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Separated sequences in Banach spaces
(Gescheiden rijtjes in Banach ruimtes)

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Wouter van Heijst

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Wouter van Heijst

Technische Universiteit Delft

Begeleider

Prof.Dr. J.M.A.M. van Neerven

Overige commissieleden

Dr. M.H.A. Haase

Dr. J.G. Spandaw

Dr. E.A. Cator

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Delft

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Introduction

When a notion of distance is available one can consider the proximity of the elements of a set relative to each other. A set B is called r-separated if no two points are closer to each other than distance r. The largest possible lower bound $\text{sep}(B)$ on the pairwise distances is called the separation constant of B. In the setting of a normed vector space the distance between vectors x and y is expressed as the length of their difference $\|x-y\|$. The separation constant of a sequence (x_n) of vectors is given by $\text{sep}(x_n) = \inf_{m\neq n}\|x_n - x_m\|$. For $r > 0$ arbitrary an infinite r-separated set is easily found in any nontrivial normed vector space: $\{nr\frac{x}{\|x\|} : n \in \mathbb{N}\}$ suffices whenever $x \neq 0$. As this doesn't tell us anything about the structure of the vector space we are led to ask if it is possible to find an r-separated set in a bounded subset. How close to the diameter of the set can we choose r?

The closed unit ball B_X of a finite-dimensional vector space X is compact. Any sequence therein will have a convergent subsequence making it futile to hope for a separated sequence. Infinite-dimensional vector spaces on the other hand bring solace. The classical ℓ_p sequence spaces have $2^{\frac{1}{p}}$-separated standard bases ($e_n(j) = 1$ if $j = n$ and 0 otherwise)

$$\|e_n - e_m\| = \left(\sum_{j=1}^{\infty} |(e_n - e_m)(j)|^p\right)^{\frac{1}{p}} = (|1|^p + |-1|^p)^{\frac{1}{p}} = 2^{\frac{1}{p}}, \quad n \neq m.$$

Explicitly giving the separation of a sequence is not as easy in all Banach spaces. Still this is an encouraging result to look for separated sequences in infinite-dimensional Banach spaces in general.

Riesz's Lemma (1918). *Let X be a normed linear space and Y a closed proper subspace of X. Then for every $\theta \in (0,1)$ there is a vector x_θ in the unit sphere S_X of X such that for all $y \in Y$ the distance $\|x_\theta - y\| \geq \theta$.*

Let X be an infinite-dimensional Banach space. Fix $\theta \in (0,1)$ and pick any $x_1 \in S_X$. The linear span $\text{Lin}\{x_1\}$ of x_1 is a finite-dimensional subspace of X and thus both closed and a proper subspace of X. Riesz's lemma allows us to find $x_2 \in S_X \setminus \text{Lin}\{x_1\}$ such that no vector in $\text{Lin}\{x_1\}$ lies closer to x_2 than θ. The linear span $\text{Lin}\{x_1, x_2\}$ is again a closed and proper subspace of X. Repeating this procedure produces a θ-separated sequence $y_\theta = \{x_1, x_2, \cdots\}$ in the unit sphere of X. As we can do this for every $\theta \in (0,1)$ we can almost get a 1-separated sequence, but not quite. We define Kottman's constant $K(X)$ to describe this notion

$$K(X) = \sup\ \{\text{sep}(x_n) : (x_n) \in B_X\}.$$

The exercise with Riesz's lemma can be summarised as saying that $K(X) \geq 1$ for any infinite-dimensional Banach space X. Finite-dimensional spaces have Kottman's constant equal to zero and the ℓ_p example shows $K(\ell_p) \geq 2^{\frac{1}{p}}$.

A small improvement on the achievable separation was given by Kottman [11] using combinatorial methods in 1975, "The unit ball of every infinite-dimensional normed space contains a sequence where every two distinct elements have distance greater than 1". This does not improve the estimate on $K(X)$ as the infimum over pairs in the sequence can still be 1. Diestel gives a short noncombinatorial proof on page 7 of [4] exclaiming that Banach himself could have done this 40 years earlier.

The real breakthrough came about in 1981 when Elton and Odell published their result,

The Elton-Odell $(1+\varepsilon)$**-separation Theorem.** *If X is an infinite-dimensional normed linear space, then there are an $\varepsilon > 0$ and a sequence $(x_n) \subset S_X$ for which $\|x_n - x_m\| > 1 + \varepsilon$ if $n \neq m$.*

Now we can confidently state that $K(X) \in (1, 2]$ for any infinite-dimensional Banach space, but finding ε for a given space can still be a challenge. In this thesis we shall give estimates for ε for some classes of Banach spaces as well as try to answer a related question that presents itself. Diestel poses the following open problem on page 254 in his notes on the Elton-Odell theorem:

> **Problem.** For which infinite-dimensional Banach spaces X is there an $\varepsilon > 0$ such that given any infinite-dimensional closed linear subspace Y of X, then one can find a $(1 + \varepsilon)$-separated sequence in B_Y?

The sequel is heavily based on two papers by Prus and Kryczka [13, 15]. The contribution in this thesis consists of filling in some details.

Section 1 contains a brief review of some functional analysis as well as introducing some notation and Ramsey theory that will be used in the following sections. Section 2 is based on [15] and deals with non-Schur Banach spaces. Section 3 in turn is based on [13] and gives an estimate for $K(X)$ for nonreflexive spaces.

1 Preliminaries and notation

1.1 Functional analysis

This section is intended to provide the necessary concepts used in the rest of this thesis without explicit explanation later on. We refer an interested reader to a basic course on functional analysis (such as Rudin[16]) for a more comprehensive treatment.

Definition 1.1 (norm). *A norm $\|\cdot\|\colon X \to [0,\infty)$ on a vector space X over a field $\mathbb{F}$ satisfies the following properties, for all $x \in X$, $\alpha \in \mathbb{F}$.*

- $\|x\| = 0 \Leftrightarrow x = 0$
- $\|\alpha x\| = |\alpha|\|x\|$
- $\|x+y\| \le \|x\| + \|y\|$ *(triangle inequality)*

A useful form of the triangle inequality is the *reverse triangle inequality.* Two vectors are written as the sum of the other plus the difference: $x = y+(x-y)$ and $y = x+(y-x)$

$$\left.\begin{array}{ll} \|x\| = \|y+(x-y)\| & \le \|y\| + \|x-y\| \\ \|y\| = \|x+(y-x)\| & \le \|x\| + |-1|\|x-y\| \end{array}\right\} \quad \Rightarrow \quad \Big|\|x\| - \|y\|\Big| \le \|x-y\|$$

Norms on spaces X and Y will be distinguished as $\|\cdot\|_X$ and $\|\cdot\|_Y$ or sometimes $|\cdot|$. Concrete examples are the Hölder p-norms on classical sequence spaces ℓ_p,

$$\|(x_1,x_2,\cdots)\|_p = \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{\frac{1}{p}}, \qquad \|(x_1,x_2,\cdots)\|_\infty = \sup_{n\in\mathbb{N}} |x_n|.$$

A sequence $(x_n)_{n=1}^\infty$ in a space X is a function $x : \mathbb{N} \to X$ from the natural numbers to X. For convenience we will usually write (x_n) unless there is call for the indexing to be explicit. Given a sequence (x_n) if for all $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that for all $n, m > N$ we have $\|x_n - x_m\| < \varepsilon$, (x_n) is called a *Cauchy sequence.* When all Cauchy sequences in a space X also converge to a limit in X, X is *complete.* The real numbers $\mathbb{R}$ are complete, but the rational numbers $\mathbb{Q}$ are not.

A complete normed vector space is called a *Banach space* after Stefan Banach who started the systematic study of such spaces with his 1932 book on the subject.

Some examples include the spaces of real and complex numbers $\mathbb{R}$ and $\mathbb{C}$, the spaces ℓ_p of absolutely p-summable sequences referenced above and c_0 the space of sequences converging to zero (equipped with the supremum norm $\|\cdot\|_\infty$) and the Sobolev spaces $W^{k,p}$ spaces containing equivalence classes of k times weakly differentiable p-integrable functions.

We say that a sequence $(x_n)_{n=1}^\infty$ has *finite support* when a finite number of terms are non-zero. The subspace c_{00} of c_0 that consists of sequences with finite support is *not* complete for any norm. In the supremum norm $(\{1/n\}_{n=1}^k)_{k=1}^\infty$ is a Cauchy-sequence in c_{00} but its limit $(1/n)_{n=1}^\infty$ does not lie in c_{00}. Hence c_{00} is not closed as a subspace of c_0.

A linear map $T\colon X \to Y$ between Banach spaces is called an operator. The operator norm $\|T\|$ of T is given by

$$\|T\| := \sup_{x\neq 0} \frac{\|Tx\|_Y}{\|x\|_X} = \sup_{\|x\|=1} \|Tx\|_Y.$$

When X is finite-dimensional $\|T\|$ will be finite also. In the infinite-dimensional case the norm of T can be unbounded. Continuity of an operator occurs precisely when the operator is bounded. We will restrict our attention to bounded operators. The two most important classes of operators for us are functionals and isomorphisms between Banach spaces.

An operator $f : X \to \mathbb{F}$ from a vector space X to the underlying scalar field $\mathbb{F}$ is called a *functional* and is usually denoted with the symbols x^* or f. The *(topological) dual* X^* is the set of all continuous linear functionals on X. As mentioned above in the finite-dimensional case all linear functionals are continuous but this is no longer true when infinite-dimensional vector spaces are concerned. The linear structure of the functionals endows X^* with a vector space structure. The operator norm as defined above is seen to turn X^* into a normed vector space in particular. Even when starting out with a normed vector space that is not complete, when its underlying scalar field is complete the dual is always a Banach space. The dual of $\mathbb{C}^n$ is isomorphic to $\mathbb{C}^n$. For $1 \leq p < \infty$ the dual of ℓ_p is ℓ_q with $\frac{1}{p} + \frac{1}{q} = 1$. The exponents p and q fulfilling this condition are called *Hölder conjugates*.

The dual X^{**} of X^* is called the *bidual* of X. The natural embedding $\iota : x \mapsto (f \mapsto f(x))$ associates with every vector $x \in X$ evaluation of a functional $f \in X^*$ at that vector x. When ι is not surjective (and hence X^{**} is strictly larger than X itself) X is said to be *nonreflexive*. When ι is surjective it is an isomorphism and X is called *reflexive*. Due to the symmetry in the Hölder conjugates the ℓ_p spaces with $1 < p < \infty$ are reflexive. The same holds for $\mathbb{C}^n$. On the other hand the dual and bidual of c_0 are respectively ℓ_1 and ℓ_∞. This makes c_0 nonreflexive. Since being reflexive is equivalent with having a reflexive dual[16, p. 111] ℓ_1 and ℓ_∞ are nonreflexive as well.

When a sequence of vectors (x_n) converges to a limit x, i.e. $\lim_{n\to\infty} \|x_n - x\| = 0$, we write $x_n \to x$. The vectors are said to converge strongly or in norm. A different mode of convergence occurs when for all functionals $x^* \in X^*$ we have that $x^*(x_n) \to x^*(x)$. The sequence is said to converge weakly to x. We may denote this as $x_n \xrightarrow{w} x$. Frequently we will concern ourselves with sequences weakly converging to the zero vector. In this case we will say that the sequence is *weakly null*.

As the terminology suggests norm convergence implies weak convergence,

$$|x^*(x_n) - x^*(x)| = |x^*(x_n - x)| \leq \|x^*\|\|x_n - x\| \to 0.$$

But the converse is not true. Spaces where the two concepts coincide are said to have the *Schur* property. A classic example is the space ℓ_1 which was proven by Schur himself in 1910.

A sequence $(x_n)_{n=1}^\infty$ is *weakly Cauchy* if for every $x^* \in X^*$ the limit $\lim_{n\to\infty} x^*(x_n)$

exists. In general a weakly Cauchy sequence need not converge weakly.

An isomorphism T between two normed vector spaces is called an *isometric* isomorphism when vectors get mapped to vectors of the same length. Equivalently $\|T\| = 1 = \|T^{-1}\|$. As an example that isomorphisms in general are not isometries the map $T\colon X \to X$ given by $T\colon x \mapsto 2x$ is linear, continuous and a bijection but $\|Tx\| = 2\|x\|$.

A sequence (e_n) is called a *Schauder basis* (or simply a basis) for a normed vector space X if for each $x \in X$ there is a unique sequence of scalars (a_n) such that $\|x - \sum_{i=1}^n a_i e_i\| \xrightarrow{n\to\infty} 0$. If a sequence $(x_n) \subset X$ is a basis for its own closed linear span $[x_n]$ it is called a *basic sequence*. Note that it need not be a basis for the entire space X.

Let $(e_n)_{n=1}^\infty$ be a basis for a Banach space X. In addition let $(p_n)_{n=1}^\infty$ be a sequence of strictly increasing integers with $p_0 = 0$ and let (a_n) be a sequence of scalars. A nonzero sequence of vectors $(u_n)_{n=1}^\infty$ of the form

$$u_n = \sum_{p_{n-1}+1}^{p_n} a_n e_n$$

is called a *block basic sequence* of $(e_n)_{n=1}^\infty$.

Two basic sequences (x_n) and (y_n) are called *equivalent* if there is an (continuous linear) isomorphism $T\colon [x_n] \to [y_n]$ between their closed linear spans that maps x_n to y_n. When one of the sequences is normalized the other will be at least seminormalized. To see that assume $\|x_n\| = 1$, then

$$\left.\begin{array}{ll} \|y_n\| = \|Tx_n\| & \le \|T\|\|x_n\| = \|T\| \\ \|x_n\| = \|T^{-1}x_n\| & \le \|T^{-1}\|\|y_n\| \end{array}\right\} \Rightarrow \frac{1}{\|T^{-1}\|} \le \|y_n\| \le \|T\|.$$

A sequence of sets (A_n) is called increasing when $A_n \subset A_{n+1}$ and decreasing when $A_n \supset A_{n+1}$. The set of all subsets of X of size k is denoted as $X^{(k)}$.

1.2 Ramsey theory

The application of combinatorial Ramsey methods to Banach space theory has been a fruitful endeavour. Results relevant to us were achieved by Elton and Odell in their seminal paper [6] and by Brunel and Sucheston to define their spreading model[2, 3, 8] that we will use in section 2.2.

The formulation of the following theorem follows that of Gowers in the Handbook of the Geometry of Banach spaces[8] chapter on Ramsey methods in Banach spaces. Other treatments can be found in chapter 10 of [1] or chapter X of [4].

Definition 1.2 (r-colouring)**.** *Let r be a positive integer. An r-colouring g of a set A is a function from A to $\{1, \ldots, r\}$.*

When a colouring on a set A is constant on a subset $B \subset A$, then B is said to be *monochromatic* with respect to that colouring.

Theorem 1.3 (Theorem of Ramsey [8])**.** *Let r and k be positive integers. For any r-colouring f of $\mathbb{N}^{(k)}$ there is an infinite subset $Y \subset \mathbb{N}$ for which $Y^{(k)}$ is monochromatic.*

Proof. For the simplest case, $k = 1$, let $\iota\colon n \mapsto \{n\}$ identify $\mathbb{N}$ with $\mathbb{N}^{(1)}$. If there is no infinite monochromatic subset of $\mathbb{N}$ then $\cup_{s\in\{1,\ldots,r\}}(f \circ \iota)^{-1}(s)$ would be a finite set, while clearly $(f \circ \iota)(\mathbb{N}) \subset \{1, \cdots, r\}$.

Induction hypothesis: *If an arbitrary colouring of $\mathbb{N}^{(k)}$ admits an infinite monochromatic set, then so will a colouring of $\mathbb{N}^{(k+1)}$.*

Let $f : \mathbb{N}^{(k+1)} \to \{1, \ldots, r\}$ be an r-colouring of subsets of size $k + 1$.
Put $x_0 = \min \mathbb{N}$ and $Y_1 = \mathbb{N} \setminus \{x_0\}$. On $Y_1^{(k)}$ we define a new colouring

$$g_1(A) = f(\{x_0\} \cup A)$$

By the induction hypothesis there is an infinite set $X_1 \subset Y_1$ on which g_1 is monochromatic. Now set $x_1 = \min X_1$, $Y_2 = X_1 \setminus \{x_1\}$. On $Y_2^{(k)}$ we define a colouring $g_2(A) = f(\{x_1\} \cup A)$. Again we can find an infinite subset X_2 that is monochromatic under g_2.
Proceeding this process we end up with a decreasing sequence of infinite sets $\mathbb{N} \supset X_1 \supset X_2 \supset \ldots$ and an increasing sequence of numbers $x_0 < x_1 < x_2 < \ldots$.
Put $Y = \{x_1, x_2, \ldots\}$ and let $A = \{x_{n_1}, \ldots, x_{n_{k+1}}\} \in Y^{(k+1)}$. Due to our construction we know that $A \setminus \{x_{n_1}\} \subset X_{n_1+1}$ which completely determines the behaviour of f on A

$$f(A) = f(\{x_{n_1}\} \cup (A \setminus \{x_{n_1}\})) = g_{n_1+1}(X_{n_1+1})$$

An equivalence relation $\sim$ on $Y^{(k+1)}$ defined by $A \sim B$ when $\min A = \min B$ induces equivalence classes that can be represented by the x_n. Each equivalence class is monochromatic under f. The colouring f on $Y^{(k+1)}$ thus behaves essentially as a colouring of $Y^{(1)}$ and admits an infinite subset Z of Y such that $Z^{(k+1)}$ is monochromatic.

□

In the sequel we will make use of Ramsey's theorem to construct subsequences with desirable properties.

Corollary 1.4. *Let $F : \mathbb{N}^{(k)} \to [a, b] \subset \mathbb{R}$ be a function assigning to a set of k indices a number in a closed interval. For any $\varepsilon > 0$ there is an infinite set Y such that for two sets of indices $\mathbf{n}_1$, $\mathbf{n}_2 \in Y^{(k)}$ their images under F are no further apart than ε:*

$$|F(\mathbf{n}_1) - F(\mathbf{n}_2)| \leq \varepsilon$$

Proof. Decompose $[a, b]$ into r intervals $I_1, \cdots, I_r$ of width less than ε and colour $\mathbf{n} \in N^{(k)}$ corresponding to which interval $F(\mathbf{n})$ lies in. According to Ramsey's theorem there is an interval I_j and an infinite $Y \subset \mathbb{N}$ for which $F(Y^{(k)}) \subset I_j$. □

2 Separated sequences in Banach spaces without the Schur property

In the paper "Constructing separated sequences in Banach spaces"[15] Stanisław Prus sets out to answer Diestel's problem: for which spaces X does

$$s(X) = \inf\{K(Y) : Y \text{ is an infinite-dimensional closed subspace of } X\} > 1,$$

where $K(Y)$ is Kottman's constant, $K(Y) = \sup\{\inf_{i \neq j} \|x_i - x_j\| : (x_n) \subset Y_X\}$. We will follow Prus' approach of considering Banach spaces without the Schur property.

2.1 Motivation

When in a Banach space weak convergence of a sequence implies also the norm convergence of that sequence we say that the space has the Schur property (or X *is* Schur).

Let X be an infinite-dimensional Banach space with the Schur property and (x_n) a sequence with a positive separation constant, contained in the unit sphere S_X of X

$$\operatorname{sep}(x_n) := \inf_{\substack{n,m \in \mathbb{N} \\ n \neq m}} \|x_n - x_m\| > 0.$$

Suppose a subsequence (y_n) of (x_n) is weakly Cauchy. A sequence of differences (e.g. $(y_n - y_{n+1})_{n=1}^{\infty}$) is then necessarily weakly null as

$$\lim_{n \to \infty} |x^*(y_n - y_{n+1})| = \lim_{n \to \infty} |x^*(y_n) - x^*(y_{n+1})| = 0$$

Since X is Schur the differences must also converge in norm, but $\lim_{n \to \infty} \|y_n - y_{n+1}\| = 0$ contradicts the separation constant being positive. We conclude that no subsequence of (x_n) can be weakly Cauchy. Rosenthal's theorem now implies that (x_n) has a subsequence equivalent to the canonical basis of ℓ_1:

Rosenthal's ℓ_1 Theorem ([1, Theorem 10.2.1])**.** *Let $(x_n)_{n=1}^{\infty}$ be a bounded sequence in an infinite-dimensional Banach space X. Then either:*

(a) $(x_n)_{n=1}^{\infty}$ has a subsequence which is weakly Cauchy, or

(b) $(x_n)_{n=1}^{\infty}$ has a subsequence which is basic and equivalent to the canonical basis of ℓ_1

Through this equivalence we can relate the separation constant of the basic sequence (e_n) of ℓ_1 to the equivalent sequence in X. Recall that the standard basis of ℓ_1 is 2-separated. Let $T\colon \ell_1 \to X$ be the isomorphism that maps (e_n) into (x_n). Then $\frac{2}{\|T^{-1}\|} \leq \|x_n - x_m\| \leq 2\|T\|$. As T need not be an isometry this doesn't give us the control we need. However, since we are dealing with an isomorphism of ℓ_1 we can improve the estimate to be nearly isometric.

James's ℓ_1 Distortion Theorem ([1, Theorem 10.3.1]). *Let $(x_n)_{n=1}^{\infty}$ be a normalized basic sequence in a Banach space X which is equivalent to the canonical ℓ_1-basis. Then given $\varepsilon > 0$ there is a normalized block basic sequence $(y_n)_{n=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$ such that*

$$\left\| \sum_{k=1}^{N} a_k y_k \right\| \geq (1 - \varepsilon) \sum_{1=k}^{N} |a_k|$$

for any sequence of scalars $(a_k)_{k=1}^{N}$.

Setting scalars $a_n = 1$, $a_m = -1$ and the rest zero we get for every $\varepsilon > 0$ a sequence (y_n^{ε}) in the unit sphere of X such that

$$\|y_n^{\varepsilon} - y_m^{\varepsilon}\| \geq (1 - \varepsilon)2.$$

Taking the supremum over all such sequences we establish the fact that $K(X) = 2$ whenever X is Schur. The following lemma shows that then also $s(X) = 2$.

Lemma 2.1. *All closed subspaces of a Schur space are Schur.*

Proof. Let X be Schur and Y a closed (infinite-dimensional) linear subspace of X. Let (y_n) be a sequence in Y that converges weakly to y_∞. In X it also converges in norm to y_∞. Since Y is closed $y_\infty \in Y$ so (y_n) converges in norm in Y, hence Y is Schur. □

In the sequel we will therefore focus our attention on Banach spaces that do *not* have the Schur property.

2.2 Banach spaces without the Schur property

Since weak and norm convergence don't coincide in Banach spaces without the Schur property there exist sequences that converge weakly to 0 but not in norm. We are interested in a subset of these sequence for which a certain limit exists.

Theorem 2.2 ([3, Proposition 1]). *Let (x_n) be a bounded sequence in a Banach space X. Then there exists a subsequence (y_n) of (x_n) such that for all scalars $\alpha_1, \ldots, \alpha_m$ the following limit exists and is well-defined:*

$$L(\alpha_1, \ldots, \alpha_m) := \lim_{\substack{n_1 < \ldots < n_m \\ n_1 \to \infty}} \left\| \sum_{i=1}^{m} \alpha_i y_{n_i} \right\| \tag{1}$$

Proof. Let (x_n) be bounded and $\alpha_1, \ldots, \alpha_m$ given. For any $\mathbf{n} \in \mathbb{N}^{(m)}$ the expression $\| \sum_{i=1}^{m} \alpha_i x_{n_i} \|$ is bounded by $U := \max \alpha_i \sup \|x_n\|$. Divide the interval $[0, U]$ in two equal subintervals. By Corollary 1.4 to Ramsey's theorem there is a subsequence (x_n^1) of (x_n) such that $F(\mathbb{N}^{(m)})$ falls entirely in one of the two intervals. Repeating the procedure with this interval and taking the diagonal sequence (x_n^n) means that for $A, B \in \mathbb{N}^{(m)}$

with $\min A, \min B > N$

$$\left| \left\| \sum_{n_i \in A} \alpha_i x_{n_i}^{n_i} \right\| - \left\| \sum_{k_i \in B} \alpha_i x_{k_i}^{k_i} \right\| \right| \leq \frac{U}{2^N}$$

Hence the limit $L(\alpha_1, \ldots, \alpha_m)$ along (x_n^n) exists for this particular choice of α_i. Moreover it does not depend on the values of $n_1 < \ldots < n_m$, but only on the fact that n_1 tends to infinity. This justifies the otherwise ambiguous notation $\lim_{\substack{n_1 < \cdots < n_m \\ n_1 \to \infty}}$.

To establish the existence for all possible combinations of scalars $\alpha_1, \ldots, \alpha_k$ note that $(\alpha_1, \ldots, \alpha_k, 0, 0, \ldots) \in c_{00}$ and that c_{00} has a countable dense subset S consisting of all sequences of rational coefficients and finite support.

Let $\{q_i\}_{i=0}^{\infty}$ be an enumeration of $\mathbb{Q}$ (or the complex numbers with rational coefficients if the scalar field is $\mathbb{C}$) with $q_0 = 0$. A correspondence ϕ between the natural numbers and the set S can be set up using the unique prime decomposition of a natural number

$$\phi(2^{k_1} 3^{k_2} 5^{k_3} \ldots) = (q_{k_1}, q_{k_2}, q_{k_3}, \ldots)$$

Thus $\phi(1)$ is the null vector and $\phi^{-1}((0, 0, q_3, q_1, 0, \ldots)) = 22$. This shows that S is countable. The fact that S is dense in c_{00} follows from $\mathbb{Q}$ being dense in $\mathbb{R}$. With an enumeration for S at hand we can take a diagonal sequence so that the limit (1) exists for all $\alpha \in S$. It remains to show that the statement is true for vectors where not all coefficients are rational.

Let $b = (b_1, \cdots, b_r, 0, 0, \cdots) \in c_{00} \setminus S$ have real coefficients. For every $\varepsilon > 0$ there is an $a = (a_1, \cdots, a_r, 0, 0, \cdots) \in S$ such that $\|b - a\|_1 \leq \varepsilon/(4U)$. Thus for $A \in \mathbb{N}^{(r)}$ the difference $\| \sum_{n_i \in A} b_i x_{n_i} - \sum_{n_i \in A} a_i x_{n_i} \| \leq \sum_{i=1}^{r} |b_i - a_i| \|x_{n_i}\| = \|b - a\|_1 U$.

For a we know that the limit $L(a)$ exists. Taking $v \in \mathbb{N}$ large enough so that for all $A \in \mathbb{N}^{(r)}$ with $v < \min A$ we have $|\| \sum_{i=1}^{r} a_i x_{n_i} \| - L(a)| \leq \varepsilon/4$

$$\sup_{v < \min \mathbf{n}} \left\| \sum_{i=1}^{r} b x_{n_i} \right\| \leq \sup_{v < \min \mathbf{n}} \left\| \sum_{i=1}^{r} a x_{n_i} \right\| + \varepsilon/4 \leq L(a) + \varepsilon/2$$
$$\inf_{v < \min \mathbf{n}} \left\| \sum_{i=1}^{r} b x_{n_i} \right\| \geq \inf_{v < \min \mathbf{n}} \left\| \sum_{i=1}^{r} a x_{n_i} \right\| - \varepsilon/4 \geq L(a) - \varepsilon/2$$

The difference between the supremum and infimum above is smaller than ε. As ε is arbitrary taking the limit shows that the difference between the limit superior and limit inferior is 0 which by definition means that the limit exists. The desired limit exists for all $a \in c_{00}$. □

By $\mathcal{N}(X)$ we denote the set of all sequences (y_n) in a Banach space X such that all limits as defined in Theorem 2.2 exist, (y_n) converges weakly to 0 and (y_n) does not have a norm-convergent subsequence. By $\mathcal{N}_1(X)$ we denote the subset of $\mathcal{N}(X)$ that consists of normalized sequences.

Clearly, when $\mathcal{N}(X)$ is non-empty, X does not have the Schur property. The converse holds as well. Let (z_n) be a sequence that converges weakly to 0, but does not converge in norm. In other words, there is an $\varepsilon > 0$ such that for all N, $\sup_{m>N} \|z_m\| > \varepsilon$. We can assume (z_n) is uniformly bounded away from zero by ε. Denote with (x_n) the normalisation $(\frac{z_n}{\|z_n\|})$ of our original sequence. Let $f \in X^*$ be an arbitrary functional, then since

$$|f(x_n)| = \frac{|f(z_n)|}{\|z_n\|} < \frac{|f(z_n)|}{\varepsilon} \xrightarrow{n\to\infty} 0,$$

(x_n) is still weakly null. Since it is normalized it does not have norm-Cauchy subsequences. As it is bounded we can apply Theorem 2.2 to obtain a subsequence for which the desired limits exist, proving that $\mathcal{N}_1(X) \subset \mathcal{N}(X)$ is non-empty.

2.2.1 A spreading model based on $\mathcal{N}_1(X)$

On the space c_{00} of finitely supported sequences with basis $\{e_i\}_{i=1}^{\infty}$ we define a norm based on $(y_n) \in \mathcal{N}_1(X)$,

$$\left|\sum_{i=1}^{m} \alpha_i e_i\right|_{(y_n)} := L(\alpha_1, \dots, \alpha_m) = \lim_{\substack{n_1<\dots<n_m \\ n_1\to\infty}} \left\|\sum_{i=1}^{m} \alpha_i y_{n_i}\right\|. \tag{2}$$

The subscript makes it explicit which sequence the norm depends on. When no confusion will arise we will not write the subscript.

The completion of c_{00} with respect to the norm $|\cdot|$ is called a spreading model for the sequence (y_n). The normalisation of (y_n) in X carries over to (e_n) in the spreading model.

Due to the freedom of choosing subsequences in the construction of the norm, the sequence (e_n) is spreading in the spreading model. That is, as long as the order in which the coefficients appear in a vector remains the same, we can spread them further apart without changing the norm,

$$\left|\sum_{i=1}^{m} \alpha_i e_{k_i}\right|_{(y_n)} = \lim_{\substack{n_1<\dots<n_m \\ n_1\to\infty}} \left\|\sum_{i=1}^{m} \alpha_i y_{n_i}\right\| = \left|\sum_{i=1}^{m} \alpha_i e_i\right|_{(y_n)}.$$

Lemma 2.3 (Increasing norms on increasing index sets)**.** *Given a sequence of scalars $(a_n)_{n=1}^{\infty}$, and finite sets $I \subset J \subset \mathbb{N}$,*

$$\left|\sum_{i\in I} a_i e_i\right| \leq \left|\sum_{i\in J} a_i e_i\right|.$$

The following proof is based on a proof from Beauzamy[2, Lemma 2].

Proof. Fix $N \in \mathbb{N}$ and scalars $a_1, \dots, a_N$. Let $J = \{n_1, \dots, n_N\}$. Since (e_n) is spreading $|\sum_{i=1}^{N} a_i e_{n_i}| = |\sum_{i=1}^{N} a_i e_i|$ and we will take J to be $\{1, \dots, N\}$. If for arbitrary $M \in J$

the statement holds for $I = J\backslash\{M\}$ then it will hold for any subset of J. We can assume $a_M \neq 0$.

As the norm in the spreading model is defined via a limit of norms along the underlying sequence, let us start by fixing $\varepsilon > 0$, and finding $v \in \mathbb{N}$ such that for all indices $v < n_1 < n_2 < \ldots < n_N$ we have:

$$\begin{cases} \Big|| \sum_{i=1}^{N} a_i e_i| - \| \sum_{i=1}^{N} a_i y_{n_i} \|\Big| < \varepsilon \\ \Big|| \sum_{\substack{i=1 \\ i\neq M}}^{N} a_i e_i| - \| \sum_{\substack{i=1 \\ i\neq M}}^{N} a_i y_{n_i} \|\Big| < \varepsilon. \end{cases} \tag{3}$$

Since the weak closure of the convex hull equals the norm-closure of the convex hull [16, Theorem 3.13, p. 67] and (y_n) is weakly null, there is a sequence of convex combinations of $(y_n)_{n>v+M}$ that converges to 0 in norm. Let $\sum_{j=1}^{k} \lambda_j y_{m_j}$ be such a convex combination that is in norm close to 0:

$$\left\| \sum_{j=1}^{k} \lambda_j y_{m_j} \right\| < \frac{\varepsilon}{|a_M|}. \tag{4}$$

Our choice of indices $n_1, \ldots, n_N$ to satisfy condition (3) will be such that $n_{M-1} < m_1$ and $m_k < n_{M+1}$. It follows that for this choice of indices and for $j \in \{1, \ldots, k\}$ one can write

$$\Big|\sum_{i=1}^{N} a_i e_i\Big| \geq \left\| \Big(\sum_{i=1}^{M-1} a_i y_{m_1 - M + i} \Big) + a_M y_{m_j} + \sum_{i=M+1}^{N} a_i y_{m_k + i - M} \right\| - \varepsilon$$

Let us introduce shorthand notation for the unwieldy sums before and after index M, $S_b := \Big(\sum_{i=1}^{M-1} a_i y_{m_1 - M + i} \Big)$ and $S_a := \sum_{i=M+1}^{N} a_i y_{m_k + i - M}$. Summing this inequality over j and weighting with λ_j we obtain:

$$\begin{aligned}
\Big(\sum_{j=1}^{k}\lambda_j\Big)\Big|\sum_{i=1}^{N}a_ie_i\Big| &\geq \sum_{j=1}^{k}\lambda_j\|S_b+a_My_{m_j}+S_a\|-\varepsilon\\
&\geq \Big\|\sum_{j=1}^{k}\lambda_j(S_b+a_My_{m_j}+S_a)\Big\|-\varepsilon\\
&= \Big\|S_b+a_M\sum_{j=1}^{k}\lambda_jy_{m_j}+S_a\Big\|-\varepsilon\\
&\geq \|S_b+S_a\|-|a_M|\Big\|\sum_{j=1}^{k}\lambda_jy_{m_j}\Big\|-\varepsilon\\
&\overset{(4)}{>} \|S_b+S_a\|-2\varepsilon\\
&\overset{(3)}{>} \Big|\sum_{\substack{i=1\\i\neq M}}^{N}a_ie_i\Big|-3\varepsilon
\end{aligned}$$

□

Proposition 2.4 (Changing signs doubles the norm at most)**.** *Let $\theta_i \in \{-1,1\}$,*

$$\Big|\sum_{i=1}^{m}\theta_i\alpha_ie_i\Big| = \Big|\sum_{\substack{i=1\\\theta_i=1}}^{m}\alpha_ie_i - \sum_{\substack{i=1\\\theta_i=-1}}^{m}\alpha_ie_i\Big| \leq \Big|\sum_{\substack{i=1\\\theta_i=1}}^{m}\alpha_ie_i\Big| + \Big|\sum_{\substack{i=1\\\theta_i=-1}}^{m}\alpha_ie_i\Big| \leq 2\Big|\sum_{i=1}^{m}\alpha_ie_i\Big|.$$

Lemma 2.5 ([15, Lemma 2])**.** *Let $\varepsilon_i \in \{-1,1\}$ for every $i\in\mathbb{N}$. Then*

$$\limsup_{m\to\infty}\Big|\sum_{i=1}^{2^m}\varepsilon_ie_i\Big|^{\frac{1}{m}} = \limsup_{m\to\infty}\Big|\sum_{i=1}^{2^m}e_i\Big|^{\frac{1}{m}} \leq 2.$$

Proof. Using $\varepsilon_i\varepsilon_i = 1$ we apply Proposition 2.4 twice

$$\frac{1}{2}\Big|\sum_{i=1}^{2^m}e_i\Big| \leq \Big|\sum_{i=1}^{2^m}\varepsilon_ie_i\Big| \leq 2\Big|\sum_{i=1}^{2^m}e_i\Big|.$$

Taking the m-th root and the lim sup both respect the ordering so for $m\to\infty$

$$\limsup_{m\to\infty}\Big|\sum_{i=1}^{2^m}e_i\Big|^{\frac{1}{m}} \leq \limsup_{m\to\infty}\Big|\sum_{i=1}^{2^m}\varepsilon_ie_i\Big|^{\frac{1}{m}} \leq \limsup_{m\to\infty}\Big|\sum_{i=1}^{2^m}e_i\Big|^{\frac{1}{m}}.$$

As $|\sum_{i=1}^{2^m} e_i|^{\frac{1}{m}} \leq (\sum_{i=1}^{2^m} |e_i|)^{\frac{1}{m}} = 2$ the lim sup is also finite. $\square$

2.2.2 The property $\lambda(X)$

Given a sequence $(x_n) \in \mathcal{N}_1(X)$ we put

$$l(x_n) = \limsup_{m\to\infty} \left|\sum_{i=1}^{2^m} e_i\right|_{(x_n)}^{\frac{1}{m}}.$$

As remarked in the proof of Lemma 2.5, $l(x_n) \leq 2$. From Lemma 2.3 we also see that $l(x_n)$ is bounded from below by 1:

$$1 = |e_1| \leq \left|e_1 + \sum_{i=2}^{2^m} e_i\right| \quad \Rightarrow \quad \left|\sum_{i=1}^{2^m} e_i\right|^{\frac{1}{m}} \geq 1.$$

Next we set

$$\lambda(X) = \inf\{l(x_n)) : (x_n) \in \mathcal{N}_1(X)\}.$$

It follows that $\lambda(X) \in [1, 2]$.

Theorem 2.6 ([15, Theorem 3])**.** *Let X be a Banach space without the Schur property. Then $s(X) \geq \lambda(X)$.*

Proof. Recall that $s(X)$ is the infimum over all infinite-dimensional subspaces of their Kottman's constant. Let Y be an infinite-dimensional closed subspace of X. If Y has the Schur property then $K(Y) = 2$. Hence we may assume Y to be non-Schur and $\mathcal{N}_1(Y)$ non-empty. Fix a sequence (y_n) from $\mathcal{N}_1(Y)$ to base our spreading model on. Put $\varepsilon_i = (-1)^i$ and consider vectors in the spreading model

$$v_m = \sum_{i=1}^{2^m} \varepsilon_i e_i \qquad \text{and} \qquad u_m = \sum_{i=1}^{2^{m-1}} \varepsilon_i e_i + \sum_{i=2^{m-1}+2}^{2^m+1} \varepsilon_i e_i.$$

For any m the norms of v_m and u_m differ by 2 at most

$$\Big||v_m| - |u_m|\Big| \leq |v_m - u_m| = |-e_{2^{m-1}+1} + e_{2^m+1}| \leq 2. \tag{5}$$

Lemma 2.5 and the definition of $\lambda(X)$ tell us that $\limsup_{m\to\infty} |v_m|^{\frac{1}{m}} = l(y_n) \geq \lambda(X)$. From Lemma 2.3 we know that the sequence $(|v_m|)$ is nondecreasing. If (v_m) is bounded, say by M, then $\limsup_{m\to\infty} |v_m|^{\frac{1}{m}} \leq \limsup_{m\to\infty} M^{\frac{1}{m}} = 1$ so $\lambda(X) = 1$. In that case as argued in the introduction based on Riesz's Lemma $K(Y) \geq 1$ for every infinite-dimensional space Y and thus the claim holds:

$$s(X) \geq \inf K(Y) \geq 1 = \lambda(X)$$

In the case that $\lambda(X) > 1$ we must have that (v_m) is unbounded. Given a $\gamma > 0$ we can find an $N_0 \in \mathbb{N}$ such that for all $n \geq N_0$ we have

$$|v_n| \geq \frac{2}{\gamma}.$$

Claim: Call $\lambda = \lambda(X)$. There exists an $N > N_0$ such that $|v_N| \geq (\lambda - \gamma)|v_{N-1}|$. If not, then for all $m > N_0$ $|v_m| < (\lambda - \gamma)|v_{m-1}|$, and we get $|v_m| < (\lambda - \gamma)^{m-N_0}|v_{N_0}|$. Taking m-th roots and the lim sup we get:

$$\begin{aligned}\lambda \leq \limsup_{m\to\infty} \sqrt[m]{|v_m|} &< \limsup_{m\to\infty} \sqrt[m]{|v_{N_0}|(\lambda-\gamma)^{m-N_0}} \\ &= \limsup_{m\to\infty} \sqrt[m]{|v_{N_0}|}(\lambda-\gamma)^{1-\frac{N_0}{m}} = (\lambda - \gamma).\end{aligned}$$

Hence there must be an $N > N_0$ such that the ratio $|v_N|/|v_{N-1}|$ is at least $\lambda - \gamma$. Combining (5) and our choice of $|v_N|$ such that $|v_N| \geq \frac{2}{\gamma}$ gives

$$\frac{|u_N|}{|v_{N-1}|} \geq \frac{|v_N| - 2}{|v_{N-1}|} = \frac{|v_N|}{|v_{N-1}|} - \frac{2}{|v_{N-1}|} \geq (\lambda - \gamma) - \gamma = \lambda - 2\gamma.$$

To estimate $K(Y)$ we need to tie our result so far to the differences of a norm 1 sequence. Put

$$x_k = \sum_{i=1}^{2^{N-1}} \varepsilon_i y_{k(2^{N-1})+i}\ , \qquad\qquad z_k = \frac{x_k}{\|x_k\|},$$

for $k \in \mathbb{N}$. The sequence (z_k) will be our candidate for estimating a lower bound on $K(Y)$.

From the definition (2) of the spreading model it is clear that $\lim_{k\to\infty} \|x_k\| = |v_{N-1}|$. We also have $\lim_{\substack{n<m \\ n\to\infty}} \|x_n - x_m\| = |u_N|$. To see this note that (e_n) is spreading in the spreading model, we still have room to shift. We can write:

$$\begin{aligned}|u_N| = \left|\sum_{i=1}^{2^{N-1}} \varepsilon_i e_i + \sum_{i=2^{N-1}+2}^{2^N+1} \varepsilon_i e_i\right| &= \left|\sum_{i=1}^{2^{N-1}} \varepsilon_i e_i + \sum_{i=2^{N-1}+1}^{2^N} \varepsilon_{i+1} e_i\right| \\ &= \left|\sum_{i=1}^{2^{N-1}} \varepsilon_i e_i - \sum_{i=2^{N-1}+1}^{2^N} \varepsilon_i e_i\right|,\end{aligned}$$

where we used that $\varepsilon_i = -\varepsilon_{i+1}$. Knowing the limiting values of $\|x_k\|$ and $\|x_n - x_m\|$ allows us to derive the limit of differences of z_k

$$\lim_{\substack{n<m \\ n\to\infty}} \|z_n - z_m\| = \lim_{\substack{n<m \\ n\to\infty}} \left\|\frac{x_n}{\|x_n\|} - \frac{x_m}{\|x_m\|}\right\| = \frac{1}{|v_{N-1}|} \lim_{\substack{n<m \\ n\to\infty}} \|x_n - x_m\| = \frac{|u_N|}{|v_{N-1}|}.$$

In general sep $(z_k) = \inf_{n \neq m} \|z_n - z_m\|$ will be smaller than the above limit. However this is no obstacle, as we can get as close as we want.

Fix $\varepsilon > 0$, then there is an N such that for all $n, m > N$ $\|z_n - z_m\| \geq \frac{|u_N|}{|v_{N-1}|} - \varepsilon$ and thus the sequence $(z_k)_{k>N}$ has a separation at least that good, that is,

$$K(Y) = \sup_{(y_n) \subset B_Y} \operatorname{sep}(y_n) \geq \sup_{(z_k)_{k>N}, N \in \mathbb{N}} \operatorname{sep}(z_k) = \frac{|u_N|}{|v_{N-1}|} \geq \lambda(X) - 2\gamma,$$

where $\gamma > 0$ was chosen arbitrarily. Thus we obtain $K(Y) \geq \lambda(X)$. □

Proposition 2.7. *The bound given by λ is invariant under isomorphisms.*

Proof. Let $T\colon X \to Y$ be an isomorphism between non-Schur Banach spaces and $(x_n) \in \mathcal{N}_1(X)$. We will show that we can find a sequence $(y_n) \in \mathcal{N}_1(Y)$ such that $l(y_n) \leq l(x_n)$. The obvious candidate (Tx_n) does not work as not all properties of (x_n) are retained under the isomorphism.

The Banach-adjoint T^* of T is defined such that for a functional y^* its operation on Tx can be expressed as a functional in X^* working on x. In other words $y^*(Tx) = (T^*y^*)(x)$. Using the adjoint we see that (Tx_n) is still weakly null,

$$|y^*(Tx_n)| = |(T^*y^*)(x_n)| \xrightarrow{n\to\infty} 0.$$

It also follows that (Tx_n) has no norm-Cauchy subsequence as it is weakly null but bounded away from zero, $\|x_n\| = \|T^{-1}Tx_n\| \leq \|T^{-1}\|\|Tx_n\|$. The remaining requirements for a sequence to be an element of $\mathcal{N}_1(Y)$ is that it consists of norm one vectors and that all limits as in (1) exist. It will turn out that it is useful to also guarantee the existence of one particular limit for the sequence (Tx_n) prior to normalising. Employing Theorem 2.2 we pass to a subsequence of (Tx_n) for which the limit $\lim_{n\to\infty} \|Tx_n\|$ exists. Normalising the resulting subsequence and passing to a further subsequence for which all the required limits exist we are ensured of a sequence $(y_n) \in \mathcal{N}_1(Y)$. We are ready to show the relation between $l(y_n)$ and $l(x_n)$, that is

$$\begin{aligned} l(y_n) &= \limsup_{m\to\infty} \lim_{\substack{k_1<\cdots<k_m \\ k_1\to\infty}} \left\| \sum_{i=1}^{2^m} y_{k_i} \right\|^{\frac{1}{m}} = \limsup_{m\to\infty} \lim_{\substack{n_1<\cdots<n_{2^m} \\ n_1\to\infty}} \left\| T \sum_{i=1}^{2^m} \frac{x_{n_i}}{\|Tx_{n_i}\|} \right\|^{\frac{1}{m}} \\ &\leq \limsup_{m\to\infty} \left(\frac{\|T\|}{\lim_{n\to\infty} \|Tx_n\|} \right)^{\frac{1}{m}} \lim_{\substack{n_1<\cdots<n_{2^m} \\ n_1\to\infty}} \left\| \sum_{i=1}^{2^m} x_{n_i} \right\|^{\frac{1}{m}} = l(x_n). \end{aligned}$$

As we can construct a corresponding sequence $(y_n) \in \mathcal{N}_1(Y)$ for every $(x_n) \in \mathcal{N}_1(X)$ the result is that $\lambda(X) = \inf\{l(x_n) : (x_n) \in \mathcal{N}_1(X)\} \geq \lambda(Y)$. Since $T^{-1}\colon Y \to X$ is of course also an isomorphism the inequality $\lambda(Y) \geq \lambda(X)$ is proved the same way.
In conclusion: when X and Y are isomorphic $\lambda(X) = \lambda(Y)$. □

2.3 Cotype, the Orlicz property and C-convexity

The theory of (Rademacher) type and cotype can be traced to early papers of Orlicz in 1933 although the current development was started in the 1970s. In this section we will follow Chapter 5 of Kadets and Kadets[10] to use a slightly different notion, M-cotype, that is convenient to work with considering our spreading model from the previous section.

Definition 2.8. *A Banach space X is of* cotype q, $2 \leq q < \infty$, *if there is a constant C such that*

$$\Big(\sum_{i=1}^{n} \|x_i\|^q\Big)^{1/q} \leq C\Big(\underset{\varepsilon=\pm 1}{Av} \|\sum_{i=1}^{n} \varepsilon_i x_i\|^q\Big)^{1/q}$$

holds for any finite collection of elements $\{x_k\}_{k=1}^n \subset X$. Here Av denotes the average.

Definition 2.9 ([5,17])**.** *For $1 \leq q < \infty$ the $(q,1)$-summing norm $\pi_{q,1}(T)$ of an operator T defined between two Banach spaces X, Y is the infimum of the numbers $C > 0$ such that*

$$\Big(\sum_{i=1}^{n} \|T(x_i)\|^q\Big)^{1/q} \leq C \max_{\varepsilon_i=\pm 1}\Big(\Big\|\sum_{i=1}^{n} \varepsilon_i x_i\Big\|^q\Big)^{1/q}$$

holds for any finite collection of elements $\{x_k\}_{k=1}^n \subset X$. The operator T is said to be $(q,1)$-summing *if $\pi_{q,1}(T)$ is finite. Moreover if the identity operator* id_X *of X is $(q,1)$-summing we say that X has the* Orlicz property with exponent q.

Having cotype q clearly implies that the identity operator id_X of a Banach space X is $(q,1)$-summing. The converse is true for $q > 2$ but not when $q = 2$ due to a counterexample by Talagrand[17, 18]. Definitions of the Orlicz property vary in the literature, sometimes restricting it only to the case $q = 2$. Where we say that X has the Orlicz property with exponent q the terminology from Kadets and Kadets[10, Definition 4.2.2, p. 49], as used in the paper by Prus, instead says that X has *M-cotype* q.

The spaces ℓ_p, $1 \leq p < \infty$, have cotype $\max\{2, p\}$ [1, Theorem 6.2.14].

Corollary 2.10. *If a Banach space X has the Orlicz property with exponent* p *for some $p \geq 1$, then $s(X) \geq 2^{\frac{1}{p}}$.*

Proof. When X is Schur $s(X) = 2 \geq 2^{\frac{1}{p}}$ for all $p \geq 1$. Assume X is not Schur and let $(x_n) \in \mathcal{N}_1(X)$ be arbitrary. By definition of $\mathcal{N}_1(X)$ the limit

$$\lim_{\substack{n_1 < \cdots < n_{2^m} \\ n_1 \to \infty}} \left\|\sum_{k=1}^{2^m} \varepsilon_k x_k\right\|$$

exists for all $\varepsilon \in \{-1, 1\}^{2^m}$. On the other hand by Corollary 1.4 there is a subsequence

(y_n) of (x_n) such that the limit

$$\lim_{\substack{n_1<\dots<n_{2^m}\\ n_1\to\infty}} \max_{\varepsilon\in\{-1,1\}^{2^m}} \left\|\sum_{k=1}^{2^m} \varepsilon_k y_k\right\|$$

exists. Fix an ordering on $\{-1,1\}^{2^m}$ and colour $\mathbb{N}^{(2^m)}$ according to the lowest ordered $\varepsilon \in \{-1,1\}^{2^m}$ that maximizes $\|\sum_{k=1}^{2^m} \varepsilon_k y_{n_k}\|$. By 1.3 we can pass to a subsequence of (y_n) and a particular $\varepsilon^{\max}$ such that we can interchange taking the maximum and the limit, that is

$$\begin{aligned}\lim_{\substack{n_1<\dots<n_{2^m}\\ n_1\to\infty}} \max_{\varepsilon_k=\pm1} \left\|\sum_{k=1}^{2^m} \varepsilon_k x_k\right\| &= \lim_{\substack{n_1<\dots<n_{2^m}\\ n_1\to\infty}} \left\|\sum_{k=1}^{2^m} \varepsilon_k^{\max} x_k\right\| \\ &\le \max_{\varepsilon_k=\pm1} \lim_{\substack{n_1<\dots<n_{2^m}\\ n_1\to\infty}} \left\|\sum_{k=1}^{2^m} \varepsilon_k x_k\right\| = \max_{\varepsilon_k=\pm1} \left|\sum_{k=1}^{2^m} \varepsilon_k e_k\right|.\end{aligned}$$

It remains to combine the Orlicz property of X with the fact that (x_n) is normalized so that

$$2^{\frac{m}{p}} C = C(\sum_{k=1}^{2^m} \|x_{n_k}\|^p)^{1/p} \le \max_{\varepsilon_k=\pm1} \left|\sum_{k=1}^{2^m} \varepsilon_k e_k\right| \le 2\left|\sum_{k=1}^{2^m} e_k\right|,$$

where the final inequality is due to Proposition 2.4. Taking the lim sup we see that $l(x_n) \ge 2^{\frac{1}{p}}$. Since (x_n) was arbitrarily chosen from $\mathcal{N}_1(X)$ we now have $\lambda(X) \ge 2^{\frac{1}{p}}$. The conclusion that $s(X) \ge 2^{\frac{1}{p}}$ follows from Theorem 2.6. □

In particular for ℓ_p, $1 \le p < \infty$, we get $s(\ell_p) \ge \min\{2^{\frac{1}{2}}, 2^{\frac{1}{p}}\}$.

Next we would like to give some lower bounds on $\lambda(X)$ using a notion called *C-convexity*. A Banach space X is *finitely representable* in a Banach space Y if for every finite-dimensional subspace E of X and every $\varepsilon > 0$ there is a finite-dimensional subspace F of Y and an isomorphism $T\colon E \to F$ with $\|T\|\|T^{-1}\| \le 1 + \varepsilon$.

When c_0 is finitely representable in a Banach space X it does not have the Orlicz property for any exponent p [10, Theorem 5.2.1, p 62]. This motivates the study of spaces in which c_0 is not finitely representable.

Definition 2.11. *A space X is said to be* C-convex *if c_0 is not finitely representable in X.*

Definition 2.12 ([10, Definition 5.2.2, p. 65])**.** *The* measure of C-convexity *of the space X is the function $C(m, X) : \mathbb{N} \to \mathbb{R}_+$ defined by the formula*

$$C(m, X) = \inf\left\{\max\left\{\left\|\sum_{k=1}^{m} \varepsilon_k x_k\right\| : \varepsilon_k \in \{-1,1\}\right\} : \|x_i\| \ge 1\right\}$$

Prus introduces a slight modification to $C(m,x)$ using our spreading model

$$C_1(m,X) = \inf\left\{ \max\left\{ \left|\sum_{k=1}^{m} \varepsilon_k e_k\right|_{(x_n)} : \varepsilon_k \in \{-1,1\} \right\} : (x_n) \in \mathcal{N}_1(X) \right\}$$

Recall that the spreading model norm $|\cdot|_{(x_n)}$ in the definition of $C_1(m,X)$ is the limit of the norms taken in X in the definition of $C(m,X)$ along a particular sequence of norm-one vectors (x_n). $C(m,X)$ takes the infimum of the same expression as $C_1(m,X)$, but over a larger set and thus $C(m,X) \leq C_1(m,X)$.

Lemma 2.13 ([15, Lemma 7])**.** *Let X be a Banach space without the Schur property. Then*

$$C_1(m\cdot k, X) \geq C_1(m,X)C_1(k,X)$$

Proof. Let m and k be positive integers and $(x_n) \in \mathcal{N}_1(X)$ arbitrary. There is a combination of $\{\varepsilon_1, \dots, \varepsilon_m\} \in \{-1,1\}^m$ for which $|\sum_{i=1}^m \varepsilon_i e_i|_{(x_n)}$ is maximal. By definition of $C_1(m,X)$ as the infimum over all sequences in $\mathcal{N}_1(X)$

$$\left|\sum_{i=1}^{m} \varepsilon_i e_i\right| \geq C_1(m,X)$$

$|\sum_{i=1}^m \varepsilon_i e_i|$ arises as the limit of similar sums taken along the sequence (x_n) in X. Put

$$y_n = \sum_{i=1}^{m} \varepsilon_i x_{mn+i}$$

Fix $\varepsilon > 0$ and $f \in X^*$, there is an $N \in \mathbb{N}$ for which (x_n) has $|f(x_n)| \leq \varepsilon/m$ for all $m \geq N$. $|f(y_n)| \leq \sum_{i=1}^m |f(x_{mn+i})| \leq m\varepsilon/m = \varepsilon$. So the sequence (y_n) is also weakly null. Moreover it retains the property that no subsequence is norm-Cauchy.

$$\lim_{n\to\infty} \|y_n\| = \lim_{n\to\infty} \|\sum_{i=1}^{m} \varepsilon_i x_{mn+i}\| = |\sum_{i=1}^{m} \varepsilon_i e_i|$$

As in the proof that $\mathcal{N}_1(X)$ is non-empty when X is Schur we can now pass to a subsequence (z_n) of the normalisation of (y_n) that is in $\mathcal{N}_1(X)$. This fact we exploit. Let $\{\theta_1, \dots, \theta_k\} \in \{-1,1\}^k$ be the signs for which $|\sum_{i=1}^k \theta_i e_i|_{(z_n)}$ is maximal. We consider a sum of k blocks of length m in the spreading model for (x_n) where the signs are chosen based on the ε_i and θ_i that stay above $C_1(m,X)$ and $C_1(k,X)$.

The resulting randomised sum of mk vectors in the spreading model is smaller or equal in length to the choice of signs that would maximise

$$\left|\sum_{j=0}^{k-1}\sum_{i=1}^{m}\theta_j\varepsilon_i e_{mj+i}\right|_{(x_n)} = \lim_{\substack{n_1<\ldots<n_k\\ n_1\to\infty}}\left\|\sum_{j=1}^{k}\sum_{i=1}^{m}\theta_j\varepsilon_i x_{mn_j+i}\right\| = \lim_{\substack{n_1<\ldots<n_k\\ n_1\to\infty}}\left\|\sum_{i=1}^{k}\theta_i y_{n_i}\right\|$$
$$= \lim_{\substack{n_1<\ldots<n_k\\ n_1\to\infty}}\left\|\sum_{i=1}^{k}\theta_i\|y_{n_i}\|z_{n_i}\right\| = \lim_{n\to\infty}\|y_n\| \lim_{\substack{n_1<\ldots<n_k\\ n_1\to\infty}}\left\|\sum_{i=1}^{k}\theta_i z_{n_i}\right\|$$
$$= \left|\sum_{i=1}^{m}\varepsilon_i e_i\right|_{(x_n)}\left|\sum_{i=1}^{k}\theta_i e_i\right|_{(z_n)} \geq C_1(m,X)C_1(k,X)$$

As (x_n) was arbitrarily chosen from $\mathcal{N}_1(X)$ the result is that $C_1(mk,X) \geq C_1(m,X)C_1(k,X)$. $\square$

Now we are finally ready to make good on our claim that the modulus of convexity can bound $\lambda(X)$ from below. As in the proof of Corollary 2.10 for $(x_n) \in \mathcal{N}_1(X)$ we apply Proposition 2.4 to get

$$C_1(2^m,X) = \inf_{(x_n)\in\mathcal{N}_1(X)}\max_{\varepsilon_k=\pm1}\left|\sum_{k=1}^{2^m}\varepsilon_k e_k\right|_{(x_n)} \leq \inf_{(x_n)\in\mathcal{N}_1(X)} 2\left|\sum_{k=1}^{2^m} e_k\right|_{(x_n)}.$$

From the previous lemma we know that $C_1(2,X)^m \leq C(2^m,X)$. Hence

$$C(2,X) \leq \limsup_{m\to\infty} C_1(2^m,X)^{1/m} \leq \lambda(X).$$

Gao[7, Theorem 3.1] shows that for $2 \leq p < \infty$ we attain $C(2,\ell_p) = 2^{\frac{1}{p}}$ but that for $1 \leq p < 2$, $C(2,\ell_p) = 2^{\frac{1}{q}}$ where q is the Hölder conjugate of p. This does not improve on Corollary 2.10.

Maluta and Papini[14, Lemma 1.3] connect $C(2,X)$ to the modulus of convexity $\delta_X(\varepsilon)$. Recall that for $\varepsilon \in [0,2]$

$$\delta_X(\varepsilon) = \inf\{1 - \frac{\|x+y\|}{2} : x,y \in S_X, \|x-y\| \geq \varepsilon\}.$$

Their result is that for every space X,

$$C(2,X) \geq \frac{1}{1-\delta_X(\sqrt{2})}.$$

3 Nonreflexive Banach spaces

In this section we will follow the paper "Separated sequences in nonreflexive Banach spaces" [13] by Andrzej Kryczka and Stanisław Prus to derive an estimate for $K(X)$ when X is a nonreflexive Banach space. The chief ingredients will be a result by James concerning the existence of certain sequences in nonreflexive Banach spaces, the familiar application of Ramsey theory by Brunel-Sucheston, and a clever choice of sequences.

In "Weak compactness and reflexivity" [9] James gives a long list of equivalent conditions for a Banach space to be reflexive. We will employ case (31) of Theorem 7 from his paper, rephrased for our context.

Theorem 3.1. *Let X be a Banach space. The following two statements are equivalent,*

1. *X is nonreflexive.*
2. *For each $0 < \theta < 1$ there exist a sequence of vectors $(x_n)_{n=1}^{\infty} \subset B_X$ and a sequence of functionals $(x_k^*)_{k=1}^{\infty} \subset B_{X^*}$ such that*

$$x_k^*(x_n) = \begin{cases} 0, & k > n \\ \theta, & \textit{if } k \leq n. \end{cases}$$

To illustrate the behaviour with an example, the functional x_3^* takes the value 0 on the first two vectors x_1 and x_2 and is constant θ on the tail $(x_n)_{n=3}^{\infty}$. This characterisation of nonreflexivity will allow us to show that sequences with a certain separation exist.

Theorem 3.2 ([13, Theorem 1])**.** *For any nonreflexive Banach space X Kottman's constant $K(X)$ is bounded from below by $\sqrt[5]{4}$.*

Proof. Let X be a nonreflexive Banach space. By Theorem 3.1 there is a sequence of vectors (x_n) in the unit ball B_X of X with a corresponding sequence of functionals (x_k^*) in B_X^* that do not see the first $k-1$ terms of the sequence (x_n) and evaluate to θ on its tail. We derive five sequences (x_n^0) to (x_n^4) from (x_n) in such a way that we can control their separation constants.

Put $\epsilon_1^1 = 1$, $\epsilon_1^2 = -1$, $\epsilon_i^j = (-1)^i$ if $1 \leq j < i$ or $j = 2i$ and $\epsilon_i^j = (-1)^{i+1}$ if $i \leq j < 2i$ where $i = 2, 3, 4$. The resulting pattern of signs is illustrated in the table below.

i \ j	1	2	3	4	5	6	7	8
1	+	-						
2	+	-	-	+				
3	-	-	+	+	+	-		
4	+	+	+	-	-	-	-	+

By Theorem 2.2 there is a subsequence of (x_n) such that all limits $\lim_{\substack{n_1<\cdots<n_{2i}\\ n_1\to\infty}} \| \sum_{j=1}^{2i} \epsilon_i^j x_{n_j} \|$ exist. Starting far enough into this subsequence we can assure that every sum is as close to the limit as we desire. Not actually being interested in the limit itself, but in an interval around it, there are positive constants a_1, a_2, a_3, a_4 such that

$$\theta a_i \leq \left\| \sum_{j=1}^{2i} \epsilon_i^j x_{n_j} \right\| \leq a_i. \tag{6}$$

We construct the elements of our sequences

$$\begin{aligned}
x_n^0 &= x_n, \\
x_n^1 &= \frac{1}{a_1}\left(x_{2n-1} - x_{2n}\right), \\
x_n^2 &= \frac{1}{a_2}\left(x_1 - x_{3n-1} - x_{3n} + x_{3n+1}\right), \\
x_n^3 &= \frac{1}{a_3}\left(-x_1 - x_2 + x_{4n-1} + x_{4n} + x_{4n+1} - x_{4n+2}\right), \\
x_n^4 &= \frac{1}{a_4}\left(x_1 + x_2 + x_3 - x_{5n-1} - x_{5n} - x_{5n+1} - x_{5n+2} + x_{5n+3}\right),
\end{aligned}$$

for $n \in \mathbb{N}$. It is clear that x_n^0 is contained in the unit ball of X. To see that $x_n^i \in B_X$ for $i = 1, 2, 3, 4$ note that $x_n^i = \frac{1}{a_i} \sum_{j=1}^{2i} \varepsilon_i^j x_{n_j}$ where $n_1 < \ldots < n_{2i}$. Our construction of (x_n) makes these sums seminormalized with the result that $\|x_n^i\| = \frac{1}{a_i} \| \sum_{j=1}^{2i} \varepsilon_i^j x_{n_j} \| \leq \frac{a_i}{a_i} = 1$. The separation of the sequence x_n^0 can be calculated directly from condition (6),

$$\operatorname{sep}(x_n^0) = \inf_{n \neq m} \|x_n^0 - x_m^0\| = \inf_{n<m} \|\varepsilon_1^1 x_n - \varepsilon_1^2 x_m\| \geq \theta a_1.$$

As to the differences $\|x_m^i - x_n^i\|$, the first $i-1$ vectors in the sum for x_m^i are constant for all m and thus cancel against each other,

$$\begin{aligned}
x_n^i - x_m^i &= \Big(\sum_{j=1}^{i-1} \varepsilon_i^j x_j + \sum_{j=i}^{2i} \varepsilon_i^j x_{(i+1)n-1+j-i} \Big) - \Big(\sum_{j=1}^{i-1} \varepsilon_i^j x_j + \sum_{j=i}^{2i} \varepsilon_i^j x_{(i+1)m-1+j-i} \Big) \\
&= \sum_{j=1}^{i+1} \varepsilon_{i+1}^j x_{(i+1)n-2+j} + \sum_{j=i+2}^{2i+2} \varepsilon_{i+1}^j x_{(i+1)m-3+j-i},
\end{aligned}$$

where the remaining $2(i+1)$ vectors have signs according to level $i+1$. Thus $\operatorname{sep}(x_n^i) \geq \inf_{n_1<\ldots<n_{2(i+1)}} \frac{1}{a_i} \| \sum_{j=1}^{2i+2} \varepsilon_{i+1}^j x_{n_j} \| \geq \frac{\theta a_{i+1}}{a_i}$ if $i = 1, 2, 3$.

The separation of x_n^4 can be estimated with the functionals (x_n^*). Assume $m < n$

$$x_m^4 - x_n^4 = \frac{1}{a_4}(-x_{5m-1} - x_{5m} - x_{5m+1} - x_{5m+2} + x_{5m+3}$$
$$+x_{5n-1} + x_{5n} + x_{5n+1} + x_{5n+2} - x_{5n+3})$$

The functional x_{5m+3}^* will not see the 4 vectors with negative sign (recall $x_m^*(x_k) = 0$ when $k < m$)

$$\begin{aligned} x_{5m+3}^*(x_m^4 - x_n^4) &= \frac{1}{a_4} x_{5m+3}^*(x_{5m+3} + x_{5n-1} + x_{5n} + x_{5n+1} + x_{5n+2} - x_{5n+3}) \\ &= \frac{5\theta - \theta}{a_4} = \frac{4\theta}{a_4} \end{aligned}$$

Due to the norm of a vector being the same as the norm of the evaluation of functionals at that vector

$$\|x_m^4 - x_n^4\| \geq x_{5m+3}^*(x_m^4 - x_n^4) = \frac{4\theta}{a_4}.$$

At this point we have 5 separated sequences contained in the unit ball but we do not know which of those has the largest estimate on the separation constant. In the case of equal separation the separation constant is equal to the 5-th root of the product of the estimates. When they are not equally separated then at least one of them will exceed this bound,

$$\max\left\{\theta a_1, \frac{\theta a_2}{a_1}, \frac{\theta a_3}{a_2}, \frac{\theta a_4}{a_3}, \frac{4\theta}{a_4}\right\} \geq \sqrt[5]{\frac{\theta^5 a_1 a_2 a_3 a_4 4}{a_1 a_2 a_3 a_4}} = \theta\sqrt[5]{4}.$$

Since θ is arbitrary we can construct a sequence for each $\theta \in (0, 1)$

$$K(X) \geq \sup_{\theta\in(0,1)} \theta\sqrt[5]{4} = \sqrt[5]{4}.$$

$\square$

The construction followed above generalises to any number of sequences not just 5. The result will then be that $K(X) \geq \sqrt[n+1]{n}$. The best we can do is when $n = 4$ which is precisely the bound used in the proof.

4 Some notes on applications

My work has mainly been to understand the background and proofs in the two studied papers and not so much applications of those results. For readers who are interested in applications I will share some of the leads I have come across, mainly through the papers of Kryczka and Prus, as well as Maluta and Papini.

Kottman introduced[12] the packing constant $P(\aleph_0, X)$ to study the problem of packing balls of equal size wholly into the unit sphere of a Banach space X. In terms of Kottman's constant $K(X)$, as used in this thesis, the packing constant can be expressed as $P(\aleph_0, X) = \frac{K(X)}{2+K(X)}$. In Lemma 1.5 of the same paper Kottman shows that $K(\ell_p) = 2^{1/p}$. Geometrically then the statement that $K(\ell_p) = 2^{1/p}$ can be interpreted as saying that the closed unit ball B of ℓ_p has enough room to accomodate disjoint open balls of radius $\frac{2^{1/p}}{2}$ with the caveat that only the centers of the balls are required to fall within B.

Separated sequences allow us to classify Banach spaces via Kottman's constant $K(X)$ and the related constant $s(X)$. For finite-dimensional spaces both are 0 whereas all values in the interval $(1, 2]$ arise as Kottman's constant for some infinite-dimensional space.

Both ℓ_1 and $C[0, 1]$ admit 2-separated sequences in their unit spheres and hence $K(\ell_1) = K(C[0, 1]) = 2$. Their infinite-dimensional subspaces show a different structure. Every separable Banach space embeds isometrically[8, p. 19] into $C[0, 1]$ so copies of ℓ_p for $1 \leq p < \infty$ can be found. Hence $s(C[0, 1]) = \inf\{2^{1/p} : 1 \leq p\} = 1$.

Broadening horizons beyond the unit ball the quantity $\beta(A) = \sup_{(x_n)\subset A} \inf_{n\neq m} \|x_n - x_m\|$ is called the *separation measure of noncompactness* in metric fixed point theory.

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