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Geometric nonlinear shape sensing using the calibration matrix method

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Abstract

Structural Health Monitoring (SHM) is increasingly being considered for predictive maintenance of aircrafts. This study investigates the use of strain and displacement measurements obtained from a limited number of sensors as input to a calibration matrix (CM) algorithm to compute the shapes of structures undergoing geometric nonlinear (GNL) structural deformation. To evaluate CM, GNL benchmark problems from the literature were expanded with sensors at specific locations on these structures. Two problems used for GNL inverse finite element methods (iFEM) in the literature were implemented as well, for comparison. Each problem was modeled using ABAQUS and subjected to significant deformations, resulting in strain and displacement measurements from simulated sensors. A novel GNL CM variant is presented that converts virtual sensor data into accurately reconstructed strain/displacement distributions and applied loads, achieving superior accuracy for the two state-of-the-art GNL iFEM problems. The accuracy was assessed for each problem based on the number and type of sensors employed. Under noiseless conditions, the presented method achieves a force reconstruction error of 1% or less for all problems utilizing 96 strain sensors and of 2% or less for all but one problem when employing 96 displacement sensors or a combination of strain and displacement sensors. A displacement error of 0.34% or less is achieved for all benchmark problems, using just 32 sensors of either type. When subjected to noise, the accuracy is comparable or better than the iFEM results with 96 sensors of either type for all but one of the benchmark problems, and superior when using a greater number of sensors, reaching median errors of less than 1%. A median force error of 2.5% is reached with 720 strain sensors when subject to 10% noise. This level of accuracy is achieved for both isotropic and composite laminate materials, and for structures undergoing buckling deformation.

Keywords

load monitoring, shape sensing, calibration matrix method, geometric nonlinear, iFEM, structural health monitoring, strain sensors, displacement sensors, composite laminates, buckling

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Introduction

The aircraft industry is always looking for ways to reduce how much time and money needs to be spent on inspection, maintenance and repair, while maintaining the same (or achieving a better) level of safety.¹ One of these methods is the Digital Twin concept,^{2,3} which aims to continuously collect sensor data and other information on a per tail number basis. This data is not only used to monitor aircraft but is also fed into the Digital Twin: a computer model of the entire aircraft that aims to predict how this complex system will behave in the future, thus enabling proactive and efficiently targeted inspection, maintenance and repair.

Part of the Digital Twin is a model of the aircraft structure. Based on sensor and flight data from parts of the structure, it has been demonstrated under research conditions^{4,5} that it is possible to make estimates about

the displacement, strain, stress and load distributions that occur throughout the entire structure. As these estimates are made repeatedly over time using various “shape sensing” algorithms, the results can be fed into other models that predict the structural integrity and life of the components. As such, these components can then be targeted individually for inspection, maintenance and repair, before the accumulated damage starts to reach potentially critical levels.

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Traditional sensors, such as metallic foil strain gages, are too limited to achieve these goals: as each sensor only measures strains at a single point and along a single axis, thus, too little data is available to make accurate estimates of the structural conditions for the entire structure. With the arrival of new types of sensors, such as fiber optic strain gages and experimental mechanics techniques like digital image correlation (DIC), strains and displacements can be determined for many points. These new technologies have allowed shape sensing algorithms to mature and deliver highly accurate estimates of displacement, strain and stress, which combined with novel inverse methods have been able to monitor the load distributions at every point within a structure.^{5,6}

The lack of good sensors did not stop researchers from trying to determine accurate reconstructions of (deformed) structural shapes. As early as 1983,⁷ shape sensing was mentioned in the literature: Carl G. Saunders used Moiré patterns and an interactive computer program to improve how prostheses would fit patients.^{8–10} Similar techniques were applied by Weathers et al.¹¹ just a few years later to study structural problems, both numerically and experimentally, using laser speckle patterns.

The fundamental theoretical challenges of shape sensing were first described by Maniatty et al.¹²: Using displacement measurements for shape sensing has the challenge that small measurement errors often lead to much larger errors when strains and stresses are calculated, because the displacement is differentiated. Using strains instead¹³ comes with its own challenge: it is an inverse problem to translate strain measurements into a complete distribution of structural displacements. Such inverse problems are ill-posed and do not, in general, have solutions that are unique, stable or even exist at all.^{14–17}

Various alternatives were proposed to solve these issues and to replace the methods of Saunders and Weathers, which were unsuitable for large scale applications. These methods included objective function minimization with regularization,^{12,18,19} Bayesian statistics,¹⁸ polynomial fitting,^{19–22} dynamic programming,²³ modal transformation²⁴ and direct integration of the strains and curvatures measured by multicore optical fibers.^{25–28} Most of these alternatives have the drawback that their parameters needed to be tuned for every individual problem, which made it challenging to apply them as a general method. The direct integration method for multicore optical fibers was an exception, as it did not require tuning, but this method only solved part of the problem: the method only determined the shape of the fiber, not the shape of the entire structure to which it was attached.

In 2001, a more general approach for shape sensing was demonstrated by Tessler et al. at NASA: the inverse Finite Element Method, or “iFEM,” aimed at reconstructing the shape of deformed aircraft based on in-flight strain measurements. In 2002, Tessler et al.

proposed to apply iFEM to Structural Health Monitoring (SHM) problems.^{4,29} Numerical demonstrations for beam, plate and shell structures in 2003³⁰ were quickly followed by experimental validation.^{31–33} Additional iFEM studies showed the use of fiber optic strain gage data to find structural anomalies³⁴ and to provide input for SHM.⁶ Real-time performance was demonstrated for SHM purposes in 2007 and 2009.^{35,36} Tessler and researchers from Politecnico di Torino applied iFEM to beam, truss and frame elements in 2011,^{37,38} dynamic loads with numerical simulations in 2012³⁹ and examined how sensor placement and mesh density affected accuracy in 2014.⁴⁰ Earlier iFEM work with simple shell elements turned out to have a severe limitations for sandwich and laminate structures, which was shown and addressed in 2013^{41,42} using Refined Zigzag Theory.^{43,44} In 2018, Tessler et al.⁴⁵ presented a linear incremental iFEM that could reconstruct GNL deformation for a cantilever plate and a clamped plate, both subjected to uniform pressure.

Kefal and Oterkus demonstrated iFEM using quadrilateral shell elements for maritime applications in 2016^{46–48} and other structures.^{49,50} Other variants of iFEM have been demonstrated on off-shore drilling platforms⁵¹ and wind turbines.⁵² In 2022, Li et al.⁵³ showed, together with Kefal, how a linearized iterative iFEM algorithm could be used to reconstruct the geometrically nonlinear deformation of a simple and a stiffened plate, both subjected to a uniform bending load. Until a few years ago, it was challenging to compare the merits of various shape sensing methods, as each method was demonstrated for different structures, materials, loads, boundary conditions, sensors and finite elements. While standardized Finite Element Method (FEM) problems have existed in the literature for many years for both linear⁵⁴ and geometric nonlinear⁵⁵ FEM algorithms, a standard set of linear shape sensing problems was presented only in 2019.⁵⁶ The first comparison of two different shape sensing algorithms in the same paper using these standard problems was presented in 2024.⁵⁷ So far, a standard set of geometric nonlinear shape sensing problems has not been documented in the open literature.

In addition to iFEM, another general approach for shape sensing is the Calibration Matrix algorithm (CM). Rudimentary versions of this algorithm date back to 1954⁵⁸ and relied on costly and time-consuming calibration processes that involve applying many individual loads at different points on the physical structure while recording the measurements made by the sensors attached to this structure. More recently, it has become feasible to replace this calibration process with a numerical alternative, where the effects of these individual loads on the structure and resulting sensor measurements are simulated instead, as demonstrated by Nakamura et al.⁵⁹ on a cantilevered thin flat plate subjected to a distributed pressure load. Numerical demonstrations of this algorithm on more complex three-dimensional structures were presented in various

papers.^{57,60–62} An experimental demonstration on an unmanned aerial vehicle (UAV) was presented by Colombo et al.⁵ The method was extended to reconstructing dynamic loads on beams, plates⁶³ and wings,⁶⁴ with an experimental demonstration for a cantilevered plate by Jiang et al.⁶⁵

In this study, a novel variant of the Calibration Matrix (CM) method is presented that can reconstruct the deformed shape and the applied loads for a structure undergoing geometric nonlinear (GNL) deformation. The method is demonstrated numerically for several new GNL shape sensing benchmark problems, which are based on well-known GNL benchmark problems for FEM algorithms that were presented by Sze et al.⁵⁵

The accuracy of this new GNL CM method is compared to two GNL shape sensing methods from the literature, specifically the linear incremental iFEM method presented by Tessler et al.⁴⁵ and the iterative GNL iFEM method by Li et al.,⁵³ to determine the most reliable method for finding the deformed shape of structures undergoing large deformations and rotations, while still experiencing only elastic strain.

The linear incremental iFEM method that was numerically evaluated and presented by Tessler et al.⁴⁵ had displacement reconstruction errors up to 6.3% under noise-free conditions. Two structures were analyzed: a wing-shaped cantilever plate and a clamped square plate, both of constant thickness and loaded by uniform distributed loads on their top surface. Each structure was instrumented with up to 200 simulated strain sensors on the top and bottom surfaces. The clamped square results, which are examined in more detail in Section ‘iFEM results from literature’, demonstrate that GNL iFEM can achieve an error of 6.3% with a sparse distribution of 29 strain sensors, which improves to 1.6% when using 100 sensors. The results then level out, improving no further when more (200, 2 for each element) sensors are used. For the wing-shaped cantilever plate, results are only presented for the case with the maximum number of sensors (88, 2 for each element), for which an error of 0.95% is reached.

The iterative GNL iFEM method that was numerically evaluated and presented by Li et al.⁵³ had displacement reconstruction errors up to 6.3% under noise-free conditions and up to approximately 6.6% with 3% noise, 7.2% with 5% noise and 7.7% with 10% noise for two structures: a cantilever plate and a stiffened cantilever plate, each loaded by a uniformly distributed load and instrumented with up to 2016 simulated strain sensors on the top and bottom surfaces. The results of Li et al. are also examined in more detail in Section ‘iFEM results from literature’ and also appear to reach their maximum accuracy with a relatively small number (60–180) of sensors, obtaining little to no improvement in accuracy from the use of larger numbers of sensors. The algorithm appears to be resilient in the face of random noise, as the average error only increases by approximately 0.17% for each additional

1% of noise. Similarly, this study will show how the accuracy of the GNL CM method is affected by the number of sensors and the level of noise for a variety of benchmark problem types.

Nonlinear benchmark problems

Seven geometric nonlinear benchmark problems were created to test the developed geometric nonlinear CM algorithm. These problems were based on existing benchmarks for geometric nonlinear FEM algorithms from the literature.^{55,66,67} Figure 1 shows the geometry of the selected problems and the direction(s) of the actual load(s). The actual load values for each problem can be found in Table 1. The calibration loads are applied in the same direction(s) as the actual load(s), in increments ranging from 0% to 100% of the actual load.

Additionally, Figure 2 shows the geometry of the problems selected from Tessler et al.⁴⁵ and Li et al.⁵³ The edges marked in red reflect the clamped boundary conditions of these structures.

The properties of each problem are listed in Table 1. Any system of consistent units can be used to analyze these benchmark problems; here, unprefix SI units were used. All structures were meshed with four node shell elements with reduced integration, specifically S4R in ABAQUS CAE, to match how they were analyzed by Sze et al.⁵⁵ Most of the materials are isotropic, with the exception of the ‘‘Laminated Shell’’ variants of the semi-cylindrical shell and the hinged cylindrical roof. While the displacements and rotations are large, no material nonlinearity is considered.

To provide input data to the GNL CM analyses, sensor data was needed. Strain and displacement sensors were used for this study, which were placed on the surfaces of the benchmark structures. The sensors were distributed in regular grid patterns, to facilitate reproduction of the sensor locations. All sensors were mounted on the top and bottom surfaces of the structure only, that is, no sensors were embedded within any of the structures. The same numbers of sensors were used for each structure: 16, 48, 96, 160, 240, 336, 448, 576 and 720, which are the total numbers of sensors resulting from $2 \times 2 \times 4$, $2 \times 3 \times 8$, $2 \times 4 \times 12$, $2 \times 5 \times 16$, $2 \times 6 \times 20$, $2 \times 7 \times 24$, $2 \times 8 \times 28$, $2 \times 9 \times 32$ and $2 \times 10 \times 36$ grid patterns, respectively.

In lieu of real sensor data, sensors were simulated based on the FEM results. This simulation process uses linear interpolation for both the displacement sensors and the strain sensors. Specifically, for the displacement sensors, it works by: (1) Finding the nearest node in the mesh; (2) Finding every element that includes that particular node; and (3) Interpolating the FEM displacements from every node of those elements based on their distance to the displacement sensor.

For strain sensors, the process is slightly different. It works by: (1) Finding the nearest Gauss node; (2)

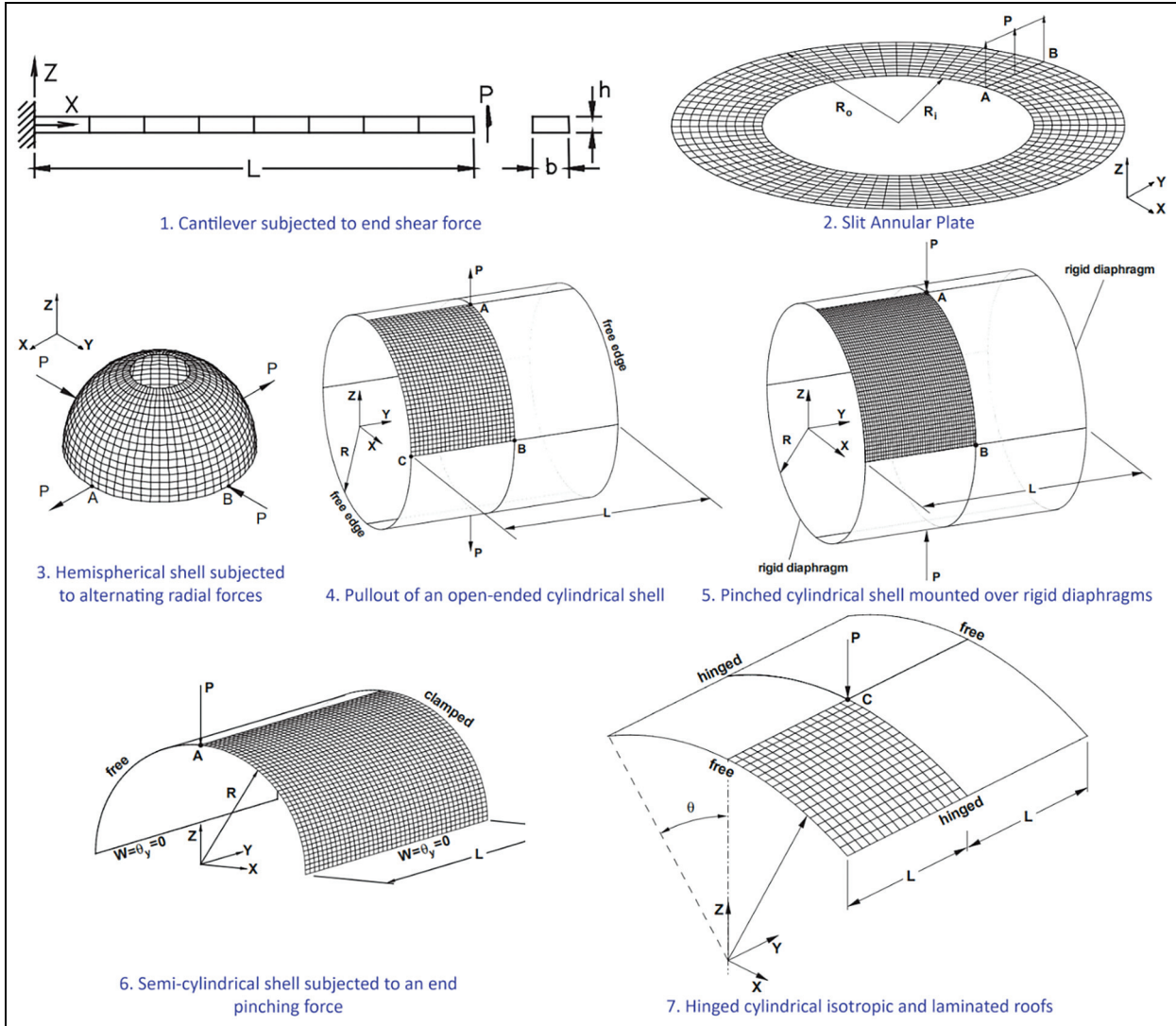


Figure 1. Selected benchmarks problems from Sze et al.⁵⁵

Finding the element that contains that Gauss node; and (3) Interpolating the FEM strains from the four closest Gauss nodes of that element based on their distance to the strain sensor.

The CM analysis also needs multiple load cases as input, as the actual load case is reconstructed as a linear combination of calibration load cases. The calibration load cases for the GNL CM algorithm are not just multiple load cases applied in different directions, but also multiple load cases applied in the same direction with a different load magnitude, including both positive and negative loads for some of the benchmark problems. The different load magnitudes of each calibration load case correspond to the loads applied at the various incremental steps that were made during the FEM analysis on the way to the final maximum load. The points and directions in which loads were applied are listed for each benchmark problem in Table 2:

Lastly, for each benchmark problem, a specific load case needed to be picked that would be reconstructed with the CM algorithm. The same choice was made for

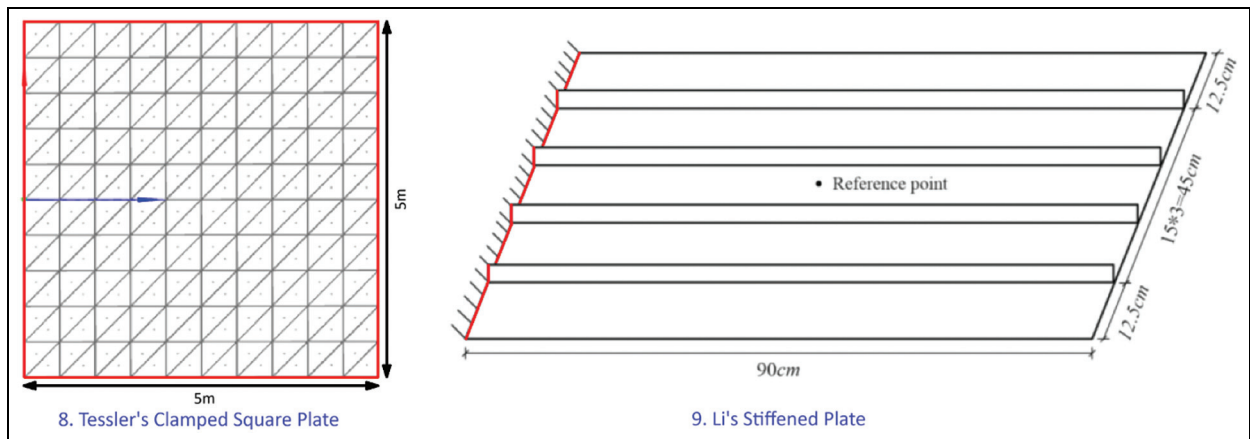
each of the benchmark problems: A load magnitude of 90% of the maximum load, as listed in Table 1 and applied in the same direction as shown in Figure 1.

Methodology

In this study, the GNL FEM analyses were carried out in ABAQUS CAE 2024 using S4R and S3R elements, which are shell elements with reduced integration, consistent with the approaches of the papers from which the problems were obtained. The results of these analyses were exported as RPT and TXT files, which were then imported into an in-house C++ programmed CM algorithm. The algorithm was developed specifically for running the GNL CM analyses on benchmark problems. The GNL FEM analyses determined how each benchmark structure responded to the applied loads, verifying the correct implementation of the standard problems and providing detailed displacement and strain results. These results are used both for simulating the sensor data that are needed as input for the

Table 1. Benchmark problem properties from Sze et al.,⁵⁵ Tessler et al.⁴⁵ and Li et al.⁵³

Problem number	Name	Element shape	E & (ν)	Dimensions	Load case & BCs	Mesh size
1	Cantilever subjected to end shear force	Regular	1.2×10^6 (0)	$L = 10$ $b = 1$ $h = 0.1$	$P_0 = EI/L^2 = 1$ $P_{\max} = 4P_0 = 4$	8 elements
2	Slit annular plate	Trapezoidal	21×10^6 (0)	$R_i = 6$ $R_o = 10$ $h = 0.03$	$P_{\max} = 0.8$ (force/length)	6×30 elements
3	Hemispherical shell subjected to alternating radial forces	Trapezoidal	6.825×10^7 (0.3)	$R = 10$ $h = 0.04$ $\theta_{\text{cutout}} = 18^\circ$	$P_{\max} = 400$	16×16 elements (one quarter)
4	Pullout of an open-ended cylindrical shell	Regular	10.5×10^6 (0.3125)	$R = 4.953$ $L = 10.35$ $h = 0.094$	$P_{\max} = 40,000$	24×36 elements (one eighth)
5	Pinched cylindrical shell mounted over rigid diaphragms	Regular	30×10^3 (0.3)	$R = 100$ $L = 200$ $h = 1$	$P_{\max} = 12,000$	40×40 elements (one eighth)
6	Semi-cylindrical shell subjected to an end pinching force	Regular	Isotropic: 2068.5 (0.3) Laminated: $E_L = 2068.5$ $E_T = 517.125$ $G_{LT} = 795.6$ (0.3) Laminations: [0°/90°/0°] or [90°/0°/90°]	$R = 101.6$ $L = 304.8$ $h = 3$	$P = 2000$	40×40 elements
7	Hinged cylindrical isotropic and laminated roofs	Regular	Isotropic: 3102.75 (0.3) Laminated: $E_L = 3300$ $E_T = 1100$ $G_{LT} = 660$ (0.25) Laminations: [0°/90°/0°] or [90°/0°/90°]	$R = 2540$ $L = 254$ $\theta = 0.1$ rad	$P_{\max} = 3000$	16×16
8	Tessler's clamped square plate	Triangular	200×10^9 (0.2)	$h = 12.7$ $L = 10$ $b = 1$ $h = 0.1$	Uniform pressure: 5 MPa	$10 \times 10 \times 2$ triangles
9	Li's stiffened plate	Regular	206×10^9 (0.3)	$L = 0.9$ $b = 0.7$ $h = 0.0125$ rib height = 0.03	Uniform pressure: 1 MPa	1152 elements

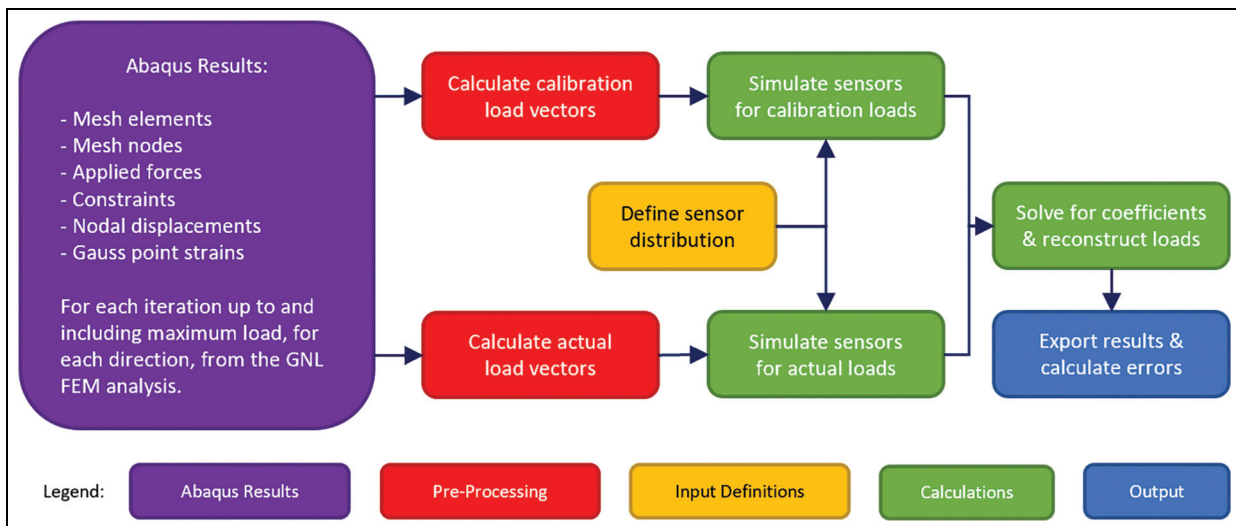
**Figure 2.** Selected problems from Tessler et al.⁴⁵ and Li et al.⁵³

GNL CM analysis and to evaluate the accuracy of the reconstructed loads and deformed shapes.

Figure 3 shows a flowchart of the GNL CM method, which uses the GNL FEM results from ABAQUS CAE

Table 2. Load directions used for the calibration load cases for each benchmark problem.

Problem number	Benchmark problem	Load directions	Description
1	Cantilever subjected to end shear force	X, Y, Z	Loads applied at tip in positive X, Y and Z directions.
2	Slit Annular Plate	X, Y, Z	Loads applied along line A-B in positive X, Y and Z directions.
3	Hemispherical shell subjected to alternating radial forces	XY−−, XY−+, XY+−, XY++	Pushing inward (−) or pulling outward (+) at points A and B along the X and Y directions, respectively.
4	Pullout of an open-ended cylindrical shell	Z−, Z+	Loads applied at point A in positive and negative Z directions.
5	Pinched cylindrical shell mounted over rigid diaphragms	Z−, Z+	Loads applied at point A in positive and negative Z directions.
6	Semi-cylindrical shell subjected to an end pinching force	X, Y, Z	Loads applied at point A in positive X, Y and Z directions.
7	Hinged cylindrical isotropic and laminated roofs	Z−, Z+	Loads applied at point C in positive and negative Z directions.
8	Tessler's Clamped Square Plate	Y−	Uniform pressure applied on top surface in negative Y direction.
9	Li's Stiffened Plate	Y−, Y+	Uniform pressure applied to bottom surface in positive and negative Y directions.

**Figure 3.** Methodology flowchart showing the GNL Calibration Matrix method based on the ABAQUS CAE GNL finite element results.

to simulate displacement and strain sensor data for the actual load cases and the calibration load cases. For some of the results, the simulated sensor data was adjusted by 1%, 3%, 5% or 10% noise, for both the calibration loads and the actual loads. This noise was uniformly distributed and randomly generated using the mt19937_64 Mersenne twister engine from the Microsoft Visual C++ standard library, version 14.36.32532. New random noise values were generated and applied to each sensor data point and for each load case.

The calibration load cases include loads applied in different directions and at a range of magnitudes, from small loads that induce only linear deformation to large loads that induce geometrically nonlinear deformation.

The algorithm then solves for the calibration load case coefficients, that is, how much does each calibration load case contribute to the actual load case that is being examined. The applied loads of each calibration load case are multiplied by these calibration load case coefficients and then summed to reconstruct an approximation of the actual load case.

The reconstructed load case is then compared to the actual load case to determine the reconstruction error, for example, how much do the reconstructed applied forces differ from the actual applied forces. Similarly, the reconstructed displacements and strains are compared to their actual values. These errors and the other results are then exported for further processing.

Whereas the result of the FEM analysis is a prediction of the structural response, the result of the CM analysis is instead an estimate of the structural response, as the number of experimental measurements on which it is based is limited. The results of both analyses are stored and compared to determine the error between the FEM results and the CM results. As such, a smaller error indicates a more accurate estimate of the deformation.

Since sensor data are not available for every point throughout the structure, the CM results are expected to not exactly match the FEM results, even for the results based on noiseless sensor data. In real life, sensors would also suffer from various sources of further inaccuracies, such as noise, bias, inaccurate placement or misalignment. In this study, no experimental errors are present because the sensor data was simulated using linear interpolation of the FEM results. The noiseless CM results, which are still affected by numerical and interpolation errors, serve as a proof of concept of the degree of accuracy that the GNL CM method can achieve under ideal conditions. The CM results based on noisy data give an impression of how the accuracy is affected by more realistic and suboptimal conditions.

To simulate the sensor data for a particular strain sensor, the element was found that contained the Gauss nodes that are closest to that sensor. Linear interpolation was then carried out using the locations and strain values of all the Gauss nodes of that element, based on the distance between the strain sensor and each Gauss node. The simulation of displacement sensors followed a similar process by finding the elements that include the node that was closest to that sensor. Linear interpolation was then carried out using the locations and displacement values of all the nodes of those elements, based on the distance between the sensor and each node. These methods of interpolation produced the most accurate results for linear iFEM and CM algorithms^{56,57} and were not modified for the nonlinear method presented here to ensure consistency.

Theory

Just like the linear variant in de Mooij and Martinez,⁵⁷ the geometric nonlinear calibration matrix method (GNL CM) works by assuming that the actual load case can be reconstructed by a linear combination of calibration load cases. The difference is that these calibration load cases include not just loads applied at various locations and in multiple directions, but also much larger loads that cause GNL deformation of the structure. The calibration load cases with these larger loads can be applied at the same location and in the same direction as other calibration load cases, as long as the load magnitude is different.

The resulting displacement and strain distributions for the reconstructed load case are a linear combination

of the displacement and strain distributions that resulted from the individual calibration load cases, that is, using the same calibration load case coefficients that are needed to combine the calibration loads to reconstruct the actual load. Similarly, the (simulated) displacement and strain measurements for the actual load case will also be equivalent linear combinations of the measurements that were made for the calibration load cases, provided that the same sensor locations and orientations are used.⁵⁷

As for the linear method, when the actual load case can be reproduced as a linear combination of the calibration load cases and there are at least as many sensor data points as calibration load cases, then it will be possible for this method to make accurate reconstructions of actual load cases.^{57,59} In general, this principle of superposition does not apply to GNL deformations: scaling and adding together small linear applied loads and the corresponding deformations generally does not result in an accurate picture of the nonlinear deformation that would occur for the equivalent nonlinear applied loads. However, as will be demonstrated here, when the collection of calibration load cases includes nonlinear loads that result in deformations that are sufficiently similar to those resulting from the actual load case, the relationship between these similar loads and the actual nonlinear load case will be approximately linear.

To be sufficiently similar, this representative set should include calibration load cases for both linear and nonlinear behavior of the structure. Linear interpolation between adjacent calibration loads is utilized to develop a piecewise linear approximation of the load curve(s). The steps along this load curve(s) should be sufficiently small: sufficient here meaning that the error between this piecewise linear approximation and the actual GNL load curve(s) is smaller than the desired level of accuracy. Here, this level of accuracy was set to 0.5% for the residual force and 1% for the displacement correction, which matches the default ABAQUS settings for nonlinear problems.

This piecewise linear approximation is not an approximation of a 2D load curve that plots the magnitude of a single measurement s of displacement (or strain, or other measurements) against the magnitude of a single applied load. Instead, it approximates one or more load curves through an n -dimensional space, where n is the number of sensor datapoints, as shown in equation (1):

$$\mathbf{s} = (s_1, s_2, \dots, s_n) \in \mathbb{R}^n \quad (1)$$

For instance, when a distributed fiber optic strain sensor is used that measures strains at 1000 points along its length, the set of strain measurements for each individual calibration load case i corresponds to a particular

point $s_i^{calibration}$ in a 1000-dimensional space, as shown in equation (2):

$$s_i^{calibration} = (s_{i-1}^{calibration} \dots s_{i-1000}^{calibration}) \in \mathbb{R}^{1000} \quad (2)$$

When that same fiber optic strain sensor is used to obtain strain measurements for the actual load case, the result is another point s_i^{actual} in that same 1000-dimensional space, as shown in equation (3):

$$s_i^{actual} = (s_{i-1}^{actual} \dots s_{i-1000}^{actual}) \in \mathbb{R}^{1000} \quad (3)$$

The calibration matrix method effectively determines which of the calibration load case points are closest to the actual load case point in this 1000-dimensional space, and performs a barycentric interpolation to find how much each calibration load case contributes to the actual load case: the closer that a calibration load case point is to the actual load case point, the higher its contribution.

The central assumption of the method is that the calibration load case coefficients that produce an accurate reconstruction of sensor measurements will also produce an accurate reconstruction of displacements, strains and applied loads. This assumption will hold (approximately) if the following condition is met: the actual load case point is within the region of validity formed by the collection of calibration load case points in the 1000-dimensional space.

To illustrate this, Figure 4 below shows a simplified and fictitious example, where strains are measured at three points instead of 1000. This makes each load case a point in a 3-dimensional space. The strain measurements for three different load curves are plotted in Figure 4(a). The region of validity in which interpolation between these measurements is assumed to result in accurate reconstructions is shown in Figure 4(b), along with three example points corresponding to the measurements for actual load cases. Point 1 is within the region, which is expected to result in a reconstruction with the desired level of accuracy. Point 2 is outside the region but still close to it, which is expected to give a result that is still reasonable, but with a lower accuracy. Point 3 is far outside the region, which is expected to result in a low degree of accuracy.

The reconstruction is approximate even when the condition of being within the region of validity is met because the set of calibration load cases are only a piecewise linear approximation of this nonlinear space: the actual relationship between applied loads and displacements and strains is still nonlinear.

A collection of calibration load cases that span the entire expected range of GNL deformations will effectively provide a piecewise linear basis for reconstructing any actual load case that is within or close to this range.⁶⁸ The quality of this approximate approach will

be evaluated here for various nonlinear benchmark problems.

For this study, it was ensured that it would be possible to reconstruct the actual load cases through linear combination of calibration load cases. This was achieved by defining the actual load case as a load equal to 90% of the maximum load. The displacements and strains of this actual load case were then found through a linear combination of the two ABAQUS CAE load increments that were evaluated for a total applied load just below and just above the 90% mark. For example, two calibration load cases at 87% and 92% would be multiplied by 0.4 and 0.6 respectively and then added together to reach an actual load case of 90%.

For each benchmark problem, GNL FEM analyses were carried out for each load case, both calibration and actual, to determine the resulting displacements and strains throughout the structure. Ideally, an actual load (or load distribution) f^{actual} can be reconstructed by a linear superposition of m calibration loads (or load distributions) $f_k^{calibration}$, each multiplied by the corresponding calibration load case coefficient c_k^{actual} , as shown in equation (4):

$$f^{actual} = \sum_{k=1}^m c_k^{actual} f_k^{calibration} \quad (4)$$

Similarly, with the same calibration load case coefficients, the actual displacement distribution u^{actual} and actual strain distribution E^{actual} can be constructed from the calibration displacement distributions $u_k^{calibration}$ and calibration strain distributions $E_k^{calibration}$, respectively, as shown in equations (5) and (6):

$$u^{actual} = \sum_{k=1}^m c_k^{actual} u_k^{calibration} \quad (5)$$

$$E^{actual} = \sum_{k=1}^m c_k^{actual} E_k^{calibration} \quad (6)$$

It is not possible in practice to measure every displacement and every strain that occurs throughout a structure. Instead, only a select number of values are measured at certain points on the structure. As these measured displacement and strain values will be a subset of the complete displacement and strain distributions, their values for the actual and calibration load cases will follow the same relationship as shown in equations (4)–(6). As these measurements can be made for both the actual and each of the calibration load cases, it would be possible for equations (5) and (6) or a combination of both, to be solved for the calibration load case coefficients c^{actual} , as if the measurements were perfect. In actuality, the coefficients that are determined with the calibration matrix method will approximate the true coefficients, $c^{reconstructed}$, due to measurement errors (for real sensor data) or interpolation errors (for simulated sensor data).

Different types of sensor data, or even a mix of different types, can be used as input for the CM algorithm,

as long as the collection of sensors and their locations and orientations are consistent between all the load cases. This makes the algorithm flexible and robust, as it can adapt to sensors failing by leaving out the missing or invalid sensor data components for each load case. For general applicability, all types of sensor data components s_k will be lumped together here into a single sensor vector s as shown in equation (7), which can be determined for both actual and calibration load cases:

$$s = [s_1 \dots s_n]^T \quad (7)$$

In equation (7), n is the number of measured or simulated sensor values. For displacement data, this would mean that s_1 through s_3 are the three displacement components u , v and w of the first displacement sensor, and so on for the subsequent triples of sensor values for the corresponding sensors. For strain data, it would mean that s_1 through s_6 are the six strain components of the first strain sensor, and so on for the subsequent sextuples for the corresponding sensors. For the simulated sensor measurements that are used here, the displacements or strains that would have been measured are determined with barycentric interpolation of the FEM results from the nearest four mesh nodes (for displacements) or the nearest four Gauss nodes (for strains).

The sensor simulations based on m calibration load cases result in a set of m calibration sensor vectors $s_k^{calibration}$. Similarly, the same type of simulation for the actual load case results in an actual sensor vector s^{actual} . Similar to equations (4)–(6), the actual sensor vector should be related, by approximation, to the calibration sensor vectors by the reconstructed calibration load case coefficients $c^{reconstructed}$ as shown in equation (8):

$$s^{actual} \cong \sum_{k=1}^m c_k^{reconstructed} s_k^{calibration} \quad (8)$$

Rewriting equation (8) in matrix notation results in equation (9):

$$s^{actual} \cong S c^{reconstructed} = \begin{bmatrix} s_{1-1}^{calibration} & \dots & s_{m-1}^{calibration} \\ \vdots & \ddots & \vdots \\ s_{1-n}^{calibration} & \dots & s_{m-n}^{calibration} \end{bmatrix} \begin{Bmatrix} c_1^{reconstructed} \\ \vdots \\ c_m^{reconstructed} \end{Bmatrix} \quad (9)$$

In equation (9), S is an n -by- m sensor matrix containing the n sensor values from each of the m calibration sensor vectors. This matrix is generally not invertible because it is not square. By making sure that the number of sensor values n is larger than the number of calibration load case coefficients m , this system

becomes overdetermined. For an overdetermined system of linear equations, the optimal approximate solution can be found using ordinary least squares,⁶⁹ as shown in equation (10):

$$c^{reconstructed} = (S^T S)^{-1} S^T s^{actual} \quad (10)$$

The reconstructed calibration load case coefficients $c^{reconstructed}$ should approximately equal the actual calibration load case coefficients c^{actual} , as shown in equation (11), provided that a sufficient number of sensors are used and that the set of calibration load cases span a range of loads that includes or comes close to the actual load:

$$c^{reconstructed} \cong c^{actual} \quad (11)$$

To evaluate the quality of the approximation, the relative two-norm error for the coefficients ϵ_{coeff} can be determined with equation (12). Smaller errors indicate a more accurate reconstruction:

$$\epsilon_{coeff} = \frac{\sqrt{\sum_{k=1}^m (c_k^{reconstructed} - c_k^{actual})^2}}{\sqrt{\sum_{k=1}^m (c_k^{actual})^2}} \quad (12)$$

The reconstructed calibration load case coefficients $c^{reconstructed}$ can be used to determine the reconstructed loads $f^{reconstructed}$ from the known calibration loads $f_k^{calibration}$, which should approximately equal the actual loads f^{actual} , as shown in equation (13):

$$f^{reconstructed} = \sum_{k=1}^m c_k^{reconstructed} f_k^{calibration} \cong f^{actual} \quad (13)$$

Similar to the error for the coefficients, a relative two-norm error ϵ_{force} can be calculated for the load reconstruction, as shown in equation (14):

$$\epsilon_{force} = \frac{\sqrt{\sum_{k=1}^m (f_k^{reconstructed} - f_k^{actual})^2}}{\sqrt{\sum_{k=1}^m (f_k^{actual})^2}} \quad (14)$$

Similarly, two-norm errors can be calculated to determine the quality of the reconstruction of the displacements, strains, or sensor values.

In an attempt to reduce the magnitude of these errors, a variant of the method was developed that applied Tikhonov regularization. Similar to the approach by de Mooij et al.⁵⁶ for linear iFEM, this type of regularization is intended to bias toward smaller solutions, as shown in equation (15). In this equation, C_0 is the regularization weight and I is the identity matrix:

$$c^{reconstructed} = (S^T S + C_0 I)^{-1} S^T s^{actual} \quad (15)$$

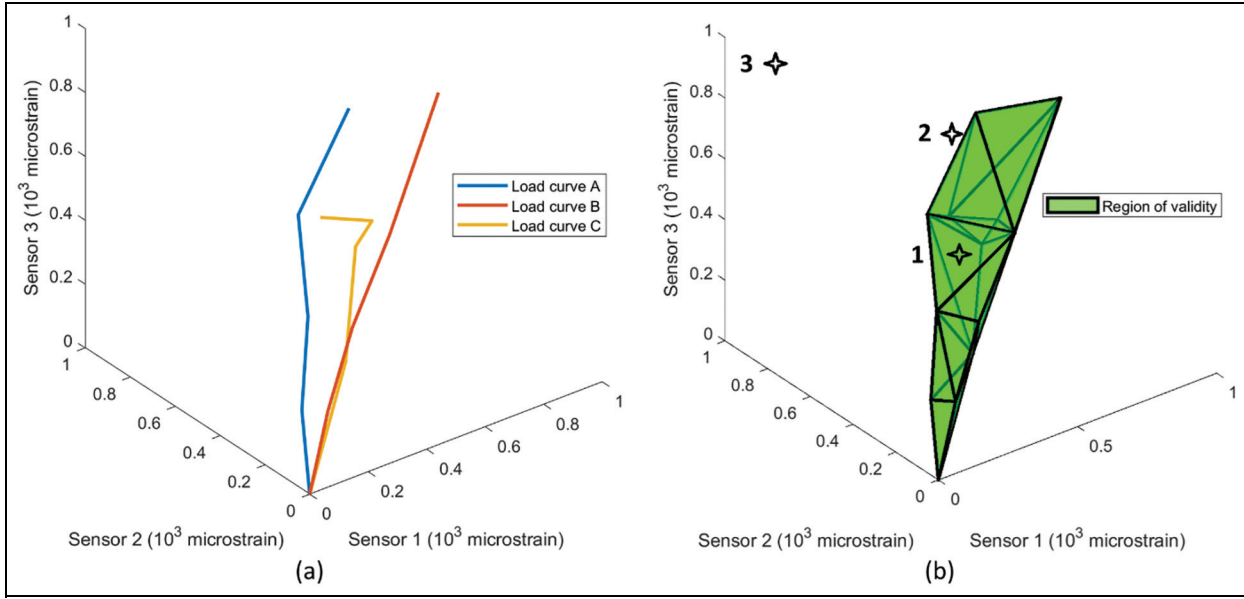


Figure 4. (a) Three fictitious calibration load curves plotted in a 3-dimensional strain space and (b) region of validity formed by interpolating between the measurements of the calibration load curves, along with three example points representing actual load cases inside the region of validity, close to it and far from it.

Results and discussion

Calibration matrix results

Using the methodology and theory as described in Sections ‘Methodology’ and ‘Theory’, the CM method uses the FEM results of each benchmark problem to determine the calibration and actual load vectors, $f_k^{calibration}$ and f_k^{actual} , to simulate the strain and/or displacement sensors for each load case, to construct the sensor matrix S and the actual sensor vector s^{actual} . These are used to determine reconstructed calibration load case coefficients $c^{reconstructed}$, which are combined with the corresponding calibration load vectors $f_k^{calibration}$ to find the reconstructed load vector $f^{reconstructed}$ and the reconstructed displacements and strains throughout the mesh. The reconstructed forces and displacements were compared to their actual counterparts to determine the reconstruction error, which enabled a comparison to the GNL iFEM algorithms from the literature.

The FEM result for the actual load of problem 1, the cantilever beam loaded in shear, can be seen in Figure 5(a), colored according to the displacement along the z -axis. Figure 5(b) shows the corresponding CM reconstruction along with the 720 displacement sensors that produced the input data. The sensors are colored according to their displacements as well, using the same color scale, but with a greater reflectivity to stand out against the backdrop of the structure itself. Note that all the actual and reconstructed load cases were performed at 90% of the maximum load, so the deformation is smaller than in the FEM results of Sze et al. This percentage was chosen instead of 100% to make it possible for a reconstruction to both underestimate or overestimate the load.

When the FEM and CM results are compared, the deformed structures and full field distributions are nearly indistinguishable to the naked eye, for both this benchmark problem and the subsequent ones. As such, these will not be shown here for the remaining problems but can instead be seen in the original Sze et al.⁵⁵ paper or in the research data published for this paper on the 4TU Research Data website through the links provided in the Data availability section (<https://data.4tu.nl/datasets/7dc3b79a-5571-4337-bde4-19c0622a5e6b>).

Instead, to assess the differences between the FEM and CM results more precisely, the reconstructed force and displacement vectors were compared to the actual force and displacement vectors using the two-norm force error ε_{force} shown in equation (14). The resulting error values were plotted against the number of sensors, as shown in Figure 6, for three different variants of the CM algorithm: a variant that uses only strain sensor data, one that uses both strain and displacement sensor data and another that uses only displacement sensor data.

Figure 6 shows the force reconstruction errors on the left and the displacement reconstruction errors on the right for the first benchmark problem, the cantilever beam loaded in shear. For each type of error, there are 15 curves grouped into three groups of 5. The three groups correspond to the type of sensors that were used: blue curves are based on strain sensors only, red curves are based on both strain and displacement sensors, and yellow curves are based on displacement sensors only. The five curves within each of these groups correspond to a particular level of noise that was applied to the sensor data, each marked by a unique symbol: circles for 0% noise, pluses for 1% noise,

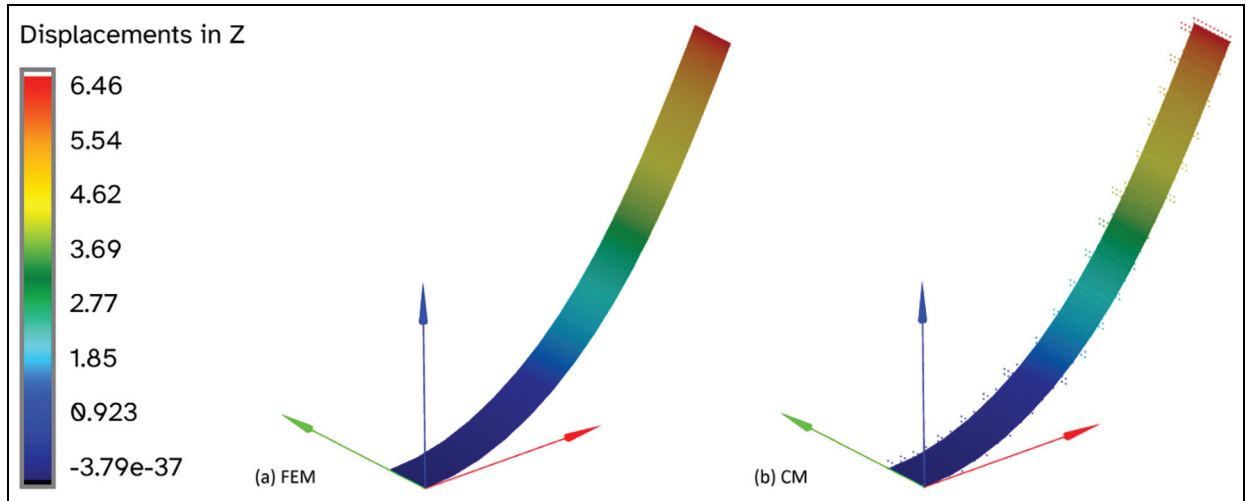


Figure 5. Benchmark problem I cantilever shear at 90° load: (a) FEM result colored by displacements in Z and (b) CM reconstruction colored by displacements in Z, with 720 displacement sensors.

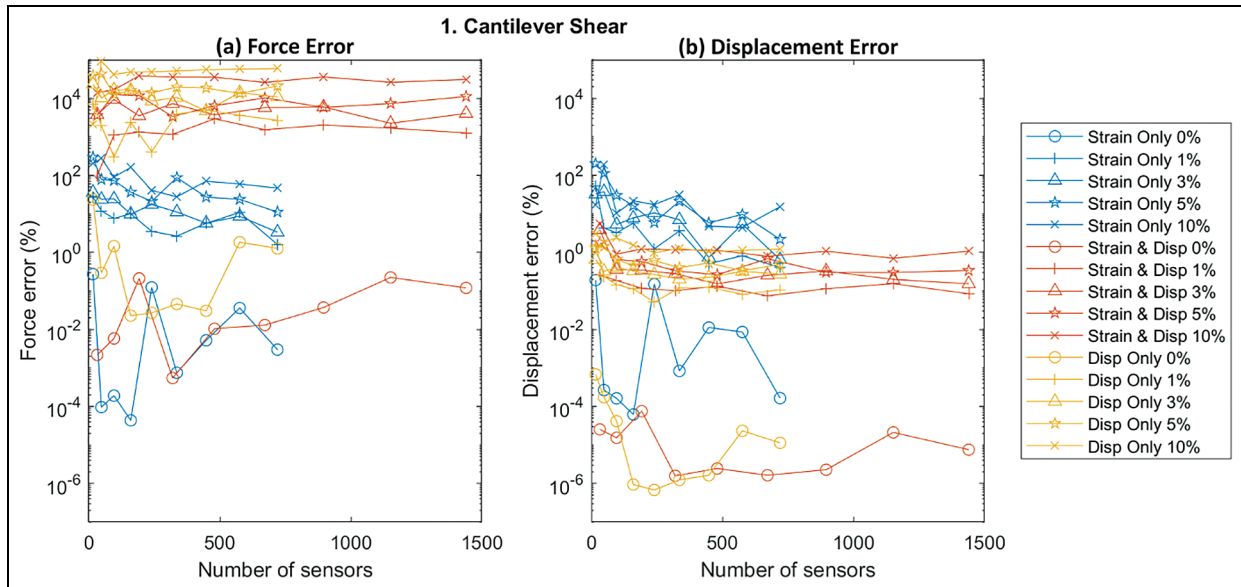


Figure 6. Force (a) and displacement (b) errors plotted versus the number of sensors for problem I, cantilever shear, for reconstructions based on only strain sensors, both strain and displacement sensors and only displacement sensors.

triangles for 3% noise, stars for 5% noise and $\times$'s for 10% noise. The curves for the intermediate noise levels generally lie between those for 0% and 10% noise for all the problems. To make the results easier to read, Figure 7 shows the same results with the intermediate curves omitted as to show only the upper and lower bounds. Later figures for the other benchmark problems follow the same format.

As can be seen in Figure 6(a), for the noiseless data, the CM algorithm quickly reaches and stays at a reconstructed force error of around 2% or less. Figure 6(b) shows a reconstructed displacement error of around 0.2% or less, with just 48 sensors of either type or a

combination of both types of sensors. However, even for just 1% of noise, both types of error jump up by more than an order of magnitude. Further increases in the level of noise result in smaller increases of the reconstruction errors. For this first problem, any level of noise significantly deteriorates the algorithm's performance to the point that noisy displacement data is not suitable for force reconstruction. Displacement reconstruction is more resilient, still achieving an error of around 1% when the data is contaminated by 10% noise, reaching good results with as few as 96 displacement sensors. Noisy strain data can be compensated for by using a larger number of sensors: with 5% noise,

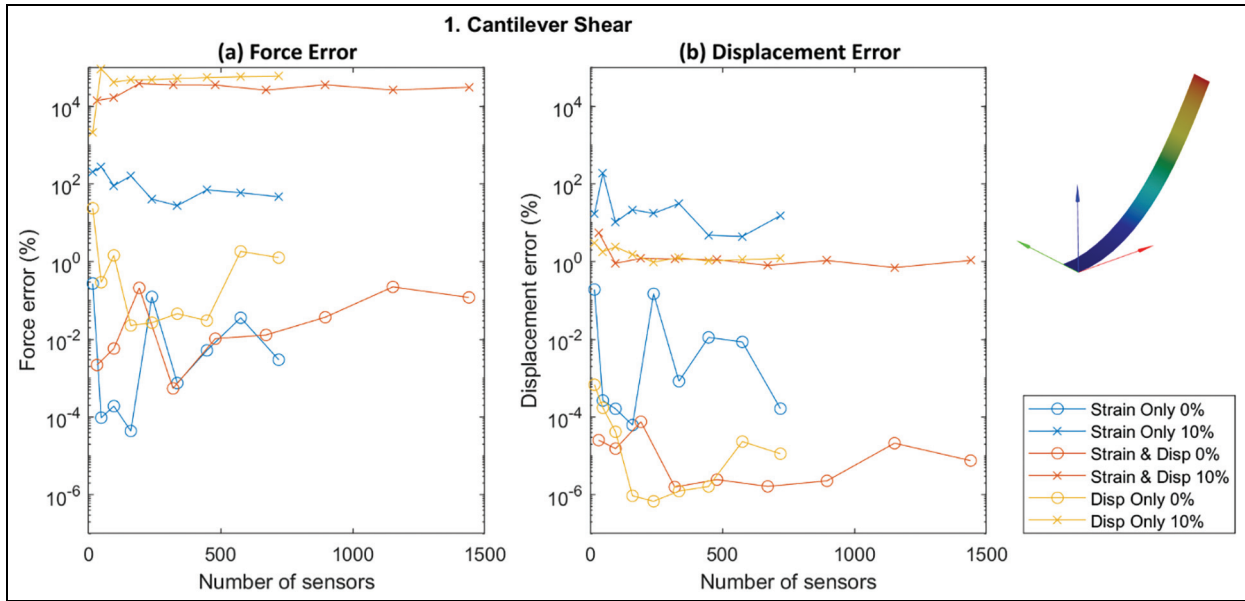


Figure 7. Force (a) and displacement (b) errors versus the number of sensors for problem 1, cantilever shear, using only strain sensors, both strain and displacement sensors and only displacement sensors, for 0% and 10% noise.

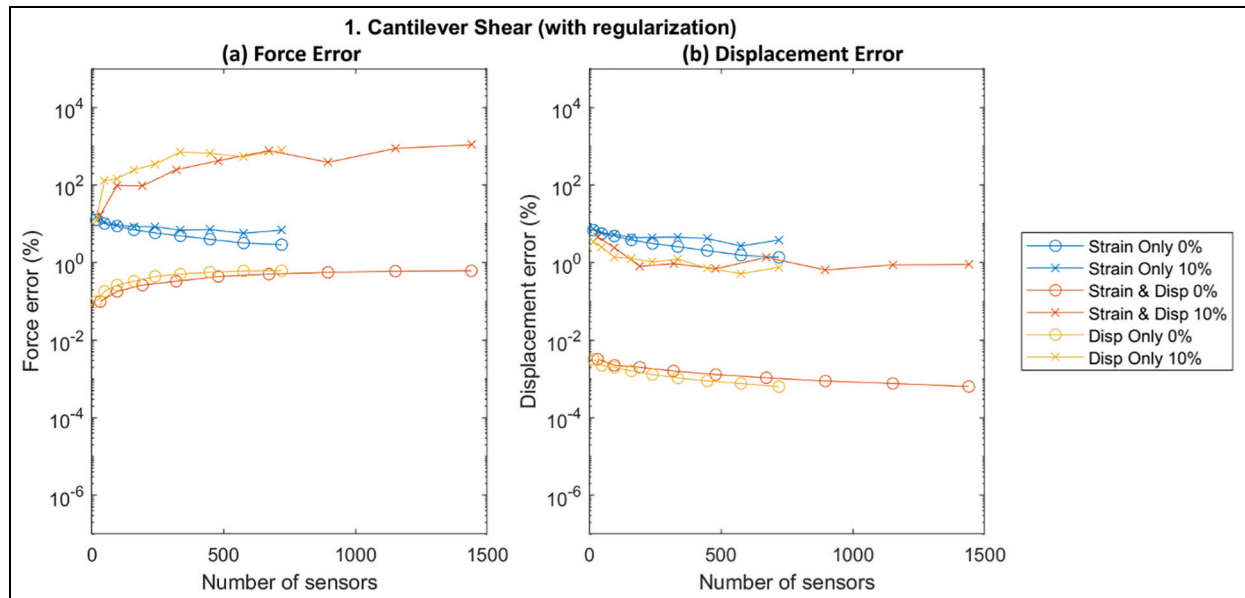


Figure 8. Force (a) and displacement (b) errors versus the number of sensors for problem 1, cantilever shear, using only strain sensors, both strain and displacement sensors and only displacement sensors, for 0% and 10% noise, with regularization.

force errors of 11% and displacement errors of 2% can be reached by increasing the number of strain sensors to 720.

All of the problems were also analyzed with the CM algorithm variant that includes Tikhonov regularization, as described in equation (15), with the regularization weight C_0 set to 10^{-4} . The results of this alternative approach are shown in Figure 8 for problem 1.

While regularization can lower the force reconstruction errors when using a small number of displacement sensors, reduce the variability of errors across different

numbers of sensors and slightly lower the errors for noisy data, it also causes a significant deterioration of the results when there is little to no noise, particularly when using strain data, and provides little to no benefit when the number of sensors is large. For brevity, this graph will be omitted for the other problems, as these results are quite representative for the effects of regularization.

Figure 9 shows the errors for problem 2, the slit annular plate. The force error based on displacement sensors only or on both types of sensors combined is

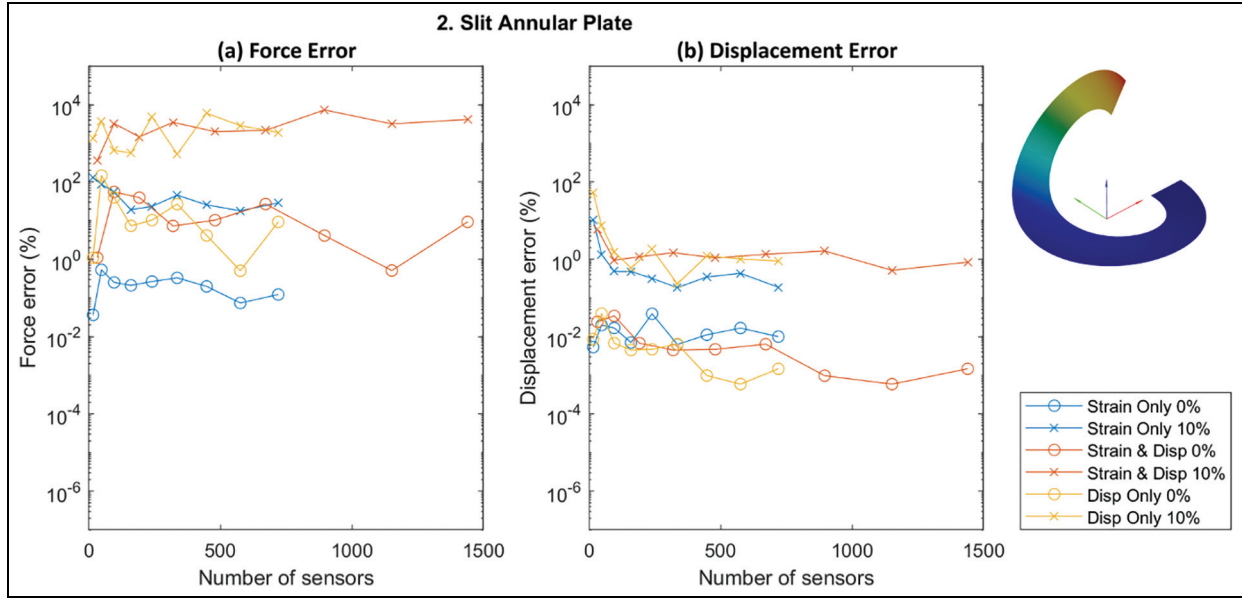


Figure 9. Force (a) and displacement (b) errors versus the number of sensors for problem 2, slit annular plate, using only strain sensors, both strain and displacement sensors and only displacement sensors, for 0% and 10% noise.

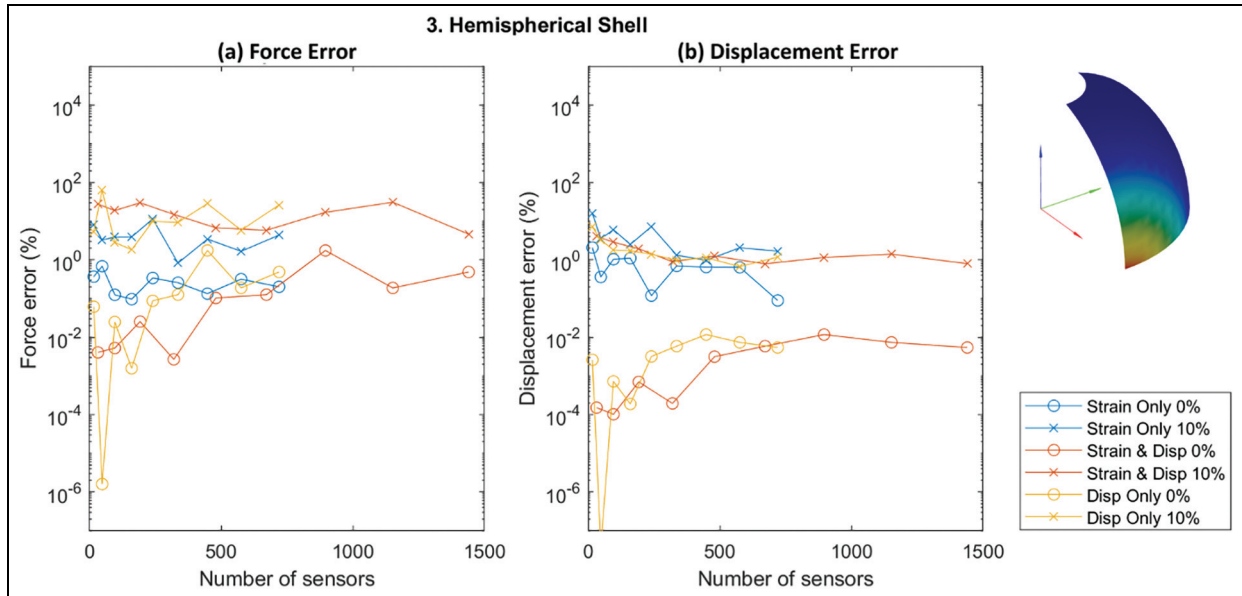


Figure 10. Force (a) and displacement (b) errors versus the number of sensors for problem 3, hemispherical shell, using only strain sensors, both strain and displacement sensors and only displacement sensors, for 0% and 10% noise.

higher than for problem 1 for the noiseless results but still reaches errors below 10% as the number of sensors is increased. The force errors for these types of data are lower than for problem 1 when noise is introduced but still become prohibitively large. The results are better for the force error based on strain sensors and for the displacement error based on any sensor type: the noiseless results reach errors that are just fractions of a percent. Unlike problem 1, these results are only gradually deteriorated by noise: force errors based on 336 or more strain sensors are 10% or lower, and the

displacement errors reach approximately 2% for 96 sensors of any type even at the highest noise levels and continue to gradually improve as the number of sensors grows. Overall, these results are more consistent than those of problem 1, converge to similar levels of quality as the number of displacement sensors increases, and reach a better quality when using strain sensors.

Figure 10 shows the errors for problem 3, the hemispherical shell, which are both 2% or lower for any number of sensors when undisturbed by noise. While noise still deteriorates the results, the effect is far less

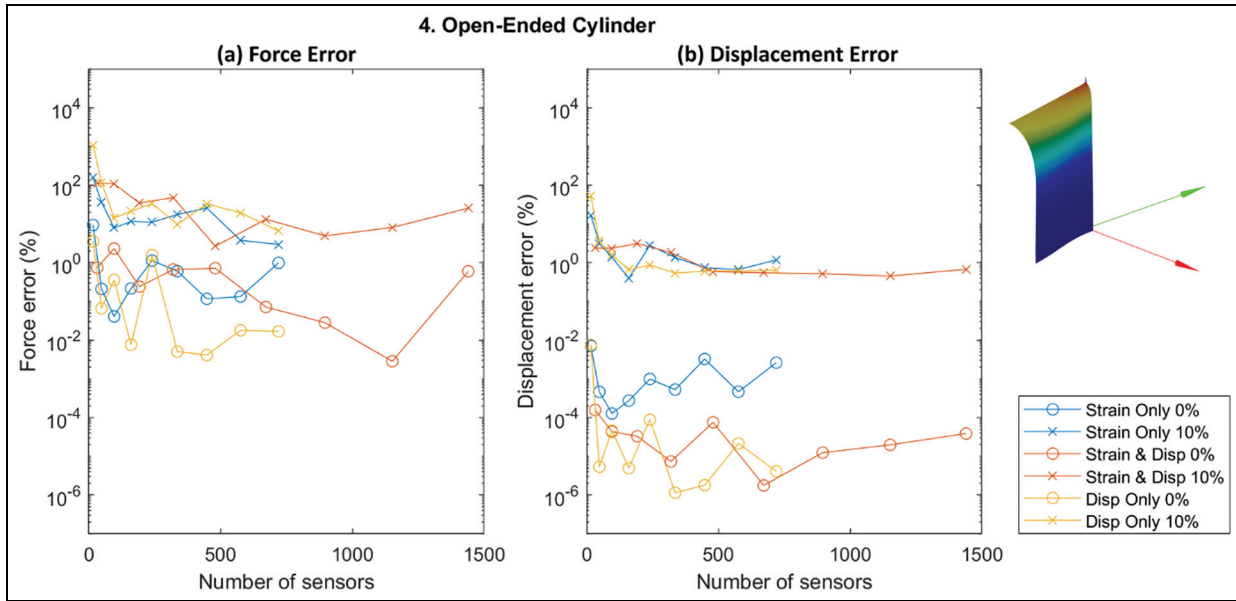


Figure 11. Force (a) and displacement (b) errors versus the number of sensors for problem 4, open-ended cylinder, using only strain sensors, both strain and displacement sensors and only displacement sensors, for 0% and 10% noise.

pronounced than for the previous two problems: for 96 sensors or more, the force errors drop and stay below 40% when using displacement data and below 10% when using strain data, even at the highest levels of noise. Similarly, the displacement errors drop and stay below 3% when using displacement data and below 10% when using strain data and continue to decrease to 1% as the number of sensors is increased. Overall, these results are better than those of problems 1 and 2, especially for force reconstruction based on noisy data.

Figure 11 shows the errors for problem 4, the open-ended cylinder, showing a pattern similar to problem 3. The noiseless results show force errors below 10% that decrease to below 1% for all approaches as the number of sensors increases, on par with the force errors of the hemispherical shell. The noiseless displacement errors are even lower, at 0.01% or less, reaching a level of quality similar to that of the slit annular plate. Like problem 3 and unlike problems 1 and 2, the results based on noisy data are relatively close to the noiseless results. The force errors are higher than for problem 3, reaching 25% or less when using noisy displacement data and 14% or less when using noisy strain data. The displacement reconstruction quality is similar, reaching errors of around 1% or less at the highest levels of noise.

Figure 12 shows the results for problem 5, the pinched cylindrical shell. The error curves show a rapid descent for all variants, reaching errors mostly below 0.1% under noiseless conditions. At the highest noise level, the force errors drop below 20% at 96 sensors or more, and below 10% at 320 sensors or more. Similarly, the displacement errors drop below 5% at 96 sensors or more and eventually reach errors around 1%. These results are similar to problems 3 and 4, with

better force reconstruction results and better results overall when using strain sensors only.

Figure 13 shows the errors for problem 6, the pinched semicylinder, for the isotropic material. The results are similar between the isotropic and laminate variants, so the laminate results are not shown separately here for brevity. Like problem 5 and unlike problems 1–4, both sets of error curves are clustered closely together, showing consistent results for different types of sensors when using data undisturbed by noise. Like problems 3–5, the noisy results are again relatively close to the noiseless results. The errors rapidly decline and stabilize as the number of sensors is increased. Most of the noiseless errors converge around 0.01%, except the force error for the isotropic variant, which hovers around 1% for the variants using only displacement sensors or both types of sensors. For the isotropic variant, the noisy results for the force reconstruction drop and stay below 3% when using 96 strain sensors or more but stay high when using displacement sensors or both sensor types, only reaching 10% when using the maximum number of sensors and noise levels of 5% or less. The displacement errors at any level of noise drop below 1% at 48 strain sensors and below 4% when using 96 displacement sensors or both types of sensors. Similar levels of quality result under noisy conditions for both laminate variants of the structure.

Figure 14 shows the errors for problem 7, the hinged cylindrical roof, for the isotropic material. The results are again similar between the isotropic and laminate variants, so the laminate results are not shown separately here. Like problem 5 and 6, the curves are again clustered closely together. Under noiseless conditions, almost all errors are below 0.01%, even lower than for the other problems. The noiseless trends are again

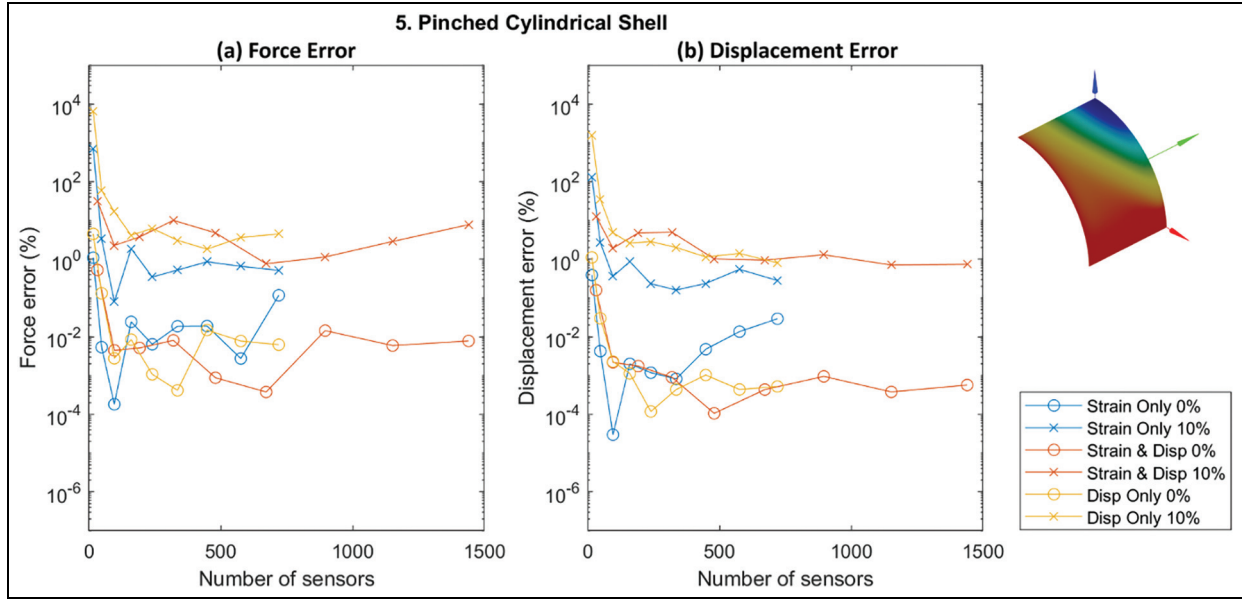


Figure 12. Force (a) and displacement (b) errors versus the number of sensors for problem 5, pinched cylindrical shell, using only strain sensors, both strain and displacement sensors and only displacement sensors, for 0% and 10% noise.

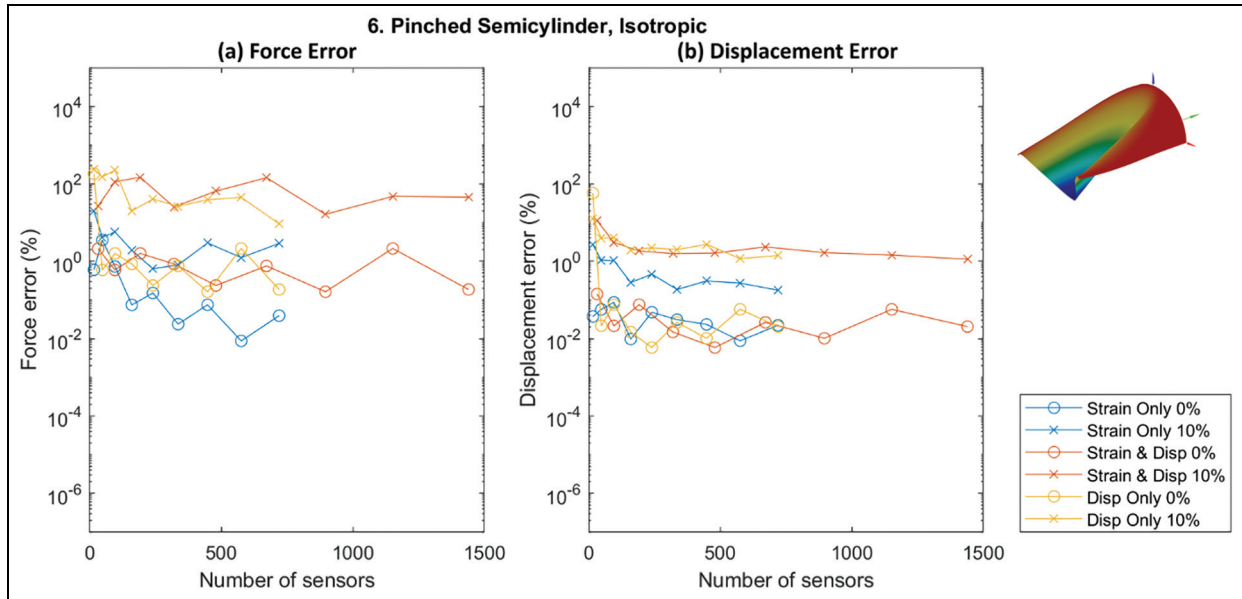


Figure 13. Force (a) and displacement (b) errors for problem 6, pinched semicylinder (isotropic), using only strain sensors, both strain and displacement sensors and only displacement sensors, for 0% and 10% noise.

similar for each material, except that the errors based on only strain sensors are slightly higher for the 90/0/90 laminate. The results based on noisy data are similar to those of problem 6, except that the force errors are lower: these reach 5% or less at 96 sensors or more for all structural variants and each type of sensor, and eventually drop to around 1%, even at the highest levels of noise. Both types of error based on only strain sensors are again better than those based on only displacement sensors or on both types of sensors.

To show the performance of the GNL CM algorithm over all the benchmark problems, the results are summarized in Figure 15 below by determining the median force and displacement errors. Medians were taken over all the benchmark problems for results that share the same type of sensor data, noise level, and number of sensors. The error bars indicate the lower and upper quartile of each group of results. In Figure 15, the noisy results (10%) are the upper three lines in each graph. Counterintuitively, these noisy results have less

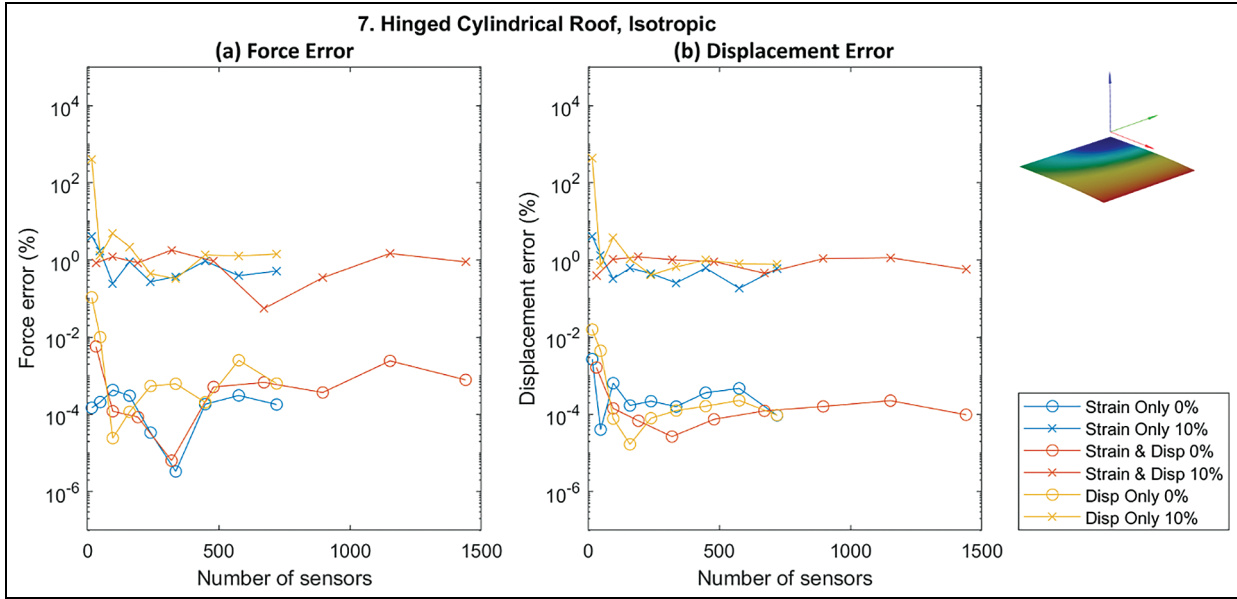


Figure 14. Force (a) and displacement (b) errors for problem 7, hinged cylindrical roof (isotropic), using only strain sensors, both strain and displacement sensors and only displacement sensors, for 0% and 10% noise.

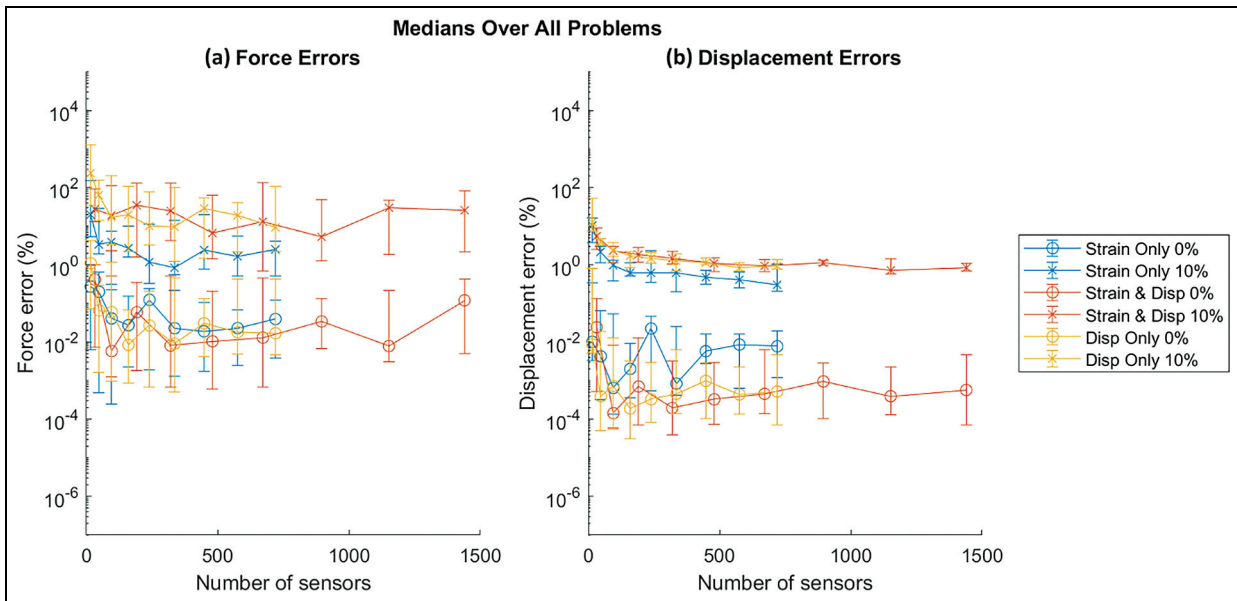


Figure 15. Median force (a) and displacement (b) errors over all the benchmark problems for 0% and 10% noise.

variation in the error magnitude between all the benchmark problems analyzed. As seen in these figures, while the median noisy results have higher reconstruction errors overall, the magnitude of those errors has less variation between different benchmark problems than the noiseless results. This is most pronounced in the displacement errors, shown in Figure 15(b), where the error bars of the upper three lines are visibly smaller than those of the lower three lines.

To separate the overall effect of noise on the quality of GNL CM reconstructions, the noiseless errors were subtracted from the corresponding errors based on noisy data. Medians were again determined over all the

benchmark problems for results that share the same type of sensor data, noise level, and number of sensors, as shown in Figure 16. The additional error introduced by noise is largest for few sensors and appears to decrease exponentially as more are used.

At a low sensor count, the magnitude of this additional error corresponds approximately to the magnitude of the noise that was introduced for most cases, except for the force reconstruction error based on displacement data only, which is approximately an order of magnitude greater. As the number of sensors increases, the additional error shrinks to the same order of magnitude as the noise that was introduced for this

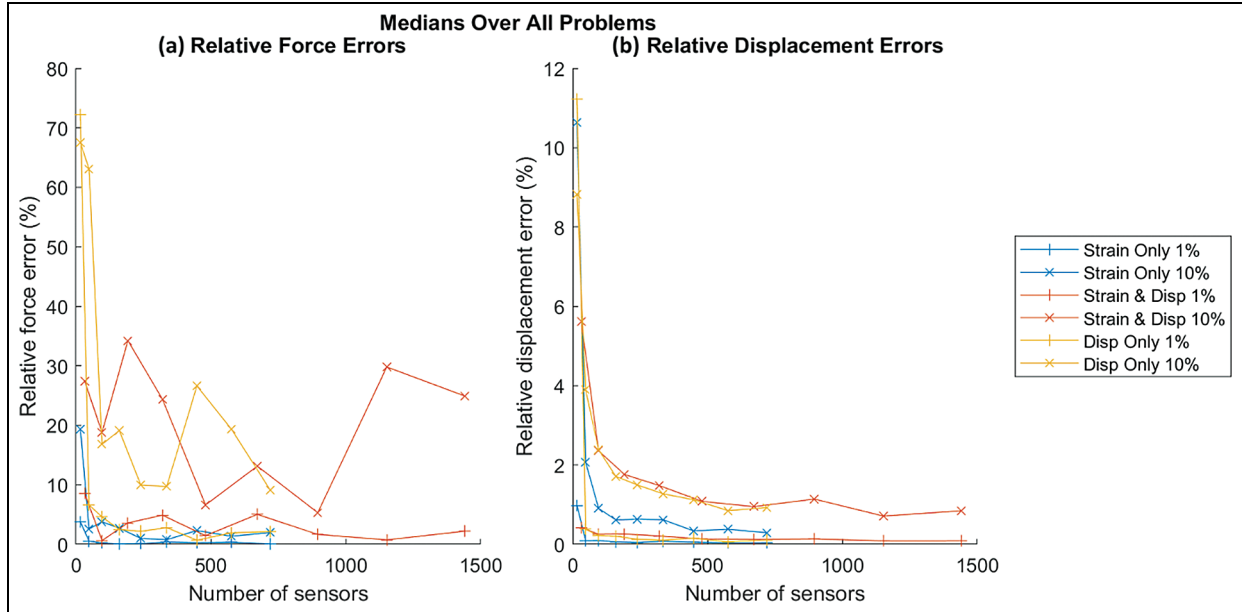


Figure 16. Median relative force (a) and displacement (b) errors over all the benchmark problems for 0% and 10% noise.

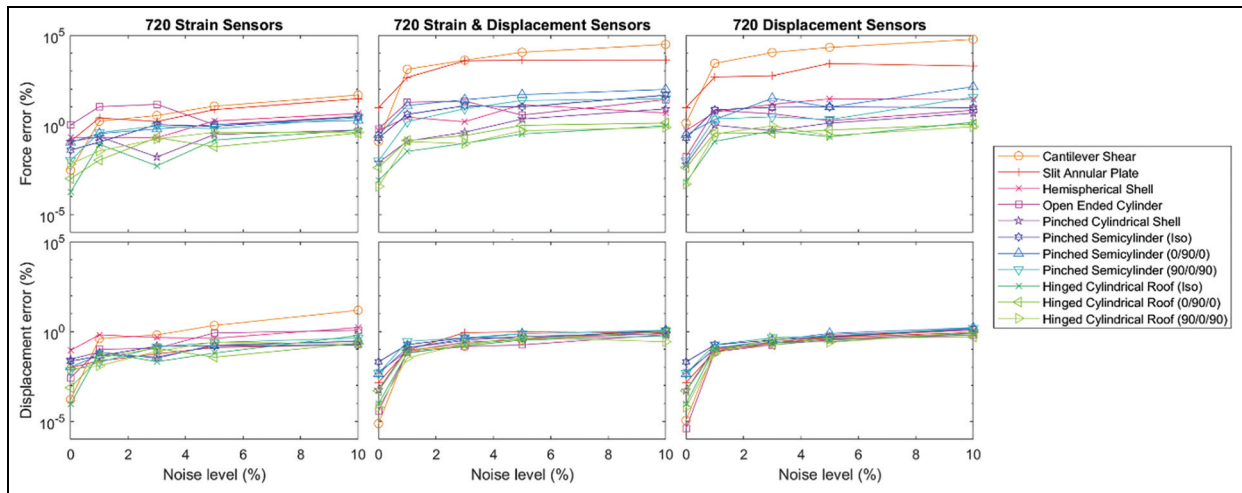


Figure 17. Force and displacement errors versus noise level for all benchmark problems, based on 720 sensors.

case and for the force error based on combined sensor types. The force error based on strain data and the displacement errors based on displacement data or combined data shrink to an order of magnitude smaller than the introduced noise, and the displacement error based on strain data shrinks by almost two orders of magnitude.

To provide additional insight into how force errors and displacement errors grow for each benchmark problem as the noise level is increased, the errors have been plotted against the noise level in Figure 17 below, showing only the results for the maximum number of sensors.

The force errors are generally higher than displacement errors, especially as the noise level is increased.

For the simpler structures, namely Cantilever Shear and Slit Annular Plate, the force error exceeds 100% even at just 1% noise when using displacement sensors or both types of sensors. When using only strain sensors instead, the force errors are more similar to the displacement errors, peaking at 47% at 10% noise for the Cantilever Shear problem. The more complex structures achieve lower force errors of around 1% at 10% noise for all types of sensor data.

The displacement errors are generally lower than the force errors and clustered more closely together. Unlike the force errors, the displacement errors reach their highest level when using strain data only, peaking at 15% at 10% noise for the Cantilever Shear problem. The displacement errors for the other problems and

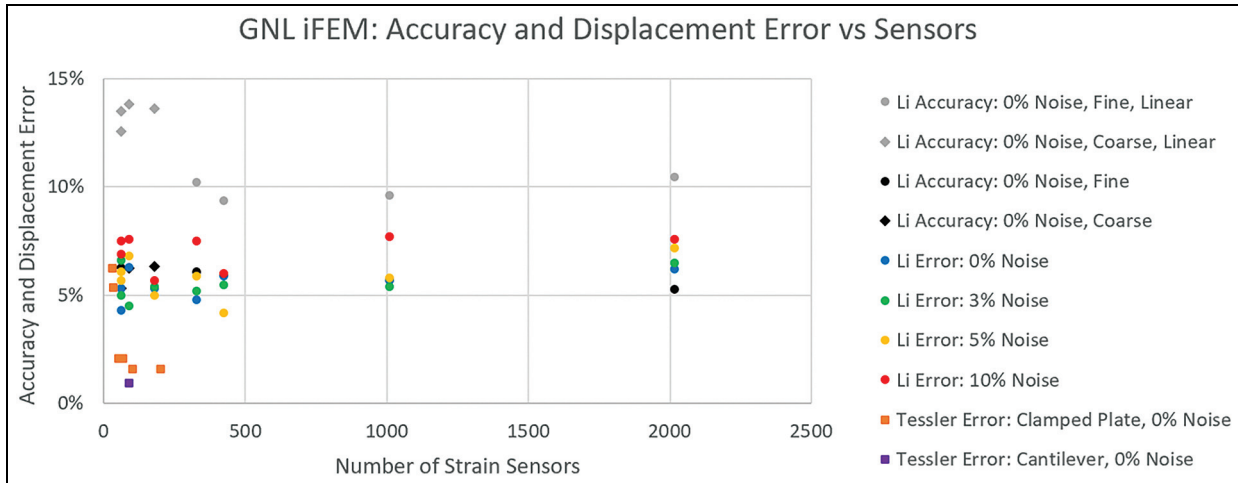


Figure 18. Reported accuracy and maximum z -displacement reconstruction errors of the linear and nonlinear iFEM results plotted against the number of simulated strain sensors that were used, for both the fine and coarse meshes of the stiffened cantilever plate of Li et al. and both structures of Tessler et al. Adapted from Table 1 and Figures 12 and 15 of Li et al.⁵³ and Figure 12 of Tessler et al.⁴⁵

sensor types peak between 0.2% and 1.7% at 10% noise.

iFEM results from literature

Tessler et al. presented results for two structures: a wing-shaped cantilever plate and a clamped square plate. Results for the first structure are only presented for 88 sensors, reaching an error of 0.95%. For the second structure, results are presented for different numbers of sensors, ranging from 29 to 200, reaching a maximum error of 6.3% that decreases to 1.6% as the number of sensors is increased, as shown in Figure 18. All of these results were obtained for sensor data undisturbed by noise, so the results of the linear incremental iFEM method can only be compared to the noiseless results of the GNL CM method.

An interesting pattern stood out in the data presented by Li et al: the stiffened cantilever plate is analyzed using both a fine and a coarse mesh, each with four different variants for the number and locations of the simulated strain sensors that are applied to the top and bottom surfaces of the plate. The listed accuracy percentages for the comparison of the final deformation between the iFEM and FEM results can be plotted against the number of sensors, as shown in Figure 18 below. Similarly, the maximum z -direction displacement reconstruction errors from the figures can also be plotted against the number of sensors, shown in the same figure. The accuracy and errors appear to be approximately constant for the nonlinear iFEM algorithm as the number of sensors increases. The linear iFEM algorithm does appear to show an improvement in accuracy, but this could also be due to the finer mesh that is used along with the larger number of sensors.

The displacement reconstruction errors of GNL iFEM do increase as the noise level increases, which is shown in Figure 19, along with how the errors of GNL CM increase with noise. A linear fit was performed for each problem to estimate how the errors increase with the level of noise, shown as dotted lines and equations in the same color as the corresponding data.

While the nonlinear iFEM results were obtained for just a few problems, some preliminary conclusions can be drawn, to which the GNL CM results of this study can be compared: The GNL iFEM algorithms appear to reach their maximum accuracy with a relatively small number (60–180) of sensors, obtaining little to no improvement in accuracy from the use of larger numbers of sensors. The algorithm appears to be resilient in the face of random noise, as the average error only increases by approximately 0.17% for each additional 1% of noise. Similarly, this study showed how the accuracy of the GNL CM method is affected by the number of sensors and the level of noise for a variety of benchmark problem types. For the same stiffened plate, the error for GNL CM increases by approximately 0.13% for each additional 1% of noise, slightly lower than the increase for GNL iFEM. This increase is 0.18% on average for the other benchmark problems presented here, or 0.05% when the 1.53% outlier of problem 1, Cantilever Shear, is excluded. For further comparison, Figure 20 shows how the displacement reconstruction error evolves as the number of strain sensors is increased at 0% and 10% noise, respectively. The GNL CM errors are significantly lower than those of GNL iFEM when reconstructing displacements for the same structures and sensors. Generally, GNL CM also reaches lower errors for most of the benchmark problems, except for the simplest structures.

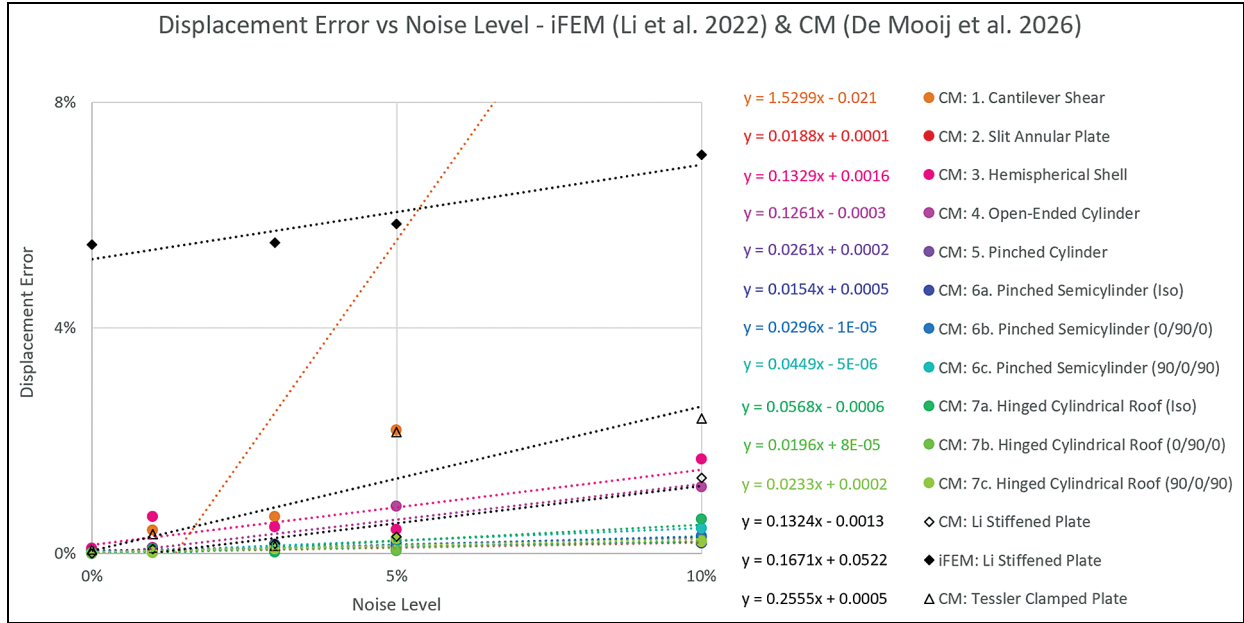


Figure 19. Displacement errors plotted against the noise level for the nonlinear iFEM results (adapted from Figures 12 and 15 of Li et al.⁵³) and the nonlinear CM results.

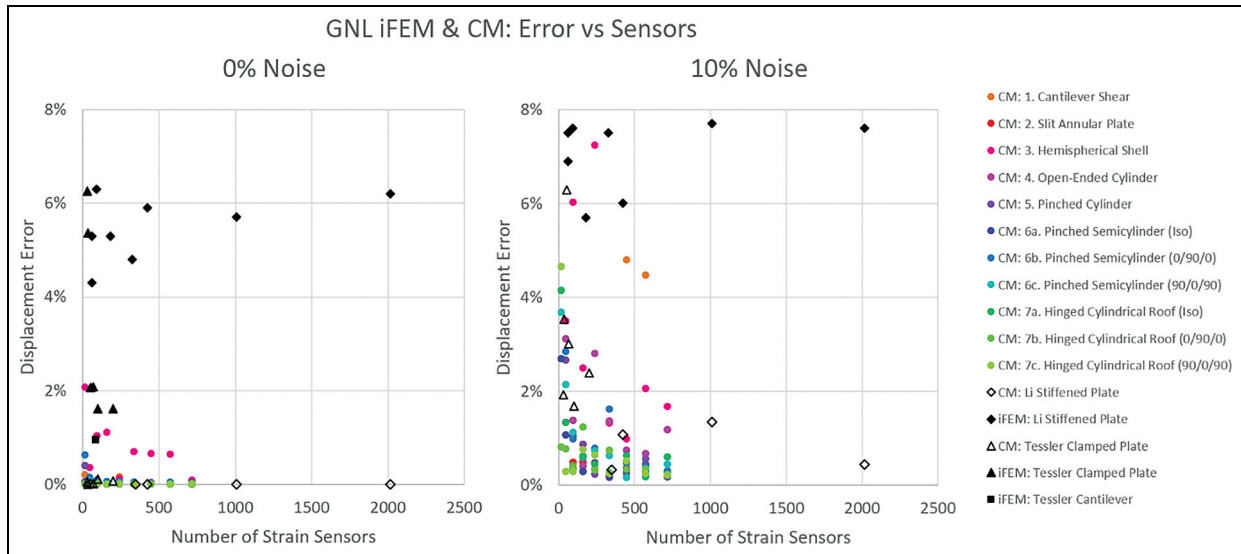


Figure 20. GNL iFEM and CM at 0%–10% noise: Displacement reconstruction errors for different numbers of strain sensors.

Discussion

The results indicate that, under noiseless conditions, the calibration matrix method can produce accurate or even highly accurate reconstructions of geometric nonlinear deformation based on data from a small number of simulated sensors for problems from the literature and several different benchmark problems, based on a representative set of calibration load cases. When noise is introduced to the sensor data, the performance deteriorates, particularly for reconstructions based on displacement data or the combination

of strain and sensor data and for the simpler structures. Still, the GNL CM algorithm can perform accurate displacement reconstructions based on each type of sensor and accurate force reconstructions when using strain sensors for all the benchmark problems, particularly for the problems with more complex deformation. Force reconstructions based on noisy data from displacement sensors or the combination of both sensor types only reach acceptable errors ($< 10\%$) for the simpler problems when a larger number of sensors is used.

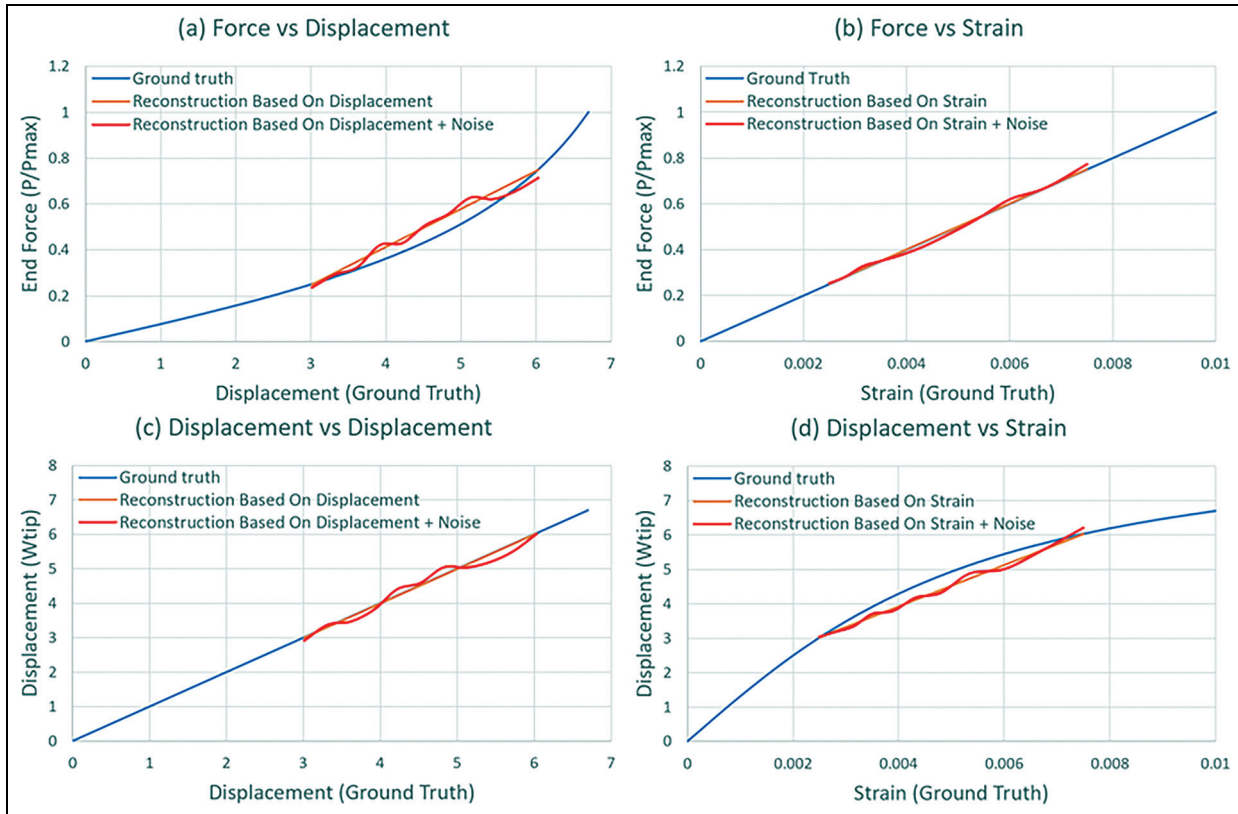


Figure 21. Simplified example of deformation curves, based on problem 1, the cantilever beam loaded in shear.

The GNL CM method uses data from a reasonable number of sensors and can reconstruct the deformation of each structure accurately. Without noise, it achieves force errors of 1% or less for all benchmark problems, using just 96 strain sensors. A force error of 2% or less is reached for all but one of the benchmark problems with 96 displacement sensors, or a combination of both types of sensors. A displacement error of 1% or less is achieved for all benchmark problems, using just 48 sensors of either type. With noise, force errors for the more complex structures reach acceptable or good ($< 1\%$) levels for higher numbers of sensors. The results based on strain data are generally better than those based on displacement data or the combination of both types of data. Counterintuitively, the force reconstructions for simpler structures only reach acceptable levels (11%) at lower noise levels (5%) and when using large numbers of strain sensors (720). The displacement reconstructions based on noisy data are better and more consistent, typically reaching acceptable results even at the highest levels of noise with just 48 displacement sensors and good results around 500 displacement sensors. Good displacement reconstructions based on strain data are achieved with just 48 strain sensors, even at the highest noise levels.

The lower accuracy for the simpler structures under noisy conditions appears to be due to the fact that their simpler deformation patterns lead to an effectively lower number of sensors: especially for problem 1, the

cantilever beam subjected to an end shear force, the sensors in each row across the width of the beam all capture the same displacements and strains. When two sensors always measure (nearly) identical values across all calibration load cases, this results in (nearly) identical rows in the sensor matrix $\mathcal{S}$, seen in equations (9) and (10). This appears to increase the sensitivity to the deteriorating effects of noise and numerical error, especially when attempting to use noisy displacement data to reconstruct applied forces for relatively simple structures undergoing geometric nonlinear deformation.

A different effect appears to be responsible for the fact that the force errors are generally higher for reconstructions based on displacement data than those based on strain data, and vice-versa for displacement reconstructions: The CM method is based on interpolation. When linear interpolation is applied between nonlinear load cases, the resulting reconstruction will have an interpolation error, which is apparent even in very simplified scenarios. Consider the scenario shown in Figure 21 of reconstructing the applied force or the vertical tip displacement for problem 1, the cantilever beam loaded in shear, based on measurements of displacement or strain at only a single point and just two calibration load cases:

The blue curves show the ground truth, along with orange curves for reconstructions without noise and red curves for reconstructions with noise (-5% to $+5\%$, homogeneously distributed). There is a clear

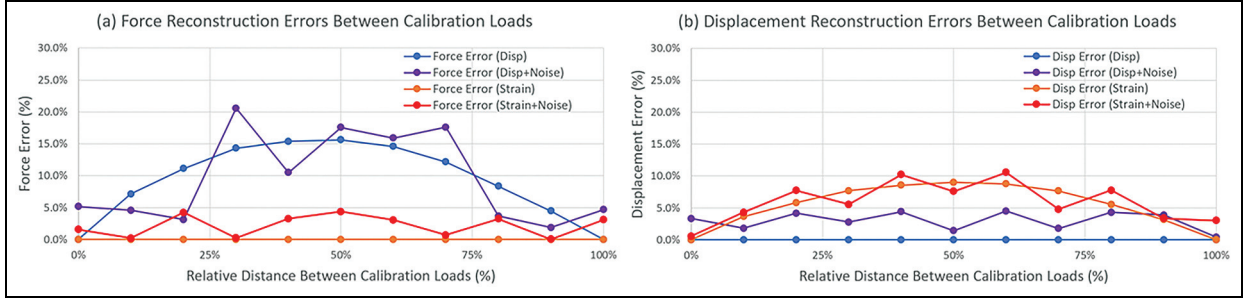


Figure 22. Force ((a) left) and displacement ((b) right) reconstruction errors plotted against the relative distance between calibration loads.

gap between ground truth and reconstructions without noise for forces based on displacement (a, top-left) and for displacements based on strain (d, bottom-right), while there is no such gap for forces based on strain (b, top-right) or displacements based on displacement (c, bottom-left).

Even though this simplified example uses only two calibration load cases, measured at applied forces of 0.25 and 0.75 (P/P_{max}) and reconstructs only a single force or displacement, the same pattern occurs. The force errors are higher for reconstructions based on displacement data than those based on strain data, and vice-versa for displacement reconstructions. These errors can be quantified with the relative two-norm error from equation (12) and plotted against the relative distance between the calibration loads, as can be seen in Figure 22.

This effect is present both with and without random noise distorting the measurements, and matches the pattern seen in the GNL CM results for most of the problems, especially those with a load-deflection curve similar to the approximately quadratic curve of this simplified example. The load-deflection curves presented by Tessler et al.,⁴⁵ Li et al.⁵³ and most of those of Sze et al.⁵⁵ fall in this category. Some of the Sze benchmark problems depart from this pattern, such as the open-ended cylinder, which exhibits more complex buckling and snapping behavior. While interpolation errors will still occur for these problems, the resulting reconstruction error will not be as predictable as for the problems that have approximately quadratic load-deflection curves.

Under noise-free conditions, the accuracy of this new GNL Calibration Matrix method is greater than the state-of-the-art GNL iFEM algorithms that were presented by Tessler et al.⁴⁵ and Li et al.⁵³: GNL CM achieves displacement reconstruction errors of 2.06% or lower with 16 sensors or more of either type for all the benchmark problems presented here. When the same problems are evaluated, the GNL CM achieves displacement reconstruction errors of 0.11% or lower, whereas GNL iFEM achieves errors ranging from 1.61% to 6.32%.

When subjected to noise, the GNL CM method reaches a displacement reconstruction accuracy similar to GNL iFEM around 96 sensors of either type, achieving median errors of 2.4% or lower and maximum errors of 8.6% or lower at all noise levels for almost all benchmark problems. The exception is the cantilever beam when using strain data, which has a displacement reconstruction error of 30% for 96 strain sensors. The GNL iFEM algorithm achieves errors up to 6.6% with 3% noise, 7.2% with 5% noise and 7.7% with 10% noise, whereas the GNL CM algorithm achieves errors of 0.17%, 0.29% and 1.34% at the same noise levels for the same structure. The GNL CM method has the additional advantage that the errors decrease further as the number of sensors is increased, reaching median errors of less than 1%, whereas the displacement errors of GNL iFEM remain approximately the same. This indicates that GNL CM is a reliable and more accurate method for finding the deformed shape of structures undergoing large deformations and rotations.

The GNL CM method also directly reconstructs applied loads, which GNL iFEM does not. While the force errors are generally larger than the displacement errors, they still reach acceptable levels for most of the benchmark problems when more sensors are used. The median error for 10% noise and 720 strain sensors is 2.5%, and 9.3% when using 720 displacement sensors. When benchmark problem 1 is excluded, these median force errors decrease to 1.8% and 6.6%.

The GNL CM method performs well overall, but it is hard to predict in advance what level of accuracy will be achieved based on the characteristics of the problem: benchmark problems with relatively simple deformation, such as problems 1, 2 and 7, can have higher or lower errors than problems with more complex buckled deformation, such as problems 5 and 6. This could be due to the fact that simpler deformation patterns lead to an effectively lower number of sensors, as (near-) identical data measured by different sensors reduces the effective rank of the sensor matrix $\mathcal{S}$. The greater variety in sensor data for more complex deformation leads to a higher effective rank, allowing more consistent results.

Additionally, it is hard to predict which type of sensor data will produce the best results. While both types of sensor or their combination produce good results, the best results are achieved with strain sensors for some problems and with displacement sensors for others. It is unclear what causes this inconsistency. What is clear is that the combination of both types of sensor does not produce results that are superior to both types of sensor in isolation for any of the problems. For the GNL CM algorithm there is no clear benefit to using a combination of strain and displacement sensors. Instead, an equivalent or better level of accuracy can be reached with a single type of sensor.

While good results have been obtained here for a range of different structures and load cases, including composite laminates and buckling, the problems and their load cases are still quite simple compared to real-life structures. The quality may be lower for real applications, due to potentially disruptive factors that have not been taken into account here. Some of these factors are non-uniform and/or skewed noise in the sensor data, fluctuating loads such as impacts or vibrations which are not captured quickly enough or at all by the sensor system or inaccurate information about the locations and/or orientations of the sensors relative to the structure. Real sensors often also do not capture all components of strain and displacement at a particular point, but only some subset along certain axes. These and other potential sources of error should be studied in future work.

Conclusions & future work

The CM method can reconstruct geometric nonlinear deformation with accuracy superior to iFEM methods from the literature, under both noiseless and noisy conditions. CM can also reconstruct applied forces directly, not just displacements and strains, and its accuracy improves as more sensors are used. The most accurate results are reached when using strain sensors and, counterintuitively, for structures with more complex deformation.

Applying regularization to CM is only beneficial with few sensors and/or high noise. Future work could automate its application, perhaps by making the regularization weight C_0 proportional to the (expected) amount of noise and the inverse of the number of sensors.

To evaluate how the GNL CM algorithm performs under more realistic circumstances, it should be applied to more complex structures undergoing a greater variety of loads, which would be more representative of real use. Additional research could also confirm whether more complex deformation always yields more accurate results. The algorithm should also be evaluated using real sensor data to validate its performance before it is deployed for real applications. Fully validating this (or any other) shape sensing algorithm would require a full-scale test on a real structure under controlled

conditions, including complete control over the applied forces.

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Author contributions

Cornelis de Mooij: Conceptualization, Methodology, Software, Validation, Formal analysis, Investigation, Data Curation, Writing–Original Draft, Visualization, Project administration. Marcias Martinez: Conceptualization, Resources, Writing–Review & Editing, Supervision, Project administration, Funding acquisition

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Data availability statement

The research data underlying this paper can be accessed at the 4TU Research Data website at: <https://data.4tu.nl/datasets/7dc3b79a-5571-4337-bde4-19c0622a5e6b> and <https://doi.org/10.4121/7dc3b79a-5571-4337-bde4-19c0622a5e6b>

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