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RESEARCH

Projected BIE: a new unbiased minimum variance estimator for the mixed-integer GNSS model

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Abstract
Mixed-integer (MI) estimation theory is fundamental to carrier-phase GNSS, where real-valued parameters are coupled with integer ambiguities. The class of integer-equivariant (IE) estimators is the largest class that properly respects this structure. Its unbiased minimum mean-squared-error member is the best integer equivariant (BIE) estimator, whose precision is never worse than that of the BLUE and whose performance is often close to that of integer least-squares. Its practical drawback, however, is the need to evaluate an infinite sum. We introduce the projected-BIE (PBIE) estimator, a new unbiased minimum-variance IE-estimator with a finite representation. It is obtained by restricting the admissible periodic corrections in the canonical IE-representation and solving the resulting optimization problem. PBIE is valid for general observation distributions, has an exactly computable variance matrix, and is obtained from normal equations. We further show that PBIE has a dual optimality property: it is both minimum-variance in its class and the closest estimator in that class to the BIE. For normally distributed data, PBIE simplifies considerably and can be computed efficiently from the BLUE of the integer ambiguities.

Keywords Global navigation satellite system (GNSS) · Mixed integer models · Integer ambiguity resolution · Integer-equivariant (IE) estimation · Best integer equivariant (BIE) estimator · Projected BIE (PBIE) estimator · Best linear unbiased estimation (BLUE)

Introduction

Mixed-integer (MI) estimation theory underpins high-precision carrier-phase Global Navigation Satellite System (GNSS) positioning, navigation, and timing (Strang and Borre 1997; Hofmann-Wellenhof et al. 2008; Leick et al. 2015; Teunissen and Montenbruck 2017; Morton et al. 2020). In carrier-phase GNSS models, the unknowns naturally split into real-valued parameters (e.g., coordinates, clocks, atmospheric and instrumental delays) and integer-valued carrier-phase ambiguities. This hybrid structure gives rise to the mixed-integer model, whose estimators must simultaneously respect the continuous real domain and the integer ambiguity domain. Modern precise GNSS methods–such as Real-Time Kinematic (RTK), Precise Point Positioning ambiguity resolution (PPP-AR), network RTK, and attitude determination–derive much of their strength from exploiting this mixed real-integer nature rather than treating all parameters as real-valued (Giorgi et al. 2010; Odijk et al. 2017; Brack 2017; Green and Humphreys 2018; Psychas and Verhagen 2020; Yong et al. 2021; Khodabandeh et al. 2021; An et al. 2024; Massarweh and Verhagen 2025; Yang et al. 2025).

A central concept in MI estimation theory is the class of integer-equivariant (IE) estimators introduced in Teunissen (2003). This is the broadest class of estimators that fully respects the integer structure of the model. Integer equivariance means that the estimator commutes with integer translations of the ambiguity parameters: if the true ambiguity vector is shifted by any integer vector, then the estimator shifts its integer component by exactly the same amount, while the estimation rule itself remains unchanged. Hence, IE-estimators preserve the intrinsic symmetry of the

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mixed-integer model. The class includes familiar estimators such as the float solution, integer least-squares, integer bootstrapping, and integer aperture estimators, while remaining sufficiently structured for rigorous optimality analysis.

Within this class, the optimal unbiased minimum mean-squared-error estimator is the best integer equivariant (BIE) estimator of Teunissen (2003). Its precision is never worse than that of the best linear unbiased estimator (BLUE), and in many practical cases it approaches the performance of integer least-squares estimation, whose ambiguity success rate is maximized under standard assumptions. This makes BIE an attractive compromise between robustness and precision, especially when full ambiguity fixing is risky or only partly reliable. Successful applications of the BIE principle have been widely reported, for instance, in RTK, PPP-AR, and GNSS attitude determination (Brack et al. 2023b, a; Yang et al. 2024; Ma et al. 2022; Miao et al. 2024; Duong et al. 2021; Odolinski and Teunissen 2022; Liu et al. 2024; Wu and Bian 2026).

Despite these advantages, the BIE-estimator has an important computational limitation: its defining formula involves an infinite sum over the *n*-dimensional space of integers. In practice, this requires truncation, screening, or approximation. Motivated by this drawback, we develop a new unbiased minimum-variance IE-estimator whose formulation is finite from the outset.

Our starting point is the representation theorem of Teunissen (2003), which shows that every IE-estimator can be written as the BLUE plus an additive periodic correction term. Building on this result, we restrict the admissible class of periodic corrections to a carefully chosen subclass that preserves the essential mixed-integer symmetry while yielding a tractable optimization problem. Solving this problem leads to a new estimator, termed the projected-BIE (PBIE) estimator.

The PBIE-estimator is valid for general observation distributions and does not rely on Gaussian assumptions. It admits a finite closed-form representation, has an exactly computable variance matrix, and can be obtained by solving a corresponding system of normal equations. We further compare PBIE with BIE and derive a geometric interpretation showing that PBIE enjoys a dual optimality property: it is the minimum-variance unbiased estimator within its restricted class and, at the same time, the estimator in that class closest to the BIE-estimator in an appropriate metric sense. Thus, PBIE can be viewed as a principled projection of the full BIE solution onto a computationally manageable estimator space.

Finally, for Gaussian observation distributions, PBIE simplifies considerably: its computation depends only on the BLUE of the integer ambiguities, leading to an efficient implementation. We derive the corresponding formulas and illustrate the estimator with numerical examples.

The remainder of this contribution is organized as follows. "Best integer equivariant (BIE) estimation" Section reviews the mixed-integer model and the class of IE-estimators. "The projected BIE (PBIE) estimator" Section formulates the restricted optimization problem and derives the PBIE-estimator. "Geometry of PBIE and BIE estimators" Section presents its variance and geometric properties, together with the comparison to BIE. "The multivariate-normal PBIE estimator" Section treats the Gaussian specialization and computational aspects. Finally, "Summary and conclusions" Section provides a summary and conclusions.

The following notation is used throughout: We denote a random variable/vector by means of an underscore; thus $\underline{x}$ is a random variable/vector, while x is not. $\mathsf{E}(.)$ and $\mathsf{D}(.)$ denote the expectation and dispersion operators, respectively, and $f_{\underline{x}}(x)$ the probability density function (PDF) of $\underline{x}$. Q_{xx} denotes the variance matrix of $\underline{x}$ and Q_{xy} the covariance matrix of $\underline{x}$ and $\underline{y}$. $\mathcal{N}_m(\mu, Q)$ denotes the m-variate normal distribution with mean μ and variance matrix Q. An IE-estimator of θ is denoted as $\underline{\theta}$, its BLUE as $\hat{\underline{\theta}}$, its BIE as $\check{\underline{\theta}}$, and its PBIE as $\check{\underline{\theta}}_{\mathrm{P}}$. $\mathbb{Z}^n$ denotes the n-dimensional space of integer numbers and $\mathbb{R}^p$ the p-dimensional space of real numbers. The determinant of a matrix Q is denoted as $|Q|$ and its trace as $\mathrm{trace}(Q)$ or $\mathrm{tr}(Q)$. The range space of matrix M is denoted as $\mathcal{R}(M)$. Matrix $M^+ = (M^T Q_{yy}^{-1} M)^{-1} M^T Q_{yy}^{-1}$ denotes the BLUE-inverse of the full column rank matrix M, and $P_M = MM^+$ and $P_M^\perp = I_m - P_M$ are the orthogonal projectors that project on the range space of M and its orthogonal complement, respectively. The squared weighted norm is denoted as $||.||_R^2 = (.)^T R^{-1}(.)$.

Best integer equivariant (BIE) estimation

To formulate our linear(ized) GNSS model of observation equations, we assume its vector of observables $\underline{y} \in \mathbb{R}^m$ to have the following expectation and dispersion (Leick et al. 2015; Teunissen and Montenbruck 2017; Morton et al. 2020),

$$\mathsf{E}(\underline{y}) = Aa + Bb, \ \mathsf{D}(\underline{y}) = Q_{yy}, \ a \in \mathbb{Z}^n, b \in \mathbb{R}^p \tag{1}$$

in which (A, B) is a given $m \times (n+p)$ matrix of full rank and Q_{yy} the given positive-definite variance matrix of the observables. The m-vector $\underline{y}$ contains the carrier-phase and pseudorange observables, the n-vector a the unknown integer-estimable ambiguities, and the real-valued p-vector b the remaining unknown parameters, such as baseline

components (coordinates), clock biases, instrumental delays and possibly atmospheric delay parameters (troposphere, ionosphere). The matrix (A, B) contains the relative receiver-satellite geometry and the wavelengths of the carrier-phase data. In the following, a and b will be referred to as the ambiguity vector and the baseline vector, respectively. Note, although it is usual to assume that $\underline{y}$ follows a multivariate normal distribution, such assumption is not yet made here.

To do justice to the mixed-integer nature of the above GNSS model, Teunissen (2003) introduced the class of *integer-equivariant* (IE) estimators. This class is defined as follows:

Definition 1 (*IE-Estimation*) With respect to model (1), the estimator $\underline{\theta} = G_\theta(\underline{y})$, $G_\theta : \mathbb{R}^m \to \mathbb{R}^q$, is said to be an integer equivariant (IE) estimator of $\theta = L_a a + L_b b$, with $L_a \in \mathbb{R}^{q\times n}$, $L_b \in \mathbb{R}^{q\times p}$, if

$$G_\theta(y + Az + B\varsigma) = G_\theta(y) + L_a z + L_b \varsigma, \tag{2}$$

for all $y \in \mathbb{R}^m, z \in \mathbb{Z}^n, \varsigma \in \mathbb{R}^p$.

This estimation principle was motivated with reference to the linearity and integer remove-restore property. When estimating integer ambiguities, for instance, it seems reasonable to ask of the estimator that if an arbitrary number of cycles is added to the phase data, that the outcome of the ambiguity estimator shifts over the same integer amount. In (ibid) it is proven that the class of IE-estimators encompasses the class of linear unbiased (LU) estimators, LU $\subset$ IE . This is an important property as it implies that optimality in IE automatically carries over in LU. LU-estimators with a certain optimality property can thus never outperform IE-estimators having the same optimality property within their class. Another important property from (ibid) is that every IE-estimator of θ can be written as a periodic correction on the BLUE of θ:

IE-representation Theorem (Teunissen 2003): Let $\hat{\underline{a}}$ and $\hat{\underline{b}}$ be the BLUEs of a and b, respectively, and $\underline{t} = D^T\underline{y}$ the misclosure vector, for which $D \in \mathbb{R}^{m\times(m-n-p)}$ is a basis matrix of the null space of $(A, B)^T$. Then every IE-estimator $\underline{\theta}$ of $\theta = L_a a + L_b b$ can be represented as

$$\underline{\theta} = L_a\hat{\underline{a}} + L_b\hat{\underline{b}} + \Pi_\theta(\hat{\underline{a}}, \underline{t}) \tag{3}$$

for some periodic function $\Pi_\theta(x + z, t) = \Pi_\theta(x, t)$, $\forall z \in \mathbb{Z}^n$. ■

Note that $\hat{\underline{\theta}} = L_a\hat{\underline{a}} + L_b\hat{\underline{b}}$ is the BLUE of θ. It is the solution one would get when the integerness of the ambiguities is not imposed. The above result shows therefore that the impact of ambiguity resolution is felt through the periodic function $\Pi_\theta(\hat{\underline{a}}, \underline{t})$. Since this function can be seen to act as a 'correction' on the BLUE $\hat{\underline{\theta}}$, one can now think of finding such a function that enables one to improve the BLUE of θ. As it is shown in (ibid), this leads to the best integer-equivariant (BIE) estimator of θ when using the criterion of minimizing the mean-squared-error (MSE) $\mathsf{E}||\check{\underline{\theta}} - \theta||^2$:

BIE Theorem (Teunissen 2003): Let $f_{\underline{y}}(y) = f(y - Aa - Bb)$ be the PDF of $\underline{y}$ of (1) and let $\mathcal{H}_\theta = \{\underline{\theta} \in \mathbb{R}^q | \underline{\theta} = \widehat{\underline{\theta}} + \Pi_\theta(\hat{\underline{a}}, \underline{t}), \Pi_\theta(x + z, t) = \Pi_\theta(x, t), \forall z \in \mathbb{Z}^n\}$ be the class of IE-estimators of $\theta(a, b) = L_a a + L_a b$. Then the best integer-equivariant (BIE) estimator $\underline{\theta} = \arg\min_{\underline{\theta}\in\mathcal{H}_\theta} \mathsf{E}||\underline{\theta} - \theta||^2$ of $\theta = \theta(a, b)$ is given as

$$\check{\underline{\theta}} = \hat{\underline{\theta}} - \sum_{z\in\mathbb{Z}^n} \int_{\mathbb{R}^p} \theta(\hat{\underline{a}} - z, \beta) w_\theta(\hat{\underline{a}} - z, \beta, \underline{t}) d\beta \tag{4}$$

with weight function

$$w_\theta(\alpha - z, \beta, \tau) = \frac{f(A(\alpha - z) + B\beta + C\tau)}{\sum_{z\in\mathbb{Z}^n} \int_{\mathbb{R}^p} f(A(\alpha - z) + B\beta + C\tau) d\beta}$$

■

Note that BIEs of linear functions of only the ambiguity vector or only the baseline vector, are obtained by setting $L_b = 0$ or $L_a = 0$, while the BIE of any linear function of the observational expectation $\mathsf{E}(\underline{y})$ can be obtained by suitable choice of L_a and L_b.

Also note, as the second term on the right-hand side of (4) is periodic in the ambiguity-argument, that the solution remains unchanged when an arbitrary integer vector is added to $\hat{\underline{a}}$. Hence, the solution also remains unchanged when $\hat{\underline{a}}$ is replaced by the ambiguity-residual $\check{\underline{\epsilon}} = \hat{\underline{a}} - \mathcal{I}(\hat{\underline{a}})$, with $\mathcal{I} : \mathbb{R}^n \mapsto \mathbb{Z}^n$ being an arbitrary integer-equivariant integer map. It is thus the fractional part of $\hat{\underline{a}}$ that counts and not so much $\hat{\underline{a}}$ itself. As a result we see that the second term on the right-hand side of (4) is essentially driven by the complete misclosure vector of the mixed-integer model, $(\underline{t}^T, \check{\underline{\epsilon}}^T)^T \in \mathbb{R}^{m-p}$. Vector $\underline{t}$ is namely only the misclosure vector of model (1) when the ambiguities are treated as real-valued parameters, i.e. the so-called 'float' model. The misclosure vector extends with the ambiguity-residual vector when the integerness of the ambiguities is taken into account. Just as this extended misclosure vector plays a pivotal role in the mixed-integer model testing of Teunissen (2025), it will also do so in the theory of the following sections.

We also remark that (4) is a very general result as it holds true for any PDF of the form $f_{\underline{y}}(y) = f(y - Aa - Bb)$. In Teunissen (2020), further expressions of the BIE-estimator $\check{\underline{\theta}}$ have been developed for the class of *elliptically contoured*

distributions (ECD), thereby also providing analytical results for the multivariate normal, the contaminated normal, and the multivariate Student distribution. And building on the periodic representation of IE-estimators, an ECD-BIE estimation theorem in spectral form was presented in Teunissen (2026c). This result established twin representations of ECD-BIE estimators in both spatial and spectral domains.

To appreciate the 'best' property of the BIE-estimator $\check{\underline{\theta}}$, it follows from LU $\subset$ IE that the mean squared error (MSE) of the best linear unbiased estimator (BLUE) is never better than that of its corresponding BIE estimator. Also, since both estimators are unbiased, it implies that the BLUE-precision is never better than the BIE-precision,

$$\mathsf{D}(\check{\underline{\theta}}) \leq \mathsf{D}(\hat{\underline{\theta}}) \tag{5}$$

In the context of GNSS it would therefore make sense to *always* compute the BIE baseline estimator, as its precision will never be poorer than that of the 'float' BLUE baseline solution $\hat{\underline{b}}$.

The projected BIE (PBIE) estimator

The BIE-estimator (4) has the smallest mean-squared-error of all IE-estimators; thus of all estimators in the class

$$\mathcal{H}_\theta = \{\underline{\theta} \in \mathbb{R}^q | \ \underline{\theta} = \hat{\underline{\theta}} + \Pi_\theta(\hat{\underline{a}}, \underline{t})\} \tag{6}$$

This is a large class of estimators, in particular since the only property required of the functions $\Pi_\theta(\hat{\underline{a}}, \underline{t})$ is their $\mathbb{Z}^n$ -periodicity in first argument. One of the consequences of the absence of other restrictions is that the BIE-estimator requires the resolution of an infinite sum for its computation (cf. 4). As this is practically impossible, methods of truncations are used in practice to obtain sufficiently precise approximations, see e.g. Teunissen (2005); Odolinski and Teunissen (2020); Wu (2022). Recently, through the development of Fourier ambiguity resolution, new and more efficient truncation methods have been developed and presented in Teunissen (2026a, 2026b).

Instead of working with the large IE-class (6), the idea of the current contribution is to restrict the IE-class such that the best integer-equivariant estimator in this reduced class can be computed without the need of having to resolve infinite sums. To put restrictions on the functions $\Pi_\theta(\alpha, \tau)$, we assume to have a span of k $\mathbb{Z}^n$-periodic, scalar base-functions available, $\{f_1(\alpha), \ldots, f_k(\alpha)\}$, such that, for any τ, linear combinations of them, $\sum_{i=1}^k F_i(\tau) f_i(\alpha)$, can play the role of $\Pi_\theta(\alpha, \tau)$. Thus, instead of working with unknown periodic functions $\Pi_\theta(\alpha, \tau)$, we will now be working with unknown linear combinations of k known periodic functions; thus with the factorization $F_\theta(\tau) f(\alpha)$, in which $f(\alpha) = [f_1(\alpha), \ldots, f_k(\alpha)]^T$ and $F_\theta(\tau) = [F_1(\tau), \ldots, F_k(\tau)] \in \mathbb{R}^{q \times k}$. The reduced class of IE-estimators that we will be working with is therefore given as

$$\mathcal{F}_\theta = \{\underline{\theta} \in \mathbb{R}^q | \ \underline{\theta} = \hat{\underline{\theta}} + F_\theta(\underline{t}) f(\hat{\underline{a}})\} \subset \mathcal{H}_\theta \tag{7}$$

That $\mathcal{F}_\theta$ is a subset of $\mathcal{H}_\theta$ follows directly from the $\mathbb{Z}^n$-periodicity of $f(\alpha) : \mathbb{R}^n \mapsto \mathbb{R}^k$.

Our goal is now to find the best integer-equivariant estimator of θ in the reduced set of IE-functions $\mathcal{F}_\theta$. This new estimator will be named the *projected BIE* estimator (PBIE), the choice of which will become clear once the geometric relationship is established (see "Geometry of PBIE and BIE estimators" Section). We have the following result:

Theorem 1 *(The PBIE estimator) Within the finite-rank class of IE-estimators* $\mathcal{F}_\theta = \{\underline{\theta} \in \mathbb{R}^q | \ \underline{\theta} = \hat{\underline{\theta}} + F_\theta(\underline{t}) f(\hat{\underline{a}})\}$, *with known* $\mathbb{Z}^n$*-periodic vector function* $f(\alpha) : \mathbb{R}^n \mapsto \mathbb{R}^k$, *but unknown* $F_\theta(\tau) \in \mathbb{R}^{q \times k}$, *the minimum mean-squared-error estimator* $\check{\underline{\theta}}_{\mathrm{P}}$ *of* θ,

$$\check{\underline{\theta}}_{\mathrm{P}} = \arg \min_{\underline{\theta} \in \mathcal{F}_\theta} \mathsf{E} ||\underline{\theta} - \theta||^2 \tag{8}$$

is given as

$$\check{\underline{\theta}}_{\mathrm{P}} = \hat{\underline{\theta}} - C_{\hat{\theta} f}(\underline{t}) C_{ff}(\underline{t})^{-1} f(\hat{\underline{a}}) \tag{9}$$

where

$$\begin{array}{rcl} C_{\hat{\theta} f}(\underline{t}) & = & \mathsf{E}[(\hat{\underline{\theta}} - \theta) f(\hat{\underline{a}})^T | \underline{t}] \\ C_{ff}(\underline{t}) & = & \mathsf{E}[f(\hat{\underline{a}}) f(\hat{\underline{a}})^T | \underline{t}] \end{array} \tag{10}$$

■

Proof See Appendix. □

This result shows the advantage the PBIE-estimator has over the BIE-estimator (4). Although the PBIE-estimator is still nonlinear in $\hat{\underline{a}}$ and $\underline{t}$, it is free from the infinite integer sum and it is easy to compute once the matrices $C_{\hat{\theta} f}(t)$ and $C_{ff}(t)$ are known. If we write the PBIE-estimate as

$$\check{\theta}_{\mathrm{P}} = \hat{\theta} - C_{\hat{\theta} f}(t) x, \ \text{with} \ C_{ff}(t) x = f(\hat{a}) \tag{11}$$

we can treat $C_{ff}(t) x = f(\hat{a})$ as a system of normal equations. Hence, a Cholesky decomposition on the 'normal' matrix $C_{ff}(t)$, followed by forward- and backward substitution, enables one to solve for x, and then for any chosen

linear function $\theta = L_a a + L_b b$, its PBIE-estimate directly as $\breve{\theta}_{\mathrm{P}} = \hat{\theta} - C_{\hat{\theta}f}(t)x$.

Note, since $\underline{\hat{\theta}}$ is unbiased, that $C_{\hat{\theta}f}(t)$ is the misclosure-conditioned covariance matrix of $\underline{\hat{\theta}}$ and $f(\underline{\hat{a}})$. Matrix $C_{ff}(t)$, however, is not a conditioned variance-matrix of $f(\underline{\hat{a}})$, as $\mathsf{E}[f(\underline{\hat{a}})] = 0$ may not hold true. Also note, for $\underline{\breve{\theta}}_{\mathrm{P}}$ to be unbiased, that the condition $\mathsf{E}[C_{\hat{\theta}f}(\underline{t})C_{ff}^{-1}(\underline{t})f(\underline{\hat{a}})] = 0$ must hold. The following Corollary provides some important properties of $\underline{\breve{\theta}}_{\mathrm{P}}$, including the condition of unbiasedness.

Corollary 1 *(Orthogonality, Mean and Variance of PBIE-estimator) The PBIE-estimator (9) has the following properties:*

1. *Orthogonality*

$$\mathsf{E}[(\underline{\breve{\theta}}_{\mathrm{P}} - \theta)f(\underline{\hat{a}})^T] = 0 \tag{12}$$

2. *Mean*

$$\mathsf{E}(\underline{\breve{\theta}}_{\mathrm{P}}) = \theta \text{ if } f_1(\underline{\hat{a}}) = 1 \tag{13}$$

3. *Variance w.r.t. $\underline{\theta} \in \mathcal{F}_\theta$*

$$\mathsf{D}(\underline{\breve{\theta}}_{\mathrm{P}}) = \mathsf{D}(\underline{\theta}) - \mathsf{D}(\underline{\breve{\theta}}_{\mathrm{P}} - \underline{\theta}) \text{ if } \mathsf{E}(\underline{\theta}) = \mathsf{E}(\underline{\breve{\theta}}_{\mathrm{P}}) \tag{14}$$

4. *Variance w.r.t. $\underline{\hat{\theta}}$*

$$\mathsf{D}(\underline{\breve{\theta}}_{\mathrm{P}}) = \mathsf{D}(\underline{\hat{\theta}}) - \mathsf{E}[C_{\hat{\theta}f}(\underline{t})C_{ff}^{-1}(\underline{t})C_{f\hat{\theta}}(\underline{t})] \text{ if } f_1(\underline{\hat{a}}) = 1 \tag{15}$$

5. *Closed under linear transformations Let $\underline{\breve{a}}_{\mathrm{P}}$, $\underline{\breve{b}}_{\mathrm{P}}$, and $\underline{\breve{\theta}}_{\mathrm{P}}$ be the PBIEs of a, b and $\theta = L_a a + L_b b$. Then*

$$\underline{\breve{\theta}}_{\mathrm{P}} = L_a \underline{\breve{a}}_{\mathrm{P}} + L_b \underline{\breve{b}}_{\mathrm{P}} \tag{16}$$

■

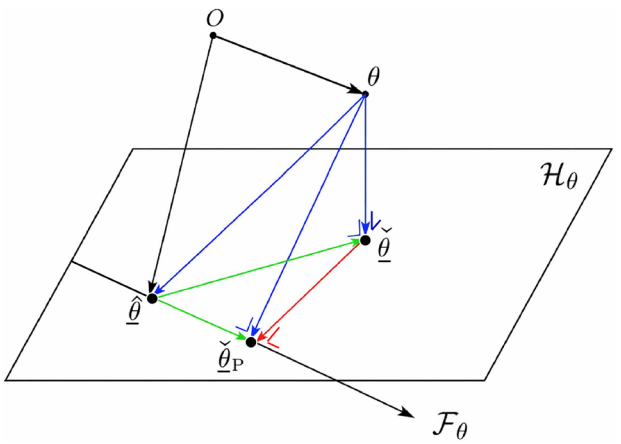

Fig. 1 Geometry of PBIE and BIE: The $\underline{\hat{\theta}}$-translated linear subspaces $\mathcal{H}_\theta \supset \mathcal{F}_\theta$, with the BIE $\underline{\breve{\theta}} = \underline{\hat{\theta}} + P_{\mathcal{H}}(\underline{\theta} - \underline{\hat{\theta}}) \in \mathcal{H}_\theta$ closest to the unknown parameter vector θ and the PBIE $\underline{\breve{\theta}}_{\mathrm{P}} = \underline{\hat{\theta}} + P_{\mathcal{F}}(\underline{\theta} - \underline{\hat{\theta}}) \in \mathcal{F}_\theta$ the second-closest to θ, but of all $\underline{\theta} \in \mathcal{F}_\theta$ closest to $\underline{\breve{\theta}}$

Proof See Appendix. □

The first property is one that we will revisit and use in the following section. The second property states that the PBIE-estimator is unbiased if $f_1(\underline{\hat{a}}) = 1$. Hence, with the standard Fourier basis, which automatically includes 1, the unbiasedness of PBIE is automatic. For another basis not having this property, unbiasedness can be enforced by having $f_1(\underline{\hat{a}}) = 1$ as an external constraint. Although this will increase the MMSE, the resulting PBIE will then still have an MMSE not larger than that of the BLUE. The third and fourth property show by how much the PBIE-estimator improves upon the precision of an arbitrary IE-estimator and the BLUE estimator. The last property of the Corollary says that the PBIE of a linear function is equal to the linear function of the PBIE. This therefore also applies to admissible ambiguity reparameterizations. Let $z = Z^T a$, with matrix Z integer and volume-preserving ($|Z| = \pm 1$). Then $\underline{\breve{z}}_{\mathrm{P}} = Z^T \underline{\breve{a}}_{\mathrm{P}}$.

Geometry of PBIE and BIE estimators

In this section we compare the PBIE-estimator $\underline{\breve{\theta}}_{\mathrm{P}}$ with the BIE-estimator $\underline{\breve{\theta}}$. We will use a geometric approach for this qualitative comparison. As the two estimators are defined as

$$\begin{array}{rcl} \underline{\breve{\theta}}_{\mathrm{P}} & = & \arg\min_{\underline{\theta} \in \mathcal{F}_\theta} E||\underline{\theta} - \theta||^2 \\ \underline{\breve{\theta}} & = & \arg\min_{\underline{\theta} \in \mathcal{H}_\theta} E||\underline{\theta} - \theta||^2 \end{array} \tag{17}$$

it is the difference between the two sets $\mathcal{H}_\theta$ and $\mathcal{F}_\theta \subset \mathcal{H}_\theta$ (cf. 6 and 7) that determines their difference. Both these sets can geometrically be seen to be $\underline{\hat{\theta}}$-*translated* linear subspaces, i.e. as $\mathcal{H}_\theta = \underline{\hat{\theta}} + \mathcal{H}$ and $\mathcal{F}_\theta = \underline{\hat{\theta}} + \mathcal{F}$. This follows since the set of periodic functions $\amalg(\alpha, \tau)$, being $\mathbb{Z}^n$-periodic in their first argument, as well as the set of functions $F(\tau)f(\alpha)$, with given $f(\alpha)$ being $\mathbb{Z}^n$-periodic, are closed under taking arbitrary linear combinations of their elements, and are therefore linear subspaces themselves, denoted as $\mathcal{H}$ and $\mathcal{F}$, respectively. With the induced inner-product $\langle \underline{u}, \underline{v} \rangle = \mathsf{E}[\underline{u}^T \underline{v}]$ for any two random vectors $\underline{u}, \underline{v} \in \mathbb{R}^q$, the two minimization problems of (17) can therefore *geometrically* be seen as the problem of finding the θ-closest positions of $\underline{\theta}$ in $\mathcal{H}_\theta$ and $\mathcal{F}_\theta \subset \mathcal{H}_\theta$, respectively, see Fig. 1. For the PBIE-estimator this leads to the following geometric characterizations.

Theorem 2a *(Geometry of PBIE-estimator) Let the operator $P_{\mathcal{F}}$ be defined as $P_{\mathcal{F}}\underline{x} := \mathsf{E}[\underline{x}f(\underline{\hat{a}})^T|\underline{t}]C_{ff}^{-1}(\underline{t})f(\underline{\hat{a}})$. Then the PBIE-estimator can be written as*

$$\underline{\breve{\theta}}_{\mathrm{P}} = \underline{\hat{\theta}} + P_{\mathcal{F}}(\underline{\theta} - \underline{\hat{\theta}}) \tag{18}$$

in which $P_{\mathcal{F}}$ is an orthogonal projector satisfying

$$\begin{array}{ll}(a) & \text{Range} : P_{\mathcal{F}}\underline{x} \in \mathcal{F} \\ (b) & \text{Idempotence} : P_{\mathcal{F}}P_{\mathcal{F}}\underline{x} = P_{\mathcal{F}}\underline{x} \\ (c) & \text{Orthogonality} : \underline{x} - P_{\mathcal{F}}\underline{x} \perp \mathcal{F}\end{array} \tag{19}$$

and its mean-squared-error and variance matrix are given, with respect to that of the BLUE $\hat{\underline{\theta}}$, as

$$\begin{array}{lllll}(d) & \mathsf{E}||\check{\underline{\theta}}_{\mathrm{P}} - \theta||^2 & = & \mathsf{E}||\hat{\underline{\theta}} - \theta||^2 - \mathsf{E}||P_{\mathcal{F}}(\hat{\underline{\theta}} - \theta)||^2 \\ (e) & \mathsf{D}(\check{\underline{\theta}}_{\mathrm{P}}) & = & \mathsf{D}(\hat{\underline{\theta}}) - \mathsf{D}(P_{\mathcal{F}}(\hat{\underline{\theta}} - \theta))\end{array} \tag{20}$$

■

Proof See Appendix. □

Note that (20*e*) is consistent with the result of Corollary 1, as $\mathsf{D}(P_{\mathcal{F}}(\hat{\underline{\theta}} - \theta)) = \mathsf{E}[C_{\hat{\theta}f}(\underline{t})C_{ff}^{-1}(\underline{t})C_{f\hat{\theta}}(\underline{t})]$.

Expression (18) shows that the periodic correction the BLUE $\hat{\underline{\theta}}$ has to undergo to get the PBIE, is itself an orthogonal projection of its error $\theta - \hat{\underline{\theta}}$ onto the linear subspace $\mathcal{F}$. We will now show that a similar geometric interpretation can be given to the BIE-estimator, albeit that the subspace onto which is projected encompasses that of $\mathcal{F}$. To obtain such similar geometric characterization for the BIE-estimator, we first reformulate its *estimation* problem (17) as one of *prediction*,

$$\check{\underline{\theta}} = \hat{\underline{\theta}} + \arg\min_{\Pi_\theta(\alpha,\tau)} \mathsf{E}||(\theta - \hat{\underline{\theta}}) - \Pi_\theta(\hat{\underline{a}}, \underline{t})||^2 \tag{21}$$

Thus, instead of estimating the unknown deterministic parameter vector θ, as in (17), the problem has now been formulated as one of predicting the zero-mean random error-vector $\theta - \hat{\underline{\theta}}$ by $\Pi_\theta(\hat{\underline{a}}, \underline{t})$. With this prediction-interpretation, it is the *conditional-mean* operator that comes as the natural solver for the above minimization problem. It is namely well-known that, with respect to the metric $\langle \underline{u}, \underline{v} \rangle = \mathsf{E}[\underline{u}^T\underline{v}]$, the conditional-mean $\mathsf{E}[\underline{u}|\underline{w}]$ is an orthogonal projector, satisfying the following geometric properties,

1. Conditional-Mean: $\mathsf{E}[\underline{u}|\underline{w}] = \arg\min_h \mathsf{E}||\underline{u} - h(\underline{w})||^2$
2. Idempotence: $\mathsf{E}[\mathsf{E}[\underline{u}|\underline{w}]|\underline{w}] = \mathsf{E}[\underline{u}|\underline{w}]$
3. Orthogonality: $\mathsf{E}[(\underline{u} - \mathsf{E}[\underline{u}|\underline{w}])^T h(\underline{w})] = 0$ for all $h : \mathbb{R}^r \mapsto \mathbb{R}^q$.

see e.g. the best-prediction review in Teunissen (2007) or Stark and Woods (1986). The first property states that of all random vectors $h(\underline{w})$, the conditional-mean $\mathsf{E}[\underline{u}|\underline{w}]$ is closest to $\underline{u}$, i.e. it is the orthogonal projection of $\underline{u}$ onto the space of random vectors $h(\underline{w})$. The second property states that adding $\underline{w}$ to a conditional expectation already determined by $\underline{w}$, will not change the expectation as no new information is provided, i.e. a repeat of the orthogonal projection does not change the result. And the third property states that the residual $\underline{u} - \mathsf{E}[\underline{u}|\underline{w}]$ is orthogonal to the space of random vectors $h(\underline{w})$.

With the conditional-mean being an orthogonal projector it is tempting to take as minimizer of (21) the conditional mean $\mathsf{E}[(\theta - \hat{\underline{\theta}})|\,\hat{\underline{a}}, \underline{t}]$. However, this is incorrect, as this conditional-mean has not yet taken into account the $\mathbb{Z}^n$-periodicity of the functions $\Pi_\theta(\alpha, \tau)$. In order to do so, we make use of the following Lemma.

Lemma 1 *Let $S_\Pi = \{\Pi_\theta : \mathbb{R}^n \times \mathbb{R}^{m-n-p} \mapsto \mathbb{R}^q |\ \Pi_\theta(\alpha + z, \tau) = \Pi_\theta(\alpha, z), \forall \alpha \in \mathbb{R}^n, \forall \tau \in \mathbb{R}^{m-n-p}, \forall z \in \mathbb{Z}^n\}$ be the set of functions that are $\mathbb{Z}^n$-periodic in their first argument and let $S_H = \{\mathrm{H}_\theta : \mathbb{R}^n \times \mathbb{R}^{m-n-p} \mapsto \mathbb{R}^q |\ \mathrm{H}_\theta(\alpha, \tau) = G(\alpha - \mathcal{I}(\alpha), \tau)$, for some G and some integer equivariant $\mathcal{I} : \mathbb{R}^n \mapsto \mathbb{Z}^n, \forall \alpha \in \mathbb{R}^n, \forall \tau \in \mathbb{R}^{m-n-p}, \forall z \in \mathbb{Z}^n\}$. Then $S_{\Pi_\theta} = S_{\mathrm{H}_\theta}$.* ■

Proof First we show $S_{\mathrm{H}_\theta} \subset S_{\Pi_\theta}$. Let $h \in \mathrm{H}_\theta$. Then a G and an $\mathcal{I}$ exist such that $h(\alpha, \tau) = G(\alpha - \mathcal{I}(\alpha), \tau)$. Since $\mathcal{I}$ is $\mathbb{Z}^n$-periodic, it follows that h is $\mathbb{Z}^n$-periodic in its first argument and therefore $h \in S_\Pi$. Now we show $S_{\Pi_\theta} \subset S_{\mathrm{H}_\theta}$. Let $h \in S_{\Pi_\theta}$. Then, as h is $\mathbb{Z}^n$-periodic in its first argument, we have $h(\alpha, \tau) = h(\alpha - \mathcal{I}(\alpha), \tau)$, thus showing that $h \in S_{\mathrm{H}_\theta}$. Therefore $S_{\Pi_\theta} = S_{\mathrm{H}_\theta}$. □

This result states that instead of working with functions that have to be $\mathbb{Z}^n$-periodic in their first argument, one may just as well work with any function, as long as $\alpha - \mathcal{I}(\alpha)$ is used in its first argument. This shows that we can use the ambiguity-residual $\check{\underline{\epsilon}} = \hat{\underline{a}} - \mathcal{I}(\hat{\underline{a}})$ and formulate (21) as

$$\check{\underline{\theta}} = \hat{\underline{\theta}} + \arg\min_{\mathrm{H}_\theta(\alpha,\tau)} \mathsf{E}||(\theta - \hat{\underline{\theta}}) - \mathrm{H}_\theta(\check{\underline{\epsilon}}, \underline{t})||^2 \tag{22}$$

but now without the constraint that $\mathrm{H}_\theta(\alpha, \tau)$ has to be $\mathbb{Z}^n$-periodic in its first argument. As a result, it now directly follows that the minimizer is given by the conditional mean $\hat{\mathrm{H}}_\theta(\check{\underline{\epsilon}}, \underline{t}) = \mathsf{E}[(\theta - \hat{\underline{\theta}})|\check{\underline{\epsilon}}, \underline{t}]$. For the BIE-estimator this leads to the following geometric characterization, see Fig. 1.

Theorem 2b *(Geometry of BIE-estimator) Let $P_{\mathcal{H}}\underline{x} := \mathsf{E}[\underline{x}|\check{\underline{\epsilon}}, \underline{t}]$. Then the BIE-estimator can be written as*

$$\check{\underline{\theta}} = \hat{\underline{\theta}} + P_{\mathcal{H}}(\theta - \hat{\underline{\theta}}) \tag{23}$$

in which the orthogonal projector $P_{\mathcal{H}}$ satisfies,

$$\begin{array}{ll}(a) & \text{Range} : P_{\mathcal{H}}\underline{x} \in \mathcal{H} \\ (b) & \text{Idempotence} : P_{\mathcal{H}}P_{\mathcal{H}}\underline{x} = P_{\mathcal{H}}\underline{x} \\ (c) & \text{Orthogonality} : \underline{x} - P_{\mathcal{H}}\underline{x} \perp \mathcal{H}\end{array} \tag{24}$$

and its mean-squared-error and variance matrix are given, with respect to that of the BLUE $\hat{\underline{\theta}}$, as

$$\begin{array}{lll} (d) & \mathsf{E}||\breve{\underline{\theta}} - \theta||^2 & = \mathsf{E}||\hat{\underline{\theta}} - \theta||^2 - \mathsf{E}||P_{\mathcal{H}}(\hat{\underline{\theta}} - \theta)||^2 \\ (e) & \mathsf{D}(\breve{\underline{\theta}}) & = \mathsf{D}(\hat{\underline{\theta}}) - \mathsf{D}(P_{\mathcal{H}}(\hat{\underline{\theta}} - \theta)) \end{array} \tag{25}$$

■

Proof See Appendix. □

This result shows that the BIE-estimator has exactly the same structure as the PBIE-estimator (cf. 18). In both cases the periodic corrections to the BLUE are orthogonal projections of its error; in case of the BIE onto the linear subspace $\mathcal{H}$, while in case of the PBIE onto the smaller linear subspace $\mathcal{F} \subset \mathcal{H}$. And as with the PBIE, we also have an exact expression for the difference in precision between the BLUE $\hat{\underline{\theta}}$ and the BIE $\breve{\underline{\theta}}$ (cf. 25*e*).

With the above geometry in mind, we can now also use the geometry to express the PBIE-estimator in the BIE-estimator, see Fig. 1. We have the following result.

Theorem 2c *(PBIE expressed in BIE) The PBIE-estimator can be expressed in the BIE-estimator as*

$$\breve{\underline{\theta}}_{\mathrm{P}} = \hat{\underline{\theta}} + P_{\mathcal{F}}(\breve{\underline{\theta}} - \hat{\underline{\theta}}) \tag{26}$$

and its mean-squared-error and variance matrix can be expressed in that of the BIE-estimator as

$$\begin{array}{lll} (a) & \mathsf{E}||\breve{\underline{\theta}}_{\mathrm{P}} - \theta||^2 & = \mathsf{E}||\breve{\underline{\theta}} - \theta||^2 + \mathsf{E}||(P_{\mathcal{H}} - P_{\mathcal{F}})(\hat{\underline{\theta}} - \theta)||^2 \\ (b) & \mathsf{D}(\breve{\underline{\theta}}_{\mathrm{P}}) & = \mathsf{D}(\breve{\underline{\theta}}) + \mathsf{D}((P_{\mathcal{H}} - P_{\mathcal{F}})(\hat{\underline{\theta}} - \theta)) \end{array} \tag{27}$$

■

Proof See Appendix. □

This result shows that PBIE's periodic correction to the BLUE is an $\mathcal{F}$-projected version of BIE's periodic correction to the BLUE.

When we combine the expressions of the mean-squared errors of the three estimators $\hat{\underline{\theta}}$, $\breve{\underline{\theta}}_{\mathrm{P}}$, and $\breve{\underline{\theta}}$ (cf. Theorem 2*a*-2*c*), we can write $\mathsf{E}||\hat{\underline{\theta}} - \theta||^2 = \mathsf{E}||\breve{\underline{\theta}}_{\mathrm{P}} - \theta||^2 + ||\breve{\underline{\theta}}_{\mathrm{P}} - \hat{\underline{\theta}}||^2 = ||\breve{\underline{\theta}} - \theta||^2 + ||\breve{\underline{\theta}} - \breve{\underline{\theta}}_{\mathrm{P}}||^2 + ||\breve{\underline{\theta}}_{\mathrm{P}} - \hat{\underline{\theta}}||^2$ (see Fig. 1), thus showing that we have the ordering

$$\mathsf{E}||\breve{\underline{\theta}} - \theta||^2 \leq \mathsf{E}||\breve{\underline{\theta}}_{\mathrm{P}} - \theta||^2 \leq \mathsf{E}||\hat{\underline{\theta}} - \theta||^2 \tag{28}$$

The same ordering we have for their variance matrices,

$$\mathsf{D}(\breve{\underline{\theta}}) \leq \mathsf{D}(\breve{\underline{\theta}}_{\mathrm{P}}) \leq \mathsf{D}(\hat{\underline{\theta}}) \tag{29}$$

Thus the PBIE-estimator is never poorer than the BLUE, but also never better than the BIE. However, although the PBIE-estimator is never better than the BIE, the following result shows that it provides the best possible approximation to the BIE, see Fig. 1.

Theorem 3 *(PBIE as best approximation of BIE) For any IE-estimator $\underline{\theta} \in \mathcal{F}_\theta$, we have the Pythagorean decomposition*

$$\mathsf{E}||\underline{\theta} - \theta||^2 = \mathsf{E}||\breve{\underline{\theta}} - \theta||^2 + \mathsf{E}||\breve{\underline{\theta}} - \underline{\theta}||^2 \tag{30}$$

and therefore

$$\breve{\underline{\theta}}_{\mathrm{P}} = \arg \min_{\underline{\theta} \in \mathcal{F}_\theta} \mathsf{E}||\breve{\underline{\theta}} - \underline{\theta}||^2 \tag{31}$$

■

Proof We have $\underline{\theta} - \theta = (\breve{\underline{\theta}} - \theta) + (\underline{\theta} - \breve{\underline{\theta}})$. Recognizing that $\breve{\underline{\theta}} - \theta \perp \mathcal{H}$ and $\underline{\theta} - \breve{\underline{\theta}} \in \mathcal{H}\ \forall \underline{\theta} \in \mathcal{F}_\theta$, the Pythagorean decomposition (30) follows. As the first term on the right-hand side of (30) does not depend on $\underline{\theta}$, the result (31) follows by recognizing that the $\underline{\theta}$-minimizer of (30) equals that of $\mathsf{E}||\breve{\underline{\theta}} - \underline{\theta}||^2$. □

Based on this result, we can now make the important conclusion that the PBIE-estimator is not only best in its class (i.e. unbiased and of minimum variance within the IE-class $\mathcal{F}_\theta$), but also the unbiased minimum-variance estimator that is closest to the BIE-estimator, which itself is best in the larger IE-class $\mathcal{H}_\theta \supset \mathcal{F}_\theta$.

With the restriction of $\mathcal{H}_\theta$ to $\mathcal{F}_\theta$, one can of course think of further restricting the subspace with the aim to obtain an even simpler unbiased minimum variance estimator than the PBIE of Theorem 1 (cf. 9). Following the same (geometric) approach as above, one can indeed show if the IE-class is restricted to $\mathcal{G}_\theta = \{\underline{\theta} \in \mathbb{R}^q |\ \underline{\theta} = \hat{\underline{\theta}} + F f(\hat{\underline{a}})\} \subset \mathcal{F}_\theta$ (i.e. F is now an unknown *constant* matrix), that the conditioning on the misclosure vector $\underline{t}$ in (9) disappears and that the estimator takes on the simpler form $\breve{\underline{\theta}}_{\mathrm{cP}} = \hat{\underline{\theta}} - \mathsf{E}[\hat{\underline{\theta}} - \theta) f(\hat{\underline{a}})^T] \mathsf{E}[f(\hat{\underline{a}}) f(\hat{\underline{a}})^T] f(\hat{\underline{a}})$. This simplification goes of course at the expense of a larger variance. Similar to (27*b*), we have then $\mathsf{D}(\breve{\underline{\theta}}_{\mathrm{cP}}) = \mathsf{D}(\breve{\underline{\theta}}_{\mathrm{P}}) + \mathsf{D}((P_{\mathcal{F}} - P_{\mathcal{G}})(\hat{\underline{\theta}} - \theta))$. As the following section will show, the restriction of $\mathcal{F}_\theta$ to $\mathcal{G}_\theta$ is not needed if the observables may be assumed to have a multivariate normal distribution.

The multivariate-normal PBIE estimator

All the results presented sofar are generally valid as no specific distributional assumptions were made. In this section, however, we will assume $\underline{y}$ to have a multivariate normal distribution, $\underline{y} \sim \mathcal{N}_m(Aa + Bb, Q_{yy})$, and develop its PBIE-estimator accordingly. As in this case the misclosure vector $\underline{t}$ is independent of the BLUEs $\underline{\hat{a}}$ and $\underline{\hat{b}}$, we obtain directly from (9) and (15) the simplified result

$$\begin{array}{rcl} \underline{\breve{\theta}}_{\mathrm{P}} & = & \underline{\hat{\theta}} - C_{\hat{\theta}f} C_{ff}^{-1} f(\underline{\hat{a}}) \\ \mathsf{D}(\underline{\breve{\theta}}_{\mathrm{P}}) & = & \mathsf{D}(\underline{\hat{\theta}}) - C_{\hat{\theta}f} C_{ff}^{-1} C_{f\hat{\theta}} \end{array} \tag{32}$$

This shows that the C-matrices are now constants and therefore only need to be computed once. A further significant simplification can be achieved if we choose the span of the periodic functions in $f(\underline{\hat{a}})$ wisely. In the current context, the Fourier basis functions are a natural choice, since it is well-known that every square-integrable periodic function possesses a Fourier expansion. Furthermore, as we will see, the required moments of the PBIE-estimator can be computed analytically in the Gaussian case. Alternative choices for the basis functions are of course also possible. One could think of periodic splines or wavelets for instance. But although these are known to have good local support, they will have the disadvantage that the required moments are not available analytically. With the choice for Fourier basis functions, the entries of $f(\underline{\hat{a}})$ are

$$f(\underline{\hat{a}}) = [1, \underline{\hat{c}}_1, \ldots, \underline{\hat{c}}_\ell, \underline{\hat{s}}_1, \ldots, \underline{\hat{s}}_\ell]^T \tag{33}$$

with

$$\begin{array}{l} \underline{\hat{c}}_i = \cos(2\pi z_i^T \underline{\hat{a}}) \\ \underline{\hat{s}}_i = \sin(2\pi z_i^T \underline{\hat{a}}) \\ z_i \in \mathbb{Z}^n, i = 1, \ldots, \ell \end{array} \tag{34}$$

Thus the periodic vector function $f(\underline{\hat{a}}) : \mathbb{R}^n \mapsto \mathbb{R}^{k=2\ell+1}$ is completely specified once the ℓ integer vectors z_i are chosen. How to choose these vectors will be discussed later.

With the choice of Fourier basis functions and the fact that PBIE best approximates the BIE (cf. Theorem 3), one might be inclined to believe that PBIE equals the truncated version of BIE's Fourier expansion. This is however false in general, since the PBIE is distributional optimized. Only in the limiting case, when the PDF of the ambiguity residuals becomes uniform over its pull-in region, will truncation of BIE's Fourier expansion provide the PBIE.

We are now in a position to determine the matrices $C_{\hat{\theta}f}$ and C_{ff} of (32). To do so, we make use of some results from Fourier-transform theory. With the PDF of the random vector $\underline{x} \sim \mathcal{N}_r(0, Q)$ denoted as $f_{\underline{x}}(x)$, we have for its Fourier-transform (Teunissen 2026a; Champeney 1987),

$$\begin{array}{l} F_{\underline{x}}(s) = \displaystyle\int_{\mathbb{R}^r} f_{\underline{x}}(x) \exp(-2\pi j s^T x) dx \\ \quad = \exp(-\frac{1}{2} 4\pi^2 ||s||^2_{Q^{-1}}) \\ \frac{1}{2\pi} j \partial_s F_{\underline{x}}(s) = \displaystyle\int_{\mathbb{R}^r} x f_{\underline{x}}(x) \exp(-2\pi j s^T x) dx \\ \quad = -2\pi j Q s \exp(-\frac{1}{2} 4\pi^2 ||s||^2_{Q^{-1}}) \end{array} \tag{35}$$

in which j denotes the imaginary unit ($j^2 = -1$) and the complex exponential can be expressed in cosines and sines as $\exp(-2\pi j s^T x) = \cos(2\pi s^T x) - j \sin(2\pi s^T x)$. Using (35), it is not difficult to verify that

$$\begin{array}{l} C_{\hat{\theta} f_1} = \mathsf{E}[(\underline{\hat{\theta}} - \theta) 1] = 0 \\ C_{\hat{\theta} f_{j+1}} = \mathsf{E}[(\underline{\hat{\theta}} - \theta) \underline{\hat{c}}_j] = 0 \\ C_{\hat{\theta} f_{j+\ell+1}} \\ \quad = \mathsf{E}[(\underline{\hat{\theta}} - \theta) \underline{\hat{s}}_j] = 2\pi Q_{\hat{\theta}\hat{a}} z_j \exp(-\frac{1}{2} 4\pi^2 ||z_j||^2_{Q^{-1}_{\hat{a}\hat{a}}}) \end{array} \tag{36}$$

Note that the ℓ terms of the last expression actually form the covariance matrix of $\underline{\hat{\theta}}$ and $\underline{\hat{s}} = [\underline{\hat{s}}_1, \ldots, \underline{\hat{s}}_\ell]^T$, for which we therefore use the notation $Q_{\hat{\theta}\hat{s}}$.

To determine matrix C_{ff}, we need to determine the entries of the $(2\ell + 1) \times (2\ell + 1)$ partitioned matrix

$$C_{ff} = \begin{bmatrix} \mathsf{E}[1] & \mathsf{E}[\underline{\hat{c}}_j] & \mathsf{E}[\underline{\hat{s}}_j] \\ \mathsf{E}[\underline{\hat{c}}_i] & \mathsf{E}[\underline{\hat{c}}_i \underline{\hat{c}}_j] & \mathsf{E}[\underline{\hat{c}}_i \underline{\hat{s}}_j] \\ \mathsf{E}[\underline{\hat{s}}_i] & \mathsf{E}[\underline{\hat{s}}_i \underline{\hat{c}}_j] & \mathsf{E}[\underline{\hat{s}}_i \underline{\hat{s}}_j] \end{bmatrix} = \begin{bmatrix} 1 & \mathsf{E}[\underline{\hat{c}}_j] & 0 \\ \mathsf{E}[\underline{\hat{c}}_i] & \mathsf{E}[\underline{\hat{c}}_i \underline{\hat{c}}_j] & 0 \\ 0 & 0 & \mathsf{E}[\underline{\hat{s}}_i \underline{\hat{s}}_j] \end{bmatrix} \tag{37}$$

The zero-entries are due to the fact that the sine-functions and the sine-cosine product functions are odd functions, while the PDF of $\underline{\hat{a}}$ is symmetric with respect to its integer mean $a \in \mathbb{Z}^n$. We now observe, when we would substitute (36) and (37) into (32), that only the terms $\mathsf{E}[\underline{\hat{s}}_i \underline{\hat{s}}_j]$ of (37) contribute. Note that these are now true covariances as $\mathsf{E}[\underline{\hat{s}}_i] = 0$ (cf. 37). To determine these covariances, we first express the sine-product as a difference of two cosines,

$$\underline{\hat{s}}_i \underline{\hat{s}}_j = \tfrac{1}{2}[\cos(2\pi(z_i - z_j)^T \underline{\hat{a}}) - \cos(2\pi(z_i + z_j)^T \underline{\hat{a}})] \tag{38}$$

Hence, to determine the expectation of the sine-product, it suffices to determine the expectation of a cosine, which is the real-part of the Fourier transform of a normally distributed PDF. From the first equation of (35) follows therefore that $\mathsf{E}[\cos(2\pi(z_i - z_j)^T \underline{\hat{a}})] = \exp(-\frac{1}{2} 4\pi^2 ||z_i - z_j||^2_{Q^{-1}_{\hat{a}\hat{a}}})$ and $\mathsf{E}[\cos(2\pi(z_i + z_j)^T \underline{\hat{a}})] = \exp(-\frac{1}{2} 4\pi^2 ||z_i + z_j||^2_{Q^{-1}_{\hat{a}\hat{a}}})$. By now collecting the above results, we have proven the following theorem:

Theorem 4 *(Multivariate-normal PBIE estimator) Let* $\underline{y} \sim \mathcal{N}_m(Aa + Bb, Q_{yy})$*, with* $a \in \mathbb{Z}^n$ *and* $b \in \mathbb{R}^p$*, and let* $\underline{\hat{\theta}}$ *and* $\underline{\hat{a}}$ *be the BLUEs of* $\theta = L_a a + L_b b$ *and* a*, respectively. Then, with choice (2), the unbiased minimum-variance PBIE-estimator of* θ*, and its variance matrix, are given as*

$$\begin{array}{rcl} \underline{\check{\theta}}_{\mathrm{P}} & = & \underline{\hat{\theta}} - Q_{\hat{\theta}\hat{s}} Q_{\hat{s}\hat{s}}^{-1} \underline{\hat{s}} \\ Q_{\check{\theta}_{\mathrm{P}}\check{\theta}_{\mathrm{P}}} & = & Q_{\hat{\theta}\hat{\theta}} - Q_{\hat{\theta}\hat{s}} Q_{\hat{s}\hat{s}}^{-1} Q_{\hat{s}\hat{\theta}} \end{array} \tag{39}$$

with $\underline{\hat{s}} = [\sin(2\pi z_1^T \underline{\hat{a}}), \ldots, \sin(2\pi z_\ell^T \underline{\hat{a}})]^T$, $(Q_{\hat{s}\hat{s}})_{ij} = \frac{1}{2}[\exp(-\frac{1}{2} 4\pi^2 ||z_i - z_j||^2_{Q_{\hat{a}\hat{a}}^{-1}}) - \exp(-\frac{1}{2} 4\pi^2 ||z_i + z_j||^2_{Q_{\hat{a}\hat{a}}^{-1}})]$*, and* $Q_{\hat{\theta}\hat{s}_j} = 2\pi Q_{\theta\hat{a}} z_j \exp(-\frac{1}{2} 4\pi^2 ||z_j||^2_{Q_{\hat{a}\hat{a}}^{-1}})$*,* $i, j = 1, \ldots, \ell$*.* ■

This result shows that in case of normally distributed data, the PBIE-estimator takes on a particularly simple form. Its correction to the BLUE is a linear function of only sine-functions. Its structure also reveals its link to *best linear prediction* (BLP) (Teunissen 2007). Indeed, since $-Q_{\hat{\theta}\hat{s}} Q_{\hat{s}\hat{s}}^{-1} = \arg\min_L \mathsf{E}||(\theta - \hat{\theta}) - L\underline{\hat{s}}||^2$, it follows that $-Q_{\hat{\theta}\hat{s}} Q_{\hat{s}\hat{s}}^{-1} \underline{\hat{s}}$ can be seen to be the BLP of the random vector $(\theta - \hat{\theta})$, with its variance matrix of the prediction error being equal to the variance matrix of PBIE $\underline{\check{\theta}}_{\mathrm{P}}$.

For our discussion in the next section it is also useful to have updating expressions available for the PBIE and its variance matrix.

Corollary 2 *(PBIE updating) Let* $\underline{\hat{s}}$ *be partitioned in an 'old' and a 'new' part as* $\underline{\hat{s}} = [\underline{\hat{s}}_{\mathrm{o}}^T, \underline{\hat{s}}_{\mathrm{n}}^T]^T$*. Then* $\underline{\check{\theta}}_{\mathrm{P}}$ *can be expressed in the 'old' PBIE* $\underline{\check{\theta}}_{\mathrm{o,\,P}}$ *as*

$$\begin{array}{rcl} \underline{\check{\theta}}_{\mathrm{P}} & = & \underline{\check{\theta}}_{\mathrm{o,\,P}} - Q_{\hat{\theta}\hat{r}_n} Q_{\hat{r}_n\hat{r}_n}^{-1} \underline{\hat{r}}_n \\ Q_{\check{\theta}_{\mathrm{P}}\check{\theta}_{\mathrm{P}}} & = & Q_{\check{\theta}_{\mathrm{o,\,P}}\check{\theta}_{\mathrm{o,\,P}}} - Q_{\hat{\theta}\hat{r}_n} Q_{\hat{r}_n\hat{r}_n}^{-1} Q_{\hat{r}_n\hat{\theta}} \end{array} \tag{40}$$

with $\underline{\check{\theta}}_{\mathrm{o,\,P}} = \underline{\hat{\theta}} - Q_{\hat{\theta}\hat{s}_o} Q_{\hat{s}_o\hat{s}_o}^{-1} \underline{\hat{s}}_o$ *the PBIE based on* $\underline{\hat{s}}_o$*,* $\underline{\hat{r}}_n = \underline{\hat{s}}_n - Q_{\hat{s}_n\hat{s}_o} Q_{\hat{s}_o\hat{s}_o}^{-1} \underline{\hat{s}}_o$ *the residual of* $\underline{\hat{s}}_n$*'s best linear prediction based on* $\underline{\hat{s}}_o$*,* $Q_{\hat{\theta}\hat{r}_n} = Q_{\hat{\theta}\hat{s}_n} - Q_{\hat{\theta}\hat{s}_o} Q_{\hat{s}_o\hat{s}_o}^{-1} Q_{\hat{s}_o\hat{s}_n}$*, and* $Q_{\hat{r}_n\hat{r}_n} = Q_{\hat{s}_n\hat{s}_n} - Q_{\hat{s}_n\hat{s}_o} Q_{\hat{s}_o\hat{s}_o}^{-1} Q_{\hat{s}_o\hat{s}_n}$*.*

Proof With $\underline{\hat{s}} = [\underline{\hat{s}}_{\mathrm{o}}^T, \underline{\hat{s}}_{\mathrm{n}}^T]^T$, the variance matrix $Q_{\hat{s}\hat{s}}$ can be written as a partitioned matrix. The result follows then directly from substituting the inverse of this partitioned matrix into (39). □

Note that we did not express the variance matrix of $\underline{\check{\theta}}_{\mathrm{P}}$ in terms of the usual conditional variance matrices. This would namely only be possible if $\underline{\hat{\theta}}$ and $\underline{\hat{s}}$ would be jointly normal, which is not the case here.

Another useful result is to show how the PBIE of θ depends on the PBIE of the integer ambiguities. We know of the BIE $\underline{\check{\theta}}$ that it can be expressed in the BIE of $a \in \mathbb{Z}^a$ as $\underline{\check{\theta}} = \underline{\hat{\theta}} - Q_{\hat{\theta}\hat{a}} Q_{\hat{a}\hat{a}}^{-1} (\underline{\hat{a}} - \underline{\check{a}})$ (Teunissen 2003). The following Corollary shows, that a similar form is also true for the PBIE.

Corollary 3 *(PBIE* $\underline{\check{\theta}}_{\mathrm{P}}$ *expressed in ambiguity PBIE* $\underline{\check{a}}_{\mathrm{P}}$*) The PBIE of* $\theta = L_a a + L_b b$ *can be expressed in the PBIE of* $a \in \mathbb{Z}^n$ *as*

$$\begin{array}{rcl} \underline{\check{\theta}}_{\mathrm{P}} & = & \underline{\hat{\theta}} - Q_{\hat{\theta}\hat{a}} Q_{\hat{a}\hat{a}}^{-1} (\underline{\hat{a}} - \underline{\check{a}}_{\mathrm{P}}) \\ Q_{\check{\theta}_{\mathrm{P}}\check{\theta}_{\mathrm{P}}} & = & Q_{\hat{\theta}|\hat{a},\hat{\theta}|\hat{a}} + Q_{\hat{\theta}\hat{a}} Q_{\hat{a}\hat{a}}^{-1} Q_{\check{a}_{\mathrm{P}}\check{a}_{\mathrm{P}}} Q_{\hat{a}\hat{a}}^{-1} Q_{\hat{a}\hat{\theta}} \end{array} \tag{41}$$

with $Q_{\hat{\theta}|\hat{a},\hat{\theta}|\hat{a}} = Q_{\hat{\theta}\hat{\theta}} - Q_{\hat{\theta}\hat{a}} Q_{\hat{a}\hat{a}}^{-1} Q_{\hat{a}\hat{\theta}}$ ■

Proof The occurrence of $Q_{\hat{\theta}\hat{a}}$ in $Q_{\hat{\theta}\hat{s}_j}$ of Theorem 3 (see also last equation of 36), implies that we can write $\underline{\check{\theta}}_{\mathrm{P}} = \underline{\hat{\theta}} - Q_{\hat{\theta}\hat{a}} \underline{v}$, with $\underline{v}$ not depending on $\hat{\theta}$. The PBIE of a follows therefore as $\underline{\check{a}}_{\mathrm{P}} = \underline{\hat{a}} - Q_{\hat{a}\hat{a}} \underline{v}$, which combined with the previous expression gives the first equation of (41). Its second equation follows from an application of the variance-covariance propagation law, thereby recognizing that $\underline{\hat{\theta}} - Q_{\hat{\theta}\hat{a}} Q_{\hat{a}\hat{a}}^{-1} (\underline{\hat{a}} - a)$ is independent of $(\underline{\check{a}}_{\mathrm{P}} - a)$. □

The second expression of (41) shows how the variance matrix of $\underline{\check{a}}_{\mathrm{P}}$ contributes to the variance matrix of $\underline{\check{\theta}}_{\mathrm{P}}$. The variance matrix $Q_{\check{a}_{\mathrm{P}}\check{a}_{\mathrm{P}}}$ depends, as we have seen, on the variance matrix of the BLUE of a, $Q_{\hat{a}\hat{a}}$, and on the choice of sinusoidal base-functions of $\underline{\hat{s}}$, which on their turn depend on the choice of the ℓ integer vectors z_i, $i = 1, \ldots, l$. In the next section we discuss how to best choose these integer vectors.

Choosing the sinusoidal base-functions

The sinusoidal base-functions $\underline{\hat{s}}_i = \sin(2\pi z_i^T \underline{\hat{a}})$ are determined once the set of integer vectors $z_1, \ldots, z_\ell$ is chosen. Let $f^T\theta$ be the parameter of interest. Then the general goal is to choose the integer vectors such that the precision of $f^T\underline{\check{\theta}}_{\mathrm{P}}$ shows a considerable improvement over $f^T\underline{\hat{\theta}}$, either in absolute terms or relative. Hence, with $Q_{\check{\theta}_{\mathrm{P}}\check{\theta}_{\mathrm{P}}} = Q_{\hat{\theta}\hat{\theta}} - Q_{\hat{\theta}\hat{s}} Q_{\hat{s}\hat{s}}^{-1} Q_{\hat{s}\hat{\theta}}$, this gives for the difference of their respective variances, $\Delta\sigma^2_{f^T\theta} = \sigma^2_{f^T\hat{\theta}} - \sigma^2_{f^T\check{\theta}_{\mathrm{P}}}$, the to be considered objective function as

$$\Delta\sigma^2_{f^T\theta} = f^T Q_{\hat{\theta}\hat{s}} Q_{\hat{s}\hat{s}}^{-1} Q_{\hat{s}\hat{\theta}} f \tag{42}$$

The goal is then ideally to choose the integer vectors so as to maximize $\Delta\sigma^2_{f^T\theta}$, or, alternatively, to make it at least sufficiently large. Is one more interested in a relative improvement, for instance to achieve a

certain percentage of improvement, then it is the ratio $\Delta\sigma^2_{f^T\theta}/\sigma^2_{f^T\hat{\theta}} = f^TQ_{\hat{\theta}\hat{s}}Q^{-1}_{\hat{s}\hat{s}}Q_{\hat{s}\hat{\theta}}f/f^TQ_{\hat{\theta}\hat{\theta}}f$ that one would like to have large. Note, in case $\ell = 1$, that this ratio has the clear statistical interpretation of being the square of the correlation coefficient between $f^T\underline{\hat{\theta}}$ and $\underline{\hat{s}}$: $\Delta\sigma^2_{f^T\theta}/\sigma^2_{f^T\hat{\theta}} = (\sigma_{f^T\hat{\theta},\hat{s}}/\sigma_{f^T\hat{\theta}}\sigma_{\hat{s}})^2 = \rho^2_{f^T\hat{\theta},\hat{s}}$. Thus a good integer vector is one which achieves a high absolute correlation between $\underline{\hat{s}}$ and $f^T\underline{\hat{\theta}}$, or, a large absolute covariance between $\underline{\hat{s}}$ and $f^T\underline{\hat{\theta}}$, with small variance for $\underline{\hat{s}}$ ($\sigma^2_{f^T\hat{\theta}}$ is independent of this choice). Using the results of Theorem 3, $f^TQ_{\hat{\theta}\hat{s}} = 2\pi f^TQ_{\hat{\theta}\hat{a}}z\exp(-2\pi^2z^TQ_{\hat{a}\hat{a}}z)$ and $Q_{\hat{s}\hat{s}} = \frac{1}{2}[1-\exp(-8\pi^2z^TQ_{\hat{a}\hat{a}}z)]$, this gives for the variance-difference (42),

$$\Delta\sigma^2_{f^T\theta} = \frac{8\pi^2(f^TQ_{\hat{\theta}\hat{a}}z)^2\exp(-4\pi^2z^TQ_{\hat{a}\hat{a}}z)}{1-\exp(-8\pi^2z^TQ_{\hat{a}\hat{a}}z)} \tag{43}$$

This shows that a good choice for z is one which makes the absolute covariance between $f^T\underline{\hat{\theta}}$ and $z^T\underline{\hat{a}}$ large and the variance of $z^T\underline{\hat{a}}$, being $z^TQ_{\hat{a}\hat{a}}z$, small. Thus for general purpose applications it is always important to aim for integer vectors having small norms with respect to the metric of the ambiguity variance matrix. The following example provides an illustration.

Example 1 (*Geometry-free GNSS model*) In the scalar ambiguity case, such as for a single-frequency, double-differenced, geometry-free GNSS model, the choice for the integers becomes very simple indeed. As they are scalars as well, and the aim is to keep $z^TQ_{\hat{a}\hat{a}}z$ small, one can start with $z_1 = 1$ and then increase it every time by 1. Figure 2 shows in cycles already how much smaller $\sigma_{\check{a}_P}$ is than $\sigma_{\hat{a}}$ for the choice $z_1 = 1$ and the choice $z_1 = 1$ and $z_2 = 2$. □

In case more than one parameter is of interest, the objective function needs to be generalized accordingly. Different such generalizations are possible. For instance, as a relative measure one can consider the determinantal ratio $|Q_{\check{\theta}_P\check{\theta}_P}|/|Q_{\hat{\theta}\hat{\theta}}|$ to reach a large gain if this ratio is small. As $|Q_{\check{\theta}_P\check{\theta}_P}|/|Q_{\hat{\theta}\hat{\theta}}| = |I_q - Q^{-1}_{\hat{\theta}\hat{\theta}}Q_{\hat{\theta}\hat{s}}Q^{-1}_{\hat{s}\hat{s}}Q_{\hat{s}\hat{\theta}}| = \prod_{i=1}^q(1-\rho_i^2)$, with ρ_i the *canonical correlations* (Teunissen 1997) p.514, larger correlations will then result in larger gains. However, in this instance, a much simpler approach is to use the difference of the mean-squared errors of the two estimators (cf. 20(*d*) of Theorem 2a), $\mathsf{E}||P_{\mathcal{F}}(\underline{\hat{\theta}}-\theta)||^2 = \mathsf{E}||\underline{\hat{\theta}}-\theta||^2 - \mathsf{E}||\underline{\check{\theta}}_P-\theta||^2$, thus leading, as a generalization of (42), to the use of the trace,

$$\mathsf{E}||P_{\mathcal{F}}(\underline{\hat{\theta}}-\theta)||^2 = \mathrm{trace}(Q_{\hat{\theta}\hat{s}}Q^{-1}_{\hat{s}\hat{s}}Q_{\hat{s}\hat{\theta}}) \tag{44}$$

Note, in case θ concerns the position parameters of the mixed-integer model (1), that (44) equals the difference of the squared PDOPs, $\Delta\mathrm{PDOP}^2_\theta = \mathrm{PDOP}^2_{\check{\theta}_P} - \mathrm{PDOP}^2_{\hat{\theta}}$.

With the objective function (44), the search for the ℓ best n-dimensional integer vectors boils then down to solving the maximization problem $\max_{z_i\in\mathbb{Z}^n_+,i=1,\ldots,l}\mathrm{trace}(Q_{\hat{\theta}\hat{s}}Q^{-1}_{\hat{s}\hat{s}}Q_{\hat{s}\hat{\theta}})$, in which $\mathbb{Z}^n_+$ is the integer set, taking only one of each nonzero integer pair $\{-z_i,+z_i\}$. This last specification is needed, since the z-dependence of the objective function does not discriminate between $-z_i$ and $+z_i$.

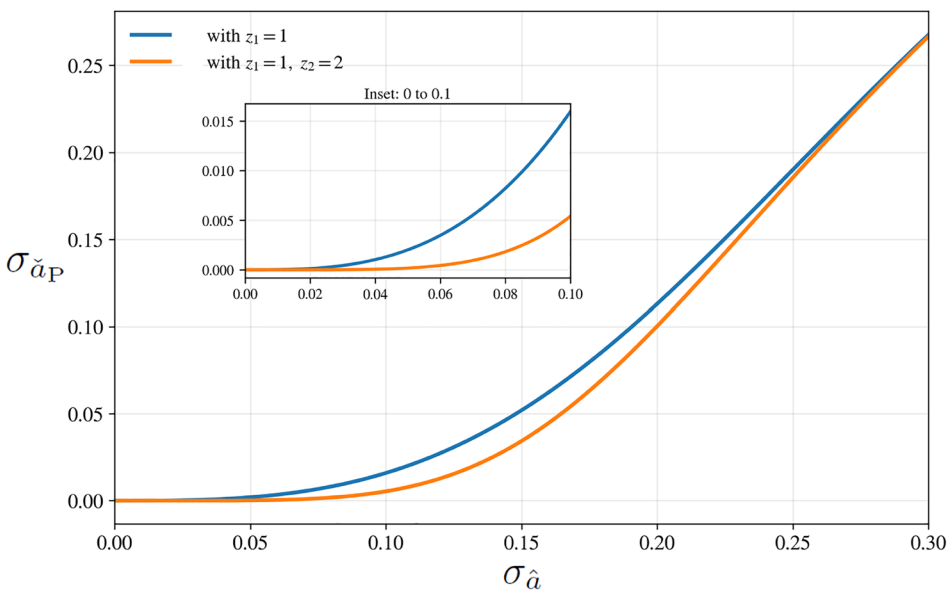

Fig. 2 PBIE ambiguity standard deviation as function of the BLUE ambiguity standard deviation for a single-frequency, double-differenced, geometry-free model, with the choice $z_1 = 1$ and the choice $z_1 = 1$ and $z_2 = 2$

As short integer vectors are to be preferred, a practical approach is to first start creating a candidate integer set

$$\mathcal{E}_r = \{z \in \mathbb{Z}^n_+ |\ z^T Q_{\hat{a}\hat{a}} z \leq r^2\} \tag{45}$$

whereby r is chosen so as to make the exponential contributions in the coefficients of Theorem 3 small enough. The collection $\mathcal{E}_r$ is particularly simple when the variance matrix is a unit matrix, $Q_{\hat{a}\hat{a}} = I_n$, since then for e.g. $n = 2$, the integer vectors, with increasing length, are

$$\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right], \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right], \left[\begin{smallmatrix}-1\\1\end{smallmatrix}\right], \left[\begin{smallmatrix}2\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}0\\2\end{smallmatrix}\right], \left[\begin{smallmatrix}2\\2\end{smallmatrix}\right], \left[\begin{smallmatrix}-2\\2\end{smallmatrix}\right], \ldots \tag{46}$$

Although in our case the ambiguity variance matrix is not a unit matrix, the requirement of working with short integer vectors, preferably with a diagonal dominant variance matrix, shows the importance of first applying the LAMBDA-decorrelation (Teunissen 1995) to $\underline{\hat{a}}$, before creating the integer set (45). Thus from now on we assume that $Q_{\hat{a}\hat{a}}$ has already been LAMBDA-transformed, using e.g. the toolbox of Massarweh et al. (2025). The same toolbox can then also be used to collect all integer vectors in $\mathcal{E}_r$, keeping one representative for each pair $\pm z$. The maximum of (44) is then found by computing for each such ℓ-set in $\mathcal{E}_r$ the function value $\mathrm{trace}(Q_{\hat{\theta}\hat{s}} Q^{-1}_{\hat{s}\hat{s}} Q_{\hat{s}\hat{\theta}})$, followed by choosing the maximizer. Instead of this direct combinatorial approach, one may also opt for an incremental approach, where first a list of best single integer vectors is made using Theorem 3, followed by an evaluation of their achieved gain using the variance updating formula of Corollary 2.

Thus for each representative integer vector z of $\mathcal{E}_r$, one evaluates what its single contribution of $\underline{\hat{s}} = \sin(2\pi z^T \underline{\hat{a}})$ is to reducing the trace of the variance matrix of $\underline{\hat{\theta}}$. Hence, ordering based on the trace-version of (43), i.e.

$$\frac{\mathrm{trace}(Q_{\hat{\theta}\hat{s}} Q_{\hat{s}\hat{\theta}})}{Q_{\hat{s}\hat{s}}} = \frac{8\pi^2 (z^T Q_{\hat{a}\hat{\theta}} Q_{\hat{\theta}\hat{a}} z) \exp(-4\pi^2 z^T Q_{\hat{a}\hat{a}} z)}{1 - \exp(-8\pi^2 z^T Q_{\hat{a}\hat{a}} z)} \tag{47}$$

is used to make a list of best single integer vectors. Starting from the best in this list, one then adds integer vectors one by one, thereby evaluating for each new $\hat{s}_n = \sin(2\pi z_n^T \underline{\hat{a}})$ its gain contribution

$$\mathrm{trace}(Q_{\check{\theta}_\mathrm{P}\check{\theta}_\mathrm{P}} - Q_{\check{\theta}_{\mathrm{o,\,P}}\check{\theta}_{\mathrm{o,\,P}}}) = \frac{\mathrm{trace}(Q_{\hat{\theta}\hat{r}_n} Q_{\hat{r}_n\hat{\theta}})}{Q_{\hat{r}_n\hat{r}_n}} \tag{48}$$

until the contribution is considered negligible. The following example illustrates the computational steps.

Example 2 (*Dual-frequency geometry-free GNSS model*) Consider the dual-frequency geometry-free, k-epoch, mixed-integer GNSS model

$$\begin{aligned} \mathsf{E}\begin{bmatrix}\underline{p}\\ \underline{\phi}\end{bmatrix} &= \begin{bmatrix} I_k \otimes e_2 & 0 \\ I_k \otimes e_2 & e_k \otimes \Lambda \end{bmatrix} \begin{bmatrix}\rho \\ a\end{bmatrix}, \mathsf{D}\begin{bmatrix}\underline{p}\\ \underline{\phi}\end{bmatrix} \\ &= \begin{bmatrix} \sigma_p^2 I_k \otimes I_2 & 0 \\ 0 & \sigma_\phi^2 I_k \otimes I_2 \end{bmatrix} \end{aligned} \tag{49}$$

with $e_k = [1, \ldots, 1]^T \in \mathbb{R}^k$, $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2)$, and $\otimes$ denoting the Kronecker product. The variance matrix of the BLUE of $a \in \mathbb{Z}^2$ follows, with the variance-ratio $\epsilon = \sigma^2_\phi / \sigma^2_p$, as $Q_{\hat{a}\hat{a}} = \frac{1}{k}\sigma^2_p \Lambda^{-1}[\epsilon I_2 + e_2 e_2^T]\Lambda^{-1}$. Its numerical entries are given as

$$Q_{\hat{a}\hat{a}} = \tfrac{1}{k}\begin{bmatrix} 2.48286 & 1.96544 \\ 1.96544 & 1.55617 \end{bmatrix} \text{ cycles}^2 \tag{50}$$

when $\lambda_1 = 19.04$cm, $\lambda_2 = 24.05$cm, $\epsilon = 10^{-4}$, and $\sigma_p = 30cm$. With the admissible ambiguity transformation $\underline{\hat{z}} = Z^T \underline{\hat{a}}$, using

$$Z^T = \begin{bmatrix} 4 & -5 \\ -7 & 9 \end{bmatrix} \tag{51}$$

the decorrelated ambiguity variance matrix becomes

$$Q_{\hat{z}\hat{z}} = Z^T Q_{\hat{a}\hat{a}} Z = \tfrac{10^{-2}}{k}\begin{bmatrix} 1.21544 & -0.10338 \\ -0.10338 & 6.36561 \end{bmatrix} \text{ cycles}^2 \tag{52}$$

Note the two orders of variance improvement and the significant reduction in absolute correlation coefficient, from 99.98 to 3.72 (cf. 50 and 52). We now compute, for $k = 1$ and $k = 2$, the step-wise improvements of the variances of the PBIE-estimator of $z = Z^T a$, when the five best integer vectors, as determined by (47), are used. These integer vectors $z_i \in \mathbb{Z}^2$, $i = 1, \ldots, 5$, are shown in Table 1. Note that only the fifth integer vector differs between the $k = 1$ and the $k = 2$ case. The step-wise trace improvements are computed using (48), with the accumulated trace of the PBIE-estimator, $\mathrm{trD}(\check{\underline{\theta}}_\mathrm{P})$, also shown for each step in Table 1. For $k = 1$, the variance-traces of the BLUE $\underline{\hat{z}}$, the BIE $\underline{\check{z}}$, and the PBIE $\underline{\check{z}}_\mathrm{P}$ (after five steps) are respectively, $\mathrm{trD}(\underline{\hat{z}}) = 0.07581$,$\mathrm{D}(\underline{\check{z}}) = 0.03568$, and $\mathrm{trD}(\underline{\check{z}}_\mathrm{P}) = 0.03644$. This shows, already with only five integer vectors, that the PBIE-estimator has a 54% improvement on the BLUE and a variance gap with the BIE-estimator of only 0.00076 cycles^2. For $k = 2$, we have $\mathrm{trD}(\underline{\hat{z}}) = 0.05361$,$\mathrm{D}(\underline{\check{z}}) = 0.01397$, and $\mathrm{trD}(\underline{\check{z}}_\mathrm{P}) = 0.01454$, thus showing a 73% improvement of the PBIE over the BLUE, and a variance gap with the BIE-estimator of only 0.00057 cycles^2. □

Table 1 Geometry-free PBIE ambiguity-precision $\mathrm{trD}(\underline{\check{z}}_{\mathrm{P}})$, for $k = 1$ (top half) and $k = 2$ (bottom half), compared with that of the BLUE, $\mathrm{trD}(\underline{\hat{z}})$, and the BIE, $\mathrm{trD}(\underline{\check{z}})$. The PBIE-precision $\mathrm{trD}(\underline{\check{z}}_{\mathrm{P}})$ is shown in dependence of the used integer vectors z_i, $i = 1, \ldots, 5$

$\mathrm{trD}(\underline{\hat{z}}) = 0.07581$	Step 1	Step 2	Step 3	Step 4	Step 5
$\mathrm{trD}(\underline{\check{z}}) = 0.03568$	$z_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$	$z_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$z_3 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$	$z_4 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$	$z_5 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
$\mathrm{trD}(\underline{\check{z}}_{\mathrm{P}})$:	0.04971	0.03799	0.03694	0.03654	0.03644
$\mathrm{trD}(\underline{\hat{z}}) = 0.05361$	Step 1	Step 2	Step 3	Step 4	Step 5
$\mathrm{trD}(\underline{\check{z}}) = 0.01397$	$z_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$	$z_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$z_3 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$	$z_4 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$	$z_5 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$
$\mathrm{trD}(\underline{\check{z}}_{\mathrm{P}})$:	0.02574	0.01731	0.01546	0.01484	0.01454

Summary and conclusions

In this contribution, we introduced the projected-BIE (PBIE) estimator as a new member of the class of integer-equivariant estimators for the mixed-integer model. The development was motivated by the computational limitations of the classical BIE-estimator, whose exact formulation is based on an infinite sum over the integer space $\mathbb{Z}^n$. By exploiting the canonical IE-representation of Teunissen (2003) as the additive admissible periodic correction of the BLUE, we reformulated the unbiased minimum-variance problem on a restricted estimator space and derived a finite alternative that preserves the essential mixed-integer structure of the original estimation problem. The resulting PBIE-estimator retains the key principles of mixed-integer estimation theory. It respects integer equivariance, is unbiased, and minimizes variance within the selected subclass of admissible estimators. Unlike the BIE-estimator, however, PBIE is finite by construction. Its representation does not require truncation of infinite series, but follows directly from the solution of a well-defined system of normal equations.

An insightful outcome of this contribution is also the explicit relationship established between PBIE and BIE. Through a geometric analysis of their estimator structure, their exact variance matrices were determined and the PBIE-estimator was shown to satisfy a dual optimality property: it is not only the minimum-variance unbiased estimator within its restricted class, but it is also the estimator that provides the best approximation to the BIE-estimator. This characterization places PBIE on firm ground and shows that it is a principled reduced-complexity counterpart of BIE.

Another important feature of the PBIE framework is its generality. The results provided are valid for very general observational distributions and therefore extend beyond the Gaussian setting that underlies many classical ambiguity-resolution methods. This makes the estimator relevant for contemporary GNSS applications in which heavy-tailed noise, or robust stochastic modeling play an increasingly important role. A further study of PBIE's statistical and numerical characteristics under such distributions will be given in a separate contribution. At the same time, it was shown that in the usual Gaussian case the PBIE-estimator simplifies considerably, whereby the periodic correction to the BLUE now depends only on the float ambiguity solution, leading to an especially efficient implementation.

Although developed here for the broad application fields of mixed-integer GNSS models, we finally note that the PBIE and its underlying theory has a wider range of applicability. Next to the regional and global satellite navigation systems, and the upcoming LEO-PNT constellations, it also applies to other carrier-phase-based interferometric systems, such as Very Long Baseline Interferometry (VLBI), Interferometric Synthetic Aperture Radar (InSAR), or underwater acoustic carrier-phase positioning.

Appendix

Proof of Theorem 1 (*The PBIE estimator*) From the decomposition

$$\begin{aligned} &\mathsf{E}||\underline{\theta} - \theta||^2 = \mathsf{E}||(\underline{\check{\theta}}_{\mathrm{P}} - \theta) + (\underline{\theta} - \underline{\check{\theta}}_{\mathrm{P}})||^2 \\ =&\mathsf{E}||\underline{\check{\theta}}_{\mathrm{P}} - \theta||^2 + 2\mathsf{E}[(\underline{\check{\theta}}_{\mathrm{P}} - \theta)^T(\underline{\theta} - \underline{\check{\theta}}_{\mathrm{P}})] + \mathsf{E}||\underline{\theta} - \underline{\check{\theta}}_{\mathrm{P}}||^2 \end{aligned} \tag{53}$$

it follows that $\mathsf{E}||\underline{\check{\theta}}_{\mathrm{P}} - \theta||^2 \leq \mathsf{E}||\underline{\theta} - \theta||^2$ for all $\underline{\theta} \in \mathcal{F}_\theta$, if the cross-term $\mathsf{E}[(\underline{\check{\theta}}_{\mathrm{P}} - \theta)^T(\underline{\theta} - \underline{\check{\theta}}_{\mathrm{P}})]$ vanishes for all $\underline{\theta} \in \mathcal{F}_\theta$. As $\underline{\theta} = \hat{\theta} + F_\theta(\underline{t})f(\underline{\hat{a}})$, we need to find the matrix function $F_\theta(\tau)$ that makes the above cross-term vanish. With $\underline{\check{\theta}}_{\mathrm{P}} = \hat{\theta} + \hat{F}_\theta(\underline{t})f(\underline{\hat{a}})$, we have

$$\begin{aligned} \mathsf{E}[(\underline{\check{\theta}}_{\mathrm{P}} - \theta)^T(\underline{\theta} - \underline{\check{\theta}}_{\mathrm{P}})] &= \mathrm{trace}\,\mathsf{E}[(\underline{\check{\theta}}_{\mathrm{P}} - \theta)(\underline{\theta} - \underline{\check{\theta}}_{\mathrm{P}})^T] \\ &= \mathrm{trace}\,\mathsf{E}[(\underline{\check{\theta}}_{\mathrm{P}} - \theta)f(\underline{\hat{a}})^T(\hat{F}_\theta(\underline{t}) - F_\theta(\underline{t}))^T] \\ &= \mathrm{trace}\,\mathsf{E}\left[\mathsf{E}[(\underline{\check{\theta}}_{\mathrm{P}} - \theta)f(\underline{\hat{a}})^T(\hat{F}_\theta(\tau) - F_\theta(\tau))^T|\underline{t} = \tau]\right] \end{aligned} \tag{54}$$

As this needs to vanish for all $\underline{\theta} \in \mathcal{F}_\theta$, and thus for all $F_\theta(\tau)$, the following 'orthogonality' condition needs to hold,

$$\mathsf{E}[(\underline{\check{\theta}}_{\mathrm{P}} - \theta)f(\underline{\hat{a}})^T|\underline{t} = \tau] = 0 \tag{55}$$

From substituting $\check{\underline{\theta}}_{\mathrm{P}} = \hat{\underline{\theta}} + \hat{F}_{\theta}(\underline{t})f(\underline{\hat{a}})$, the result (9) follows. □

Proof of Corollary 1 (*Orthogonality, Mean and Variance of PBIE-estimator*) (1) Making use of the property of iterated expectations, substitution of expression (9) into (12) gives

$$\mathsf{E}\left[\mathsf{E}[(\hat{\underline{\theta}}-\underline{\theta})f(\underline{\hat{a}})^T - C_{\hat{\theta}f}(\tau)C_{ff}^{-1}(\tau)f(\underline{\hat{a}})f(\underline{\hat{a}})^T|\underline{t}=\tau]\right] = 0 \tag{56}$$

which proves the result stated. (2) Substitution of $f_1(\underline{\hat{a}}) = 1$ into (12) proves (13). (3) If $\mathsf{E}(\underline{\theta}) = \mathsf{E}(\check{\underline{\theta}}_{\mathrm{P}})$, then $\mathsf{D}(\check{\underline{\theta}}_{\mathrm{P}}) = \mathsf{D}(\underline{\theta} + \check{\underline{\theta}}_{\mathrm{P}} - \underline{\theta}) = \mathsf{D}(\underline{\theta}) + \mathsf{E}[\underline{\theta}(\check{\underline{\theta}}_{\mathrm{P}} - \underline{\theta})^T] + \mathsf{E}[\underline{\theta}\,(\check{\underline{\theta}}_{\mathrm{P}} - \underline{\theta})^T]^T + \mathsf{D}(\check{\underline{\theta}}_{\mathrm{P}} - \underline{\theta})$. The result is then proven by showing that $\mathsf{E}[\underline{\theta}(\check{\underline{\theta}}_{\mathrm{P}} - \underline{\theta})^T] = \mathsf{D}(\check{\underline{\theta}}_{\mathrm{P}} - \underline{\theta})$ or equivalenty that $\mathsf{E}[\check{\underline{\theta}}_{\mathrm{P}}(\check{\underline{\theta}}_{\mathrm{P}} - \underline{\theta})^T] = 0$. We have, since $\mathsf{E}(\check{\underline{\theta}}_{\mathrm{P}}) = \mathsf{E}(\underline{\theta}) = \mathsf{E}(\underline{\theta})$, indeed that $\mathsf{E}[\check{\underline{\theta}}_{\mathrm{P}}(\check{\underline{\theta}}_{\mathrm{P}} - \underline{\theta})^T] = \mathsf{E}[(\check{\underline{\theta}}_{\mathrm{P}} - \theta)(\check{\underline{\theta}}_{\mathrm{P}} - \underline{\theta})^T] = \mathsf{E}[(\check{\underline{\theta}}_{\mathrm{P}} - \theta)f(\underline{\hat{a}})^T(\hat{F}(\underline{t}) - F(\underline{t}))] = 0$ due to the orthogonality property (12). (4) If $f_1(\underline{\hat{a}}) = 1$, then $\mathsf{E}(\hat{\underline{\theta}}) = \theta = \mathsf{E}(\hat{\underline{\theta}})$, which implies that (14) can be used for $\underline{\theta} := \hat{\underline{\theta}}$. The stated result (15) follows then from using $\mathsf{D}(\check{\underline{\theta}}_{\mathrm{P}} - \hat{\underline{\theta}}) = \mathsf{E}[(\check{\underline{\theta}}_{\mathrm{P}} - \hat{\underline{\theta}})(\check{\underline{\theta}}_{\mathrm{P}} - \hat{\underline{\theta}})^T] = \mathsf{E}[\mathsf{E}[C_{\theta f}(\tau)C_{ff}^{-1}(\tau)f(\underline{\hat{a}})f(\underline{\hat{a}})^T C_{ff}^{-1}(\tau)C_{f\theta}(\tau)]\underline{t} = \tau]]$. (5) The result follows from combining $\hat{\underline{\theta}} = L_a\underline{\hat{a}} + L_b\underline{\hat{b}}$ and $C_{\hat{\theta}f}(\underline{t}) = L_aC_{\hat{a}f}(\underline{t}) + L_bC_{\hat{b}f}(\underline{t})$ with $\check{\underline{\theta}}_{\mathrm{P}} = \hat{\underline{\theta}} - C_{\hat{\theta}f}(\underline{t})C_{ff}^{-1}(\underline{t})f(\underline{\hat{a}})$. □

Proof of Theorem 2a (*Geometry of PBIE-estimator*) Equation (18) follows directly from Theorem 1 and the definition of the operator $P_{\mathcal{F}}$; (*a*) For any $\underline{x} \in \mathbb{R}^q$, $P_{\mathcal{F}}\underline{x}$ is of the form $F(\underline{t})f(\underline{\hat{a}})$ and therefore an element of $\mathcal{F}$; (*b*) According to the definition $P_{\mathcal{F}}P_{\mathcal{F}}\underline{x} = \mathsf{E}[P_{\mathcal{F}}\underline{x}f(\underline{\hat{a}})^T|\underline{t}]C_{ff}^{-1}(\underline{t})f(\underline{\hat{a}})$. Substitution of $P_{\mathcal{F}}\underline{x} = \mathsf{E}[\underline{x}f(\underline{\hat{a}})^T|\underline{t}]C_{ff}^{-1}(\underline{t})f(\underline{\hat{a}})$ gives $P_{\mathcal{F}}P_{\mathcal{F}}\underline{x} = \mathsf{E}[\underline{x}f(\underline{\hat{a}})^T|\underline{t}]C_{ff}^{-1}(\underline{t})f(\underline{\hat{a}}) = P_{\mathcal{F}}\underline{x}$; (*c*) To show that $\mathsf{E}[(\underline{x} - P_{\mathcal{F}}\underline{x})^T\underline{h}] = 0$ for any $\underline{h} \in \mathcal{F}$, we substitute $\underline{h} = F(\underline{t})f(\underline{\hat{a}})$ to get $\mathsf{E}[(\underline{x} - P_{\mathcal{F}}\underline{x})^T F(\underline{t})f(\underline{\hat{a}})] = \mathsf{E}[\mathsf{E}[(\underline{x} - P_{\mathcal{F}}\underline{x})^T F(t)f(\underline{\hat{a}})|\underline{t} = t]] = \mathrm{trace}\ \mathsf{E}[\mathsf{E}[(\underline{x}P_{\mathcal{F}}\underline{x}) - f(\underline{\hat{a}})^T F(t)^T|\underline{t} = t]]$, which shows that it is sufficient to show that $\mathsf{E}[(\underline{x} - P_{\mathcal{F}}\underline{x})f(\underline{\hat{a}})^T|\underline{t} = t] = 0$. This indeed follows once $P_{\mathcal{F}}\underline{x} = \mathsf{E}[\underline{x}f(\underline{\hat{a}})^T|\underline{t}]C_{ff}^{-1}(\underline{t})f(\underline{\hat{a}})$ is substituted; (*d*) We only prove (*e*), as (*d*) follows from taking the trace of (*e*); (*e*) In order to prove (20) (*e*), it follows, from applying the variance-propagation law to (18), that we need to prove

$$\begin{aligned}&\mathsf{E}[(\hat{\underline{\theta}}-\underline{\theta})\underline{e}^T] = \mathsf{E}[\underline{e}\underline{e}^T] \text{ for } \underline{e} \\ &\quad = P_{\mathcal{F}}(\hat{\underline{\theta}}-\underline{\theta}) = \mathsf{E}[(\hat{\underline{\theta}}-\underline{\theta})f(\check{\underline{\epsilon}})^T|\underline{t}]C_{ff}^{-1}(\underline{t})f(\check{\underline{\epsilon}})\end{aligned} \tag{57}$$

When conditioning $\underline{e}$ on $\check{\underline{\epsilon}}, \underline{t}$, we may take it outside the expectation operator and write for the conditional expectation $\mathsf{E}[(\hat{\underline{\theta}} - \underline{\theta})\underline{e}^T|\check{\underline{\epsilon}}, \underline{t}] = \mathsf{E}[(\hat{\underline{\theta}} - \underline{\theta})|\check{\underline{\epsilon}}, \underline{t}]\underline{e}^T = \underline{e}\underline{e}^T$. Using this result in the expression for the unconditional expectation, gives $\mathsf{E}[(\hat{\underline{\theta}} - \underline{\theta})\underline{e}^T] = \mathsf{E}[\mathsf{E}[(\hat{\underline{\theta}} - \underline{\theta})\underline{e}^T|\check{\underline{\epsilon}}, \underline{t}]] = \mathsf{E}[\underline{e}\underline{e}^T]$, which proves the result. □

Proof of Theorem 2b (*Geometry of BIE-estimator*) Expression (23) follows from recognizing that the conditional mean $\mathsf{E}[(\theta - \hat{\underline{\theta}})|\check{\underline{\epsilon}}, \underline{t}]$ is the minimizer of (22). The proof of (24) goes along same lines as that of (19). We only prove (*e*), as (*d*) follows from taking the trace of (*e*); In order to prove (25*e*), it follows, from applying the variance-propagation law to (23), that we need to prove

$$\mathsf{E}[(\hat{\underline{\theta}}-\underline{\theta})\underline{e}^T] = \mathsf{E}[\underline{e}\underline{e}^T] \text{ for } \underline{e} = P_{\mathcal{F}}(\hat{\underline{\theta}}-\underline{\theta}) = \mathsf{E}[(\hat{\underline{\theta}}-\underline{\theta})|\check{\underline{\epsilon}}, \underline{t}] \tag{58}$$

When conditioning $\underline{e}$ on $\check{\underline{\epsilon}}$,$\underline{t}$, we may take it outside the expectation operator and write for the conditional expectation $\mathsf{E}[(\hat{\underline{\theta}} - \underline{\theta})\underline{e}^T|\check{\underline{\epsilon}},\underline{t}] = \mathsf{E}[(\hat{\underline{\theta}} - \underline{\theta})|\check{\underline{\epsilon}},\underline{t}]\underline{e}^T = \underline{e}\underline{e}^T$. Using this result in the expression for the unconditional expectation, gives $\mathsf{E}[(\hat{\underline{\theta}} - \underline{\theta})\underline{e}^T] = \mathsf{E}[\mathsf{E}[(\hat{\underline{\theta}} - \underline{\theta})\underline{e}^T|\check{\underline{\epsilon}},\underline{t}]] = \mathsf{E}[\underline{e}\underline{e}^T]$, which proves the result. □

Proof of Theorem 2c (*PBIE expressed in BIE*) As $\mathcal{F} \subset \mathcal{H}$, the two projectors $P_{\mathcal{F}}$ and $P_{\mathcal{H}}$ have the property $P_{\mathcal{F}}P_{\mathcal{H}} = P_{\mathcal{H}}P_{\mathcal{F}} = P_{\mathcal{F}}$. Therefore $\check{\underline{\theta}}_{\mathrm{P}} - \hat{\underline{\theta}} = P_{\mathcal{F}}(\theta - \hat{\underline{\theta}}) = P_{\mathcal{F}}P_{\mathcal{H}}(\theta - \hat{\underline{\theta}}) = P_{\mathcal{F}}(\check{\underline{\theta}} - \hat{\underline{\theta}})$,from which the result (26) follows. We only prove (*b*), as (*a*) follows from taking the trace of (*b*). We have $\check{\underline{\theta}}_{\mathrm{P}} - \theta = \check{\underline{\theta}} - \theta + (P_{\mathcal{H}} - P_{\mathcal{F}})(\hat{\underline{\theta}} - \theta)$. The result (*b*) is proven if we can show that $\mathsf{E}[(\check{\underline{\theta}} - \theta)\underline{h}(\check{\underline{\epsilon}}, \underline{t})^T] = 0$, for $\underline{h}(\check{\underline{\epsilon}}, \underline{t}) = (P_{\mathcal{H}} - P_{\mathcal{F}})(\hat{\underline{\theta}} - \theta)$. By using the property of iterated expectations, we obtain

$$\begin{aligned}\mathsf{E}[(\check{\underline{\theta}} - \theta)\underline{h}(\check{\underline{\epsilon}}, \underline{t})^T] &= \mathsf{E}[\mathsf{E}[(\check{\underline{\theta}} - \theta)\underline{h}(\check{\underline{\epsilon}}, \underline{t})^T]|\check{\underline{\epsilon}}, \underline{t}] \\ &= \mathsf{E}[\mathsf{E}[(\check{\underline{\theta}} - \theta)|\check{\underline{\epsilon}}, \underline{t}]\underline{h}(\check{\underline{\epsilon}}, \underline{t})^T] \\ &= 0\end{aligned} \tag{59}$$

since

$$\begin{aligned}\mathsf{E}[(\check{\underline{\theta}} - \theta)|\check{\underline{\epsilon}}, \underline{t}] &= \mathsf{E}[\hat{\underline{\theta}} - \theta - \mathsf{E}[(\hat{\underline{\theta}} - \theta)|\check{\underline{\epsilon}}, \underline{t}]|\check{\underline{\epsilon}}, \underline{t}] \\ &= \mathsf{E}[(\hat{\underline{\theta}} - \theta)|\check{\underline{\epsilon}}, \underline{t}] - \mathsf{E}[(\hat{\underline{\theta}} - \theta)|\check{\underline{\epsilon}}, \underline{t}] \\ &= 0\end{aligned} \tag{60}$$

□

Author contributions PT developed theory, method and manuscript text

Data availability No datasets were generated or analysed during the current study.

Declarations

Conflict of interest The author declares no conflict of interest.

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