mu)$. The *extreme value of* y_F can be calculated by taking the derivative of y_k or of f (μ). We obtain:

$$f'(\mu) = \frac{1}{(a - rm\sin^2\mu)^2}\Big[(a - rm\sin^2\mu)\,[\cos\mu\,\{p\cos\lambda +$$

$$+ (am - r)\cos\mu\} + \sin\mu\,\{- p\sin\lambda\frac{d\lambda}{d\mu} - (am - r)\sin\mu\}] -$$

$$- \sin\mu\,\{p\cos\lambda + (am - r)\cos\mu\}\,(- 2\,rm\sin\mu\cos\mu)\Big]$$

We suppose: $a - r\sin\mu\sin\lambda \neq o$

that is, $a - rm\sin^2\mu \neq o$

As sin λ = m sin μ, we get: cos λ . dλ = m cos μ dμ and we obtain if f'(μ) = o:

$$(a - rm\sin^2\mu)\,[\cos\mu\,\{p\cos\lambda + (am - r)\cos\mu\} +$$

$$+ \sin\mu\,\{- p\sin\lambda\frac{m\cos\mu}{\cos\lambda} - (am - r)\sin\mu\}] +$$

$$+ 2rm\sin^2\mu\cos\mu\,\{p\cos\lambda + (am - r)\cos\mu\} = o$$

This equation is reducible to:

$$\{(am^2 + a - mr)\sin^2\mu - a\}\,(mr\sin^2\mu - a)^2 = o$$

or: $$\sin^2\mu = \frac{a}{a + am^2 - mr}$$

as $$mr\sin^2\mu - a \neq o$$

or $$|\sin\mu| = \sqrt{\frac{a}{N}}$$

2. If $\sin\mu = \sqrt{\frac{a}{N}}$, we obtain:

$$y_{F_{extr}} = \frac{r\cos\alpha}{1 - rm\frac{a}{N}}\,\{\sqrt{\frac{(am - r)(a - rm)}{m}}\cdot\sqrt{\frac{a - rm}{N}} +$$

$$+ (am - r)\sqrt{\frac{m(am - r)}{N}}\}\sqrt{\frac{a}{N}}$$

This expression is reducible to:

$$y_{F_{extr}} = r\cos\alpha\sqrt{\frac{am - r}{am}}$$

As
$$\frac{y}{z} = -\frac{\cot\alpha}{a - mr}\sqrt{(am - r)(\frac{a}{m} - r)}$$

we get:
$$z_{extr} = -y_{extr}\tan\alpha\frac{(a - mr)\sqrt{m}}{\sqrt{(a - mr)(am - r)}}$$

$$= -r\sin\alpha\sqrt{\frac{a - mr}{a}}$$

The extreme value of FG becomes:

$$FG^2_{extr} = y^2_{extr} + z^2_{extr} = r^2\{\cos^2\alpha\frac{am - r}{am} + \sin^2\alpha\frac{a - mr}{a}\}$$

$$= r^2\{\cos^2\alpha + \sin^2\alpha - \frac{r}{am}(\cos^2\alpha + m^2\sin^2\alpha)\}$$

$$= r^2(1 - \frac{r}{am}\frac{2am}{r})$$

$$= -r^2$$

or FG_{extr} is *imaginary* namely:

$$FG_{extr} = ir$$

3. If T is the midpoint of A'B' we shall calculate the distance TF if GF has its extreme value.

The coordinates of T are:

$$\{r\cos\lambda\cos\mu\ ;\ r\cos\alpha\sin\lambda\cos\mu\ ;\ -r\sin\alpha\sin\mu\cos\lambda\}$$

and the coordinates of F are:

$$\{p\ ;\ r\cos\alpha\sqrt{\frac{am - r}{am}}\ ;\ -r\sin\alpha\sqrt{\frac{a - mr}{a}}\}$$

where:

$$p = \sqrt{(am - r)(\frac{a}{m} - r)}\ ; \qquad \sin\mu = \sqrt{\frac{a}{N}}\ ;\ \sin\lambda = m\sqrt{\frac{a}{N}}\ ;$$

$\cos \mu = \sqrt{\dfrac{m(am - r)}{N}}$; $\cos \lambda = \sqrt{\dfrac{a - mr}{N}}$ and $N = a + am^2 - mr$

The coordinates of T are reducible to:

$$\{\frac{rpm}{N}; \quad r \cos \alpha \frac{m}{N} \sqrt{am(am - r)}; \quad - r \sin \alpha \frac{1}{N} \sqrt{a(a - mr)}\}$$

We obtain:

$$TF^2 = (p - \frac{rpm}{N})^2 + r^2 \cos^2 \alpha \; am(am - r)(\frac{1}{am} - \frac{m}{N})^2 +$$
$$+ r^2 \sin^2 \alpha \; a(a - mr)(\frac{1}{a} - \frac{1}{N})^2$$

This expression is reducible by aid of the relation $r \cos^2 \alpha + r m^2 \sin^2 \alpha = 2am$ to: $TF^2 = p^2$

or $TF = p$

We obtain the theorem:

If $A'B'$ is tangent in one of its points F, this point F is a point of one of the double-lines d_1 of the surface Π generated by $A'B'$. The distance of F to the midpoint T of $A'B'$ is equal to the distance of d_1 to the midpoint O of AB. The corresponding position of the quadrilateral, however, is not real.

§ 7. The case in which A'B' is characteristic

The tangent in A' is perpendicular to A'B and intersects the X-axis in the point D (fig. 60). Similarly the tangent in B' is perpendicular to AB' and intersects the X-axis in E. A'B' is the characteristic of a plane through A'B' if *the tangents in A' and B' lie in one plane*. Therefore it is necessary that the points E and D coincide.

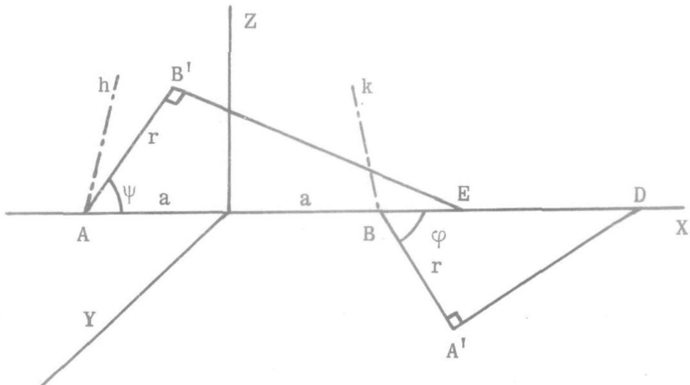

Figure 60

We get:
$$OE = AE - a = r \sec \psi - a$$
$$OD = BD + a = r \sec \varphi + a$$

The points E and F coincide if OE = OD.

or:
$$r \sec \psi - a = r \sec \varphi + a$$

or:
$$r \frac{\cos \varphi - \cos \psi}{\cos \varphi \cos \psi} = 2a$$

or:
$$- r \frac{2 \sin \lambda \sin \mu}{\frac{1}{2} (\cos 2 \lambda + \cos 2 \mu)} = 2a$$

or:
$$\sin^2 \mu = \frac{a}{am^2 + a - rm}$$

which is *in accordance with the condition that A'B' is a tangent.*

§ 8. The line conjugated to A'B'

1. The line $(A'B')^p$ is defined as the line of intersection of the planes α and β through A' and B' respectively and normal to the tangents at these points.

The plane α is the plane through A' and k or through A'B and k and the plane β is the plane through B' and h or through AB' and h.

The pencil of planes through the hinge-axis h is:

$$y + z \tan \alpha + \lambda(x + a) = o$$

Substitution of the coordinates of B' namely

$$\{-a + r \cos \psi ; r \sin \psi \cos \alpha ; r \sin \psi \sin \alpha\}$$

gives:
$$\lambda = - \tan \psi : \cos \alpha$$

The equation of the plane β becomes:

$$y \cos \alpha + z \sin \alpha = (x + a) \tan \psi$$

Similarly the equation of the plane α is:

$$y \cos \alpha - z \sin \alpha = (x - a) \tan \varphi$$

The line $(A'B')^p$ conjugated to A'B' is given by these two equations.

2. The direction numbers of A'B' are (§ 1):

$$2a + r(\cos\varphi - \cos\psi)\ ; \qquad r\cos\alpha(\sin\varphi - \sin\psi)\ ;$$
$$- r\sin\alpha(\sin\varphi + \sin\psi)$$

or:
$$a - r\sin\lambda\sin\mu\ ; \qquad r\cos\alpha\sin\mu\cos\lambda\ ;$$
$$- r\sin\alpha\sin\lambda\cos\mu$$

The direction numbers of $(A'B')^P$ are given by:

$$\begin{Vmatrix} -\tan\psi & \cos\alpha & \sin\alpha \\ -\tan\varphi & \cos\alpha & -\sin\alpha \end{Vmatrix}$$

or: $-2\sin\alpha\cos\alpha\ ;\ -\sin\alpha(\tan\varphi + \tan\psi)\ ;\ \cos\alpha(\tan\varphi - \tan\psi)$

or:
$$-2\sin\alpha\cos\alpha\cos\varphi\cos\psi\ ; \qquad -\sin\alpha\sin(\varphi+\psi)\ ;$$
$$\cos\alpha\sin(\varphi-\psi)$$

or:
$$-\sin\alpha\cos\alpha(\cos 2\lambda + \cos 2\mu)\ ; \qquad -\sin\alpha\sin 2\lambda\ ;$$
$$\cos\alpha\sin 2\mu$$

The line A'B' is *perpendicular to its conjugated line* $(A'B')^P$ if:

$$-(a - r\sin\lambda\sin\mu)\sin\alpha\cos\alpha(\cos 2\lambda + \cos 2\mu) -$$
$$- r\sin\alpha\cos\alpha\sin\mu\cos\lambda\sin 2\lambda -$$
$$- r\sin\alpha\cos\alpha\sin\lambda\cos\mu\sin 2\mu = 0$$

The equation is reducible to:

$$\sin^2\mu = \frac{a}{a + am^2 - rm}$$

which is the same equation as given in § 7. Consequently *the conditions that A'B' is a tangent and that $(A'B')^P$ is perpendicular to A'B' are identically equal.* This is in accordance with theorem XIV (chapter I, § 5): If a line l is perpendicular to its conjugated line l^P, l is a tangent.

§ 9. Crossed and not-crossed quadrilaterals

1. The orientation of the angles φ and ψ, that is, the angles between the rotating sides and the positive X-axis, is given in fig. 61.

From $\sin\lambda = m\sin\mu$ where $\lambda = \frac{1}{2}(\varphi + \psi)$ and $\mu = \frac{1}{2}(\varphi - \psi)$ follows:

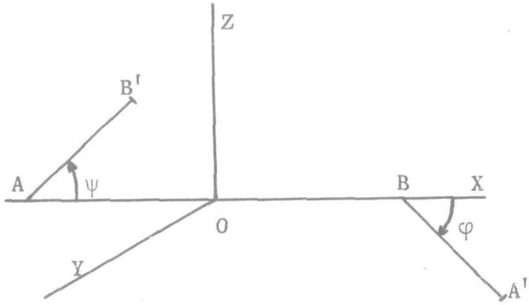

Figure 61

$$\sin \tfrac{1}{2}(\varphi + \psi) = m \sin \tfrac{1}{2}(\varphi - \psi)$$

or:

$$(m + 1) \tan \tfrac{1}{2} \psi = (m - 1) \tan \tfrac{1}{2} \varphi \qquad (1)$$

where m is given by the equation:

$$m^2 \sin^2 \alpha - km + \cos^2 \alpha = o \qquad (2)$$

Let m_1 be one of the roots of this equation and ψ_1 any value of ψ. We get:

$$\tan \tfrac{1}{2}\varphi = \frac{m_1 + 1}{m_1 - 1} \tan \tfrac{1}{2} \psi_1$$

This equation gives only one value of $\varphi(\pm\, n \,.\, 2\pi)$. Consequently if m_1 and m_2 are the roots of (2), a surface Π_1 generated by A'B' corresponds to $m = m_1$ and a surface Π_2 corresponds to $m = m_2$. Only if $m_1 = m_2$ the surfaces Π_1 and Π_2 are identical (chapter IV, § 1.4).

2. The isogram is called a *crossed* quadrilateral if an increase of φ corresponds to a decrease of ψ. If an increase of φ corresponds to an increase of ψ, the quadrilateral is called not-crossed.

We consider φ and ψ as functions of the time t. If we take the derivative with respect to t in the equation (1), we obtain:

$$\frac{m+1}{2 \cos^2 \tfrac{1}{2} \psi} \,.\, \dot{\psi} = \frac{m-1}{2 \cos^2 \tfrac{1}{2} \varphi} \,.\, \dot{\varphi}$$

The mechanism is crossed if $\dot{\psi} : \dot{\varphi} < o$ and not-crossed if $\dot{\psi} : \dot{\varphi} > o$ or $\frac{m-1}{m+1} < o$ and $\frac{m-1}{m+1} > o$ respectively.

From $m_1 + m_2 = k : \sin^2 \alpha$ and $m_1 \,.\, m_2 = \cot^2 \alpha$ follows that if m_1 and m_2 are real ($k > \sin 2\alpha$), they are positive. Therefore *the condition of the crossed mechanism* is $m - 1 < o$ or $m < 1$ and of *the not-crossed mechanism* $m > 1$.

The mechanism corresponding to $m = m_1$ is denoted by M_1 and the one corresponding to $m = m_2$ by M_2.

If we define $m_1 > m_2$ we distinguish the following cases:

A. $m_1 > 1$ and $m_2 \geqslant 1$; M_1 and M_2 are both not-crossed.
B. $m_1 > 1$ and $m_2 < 1$; M_1 is not-crossed and M_2 is crossed.
C. $m_1 < 1$ and $m_2 < 1$; M_1 and M_2 are both crossed.

and further we have the limiting cases:

D. $m_1 > 1$ and $m_2 = 1$
E. $m_1 = 1$ and $m_2 < 1$
F. $m_1 = 1$ and $m_2 = 1$

3. A. m_1 and m_2 are both > 1 if:

a) m_1 and m_2 are real, that is, $k > \sin 2\alpha$
b) $m_1 + m_2 > 2$, that is, $k > 2 \sin^2 \alpha$
c) $(m_1 - 1)(m_2 - 1) > o$ or $m_1 \cdot m_2 - (m_1 + m_2) + 1 > o$, that is, $k < 1$

Hence:

$$\begin{cases} 2 \sin^2 \alpha < k < 1 \\ \sin 2\alpha < k < 1 \end{cases}$$

The first condition is only satisfied if $o < \alpha < \pi/4$ and for these values of α we have $2 \sin^2 \alpha < \sin 2\alpha$.

We obtain: *The two mechanisms M_1 and M_2 are not-crossed* if

$$\sin 2\alpha < k < 1 \quad \text{and} \quad o < \alpha < \pi/4$$

B. $m_1 > 1$ and $m_2 < 1$ if:

$$(m_1 - 1)(m_2 - 1) < o$$

or: $m_1 m_2 - (m_1 + m_2) + 1 < o$ or $k > 1$ or $2a > r$

We obtain: *If the fixed link AB is longer than the links AB' and A'B the mechanisms M_1 and M_2 are not-crossed and crossed respectively.*

C. m_1 and m_2 are both < 1 if:

a) m_1 and m_2 are real, that is, $k > \sin 2\alpha$
b) $m_1 + m_2 < 2$, that is, $k < 2 \sin^2 \alpha$
c) $(m_1 - 1)(m_2 - 1) > o$, that is, $k < 1$

Hence:

$$\begin{cases} \sin 2\alpha < k < 1 \\ \sin 2\alpha < k < 2 \sin^2 \alpha \end{cases}$$

The second condition is only fulfilled if $\alpha > \pi/4$. For the values of α between $\pi/4$ and $\pi/2$ we have $2 \sin^2 \alpha > 1$ and therefore we obtain:

The two mechanisms M_1 and M_2 are crossed if

$$\sin 2\alpha < k < 1 \qquad \text{and} \qquad \pi/4 < \alpha < \pi/2$$

D. $m_2 = 1$. Subsitution into the equation (2) gives:

$$\sin^2 \alpha - k + \cos^2 \alpha = o \qquad \text{or} \qquad k = 1$$

As $m_1 \cdot m_2 = \cot^2 \alpha$ we get $m_1 = \cot^2 \alpha$. As we supposed $m_1 \geqq m_2$, we conclude $\cot^2 \alpha > 1$ or $o < \alpha < \pi/4$ and from $m_1 > 1$ follows that the mechanism M_1 is not-crossed. The mechanism M_2 with $m_2 = 1$ is a degenerated mechanism which follows from the equation (1): $\tan \frac{1}{2} \psi = o$ or $\psi = o$ $(\pm n \cdot 2\pi)$ and φ is undetermined.
As $\psi = o$ the point B' coincides with B (fig. 62). *The surface Π_2* generated by A'B' if $m = m_2 = 1$ *is the plane through the circles described by A'.*

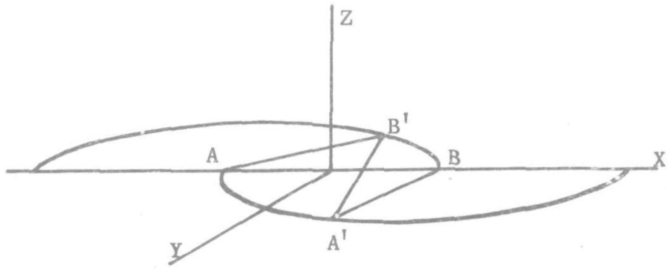

Figure 62

E. $m_1 = 1$. Substitution into (2) gives again $k = 1$. Further we get $m_2 = \cot^2 \alpha$ and as we supposed $m_1 \geqq m_2$ we have $\cot^2 \alpha < o$ or $\pi/4 < \alpha \leqq \pi/2$.

As $m_2 < 1$, the mechanism M_2 is crossed. The value $m_1 = 1$ gives the *same degenerated mechanism* M_1 as mentioned in D.

F. $m_1 = m_2 = 1$.
Hence we obtain: $k = \sin 2\alpha$; $k = 1$; $\alpha = \pi/4$, that is, *the two mechanisms M_1 and M_2 are degenerated.*

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STELLINGEN

I

Koppelt men twee niet-evenwijdige assen door middel van de ruimtelijke stangenvierhoek van *Bennett*, dan verkrijgt men een overbrenging zonder dood punt. De nadelen van deze koppeling zijn:

1. het verschil in momentane omwentelingssnelheid van de assen;
2. de wringing in de koppelingsstang.

Diss. pag. 46.
Bennett, A new mechanism. Engineering **4**, **12**, 777 (1903).

II

Zij van een orthogonale paraboloïde a één der beschrijvenden door de top en b een beschrijvende die a snijdt in een punt P, dat niet met de top samenvalt, dan kan men vier punten A, B, C en D op b aannemen, zodanig dat AB = CD en BP = PC, terwijl de punten de volgorde A, B, C, D hebben. Noemt men de beschrijvenden door deze punten respectievelijk a_1, a_2, a_3, a_4 en de normalen in deze punten n_A, n_B, n_C, n_D, dan geeft spiegeling van A met n_A ten opzichte van a_2 en van C met n_C ten opzichte van a_4, de punten A' met $n_{A'}$, en C' met $n_{C'}$. De vierhoek A'BDC' blijkt dan een vierhoek van *Bennett* te zijn met $n_{A'}$, n_B, n_D, $n_{C'}$, als respectievelijke scharnierassen.

Bennett, The skew isogram mechanism.
J.London Math. Soc. **13**, 151 (1914).

III

Dat de door *Schoenflies* gegeven theorie rechtstreeks kan worden toegepast op de beweging van de aan de verbindingsstang van de vierhoek van *Bennett* gekoppelde ruimte berust op het feit dat de scharnierassen twee aan twee toegevoegde rechten zijn.

Schoenflies, Geometrie der Bewegung (Leipzig, 1886) pag. 87.

IV

In Euclides **III-IV**, 140-141 (1953) kiest *Haantjes* in zijn artikel over de kinematische methode in de meetkunde als voorbeeld de rechte van Wallace. Zijn oplossing kan kinematisch fraaier worden gegeven door een bijzondere wenteling om P.

V

Door *Grüss* is de rol-glijbeweging gedefinieerd als een beweging van een vlak V_1, ten opzichte van een vast vlak V, waarbij een kromme C_1 uit V_1 voortdurend raakt aan een kromme C uit V, terwijl tussen overeenkomstige booglengten op deze krommen een constante verhouding λ bestaat.

De rol-glijbeweging is geen speciale beweging; elke beweging van vlak V_1 ten opzichte van V kan als zodanig worden beschouwd. Daarbij kan de factor λ nog willekeurig worden gekozen.

Grüss, Zur Kinematik des Rollgleitens, Z. Angew. Math. Mech. **31**, 97-103 (1951).

VI

Bekend is de volgende eigenschap. Een kegelsnede raakt aan de zijden van de driehoek $A_1A_2A_3$; verbindt men de raakpunten op de beide in A_i samenkomende zijden en snijdt men de verbindingslijn met de overstaande zijde van A_i, dan liggen de drie snijpunten (volgens het theorema van Pascal) op één rechte.

Deze eigenschap laat een meerdimensionale uitbreiding toe; deze heeft betrekking op de figuur van een quadratische variëteit, die aan de ribben van een simplex raakt.

VII

De methode van *Alders*, volgens welke de oppervlakte van een driehoek wordt gedefinieerd als het halve product van hoogte en basis biedt een goede gelegenheid tot de formele opbouw van het begrip oppervlakte. Zijn analoge methode voor de inhoud daarentegen heeft ernstige bezwaren.

Alders, Vlakke Meetkunde (1952) pag. 109.
Alders, Stereometrie (1950) pag. 53.

VIII

De door *De Kok* ontwikkelde en door hem als nieuw beschouwde methode van numerieke quadratuur, volgens welke bij integratie over het interval $m - a < x < m + a$ de functiewaarden in de punten $x = m \pm a$ en $x = m \pm a : \sqrt{5}$ worden gebruikt, is een bijzonder geval van een algemene integratiemethode, die op de eigenschappen der polynomen van Legendre berust.

De Kok, Numerieke Integratie, Euclides **25**, 271-273 (1950)

IX

Schuh geeft een kenmerk voor convergentie van uitgebreide kettingbreuken, waarbij ondersteld is dat de gedeeltelijke tellers $b_1, b_2, \ldots$ en de gedeeltelijke noemers $a_1, a_2, \ldots$ alle positief zijn. Dit kenmerk kan als volgt verscherpt worden:

$$\frac{b_n}{a_n a_{n-1}} \leqq n$$

Schuh, Lessen over Hoogere Algebra III, 683 (1926)

X

De didactische waarde van films over de wiskunde is belangrijk minder dan die van films over de natuurkunde.

XI

De omzetting van de H.B.S. met vijfjarige cursus in een Atheneum met zesjarige cursus heeft grote nadelen.

XII

Het is wenselijk dat onderwijzers en leraren kennis nemen van de industrie door tenminste één jaar in één of meer fabrieken te werken.

XIII

De nomografie kan, ondanks de ontwikkeling van de rekenmachines, veelvuldiger toegepast worden bij de bedrijfsefficiëntie.

XIV

Nu ons land, noodgedwongen, in een snel tempo wordt geïndustrialiseerd, is het gewenst de toekomstige fabrieksarbeider in de laatste jaren van zijn schoolopleiding begrip bij te brengen van deze noodzaak en van het belang der arbeidsproductiviteit voor het nationale, materiële bestaansniveau.

XV

Het is aan te bevelen, grote industriële ondernemingen onder te verdelen is niet te grote, in zichzelf zoveel mogelijk afgeronde, zelfstandige eenheden, die alle phasen van het industriële proces verzorgen voor een beperkte variatie van producten en die geografisch gespreid zijn.