Minimal Surfaces

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Chapter 1

Abstract

In this report we will consider a method for minimising surfaces with given boundaries, which makes this problem convex and converges without pre-requisite knowledge of the topology of the minimal surface. We will translate the method, which was originally described in the language of differential forms, to vector fields. We will also discuss some applications of minimal surfaces.

Chapter 2

Introduction

In this report we will study the following problem: Finding a surface of minimal surface area when only the boundary is given. This problem is called the Plateau problem [1] and it has been studied widely since it has been formulated by Lagrange ([2], [3]).

We will especially study the method uses by Stephanie Wang and Albert Chern in the article: "Computing minimal surfaces with Differential forms" [4]. They propose a general method to find a surface with minimal area given an arbitrary closed boundary, this surface is called a minimal surface. The largest strength of their method is that their numerical method converges to the minimal surface regardless of the topology of the minimal surface.

2.1 Purpose of this report and research

The purpose of this report will be to show that the method that Wang and Chern have introduced in terms of differential forms can also be described in terms of vector fields. We will also highlight in which situations their method is applicable and when it fails.

The translation to vector fields is aimed to help people who are more comfortable with vector fields to understand the methodology of Wang and Chern's paper.

2.2 What is a minimal surface

Let Σ be a surface which has the boundary curve Γ. This surface will be referred to as minimal if it has the property that its area is smaller or equal to all other surfaces which have the boundary curve Γ.

A minimal surface is therefore defined by the boundary curve of which it is a minimal surface. One very simple example of minimal surfaces are disks being the minimal surface given a circle as a boundary. As long as the boundaries remain two dimensional the minimal surface will simply be the surface enclosed by the boundary curve. It becomes interesting when the boundaries are embedded in 3 dimensions. Take two circles for example as shown in figure 2.1. If the circles are close together the area of the surface connecting the two circles is smaller than the surface area of two disks. It can also be seen that the surface is no longer just a straight connection between boundaries.

Figure 2.1: The minimal surface between two circles which are close to each other is a so called catenoid. Note that the boundary is made up of two unconnected curves.

2.3 Where do we see applications of minimal surfaces

Minimal surfaces occur in several different fields. They are mainly a result of a process where a large surface area is unfavourable. This is true in the well known case of soap films, where surface tension is applied along the surface, which therefore takes the shape that minimises the surfaces area.

One field where minimal surfaces are used is architecture [5]. Sometimes it is used simply because it results in nice shapes, but it can also be useful for using the minimal material to connect the structural framework of a building.

Another field in which minimal surfaces pop up are models for microscopic structures in nature. For example [6], [7] show that the interface between skin keratins and water is a minimal surface. Since the keratins are hydrophobic, the interface between the two has a certain cost associated with its surface, therefore the optimal shape will be the one such that the interface surface is minimal.

2.4 Layout of this report

The next chapter of the report will cover theory on vector currents and the Helmholtz decomposition, which will be used in the rest of the paper. In chapter 4 we will explain the method for minimising surfaces and currents in the language of vector fields. In chapter 5 we will show a possible way for implementing the numerical method and show some results. Lastly we will discuss how well this method can be applied to certain numerical problems and how successful the translation to vector fields is.

Chapter 3

Theory

In this section the necessary prerequisite knowledge for the report will be discussed. In their paper Wang and Chern use the language of differential forms, and therefore all the theoretical tools are also results from differential geometry. In this chapter we will reformulate the important concepts in terms of vector calculus so we can use them as such in the rest of the paper.

3.1 Vector currents

The first concept we will discuss are vector currents. These vector currents can be used to represent curves and surfaces, while they also behave similarly to vector fields. While currents are a well studied concept in geometric measure theory [8] [9], in this report we will only use vector currents, which are generalised vector fields.

Definition 1. *A vector current is a linear functional* $\overrightarrow{\phi} : C_0^\infty\left(U, \mathbb{R}^3\right) \to \mathbb{R}$

Here U is a compact subset of $\mathbb{R}^3$ and $C_0^\infty\left(U, \mathbb{R}^3\right) \to \mathbb{R}$ refers to a mapping from smooth functions, which are 0 outside of a compact domain, to a scalar value.

These so called vector currents can be seen as a higher dimensional analogue of the Dirac-δ distribution. Recall that a Dirac-δ (in 3 dimensions) has the following defining property: Dirac-δ centered at the origin is a linear functional $\delta : C_0^\infty(U, \mathbb{R}) \to \mathbb{R}$

$$\delta(f) = f(\mathbf{0}). \tag{3.1}$$

One example of a linear functional on vector fields is the inner product with another vector field:

Example 1. *The vector field* $\overrightarrow{G}$ *can be associated to the linear functional* $\langle \overrightarrow{G}, \cdot \rangle$: $C_0^\infty\left(U, \mathbb{R}^3\right) \to \mathbb{R}$:

$$\left\langle \overrightarrow{G}, \overrightarrow{F} \right\rangle = \iiint_U \overrightarrow{G} \cdot \overrightarrow{F} dV. \tag{3.2}$$

Geometric structures can also define linear functionals, these are the functionals we will be using to represent surfaces and curves.

Piecewise continuous and oriented curves can define a vector current. A vector current along the curve Γ has the following associated vector current:

Example 2. *The vector current associated to the curve* Γ*,*
$\overrightarrow{\delta_\Gamma} : C_0^\infty\left(U, \mathbb{R}^3\right) \to \mathbb{R}$:

$$\overrightarrow{\delta_\Gamma}\left(\overrightarrow{F}\right) = \int_\Gamma \overrightarrow{F}(\mathbf{x}) \cdot d\mathbf{s}. \tag{3.3}$$

A piecewise continuous and oriented surface Σ also has an associated vector current:

Example 3. *The associated vector current to the surface* Σ*,*
$\overrightarrow{\delta_\Sigma} : C_0^\infty\left(U, \mathbb{R}^3\right) \to \mathbb{R}$:

$$\overrightarrow{\delta_\Sigma}\left(\overrightarrow{F}\right) = \iint_\Sigma \overrightarrow{F}(\mathbf{x}) \cdot d\mathbf{S}. \tag{3.4}$$

However these currents are functionals, while we want their behaviour to represent that of generalised vector fields. In general all functionals can be represented as the limit of some row of functionals defined by vector fields as in example 1.

Theorem 1. *For each linear functional* $\overrightarrow{\phi} : C_0^\infty\left(U, \mathbb{R}^3\right) \to \mathbb{R}$*, there exists a sequence of vector fields* $\left\{\overrightarrow{G_\epsilon}\right\}$ *such that:*

$$\overrightarrow{\phi}\left(\overrightarrow{F}\right) = \lim_{\epsilon \to 0} \left\langle \overrightarrow{G_\epsilon}, \overrightarrow{F} \right\rangle \tag{3.5}$$

We will show that this is true for the vector currents which are related to geometric structures. As a motivation we will show the same behaviour for a standard Dirac-δ distribution.

Often the Dirac-δ is said to be the limit of a function that gets narrower and taller that is:

$$\delta\left(\mathbf{x}\right) = \lim_{\epsilon \to 0} \chi_\epsilon\left(\mathbf{x}\right). \tag{3.6}$$

Here $\chi_\epsilon(\mathbf{x})$ can be any function that is zero for $\mathbf{x}$ outside of the ball $\mathcal{B}_\epsilon(\mathbf{0})$ as long as an integral over the complete ambient space returns one: $\iiint \chi_\epsilon(\mathbf{x}) = 1$. It doesn't matter which function is used to approach the Dirac-δ distribution, but one example of a smooth function which can be used is the following (in spherical coordinates):

$$\chi_\epsilon(r, \theta, \phi) = \begin{cases} c_\epsilon \left(\cos(\frac{\pi}{\epsilon} r) + 1\right) & \text{if } r \leq \epsilon, \\ 0 & \text{else.} \end{cases}$$

The value of the constant $c_\epsilon = \frac{3\pi}{4(\pi^2 - 6)\epsilon^3}$. It must be noted now that the limit in the equation 3.6 diverges so it is not well defined, however it captures the properties of a dirac delta function when evaluated under an integral over a compact space U first:

$$\delta(f) = \lim_{\epsilon \to 0} \iiint_U f\left(\mathbf{x}\right) \chi_\epsilon\left(\mathbf{x}\right) dx^3 = f\left(\mathbf{0}\right). \tag{3.7}$$

Similarly the behaviour of vector currents can be captured by functions as long as we first evaluate the integral over a compact space before taking the limit. The vector current defined over a piecewise smooth oriented curve Γ with a parametrisation $\gamma(t)$ that is piecewise-differentiable may be loosely thought of as follows, $\overrightarrow{\delta_\Gamma}$: $U \mapsto \mathbb{R}^3$:

$$\overrightarrow{\delta_\Gamma}(\mathbf{x}) = \lim_{\epsilon \to 0} \int_\Gamma \overrightarrow{T_\Gamma}(\mathbf{s}) \chi_\epsilon(\mathbf{x} - \mathbf{s}) ds, \tag{3.8}$$

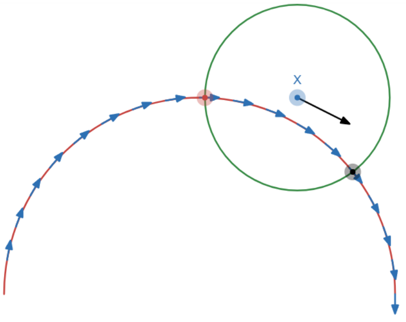

Figure 3.1: The vector $\overrightarrow{\delta_\Gamma}$ evaluated at a point x near a curve. the direction of the vector is a weighted average over all vectors along Γ within the distance ϵ

where $\overrightarrow{T_\Gamma}$ is the unit tangent vector to Γ:

$$\overrightarrow{T_\Gamma}(\mathbf{x}) = \begin{cases} \frac{d\overrightarrow{\gamma}}{dt}(t)\left|\frac{d\overrightarrow{\gamma}}{dt}(t)\right|^{-1} & \text{where it is defined,} \\ \mathbf{0} & \text{else.} \end{cases}$$

Once again $\overrightarrow{\delta_\Gamma}$ is not well defined since it diverges everywhere on the curve Γ. But taking the product with this "function" will yield the same result as applying the vector current as long as we move the limit outside an integral over the compact space U. The expression 3.8 is a sum of all the tangent vectors in a small distance ϵ centered around the point x. An example of this is shown in figure 3.1

We will now show that the current $\overrightarrow{\delta_\Gamma}$ can in fact be written as the limit of inner products with the sequence of vector fields $\overrightarrow{\delta_\Gamma^\epsilon}$, which are defined as follows:

$$\overrightarrow{\delta_\Gamma^\epsilon}(\mathbf{x}) = \int_\Gamma \overrightarrow{T_\Gamma}(\mathbf{s})\chi_\epsilon(\mathbf{x}-\mathbf{s})ds. \tag{3.9}$$

Theorem 2. *We can rephrase the functional $\overrightarrow{\delta_\Gamma}$ acting on the vector field F as follows*

$$\overrightarrow{\delta_\Gamma}\left(\overrightarrow{F}\right) = \lim_{\epsilon\to 0}\left\langle \overrightarrow{\delta_\Gamma^\epsilon}, \overrightarrow{F}\right\rangle.$$

Proof. First we take the definition of $\overrightarrow{\delta_\Gamma}$:

$$\overrightarrow{\delta_\Gamma}\left(\overrightarrow{F}\right) = \int_\Gamma \overrightarrow{F}\cdot d\mathbf{s} = \int_\Gamma \overrightarrow{F}\cdot\overrightarrow{T_\Gamma}\, ds. \tag{3.10}$$

We now use the property of the standard Dirac-δ distribution to introduce another integral:

$$\int_\Gamma \overrightarrow{F}(\mathbf{s})\cdot\overrightarrow{T_\Gamma}(\mathbf{s})ds = \int_\Gamma \lim_{\epsilon\to 0}\iiint_U \overrightarrow{F}(\mathbf{x})\cdot\overrightarrow{T_\Gamma}(\mathbf{s})\chi_\epsilon(\mathbf{x}-\mathbf{s})dx^3 ds. \tag{3.11}$$

We now define the function g_ϵ as follows:

$$g_\epsilon(\mathbf{y}) = \iiint_U \overrightarrow{F}(\mathbf{x})\cdot\overrightarrow{T_\Gamma}(\mathbf{y})\chi_\epsilon(\mathbf{x}-\mathbf{y})dx^3 \tag{3.12}$$

and the function $g_n(\mathbf{y})$ is defined by taking $\epsilon = \frac{1}{n}$. The following are true for almost all $\mathbf{y}$:

1. $\lim_{n\to\infty} g_n(\mathbf{y}) = \overrightarrow{F}(\mathbf{y}) \cdot \overrightarrow{T_\Gamma}(\mathbf{y})$ this is only not true for points where $\overrightarrow{T_\Gamma}$ is not continuous but all these points together have measure zero.
2. $|g_n(\mathbf{y})| \leq \sup_M \left|\overrightarrow{F}(\mathbf{y})\right|$ which is a finite constant since $\overrightarrow{F}$ itsself is a finite vector field.

As a result using the dominated convergence theorem we can move the limit out of the integral over Γ, so the righthand side of equation 3.11 becomes:

$$\lim_{\epsilon\to 0} \int_\Gamma \iiint_U \overrightarrow{F}(\mathbf{x}) \cdot \overrightarrow{T_\Gamma}(\mathbf{s}) \chi_\epsilon(\mathbf{x}-\mathbf{s}) dx^3 ds. \tag{3.13}$$

By Fubini's theorem we can reverse the order of integration, so 3.13 becomes:

$$\lim_{\epsilon\to 0} \iiint_U \int_\Gamma \overrightarrow{F}(\mathbf{x}) \cdot \overrightarrow{T_\Gamma}(\mathbf{s}) \chi_\epsilon(\mathbf{x}-\mathbf{s}) ds dx^3 \tag{3.14}$$

and we can remove the vector field $\overrightarrow{F}$ out of the inner integral so 3.14 becomes:

$$\lim_{\epsilon\to 0} \iiint_U \overrightarrow{F}(\mathbf{x}) \cdot \int_\Gamma \overrightarrow{T_\Gamma}(\mathbf{s}) \chi_\epsilon(\mathbf{x}-\mathbf{s}) ds dx^3. \tag{3.15}$$

The right hand integral is precisely $\overrightarrow{\delta_\Gamma^\epsilon}$ and therefore we can conclude the proof:

$$\overrightarrow{\delta_\Gamma}\left(\overrightarrow{F}\right) = \lim_{\epsilon\to 0} \iiint_U \overrightarrow{F} \cdot \overrightarrow{\delta_\Gamma^\epsilon}\, dx^3 = \lim_{\epsilon\to 0} \left\langle \overrightarrow{\delta_\Gamma^\epsilon}, \overrightarrow{F} \right\rangle. \tag{3.16}$$

□

Before we talk about operations on vector currents we still want show the vector current on surfaces has the same property as shown above for the curve vector current. We begin by directly introducing $\overrightarrow{\delta_\Sigma^\epsilon}$ which is very similar to $\overrightarrow{\delta_\Gamma^\epsilon}$:

$$\overrightarrow{\delta_\Sigma^\epsilon}(\mathbf{x}) = \iint_\Sigma \overrightarrow{N_\Sigma}(\mathbf{y}) \chi_\epsilon(\mathbf{x}-\mathbf{y}) d\mathbf{y}, \tag{3.17}$$

where $\overrightarrow{N_\Sigma}$ is the unit normal vector to the surface Σ on all points where it is well defined. Note that Σ has to be an oriented surface so that it has only one defined normal vector.

We will now concisely repeat the proof for the following theorem which is almost the same as theorem 2

Theorem 3. *We can rephrase the functional $\overrightarrow{\delta_\Sigma}$ working on the vector field $\overrightarrow{F}$ as follows*

$$\overrightarrow{\delta_\Sigma}\left(\overrightarrow{F}\right) = \lim_{\epsilon\to 0} \left\langle \overrightarrow{\delta_\Sigma^\epsilon}, \overrightarrow{F} \right\rangle.$$

Proof. We follow the same steps as earlier:

$$
\begin{aligned}
\overrightarrow{\delta_\Sigma}\left(\overrightarrow{F}\right) &= \iint_\Sigma \overrightarrow{F} \cdot \overrightarrow{d\mathbf{S}} \\
\iint_\Sigma \overrightarrow{F} \cdot \overrightarrow{d\mathbf{S}} &= \iint_\Sigma \overrightarrow{F} \cdot \overrightarrow{N_\Sigma}\, dS \\
&\overset{Dirac-\delta}{=} \iint_\Sigma \lim_{\epsilon\to 0} \iiint_U \overrightarrow{F}(\mathbf{x}) \cdot \overrightarrow{N_\Sigma}(\mathbf{y}) \chi_\epsilon(\mathbf{x}-\mathbf{y}) dx^3 dS \\
&\overset{D.C.}{=} \lim_{\epsilon\to 0} \iint_\Sigma \iiint_U \overrightarrow{F}(\mathbf{x}) \cdot \overrightarrow{N_\Sigma}(\mathbf{y}) \chi_\epsilon(\mathbf{x}-\mathbf{y}) dx^3 dS \qquad (3.18) \\
&\overset{Fubini's}{=} \lim_{\epsilon\to 0} \iiint_U \iint_\Sigma \overrightarrow{F}(\mathbf{x}) \cdot \overrightarrow{N_\Sigma}(\mathbf{y}) \chi_\epsilon(\mathbf{x}-\mathbf{y}) dS dx^3 \\
&= \lim_{\epsilon\to 0} \iiint_U \overrightarrow{F}(\mathbf{x}) \cdot \iint_\Sigma \overrightarrow{N_\Sigma}(\mathbf{y}) \chi_\epsilon(\mathbf{x}-\mathbf{y}) dS dx^3 \\
&= \lim_{\epsilon\to 0} \left\langle \overrightarrow{\delta_\Sigma^\epsilon}, \overrightarrow{F} \right\rangle .
\end{aligned}
$$

□

We have now proven that both surfaces and curves can be represented by linear functionals and that these functionals can in turn be thought of as generalised vector fields.

3.1.1 Operations on vector currents

We have now shown how a vector currents represents a surface and has an associated set of vector fields $\overrightarrow{\delta^\epsilon}$. We have yet to show that operations on vector fields can be translated to vector currents.

For the purpose of this report we will be working on the compact space $\mathbb{T}^3$ so we will only be looking at operations defined on that space. $\mathbb{T}^3$ is the cube with periodic boundary conditions. It is important to note that vector currents are still functionals so their vector field counterparts only make sense in the limit of the integral.

First we introduce the notation:

$$
\left\langle \overrightarrow{\delta}, \overrightarrow{F} \right\rangle = \lim_{\epsilon\to 0} \iiint_{\mathbb{T}^3} \overrightarrow{\delta^\epsilon} \cdot \overrightarrow{F} dx^3 = \overrightarrow{\delta}\left(\overrightarrow{F}\right). \qquad (3.19)
$$

This gives us a new way to write the action of a current working on a vector field. Next we will state that an operation on a vector current is defined by looking at the result of applying the operation to the associated vector fields.

Definition 2. *If G is an operation that sends vector fields to vector fields, the application of G on a vector current gives a new vector current or functional with the following property:*

$$
G \circ \overrightarrow{\delta}\left(\overrightarrow{F}\right) = \lim_{\epsilon\to 0} \iiint_{\mathbb{T}^3} \overrightarrow{F} \cdot G\left(\overrightarrow{\delta^\epsilon}\right) dx^3. \qquad (3.20)
$$

An operation is only defined on a vector current if this definition is satisfied for any vector field which $\overrightarrow{\delta}$ can take as an argument.

Note that definition 2 does not allow for operations that send vector fields to functions. In that case we must use another definition:

Definition 3. *If G is an operation that sends vector fields to functions, the application of G on a vector current gives a functional with the following property:*

$$G \circ \overrightarrow{\delta}(f) = \lim_{\epsilon \to 0} \iiint_{\mathbb{T}^3} f \; G\left(\overrightarrow{\delta^\epsilon}\right) dx^3. \tag{3.21}$$

We will now look at the operations divergence and curl. To say anything meaningful about what they do we still need the resulting functional $\nabla \cdot \overrightarrow{\delta}$ or $\nabla \times \overrightarrow{\delta}$ to work on a vector field or a function. Let us first look what the curl operator does to a vector current:

Theorem 4. *Taking the curl of a vector current is the same as taking the curl of the vector field which it works upon:*

$$\nabla \times \overrightarrow{\delta}\left(\overrightarrow{F}\right) = \overrightarrow{\delta}\left(\nabla \times \overrightarrow{F}\right). \tag{3.22}$$

Proof. We look at $\nabla \times \overrightarrow{\delta}$ working on an arbitrary field vector field $\overrightarrow{F}$.

$$\nabla \times \overrightarrow{\delta}\left(\overrightarrow{F}\right) = \lim_{\epsilon \to 0} \iiint_{\mathbb{T}^3} \overrightarrow{F} \cdot \nabla \times \overrightarrow{\delta^\epsilon} dx^3. \tag{3.23}$$

We can add an integral over any divergence since $\mathbb{T}^3$ has no boundary so it will equal zero by Gauss' theorem:

$$= \lim_{\epsilon \to 0} \iiint_{\mathbb{T}^3} \nabla \cdot \left(\overrightarrow{F} \times \overrightarrow{\delta^\epsilon}\right) + \overrightarrow{F} \cdot \nabla \times \overrightarrow{\delta^\epsilon} dx^3 \tag{3.24}$$

$$= \lim_{\epsilon \to 0} \iiint_{\mathbb{T}^3} \nabla \times \overrightarrow{F} \cdot \overrightarrow{\delta^\epsilon} dx^3 = \iiint_{\mathbb{T}^3} \nabla \times \overrightarrow{F} \cdot \overrightarrow{\delta} dx^3. \tag{3.25}$$

$\overrightarrow{F}$ was arbitrary, so in general taking the curl of a current is the same as taking the curl of the field it is working on. □

This also means the following is true:

Theorem 5. *The curl of any surface vector current defined on Σ is the same as the curve vector current over the boundary $\partial\Sigma$:*

$$\nabla \times \overrightarrow{\delta_\Sigma} = \overrightarrow{\delta_{\partial\Sigma}}. \tag{3.26}$$

Proof. To prove this we show that for any arbitrary vector field F the two functionals are the same:

$$\overrightarrow{\delta_{\partial\Sigma}}\left(\overrightarrow{F}\right) \stackrel{def}{=} \oint_{\partial\Sigma} \overrightarrow{F} \cdot d\mathbf{s} \stackrel{Stokes'}{=} \iint_{\Sigma} \left(\nabla \times \overrightarrow{F}\right) \cdot d\mathbf{S}$$
$$\stackrel{def}{=} \iiint_{\mathbb{T}^3} \left(\nabla \times \overrightarrow{F}\right) \cdot \overrightarrow{\delta_\Sigma} dx^3 = \nabla \times \overrightarrow{\delta_\Sigma}\left(\overrightarrow{F}\right). \tag{3.27}$$

For the last step we use theorem 4. Since $\overrightarrow{F}$ was arbitrary this means that the currents $\nabla \times \overrightarrow{\delta_\Sigma}$ and $\overrightarrow{\delta_{\partial\Sigma}}$ are the same. □

When we look at the divergence we expect it to turn a vector field into a scalar field, so we expect it to turn a vector current into a Dirac-δ, therefore it should work on a scalar field. The result of applying the divergence on a current is as follows:

Theorem 6. *Taking the divergence of a current is the same as taking the negative gradient of the function it is working on:*

$$\nabla \cdot \overrightarrow{\delta}(f) = \overrightarrow{\delta}(-\nabla f) \tag{3.28}$$

Proof.

$$\nabla \cdot \overrightarrow{\delta}(f) = \lim_{\epsilon \to 0} \iiint_{\mathbb{T}^3} f \ \nabla \cdot \overrightarrow{\delta^\epsilon} dx^3 \tag{3.29}$$

$$= \lim_{\epsilon \to 0} \iiint_{\mathbb{T}^3} \nabla \left(f \cdot \overrightarrow{\delta^\epsilon} \right) - \nabla f \cdot \overrightarrow{\delta^\epsilon} dx^3 \tag{3.30}$$

$$= \lim_{\epsilon \to 0} \iiint_{\mathbb{T}^3} -\nabla f \cdot \overrightarrow{\delta^\epsilon} dx^3 = \overrightarrow{\delta}(-\nabla f). \tag{3.31}$$

□

3.2 Helmholtz decomposition

Another tool we will use is the Helmholtz decomposition, which is a method for decomposing a vector field into a curl-free and a divergence free part.

Theorem 7. *A vector field $\overrightarrow{F} : C^1(V, \mathbb{R}^n)$ defined on a compact domain $V \subset \mathbb{R}^n$ there is a Helmholtz decomposition consisting of the vector fields $\overrightarrow{G} \in C^1(V, \mathbb{R}^n)$ and $\overrightarrow{R} \in C^1(V, \mathbb{R}^n)$ such that:*

$$\begin{aligned} \overrightarrow{F}(\mathbf{x}) &= \overrightarrow{G}(\mathbf{x}) + \overrightarrow{R}(\mathbf{x}) \\ \overrightarrow{G}(\mathbf{x}) &= \nabla \phi(\mathbf{x}) \\ \nabla \cdot \overrightarrow{R}(\mathbf{x}) &= 0, \end{aligned} \tag{3.32}$$

where $\phi(\mathbf{x}) \in C^2(V, \mathbb{R}^n)$ is a scalar function. This decomposition is generally not unique.

We will be using a variant of the Helmholtz decomposition on $\mathbb{T}^3$. More specifically, we will use a variant of the Helmholtz-Hodge decomposition as it appears in Schwarz [10].

Theorem 8. *Any vector field $\overrightarrow{F} : C^2(\mathbb{T}^3, \mathbb{R}^3)$ has an orthogonal decomposition consisting of the fields $\overrightarrow{G}$, $\overrightarrow{R}$ and $\overrightarrow{H}$:*

$$\begin{aligned} \overrightarrow{F}(\mathbf{x}) &= \overrightarrow{G}(\mathbf{x}) + \overrightarrow{R}(\mathbf{x}) + \overrightarrow{H}(\mathbf{x}) \\ \overrightarrow{G}(\mathbf{x}) &= \nabla \phi(\mathbf{x}) \\ \overrightarrow{R}(\mathbf{x}) &= \nabla \times \overrightarrow{A}(\mathbf{x}) \\ \nabla^2 \overrightarrow{H}(\mathbf{x}) &= 0. \end{aligned} \tag{3.33}$$

Here the field $\overrightarrow{G}$ is called a gradient field, $\overrightarrow{R}$ is called a rotational field and $\overrightarrow{H}$ is called a harmonic field

Because this decomposition is orthogonal it is also unique and it can be very useful for limiting boundary conditions to only one component of a vector field while letting the other parts be boundaryless. It is important to note that harmonic fields on a manifold with periodic boundary conditions can only be constant.

This decomposition also has a variant in two dimensions as it is mentioned by Ribiero [11]:

Theorem 9. *Any two dimensional vector field* $\overrightarrow{F} : C^1(\mathbb{T}^2, \mathbb{R}^2)$ *has an orthogonal decomposition consisting of a gradient field* $\overrightarrow{G}$*, a rotational field* $\overrightarrow{R}$*, and a harmonic field* $\overrightarrow{H}$*:*

$$\begin{aligned} \overrightarrow{F}(\mathbf{x}) &= \overrightarrow{G}(\mathbf{x}) + \overrightarrow{R}(\mathbf{x}) + \overrightarrow{H}(\mathbf{x}) \\ \overrightarrow{G}(\mathbf{x}) &= \nabla\phi(\mathbf{x}) \\ \overrightarrow{R}(\mathbf{x}) &= \nabla \times r(\mathbf{x}). \\ \nabla^2\overrightarrow{H}(\mathbf{x}) &= 0. \end{aligned} \tag{3.34}$$

Two things to note are the following, the curl in equation 3.34 is working on a scalar function, it is defined as follows:

Definition 4. *Curl on a scalar function is defined as follows:*

$$\nabla \times f(\mathbf{x}) = -\frac{\partial}{\partial y} f\hat{x} + \frac{\partial}{\partial x} f\hat{y}. \tag{3.35}$$

Furthermore the harmonic fields are again constant fields.

Chapter 4

Minimising surfaces and curves using vector calculus

In this chapter we will explain the method used by Wang and Chern [4] to solve the Plateau problem. We will do this using the language of vector fields, as opposed to the differential forms used in their paper. We state the Plateau problem:

Problem 1. *Given a boundary curve Γ on an ambient manifold M, find a oriented surface Σ on M with minimal area, that is:*

$$\underset{\Sigma:\partial\Sigma=\Gamma}{\text{minimise}} \text{ Area}(\Sigma). \tag{4.1}$$

One difficulty this problem has is that the area of surfaces with a specified boundary can have local minima, making it hard to numerically solve the problem. an example of a boundary which permits surfaces with local minima is shown in figure 4.1. Another difficulty is that not all surfaces with the same boundary have the same topology. An example of this is the catenoid, shown in figure 2.1, which has the same boundary as two disks. For some numerical approaches it can be hard to switch between topologies. The method we will explain solves both of these problems, by translating the Plateau problem to a minimisation problem of vector fields.

The method proposed by Wang and Chern also has more general applications as it can be used to find any minimal geometric object given an n-dimensional boundary. In this report we will be showing how this works for the following cases: minimal surfaces given a boundary curve (section 4.1) and minimal curves given a set of boundary points (section 4.2).

4.1 Finding minimal surfaces using vector fields

In this section we will explain the method for finding minimal surfaces given one or multiple boundary curves. The most important goal will be to define a convex problem so that it can be easily solved by a numerical method. This will be done in sections 4.1.1 to 4.1.4. After defining a convex problem we will choose to solve the problem on the domain $\mathbb{T}^3$. This is chosen because it is compact and will allow using fourier transforms,

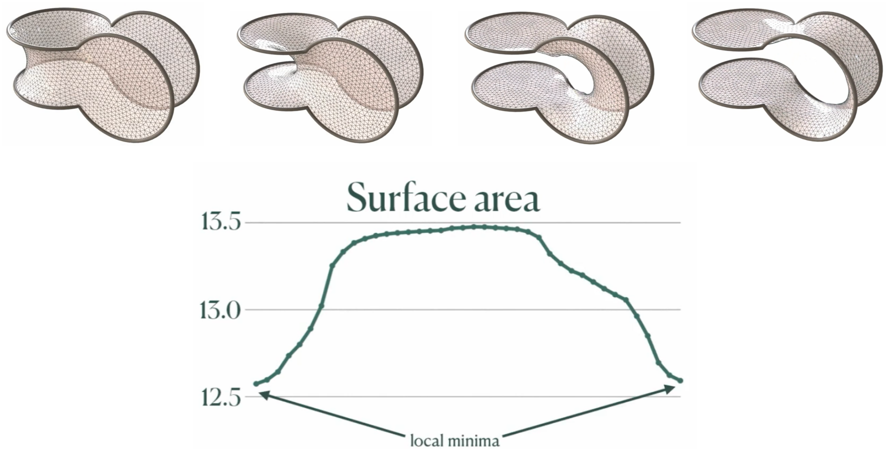

Figure 4.1: The transformation from one minimal surface to another, respectively the leftmost and rightmost surfaces. The area of all surfaces in the transformation is shown in the graph. Images are created by Wang and Chern [4]

however it will also impose some new problems. Solving these new complications and further simplifying the problem will be done in sections 4.1.5 to 4.1.7

4.1.1 Defining the surfaces and boundaries as vector currents

To solve the Plateau problem using vector currents we must redefine the problem in terms of vector currents. Let us first restate the Plateau problem:

Problem 2. *Given a oriented boundary curve Γ on M, find a oriented surface Σ on M with minimal area, that is:*

$$\underset{\Sigma:\partial\Sigma=\Gamma}{\text{minimise}}\ \text{Area}(\Sigma) \tag{4.2}$$

The orientation of the fields will be important since vectors must have a direction.

We will now use the vector currents as defined in section 3.1. That means we will replace the surface Σ by the vector current $\overrightarrow{\delta_\Sigma}$. We will also replace the boundary Γ by the vector current $\overrightarrow{\delta_\Gamma}$. However we can now only rephrase our problem after also changing the functional and the boundary condition to the language of vector fields. We will do this in sections 4.1.2 and 4.1.3

4.1.2 A norm that corresponds to area

First we will rephrase the Area functional. We will choose a norm that has the special property that it returns the area of a surface Σ when applied to the vector current $\overrightarrow{\delta_\Sigma}$. This norm is called the mass-norm and is defined as follows:

$$\|\overrightarrow{\eta}\| = \sup_{\overrightarrow{V}:M\to\mathbb{R}^3, \forall \mathbf{x}\in M: \left|\overrightarrow{V}(\mathbf{x})\right|\leq 1} \iiint_M \overrightarrow{V}(\mathbf{x}) \cdot \overrightarrow{\eta}\, dx^3, \tag{4.3}$$

for any arbitrary vector field $\overrightarrow{\eta} : M \to \mathbb{R}^3$. We now want to show that this both corresponds to area and that it is indeed a norm.

Theorem 10. $\left\|\overrightarrow{\delta_\Sigma}\right\| = \text{Area}(\Sigma)$.

Proof.

$$\text{Area}(\Sigma) = \iint_\Sigma 1 dS = \sup_{\overrightarrow{V}:M\to\mathbb{R}^3, \forall \mathbf{x}\in M: \left|\overrightarrow{V}(\mathbf{x})\right|\le 1} \iint_\Sigma \overrightarrow{V}(\mathbf{x})\cdot d\Sigma, \tag{4.4}$$

this is true because in the supremum the vector $\overrightarrow{V}(\mathbf{x})$ will assume the maximum allowed length of 1 therefore returning the value one integrated over the surface Σ.

$$\sup_{\overrightarrow{V}:M\to\mathbb{R}^3, \forall \mathbf{x}\in M: \left|\overrightarrow{V}(\mathbf{x})\right|\le 1} \iint_\Sigma \overrightarrow{V}(\mathbf{x})\cdot d\Sigma = \sup_{\overrightarrow{V}:M\to\mathbb{R}^3, \forall \mathbf{x}\in M: \left|\overrightarrow{V}(\mathbf{x})\right|\le 1} \iiint_M \overrightarrow{V}(\mathbf{x})\cdot \overrightarrow{\delta_\Sigma}(\mathbf{x}) dx^3, \tag{4.5}$$

this follows from the property of a surface vector current as shown in section 3.1. □

Theorem 11. $\|\overrightarrow{\eta}\|$ *defines a norm on continuous vector fields.*

Proof. A norm is defined by three properties:

- Triangle inequality: $p(x+y) \le p(x) + p(y)$,
- homogenity: $p(sx) = |s|p(x)$ where s is a scalar,
- point seperating: if $p(x) = 0$ then $x = 0$.

We will now show that the mass norm as it is defined here fulfills all three properties.

Triangle inequality:

$$\sup_{\overrightarrow{V}:\left|\overrightarrow{V}\right|\le 1} \iiint_M \overrightarrow{V}\cdot(\overrightarrow{X}+\overrightarrow{Y})dx^3 = \sup_{\overrightarrow{V}:\left|\overrightarrow{V}\right|\le 1} \iiint_M \overrightarrow{V}\cdot\overrightarrow{X} + \overrightarrow{V}\cdot\overrightarrow{Y} dx^3 \le \sup_{\overrightarrow{V}:\left|\overrightarrow{V}\right|\le 1} \iiint_M \overrightarrow{V}\cdot\overrightarrow{X} dx^3 + \sup_{\overrightarrow{V}:\left|\overrightarrow{V}\right|\le 1} \iiint_M \overrightarrow{V}\cdot\overrightarrow{Y} dx^3. \tag{4.6}$$

Here the inequality follows from the fact that the vector field $\overrightarrow{V}$ can only be pointed in one direction so it cannot in general give a maximal product with both $\overrightarrow{X}$ and $\overrightarrow{Y}$.

Homogenity:

$$\sup_{\overrightarrow{V}:\left|\overrightarrow{V}\right|\le 1} \iiint_M \overrightarrow{V}\cdot s\overrightarrow{X} dx^3 = |s| \sup_{\overrightarrow{V}:\left|\overrightarrow{V}\right|\le 1} \iiint_M \overrightarrow{V}\cdot\overrightarrow{X} dx^3. \tag{4.7}$$

This is a result of the linearity of the integral together with the fact that scalar multiplication does not change the orientation of a vector field, save for a possible change in sign.

Point seperating:
This property follows from the fact that every non zero part of the vector field $\overrightarrow{X}$ has a positive contribution to the norm and continuous fields cannot be nonzero on single points.

$$\mathbf{x} \neq 0 \implies \overrightarrow{V}\cdot\mathbf{x} > 0.$$

This in turn means the following is true:

$$\overrightarrow{X} \neq 0 \implies \sup_{\overrightarrow{V}:\left|\overrightarrow{V}\right|\leq 1} \iiint_M \overrightarrow{V} \cdot \overrightarrow{X} dx^3 > 0. \tag{4.8}$$

Therefore the contrapositive also holds: $\|\overrightarrow{X}\| = 0 \implies \overrightarrow{X} = 0$. $\square$

This means we have now defined the mass-norm on vector fields. This norm works as an "area functional" on currents. Since it is a norm it is automatically a convex functional, which is necessary for defining a convex minimisation problem.

4.1.3 Changing the boundary constraint

As we now have defined the variable and the functional in terms of vector calculus we also need to define the boundary conditions in terms of vector calculus. We will now show that the statements $\partial\Sigma = \Gamma$ is equivalent to $\overrightarrow{\delta_\Gamma} = \nabla \times \overrightarrow{\delta_\Sigma}$.

Theorem 12. $\partial\Sigma = \Gamma \iff \overrightarrow{\delta_\Gamma} = \nabla \times \overrightarrow{\delta_\Sigma}$.

Proof. The equivalency follows from the fact that $\overrightarrow{\delta_{\partial\Sigma}} = \nabla \times \overrightarrow{\delta_\Sigma}$, as stated in theorem 5. If we use $\partial\Sigma = \Gamma$ we can turn $\overrightarrow{\delta_{\partial\Sigma}}$ into $\overrightarrow{\delta_\Gamma}$. Therefore $\partial\Sigma = \Gamma \iff \overrightarrow{\delta_\Gamma} = \nabla \times \overrightarrow{\delta_\Sigma}$. $\square$

Now we can change the boundary constraint to $\overrightarrow{\delta_\Gamma} = \nabla \times \overrightarrow{\delta_\Sigma}$. At last we can rephrase the problem fully in terms of vector calculus:

Problem 3. *Given a boundary curve $\overrightarrow{\delta_\Gamma}$ find a $\overrightarrow{\delta_\Sigma}$ with minimal mass norm:*

$$\underset{\overrightarrow{\delta_\Sigma}:\overrightarrow{\delta_\Gamma}=\nabla\times\overrightarrow{\delta_\Sigma}}{\text{minimise}} \|\overrightarrow{\delta_\Sigma}\|_{\text{mass}}. \tag{4.9}$$

4.1.4 making the domain convex

The last step to take is making sure that the domain is a convex set. So instead of only looking at surface vector currents we want to look at all possible vector currents, or generalised vector fields. these are linear functionals so they define a convex set. The only possible problem is whether this yields the same result, but this is granted by a proof from Federer and Fleming 1960. One part of that theorem states [9]:

Theorem 13. *If X is a surface then there exists a Y which is also a vector current such that $\partial X = \partial Y$ and the vector current representation of Y has minimal mass norm for its given boundary.*

This means that if we find any generalised vector field with minimal mass norm, given a certain boundary, it has to correspond to a surface vector current.
This means our problem is now defined on a convex set, with a convex functional and has linear boundary constraints. Our problem now looks as follows:

Problem 4. *Given a boundary curve $\overrightarrow{\delta_\Gamma}$ find a vector current $\overrightarrow{\eta}$ with minimal mass norm:*

$$\underset{\overrightarrow{\eta}:\overrightarrow{\delta_\Gamma}=\nabla\times\overrightarrow{\eta}}{\text{minimise}} \|\overrightarrow{\eta}\|_{mass}. \tag{4.10}$$

Since all vector currents can be approached by vector fields this problem is numerically similar to solving for a vector field.

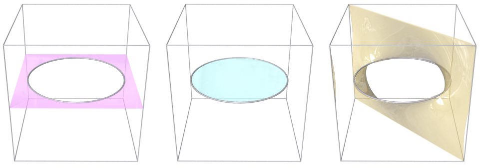

Figure 4.2: This image shows 3 possible surfaces given a circle as a boundary. When searching for a minimal surface we want to find the blue surface but without further constraints a numerical method can converge to the pink surface as it is minimal. This image was taken from Wang and Chern [4]

4.1.5 Solving issues in the periodic domain

We will be solving this problem in $\mathbb{T}^3$. Working on a periodic domain gives a new problem. It is now possible to define a surface through the boundaries, which is not the result we are looking for however it may be a minimal surface. One example of such surfaces is given in figure 4.2.

One way to solve this problem would be to make sure the boundaries of our ambient manifold are sufficiently far away from the boundary to which we want to find a surface, that way any surface that does cross the boundaries will automatically be bigger than one that does not cross the boundaries. However another method involves calculating how big the expected "flux" of the surface is and then adding that flux as a constraint in all three principle directions. Although the second method may sound somewhat complicated it will also give some advantages to be used later on.

To get familiar with the approach we once again look at the surfaces in figure 4.2. Say the size of the cube is $1 \times 1 \times 1$ in each dimension and the radius of the circle is 0.45. This means the flux in the z-direction (upwards) is $\pi \cdot 0.45^2 \approx 0.64$. The area of the pink surface is then 0.36, so if one constraint would be

$$Flux, z = 0.64,$$

that would mean that the pink surface would not be a valid result. We can then add a similar constraint for the flux in the x and y direction. (In this specific case their flux would be 0.)

Now we need to find a linear constraint that enforces this property. The flux in any specific direction is the inner product between that particular coordinate vector field and the surface normal vector.

$$\begin{aligned}\mathbf{F} &= \iint_{\Sigma} \overrightarrow{d\Sigma} = \iint_{\Sigma} \mathbf{N}_{\Sigma} dS \\ F_x &= \iint_{\Sigma} \mathbf{N}_{\Sigma} \cdot \hat{x} dS = \iint_{\Sigma,x} dS,\end{aligned} \tag{4.11}$$

where $\hat{x}$ is the unit vector field in the x direction.

If we know that the surface Σ (and therefore also the boundary Γ) do not pass through the periodic boundaries we can calculate the last integral using Stokes' theorem. Note that using Stokes' theorem is not generally applicable on $\mathbb{T}^3$ but only on $\mathbb{R}^3$.

However the fact that the boundary Γ does not pass through the periodic boundaries of $\mathbb{T}^3$ means that there is a neighbourhood around Γ and Σ that is equivalent to $\mathbb{R}^3$. Therefore the result should be the same:

$$F_x = \iint_{\Sigma,x} dS \overset{Stokes'}{=} \frac{1}{2}\oint_{\Gamma} y\ dz - z\ dy. \tag{4.12}$$

Note that this can be calculated using only the boundary Γ. If we repeat this step for the y and z direction we get the following:

$$\mathbf{F} = \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix} = \frac{1}{2}\oint_{\Gamma} \begin{pmatrix} y\ dz - z\ dy \\ z\ dx - x\ dz \\ x\ dy - y\ dx \end{pmatrix} = \frac{1}{2}\oint_{\Gamma} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \times \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \frac{1}{2}\oint_{\Gamma} \gamma \times d\gamma. \tag{4.13}$$

We now have a way to calculate the desired value of the flux in each direction. To add this as a constraint on the vector field we need to take an integral over each seperate coordinate of the vector field giving the following boundary constraint for $x_i = x, y, z$:

$$\iiint_M \overrightarrow{\eta}_{x_i} dx^3 = F_{x_i}. \tag{4.14}$$

We now add this constraint to our problem:

Problem 5. *Given a boundary curve* $\overrightarrow{\delta_\Gamma}/\gamma(t)$ *with the flux* $\mathbf{F} = \frac{1}{2}\oint_\Gamma \gamma \times d\gamma$ *find an* $\overrightarrow{\eta}$ *with minimal mass norm:*

$$\underset{\overrightarrow{\eta}:\overrightarrow{\delta_\Gamma}=(\nabla\times\overrightarrow{\eta}),\iiint_{\mathbb{T}^3}\overrightarrow{\eta}_{x_i}dx^3=F_{x_i}}{\text{minimise}} \|\overrightarrow{\eta}\|_{mass} \tag{4.15}$$

4.1.6 Helmholtz decomposition of $\overrightarrow{\eta}$

As explained in section 3.2 any sufficiently smooth vector field on $\mathbb{T}^3$ has a Helmholtz decompostion which is as follows:

$$\overrightarrow{\eta} = \overrightarrow{G} + \overrightarrow{R} + \overrightarrow{H}, \tag{4.16}$$

where $\overrightarrow{G} = \nabla\phi$, $\overrightarrow{R} = \nabla \times \overrightarrow{A}$ and $\overrightarrow{H}$ is harmonic. This decomposition is unique.

We will show that the constraint $\overrightarrow{\delta_\Gamma} = \nabla \times \overrightarrow{\eta}$ fully removes the degrees of freedom in $\overrightarrow{R}$.

Proposition 1. *Any two vector fields* $\overrightarrow{\eta_1}$,$\overrightarrow{\eta_2}$ *for which* $\overrightarrow{\delta_\Gamma} = \nabla \times \overrightarrow{\eta_i}$ *holds, have the same rotational field* $\overrightarrow{R}$ *in the Helmholtz decomposition.*

Proof. Take two vector fields $\overrightarrow{\eta_1}$ and $\overrightarrow{\eta_2}$ for which the following holds:

$$\overrightarrow{\delta_\Gamma} = \nabla \times \overrightarrow{\eta_1} = \nabla \times \overrightarrow{\eta_2}. \tag{4.17}$$

Then by linearity of the curl operator we know the following:

$$\nabla \times (\overrightarrow{\eta_1} - \overrightarrow{\eta_2}) = 0 \tag{4.18}$$

$$\nabla \times \left(\overrightarrow{G_1} + \overrightarrow{R_1} + \overrightarrow{H_1} - \left(\overrightarrow{G_2} + \overrightarrow{R_2} + \overrightarrow{H_2}\right)\right) = 0, \tag{4.19}$$

but since $\overrightarrow{G}$ is a gradient field and $\overrightarrow{H}$is harmonic they both have no curl. It follows that:

$$\nabla \times \left(\overrightarrow{R_1} - \overrightarrow{R_2}\right) = 0. \tag{4.20}$$

Both fields are can be expressed as the curl of another field which we will call $\overrightarrow{A_1}$ and $\overrightarrow{A_2}$ respectively. we will call $\overrightarrow{A_1} - \overrightarrow{A_2} = \overrightarrow{B}$. We know have the following:

$$\nabla \times \left(\nabla \times \overrightarrow{A_1} - \nabla \times \overrightarrow{A_2}\right) = \nabla \times \left(\nabla \times \overrightarrow{B}\right) = 0. \tag{4.21}$$

$$0 = \iiint \nabla \times \left(\nabla \times \overrightarrow{B}\right) \cdot \overrightarrow{B} dx^3 = \iiint \nabla \cdot \left(\left(\nabla \times \overrightarrow{B}\right) \times \overrightarrow{B}\right) + \left(\nabla \times \overrightarrow{B}\right) \cdot \left(\nabla \times \overrightarrow{B}\right). \tag{4.22}$$

The first part vanishes because of Gauss' theorem and the fact that $\mathbb{T}^3$ has no boundary. The second part is exactly the inner product of $\nabla \times \overrightarrow{B}$ with itsself so $\nabla \times \overrightarrow{B} = 0$. But since $\nabla \times \overrightarrow{B} = \overrightarrow{R_1} - \overrightarrow{R_2}$ it means they are the same. □

We will now similarly show that the constraints $\iiint_M \overrightarrow{\eta}_{x_i} dx^3 = F_{x_i}$ fully remove the degrees of freedom from the harmonic part of $\overrightarrow{\eta}$

Proposition 2. *Any two vector fields $\overrightarrow{\eta_1}$,$\overrightarrow{\eta_2}$ for which $\iiint_M \overrightarrow{\eta}_{x_i} dx^3 = F_{x_i}$ holds have the same $\overrightarrow{H}$ in the Helmholtz decomposition.*

Proof. We will show that this holds for $\overrightarrow{H}_x$ but the proof for the other dimensions works similarly.

First we will show that the constraint does not affect the fields $\overrightarrow{G}$ and $\overrightarrow{R}$:

$$\iiint_{\mathbb{T}^3} \eta_x \; dx^3 = \iiint_{\mathbb{T}^3} G_x + R_x + H_x dx^3 = F_x, \tag{4.23}$$

but $G_x = \frac{\partial}{\partial x}\phi$ and $R_x = \frac{\partial}{\partial y} B_z - \frac{\partial}{\partial z} B_y$ so the integral over these functions vanishes.

Now assume there are two solutions to the constraints $\overrightarrow{\eta_1}$ and $\overrightarrow{\eta_2}$. We have:

$$\iiint_{\mathbb{T}^3} \eta_{x,1} - \eta_{x,2} \; dx^3 = \iiint_{\mathbb{T}^3} H_{x,1} - H_{x,2} \; dx^3 = F_x - F_x = 0. \tag{4.24}$$

But since harmonic functions are constant the only way the integral can be zero is if $H_{x,1} - H_{x,2}$ is constantly zero. Therefore they must be the same function.

□

Now that we know these constraints precisely define the fields $\overrightarrow{R}$ and $\overrightarrow{H}$ we can find the corresponding fields and solve for $\overrightarrow{G}$. Our problem now looks like this:

Problem 6. *Given an initial guess $\overrightarrow{\eta_0}$ such that $\overrightarrow{\delta_\Gamma} = (\nabla \times \overrightarrow{\eta_0})$ and $\int_M \overrightarrow{\eta_0}_{x_i} dx^3 = F_{x_i}$ find a ϕ such that it minimises the mass norm of $\overrightarrow{\eta_0} + \nabla\phi$:*

$$\underset{\phi \in \mathbb{T}^3 \mapsto \mathbb{R}}{\text{minimise}} \|\overrightarrow{\eta_0} + \nabla\phi\|_{\text{mass}}. \tag{4.25}$$

This makes the number of variables 3 times as small as we are now minimising a scalar field instead of a vector field. Furthermore computing the field $\overrightarrow{\eta_0}$ can be easily done as will be shown in section 4.1.7.

4.1.7 Computing $\overrightarrow{\eta_0}$

We will now focus on finding the initial guess $\overrightarrow{\eta_0}$. To find $\overrightarrow{\eta_0}$ we need a function that solves both $\overrightarrow{\delta_\Gamma} = (\nabla \times \overrightarrow{\eta_0})$ and $\int_M \overrightarrow{\eta_0}_{x_i} dx^3 = F_{x_i}$. We choose to solve for the rotational field first and the harmonic fields last, because the latter are constant and therefore more straightforward to add to the solution for the first constraint.

To solve $\overrightarrow{\delta_\Gamma} = (\nabla \times \overrightarrow{\eta_0})$ we search for a rotational field $\overrightarrow{R} = \nabla \times \overrightarrow{\psi}$ that fulfils that constraint. We have shown in the previous section why only the rotational field has to be found. Now we are in fact searching for the field $\overrightarrow{\psi}$ which has the constraint $\overrightarrow{\delta_\Gamma} = \nabla \times \left(\nabla \times \overrightarrow{\psi}\right)$. However this is a relatively difficult equation to solve and it has a large solution set. To remove degrees of freedom from this solution we add the following constraint: $\nabla\left(\nabla \cdot \overrightarrow{\psi}\right) = 0$. This gives the following equation to solve:

$$\overrightarrow{\delta_\Gamma} = \nabla \times \left(\nabla \times \overrightarrow{\psi}\right) - \nabla\left(\nabla \cdot \overrightarrow{\psi}\right) = \nabla^2 \overrightarrow{\psi}. \tag{4.26}$$

We claim that this still gives a solution for the original problem.

Proposition 3. *If $\overrightarrow{\psi}$ meets the requirement $\overrightarrow{\delta_\Gamma} = \nabla^2 \overrightarrow{\psi}$ and $\nabla \times \overrightarrow{\eta} = \overrightarrow{\delta_\Gamma}$ it also holds that $\overrightarrow{\delta_\Gamma} = \nabla \times \left(\nabla \times \overrightarrow{\psi}\right)$*

Proof. Since $\overrightarrow{\delta_\Gamma} = \nabla \times \overrightarrow{\eta}$ for some $\overrightarrow{\eta}$ (because it is a boundary curve) we get:

$$\overrightarrow{\delta_\Gamma} = \nabla \times \left(\nabla \times \overrightarrow{\psi}\right) - \nabla\left(\nabla \cdot \overrightarrow{\psi}\right) = \nabla \times \overrightarrow{\eta} \tag{4.27}$$

$$\nabla \times \left(\nabla \times \overrightarrow{\psi} - \overrightarrow{\eta}\right) = \nabla\left(\nabla \cdot \overrightarrow{\psi}\right). \tag{4.28}$$

We can calculate the inner product of $\nabla\left(\nabla \cdot \overrightarrow{\psi}\right)$ with itsself:

$$\left\langle \nabla\left(\nabla \cdot \overrightarrow{\psi}\right), \nabla\left(\nabla \cdot \overrightarrow{\psi}\right)\right\rangle = \left\langle \nabla \times \left(\nabla \times \overrightarrow{\psi} - \overrightarrow{\eta}\right), \nabla\left(\nabla \cdot \overrightarrow{\psi}\right)\right\rangle \tag{4.29}$$

$$\begin{aligned}&\iiint_{\mathbb{T}^3} \nabla \times \left(\nabla \times \overrightarrow{\psi} - \overrightarrow{\eta}\right) \cdot \nabla\left(\nabla \cdot \overrightarrow{\psi}\right) dx^3 = \\ &\quad \iiint_{\mathbb{T}^3} \nabla \cdot \left(\left(\nabla \times \overrightarrow{\psi} - \overrightarrow{\eta}\right) \times \nabla\left(\nabla \cdot \overrightarrow{\psi}\right)\right) + \left(\nabla \times \overrightarrow{\psi} - \overrightarrow{\eta}\right) \cdot \nabla \times \nabla\left(\nabla \cdot \overrightarrow{\psi}\right) dx^3.\end{aligned} \tag{4.30}$$

The second part vanishes because the curl of a gradient is zero, the first part is a divergence over an interior so it can be expressed as a integral over the boundary via Gauss. however $\mathbb{T}^3$ has no boundary so that part vanishes as well. But we were calculating the inner product of $\nabla(\nabla \cdot \overrightarrow{\psi})$ with itself, so if that is zero, $\nabla(\nabla \cdot \overrightarrow{\psi})$ itself must be zero. Therefore:

$$\overrightarrow{\delta_\Gamma} = \nabla \times (\nabla \times \overrightarrow{\psi})$$

and our claim is proven. □

The reason we want to solve the Poisson equation is because it is a relatively easy operator to use. Solving the Poisson equation is a pointwise operation in the Fourier domain. After solving for $\overrightarrow{\psi}$ we can simply find $\overrightarrow{R}$ by computing $\nabla \times \overrightarrow{\psi}$.

The Poisson equation is solved up to a constant, so we can add the right harmonic field by choosing the correct constant such that $\int_{\mathbb{T}^3} \overrightarrow{H}_{x_i} dx^3 = F_{x_i}$. ($\overrightarrow{R}$ has no contribution to the integral.) Then $\overrightarrow{\eta_0} = \overrightarrow{R} + \overrightarrow{H}$.

4.1.8 The final problem

We have shown that for any piecewise continuous and orientable boundary curve it is possible to formulate a convex numerical problem which solves for the minimal surface. It is as follows:

Problem 7. *Given an initial guess $\overrightarrow{\eta_0}$ such that $\overrightarrow{\delta_\Gamma} = (\nabla \times \overrightarrow{\eta_0})$ and $\int_M \overrightarrow{\eta_0}_{x_i} dx^3 = F_{x_i}$ find a ϕ such that it minimises the mass norm of $\overrightarrow{\eta_0} + \nabla\phi$:*

$$\underset{\phi \in \mathbb{T}^3 \mapsto \mathbb{R}}{\text{minimise}} \|\overrightarrow{\eta_0} + \nabla\phi\|_{\text{mass}}, \tag{4.31}$$

where the initial guess is computable with only information about the boundary curve.

4.2 Finding minimal curves between boundary points

In this section we will show a similar approach to solve for minimal curves between boundary points. This problem is similar to the Plateau problem only its boundaries and variables have one less dimension. Showing how this works is done as a proof of concept, to show that the approach can be generalised to other situations with different dimensions. For this section we will be working on the manifold $M = \mathbb{T}^2$. This problem can also be solved on $\mathbb{T}^3$, however we choose to work in two dimensions to show this is not a limitation of the method.

We begin by stating the problem:

Problem 8. *Given two sets of boundary points U and V on $\mathbb{T}^2$, find an oriented curve Λ on $\mathbb{T}^2$ with minimal area, that is:*

$$\underset{\Lambda : \partial\Lambda = U \cup V}{\text{minimise}} \text{ Length}(\Lambda). \tag{4.32}$$

Here U is the set of starting points and V is the set of ending points. Here the sets U and V are the same size since each begin point must also have an endpoint, however points may overlap. Once again we will replace the curves and their boundaries by vector currents. In this case the curve(s) which we are minimising becomes the current $\overrightarrow{\delta_\Lambda}$ and the boundary points become the normal Dirac-δ's.

We will once again use the mass-norm which is similar to the one defined in section 4.1.2. In two dimensions it looks as follows:

$$\|\overrightarrow{\eta}\| = \sup_{V: M \to \mathbb{R}^2, \forall \mathbf{x} \in M : \|V(\mathbf{x})\| \leq 1} \iint_M V \cdot \overrightarrow{\eta}\, dx^2 \tag{4.33}$$

4.2.1 changing boundary constraint

In section 3.1 it was shown that:

$$\delta_{\partial\Lambda} = \nabla \times \overrightarrow{\delta_\Lambda} \tag{4.34}$$

So that means the statement $\partial\Lambda = U \cup V$ is equivalent to saying $\rho_{U,V} = \nabla \times \overrightarrow{\delta_\Lambda}$. Here $\rho_{U,V}$ is the sum of positive Dirac-δ's on the set U and negative ones on the set V:

$$\rho_{U,V} = \sum_{u\in U} \delta_u - \sum_{v\in V} \delta_v. \tag{4.35}$$

We can now redefine problem 8:

Problem 9. *Given boundary measure $\rho_{U,V}$ find a current $\overrightarrow{\delta_\Lambda}$ with minimal mass norm:*

$$\underset{\overrightarrow{\delta_\Lambda}:\rho_U - \rho_V = \nabla\times\overrightarrow{\delta_\Lambda}}{\text{minimise}} \|\overrightarrow{\delta_\Lambda}\|_{\text{mass}}. \tag{4.36}$$

We can once again use the theorem from Federer and Fleming [9] to show that the variable $\overrightarrow{\delta_\Lambda}$ can be replace by an arbitrary vector current $\overrightarrow{\zeta}$ while still can converging to the same result. The problem then looks like this:

Problem 10. *Given a boundary measure $\rho_{U,V}$ find a current $\overrightarrow{\zeta}$ with minimal mass norm:*

$$\underset{\overrightarrow{\zeta}:\rho_U - \rho_V = \nabla\times\overrightarrow{\zeta}}{\text{minimise}} \|\overrightarrow{\zeta}\|_{\text{mass}}. \tag{4.37}$$

4.2.2 Solving further issues and simplifying the problem

Once again we will encounter certain issues in the periodic domain which we will solve in this section. Because of the periodic domain a solution curve could connect through the edges of the torus. To prevent the solution from going through the edges we can calculate the expected length in both principle directions under the assumption that the solution does not cross the edges. Then we add the expected length as a condition on the solution.

Assuming the solution curve or curves do not pass through the edges the length in a principle direction, say the x direction is exactly the x-coordinate of the endpoint minus that of the starting point so the total expected length of the solution is as follows:

$$\mathbf{L}_x = \sum_V x_v - \sum_U x_u. \tag{4.38}$$

This correspond with the full expression for the total length:

$$\mathbf{L} = \sum_V \begin{bmatrix} x_v \\ y_v \\ z_v \end{bmatrix} - \sum_U \begin{bmatrix} x_u \\ y_u \\ z_u \end{bmatrix}. \tag{4.39}$$

An example for a single set of points is shown in figure 4.3.

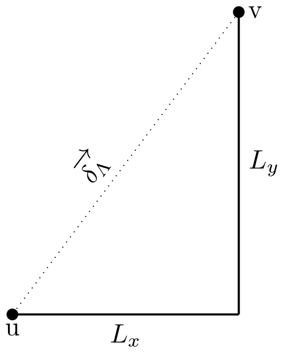

Figure 4.3: This figure shows a possible solution $\overrightarrow{\delta_\Lambda}$ between points u and v. It is easy to see that L_x and L_y can be calculated using only the points u and v.

Now we can add a constraint by measuring the coordinate fields of the solution:

$$\iint_M \overrightarrow{\zeta}_{x_i} dx^2 = L_{x_i} \tag{4.40}$$

We now have the following problem:

Problem 11. *Given a set of boundary points V and U with the precalculated length $\mathbf{L} = \sum_V v - \sum_U u$ find a $\overrightarrow{\zeta}$ with minimal mass norm:*

$$\underset{\overrightarrow{\zeta}:\rho_U - \rho_V = \nabla \times \overrightarrow{\zeta}, \iint_M \overrightarrow{\zeta}_{x_i} dx^2 = L_{x_i}}{\text{minimise}} \|\overrightarrow{\zeta}\|_{\text{mass}}. \tag{4.41}$$

4.2.3 Helmholtz decomposition of $\overrightarrow{\zeta}$

Once again we can use the Helmholtz decomposition of $\overrightarrow{\zeta}$ to show that the constraints limit the harmonic and the gradient fields. The Helmholtz decomposition of $\overrightarrow{\zeta}$ is as follows:

$$\overrightarrow{\zeta} = \overrightarrow{G} + \overrightarrow{R} + \overrightarrow{H}, \tag{4.42}$$

where $\overrightarrow{G} = \nabla g$ where g is a scalar field. $\overrightarrow{R} = \nabla \times r$ where r is a scalar field and the curl over a scalar field is defined as $\nabla \times r = -\frac{\partial r}{\partial y}\hat{x} + \frac{\partial r}{\partial x}\hat{y}$. The field $\overrightarrow{H}$ is harmonic.

The first constraint, $\overrightarrow{\zeta} : \rho_U - \rho_V = \nabla \times \overrightarrow{\zeta}$, only limits the rotational field, since both harmonic and gradient fields have no curl. furthermore:

$$\nabla \times (\nabla \times u) = \nabla^2 u, \tag{4.43}$$

so solving for the rotation of a rotational field is the equivalent to solving the Poisson equation.

Proposition 4. *Any two vector fields $\overrightarrow{\zeta_1}$ and $\overrightarrow{\zeta_2}$ for which $\rho_{U,V} = \nabla \times \overrightarrow{\zeta}$ holds have the same $\overrightarrow{R}$ in the Helmholtz decomposition*

Proof.

$$\rho_{U,V} = \nabla \times \overrightarrow{\zeta} \tag{4.44}$$

$$\rho_{U,V} = \nabla \times \left(\overrightarrow{G} + \overrightarrow{R} + \overrightarrow{H}\right) \tag{4.45}$$

$$\rho_{U,V} = \nabla \times \overrightarrow{R} \tag{4.46}$$

Assume there are two fields $\overrightarrow{\zeta_1}$ and $\overrightarrow{\zeta_2}$ for which this holds then:

$$\nabla \times \overrightarrow{\zeta_1} - \nabla \times \overrightarrow{\zeta_2} = \nabla \times \left(\overrightarrow{R_1} - \overrightarrow{R_2}\right) \tag{4.47}$$

$$0 = \nabla \times (\nabla \times r_1 - \nabla \times r_2) \tag{4.48}$$

$$0 = \nabla^2 (r_1 - r_2) \tag{4.49}$$

So the difference between r_1 and r_2 is at most a constant which means that the difference between $\overrightarrow{R_1}$ and $\overrightarrow{R_2}$ is zero. Therefore the constraint $\overrightarrow{\zeta} : \rho_U - \rho_V = \nabla \times \overrightarrow{\zeta}$ limits the field $\overrightarrow{R}$ to one specific field.

□

Furthermore the harmonic field is limited by the second constraint: $\iint_M \overrightarrow{\zeta}_{x_i} dx^2 = L_{x_i}$. This is because both the gradient and the curl field have no contribution to an integral over the whole field, since they are derivatives on a torus. The harmonic field is constant in all directions and therefore the integral over the field in each direction precisely limits the value of the harmonic field.

We can now search for a field $\overrightarrow{\zeta_0}$ for which the constraints hold and then only solve for the gradient field. finding the field to find $\overrightarrow{\zeta_0}$ first solve for r:

$$\nabla^2 r = \rho_{U,V}, \tag{4.50}$$

then $\overrightarrow{\zeta_0}$ is

$$\overrightarrow{\zeta_0} = \nabla \times r + L_x \hat{x} + L_y \hat{y}. \tag{4.51}$$

Then the problem is as follows:

Problem 12. *Given a field $\overrightarrow{\zeta_0}$ for which the constraints $\overrightarrow{\zeta_0} : \rho_{U,V} = \nabla \times \overrightarrow{\zeta_0}$, $\iint_M \overrightarrow{\zeta_0}_{x_i} dx^2 = L_{x_i}$ hold. find a g such that $\nabla g + \overrightarrow{\zeta_0}$ has minimal mass norm:*

$$\underset{g:\mathbb{T}^2 \to \mathbb{R}}{\text{minimise}} \; \|\nabla g + \overrightarrow{\zeta_0}\|_{\text{mass}}. \tag{4.52}$$

Chapter 5

Numerical implementation and results

We will go through the used numerical methods in this chapter. First we will discuss the three dimensional case and then the two dimensional case.

5.1 Numerically minimising surfaces

The problem of minimising the area of a surface with a given boundary has been shown in chapter 4 to be equivalent to solving the following problem:

Problem 13. *Given an initial guess $\overrightarrow{\eta_0}$ such that $\overrightarrow{\delta_\Gamma} = (\nabla \times \overrightarrow{\eta_0})$ and $\int_M \overrightarrow{\eta_0}_{x_i} dV = F_{x_i}$ find a ϕ such that it minimises the mass norm of $\overrightarrow{\eta_0} + \nabla\phi$:*

$$\underset{\phi:\mathbb{T}^3\to\mathbb{R}}{\text{minimise}} \|\overrightarrow{\eta_0} + \nabla\phi\|_{\text{mass}}. \tag{5.1}$$

To solve the problem numerically we must take a few steps. First we must discretize the problem.

5.1.1 Discretisation

We will be working with an $n \times n \times n$ grid, we choose the size of one cell to be $1 \times 1 \times 1$. indexing of points on the grid will be done by indices i, j, k for the directions x, y, z respectively.

The curve γ will be divided into m steps, where m has to be large enough so the curve doesn't skip over grid points. So γ is a matrix of dimensions $m \times 3$.

We will describe the process of discretising the vector current $\overrightarrow{\delta_\Gamma}$, a 2 dimensional case is shown in figure 5.1. The vector current $\overrightarrow{\delta_\Gamma}$ is discretised as follows: we look at all the grid cells which the curve γ passes through and label them in the order in which they are passed through. We will call these cells $c_0, ..., c_l$. Each time the curve enters a new grid cell c_{i+1} we add a vector to the grid point associated with this new grid cell (the cell $[0,1] \times [0,1] \times [0,1]$ is associated with point $(0,0,0)$). This vector is exactly the translation vector from the last cell, c_i to the new cell c_{i+1}.

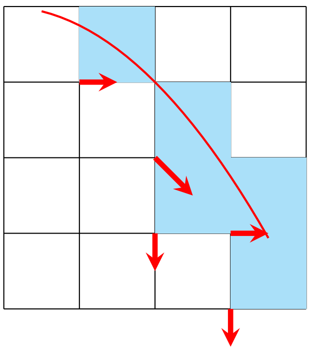

Figure 5.1: The red curve passes through the highlighted light-blue cells, a red vector represents the difference vector between the origin (lower left vertex) of the present and that of the previous cell.

The operation of integrating over volume is simply taking the sum over all grid points in the space over which one is integrating:

$$\int_M f dV = \sum_{i,j,k=0}^{n,n,n} f_{i,j,k}. \tag{5.2}$$

We will define derivatives both as forward and backward difference. A scalar function has derivatives in the x, y and z directions. The forward derivatives are as follows (ommited indices remain unchanged):

$$\frac{\partial f}{\partial x}^F_{i,j,k} = f_{i+1} - f_i \qquad \frac{\partial f}{\partial y}^F_{i,j,k} = f_{j+1} - f_j \qquad \frac{\partial f}{\partial z}^F_{i,j,k} = f_{k+1} - f_k. \tag{5.3}$$

The backwards derivatives are as follows:

$$\frac{\partial f}{\partial x}^B_{i,j,k} = f_i - f_{i-1} \qquad \frac{\partial f}{\partial y}^B_{i,j,k} = f_j - f_{j-1} \qquad \frac{\partial f}{\partial z}^B_{i,j,k} = f_k - f_{k-1}. \tag{5.4}$$

The second derivatives are always a backward and forward derivative combined:

$$\frac{\partial^2 f}{\partial x^2}_{i,j,k} = f_{i+1} - 2f_i + f_{i-1}. \tag{5.5}$$

The gradient of a scalar function f is defined as the forward derivative in all three directions:

$$\nabla f_{i,j,k} = \frac{\partial f}{\partial x}^F_{i,j,k} \hat{x} + \frac{\partial f}{\partial y}^F_{i,j,k} \hat{y} + \frac{\partial f}{\partial z}^F_{i,j,k} \hat{z} \tag{5.6}$$

The curl of a field G is defined as using the backwards derivatives so the x component of the curl is as follows:

$$\nabla \times G_{x;i,j,k} = \frac{\partial G_z}{\partial y}^B_{i,j,k} - \frac{\partial G_y}{\partial z}^B_{i,j,k} \hat{x}. \tag{5.7}$$

The divergence of a field G is calculated with backwards derivatives, as follows:

$$\nabla \cdot G_{x;i,j,k} = \left.\frac{\partial G_x}{\partial x}^B\right._{i,j,k} + \left.\frac{\partial G_y}{\partial y}^B\right._{i,j,k} + \left.\frac{\partial G_z}{\partial z}^B\right._{i,j,k}. \tag{5.8}$$

The vector Laplace operator on a vector field G is defined as the second derivate for each component in its associated direction:

$$\Delta G_{i,j,k} = \left.\frac{\partial^2 G_x}{\partial x^2}\right._{i,j,k} \hat{x} + \left.\frac{\partial^2 G_y}{\partial y^2}\right._{i,j,k} \hat{y} + \left.\frac{\partial^2 G_z}{\partial z^2}\right._{i,j,k} \hat{z}. \tag{5.9}$$

5.1.2 Finding η_0

In chapter 4 we have shown we can compute the initial guess using the following steps:

- First we solve the Poisson equation $\Delta \psi = \overrightarrow{\delta_\Gamma}$, and $\eta_0 = \nabla \times \psi$
- Then we add a constant (harmonic) field such that its contribution is exactly: $\mathbf{F} = \frac{1}{2} \oint_\Gamma \gamma \times d\gamma$.

We solve the Poisson equation in the Fourier domain, where solving the Poisson equation is the same as dividing by the following factor:

$$w_{i,j,k} = 4\left(\sin^2\left(\frac{\pi i_x}{n_x}\right) + \sin^2\left(\frac{\pi i_y}{n_y}\right)\sin^2\left(\frac{\pi i_z}{n_z}\right)\right) \tag{5.10}$$

for each value v on vertex i_x, i_y, i_z. So we take the FFT of $\overrightarrow{\delta_\Gamma}$ multiply each point by the factor $w_{i,j,k}$ and use the inverse FFT to get the solution for ψ After doing this for the x, y and z components and going back to the normal domain we have the function ψ now we only need to take the rotation to find the solution to constraint $\nabla \times \eta_0 = \overrightarrow{\delta_\Gamma}$.

We can now find the harmonic field by solving the integral $\frac{1}{2} \oint_\Gamma \gamma \times d\gamma$. This integral needs to be calculated per component and the discretisation looks as follows:

$$\mathbf{F}_x = \frac{1}{2} \oint_\Gamma \gamma_y dz - \gamma_z dy \tag{5.11}$$

$$\mathbf{F}_x = \sum_{i=0}^{m} \frac{\gamma_{y;i+1} + \gamma_{y;i}}{2} \left(\gamma_{z;i+1} - \gamma_{z;i}\right) - \frac{\gamma_{z;i+1} + \gamma_{z;i}}{2} \left(\gamma_{y;i+1} - \gamma_{y;i}\right), \tag{5.12}$$

here γ is discretised into m points. Afterwards we can add $\mathbf{F}_x/n^3$ to each grid point for the correct harmonic field. We repeat this for the y and z directions.

5.1.3 Solving the unconstrained problem

We have now reconstructed the field η_0 and can start minimising the following function:

$$\text{minimise } \|\nabla\phi + \overrightarrow{\eta_0}\|_{\text{mass}}. \tag{5.13}$$

Here the variable ϕ becomes a vector with length n^3, one entry for each grid point. The gradient is composed of 3 $n^3 \times n^3$ matrices, for each direction (x, y, z) there is one matrix which calculates the gradient of ϕ. For each principle direction η_0 has n^3 entries, so it is stored as three length n^3 vectors.

The norm can be written as the following function:

$$f(x) = \sum_{i=0}^{n^3} \sqrt{\left(\frac{\partial}{\partial x}\phi + \eta_{0;x}\right)_i^2 + \left(\frac{\partial}{\partial z}\phi + \eta_{0;y}\right)_i^2 + \left(\frac{\partial}{\partial z}\phi + \eta_{0;z}\right)_i^2}, \tag{5.14}$$

where the derivatives are $n^3 \times n^3$ matrices and the other terms are length n^3 vectors.

Since this is a convex problem a multitude of minimisation methods can be used. For the results shown below the method "L-BFGS" has been used.

5.2 Finding minimal curves

While minimal curves can be easily found using analytical methods and will result in straight lines, as long as the metric is euclidean, we will show how the same principle can be used to find minimal curves numerically.

The problem we are solving is as follows:

Problem 14. *Given a field $\overrightarrow{\zeta_0}$ for which the constraints $\overrightarrow{\zeta_0} : \rho_U - \rho_V = \nabla \times \overrightarrow{\zeta_0}, \iint_M \overrightarrow{\zeta_0}_{x_i} dV = L_{x_i}$ hold. find a g such that $\nabla g + \overrightarrow{\zeta_0}$ has minimal mass norm:*

$$\text{minimise } \left\| \nabla g + \overrightarrow{\zeta_0} \right\|_{\text{mass}}. \tag{5.15}$$

5.2.1 Discretisation

Computing minimal curves works similarly to computing minimal surfaces. Our grid is now an $n \times n$ grid, once again the distance between grid points has size 1. indices i, j will be used for the directions x, y respectively.

the boundary points will be stored as single grid points with value 1 for starting points and value -1 for ending points.

Integrating over areas will be the same as taking a sum over all grid points just as in three dimensions. While derivatives in 1 direction stay the same operators such as gradients curl and divergence change slightly.

Let p be a scalar function and P a two dimensional vector field, then

$$\nabla p_{i,j} = \frac{\partial p}{\partial x}_{i,j}^{F} \hat{x} + \frac{\partial p}{\partial y}_{i,j}^{F} \hat{y} \qquad (\nabla \times)_s p_{i,j} = -\frac{\partial p}{\partial y}_{i,j}^{F} \hat{x} + \frac{\partial p}{\partial x}_{i,j}^{F} \hat{y} \tag{5.16}$$

$$\nabla \cdot P_{i,j} = \frac{\partial P_x}{\partial x}_{i,j}^{B} + \frac{\partial P_y}{\partial y}_{i,j}^{B} \qquad (\nabla \times)_v P_{i,j} = \frac{\partial P_y}{\partial x}_{i,j}^{B} - \frac{\partial P_x}{\partial y}_{i,j}^{B}. \tag{5.17}$$

The Laplace operator on a scalar function p in two dimensions is as follows:

$$\Delta p_{i,j} = \frac{\partial^2 p}{\partial x^2}_{i,j} + \frac{\partial^2 p}{\partial y^2}_{i,j}. \tag{5.18}$$

5.2.2 Finding ζ_0

We start again by computing the initial guess ζ_0. For this we need to solve the Poisson equation $\Delta\xi = (\rho_U - \rho_V)$. This equation can then be solved in the Fourier domain precisely as in the 3-d version by dividing by the factor

$$4\left(\sin^2\left(\frac{\pi i_x}{n_x}\right) + \sin^2\left(\frac{\pi i_y}{n_y}\right)\sin^2\right) \tag{5.19}$$

Note that in this case both ξ and $(\rho_U - \rho_V)$ have a one dimensional output. We then take the scalar curl of ξ to find ζ_0

Next we calculate the harmonic field that needs to be added. For this we calculate the following:

$$\mathbf{L}_x = \sum_U u_x - \sum_V v_x \tag{5.20}$$

We once again divide by the number of grid points (n^2) and add the constant value to the x component of ζ_0. We then repeat these steps for the y direction. We can now start minimising the norm once again.

5.2.3 Solving the unconstrained problem

in 2d the discretization of the norm is as follows:

$$f(x) = \sum_{i=0}^{n^2}\sqrt{\left(\frac{\partial}{\partial x}g + \eta_{0;x}\right)_i^2 + \left(\frac{\partial}{\partial z}g + \eta_{0;y}\right)_i^2} \tag{5.21}$$

5.3 Results

Figure 5.2 shows the result of finding the minimal surface given a the following boundary Γ, it creates a saddle shaped surface:

$$x = cos(t) \quad y = sin(t) \quad z = sin(2t). \tag{5.22}$$

Figure 5.3 shows a minimised line between two coordinates (5,5) and (10,13). Both sets of points are retrieved by setting a threshold above which size vectors contribute to the surface or the line.

Code used is can be accessed via the following link: [12].

Figure 5.2: A point cloud of the minimal surface for the boundary $\gamma : x = cos(t) \quad y = sin(t) \quad z = sin(2t)$.

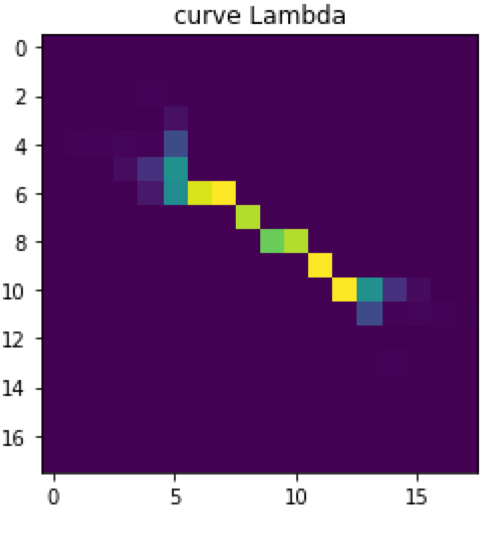

Figure 5.3: A graph of points on the line which was found between the coordinates (5,5) and (10,13).

Chapter 6

Discussion and conclusion

In this chapter we will discuss how successful the method described in Wang and Chern is for certain applications and how successful the translation to vector fields have been.

The method by Wang and Chern is applicable to many situations Its conditions are as follows the boundary has to be a piecewise smooth m dimensional object so that it can be represented by a current. the resulting m+1 dimensional object will then be either the unique minimisation or the superposition of multiple minimisations. The method relies on the boundary object not intersecting the edges of the manifold on which the problem is solved and therefore the objects have to be bounded. The method does not rely on topological correctness of the initial guess. The orientation of boundaries does matter, for example two circles which are turn in the same direction can never have a surface that connects the two.

6.1 Triply periodic minimal surfaces

Triply periodic minimal surfaces occur often in nano-biology [7] They often appear in large repeating lattice structures such as butterfly wings and exoskeletons of insects.

since triply periodic minimal surfaces do not have boundaries but are continuous surfaces one might think these cannot be found using the methods of Wang and Chern. However by looking at a small lattice element we get a boundary that does have edges, and as long as those edges can be modelled by a piecewise smooth current there is a way to find the minimal surface. The question now becomes how often the problem occurs of finding this minimal surface in nano-biology and how often in those cases is it possible to easily reconstruct the boundary edges given the lattice in which the minimal surface is sought.

The question is also if only the surface is minimised or if the volume of the lattice structure also influences the result. In the former case the method described in this paper should be sufficient, in the latter case more sophisticated methods should be found or used.

6.2 Success of translation

Mathematically and numerically there should be no difference between using the method described by Wang and Chern with differential forms, or using the method described in this paper with vector fields. The difference lies in how comfortable one is with certain notation or tools. While vector calculus may be easier to understand for many, differential forms have the large advantage of very concise notation and calculations.

6.3 Conclusion

The method described by Wang and Chern to compute minimal surfaces can be applied to almost any problem for minimising surfaces, the only drawback being that the orientation of forms or vector fields can play an important role for the convergence. As long as the problem only depends on the boundary this method will suffice. The method can be described in vector fields in stead of differential forms, in which case we trade an easier to understand language for a more concise one.

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