Technische Universiteit Delft
Faculteit Elektrotechniek, Wiskunde en Informatica
Delft Institute of Applied Mathematics

The pricing of freight options

Verslag ten behoeve van het
Delft Institute of Applied Mathematics
als onderdeel ter verkrijging

van de graad van

BACHELOR OF SCIENCE
in
TECHNISCHE WISKUNDE

door

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Delft, Nederland
July 2013

BSc verslag TECHNISCHE WISKUNDE

"The pricing of freight options"

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July, 2013

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Acknowledgements

My interest in financial mathematics increased, because of the courses I followed in the Minor Finance. That is why I decided to choose the subject of my Bachelor thesis in this area. I would like to thank my supervisor Kees Oosterlee for the advice and guidelines he provided me with throughout the process of writing this thesis. In this process, I have collaborated with Mercurious, located in The Netherlands. My thanks go to the whole organization and all of its staff, and more specifically to Jerry de Leeuw, managing director, for their practical support, professional views and network of experts throughout the process of writing this document. The collaboration with Mercurious has been very fruitful to me. Next to my thesis I have added value in return. Documentation, exercises and simulations were created by my hands which can be used in Mercurious education business. I also rewrote some Matlab codes I developed for this thesis in Excel to be used during the training courses Mercurious offers.

Contents

1 Introduction

The global economy depends on many different factors. Of great importance to the economy are commodities, such as grain, crude oil, iron ore and chemicals. These form the basis for almost all products in the world. Without them, our life would be a lot less 'luxurious'. It is not difficult to conclude that a lot of money is being invested in commodities and the commodity value chain. Not every country has excess to these commodities directly. That is why they have to be transported from one country to another. Sometimes they are even transported various times, before finally being consumed. Moreover, commodities have to be carried over the entire world. Generally, this is done by seaborne ships, although other ways of transport are also used, such as pipelines for oil and gas, cables for electricity and trains and trucks onshore. More than 95% of global trade (in volume), however, is transported by the use of vessels (see [3, p. 1]).
The great importance of transport by floating transportation equipment can now be concluded. This transport by vessels such as tankers, barges, capesize and others vessels is also called *freight*.

The freight market is a relatively new market, but it has become an increasingly important factor in the global economy. Therefore, the interest in freight and freight-related products is growing larger every year, not only for organizations who have a direct interest in maritime transportation, but also for financial institutions who see great financial opportunities in this market. As this market is relatively new, not much scientific research has been done in this area. In chapter 2 some general information about freight will be considered. In chapter 3 the ways in which freight can be traded and the factors the price of freight depends on will be discussed. Chapter 4 covers the properties of financial products related to freight, also known as *freight derivatives*, with special focus on *freight options*. Since the goal in this document is to price such a freight option, in chapter 5 the problem and approach of pricing a freight option are set out, together with some necessary assumptions. In chapter 6 an *explicit formula* for the price of a freight option is derived and the price of one particular freight option is calculated using this formula. In chapter 7 the validity of this explicit formula is checked by also calculating the prices of this freight option using *Monte Carlo simulations*. Chapter 8 then discusses whether the assumptions and approximations which were made are realistic and whether the explicit formula can be used for all freight options. Finally, a conclusion is given in chapter 9.

2 Freight in general

Before discussing the trading of freight, a closer look is taken at the concept '*freight*'. As mentioned in the introduction, freight covers the transport of goods by vessels. It plays an important role in the economy. Imagine that suddenly no more ships are available to transport commodities. This would mean that one would only have excess to products which are already in a specific region, which would limit product- and food supply enormously. To make sure everyone in the world has excess to the products they need or want, people and organizations rely greatly on freight.

Freight has to be contracted, just like commodities. The only difference is that most commodities are *real products*, while freight is a *service*. So when freight is 'bought', the service of products being transported from place A to place B is contracted. When commodities are contracted, the physical product is purchased. For example, one gives cash and receives coal.
As a result of the fact that freight is a service instead of a physical product, freight is also *non-storable*. It will be seen later that this makes the pricing of financial products related to

freight more complex.

With trading, there is normally someone who owns the product which is being traded and someone who wants to buy it, like a coal producer and a coal costumer. In the case of freight, these are called the *ship-owner* and the *charterer*. Mostly, there is another person or entity, standing between the ship-owner and the charterer, namely the *shipbroker*. This person or organization forms the connection, i.e. delivers mediation services, between the owner and the charterer. The price for which the ship-owner sells the freight to the charterer is called the *freight rate*. This freight rate is very volatile. This is due to various factors freight depends on, which is discussed in section 3.2.

2.1 Types of commodities and vessels

Trade which is transported over sea, also called *seaborne trade*, can roughly be divided into five groups: *dry bulk*, *oil tanker*, *container*, *gas tanker* and *other*. In this document focus is only put on *dry bulk commodities*, because with a market share of 38% these form a large segment of the entire seaborne trade market. The additional 62% exists of *wet bulk commodities*, which is liquid cargo in contrast to dry bulk commodities. Dry bulk commodities are commodities which are shipped in large, unpacked amounts. Examples are coal, ore and grain. Dry bulk commodities can be divided into *major bulks* and *minor bulks*. Major bulks are carried in very large loads, while minor bulks do not necessarily fill an entire vessel when being transported. Major bulks form about two thirds of the dry bulk market and therefore play a more important role in the freight market than minor bulks.
There are vessels which are specifically suitable for the transportation of dry bulk. These can be subcategorized according to their length. The four main categories are *Handysize*, *Handymax*, *Panamax* and *Capesize*, with Handysize being the smallest and Capesize the largest. Panamax and Capesize, which carry major dry bulks, are most important for the freight market, because they represent the biggest market share. Panamax vessels are the largest ships that fit through the slots of the Panama canal and they sail from the Pacific to the Atlantic Ocean. Capesize vessels are too large to fit through the Panama canal, so they travel round Cape Horn (Argentina) or Cape of Hope (South-Africa) between the Pacific and the Atlantic Ocean. Panamax ships can transport up to $70,000$ DWT[1], while Capesize vessels typically carry more than $150,000$ DWT.

[1]DWT stands for deadweight ton, which is the sum of the weights of cargo, fuel, fresh water, ballast water, provisions, passengers, and crew of a ship.

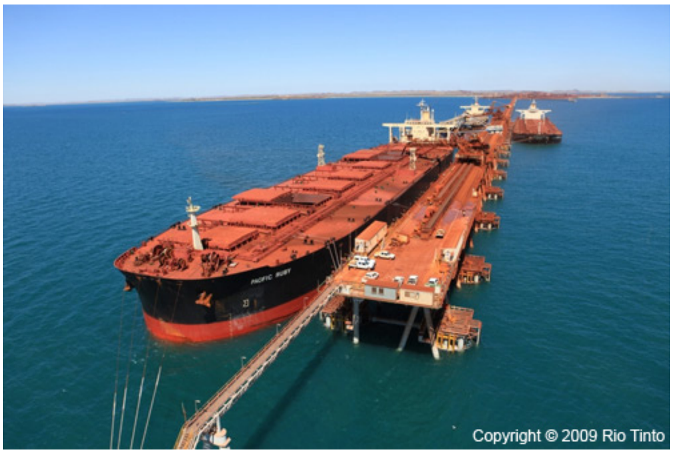

Figure 1: A Capesize vessel lying at Capre Lambert port in Australia.

Figure 1 pictures the size of a Capesize vessel. In table 1 the size in DWT of four different types of vessels, the percentage of ships in the world of each kind, the percentage of cargo each type of ship transports (measured in the weight of the cargo together with the distance traveled), the price of a new ship and the price of a five year old ship are displayed.

Name	Size in DWT	Ships	Transport	New price	Old price
Handysize	10000 to 35000	34%	together 18%	\$25 million	\$20 million
Handymax	35000 to 59000	37%		\$25 million	\$20 million
Panamax	60000 to 80000	19%	20%	\$35 million	\$25 million
Capesize	80000 and over	10%	62%	\$58 million	\$54 million

Table 1: Properties of the four main types of dry bulk vessels (as found on *www.wikipedia.org*).

There are many more Handysize and Handymax than Panamax and Capesize vessels, yet the percentage of cargo that especially Capesize vessels transport, is significantly higher than that of Handysize and Handymax vessels. This is due to the fact that Capesize vessels can carry significantly more cargo and mostly travel longer routes than smaller vessels. Furthermore, these large Capesize vessels can cost a multiple of the smaller Handysize, Handymax and Panamax vessels. In other words, buying a new Capesize vessel costs a lot more than buying a Handysize, Handymax or Panamax vessel.

2.2 Chartering

Now that a global idea of the different types of ships is provided and the enormous amounts of cargo they can carry is set out, the *chartering* of these vessels is discussed. Chartering is the term used when the owner of a ship lets it to others for transporting cargo. The charterer can use the ship to carry its own cargo or is able to re-let the ship to other charterers for a higher price, so that the charterer can make profit.
There are different ways to charter a ship, below the most common ways:

- With a *voyage charter* the ship ánd the crew are hired for a one-way trip with a specified cargo and a specified freight rate. The owner of the ship pays for the fuel, the crew and the ports. The charterer pays the owner on a per-ton or lump-sum basis[2].

- With a *time charter* the ship is being hired for a stated period of time. In such a case the charterer pays all the port costs, fuel costs and a daily hire. In 'exchange' the charterer can choose the route the ship is going to take, also the charterer can decide which ports will be passed.
- With a *bareboat charter* the owner lets the boat for a specified time without any provision like insurance, stores or crew.

To get an idea of the height of freight rates for the various types of vessels, let's have a look at the graph in figure 2, which shows the average charter rates per day in US dollars for different kinds of vessels in the period 2001-2012:

Figure 2: Average time charter rates per day in USD for different vessels (as found on *www.wikipedia.org*).

From 2001 until 2012 the rates have differed from $\$2,000$ until $\$256,000$. Since the rate is the price per day that has to be paid, also the daily rate for chartering a ship, the total amount of money circling in the freight market is enormous. This again shows the great importance of the freight market in the global economy. Furthermore, the graph shows that in 2008 the rates drastically dropped, which is due to the credit crisis.

2.3 Routes

Not only can ships be chartered in different ways, as described in the previous section, there are also different *routes* ships can be chartered for. There are many different routes vessels have to

[2] When payment is accomplished on a lump-sum basis, a specified sum of money is paid for the trip, while on a per-ton basis, payment is due per ton cargo.

take, as cargo has to be shipped over the entire world. Some routes are used very frequently. A few of these routes for Panamax and Capesize vessels are now considered. Two major Capesize voyage charter routes are Richards Bay to Rotterdam (reflected by C4, whereby the C stands for Capesize and 4 for the particular route) and Bolivar to Rotterdam (reflected by C7). Two major Panamax time charter routes are Gibraltar to Far East (reflected by P2A, whereby the P stands for Panamax and 2A for the particular route) and Pacific round (reflected by P3A). Sometimes a vessel is chartered for one single route, because this route covers the precise places the cargo has to be shipped in between. However, it is possible that cargo needs to be shipped between two places for which no specified route exists. A vessel can then be chartered for a collection of single routes, so that the resulting route covers the places the cargo has to be taken from/to.

3 Trading of freight

Until now, only the fact that freight can be contracted by trading the actual service of transporting goods was mentioned. However, there are other ways to trade freight. Also organizations that are not directly interested in the service of transporting goods can still have a great interest in the freight market. This is because *financial products related to freight* are provided by the freight market. Different kinds of financial contracts exist, in which agreements are made about paying and receiving a certain amount of money which is related to the freight rates. To give a very simple example, one could trade a contract which states that three days from now one will pay the seller of that contract the value which the freight rate of a certain route and charter-type had two days ago and one will receive the value it will have two days from now. This example gives a good insight in how freight can be traded by not actually trading the service itself, but by 'trading the freight rates'. There are many financial contracts like this circling in the freight market. In fact, these contracts form a much larger freight market share than the actual trading of the freight service. In section 4 some of these contracts and reasons why the market provides them are discussed.

3.1 Exchanges

Organizations, traders and analysts who are interested in freight are keen on data and information regarding the freight market. Transparency is important in the process of price discovery. There are exchanges and trading platforms for freight, just as there are exchanges for commodities and stocks. An *exchange* is a highly organized, regulated market where (financial) contracts can be bought and sold. An exchange supports the process of price discovery by providing information for everyone with interest in the contracts being traded.

3.1.1 The Baltic Exchange

The most well-known freight exchange[3] is *the Baltic Exchange* in London, which is an independent source of maritime market information. It was founded in 1744. The organization provides independent daily shipping market information, maintains professional ship-broking standards and resolves disputes. Parties with interest in trading freight contracts[4] can become a member of this foundation. A lot of the information about freight in this document has been found in paper [5] about the Baltic Exchange.

[3]In fact, the Baltic Exchange is a Multilateral Trrading Facility (MTF) rather than an exchange.

[4]Ship-owners, charterers, shipbrokers, but also financial institutions, maritime lawyers, educators, insurers and related associations

Just like the S&P500 shows the development of the stock rates from the 500 largest companies on the U.S. stock exchange, the Baltic Exchange developed an index for freight. This *Baltic Freight Index* BFI shows the development of freight rates for sea routes that are most used. At the beginning, in 1985, the Baltic Freight Index was based on 13 different routes. In time, more routes were added and after a while, the Baltic Freight Index got split into subcategories. There are now indices which reflect freight rates for dry bulk only (Baltic Dry Index, or BDI), for Capesize vessels (Baltic Capesize Index, or BCI), for Panamax vessels (Baltic Panamax Index, or BPI), and so on. The Baltic Dry Index is most used in the freight market, because of the large market share of dry bulk commodities.

3.1.2 IMAREX

Another well-known freight trading facility is offered by *the International Maritime Exchange (IMAREX)* in Oslo. It was founded in 2000 and began as a small association, but nowadays it handles financial freight products worth over USD 200 billion per year.

Figure 3: Baltic Dry Index and S&P500 during the last year (as found on *www.bloomberg.com*).

In figure 3 a relative chart with the Baltic Dry Index and the S&P500 over 1 year are presented. The graphs show the percentage changes from, respectively, the Baltic Dry Index and the S&P500, compared to the values of these two indices at the beginning of June 2013. The absolute percentage changes of the Baltic Dry Index are much higher than those of the S&P500. In other words, the Baltic Dry Index varies much more than the S&P500. This is in line with the statement in subsection 1, that freight rates are very volatile.

3.2 Factors which effect freight rates

In the beginning of section 2 the high volatility of freight rates was mentioned. There are many different events which can have an effect on the cost of sea transport. A distinction can be made between *internal* and *external factors*. *Internal factors* cover factors directly related to freight, like fleet supply and commodity demand, while *external factors*, such as seasonal pressures, do not influence freight directly. When looking at some examples of this second type of factors more closely, it can be seen that they can have an enormous influence on the freight rates. This makes external factors at least as important as internal drivers.

3.2.1 Internal factors

The first internal factor that drives freight rates is *fleet supply*. When few ships are available for chartering, while at the same time on a relative basis many parties wish to charter a vessel, freight rates will rise. When, on the other hand, many ships are available, freight rates will decrease. As with all products and markets it is a matter of supply and demand. For the reason described above, it is important for both ship owners and ship charterers to spend sufficient time on fleet analysis. Awareness of the global and regional number of vessels per type or class is crucial, as much as the age of those ships. When, for example, globally 50 Capesize vessels are available, out of which 40 are over 20 years old (whereas the expected lifetime of a Capesize vessel is only 25 years), it is reasonable to expect that soon there will be a shortage of those ships. Therewith, tension in the market is expected. This will then most likely put upward pressure on the freight rates for Capesize vessels. This way, ship owners, charterers and financial institution try to forecast freight rates (at least that part which depends on fleet supply).
A logical continuation of freight rate evaluation, after looking at the fleet supply, is to look at the demand-side, the *commodity demand*. When commodities have to be transported, but the availability of ships is relatively low, freight rates will increase. Conversely, when the demand is relatively low, freight rates will decrease. So, when, for example, the grain harvest has been very successful, a lot of grain has to be shipped, which results in high freight rates (of course under the condition of ceteris paribus). Market participants with interest in the freight market usually try to forecast commodity demand in some way, because they want to make an accurate estimation of how the freight rates will behave in the future.

3.2.2 External factors

Next to internal factors there are external factors which affect freight rates. First of all, *seasonal pressures* can have a great effect on the rates. In winter time, low temperatures (frost) can cause ice, which affects the routes ships can take. When detours have to be taken, not only the total freight costs rise, due to longer travel time, but also the freight rates will rise, because longer traveling time results in less availability of ships. Season or weather can also influence the harvest. When the weather is optimal, harvest will be successful, which results in a large demand of transport, and therefore high freight rates.
Another factor which affects freight rates, is the *price of bunker fuel*. When the price of bunker fuel rises, the costs for transport will increase as well. Therefore, high fuel prices result in high freight rates.
Currency exchange rates are also of influence on freight rates. Consider an Australian ship charterer who wants to transport its goods from the U.S. to Australia. The Australian company wants to charter a ship from a U.S. charterer. Doing so, he will first need to exchange Australian dollars for U.S. dollars. When the exchange rate between Australian and U.S. dollars is favorable for the Australian company, freight rates that the U.S. company will charge the Australian company for, will be relatively low. Low freight costs will support high demand, which then in time results in higher freight rates. Al in all this means that foreign exchange rates cause freight rates to fluctuate (volatility).

Above, only a few of the widespread factors that influence freight rates have been mentioned. It is not difficult to conclude that freight rates fluctuate a lot, i.e. are very volatile. They depend on so many factors, that it is impossible for the rates to stay constant. This high volatility causes some concern. Ship-owners and charterers are not able to precisely forecast the freight rates, so they carry a lot of price risk. When a charterer wants to transport goods next month, little can be said about the price the organization will have to pay the ship-owner at that time. Also the ship-owner is exposed to a lot of price risk; this market participant cannot forecast what the future spot rates will be, so he cannot say anything about the amount of his future profit or loss. To this extent, *freight derivatives* have been provided.

4 Freight derivatives

4.1 Derivatives in general

Because freight rates are very volatile, the trading of freight comes with a lot of price risk. To protect ship-owners and charterers against this risk, the market provides them with financial products related to freight, also called *freight derivatives*, like freight futures, forwards and options. Ship-owners and charterers can use freight derivatives to hedge their exposures. Derivatives are mostly used for hedging or speculating.[5] In other words, they can be used for different business motives, namely to take away risk (hedging) or to make profit (speculation). Speculating is the reason why the freight market is so interesting for non-physical players, like banks. These organizations try to predict the future freight rates, to buy a suitable freight contract, with which they expect to make a profit.

In sections 4.1.2 until 4.1.4 futures, forwards and options in general are briefly discussed, before looking at freight derivatives in particular.

4.1.1 Terminology derivatives

Throughout this document some terminology according to forwards, futures and option will be used. First, terminology for forwards/futures is discussed:

- The *underlying asset* is the asset which is going to be exchanged.
- The *contract price* F is the price at which the underlying asset is exchanged.
- The *trade date* T_1 is the date at which the amount to be paid/received by the parties is calculated.
- The *settlement date* T_N is the date at which the actual exchange will take place, so the date at which the money has to be exchanged.
- The *settlement price* S of the underlying asset is the price at which the underlying asset is settled at the settlement date.
- The *settlement period* $[T_1, T_N]$ is the period between the trade date and the settlement date.
- The party who *buys* a forward/future is called the *holder* and has to take delivery. The holder of a forward/future is said to be *long* the forward/future. The seller of the forward/future is said to be *short* the forward/future.

[5] They can also be used for arbitrage or physical trading

For options, almost the same terminology holds. There are, however, a few differences. An option has no settlement period, so no trade/settlement dates are considered and instead of a *contract price* one speaks of a *strike price*.

- The *strike price* K of an option is the price at which the underlying asset is exchanged when the contract is exercised.
- The *exercise date* of an option is the date at which the option is exercised, also the date at which the underlying asset is exchanged. This equals the trade date and the settlement date for forwards/futures.
- The *settlement price* S of the underlying asset is the price at which the underlying asset is settled at the exercise date.
- The *expiration date or maturity* T of an option is the last point in time at which the option can be exercised. After this, the option has no value anymore.
- The party who *buys* an option is called the *holder*. The *seller* of the contract is called the *writer*. The holder of an option is said to be *long* the option. The seller of the option is said to be *short* the option.
- A *call option* gives the holder the right to *buy* the underlying asset, while a *put option* gives the right to *sell* it. With a call option the writer has the obligation to sell the underlying asset if the holder decides to exercise, while with a put option the writer has the obligation to buy it.

4.1.2 Forwards

Definition 1. *A forward is a financial contract in which the holder and the writer agree to exchange the underlying asset at the settlement date for the contract price.*

Using a forward contract, a trader can lock in the price (s)he will pay/receive for a certain asset in the future and (s)he does not have to worry about the risk that the price of the asset will increase/decrease. However, once the future contract starts, the two parties are obliged to exchange the underlying asset, even if it will not be profitable for (one of) them.
In figure 4 the payoff of a forward for both the buyer and the seller of the contract are shown, where the variables F and $S(T)$ have the same meaning as mentioned in the terminology[6] ($S(T_1)$ is the settlement price at the trade date).

[6] From now on, these variables will be used to indicate the corresponding terminology throughout this document

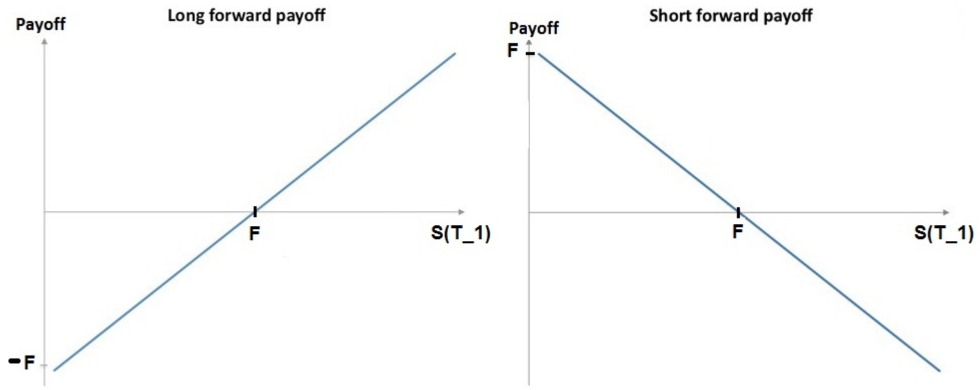

Figure 4: Payoffs from a forward contract for both the buyer and the seller.

4.1.3 Futures

A *future* is basically the same as a forward, however, there are two differences:

1. A future is *exchange-traded*, while a forward is traded *over the counter* (OTC). This means that a forward is a private contract between two parties, which implies there is a relatively high credit risk, also a probability of default. A future, on the other hand, is a more formal contract, whereby the clearing house[7] takes over the counterparty risk (and therewith assures there is no credit risk for a party when the counterpart defaults).

2. Exchange-traded contracts (futures) require the two parties to settle price changes daily, so that no party ever has a large obligation to the other. Otherwise, when one party defaults while having a large obligation to the other party, the clearing house has to pay a large amount to cover this obligation. By applying this so-called *marking-to-market* principle, the clearing house only has small amounts of money it might has to cover. With a forward, no daily settlement is required. Note that the marking-to-market principle does not change the amount the contract pays off, it only spreads the payoff over more points in time.

4.1.4 Options

Definition 2. *An option is a financial contract in which the holder has the right, but not the obligation, to buy or sell the underlying asset at the strike price on or before the expiration date.*

So, the main difference between an option and a future/forward is the fact that a future gives the holder (and its counterpart) an obligation, while an option gives the holder a right (but the writer also a potential obligation).
Figure 5 shows the payoffs for both a put and a call option with strike price K, when both long and short.

[7]A clearing house is a financial institution that provides clearing (all activities from the time a commitment is made for a transaction until it is settled) and settlement services for financial and commodities derivatives and securities transactions.

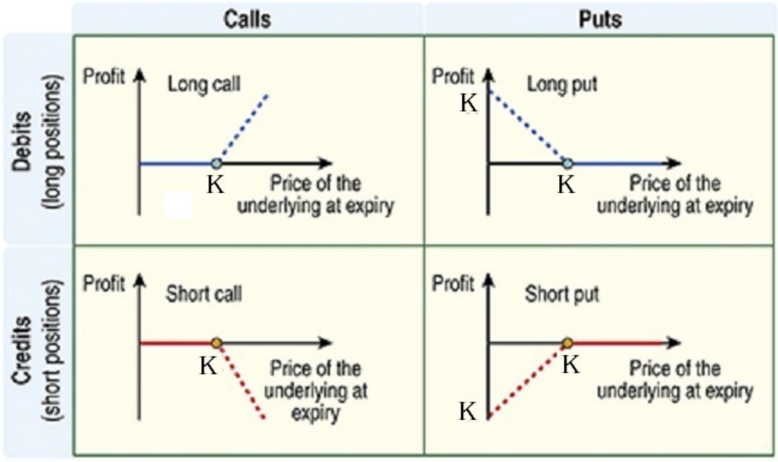

Figure 5: Payoffs from a put and call option for both the buyer and the seller.

4.2 Freight derivatives

Now that standard knowledge about derivatives is set out, focus is put on *freight derivatives* in particular. The first thing to notice, is that with freight derivatives the underlying asset can be, for example, the Baltic Freight Index[8] or another index which represents the spot freight rates. Furthermore, freight derivatives are *cash settled*. In other words, instead of actually receiving the underlying asset at delivery time, the owner of a freight derivative receives cash. When, for example, an equity future (a future with a stock as underlying asset) is *cash settled*, the owner of the future receives the difference between the value of the stock at settlement minus the agreed price at the conclusion of the deal. When the same future is *physically settled*, the owner receives the actual stock at expiration and pays the price which was fixed at the expiration date. So, in the case of freight derivatives, the owner of the derivative receives at expiration the value of the Baltic Freight Index/freight rate (settlement price) minus the price which was fixed at the conclusion of the deal.

Furthermore, in practice, all freight derivatives consist of more than one settlement period. In theory, first a single settlement period is considered and then a few of these single contracts are summed up to form a contract with more settlement periods, as will be seen later when pricing a freight option.

4.2.1 Freight futures or Forward Freight Agreements

The first freight futures to be traded were introduced on the Baltic International Freight Futures Exchange (BIFFEX) in May 1985. This shows that the financial freight market is, with an age of around 30 years, indeed rather new, as was stated in the beginning of this document.

Freight futures are mostly referred to as *Forward Freight Agreements (FFA)*. This might be a bit misleading, since the word 'forward' is involved in this, but nowadays most of the time these FFA's are traded as futures, so through a clearing house.

Definition 3. *A freight future is a financial contract which states that at a certain time in the future one party will pay the contract price and receive the value of the Baltic Freight Index at that time from the other party.*

[8]As mentioned before, this index is now subcategorized into different indices, but we will just refer to it as the Baltic Freight Index from now on.

A freight future is settled against the *average of the spot freight rates*[9] (also, the average values of the Baltic Freight Index) during the settlement period. This averaging is done for two reasons:

1. First of all, settling against the average values of the Baltic Freight Index limits the chance of price manipulation by large participants. When the FFA's would be settled against one single freight rate, participants may try to increase or decrease the freight rate in the future by trading a lot or very little in freight rates just before the time at which they want the freight rate to have a certain value. When the FFA's are settled against an average of the freight rates, the chance of this happening, decreases significantly[10].

2. Secondly, the transportation of goods generally takes more than just one day, so charterers are exposed to freight rates for some period of time. When settling against the average of freight rates during this period, freight rates during the entire length of the transportation are considered, which seems more fair than just looking at a freight rate on one of the days during this period.

Moreover, FFA's can be based on a per day contract price or a per ton contract price. When the FFA is based on a time-charter, the contract price will be expressed by an amount per day, while with voyage charters, the contract price will be expressed by an amount per ton.

Example 1. *Consider a ship-owner who is looking to protect the company against a possible decrease of the freight rates. To this extent, the company sells an FFA contract for its Panamax ship based on time chartering, with trade date* $T_1 = 1$ *in years, contract price* $F = \$18,500$ *per day, 12 settlement periods, where every settlement period has the length of one month (assume there are 30 days in a month) and with the first settlement at 31 January 2014. If the average of the freight rates in January, also the settlement price, turns out to be* $\$15,000$*, the ship-owner receives the difference of* $\$18,500 - \$15,000 = \$3,500$ *per day during 30 days. This sums up to a total amount of* $\$3,500 \times 30 = \$105,000$*. When the settlement price in February is* $\$20,000$*, the ship-owner has to pay the difference* $\$20,000 - \$18,500 = \$1,500$ *per day to the holder of the FFA, which sums up to a total amount of* $\$1,500 \times 30 = \$45,000$*. After paying or receiving these differences every month, at the end of the year the ship-owner can check whether this FFA has been profitable for them.*

In figure 6 an example of the payoff of an FFA sold at WS105[11] is shown, where the blue line indicates the realized average freight rates. The pink constant line is the contract price (so the price to be paid for the blue line). The blue and pink volumes respectively indicate the profits and losses made by the seller of the FFA contract. Note that two vertical scales are used.

[9]We make a difference between *spot freight rates* and *forward freight rates.* Spot freight rates are the prices you have to pay for immediate settlement of freight, while forward freight rates are the expected prices you will have to pay for freight in the future.

[10]These large market participants would have to buy/sell an enormous amount not just once, but various times, to be able to have a significant influence on the average of the freight rates instead of on just one single rate.

[11]WS stands for World Scale, which is a unified system of payment for freight in the tanker cargo world.

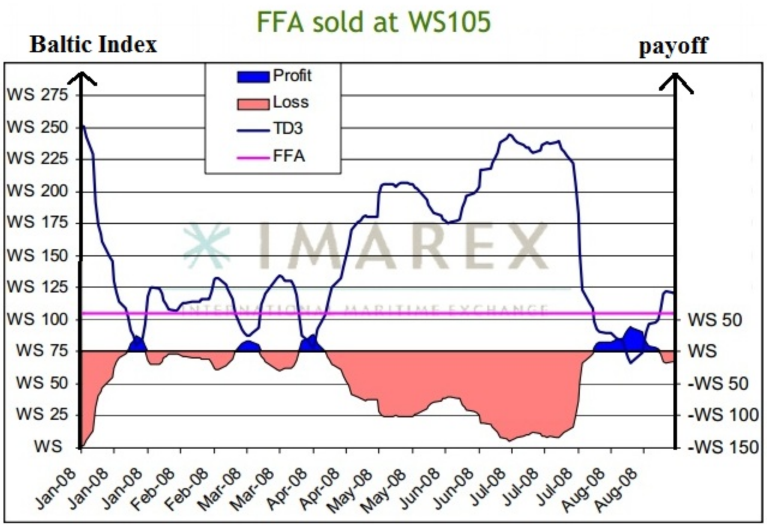

Figure 6: Profits and losses made by the seller of a FFA contract for tanker cargo sold at WS105 (as found on *www.balticexchange.com*).

It can be seen that when the blue line, also the average Baltic Index[12], transcends the pink FFA contact price, the seller of the FFA loses money. This is because the seller now earns less than he could have when just selling his freight against the spot freight rate. On the other hand, when the blue line lies beneath the pink contract price, the seller makes a profit, for he can sell his freight against a higher price than he could have, when just selling it against the spot freight rate. Logically, when the Baltic Index line and the contract price line intersect, the seller makes neither a profit, nor a loss.

We can of course also look at the same FFA, but now focus on the viewpoint of the buyer. In this case, the buyer makes a profit when the seller makes a loss and conversely makes a loss when the seller makes a profit. Figure 7 shows these profits and losses in a graph, which can be interpreted in the same way as figure 6.

[12]Notice that the words 'freight rates' and 'Baltic Freight Index' are used interchangeable

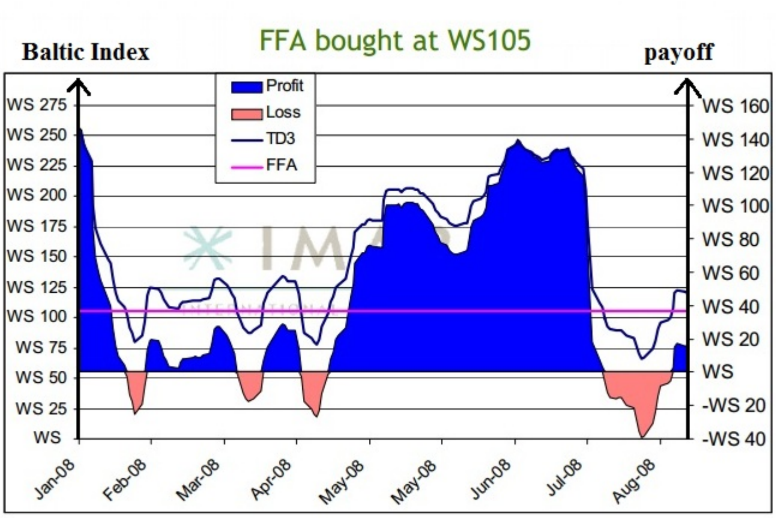

Figure 7: Profits and losses made by the buyer of a FFA contract for tanker cargo bought at WS105 (as found on *www.balticexchange.com*).

4.2.2 Freight options

Apart from freight futures, the freight market also deals with *freight options*, which are insurances against freight rates moving beyond a specified price level. There are different kinds of options. In this document, only so-called *European* and *Asian* options are important.

Definition 4. *A European option is a type of option which can only by exercised at the exercise date of the option. Its payoff is calculated by settling against the value of the underlying asset at exercise time. In other words, the payoff of a European (call) option with strike price K equals* $\max\{S(T) - K, 0\}$.

Definition 5. *An Asian option is a type of option from which the payoff is calculated by settling against the average values of the underlying asset during some period of time* $[t_1, t_N]$. *In other words, the payoff of an Asian (call) option with strike price K equals* $\max\{\frac{1}{N}\sum_{i=1}^{N} S(t_i) - K, 0\}$.

A freight option is an Asian option with the spot freight rates/Baltic Freight Index as underlying asset.

Definition 6. *A freight option is a financial contract which states that the holder has the right to pay/receive the average of the values of the freight rates during some period on or before the expiration date and receive/pay strike price. The writer then has the obligation to receive/pay this average and pay/receive the strike price when the holder decides to exercise.*

In section 5.2.2, it will be seen that a freight option can not only be seen as an *Asian option* with the spot freight rates/Baltic Freight Index as underlying asset, but also as a *European option* with an FFA as underlying asset.

With an option in his portfolio, the holder has no risk of losing any money due to high or low prices of the underlying asset, because he always has the possibility not to exercise the option and thereby have zero payoff. This way, the (expected) payoff from an option can never be negative. This differs from a future, where the holder has the obligation to exercise the contract, which can lead to a negative payoff for the holder. However, with an option, the holder has to pay an upfront premium to the seller. This is the only way the seller of the option can earn money, because the holder can never have a negative payoff. In other words, the seller can never have a positive payoff (if this premium would not be taken into account).

Example 2. *Consider a ship-owner who is looking to protect the company against possible decreases in the freight rates. To this extent, the company buys a freight put option for its Panamax ship based on time-chartering, with expiration date* $T = 1$ *in years, strike price* $K = \$18,500$ *per day, a premium of* $\$2400$ *per day, 12 settlement periods, where every settlement period has the length of one month (assume there are 30 days in a month) and with the first settlement at 31 January 2014. The total premium the ship-owner has to pay the writer of the put option is* $\$2400 \times 30 \times 12 = \$864,000$*. He has to pay the daily premium everyday upfront. If the average of the freight rates in January, also the settlement price, turns out to be* $\$15,000$*, the ship-owner receives the difference of* $\$18,500 - \$15,000 = \$3,500$ *per day. This sums up to a total amount of* $\$3,500 \times 30 = \$105,000$*. When the settlement price in February is* $\$20,000$*, in contrast to the FFA in example 1, the ship-owner will decide not to exercise his put option, so in this month he will not receive any money with this option. For the writer of this put option to determine whether the sale of this option was profitable for him, he has to add up the total amount he had to pay the holder of the option (the ship-owner) and subtract this from the total premium he received. If this resulting amount would be positive, the trade would have been profitable for him, and the holder of the option would have had a negative profit.*

If making profit was the purpose of the writer for writing this option in example 2, it is called speculation. Instead of using the option for speculation, the writer of this option could have also used it for hedging. For example, when the writer would have been long a similar freight put option, by writing this freight put option, he would hedge his position. No matter what happens, the writer would always end up with no losses, respectively profits (caused by these two options); If the put option would end in the money, the writer would exercise his put option, whereby he would receive a positive payoff. On the other hand, the holder of the put option he has written, will also exercise his option, also the writer has to pay this holder the exact same amount as he received himself with his put option. Conversely, if the option would end out of the money, no one would exercise his/her put option, so the holder (writer) of the first option has a loss (profit) equal to the premium of the option and, similarly, the writer in this example has a loss equal to the premium, caused by the second put option.

We can conclude that the main difference between the payoff of an option and the payoff of an FFA, is that with an option, the negative payoff cannot grow larger than the amount of the premium, while the negative payoff of an FFA in theory can grow until the negative value of the contract price. This is summarized in figure 8.

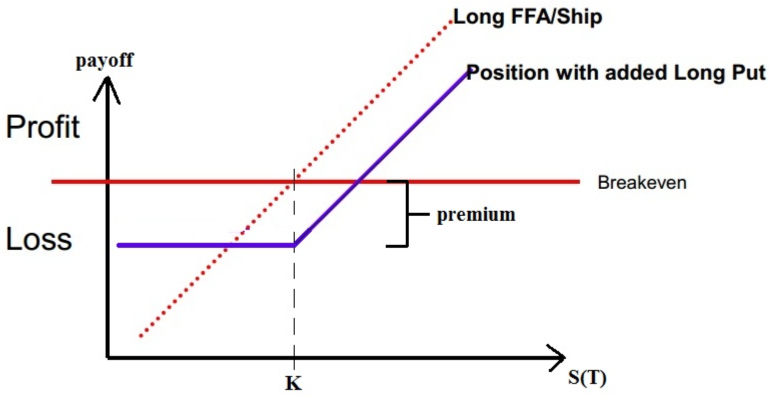

Figure 8: Payoffs from an FFA and a put option.

The payoff of the option has a lower bound, while the payoff of the future keeps on decreasing as $S(T)$ decreases. The FFA makes profit with a lower $S(T)$ than the put option does, because of the premium that has to be covered with an option.

Now, consider figure 9, similar to figures 6 and 7, to see how hedging using a freight option works. Again, the blue line indicates the average Baltic Index or the spot freight rates, the blue and pink volumes equal the profits and losses and now the pink line indicates the value of a call option.

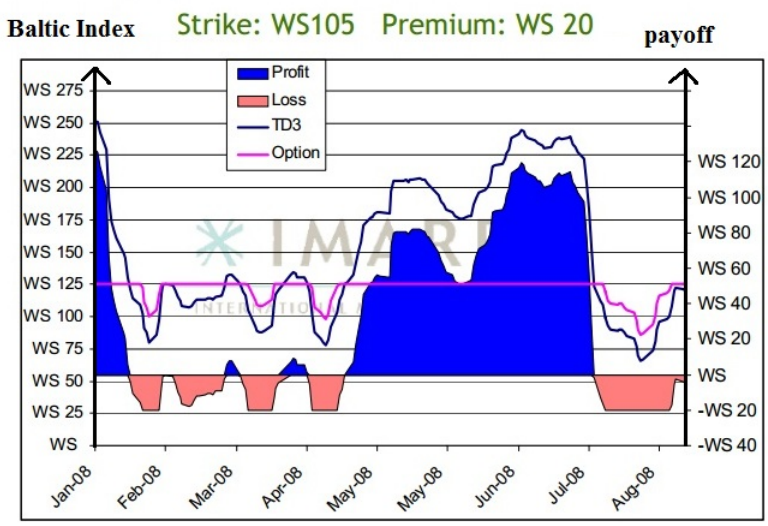

Figure 9: Profits and losses made by the buyer of a call option with strike price WS105 and premium WS20 for tanker cargo (as found on *www.balticexchange.com*).

The losses indeed never transcend the premium of WS20, because otherwise the buyer of the call option would not exercise his right. Furthermore, when the blue line exceeds the pink option price line, the buyer makes a profit, while when the blue line lies beneath the pink line, a loss is made. Logically, when the blue and the pink line intersect, neither a profit, nor a loss is made.

Similarly, we can look at a put option with the same strike price and premium. Figure 10 shows the profits and losses made by the buyer of this put option. The realized average freight rates turned out to be significantly higher than the expected average freight rates. Therefore, the strike price of the put option was determined too low, which resulted in very few profits for the buyer of the put option. Again, we see that the losses made, never exceed the premium of WS20.

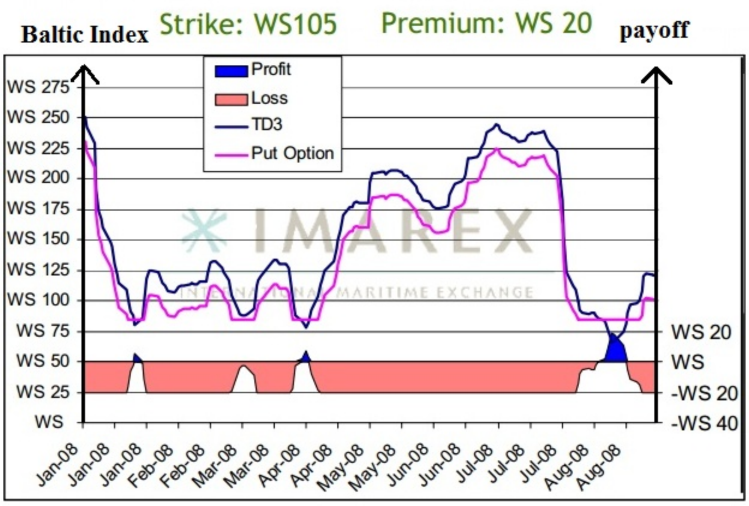

Figure 10: Profits and losses made by the buyer of a put option with strike price WS105 and premium WS20 for tanker cargo (as found on *www.balticexchange.com*).

5 The pricing of freight options

After discussing the properties of Forward Freight Agreements and freight options, we start with the actual pricing of a freight option. We discuss what the prices of freight options depend on, how they vary and how they can be calculated.

5.1 Assumptions

First of all, some assumptions have to be made:

1. It is possible to borrow or lend money at the same constant, deterministic and continuously compounded interest rate r with no transaction costs.
2. At each time t a spot freight rate exists, denoted by $S(t)$. This is a non-tradable asset and is typically a value of the Baltic Freight Index.
3. The settlement period is denoted by $[T_1, T_N]$, where $T_1 < T_N$. Take N fixings at time points T_n, $n = 1, \ldots, N$, which are chosen to be days. Set $N = 21$, so that one settlement period consists of 21 days (which is assumed to be the number of working days in one month). Holidays and weekends are ignored.

4. All FFA's considered in this document are traded through a clearing house, so credit risk is not taken into account.

5. The market for FFA's is liquid, so the FFA's can be traded continuously at each time until the end of the settlement period T_N.

6. The price of an FFA at time t, $F(t, T_1, T_N)$, is a martingale under the real-world measure P[13]. The arguments between brackets are values the price of an FFA depends on; its price depends on the current time t and the length of the settlement period.

   Definition 7. *A stochastic process X is called a martingale if it satisfies*

   (a) $E[X_{n+1}|X_1, \ldots, X_n] = X_n \ \forall n$,
   (b) $E[|X_n|] < \infty \ \forall n$

   So, when using the real probabilities in the freight market, the expected price of the FFA at time $n+1$ equals the price at time n, for all n.

7. There are no arbitrage probabilities[14] in the freight market.

   Theorem 1 (Fundamental theorem of asset pricing)**.** *A market has no arbitrage probabilities if and only if there exists a risk-neutral probability measure Q such that every discounted price process in this market is a martingale under the probability measure Q.*

   From theorem 1 it can be concluded that there exists a risk-neutral probability measure Q such that the price of an FFA at time t is a martingale not only under the real-world probability measure P (assumption 6), but also under the risk-neutral probability measure Q. More formally:

   Result 1. *The price of an FFA equals the discounted expected payoff under the risk-neutral probability Q.*

   Further explanation of Result 1 is given in section 5.1.1.

8. The payoff of an FFA starting at time t equals the difference between the average freight rates during settlement and the price $F(t, T_1, T_N)$, multiplied by a constant D, which refers to the number of days the FFA contract covers or an agreed cargo size. This depends on whether the FFA is based on a time-charter service or a voyage-charter service. More formally, the payoff of a future equals

$$D\left(\frac{1}{N}\sum_{i=1}^{N} S(T_i) - F(t, T_1, T_N)\right) \tag{1}$$

9. Spot freight rates are assumed to be log-normally distributed with drift μ and volatility σ.

[13] In section 5.1.1 this term is explained

[14] arbitrage is the phenomenon of making a risk-free profit by investing a certain amount of money in an asset higher than the amount of money made when putting this amount on a savings account with risk-free interest rate.

5.1.1 Real-world versus risk-neutral measure

In the assumptions in section 5.1, the words *real-world measure* and *risk-neutral measure* were used. Some explanation is needed for these terms. Financial assets entail risk; it is not known how, for example, the underlying asset of an option will behave in the future. To cover this risk, investors require risk premiums. These risk premiums do not only differ for different assets, since not every asset is equally risky, they also differ for different investors, since not every investor requires the same amount of 'risk-protection'. Furthermore, the risk belonging to a specific asset is difficult to predict. In other words, pricing assets when considering these actual risk premiums, also under the real-world probabilities, is difficult. To simplify the pricing of financial assets, the risk-neutral measure has been constructed. When using the risk-neutral measure, financial assets are assumed to be riskless. In other words, investors do not require risk premiums. This way, every asset can be priced in the same way, instead of needing different pricing methods for every asset, because of the different risk premiums. Therefore, the risk-neutral measure, instead of the real-world measure is used in asset pricing. The real-world measure can, however, be used to calibrate risk-neutral asset pricing models by considering historical data.
Result 1 can now be explained as follows: Under the risk-neutral probabilities, assets are assumed to be risk-free. In other words, their expected payoff is assumed to equal the real payoff. Furthermore, the freight market is assumed to have no arbitrage probabilities. Therefore, the discounted[15] expected payoff of an asset should equal its price; if the price is higher than the discounted expected payoff, which equals the discounted real payoff under the risk-neutral probabilities, no one will buy the asset, since a loss will be made. Conversely, if the price is lower than the discounted expected payoff, also the discounted real payoff under the risk-neutral probabilities, a definite profit will be made. This way, one can make a risk-less profit higher than the amount of money made by putting an amount of money equal to the asset price on a savings account with the risk-free interest rate. This is known as *arbitrage*, which was assumed not to exist. This is why the price of an asset should always equal its discounted expected payoff under the risk-neutral probabilities.

5.2 Dependences of FFA prices and freight option prices

5.2.1 Relationship freight rates and FFA prices

The discounted expected payoff of a future under the risk-neutral measure should always equal zero, to avoid arbitrage opportunities[16]. More formally, as given in [1, p. 6], the conditional expectation under the risk-neutral probability measure Q of the discounted version of equation (1) should equal zero:

$$E_t^Q \left[e^{-r(T_N - t)} D \left(\sum_{i=1}^{N} \frac{S(T_i)}{N} - F(t, T_1, T_N) \right) \right] = 0 \tag{2}$$

In equation (2) $e^{-r(T_N - t)}$ is the discount factor.
Later on, it will be shown that this equation makes sure the price of the FFA $F(t, T_1, T_N)$ equals the expected average freight rates during the settlement period.
To avoid confusion, from now on the contract price of an FFA F will be called the price of the

[15] Discounting is calculating the amount of money today which will equal a certain amount of money in the future when putting it on a savings account with risk-free interest rate r

[16] If the discounted expected payoff would equal a positive amount, 'money could be made for free'. No premium has to be paid for the future, so the discounted expected profit equals the discounted expected payoff, which would be positive in this case.

FFA. The value of the FFA, also its discounted payoff, will simply be called its value.
When time goes by, the value of a futures contract may change. This is because the underlying index changes in time. When the index increases, it is likely that the settlement price of the future, also the average of the index, will increase. In that case, a profit will be made by the holder of the future, since the contract price to be paid will be less than the settlement price. This leads to an increase in the value of the future.
The price of a futures contract may also change in time. The price of a particular future contract is fixed until the end of the contract. However, if for the same future the contract price would be determined one day later, this contract price may differ from the one the day before. That is because the contract price depends on the underlying freight rates, which change in time. That is why it is interesting to trade future contracts. If the price of a particular future contract turns out to be higher the day after the contract started, the buyer of the contract can decide to sell his contract, whereby he receives the positive difference between the fixed future price at the beginning of the contract and the price one day later.

5.2.2 Relationship FFA prices and freight options

Freight options can be seen as both arithmetic[17] price Asian options on the Baltic Freight Index or, similarly, as European options on Forward Freight Agreements. Why these two types of options are the same, will now be discussed.

Arithmetic Asian options on freight rates have a payoff equal to the maximum of the difference between the strike price and the average of the freight rates during the settlement period, and zero (since the holder of the option will not exercise when the payoff is less than zero). More formally, the payoff of a freight call option with strike price K equals

$$C(T_N, T_N) = D\left[\frac{1}{N}\sum_{i=1}^{N} S(T_i) - K\right]^+ \tag{3}$$

where $C(T_N, T_N)$ is the notation for the price of a call option at time T_N with exercise time T_N, also the payoff.
Similarly, the payoff of a freight put option with strike price K can be written as

$$P(T_N, T_N) = D\left[K - \frac{1}{N}\sum_{i=1}^{N} S(T_i)\right]^+ \tag{4}$$

where $P(T_N, T_N)$ is the notation for the price of a put option at time T_N with exercise time T_N, also the payoff. The factor D is added for the same reason as for the payoff of a future contract.

European options on Forward Freight Agreements have a payoff equal to the maximum of the difference between the strike price and the price of the Forward Freight Agreement at exercise time, and zero. More formally, the payoff of a freight call option with strike price K equals

$$C(T_N, T_N) = D\left[F(T_N, T_1, T_N) - K\right]^+ \tag{5}$$

Similarly, the payoff of a freight put option with strike price K can be written as

$$P(T_N, T_N) = D\left[K - F(T_N, T_1, T_N)\right]^+ \tag{6}$$

[17]The term arithmetic indicates the use of an arithmetic average of the freight rates, also just the sum of the freight rates during the settlement period, divided by the number of freight rates considered

Because the price of the Forward Freight Agreement at exercise time equals the average of the freight rates during the settlement period, the payoff in equation (5) is the same as the maximum of the difference between the options strike price and the average of the freight rates during the settlement period, and zero. In other words, equation (3) and (5) (and (4) and (6)) are the same. So, as stated, a freight option is both an average price Asian option on the spot freight rates as well as a European option on the future price.

Because call options gain value as their underlying asset does, as shown in figure 5, freight call options also gain value when the underlying Forward Freight Agreement does. In other words, when the price of the FFA increases, the price of the freight call option also increases. The exact dependence between these two will eventually be shown in an explicit formula in section 6.2.2. Similarly, the price of a freight put option increases, when the price of the underlying FFA decreases.

5.3 The problem

The price of an option[18] with payoff (3) is desired. This is the same as finding the price of an option with payoff (5). For $F(t, T_1, T_N)$ equation (2) must hold.

When treating the option as an Asian option, also when looking at the payoff in equation (3), no explicit formula is known for the price of the option. Therefore, this option can only be priced using Monte Carlo simulations. However, it is desirable to price this option by a method that requires less computation time, since time is a valuable quantity. Therefore, this option will be treated as a European option on the future price, so the payoff in equation (5) will be considered. Now, it is in fact possible to derive an explicit formula for the price of the option.

5.4 Approach

When deriving an analytic expression for the freight option price, an approximation for the volatility of the future prices will need to be made. Compared to Monte Carlo simulations, this analytic expression can be seen as a faster, but maybe less reliable way of pricing a freight option. Fast, due to the fact that no time-demanding simulations need to be made, since an explicit expression is available. Less reliable, since an approximation for the volatility of the future prices is made, which may not exactly equal the real volatility.
Next, the validity of this expression will be checked by treating the option as an Asian option; No explicit formula for the pricing of Asian options can be derived, so now, Monte Carlo simulations will have to be used to find the option price. However, now, no approximation for the volatility of the future price has to be made, so the prices found using these simulations will be more reliable than using the explicit formula. However, this way is in fact slower than the explicit formula, since time-demanding simulations have to be made.
The first approach is treated in section 6 and the Monte Carlo approach is discussed in section 7.

For the first method, a model for the underlying FFA prices has to be derived. After this, the technical and mathematical aspects of options will be discussed. Finally, after making an approximation for the volatility of the future prices, an explicit solution for the price of freight options will be derived.

[18] A freight call option is considered

6 An explicit pricing formula for freight options

6.1 A model for Forward Freight Agreements

As mentioned, it is desirable to treat the freight option as a European option on the futures price, for which an explicit pricing formula can be found. Doing so, a model for the underlying future (FFA) price is needed. The model for $F(t, T_1, T_N)$ depends on $S(t)$.

Consider equation (2). In Appendix A.1 it is shown that with some basic properties of expectations, equation (2) can be transformed into

$$F(t, T_1, T_N) = \frac{1}{N} \sum_{i=1}^{N} E_t^Q \left[S(T_i)\right]. \tag{7}$$

The goal is to find an analytic expression for $F(t, T_1, T_N)$. In other words, the conditional expectation in (7) has to be determined. This can be done, based on a stochastic differential equation for the spot freight rates $S(t)$.

Definition 8. *If $X(t)$ is a lognormal random variable with drift μ and volatility σ, $X(t)$ is the solution of the following stochastic differential equation:*

$$dX(t) = \mu X(t)dt + \sigma X(t)dW(t),$$

where W denotes a Wiener process.

Definition 9. *A Wiener process W is a stochastic process with the following properties:*

1. *$W(0) = 0$;*
2. *$W(t)$ does not depend upon $W(t-1)$;*
3. *The increments $W(t+dt) - W(t)$ follow a normal distribution with mean zero and variance $(t+dt) - t = dt$ for each t.*
   More formally,
   $$W(t) - W(s) \sim N(0, t-s) \quad \forall\, 0 \le s \le t.$$

By assumption 9 in section 5.1 and definition 8 the spot freight rates $S(t)$ are assumed to be a solution of the following stochastic differential equation:

$$dS(t) = \mu S(t)dt + \sigma S(t)dW^P(t). \tag{8}$$

The notation dS indicates an increment of S, also $S(t+dt) - S(t)$. Furthermore, $dt = t_{i+1} - t_i$ and $dW = W(t+dt) - W(t)$. The P in the superscript of W, indicates that this equation holds under the real-world probability measure. Because of property three of a Wiener process, we can set

$$dW \sim Y,$$

where $Y \sim N(0, dt)$, with dt the length between two time steps t_i and t_{i+1}.
Since $N(0,k) \sim \sqrt{k}N(0,1)$, we can also write

$$dW \sim \sqrt{dt}Z,$$

where $Z \sim N(0,1)$.

Equation (8) can be written as

$$dS(t) = \mu S(t)dt + \sigma S(t)\sqrt{dt}Z^P, \tag{9}$$

where μ is the *drift* of the freight rates and σ is the *volatility*. In other words, μ can be seen as the expectation of the freight rates. Z models the random fluctuations of the freight rates. If μ is positive, the freight rates on average tend to increase, while a negative μ leads towards an average decrease of the rates. Parameter σ can only be positive, because a negative volatility does not exist. The larger σ is, the bigger random movements the freight rates tend to have.

There are different ways of solving stochastic differential equation (9). The derivation is added in Appendix A.2. As given in [10, p. 54-56], the result is the following solution for equation (9)

$$S(t) = S(0)e^{(\mu-\frac{\sigma^2}{2})t+\sigma\sqrt{t}Z^P}, \tag{10}$$

where $S(0)$ is the freight rate at time 0, $t \geq 0$ and $Z \sim N(0,1)$.
Since the t in the exponent of equation (10) is obtained from $t - 0 = t$, where t is associated with $S(t)$ and 0 with $S(0)$, the desired expression for $S(T_i)$ can now be derived in the same way as (10) was in [10, p. 54-56]:

$$S(T_i) = S(t)e^{(\mu-\frac{\sigma^2}{2})(T_i-t)+\sigma\sqrt{T_i-t}Z^P}, \tag{11}$$

where $t \leq T_i$ with T_i a fixed point in the settlement period.
Now a stochastic equation for the freight rates $S(T_i)$ under the real-world probability measure is derived. But since these real probabilities are not known, a similar equation under the risk-neutral probability measure has to be found.

Theorem 2 ((Version of) Girsanov's Theorem)**.** *Let W be a Brownian motion under probability measure P. We define $W^*(t) = W(t) + \int_0^t \gamma(s)ds$ or $dW^*(t) = dW(t) + \gamma(t)dt$, where γ is the so-called market price of risk. Then W^* is a Brownian motion under the risk-neutral probability measure.*

Using *Girsanov's theorem* and substituting $dW(t) = dW^*(t) - \gamma(t)dt$ into equation (8) and (9), the following equations can be found:

$$dS(t) = \lambda S(t)dt + \sigma S(t)dW^Q(t), \tag{12}$$

or

$$dS(t) = \lambda S(t)dt + \sigma S(t)\sqrt{dt}Z^Q(t), \tag{13}$$

where $\lambda = \mu - \sigma\gamma$. Here γ is a real-valued function, which depends on time and it is called the *market price of risk*. This is the return that is desired as compensation for taking risk. It is clear that the higher the risk one is willing to take, the higher return one requires. So, when γ is large, the market requires a high return to compensate for the risk it faces.

It was already mentioned that the spot freight rates $S(t)$ are non-tradable assets, also they cannot be bought or sold. When they would have been tradable, for example when they would be storable, λ would equal the risk-free rate r. This is because with tradable assets, under the risk-neutral probability measure, the return equals the risk-free rate, based on a hedging argument[19]. But since freight rates are non-tradable assets, the assumption $\lambda = r$ cannot be used. Also, the slightly more involved equation $\lambda = \mu - \sigma\gamma$ has to be used.

Note, that setting $\gamma = 0$, which means that the market does not require a compensation for the risk it faces, makes sure the SDE for the freight rates under the risk-neutral probability measure equals the one under the real-world probability measure. However, a risk premium is required by the freight market, so equation (13) is used from here. The solution of this equation is obtained in a similar way as for equation (9).
This leads towards

$$S(T_i) = S(t)e^{(\lambda - \frac{\sigma^2}{2})(T_i - t) + \sigma\sqrt{T_i - t}Z^Q}, \tag{14}$$

where $t \leq T_i$ with T_i is a fixed time point in the settlement period.

The desired solution for $S(T_i)$ by which the expectation in equation (7) can be determined, is now obtained. Substituting (14) into (7) results in

$$F(t, T_1, T_N) = \frac{1}{N}\sum_{i=1}^{N} E_t^Q\left[S(t)e^{(\lambda - \frac{\sigma^2}{2})(T_i - t) + \sigma\sqrt{T_i - t}Z^Q}\right]. \tag{15}$$

This equation can be simplified. First, assume that the distance between two fixed time points T_i and T_{i+1} in the settlement period is the same for all i. This distance $\frac{T_N - T_1}{N-1}$ is denoted by Δ. In Appendix A.3 it is shown that equation (15) can be simplified into

$$F(t, T_1, T_N) = S(t)\frac{e^{\lambda(T_N - t)}}{N}\frac{e^{-\lambda N\Delta} - 1}{e^{-\lambda\Delta} - 1}, \tag{16}$$

as given in [1, p. 7]. An analytic equation for the future price $F(t, T_1, T_N)$ as a function of the spot freight rates $S(t)$ is now obtained. So, to know the future price at time t, the spot freight rate at time t, the drift, the volatility and the market price of risk of the spot freight rates need to be known. This model for future prices will be used to build a method for valuing freight options.

6.2 Pricing method for freight options

Now that a model for future prices is established, the valuation of a freight option can start. To be able to value an option, the parameters for the drift, the volatility and the market price of risk from the spot freight rates need to be known. These parameters are generally known by looking at the option market:
When the market price of a certain option is known, the parameters which belong to this option, such as the freight rate volatility, can be found by letting the difference between the real market option price and the option price obtained by the pricing model, equal zero. This may not always be possible, in which case the difference has to be minimized instead of setting it equal to zero. The values of the parameters should equal those values which lead to an option price equal to the market option price, when used as input in the pricing model. Basically, various values of the parameters have to be used in the pricing model, until the obtained option price equals the option price in the market. The parameters used to find this price are now the right

[19] If the return would be higher than the risk-free rate, everyone would invest in this asset, since it always yields more than when putting your money on the bank. If the return, on the other hand, would be less than the risk-free rate, nobody would invest in this asset, since putting your money on the bank always yields more than investing in this asset.

parameters for this option. When the volatility of the freight rates is obtained in such a way, it is called *implied volatility.*

Furthermore, the volatility of the futures has to be known. Unfortunately, this cannot be obtained from the option market as easily as the other parameters. Estimating this volatility can be seen as the most important ánd difficult task in freight option pricing.

6.2.1 Volatility of future prices

An expression for the volatility of the future prices is desired. This can be obtained by using the dynamics (stochastic differential equations) for the future prices and the spot freight rates. Dynamics for the spot freight rates are already provided in equation (12) or (13). Consider the following theorem.

Theorem 3 (Itô's Lemma)**.** *Let $X(t)$ be a solution of the following stochastic differential equation:*

$$dX(t) = a(X(t),t)dt + b(X(t),t)dW(t).$$

Then $f(X(t),t)$ is a solution of the stochastic differential equation below:

$$df(X(t),t) = \left(a(X(t),t)\frac{\partial f}{\partial X} + \frac{\partial f}{\partial t} + \frac{1}{2}b(X(t),t)^2\frac{\partial^2 f}{\partial X^2} \right) dt + b(X(t),t)\frac{\partial f}{\partial X}dW(t).$$

Using equation (7) and *Itô's Lemma*, as given in [1, p. 7] the following dynamics for the future prices can be found:

$$dF(t,T_1,T_N) = \sigma(t)F(t,T_1,T_N)dW^Q(t), \tag{17}$$

where

$$\sigma(t) = \begin{cases} \sigma & \text{if } t \leq T_1, \\ \sigma\frac{\frac{S(t)}{N}\sum_{i=M+1}^{N} e^{\lambda(T_i - t)}}{F(t,T_1,T_N)} & \text{if } T_M < t < T_{M+1}, M = 1,2,\ldots,N-1. \end{cases}$$

The derivation of these dynamics is provided in Appendix A.4.

From this expression, it can be seen that the future price is lognormally distributed for $t \leq T_1$, but not lognormally distributed for $T_M < t < T_{M+1}$, $M = 1,2,\ldots,N-1$. However, it is desirable to have a lognormal distribution for $F(t,T_1,T_N)$ $\forall t$, because then *Black's option pricing formula* can be used to find a closed-form solution for the option price. Therefore, a *lognormal approximation* for the future price in the settlement period is suggested. This lognormality is obtained by adjusting the volatility of the future prices in equation (17), in other words, by making an approximation for the future price volatility.
Due to this volatility approximation it is not sure whether our final option pricing formula will be accurate.

As in [1, p. 7-8] the following dynamics are suggested:

$$dF(t,T_1,T_N) = \sigma_a(t)dW^Q(t) \tag{18}$$

where

$$\sigma_a(t) = \begin{cases} \sigma & \text{if } t \leq T_1 \\ \sigma\frac{N-M}{N} & \text{if } T_M < t < T_{M+1}, M = 1,2,\ldots,N-1 \end{cases}$$

We will argue, why this volatility function is a satisfactory assumption for the volatility of the future prices. The assumption $\sigma_a(t) = \sigma$ for $t \leq T_1$ is the same as in equation (17), since in this period the volatility already leads towards lognormally distributed future prices. Therefore, we consider $T_M < t < T_{M+1}, M = 1, 2, \ldots, N-1$.
From equation (7) it is known that the future price equals the expected average of the spot freight rates during the settlement period $[T_1, T_N]$. If we are far headed in the settlement period, many spot freight rates during settlement are already known. In other words, a large part of the average spot freight rates during settlement is already known, so the future price is unlikely to change significantly after this. Therefore, the volatility of the future price will be relatively small if time to maturity is small. In terms of N and M, we can state this as:
"If $N - M$ is small, then σ_a is small as well."
Conversely, "if $N - M$ is large, then σ_a is large", because only a small part of the average spot freight rates during the settlement period is known yet, so the future price can still vary significantly.

It is now checked whether this is in accordance with the definition for $\sigma_a(t)$ with $T_M < t < T_{M+1}, M = 1, 2, \ldots, N-1$, given in (18). This is checked by letting $N - M$ approach, respectively, 0 and N. The first case describes zero time to maturity, so $t \to T_N$, while the second case considers the current time t to approach T_1. In the first case the volatility is expected to be equal to zero, since all information needed to exactly calculate the future price is known. In the second case the volatility is expected to be equal to the volatility for $t \leq T_1$.

Consider

$$\lim_{(N-M)\to 0} \sigma \frac{N-M}{N} = \lim_{Z\to 0} \sigma \frac{Z}{Z+M} = 0,$$

and

$$\lim_{(N-M)\to N} \sigma \frac{N-M}{N} = \sigma.$$

To be complete, the behaviour of σ_a when $N - M$ equals L is now checked, with L chosen arbitrarily between 0 and N, to see whether the result is smaller than $N - M \to 0$, but larger than $N - M \to N$.
Consider

$$\lim_{(N-M)\to L} \sigma \frac{N-M}{N} = \lim_{Z\to L} \sigma \frac{Z}{Z+M} = \sigma \frac{L}{L+M} < \sigma,$$

for $M > 0$, in other words $L < N$. Furthermore,

$$\lim_{(N-M)\to L} \sigma \frac{N-M}{N} = \lim_{Z\to L} \sigma \frac{Z}{Z+M} = \sigma \frac{L}{L+M} > 0,$$

for $L > 0$.

In figure 11 a graphical representation of the volatility function σ_a is shown.

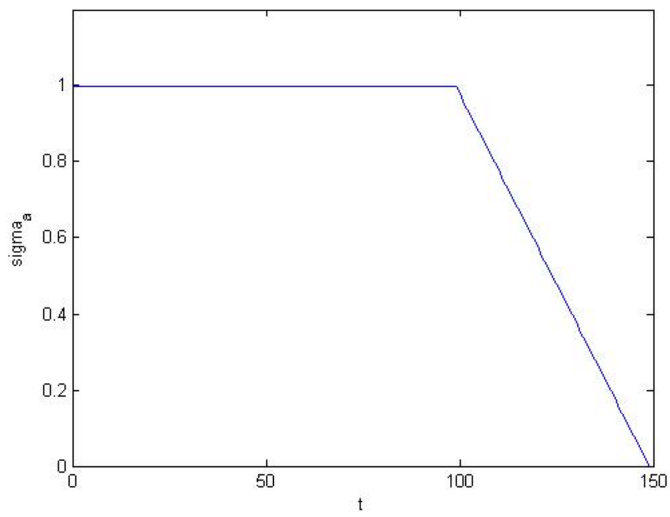

Figure 11: Volatility of the future price with $T_1 = 100$, $T_N = 150$, $\sigma = 1$ and $N = 50$

Figure 11 also shows that the fewer time left in the settlement period, the smaller the volatility, and vice versa. It has now been shown that the volatility in equation (18) satisfies the required properties, so from now on, this approximating volatility $\sigma_a(t)$ will be used.

Using this volatility, F now also follows a lognormal distribution in the settlement period.

Result 2. *If $X \sim \log N(\mu, \sigma^2)$, then $\ln(X) \sim N(\mu, \sigma^2)$ (as obtained from www.wikipedia.org).*

Result 3. *A lognormally distributed variable X has mean $\mu = \ln(E[X]) - \frac{1}{2}\sigma^2$ (as obtained from www.wikipedia.org).*

Using Results 2 and 3, it can be shown that $F(t, T_1, T_N)$ is lognormally distributed as

$$\ln(F(t, T_1, T_N)) \sim N\left(\ln(F(t, T_1, T_N)) - \frac{1}{2}\int_t^{T_N} \sigma_a(s)^2 ds, \int_t^{T_N} \sigma_a^2(s) ds\right).$$

The variance is presented by $\int_t^{T_N} \sigma_a(s)^2 ds$, since the entire volatility from t until T_N has to be considered, not just the volatility at one time point. That is because the contract price of a future at time t, $F(t, T_1, T_N)$, is determined by taking into account the entire contract period; it does not only depend on factors at time t, it depends on factors in the entire contract period. For example, if the underlying value of the future is expected to show a significant increase at time $k > t$ in the contract period, the contract price of the future will be adjusted to this. In the same way, the contract price does not only depend on its volatility at time t, but on its volatility during the entire contract period.

We define $\sigma_F^2 = \int_t^{T_N} \sigma_a^2(s) ds$. In Appendix A.5 it is shown that

$$\sigma_F^2 = \sigma^2(T_1 - t) + \sigma^2(T_N - T_1)R(N), \tag{19}$$

where σ is the volatility of the spot freight rates and $R(N) = \frac{1 - \frac{3}{2N} + \frac{1}{2N^2}}{3 - \frac{3}{N}}$.[20]

[20] In paper [1, p. 8] the formula for σ_F^2 is incorrect!

A function for the volatility of the future price σ_F is now obtained, which holds for all t. Notice, that $\sigma_F^2 \to \sigma^2(T_1 - t) + \frac{1}{3}\sigma^2(T_N - T_1)$, when $N \to \infty$, so when the future price equals the continuous expected average freight rates during settlement period, instead of working with discrete time points.

6.2.2 An explicit equation for freight option prices

With an expression for the volatility of the future price, a method of valuing a freight option can be derived. It is known that freight options can be seen as European options with a future (FFA) as underlying asset. Equation (5) showed the payoff when treating a freight call option as a European option. Because it is known from the fundamental theorem of asset pricing that the price of an asset equals its discounted expected payoff under risk-neutral probability, an equation for the price of a freight call option with strike price K at time t, $C(t, T_N)$ can now be set up, as

$$C(t, T_N) = e^{-r(T_N - t)} D E_t^Q \left[F(T_N, T_1, T_N) - K \right]^+ . \tag{20}$$

Similarly, the price of a freight put option with strike price K at time t equals

$$P(t, T_N) = e^{-r(T_N - t)} D E_t^Q \left[K - F(T_N, T_1, T_N) \right]^+ . \tag{21}$$

Based on equations (20) and (21), freight option prices can be calculated. It is, however, more desirable to have an analytic equation. Therefore, the expectations in equations (20) and (21) need to be calculated. Just as when the solution of the expectation in equation (7) was desired, now, the stochastic differential equation (18) for the future prices $F(t, T_1, T_N)$ is needed. It was shown that the future prices are approximately lognormally distributed. Therefore, *Black's formula* for option pricing can be used.

Theorem 4 (Black (1976))**.** *Let $C(F, t)$ be the price of a call option on a future, as a function of the underlying futures price F and time t. Let r be the risk-free interest rate, σ the volatility of the underlying future, T the exercise time and K the strike price.*
Then $C(F, t)$ can be stated as

$$C(F, t) = e^{-r(T-t)} \left(F\phi(d_1) - K\phi(d_2) \right)$$

where

$$d_1 = \frac{\ln\left(\frac{F}{K}\right) + \frac{1}{2}\sigma_a^2(T - t)}{\sigma_a\sqrt{T - t}}$$

and

$$d_2 = d_1 - \sigma_a\sqrt{T - t}$$

and ϕ is the cumulative distribution function of the standard normal distribution.

Using this expression, since $\sigma_a^2(T_N - t) = \int_t^{T_N} \sigma_a(s)^2 ds = \sigma_F^2$, the following expression for the price of a freight call option with strike price K and maturity time T_N can be employed:

$$C(t, T_N) = e^{-r(T_N - t)} D(F(t, T_1, T_N)\phi(d_1) - K\phi(d_2)) \tag{22}$$

where

$$d_1 = \frac{\ln\left(\frac{F(t, T_1, T_N)}{K}\right) + \frac{1}{2}\sigma_F^2}{\sigma_F}$$

and

$$d_2 = d_1 - \sigma_F.$$

Here $F(t, T_1, T_N)$ equals the future price in equation (16) and σ_F equals the future price volatility in equation (19).

Similarly, the following formula can be found for the price of a freight put option with strike price K and maturity time T_N:

$$P(t, T_N) = e^{-r(T_N - t)} D(K\phi(-d_2) - F(t, T_1, T_N)\phi(-d_1)) \tag{23}$$

6.3 Implementation explicit formula for freight option pricing

Now, using explicit formula (22), a freight call option, from which all parameters are known, will be valued. We consider a freight call option with initial spot freight rate $S_0 = \$22,500$ per day, $\lambda = 0.03$, strike price $K = \$25,000$ per day and volatility of the freight rates $\sigma = 30\%$.
As mentioned, in practice, most freight options consist of multiple settlement periods instead of just one. We consider 12 settlement periods all with a length of 21 days (this is in fact the entire length of the trading month). For all settlement periods, the corresponding option prices are calculated. These prices are called the prices of the *caplets* (*floorlets*, when considering a put option). When adding these 12 prices, the price of the corresponding *cap* (*floor*), also the price of the option consisting of 12 settlement periods, is found. Since the settlement period is considered to have a length of 21 business days, we set $T_1 = \frac{1}{252}$ and $T_N = \frac{21}{252}$, where 252 is the number of trading days in a year. If, for example, the settlement periods would consist of only the last seven trading days, $T_1 = \frac{21-7}{252} = \frac{15}{252}$ and $T_N = \frac{21}{252}$ would hold. In figures 12 and 13 a graphical representation of the division of the length of the option, so one year, into 12 settlement periods is presented.

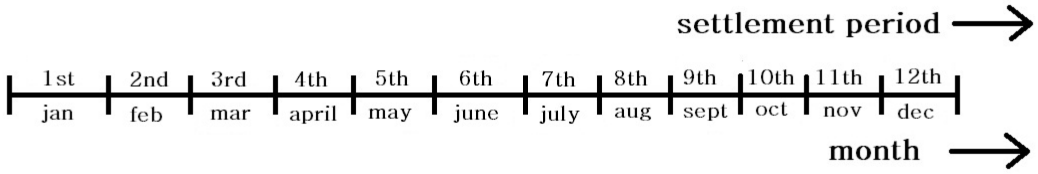

Figure 12: Graphical representation of the length of the option (one year) divided up in twelve settlement periods.

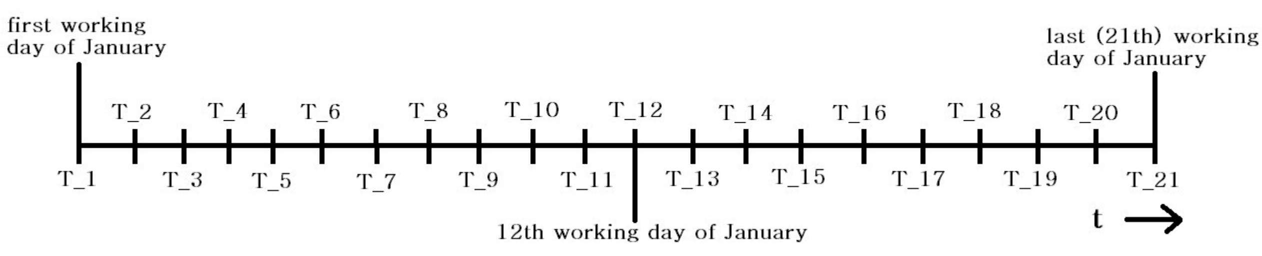

Figure 13: Graphical representation of the length of the first (January) caplet (one month, so 21 working days) divided up in 21 fixed points.

Figure 13 is zoomed in on the 'January' part of figure 12. Figure 13 is the same for every month, so for every caplet. Furthermore, since the initial spot freight rate and the strike price are expressed by a per day amount, the factor D has to express the total number of days in a month (since the caplets each last one month). Also, in January D equals 31, in February 28 (we ignore leap years), and so on.

In Appendix B.1 the Matlab code which is used to implement this into our formula is shown. Table 2 reflects, respectively, the future prices for the beginning of each month at the first of January found by using equation (16), the volatility of the future prices found by using equation (19) and the caplet prices and the cap price at the first of January found by using equation (22). So, for example, the caplet price in the 'February' row shows the price one has to pay at the first of January for the freight caplet which starts at the first of February and ends at the 28^{th} of February. The price of the cap is also the price that has to be paid the first of January for the freight cap contract which runs from the first of January until the 31^{th} of December.

Month	**FFA**	**Volatility σ_F**	**Caplet price**
Jan	\$22,529	5.2%	\$315
Febr	\$22,586	10.1%	\$5,493
March	\$22,642	13.3%	\$12,864
April	\$22,699	15.9%	\$18,648
May	\$22,756	18.1%	\$25,242
June	\$22,813	20.1%	\$29,845
July	\$22,870	21.8%	\$36,121
August	\$22,927	23.5%	\$41,136
Sept	\$22,985	25.0%	\$44,441
Oct	\$23,042	26.5%	\$50,510
Nov	\$23,100	27.9%	\$53,153
Dec	\$23,158	29.2%	\$59,185
		Cap	\$376,950

Table 2: Future prices, volatility of future prices and caplet prices for a freight call option with initial spot freight rate $S_0 = \$22,500$ per day, $\lambda = 0.03$, strike price $K = \$25,000$ per day and volatility of the freight rates $\sigma = 30\%$, see also [1, p. 22].

Note that the caplet prices in paper [1, p. 22] differ slightly from the ones we found, since the volatility function σ_F in paper [1, p. 8] is incorrect.
As shown in table 2, the volatility σ_F[21] increases with time. The increasing volatility can be explained intuitively: The internal and external factors which affect freight rates can be more accurately forecasted for the near future than for a period of time far away. For example, at the first of January it can be forecasted more accurately if there will be a hurricane in January, which will influence the freight rates, than whether there will be a sign of a hurricane in December. Therefore, the volatility for the future prices in December will be higher than that in January.

The increase in volatility can also be explained more mathematically. A short settlement period gives rise to a high volatility: a change in a freight rate during settlement has a large impact, since the settlement period consists of only few fixed freight rates. Therefore, this change has a high impact on the average freight rates during settlement. In other words also, a high impact on the future price. If the settlement period, however, is relatively long, also consists of many fixed

[21] This volatility is the volatility over the entire length of a contract, so, in contrast to the volatility in equation (18), not just at one point in time.

freight rates, a change in one of these values does not have a big impact on the average of the freight rates during settlement. Therefore, a long settlement period gives rise to a low volatility. When considering the last settlement period of the option (December), all other settlement periods have already past, so the last settlement period is relatively short compared to the past time of the option (also compared to the whole year). This leads to a relatively high volatility for the last caplet. Conversely, when considering the first settlement period (January), no other settlement periods have past yet, so the first settlement period is relatively long compared to the past time of the option (also compared to January itself). This leads to a relatively low volatility for the first caplet.

Furthermore, the caplet prices increase in time. This is both due to the increasing future prices and the increasing volatility of the future prices. First of all, high volatility means high risk, leading to high values. Secondly, in the case of a call option, a high future price means a high payoff (since the difference between the future price and the strike price will be relatively large), so also a high value.
It can be seen that after April, the option prices even exceed the future prices. This means that the average of the freight rates during settlement are expected to vary drastically; only when these average freight rates may rise above the sum of the option price and the strike price, the option will be interesting. If not, the profit of the holder of the option will always be negative. For example, the average freight rates during settlement at the end of December are expected to be able to exceed $\$59,185 + \$25,000 = \$84,185$, since otherwise no one would buy this option. Reviewing figure 2, it can be seen that these expectations, unlikely as they may seem, do match reality; freight rates in one year can in fact vary from $\$2,000$ until $\$256,000$.
However, when the future price is lower than the caplet price, it can be seen that it is more profitable to buy the future instead of the caplet; the future will always yield more profit or less loss than the caplet. Conversely, if the caplet is cheaper than the future, the caplet is preferred. So, until April it is more wisely to buy the caplets instead of the futures, while from May until December the futures are preferred.
Checking whether buying the option contract of one year (also the twelve caplets) is more profitable than buying the futures every month, cannot be done beforehand. It is known that the futures will make more profit, respectively, less loss after April, but it is not known how much exactly. Similarly, it is known that from January until April the caplets are preferred, but it is not known how much their profit is exactly compared to the futures. So, whether buying the option contract of one year is more profitable than buying the futures every month, can only be known after the contract period of the option has ended.
In reality it could be the case that the futures are not tradable (anymore). In that case, the option contract is definitely interesting.

Note that the *computational time* of the Matlab code which gives the prices in table 2, is 0.021 seconds. In the next section, it will be seen that the computational time of the Monte Carlo Matlab code is significantly longer. For this reason, it is desirable for the explicit formula to also be accurate for pricing freight options, since time is a valuable quantity.

7 Freight option pricing using Monte Carlo Simulation

In the previous section, the price of a freight call option consisting of 12 settlement periods, all containing 21 trading days, was obtained. However, this price was calculated using the explicit formula for which an approximation for the volatility of the future prices had to be made, so it is not guaranteed that this price is accurate. To check the validity of the approximating formula, we consider the option to be also the Asian option on the spot freight rates. This way, the volatility of the spot freight rates, which is known, has to be used, so no approximation for the volatility as for the future prices is made. Therefore, this way of pricing is more reliable. However, no explicit formula for the price of an Asian option is known. Therefore, *Monte Carlo simulations* have to be used to be able to price such an option.

The Monte Carlo method, although quite accurate, costs a large amount of simulations and is thus costly. It is important for a method to be both accurate and fast; Time is a valuable quantity, so a method which requires small time is preferable compared to a method which needs a lot of time. However, if this rapid method is not accurate, the method is useless. Therefore, the accuracy of our explicit formula needs to be checked.

First of all, in section 7.1, usual Monte Carlo simulations will be used. Next, in section 7.2, the variance will be reduced by using *control variates*. After this, in section 7.3 and 7.4 the variance will also be reduced by using *antithetic variates* and the *combination* of control and antithetic variates.
In all of the simulations, $5,000,000$ freight rate paths are used and
`randn('state',100)` is set to make sure the same random standard normal variables are used each time we run the Matlab code.

7.1 Monte Carlo Simulations

Consider equation (3) with the same parameters as in subsection 6.3. The price of an option with this payoff will now be calculated using Monte Carlo simulations. The basic idea is to construct many spot freight rate paths (all with initial value S_0) and for each of these paths the average value of the rates between T_1 and T_N is computed. Next, the payoff of the Asian options corresponding to each of these average freight rate values is computed. After discounting this payoff, $5 \cdot 10^6$ prices for the options, belonging to the freight rate paths which were constructed, have been found. Taking the average value of all prices, results in the desired option price.
If only one freight rate path would be used, the result would be unreliable, since the paths are constructed using *random* standard normal variables. This is known as the *Law of large numbers*: the average of the results from a large number of simulations is close to the expected value and gets closer as more simulations are made.

It is important to notice that the price found using Monte Carlo simulations, is an *estimate*, as Monte Carlo is a stochastic simulation technique. For the estimate to be as accurate as possible, many freight rate paths should be constructed, according to the Law of large numbers. The more paths, the more accurate the estimate may be. In sections 7.2 until 7.4, some other methods for reducing the standard error in the estimate are reported. A balance has to be made between the demand of high accuracy (many freight rate paths) and the demand of fast computation (few freight rate paths).

The Matlab code for the Monte Carlo simulations is added in Appendix B.2. In table 3 the prices found for the freight call option are shown, as well as the corresponding standard errors (which arise due to the use of random variables) and the relative standard errors. Also, the prices found in the previous section, using our explicit formula, are shown, together with the percentage difference between the prices found by the two methods.

Month	MC price	Std. error	Rel. std. err.	Formula price	% diff.
Jan	\$336	1.28	0.38%	\$315	-6.25%
Febr	\$5,515	8.29	0.15%	\$5,493	-0.40%
March	\$12,862	15.78	0.12%	\$12,864	0.02%
April	\$18,650	20.83	0.11%	\$18,648	-0.01%
May	\$25,242	26.62	0.11%	\$25,242	0.0%
June	\$29,817	30.40	0.10%	\$29,845	0.09%
July	\$36,178	35.92	0.10%	\$36,121	-0.16%
August	\$41,128	40.08	0.10%	\$41,136	0.02%
Sept	\$44,387	42.65	0.10%	\$44,441	0.12%
Oct	\$50,524	47.99	0.09%	\$50,510	-0.03%
Nov	\$53,196	50.14	0.09%	\$53,153	-0.08%
Dec	\$59,205	55.40	0.09%	\$59,185	-0.03%
Cap	\$377,040	55.40	0.01%	\$376,950	-8.77%

Table 3: Monte Carlo prices for a freight call option with initial spot freight rate $S_0 = \$22,500$ per day, $\lambda = 0.03$, strike price $K = \$25,000$ per day and volatility of the freight rates $\sigma = 30\%$, the standard errors, the prices found in subsection 5.5 and the percentage differences. $5 \cdot 10^6$ paths were used and 21 time steps per month were taken.

Note that the standard errors increase with time, since for longer time periods, the constructed lognormal freight rate paths can vary significantly.

The percentage differences between the prices using the two methods should be as small as possible, since then, it can be concluded that the explicit formula, is not only fast, but also accurate. It can be seen that the differences are not very high, which is a positive sign. However, since the standard errors (also the accuracy) of the prices found using Monte Carlo simulations are not as small as desired, the validity of the prices found using our explicit formula cannot yet be concluded. These standard errors should be reduced first, before making any conclusions about the percentage differences.

Note that the computational time of the Matlab code which gives the prices in table 3, is 1176 seconds, also about 20 minutes. This is, as predicted in section 6.3, much higher than the computational time of the explicit formula of only 0.021 seconds, which highlights the advantage of this formula.

7.2 Variance reduction using control variates

There are different ways to reduce the variance with Monte Carlo simulations. In this section, so-called *control variates* are used. Basically, the idea is to use estimates of the value of a known quantity and look at the corresponding errors. These errors can now be used to reduce the errors in the option prices. As a control variate, a variable which is 'close' to the required option price should be used, so a variable which has values comparable to the option values. In this case, for example, a European option or a geometric Asian option with the same parameters as the desired option, could be used.
Let X be the value of the desired option price obtained by Monte Carlo simulations and let $E[X]$ be the expectation of this price. Now, a variable Y is needed, from which we have a method to calculate its value other than using Monte Carlo, for example, by the use of some explicit formula. The value of Y calculated with this method is called $E[Y]$. Next, the value of Y is

estimated by Monte Carlo simulations and the error between the value $E[Y]$, obtained by some explicit formula, and the estimated value Y is calculated. Now define

$$Z = X + E[Y] - Y.$$

Notice, that

$$E[Z] = E[X + E[Y] - Y] = E[X] + E[Y] - E[Y] = E[X].$$

Also, instead of calculating the desired $E[X]$, $E[Z]$ can be used.
So, we obtain our desired $E[X]$ by calculating $E[Z]$ instead. In other words, we apply Monte Carlo to $Z = X + E[Y] - Y$.
We now argue why this method reduces the variance. It is known that $\text{var}[a] = 0$ if $a \in \Re$, also

$$\text{var}[Z] = \text{var}[X - Y],$$

since $E[Y] \in \Re$.
Also, to reduce variance, $\text{var}[X - Y]$ has to be smaller than $\text{var}[X]$ (this is the reason why Y should be 'close' to X). Now, look at

$$Z_\theta = X + \theta(E[Y] - Y)$$

instead of the original Z. Notice, that $E[X] = E[Z_\theta]$ still holds. Furthermore,

$$\text{var}[Z_\theta] = \text{var}[X - \theta Y] = \text{var}[X] - 2\theta\text{cov}[X, Y] + \theta^2\text{var}[Y],$$

since X and Y are not independent. This variance has to be minimized, which is done by using

$$\theta_{optimal} = \frac{\text{cov[X, Y]}}{\text{var}[Y]}.$$ [22]

7.2.1 European option as control variate

First of all, a European option with the same parameters as the desired option is used as control variate. In this case, Y is this European call option, so for $E[Y]$ the well-known *Black-Scholes* formula for a call option is used. This gives the exact value $E[Y]$ for the European call option, namely

$$C(S, t) = \phi(d_1)S - \phi(d_2)Ke^{-r(T-t)},$$

where

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + (r + \frac{\sigma^2}{2})(T - t)}{\sigma\sqrt{T - t}},$$

and

$$d_2 = d_1 - \sigma\sqrt{T - t}.$$

The estimate of this option, Y is obtained by simply constructing freight rate paths, as usual, deriving the (European call) payoff belonging to all of these paths, discounting these payoffs and, finally, taking the average of these values (Monte Carlo).

The Matlab code used to implement this method is added in Appendix B.3. In table 4 the Monte Carlo prices and the (relative) standard errors are shown, which are again compared to our explicit formula prices.

[22] This value is found by taking the derivative of $\text{var}[Z_\theta]$ and setting this equal to zero.

Month	**MC price**	**Std. error**	**Rel. std. er.**	**Formula price**	**% diff.**
Jan	\$334	1.11	0.33%	\$315	-5.69%
Febr	\$5,507	4.34	0.08%	\$5,493	-0.25%
March	\$12,873	6.62	0.05%	\$12,864	-0.07%
April	\$18,873	7.50	0.04%	\$18,648	-1.21%
May	\$25,260	8.54	0.03%	\$25,242	-0.07%
June	\$29,846	8.84	0.03%	\$29,845	-0.003%
July	\$36,131	9.63	0.03%	\$36,121	-0.03%
August	\$41,146	10.03	0.02%	\$41,136	-0.02%
Sept	\$44,445	10.04	0.02%	\$44,441	-0.01%
Oct	\$50,513	10.71	0.02%	\$50,510	-0.01%
Nov	\$53,146	10.62	0.02%	\$53,153	0.01%
Dec	\$59,172	11.21	0.02%	\$59,185	0.02%
Cap	\$377,246	11.21	0.003%	\$376,950	-0.08%

Table 4: Monte Carlo prices, using a European option as control variate, for a freight call option with initial spot freight rate $S_0 = \$22,500$ per day, $\lambda = 0.03$, strike price $K = \$25,000$ per day and volatility of the freight rates $\sigma = 30\%$, the standard errors, the prices found in subsection 5.5 and the percentage differences. $5 \cdot 10^6$ paths were used and 21 time steps per month were taken.

When comparing these (relative) standard errors to those in table 3, it can be seen that indeed the (relative) errors are smaller now. However, instead of using a European option as a a control variate, there are also other possibilities to reduce the variance.

7.3 Variance reduction using antithetic variates

Another way to reduce the variance with Monte Carlo simulations is by the use of *antithetic variates*. Basically, the idea is to reduce the variance by calculating the mean of $\frac{X+Y}{2}$ instead of X, where Y needs to have the same distribution as X. For example, if $X \sim U(0,1)$, then $Y = 1 - X$ could be used, since $1 - U(0,1) \sim U(0,1)$.
This technique reduces the variance of our estimate, because the variance of $\frac{X+Y}{2}$ is smaller than that of X. Notice, that

$$\text{var}\left(\frac{X+Y}{2}\right) = \frac{1}{4}\left(\text{var}\left(X+Y\right) + 2\text{cov}\left(X,Y\right)\right) = \frac{1}{2}\left(\text{var}(X) + \text{cov}(X,Y)\right),$$

since X and Y have the same distribution, they will also the same variance. This should be smaller than var(X). In other words,

$$\text{cov}(X,Y) < \text{var}(X),$$

or, even,

$$\text{cov}(X,Y) \leq 0.$$

The standard Monte Carlo estimate is

$$\frac{1}{M}\sum_{i=1}^{M} f(X_i).$$

In [10, p. 220-221] it is shown that $\text{cov}(X, Y) \leq 0$ whenever f is monotonic[23]. So, for a monotonic function f,

$$\text{var}\left(\frac{(f(X) + f(Y))}{2}\right) \leq \text{var}\left(f(X)\right),$$

where X and Y have the same distribution.
In our case, function f equals equation (14), which is a monotonic function (if $Z_1 \leq Z_2$, then $S(Z_1) \leq S(Z_2)$ and vice versa, when defining $S(Z_1) = S(t)e^{(\mu - \frac{\sigma^2}{2})(T-t)+\sigma\sqrt{T-t}Z_1}$). Since $Z \sim N(0,1)$, $Y = -Z$ is used, so that $Y \sim N(0,1)$. This will lead to a reduction of the variance.
The Matlab code used to implement this method is added in Appendix B.5. In table 5 the Monte Carlo prices and the (relative) standard errors are shown, which are again compared to the explicit formula prices.

Month	**MC price**	**Std. error**	**Rel. std. er.**	**Formula price**	**% diff.**
Jan	334$	0.97	0.29%	$315	-5.69%
Febr	5,504$	5.59	0.10%	$5,493	-0.20%
March	12,859$	10.39	0.08%	$12,864	0.04%
April	18,646$	13.48	0.07%	$18,648	0.01%
May	25,244$	17.07	0.07%	$25,242	-0.01%
June	29,824$	19.32	0.06%	$29,845	0.07%
July	36,129$	22.66	0.06%	$36,121	-0.02%
August	41,153$	25.19	0.06%	$41,136	-0.04%
Sept	44,409$	26.71	0.06%	$44,441	0.07%
Oct	50,514$	29.95	0.06%	$50,510	-0.01%
Nov	53,175$	31.19	0.06%	$53,153	-0.04%
Dec	59,203$	34.40	0.06%	$59,185	-0.03%
Cap	376,994$	34.40	0.01%	$376,950	-0.01%

Table 5: Monte Carlo prices, using antithetic variates, for a freight call option with initial spot freight rate $S_0 = \$22,500$ per day, $\lambda = 0.03$, strike price $K = \$25,000$ per day and volatility of the freight rates $\sigma = 30\%$, the standard errors, the prices found in subsection 5.5 and the percentage differences. $5 \cdot 10^6$ paths were used and 21 time steps per month were taken.

Indeed, the standard errors using antithetic variates turn out to be smaller then when using usual Monte Carlo. However, comparing these standard errors to those when using a European option as control variate, show that the latter method reduces the variance more then using antithetic variates.

7.4 Variance reduction using both control variates and antithetic variates

Two methods for reducing the variance with Monte Carlo simulations have been discussed so far. Combining these two methods, may lead towards even further reduction of variance.
First of all, we use antithetic variates, so we define $2M = 10 \cdot 10^6$ freight rate paths, were the first M depend on $Z \sim N(0,1)$ and the second M depend on $-Z \sim N(0,1)$. For all of these paths we compute the corresponding option prices and take the average values of the first and the $M + 1^{th}$ option price, the second and the $M + 2^{th}$, and so on. Next, we use a European call

[23] A monotonic function is a function which is decreasing, respectively increasing on its entire domain, also $\forall x \leq y \; f(x) \leq f(y)$ (monotonic increasing) or $\forall x \leq y \; f(x) \geq f(y)$ (monotonic decreasing)

option as control variate, also we estimate the values of European call options corresponding to the freight rate paths we constructed. Since we had $2M$ freight rate paths, we obtain $2M$ European call option prices. Now, again we take the average values of the first and the $M + 1^{th}$ option price, the second and the $M + 2^{th}$, and so on. We also derive the Black-Scholes value for this European call option. After calculating the optimal θ, we calculate Z which equals the Monte Carlo option values obtained with antithetic variates plus $\theta \times$(the Black-Scholes European call option value minus the Monte Carlo European call option values obtained using antithetic variates). It can be seen that now both antithetic variates and control variates have been used. The Matlab code used to implement the combination of these two methods is added in Appendix B.6. In table 6 the Monte Carlo prices and the (relative) standard errors are shown, which are again compared to the explicit formula prices.

Month	**MC price**	**Std. error**	**Rel. std. er.**	**Formula price**	**% diff.**
Jan	334$	0.78	0.23%	$315	-5.69%
Febr	5,503$	3.06	0.06%	$5,493	-0.18%
March	12,871$	4.68	0.04%	$12,864	-0.05%
April	18,648$	5.30	0.03%	$18,648	0.0%
May	25,245$	6.03	0.02%	$25,242	-0.01%
June	29,840$	6.24	0.02%	$29,845	0.02%
July	36,122$	6.80	0.02%	$36,121	-0.003%
August	41,137$	7.08	0.02%	$41,136	-0.002%
Sept	44,446$	7.10	0.02%	$44,441	-0.01%
Oct	50,513$	7.56	0.01%	$50,510	-0.006%
Nov	53,148$	7.50	0.01%	$53,153	0.009%
Dec	59,181$	7.93	0.01%	$59,185	-0.007%
Cap	376,988$	7.93	0.002%	$376,950	-0.01%

Table 6: Monte Carlo prices, using both control and antithetic variates, for a freight call option with initial spot freight rate $S_0 = \$22,500$ per day, $\lambda = 0.03$, strike price $K = \$25,000$ per day and volatility of the freight rates $\sigma = 30\%$, the standard errors, the prices found in subsection 5.5 and the percentage differences. $5 \cdot 10^6$ paths were used and 21 time steps per month were taken.

Indeed, comparing these standard errors to those in tables 4 and 5, show that these standard errors are smaller than when using only control variates, respectively antithetic variates. Now that the standard errors of the prices found using Monte Carlo are negligible, the percentage differences between the prices found by Monte Carlo simulations and by the explicit formula can be considered. The difference between the two cap values is -0.01% and the largest absolute difference between two caplet values is 5.69%. These differences are, just as the standard errors, negligible, which leads towards the conclusion that the explicit formula is valid to price freight options under this parameter set. There is no immediate reason to assume why the formula will not be accurate to price freight options with other parameter sets, but this will be discussed more thoroughly in section 8.

8 Discussion

It was shown in the previous section that the exact formula is an accurate approximation for the desired option price, since the differences between the prices found using this formula and the ones found using Monte Carlo simulations are relatively small. However, it is only shown for one particular example that the formula satisfies. It is questionable whether it works for each example. A closer look will be taken at the approximation steps made to derive the formula and it is discussed why and when these kind of approximations are allowed to be made.

In section 5.1 a few assumptions were made. This should always be kept in mind, since the accuracy of our formula is based on these assumptions. Consider assumption 1: in reality, it is not always possible to borrow and lend money at the same constant interest rate. One of the most important assumptions made, is assumption 9, which states that the spot freight rates are lognormally distributed. This is an assumption on which all derivations are based.

8.1 Geometric Brownian motion versus mean reversion

It was assumed that the spot freight rates are lognormally distributed, also that they follow a geometric Brownian motion. This assumption can be seen as the basic assumption of the model, since using this assumption, a model for the spot freight rates is created, by which then a model for the future prices is derived. In other words, if another model for the spot freight rates would have been used, another model for the future prices and therefore another formula for the option prices would have been found. It is questionable whether this assumption of lognormality is realistic in practice. Based on freight data, it may be more realistic to use another distribution for the freight rates. A more suitable assumption for the freight rates would be that of *mean-reversion*, which implies that the spot freight rates always return to some long-term mean. If spot freight rates would be modeled by mean-reversion, equation (8) could change into

$$dS(t) = \kappa(\mu - S(t))dt + \sigma S(t)dW(t),$$

as given in [3, p. 9]. Here μ is the long-term mean of the freight rates and κ is the rate at which the freight rates return to the long-term mean. Then, another analytic formula for $S(t)$ would be found instead of equation (14). Consequently, instead of equation (17) another formula for $F(t, T_1, T_N)$ would be derived. Furthermore, another future price volatility opposed to the one in equation (19) would be found. This all would result in different values found by the explicit option pricing formula in equation (22) than the ones found in section 6.3.
This risk of choosing a 'wrong' model, also a model which does not suit reality, is called *model risk*.

8.1.1 Volatility term structure

By assuming another distribution for the spot freight rates, not only another formula for $S(t)$ and $F(t, T_1, T_N)$ will be found, but also another volatility for the future prices will be derived. When the spot freight rates show mean-reversion, the volatility of the future prices will have a term structure, also it will depend on the time to maturity. This can be explained as follows:
If the spot freight rates show a decrease in value, while there is a relatively short time to maturity left, the future prices will decrease as well, since there is small time left for the spot freight rates to return to their long-term value. On the other hand, if the spot freight rates show a decrease in value, while there is still a relatively long time to maturity, the rates have time to return to their long-term mean, so the future prices will not show a large decrease. In other words, when the time to maturity is short, the volatility of the future prices is relatively large, while long time to maturity leads to low volatility values. This volatility term structure is not present when the freight rates follow a lognormal distribution, since in that case, the freight rates will not return to some long-term mean.

As given in [2, p. 7], the following volatility formula satisfies the required term structure:

$$\sigma_{m.r.}(t,T_1,T_N) = \begin{cases} (\sigma_S-\sigma_A)\left(\frac{e^{-\lambda(T_1-t)}-e^{-\lambda(T_N-t)}}{\lambda(T_N-T_1)}\right)+\sigma_A & \text{if } t\leq T_1 \\ (\sigma_S-\sigma_A)\left(\frac{1-e^{-\lambda(T_N-t)}}{\lambda(T_N-T_1)}\right)+\sigma_A\frac{T_N-t}{T_N-T_1} & \text{if } T_1<t<T_N \end{cases} \tag{24}$$

Here, σ_A is the asymptotic volatility of the spot freight rates, also the value towards which the volatility of the future prices converges when there is a lot of time to maturity left (also, when $T_1-t\to\infty$). σ_S is the short-term volatility, also the value towards which the volatility converges when there is little time to maturity left (also, when $t\to T_1$). In contrast to the volatility in equation (18), σ is now not only a function of time t, but also of the settlement period $[T_1, T_N]$, since, as explained above, the volatility now depends on the time to maturity. In the settlement period, the volatility must have the same properties as the volatility in equation (18), also, the further ahead in the settlement period we are, the lower the volatility must be, and vice versa.

The required properties of $\sigma_{m.r.}(t,T_1,T_N)$ are now checked:

- If time to maturity is large, so $T_1-t\to\infty$, $\sigma_{m.r.}=\sigma_A$ is expected.

$$\lim_{T_1-t\to\infty}\sigma_{m.r.} = \lim_{T_1-t\to\infty}(\sigma_S-\sigma_A)\left(\frac{e^{-\lambda(T_1-t)}-e^{-\lambda(T_N-t)}}{\lambda(T_N-T_1)}\right)+\sigma_A=\sigma_A$$

- If time to maturity is small, so $t\uparrow T_1$, $\sigma_{m.r.}=\sigma_S$ is expected.

$$\lim_{t\uparrow T_1}\sigma_{m.r.} = \lim_{t\uparrow T_1}(\sigma_S-\sigma_A)\left(\frac{e^{-\lambda(T_1-t)}-e^{-\lambda(T_N-t)}}{\lambda(T_N-T_1)}\right)+\sigma_A = (\sigma_S-\sigma_A)\left(\frac{1-e^{-\lambda(T_N-T_1)}}{\lambda(T_N-T_1)}\right)+\sigma_A$$

  This last term approximately equals σ_a, which can be seen by using the Taylor expansion of $e^{-\lambda(T_N-T_1)}$.

- Inside the settlement period, if $t\downarrow T_1$, $\sigma_{m.r.}=\sigma_S$ is expected.

$$\lim_{t\downarrow T_1}\sigma_{m.r.} = \lim_{t\downarrow T_1}(\sigma_S-\sigma_A)\left(\frac{1-e^{-\lambda(T_N-t)}}{\lambda(T_N-T_1)}\right)+\sigma_A\frac{T_N-t}{T_N-T_1} = (\sigma_S-\sigma_A)\left(\frac{1-e^{-\lambda(T_N-T_1)}}{\lambda(T_N-T_1)}\right)+\sigma_A$$

  Again, this last term approximately equals σ_a.

- Inside the settlement period, if $t\uparrow T_N$, $\sigma_{m.r.}=0$ is expected.

$$\lim_{t\uparrow T_N}\sigma_{m.r.} = \lim_{t\uparrow T_N}(\sigma_S-\sigma_A)\left(\frac{1-e^{-\lambda(T_N-t)}}{\lambda(T_N-T_1)}\right)+\sigma_A\frac{T_N-t}{T_N-T_1}=0$$

So, the volatility function $\sigma_{m.r.}(t,T_1,T_N)$ in equation (24) matches the requirements of the volatility of the future prices.

In figure 14, a graph of the volatility of the future prices based on mean-reverting freight rates is shown, together with the graph of the volatility based on lognormal freight rates.

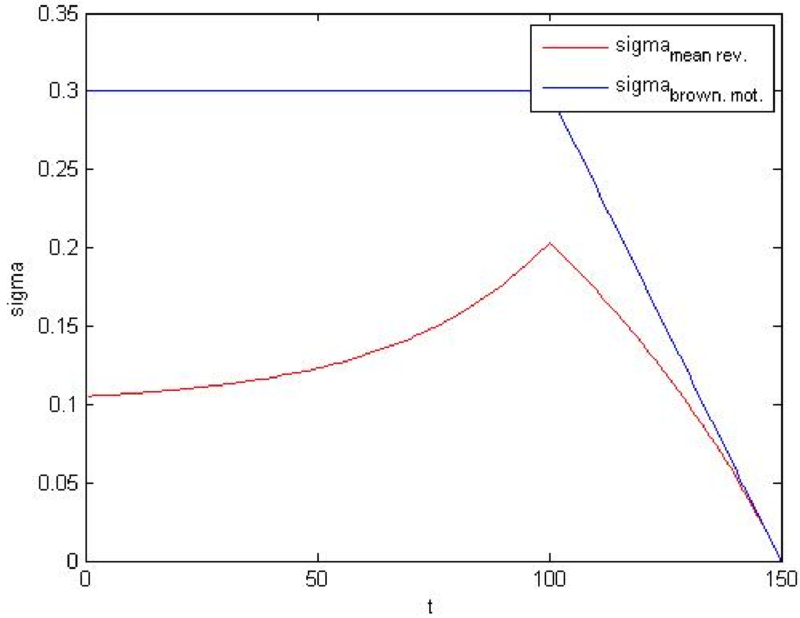

Figure 14: Volatility of the future price when the freight rates are, respectively, mean-reverting and lognormal, with $T_1 = 100$, $T_N = 150$, $N = 50$, $\sigma = 0.3$, $\sigma_a = 0.1$ and $\sigma_s = 0.3$

Indeed, outside the settlement period, the volatility based on mean-reverting freight rates always lies between σ_A and σ_S. The volatility approaches zero when $t \to T_N$ and σ_A when $T_1 - t \to \infty$.

Just as in equation (19), the volatility of the future prices from t until T_N is now required, to find an explicit formula for the freight option prices. In Appendix A.6, a formula for $\sigma^2_{F,m.r.}(t, T_1, T_N) = \int_t^{T_N} \sigma^2_{m.r.}(s, T_1, T_N) ds$ is shown in equation (A.14).

The volatility of the future prices is derived, when the spot freight rates exhibit mean-reversion. Also the formula for $F(t, T_1, T_N)$, equation (16), will change when considering mean-reverting freight rates. The derivation of this new formula is a subject for later studies.

8.2 Validity of our explicit formula

In section 7 it was shown that the prices found by using the explicit formula are quite accurate. However, this was shown only for one example. It is questionable whether this formula will still be as accurate when other values for the parameters are used. Furthermore, we argue why the approximations made to derive this formula are allowed.

8.2.1 Lognormal approximation

When deriving the explicit formula, an approximation for the volatility of future prices was used. This approximating volatility was derived by making a lognormal approximation for the future prices. The question is, was this approximation allowed to be made?
The future price can be seen as the expectation of the average of the spot freight rates, also the expectation of the sum of lognormally distributed variables. To be allowed to make the lognormal

approximation, it must therefore hold that the sum of lognormally distributed variables is again lognormally distributed. In finance it is a common thing to make a lognormal approximation for the sum of lognormal variables, which was suggested by E. Levy (see [9]). The parameters of this resulting lognormal distribution are found by so-called *Levy's lognormal moment matching.* In this method, the first two moments of the resulting lognormal distribution and the true first two moments of the average spot freight rates are set equal, out of which the mean and the variance of our approximating lognormal distribution can be found. Also, the lognormal approximation is permissible.

8.2.2 Different parameter values

When checking the steps which were taken to find the final pricing formula, it can be seen that an important assumption has been made in Appendix A.3. There, it was assumed that $e^{-\lambda\Delta} \neq 1$, after which $\sum_{i=0}^{N-1} e^{-\lambda\Delta i}$ could be rewritten as $\frac{1-e^{-\lambda\Delta N}}{1-e^{-\lambda\Delta}}$ using theorem 5. However, $\sum_{i=0}^{N-1} e^{-\lambda\Delta i}$ cannot be rewritten like this if $e^{-\lambda\Delta} = 1$, since then, the condition in theorem 5 would not hold anymore. In other words, if $\lambda = 0$ or $\Delta = 0$, $\sum_{i=0}^{N-1} e^{-\lambda\Delta i} \neq \frac{1-e^{-\lambda\Delta N}}{1-e^{-\lambda\Delta}}$. If this equality does not hold anymore, the model for $F(t, T_1, T_N)$ in equation (18) would not be valid anymore. Therefore, if either $\lambda = 0$, also $\mu = \sigma\gamma$, or $\Delta = 0$, also $\frac{T_N - T_1}{N-1} = 0 \rightarrow T_N = T_1$ or $N = \infty$, the formula in equation (22) would not produce the correct option prices anymore. This can be checked by setting, respectively, $\lambda = 0$, $T_N = T_1$ and $N = \infty$ in the Matlab code in Appendix B.1. Matlab produces NaN values in all of these cases. Also, if one of the three above cases holds, the explicit formula in equation (22) does not work anymore. In other words, the explicit formula does not work when either the mean of the spot freight rates equals the volatility of the rates times the market price of risk, when the exercise date equals the settlement date, also there is no settlement period, or when continuous time in the settlement period would be considered instead of fixed time points.

Also, precautions have to be taken when using the explicit formula, since for most parameter values it does give accurate prices, but, unfortunately, not for all values. For these few parameter values for which the formula does not work, prices have to be derived by using the Monte Carlo method. Furthermore, keep in mind that the lognormal approximation of the future prices is not completely accurate and the assumptions being made are not entirely realistic.

9 Conclusion

In this document two ways of pricing a freight option are discussed. First of all, a freight option can be valued using equation (22). To obtain this formula, spot freight rates were assumed to have a lognormal distribution. It followed that the corresponding future prices are also lognormally distributed prior to the settlement period. However, inside the settlement period they did not have a lognormal distribution. Next, an approximation for the volatility of the future prices was made, whereby the future prices were now not only lognormally distributed prior to, but also inside the settlement period. Now, Black's formula could be used to derive an analytic pricing formula for freight options, equation (22). Due to the volatility approximation made to derive equation (22), it is not sure whether the prices this formula calculates are valid. To check the validity of the formula, one particular freight option is priced using both equation (22) and Monte Carlo simulations. After reducing the variance in the Monte Carlo simulations using control and antithetic variates, the prices found using both the explicit formula and Monte Carlo simulations seemed to have insignificantly small differences. It could now be concluded that equation (22) is valid to price freight options (for almost all parameter sets). Preference is now given to this formula to price freight options instead of the Monte Carlo method, since this first one has a 1000 times smaller computational time than the Monte Carlo method. Keep in mind, however, that assumptions, like the lognormality of spot freight rates, have been made, by which the prices found using equation (22) (and the Monte Carlo method) may differ slightly from the real market option prices. According to freight data, the assumption of mean-reverting freight rates would have been a more realistic assumption than lognormal freight rates. Finding an explicit formula based on mean-reverting freight rates is a subject for later studies.

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A Appendix A

A.1 Derivation of equation (7)

Proof. Consider

$$E_t^Q\left[e^{-r(T_N-t)}D\left(\sum_{i=1}^{N}\frac{S(T_i)}{N}-F(t,T_1,T_N)\right)\right]=0 \tag{A.1}$$

Now, since constant factors can be taken outside the expectation:

$$e^{-r(T_N-t)} D E_t^Q \left[\left(\sum_{i=1}^{N} \frac{S(T_i)}{N} - F(t, T_1, T_N) \right) \right] = 0$$

Multiplying both sides by $\frac{e^{r(T_N-t)}}{D}$ gives

$$E_t^Q \left[\left(\sum_{i=1}^{N} \frac{S(T_i)}{N} - F(t, T_1, T_N) \right) \right] = 0$$

Using the fact that the expectation of a sum equals the sum of an expectation, leads towards

$$\sum_{i=1}^{N} E_t^Q \left[\left(\frac{S(T_i)}{N} - F(t, T_1, T_N) \right) \right] = 0$$

Furthermore, it is known that $E_t[X] = X$ if X depends only on the first t time steps, so $E_t[F(t, T_1, T_N)] = F(t, T_1, T_N)$. This leads to the equation

$$\sum_{i=1}^{N} E_t^Q \left[\left(\frac{S(T_i)}{N} \right) \right] = F(t, T_1, T_N)$$

Now taking out the factor $\frac{1}{N}$ gives the required equation

$$F(t, T_1, T_N) = \frac{1}{N} \sum_{i=1}^{N} E_t^Q [S(T_i)] \tag{A.2}$$

□

A.2 Proof of equation (12) and (13)

See [10, p. 54-56].

A.3 Derivation of equation (16)

Proof. Consider

$$F(t, T_1, T_N) = \frac{S(t)}{N} \sum_{i=1}^{N} E_t^Q \left[e^{(\lambda - \frac{\sigma^2}{2})(T_i - t) + \sigma \sqrt{T_i - t} Z^Q} \right] \tag{A.3}$$

$S(t)$ can be taken out of the expectation, due to $E_t[X_t] = X_t$, when X only depends on the first t time steps. The argument in equation (A.3) inside the expectation is a lognormally distributed variable (check assumption 9). Furthermore, it is known that the expectation of a lognormally distributed variable, when the corresponding normal variable has mean μ and variance σ, equals $e^{\mu + \frac{1}{2}\sigma^2}$. Since our normal distribution has mean $(\lambda - \frac{1}{2}\sigma^2)(T_i - t)$ and variance $\sigma^2(T_i - t)$

$$F(t, T_1, T_N) = \frac{S(t)}{N} \sum_{i=1}^{N} e^{(\lambda - \frac{1}{2}\sigma^2)(T_i - t) + \frac{1}{2}\sigma^2(T_i - t)} \tag{A.4}$$

also

$$F(t, T_1, T_N) = \frac{S(t)}{N} \sum_{i=1}^{N} e^{\lambda(T_i - t)}. \tag{A.5}$$

Equation (A.5) can be transformed into

$$\frac{S(t)}{N} e^{\lambda(T_N - t)} \left(1 + \sum_{i=1}^{N-1} e^{\lambda(T_i - t)} \right)$$

Now, use the assumption of equidistant observations $\Delta = \frac{T_N - T_1}{N-1}$. This leads towards

$$\lambda(T_1 - T_N) = -\lambda(T_N - T_1) = -\lambda\Delta(N-1).$$

Similarly

$$\lambda(T_2 - T_N) = -\lambda(T_N - T_2) = -\lambda\Delta(N-2),$$

$$\lambda(T_3 - T_N) = -\lambda(T_N - T_3) = -\lambda\Delta(N-3),$$

and so on. So, in other words,

$$\lambda(T_i - T_N) = -\lambda(T_N - T_i) = -\lambda\Delta(N-i).$$

Since $e^{-\lambda\Delta \times 0} = 1$, the previous equation can be transformed into

$$\frac{S(t)}{N} e^{\lambda(T_N - t)} \left(\sum_{i=0}^{N-1} e^{-\lambda\Delta i} \right) \tag{A.6}$$

Consider the following theorem.

Theorem 5. *For $r \neq 1$, the sum of the first N terms of a geometric series is*

$$\sum_{i=0}^{N-1} ar^i = a\frac{1 - r^N}{1 - r}$$

Since $\sum_{i=0}^{N-1} e^{-\lambda\Delta i}$ is a geometric series with $r = e^{-\lambda\Delta} \neq 1$ and $a = 1$, it can be rewritten as $\frac{1-e^{-\lambda\Delta N}}{1-e^{-\lambda\Delta}}$.
So, our final expression for the future price is

$$F(t, T_1, T_N) = S(t)\frac{e^{\lambda(T_N - t)}}{N} \frac{1 - e^{-\lambda\Delta N}}{1 - e^{-\lambda\Delta}} \tag{A.7}$$

□

A.4 Derivation of equation (17)

Proof. Consider equation (7). From (A.5) it is known that when $t < T_1$,

$$F(t, T_1, T_N) = \frac{S(t)}{N} \sum_{i=1}^{N} e^{\lambda(t_i - t)}$$

Assume now that the first $M < N$ time points of the settlement period have already passed. In other words, $T_M < t < T_{M+1}$ for $M = 1, 2, \ldots, N-1$. Again from (A.5) it can then be concluded that

$$F(t, T_1, T_N) = \frac{1}{N} \sum_{i=1}^{M} S(T_i) + \frac{S(t)}{N} \sum_{i=M+1}^{N} e^{\lambda(t_i - t)}$$

since $E_t[X] = X$ if X only depends on the first t time points.

Now, refer to *Itô's Lemma* stated in subsection 6.2.1. In our case, $X(t) = S(t)$, $a(X,t) = \lambda S(t)$ and $b(X,t) = \sigma S(t)$. Since $F(t, T_1, T_N)$ is a function of $S(t)$ and t, Itô's lemma can be applied now:

$$dF(S(t),t) = \left(\lambda S(t)\frac{\partial F}{\partial S} + \frac{\partial F}{\partial t} + \frac{1}{2}(\sigma S(t))^2\frac{\partial^2 F}{\partial S^2}\right)dt + \sigma S(t)\frac{\partial F}{\partial S}dW(t) \tag{A.8}$$

First consider the case where $t < T_1$, so where

$$F(t, T_1, T_N) = \frac{S(t)}{N}\sum_{i=1}^{N} e^{\lambda(t_i - t)}$$

Since then

$$\frac{\partial F}{\partial S} = \frac{1}{N}\sum_{i=1}^{N} e^{\lambda(t_i-t)}, \frac{\partial F}{\partial t} = -\frac{\lambda S(t)}{N}\sum_{i=1}^{N} e^{\lambda(t_i-t)}, \frac{\partial^2 F}{\partial S^2} = 0,$$

it can be seen that

$$dF(t, T_1, T_N) = \left(\lambda S(t)\frac{1}{N}\sum_{i=1}^{N} e^{\lambda(t_i-t)} - \frac{\lambda S(t)}{N}\sum_{i=1}^{N} e^{\lambda(t_i-t)} + \frac{1}{2}(\sigma S(t))^2 \times 0\right)dt$$
$$+\sigma S(t)\frac{1}{N}\sum_{i=1}^{N} e^{\lambda(t_i-t)}dW(t)$$

This equals

$$dF(t, T_1, T_N) = \sigma S(t)\frac{1}{N}\sum_{i=1}^{N} e^{\lambda(t_i-t)}dW(t) = \sigma F(t, T_1, T_N)dW(t) \tag{A.9}$$

Now, consider the case where $T_M < t < T_{M+1}$ for $M = 1, 2, \ldots, N-1$, so where

$$F(t, T_1, T_N) = \frac{1}{N}\sum_{i=1}^{M} S(T_i) + \frac{S(t)}{N}\sum_{i=M+1}^{N} e^{\lambda(t_i-t)}$$

Since then

$$\frac{\partial F}{\partial S} = \frac{1}{N}\sum_{i=M+1}^{N} e^{\lambda(t_i-t)}, \frac{\partial F}{\partial t} = -\frac{\lambda S(t)}{N}\sum_{i=M+1}^{N} e^{\lambda(t_i-t)}, \frac{\partial^2 F}{\partial S^2} = 0,$$

it can be seen that

$$dF(t, T_1, T_N) = \left(\lambda S(t)\frac{1}{N}\sum_{i=M+1}^{N} e^{\lambda(t_i-t)} - \frac{\lambda S(t)}{N}\sum_{i=M+1}^{N} e^{\lambda(t_i-t)} + \frac{1}{2}(\sigma S(t))^2 \times 0\right)dt$$
$$+\sigma S(t)\frac{1}{N}\sum_{i=M+1}^{N} e^{\lambda(t_i-t)}dW(t)$$

This equals

$$dF(t, T_1, T_N) = \sigma S(t)\frac{1}{N}\sum_{i=M+1}^{N} e^{\lambda(t_i-t)}dW(t) \tag{A.10}$$

Summing all these results, leads towards

$$dF(t,T_1,T_N) = \sigma(t)F(t,T_1,T_N)dW_t^Q \tag{A.11}$$

where

$$\sigma(t) = \begin{cases} \sigma & \text{if } t \leq T_1 \\ \sigma \frac{\frac{S(t)}{N}\sum_{i=M+1}^{N} e^{\lambda(T_i - t)}}{F(t,T_1,T_N)} & \text{if } T_M < t < T_{M+1}, M = 1,2,\ldots,N-1 \end{cases}$$

□

A.5 Derivation of equation (19)

Proof. Consider

$$\sigma_F^2 = \int_t^{T_N} \sigma_a^2(s)ds =$$

$$\int_t^{T_1} \sigma^2 ds + \int_{T_1}^{T_2} \sigma^2 \left(\frac{N-1}{N}\right)^2 ds + \int_{T_2}^{T_3} \sigma^2 \left(\frac{N-2}{N}\right)^2 ds + \ldots$$

$$+ \int_{T_M}^{T_{M+1}} \sigma^2 \left(\frac{N-M}{N}\right)^2 ds + \ldots + \int_{T_{N-1}}^{T_N} \sigma^2 \left(\frac{1}{N}\right)^2 ds$$

This equals

$$\sigma^2(T_1 - t) + \sigma^2(T_2 - T_1)\frac{(N-1)^2}{N^2} + \sigma^2(T_3 - T_2)\frac{(N-2)^2}{N^2} + \ldots$$

$$+\sigma^2(T_{M+1} - T_M)\frac{(N-M)^2}{N^2} + \ldots + \sigma^2(T_N - T_{N-1})\frac{1}{N^2}$$

This can be simplified into

$$\sigma_F^2 = \sigma^2(T_1 - t) + \sigma^2\Delta\left(\frac{(N-1)^2}{N^2} + \frac{(N-2)^2}{N^2} + \ldots + \frac{(N-M)^2}{N^2} + \ldots + \frac{1}{N^2}\right)$$

It can be shown that the term between brackets equals $\frac{1}{3}N - \frac{1}{2} + \frac{1}{6N}$. In other words,

$$\sigma_F^2 = \sigma^2(T_1 - t) + \sigma^2\Delta(\frac{1}{3}N - \frac{1}{2} + \frac{1}{6N})$$

Since $\Delta = \frac{T_N - T_1}{N-1}$, this can be rewritten as

$$\sigma_F^2 = \sigma^2(T_1 - t) + \sigma^2(T_N - T_1)\frac{\frac{1}{3}N - \frac{1}{2} + \frac{1}{6N}}{N-1}$$

This equals

$$\sigma_F^2 = \sigma^2(T_1 - t) + \sigma^2(T_N - T_1)R(N) \tag{A.12}$$

where $R(N) = \frac{1 - \frac{3}{2N} + \frac{1}{2N^2}}{3 - \frac{3}{N}}$. □

A.6 Derivation of formula for $\sigma^2_{F,m.r.}(t,T_1,T_N)$

$$\sigma^2_{F,m.r.}(t,T_1,T_N) = \int_t^{T_N} \sigma^2_{m.r.}(s,T_1,T_N)ds \tag{A.13}$$

This equals

$$\int_t^{T_1}\left((\sigma_S-\sigma_A)\left(\frac{e^{-\lambda(T_1-t)}-e^{-\lambda(T_N-t)}}{\lambda(T_N-T_1)}\right)+\sigma_A\right)^2 ds+$$

$$\int_{T_1}^{T_N}\left((\sigma_S-\sigma_A)\left(\frac{1-e^{-\lambda(T_N-t)}}{\lambda(T_N-T_1)}\right)+\sigma_A\frac{T_N-t}{T_N-T_1}\right)^2 ds=$$

$$\int_t^{T_1}(\sigma_S-\sigma_A)^2\left(\frac{e^{-\lambda(T_1-t)}-e^{-\lambda(T_N-t)}}{\lambda(T_N-T_1)}\right)^2+2\sigma_A(\sigma_S-\sigma_A)\frac{e^{-\lambda(T_1-t)}-e^{-\lambda(T_N-t)}}{\lambda(T_N-T_1)}+\sigma_A^2 ds+$$

$$\int_{T_1}^{T_N}(\sigma_S-\sigma_A)^2\left(\frac{1-e^{-\lambda(T_N-t)}}{\lambda(T_N-T_1)}\right)^2+2\sigma_A(\sigma_S-\sigma_A)\left(\frac{T_N-t}{T_N-T_1}\right)\left(\frac{1-e^{-\lambda(T_N-t)}}{\lambda(T_N-T_1)}\right)+\sigma_A^2\left(\frac{T_N-t}{T_N-T_1}\right)^2 ds$$

Using some basic integration techniques, it can be shown that this equals

$$\frac{1}{2}(\sigma_s-\sigma_A)^2(e^{\lambda T_N}-e^{\lambda T_1})^2(1-e^{-2\lambda(T_1-t)})\frac{e^{-2\lambda T_N}}{\lambda^3(T_N-T_1)^2}+$$

$$2(\sigma_S-\sigma_A)\sigma_A\frac{1-e^{-\lambda(T_N-T_1)}-e^{-\lambda(T_1-t)}+e^{-\lambda(T_N-t)}}{\lambda^2(T_N-T_1)}+\sigma_A^2(T_1-t)+$$

$$\frac{(\sigma_S-\sigma_A)^2}{2\lambda^3(T_N-T_1)^2}\left(1-(2-e^{-\lambda(T_N-T_1)})^2+2\lambda(T_N-T_1)\right)+$$

$$\frac{(\sigma_S-\sigma_A)\sigma_A}{\lambda^3(T_N-T_1)^2}(2(e^{-\lambda(T_N-T_1)}-1)+$$

$$\lambda^2(T_N-T_1)^2+2\lambda e^{-\lambda(T_N-T_1)}(T_N-T_1))+\frac{1}{3}\sigma_A^2(T_N-T_1) \tag{A.14}$$

B Appendix B (Matlab code)

B.1 Explicit formula

```
%VALUE FREIGHT CALL OPTION USING BLACK'S FORMULA
%PARAMETERS:
S=22500; %Initial spot freight rate
E=25000; %Strike price
lambda=0.03; %See thesis
sigma=0.3; %Volatility
N=21; %number of timesteps in settlement period
D=[31,28,31,30,31,30,31,31,30,31,30,31]; %See thesis

for j= 1:12 %There are 12 settlement periods

    T_1(j)=1/252 + N/252*(j-1); %Start of each settlement period
    T_N(j)=N/252 + N/252*(j-1); %End of each settlement period
```

```
    Dt(j)=(T_N(j)-T_1(j))/(N-1); %Length timestep of each settl. period
    R=(1-3/(2*N)+1/(2*N^2))/(3-3/N); %See thesis

    sigmaF(j)=sigma^2*T_1(j)+sigma^2*(T_N(j)-T_1(j))*R; %Volatility of
    %future prices

    F(j)=S*((exp(lambda*(T_N(j)-0))/N)*((exp(-lambda*N*Dt(j))-1)/...
        (exp(-lambda*Dt(j))-1))); %Future prices at t=0

    d1(j)=(log(F(j)/E)+0.5*sigmaF(j))/(sqrt(sigmaF(j)));
    d2(j)=d1(j)-sqrt(sigmaF(j));
    N1(j)=normcdf(d1(j));
    N2(j)=normcdf(d2(j));
    C(j)=exp(-lambda*T_N(j))*D(j)*(F(j)*N1(j)-E*N2(j)); %Black's formula

end

C %Value caplets using Black's formula
```

B.2 Monte Carlo

```
%MONTE CARLO FOR VALUE OF A CAP WITH 12 SETTLEMENT PERIODS OF 21 DAYS

%PARAMETERS:
randn('state',100)
S=22500; %Initial spot freight rate
E=25000; %Strike price
lambda=0.03; %See thesis
sigma=0.3; %Volatility
M=5e6; %number of paths
N=21; %number of timesteps in settlement period
D=[31,28,31,30,31,30,31,31,30,31,30,31]; %See thesis

T_1=zeros(12,1);
T_N=zeros(12,1);
Dt=zeros(12,1);
schattingV=zeros(12,1);
bM=zeros(12,1);
standarderror=zeros(12,1);

for j= 1:12 %There are 12 settlement periods

    T_1(j)=1/252 + N/252*(j-1); %Start of each settlement period
    T_N(j)=N/252 + N/252*(j-1); %End of each settlement period
    Dt(j)=(T_N(j)-T_1(j))/(N-1); %Length timestep for each settl. period

    %MONTE CARLO:
    V = zeros(M,1);
    for i = 1:M
```

```
        samples = randn(N+(j-1)*N,1); %Random standard normal variables
        Svals = S*cumprod(exp((lambda-0.5*sigma^2)*Dt(j)+sigma*sqrt(Dt(j))*...
                samples)); %Freight rate paths
        Smean=1/N*sum(Svals(round(T_1(j)/Dt(j)):round(T_N(j)/Dt(j))));
        %Average freight rates during settlement
        V(i) = exp(-lambda*T_N(j))*D(j)*max(Smean-E,0); %Value caplets
        %at t=0

    end

    %VALUES:
    schattingV(j) = mean(V) %Estimate caplets at t=0
    bM(j)=std(V); %Standarddeviation caplet prices
    standarderror(j)=bM(j)/sqrt(M); %Standarderror caplets
    cap=sum(schattingV); %Value cap at t=0

end
```

B.3 Monte Carlo using European option as control variate

```
%MONTE CARLO WITH EUROPEAN OPTION AS CONTROL VARIATE FOR VALUE OF A CAP
%WITH 12 SETTLEMENT PERIODS OF 21 DAYS

%PARAMETERS:
randn('state',100)
S=22500; %Initial spot freight rate
E=25000; %Strike price
lambda=0.03; %See thesis
sigma=0.3; %Volatility
M=5e6; %number of paths
N=21; %number of timesteps in settlement period
D=[31,28,31,30,31,30,31,31,30,31,30,31]; %See thesis

T_1=zeros(12,1);
T_N=zeros(12,1);
Dt=zeros(12,1);
schattingV=zeros(12,1);
bM=zeros(12,1);
standarderror=zeros(12,1);
Creal=zeros(12,1);
Zmean=zeros(12,1);
Zstd=zeros(12,1);
standarderrorZ=zeros(12,1);

for j= 1:12 %There are 12 settlement periods

    T_1(j)=1/252 + 21/252*(j-1); %Start of each settlement period
    T_N(j)=21/252 + 21/252*(j-1); %End of each settlement period
```

```
    Dt(j)=(T_N(j)-T_1(j))/(N-1); %Length timestep of each settl. period

    %Black scholes value European call option:
    Creal(j)=ch08(S,E,lambda,sigma,T_N(j))*D(j); %the function ch08 is
    %obtained from paper [10] from Higham.

    V = zeros(M,1);
    Cmc=zeros(M,1);

    for i = 1:M

        %NORMAL MONTE CARLO:
        samples = randn(21+(j-1)*21,1); %Random standard normal variables
        Svals = S*cumprod(exp((lambda-0.5*sigma^2)*Dt(j)+sigma*...
                sqrt(Dt(j))*samples)); %Freight rate paths
        Smean=1/N*sum(Svals(round(T_1(j)/Dt(j)):round(T_N(j)/Dt(j))));
        %Average freight rates during settlement
        V(i) = exp(-lambda*T_N(j))*D(j)*max(Smean-E,0); %Value caplets
        %at t=0

        %MONTE CARLO USING CONROL VARIATES:
        Cmc(i) = exp(-lambda*T_N(j))*D(j)*...
                 max(Svals(round(T_N(j)/Dt(j)))-E,0); %Estimate European
                 %call option

    end

    Theta=cov(V,Cmc);
    theta=Theta(1,2)/Theta(2,2); %Optimal value of theta

     for i=1:M

         Z(i) = V(i) + theta*(Creal(j) - Cmc(i)); %Construction of Z

     end

    %VALUES USING NORMAL MONTE CARLO:
    schattingV(j) = mean(V); %Values caplets at t=0
    bM(j)=std(V); %Standarddeviation caplets
    standarderror(j)=bM(j)/sqrt(M); %Standard errors caplets
    cap=sum(schattingV); %Value cap at t=0

    %VALUES USING CONTROL VARIATES:
    Zmean(j) = mean(Z); %Values caplets at t=0
    Zstd(j) = std(Z); %Standarddeviation caplets
    standarderrorZ(j)=Zstd(j)/sqrt(M); %Standard errors caplets
    capcv=sum(Zmean); %Value cap at t=0

end
```

B.4 Monte Carlo using antithetic variates

```
%MONTE CARLO WITH ANTITHETIC VARIATES FOR VALUE OF A CAP WITH
%12 SETTLEMENT PERIODS OF 21 DAYS

%PARAMETERS:
randn('state',100)
S=22500; %Initial spot freight rate
E=25000; %Strike price
lambda=0.03; %See thesis
sigma=0.3; %Volatility
M=5e6; %number of paths
N=21; %number of timesteps in settlement period
D=[31,28,31,30,31,30,31,31,30,31,30,31]; %See thesis

T_1=zeros(12,1);
T_N=zeros(12,1);
Dt=zeros(12,1);
schattingV=zeros(12,1);
bM=zeros(12,1);
standarderror=zeros(12,1);

for j= 1:12 %There are 12 settlement periods

    T_1(j)=1/252 + 21/252*(j-1); %Start of each settlement period
    T_N(j)=21/252 + 21/252*(j-1); %End of each settlement period
    Dt(j)=(T_N(j)-T_1(j))/(N-1); %Length timestep for each settl. period

    %MONTE CARLO USING ANTITHETIC VARIATES:
    V = zeros(M,1);
    V2=zeros(M,1);

    for i = 1:M

        samples = randn(N+(j-1)*N,1); %Random standard normal variables
        Svals = S*cumprod(exp((lambda-0.5*sigma^2)*Dt(j)+sigma*sqrt(Dt(j))...
                *samples)); %Freight rate paths 1
        Svals2 = S*cumprod(exp((lambda-0.5*sigma^2)*Dt(j)-sigma*sqrt(Dt(j))...
                *samples)); %Freight rate paths 2, now using -samples
        Smean=1/N*sum(Svals(round(T_1(j)/Dt(j)):round(T_N(j)/Dt(j))));
        %Average freight rates 1 during settlement
        Smean2= 1/N*sum(Svals2(round(T_1(j)/Dt(j)):round(T_N(j)/Dt(j))));
        %Average freight rates 2 during settlement
        V(i) = exp(-lambda*T_N(j))*D(j)*max(Smean-E,0); %Values caplets at
        %t=0 for the first freight rate paths
        V2(i)=exp(-lambda*T_N(j))*D(j)*max(Smean2-E,0); %Values caplets at
        %t=0 for the second freight rate paths
        Veind(i)=0.5*(V(i)+V2(i)); %Average values caplets at t=0
```

```
    end

    %VALUES:
    schattingV(j) = mean(Veind) %Values caplets at t=0
    bM(j)=std(Veind); %Standarddeviation caplets
    standarderror(j)=bM(j)/sqrt(M) %Standard errors caplets
    cap=sum(schattingV) %Value cap

end
```

B.5 Monte Carli using the combination of control and antithetic variates

```
%MONTE CARLO WITH CONTROL AND ANTITHETIC VARIATES FOR VALUE OF A CAP
%WITH 12 SETTLEMENT PERIODS OF 21 DAYS

%PARAMETERS:
randn('state',100)
S=22500; %Initial spot freight rate
E=25000; %Strike price
lambda=0.03; %See thesis
sigma=0.3; %Volatility
M=5e6; %number of paths
N=21; %number of timesteps in settlement period
D=[31,28,31,30,31,30,31,31,30,31,30,31]; %See thesis

T_1=zeros(12,1);
T_N=zeros(12,1);
Dt=zeros(12,1);
schattingV=zeros(12,1);
bM=zeros(12,1);
standarderror=zeros(12,1);
Creal=zeros(12,1);
Zmean=zeros(12,1);
Zstd=zeros(12,1);
standarderrorZ=zeros(12,1);

for j= 1:12 %There are 12 settlement periods

    T_1(j)=1/252 + 21/252*(j-1); %Start of each settlement period
    T_N(j)=21/252 + 21/252*(j-1); %End of each settlement period
    Dt(j)=(T_N(j)-T_1(j))/(N-1); %Length timestep of each settl. period

    %Black scholes value European call option:
    Creal(j)=ch08(S,E,lambda,sigma,T_N(j))*D(j);

    V = zeros(M,1);
    Cmc=zeros(M,1);
    V2=zeros(M,1);
    Veind=zeros(M,1);
```

```
Cmc2=zeros(M,1);
Cmceind=zeros(M,1);

for i = 1:M

    %MONTE CARLO USING ANTITHETIC VARIATES:
    samples = randn(21+(j-1)*21,1); %Random standard normal variables
    Svals = S*cumprod(exp((lambda-0.5*sigma^2)*Dt(j)+sigma*sqrt(Dt(j))*...
            samples)); %Freight rate paths 1
    Svals2 = S*cumprod(exp((lambda-0.5*sigma^2)*Dt(j)-sigma*sqrt(Dt(j))*...
            samples)); %Freight rate paths 2
    Smean=1/N*sum(Svals(round(T_1(j)/Dt(j)):round(T_N(j)/Dt(j))));
    %Average freight rates 1 during settlement
    Smean2= 1/N*sum(Svals2(round(T_1(j)/Dt(j)):round(T_N(j)/Dt(j))));
    %Average freight rates 2 during settlement
    V(i) = exp(-lambda*T_N(j))*D(j)*max(Smean-E,0); %Value caplets
    %at t=0 for the first freight rate paths
    V2(i)=exp(-lambda*T_N(j))*D(j)*max(Smean2-E,0); %Value caplets
    %at t=0 for the second freight rate paths
    Veind(i)=0.5*(V(i)+V2(i)); %Average values caplets at t=0

    %CONTROL & ANTITTHETIC VARIATES:
    Cmc(i) = exp(-lambda*T_N(j))*D(j)*...
             max(Svals(round(T_N(j)/Dt(j)))-E,0); %Estimate European
             %call options at t=0 for the first freight rate paths
    Cmc2(i) = exp(-lambda*T_N(j))*D(j)*...
              max(Svals2(round(T_N(j)/Dt(j)))-E,0); %Estimate European
              %call options at t=0 for the second freight rate paths
    Cmceind(i)=0.5*(Cmc(i)+Cmc2(i)); %Average values options at t=0

end

Theta=cov(Veind,Cmceind);
theta=Theta(1,2)/Theta(2,2); %Optimal value of theta

 for i=1:M

     Z(i) = Veind(i) + theta*(Creal(j) - Cmceind(i)); %Construction of Z

 end

%VALUES USING ANTITHETIC VARIATES:
schattingV(j) = mean(Veind); %Values caplets at t=0
bM(j)=std(V); %Standarddeviation caplets
standarderror(j)=bM(j)/sqrt(M); %Standard errors caplets
cap=sum(schattingV) %Value cap at t=0

%VALUES USING ANTITHETIC & CONTROL VARIATES:
Zmean(j) = mean(Z) %Values caplets at t=0
```

```
    Zstd(j) = std(Z); %Standarddeviation caplets
    standarderrorZ(j)=Zstd(j)/sqrt(M) %Standard errors caplets
    capcv=sum(Zmean) %Value cap at t=0

end
```