Bounds on Mixed-Weight Equilibria from Cooperative Systems

by

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to obtain the degree of Bachelor of Science
at the
Delft University of Technology

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Abstract

The analysis of equilibria of complex systems is challenging due to the high dimensionality and non-linear interactions. There are different kinds of complex systems such as cooperative systems in which all interactions help growth and mixed-weight systems, in which some interactions help growth, while others hinder it. In this thesis, we focus on the correlation between the equilibria of cooperative and mixed-weight systems. We simplified the equilibrium equations by combining the attributes of a system into a one-dimensional equation. This reduced equation is easy to compute and provides an upper bound on the equilibrium value of each node. Although this bound may exceed many actual equilibrium values, it still defines the subspace in which all equilibria must lie.

For cooperative systems, we presented a theorem that provides constraints on two vectors. If these vectors satisfy the given conditions, then there exists an equilibrium between the components. We also discussed methods to find such vector pairs.

We applied this theorem to relate the equilibria of mixed-weight and cooperative systems. The equilibria of the mixed-weight system are always less than or equal to some equilibrium in the cooperative system.

We introduced a framework for classifying cooperative equilibria. On any subset of nodes, an equilibrium may have entries that are maximal compared to all other equilibria on that subset. This leads to a single equilibrium that is the largest at every entry, called the principal equilibrium, which is component-wise maximal. The principal equilibrium upper bounds all equilibria of the mixed-weight system.

Finally, we discussed the inherent difficulty of translating cooperative equilibria into the mixed-weight system, which stems from high dimensionality and non-linearity. We stated the conditions that mixed-weight equilibria must satisfy and provided constraints determining if a cooperative-system equilibrium remains valid when competitive interactions are added.

This concludes the comparison by showing that the principal equilibrium provides a component-wise upper bound for all equilibria of the mixed-weight system.

Contents

1 Introduction

When I was a boy, I wondered if it might be possible to write a program that simulates the entire universe, down to every particle. Of course, this would require infinite memory. Instead, we use abstractions and mathematics to capture essential behaviour. There is a vast range of such behaviours in the universe, of which only some can be described by the so-called "complex systems". Many of these complex systems, with N nodes, can be written in the following form:

$$\frac{dx_i}{dt} = F(x_i) + \sum_{j=1}^{N} w_{ij} G(x_i, x_j),$$

these are also complex systems, which we will consider in this thesis. In the above,

- x_i is the quantity of node i [1].
- $F(x_i)$ is the intrinsic (self-induced) contribution.
- $\sum_{j=1}^{N} w_{ij} G(x_i, x_j)$ is the contribution of the interactions.

One might wonder if such complex systems are in abundance around us. They can be found everywhere:

- Lotka-Volterra systems to describe predator-prey dynamics, in which x_i is the population of the species i and w_{ij} is the interaction with other populations. Generally $w_{ij} > 0$ is commensalism (beneficial) and $w_{ij} < 0$ predation (disadvantageous) [1].
- Wilson-Cowan system for neural populations. The value of x_i is the firing rate activity of the neurone associated with node i, where $w_{ij} > 0$ represents an excitatory neurone and $w_{ij} < 0$ represents an inhibitory neurone [1].
- SIS systems for epidemic spread. The nodes represent different hosts, while the x_i in this model has a probabilistic interpretation; its value corresponds to the probability of being infected. In general, $w_{ij} > 0$, and the magnitude of w_{ij} indicates the contagiousness of other hosts [2].

We can even further categorise the systems by distinguishing between cooperative and competitive systems. If $w_{ij} \partial G(x_i, x_j)/\partial x_j \geq 0$, the interaction is cooperative, while if $w_{ij} \partial G(x_i, x_j)/\partial x_j \leq 0$ it is competitive [1]. Generally, we take $\partial G(x_i, x_j)/\partial x_j \geq 0$ and choose the weights w_{ij}, depending on the interaction, with a positive sign for cooperative and a negative sign for competitive. For example, the SIS system is cooperative as interaction with other hosts can only lead to the spread of the disease, increasing the probability of sickness. An

[1] The meaning of a node depends on the model it describes. As the examples on this page illustrate, it could represent a species, a neurone or a host.

example of a competitive system is certain Lotka-Volterra systems. If the environment has scarce resources needed by all species, the growth of one species will hinder the growth of all other species.

Rather than solving the system of N differential equations in time, we focus on equilibria; configurations of the quantities of each node such that they will not change over time, if the system is not disrupted.

A substantial amount of research has been done on cooperative systems. For instance, Wu *et al.* [2] simplified the system to a one-dimensional equation, making it possible to determine the existence of non-trivial equilibria analytically. In Laurence *et al.* [1], rather than determining the quantity of each node at an equilibrium, the author approximates the weighted sum of the equilibrium, which, although it contains less information, is analytically doable. It is also common for functions in systems to depend on different parameters, which brings its difficulties. In Jiang *et al.* [3], tipping points are predicted when a system, which is dependent on multiple variables, is at the transition between having a non-trivial equilibrium and not for a plant-pollinator system. Much less research has been done on competitive systems or systems that incorporate cooperative and competitive interactions.

In this thesis, our objective is to establish a connection between strictly cooperative systems and systems that incorporate cooperative and competitive elements. There are various ways to define such systems; we focus on mixed-weight systems, although two other systems are described in Appendix A.

The **mixed-weight system** is an extension of the cooperative system, where we relaxed the restriction on the weights, which may now take negative values. Their weights no longer represent the strength of positive interaction. Instead, it depends on the sign and magnitude of the weight, where a negative weight is competitive and a positive weight is cooperative. It is defined as:

$$\begin{aligned}
&\forall\, i \in \{1, \dots, N\}, && \frac{dx_i}{dt} = F(x_i) + \sum_{j=1}^{N} w_{ij}\, G(x_i, x_j) \\
&\forall\, x \in \mathbb{R}_{\geq 0}, && F(x) \leq 0 \\
&\forall\, x, y \in \mathbb{R}_{\geq 0}, && G(x, y) \geq 0 \\
&\exists\, i, j, k, l \in \{1, \dots, N\}, && w_{ij} > 0, \quad w_{kl} < 0.
\end{aligned}$$

In this thesis, due to its close relationship with cooperative systems, we focus on mixed-weight systems. We aim to create a bridge from the information on cooperative systems to the equilibria of mixed-weight systems. We want to answer two questions:

1. If the mixed-weight system has a non-trivial equilibrium, does the corresponding cooperative system also have a non-trivial equilibrium?

2. If the cooperative system has a non-zero equilibrium, does the mixed-weight system also have a non-trivial equilibrium?

We do not expect a straightforward yes or no answer to either question; therefore, we first describe the region where the equilibria exist, which is done in section 3. In section 4 we illustrate how the equilibria of the mixed-weight system correlate to those of the cooperative systems. To refine this correlation, in section 5, we will group the equilibria of the cooperative system, which reveals a special equilibrium, the principal equilibrium. In section 6 we cover a system, but instead of focussing on the equilibria, an interpretation of the whole system is given. Finally, in section 7 we discuss why we cannot translate the equilibria of the cooperative system to those of the mixed-weight system. However, we can formalise conditions on the mixed-weight equilibria and criteria in which an equilibrium of the cooperative system is also an equilibrium of the mixed-weight system. In each section, we give brief examples on how to apply theorems and corollaries except for the last section, in which we cover everything. For more elaborate examples, we refer throughout the text to the appendices. In each section, we illustrate the knowledge obtained so far with a simplified depiction. The starting point is illustrated in Figure 1.

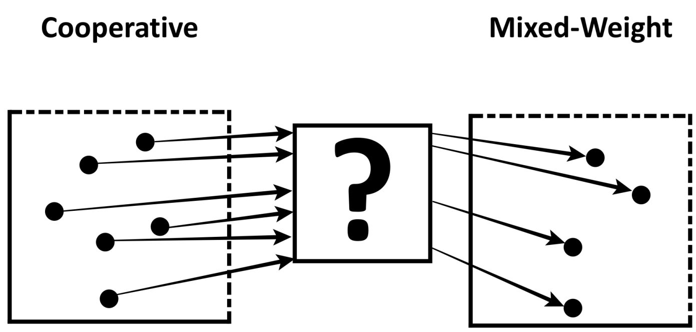

Figure 1: A conceptual illustration of the knowledge obtained. The dots represent the equilibria. The equilibria are bounded by the left and bottom solid lines, which represent the zero crossings. The dashed lines visualise the unknown extent of the equilibria. The question mark in the middle indicates that we have not yet discovered the correlation between the equilibria of both systems.

2 Assumptions and Notations

In this work, we will often switch between a mixed-weight system and its corresponding cooperative system. To simplify the notation, we differentiate between cooperative and competitive interactions.

Definition 1. *Let w_{ij} denote the weight of the directed edge from node j to node i in a network of N nodes.*

- *The node j is a **neighbour** of node i if $w_{ij} \neq 0$.*
- *The **positive interaction set** is:*

$$A^+ := \left\{(i,j) \in \{1,\dots,N\}^2 \mid w_{ij} > 0\right\}.$$

- *The **negative interaction set** is:*

$$A^- := \left\{(i,j) \in \{1,\dots,N\}^2 \mid w_{ij} < 0\right\}.$$

- *The **positive neighbouring set** of node i is:*

$$S_i^+ := \left\{j \mid (i,j) \in A^+\right\},$$

*and the **negative neighbouring set** is:*

$$S_i^- := \left\{j \mid (i,j) \in A^-\right\}.$$

- *The **dynamic function** of node i at $\mathbf{v}$ is:*

$$\left.\frac{dx_i}{dt}\right|_{\mathbf{x}=\mathbf{v}} = F(v_i) + \sum_{j\in S_i^+}^{N} w_{ij}\, G(v_i, v_j) + \sum_{j\in S_i^-}^{N} w_{ij}\, G(v_i, v_j).$$

For cooperative systems, we ignore the sum over negative interactions.

- *The **positive weighted in-degree** of node i in a system is equal to the sum of all the positive weights of the neighbours coming into node i:*

$$k_i^+ := \sum_{j\in S_i^+} w_{ij}.$$

For example, consider the network on the right of Figure 2. For node 1, the neighbours are nodes 2, 3, 4 and 5, but not 3, as node 3 does not influence node 1 (node 1 does, however, influence node 3). The positive neighbouring set of node 1, S_1^+, is $\{2,4\}$, while the negative neighbouring set, S_1^-, is $\{5\}$. The positive weighted in-degree, $k_1^+ = 16$.

The systems we will examine satisfy the assumptions commonly used to describe cooperative networks with N nodes.

Assumptions 1.

A1. Each node's value is always non-negative:

$$\forall i \in \{1, \dots, N\} \quad x_i \geq 0.$$

A2. Both functions are continuous on the non-negative real values:

$$F(x_i) \in \mathcal{C}(\mathbb{R}_{\geq 0}) \quad G(x_i, x_j) \in \mathcal{C}(\mathbb{R}^2_{\geq 0}).$$

A3. A node's self-interaction suppresses its growth, and the neighbours can only enhance it. This results in $F(x_i)$ being non-positive and $G(x_i, x_j)$ being non-negative:

$$F(x_i) : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\leq 0}, \quad G(x_i, x_j) : \mathbb{R}^2_{\geq 0} \to \mathbb{R}_{\geq 0}.$$

In addition, we assume that $F(x_i)$ is strictly negative for positive arguments:

$$\forall x_i > 0, \quad F(x_i) < 0.$$

We also require that $G(x_i, x_j) = 0$ whenever the neighbour is zero:

$$\forall x \in \mathbb{R}_{\geq 0}, \quad G(x, 0) = 0.$$

A4. If a quantity of node i, x_i, has a value of zero and all the neighbours of this node also have a value of zero, then this node will not change. Therefore, $F(x_i)$ and $G(x_i, x_j)$ are 0 if the node and the neighbours of the node are 0:

$$F(0) = 0, \quad G(0, 0) = 0.$$

A5. An increase in the value of the neighbours of a node will only result in a more positive interaction. $G(x_i, x_j)$ is non-decreasing in its second variable:

$$\frac{\partial G(x_i, x_j)}{\partial x_j} \geq 0.$$

A6. If we keep increasing the collective quantities of the nodes in a system, the system's functions ensure that as the total quantity grows, the overall system will eventually decrease. Let $\mathbf{x}$ be the vector, where the i-th entry is the quantity of node i:

$$\lim_{\|\mathbf{x}\| \to \infty} \frac{d \|\mathbf{x}\|}{dt} < 0.$$

A7. NO node has a interaction with itself:

$$\forall i \in \{1, \dots, N\}, \quad w_{ii} = 0.$$

Remark 1. From the above assumptions, we see that if the quantity of each node is equal to zero, then, independent of the weights w_{ij}, we have an equilibrium. This equilibrium is often called the trivial zero equilibrium.

Before moving on to the next section, we clarify how we transition between cooperative and mixed-weight systems. The corresponding mixed-weight system of a cooperative system has the same network with the addition of the negative interaction. In contrast, the cooperative system of a mixed-weight system is obtained by setting all negative interactions in the network to zero. We therefore assume that for all nodes i, the positive and negative interaction sets are disjoint:

$$S_i^+ \cap S_i^- = \emptyset.$$

An example of a cooperative network and a corresponding mixed-weight network is given in Figure 2. Lastly, all numerically calculated values are given with a precision of three decimals. The code used for these computations is available at `https://github.com/NorthOfhell/BEP.git`.

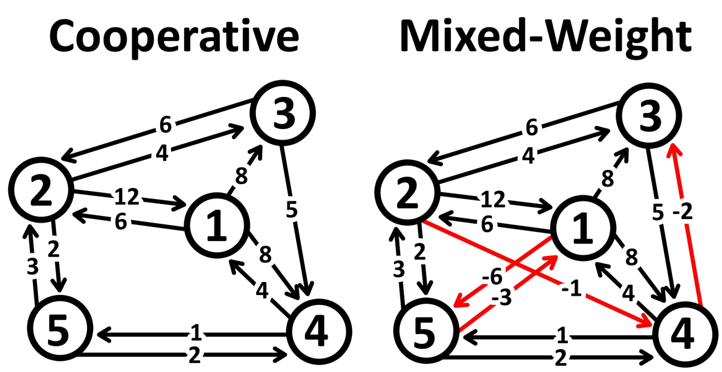

Figure 2: The network of a cooperative system on the left, and a corresponding mixed-weight network on the right. An example of a system with this network is covered in subsection 7.3. The number of each node indicates the index.

3 Bounds

Common practice for extracting information from the system is to reduce dimensionality [1][2]. In this section, we employ a similar approach to obtain an upper bound on the equilibrium values. Specifically, the upper bound represents the maximum value that any node can attain at equilibrium. From this, we determine the minimum positive weighted in-degree a node must have to not be zero in equilibrium.

However, before doing so, we would like to clarify something about the systems we examine. Consider, instead of the whole network, only a single node in the network. For such a node, if the neighbours all have the same quantity, we could instead consider one neighbour, which has a weight equal to the sum of all the previous neighbours. This follows since for an arbitrary value of the neighbours y, the dynamic function of a node i is:

$$\begin{aligned}\frac{dx_i}{dt} =& F\left(x_i\right) + \sum_{j \in S_i^+} w_{ij} G(x_i, y) + \sum_{j \in S_i^-} w_{ij} G(x_i, y) \\ =& F\left(x_i\right) + G(x_i, y) \sum_{j \in S_i^+} w_{ij} + G(x_i, y) \sum_{j \in S_i^-} w_{ij} \\ =& F\left(x_i\right) + G(x_i, y) \left(\sum_{j \in S_i^+} w_{ij} + \sum_{j \in S_i^-} w_{ij} \right).\end{aligned}$$

This is also visually shown in Figure 3. If all surrounding nodes have the same quantity, then for the middle node, there is no difference between the left and right cases. We will leverage this to conclude the information from our reduction method.

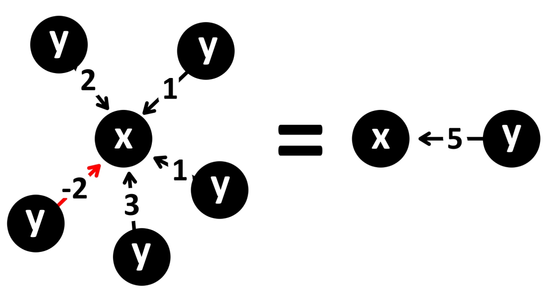

Figure 3: The symbols in each node represent the value of that node. If all the neighbouring nodes have the same value, then the dynamic function of node 1 in the left case is the same as in the right case.

The reduction method consists of two parts; first, we prove that for a reduced equation, we always have an upper bound, such that the equation is always negative above the upper bound. Secondly, combining this upper bound and the fact that for a node we can reduce the dynamic function if the neighbours have the same value, we can upper bound all the nodes of the system.

Theorem 1. *Consider a cooperative system with N nodes that meet Assumptions 1. Define*

$$W_{\max} = \max_{i \in \{1,\dots,N\}} \left\{ \sum_{j \in S_i^+} w_{ij} \right\},$$

and define x_{upper} as the supremum over the real non-negative numbers x such that:

$$F(x) + W_{\max} G(x, x) = 0,$$

then, for any value y above x_{upper}, we have:

$$F(y) + W_{\max} G(y, y) < 0.$$

Proof. x_{upper} is defined as we always have that 0 is a solution to the equation. Define the function:

$$H(x_i, x_j) = F(x_i) + W_{\max} G(x_i, x_j).$$

If we consider a network with two nodes, links in both directions with weight $W_{\max}$, then on the line $x_1 = x_2 = x$ we have:

$$\lim_{x \to \infty} H(x, x) < 0,$$

which follows from A. 6. This implies that there exists $M \in \mathbb{R}$ such that for all $x \geq M$,

$$H(x, x) \leq 0,$$

with equality if and only if $x = M$. M exists, as implied by the continuity of $H(x_i, x_j)$, which itself is a consequence of both $F(x_i)$ and $G(x_i, x_j)$ being continuous (see A. 2). We deduce that $x_{\text{upper}} = M$ and for any number $y > x_{\text{upper}}$:

$$H(y, y) = F(y) + W_{\max} G(y, y) < 0.$$

□

3.1 Upper Bound of Both Systems

We will now, with x_{upper}, perform the second step in the reduction method. For this, we will consider an arbitrary equilibrium of the system and show that the highest element of this equilibrium can be upper bounded. We do this first for cooperative systems, after which we do the same for mixed-weight systems.

Theorem 2. *Consider a cooperative system with N nodes that satisfy Assumptions 1. Let $\mathbf{v}$ be an equilibrium of the cooperative system. Each element i of the equilibrium is upper bounded:*

$$v_i \leq x_{\text{upper}}.$$

Proof. Let $\mathbf{v}$ be an arbitrary equilibrium of the system. Then there exists an index k such that:

$$v_k = \max\{v_1, \ldots, v_N\}.$$

We will show: $v_k \leq x_{\text{upper}}$. For node k, the dynamic function at equilibrium $\mathbf{v}$ is:

$$\begin{aligned}
\left.\frac{dx_k}{dt}\right|_{\mathbf{x}=\mathbf{v}} &= F(v_k) + \sum_{j \in S_k^+} w_{kj} G(v_k, v_j) \\
&\leq F(v_k) + \sum_{j \in S_k^+} w_{kj} G(v_k, v_k) \qquad (1) \\
&= F(v_k) + G(v_k, v_k) \sum_{j \in S_k^+} w_{kj} \\
&\leq F(v_k) + W_{\max} G(v_k, v_k). \qquad (2)
\end{aligned}$$

In step 1 we reduce the system. The inequality follows as the function $G(x_i, x_j)$ is non-decreasing in its second variable (see A. 5). Inequality 2 follows from the definition of $W_{\max}$ (see, Theorem 1). By inequality 2 and the definition of x_{upper} (see, Theorem 1), we can conclude that if $v_k > x_{\text{upper}}$ we must have:

$$\left.\frac{dx_k}{dt}\right|_{\mathbf{x}=\mathbf{v}} < 0.$$

This would contradict $\mathbf{v}$ being an equilibrium, thus we conclude that for each index i:

$$v_i \leq v_k \leq x_{\text{upper}}.$$

□

For mixed-weight systems, we can make a similar statement and proof.

Corollary 1. *Suppose we have a mixed-weight system that satisfies Assumptions 1. All the equilibria of this system are element-wise upper bounded by* x_{upper}*.*

Proof. A full proof is provided in Appendix Proof B.1, but we will sketch the general idea below. We proceed similarly to the proof of Theorem 2, except that the dynamic function of node k at $\mathbf{v}$ no longer is:

$$F(v_k) + \sum_{j \in S_k^+} w_{kj} G(v_k, v_j).$$

For mixed-weight systems, we have to add the extra negative interaction term, which results in:

$$\begin{aligned}
\left.\frac{dx_k}{dt}\right|_{\mathbf{x}=\mathbf{v}} &= F(v_k) + \sum_{j \in S_k^+} w_{kj} G(v_k, v_j) + \sum_{j \in S_k^-} w_{kj} G(v_k, v_j) \\
&\leq F(v_k) + \sum_{j \in S_k^+} w_{kj} G(v_k, v_j).
\end{aligned}$$

At this point, we could perform the same steps as in the proof of Theorem 2, resulting in the same conclusion. □

Remark 2. If a system with N nodes satisfies Assumptions 1, then for every equilibrium $\mathbf{v}$ of the cooperative system and for every equilibrium $\mathbf{u}$ of the mixed-weight system, we have:

$$\mathbf{v}, \mathbf{u} \in [0, x_{\text{upper}}]^N,$$

which is a direct consequence of the upper bound in cooperative systems (see Theorem 2) and mixed-weight systems (see Corollary 1).

Remark 3. If $x_{\text{upper}} = 0$, then the only equilibrium is the trivial zero equilibrium.

The deviation of the upper bound and the elements of an equilibrium depend on several factors: the variation of positive weighted in-degrees, the functions $F(x_i)$ and $G(x_i, x_j)$ and whether the system is cooperative or mixed-weight. In Appendix Example F.1 we give a cooperative system that has equilibria, in which the quantities of each node are below the upper bound, while in Appendix Example F.2 we give a cooperative system that has an equilibrium with entries equal to the upper bound.

We can conclude the first piece of the puzzle. The space in which the equilibria for both systems exist is not arbitrarily large but can be determined by solving an equation in one variable, which can also be seen in the visual graphic Figure 4.

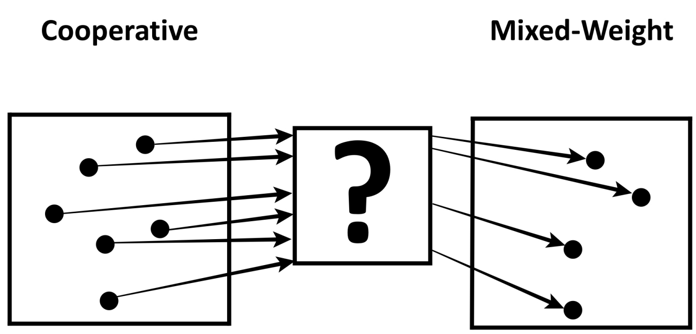

Figure 4: A conceptual illustration of the knowledge obtained. The dashed lines in Figure 1 are now full, as we can lower and upper bound every equilibrium, creating a box in which every equilibrium exists. We still miss a relationship between the equilibria of the cooperative system and the mixed-weight system.

3.2 Lower Bound of Positive Weighted In-Degree

In a cooperative system, the functions $F(x_i)$ and $G(x_i, x_j)$ counteract each other's influence. For an equilibrium, we require the negative influence, due to $F(x_i)$, to be opposite to the positive influence of the weighted sum over all the $G(x_i, x_j)$. To maximise the weighted sum, dependent on x_i, it follows by A. 5, that all x_j are equal to the upper bound. What if, even if we maximise the sum, the dynamic function of node i, with its neighbours, is still always negative;

$$\forall x_i \in \mathbb{R}_{>0} \quad F(x_i) + G(x_i, x_{\text{upper}}) \sum_{j \in S_i^+} w_{ij} < 0.$$

The negative influence is, in this case, always stronger than the positive influence, which implies that this node is zero for all equilibria. To remedy this, we could increase the positive influence by increasing the weights. Consequently, there is a minimum of the weights such that the above function is not always negative. We define this minimum as **the minimum weighted in-degree**.

Theorem 3. ***minimum weighted in-degree***
Suppose we have a system that meets Assumptions 1. We define $W_{\min}$ as the infimum of all non-negative $W_{\min}$, such that there exists a positive x for the equation:

$$F(x) + G(x, x_{\text{upper}})W_{\min} = 0.$$

Every node i of the system that has a positive weighted in-degree less than W_{min} will have a value of zero for all the equilibria of the system.

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Proof. For the proof, we refer to Appendix Proof B.3. □

If we have a node that has a positive weighted in-degree smaller than $W_{\min}$, it is, for all equilibria, zero. We might at this point wonder if we even need this node, if we are only interested in finding the equilibria; the answer is no. We could instead consider the subnetwork without the node, which is zero for all equilibria. We can link the equilibria of the subnetwork to the original network by setting the deleted node to zero. Since the interaction function $G(x_i, x_j)$ is zero if the second argument is zero, it will not diminish the equilibrium. We can categorise systems that have a node that is zero in all equilibria.

Definition 2. *A system with N nodes is a* ***participatory*** *system if for every node there exists an equilibrium for which the node has a non-vanishing entry. If for all equilibria $\mathbf{v_1}, \ldots, \mathbf{v_M}$, there is a node i such that:*

$$\forall k \in \{1, \ldots, M\}, \quad [\mathbf{v_k}]_i = 0,$$

we call the system ***partially redundant****.*

As stated above, we do not need to consider this the node that is zero in all equilibria. This implies that we can transform every partially redundant system by considering a subnetwork, which is participatory.

Corollary 2. *Every partially redundant system that satisfies Assumptions 1 can be made into a participatory system. The equilibria of the new participatory system are also the equilibria of the partially redundant system.*

Proof. For a detailed proof, we refer to Appendix Proof B.2. □

Remark 4. If a system that meets Assumptions 1 has a node that has a positive weighted in-degree smaller than $W_{\min}$, then this system is partially redundant.

In Appendix Example F.3 we illustrate a system in which one of the nodes has a positive weighted in-degree smaller than $W_{\min}$. The same holds for the reduced network, and this pattern continues recursively until we conclude that the only equilibrium is the trivial zero equilibrium.

3.3 Example: Bounds

We will now briefly cover an example of a cooperative system in which we apply the reduction method and check if one of the nodes has a weighted in-degree smaller than the minimum positive weighted in-degree. Consider the system with the following network and functions:

$$\begin{bmatrix} 0 & 4 & 0 \\ 4 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix},$$

$$F(x_i) = -4x_i + 4x_i^2 - 4x_i^3, \quad G(x_i, x_j) = x_i x_j.$$

The network is also visually shown in Figure 5. $W_{\max}$ of this network is 4. This results in the following equation for the upper bound:

$$-4x + 4x^2 - 4x^3 + W_{\max} x^2 = 0.$$

The roots are $x = 0$ and $x = 1$, hence $x_{\text{upper}} = 1$. The minimum weighted in-degree $W_{\min}$, is the smallest c, such that there is still a positive x of the equation:

$$-4x + 4x^2 - 4x^3 + cx = 0.$$

This results in $W_{\min} = 3$. Node 3 has a positive weighted in-degree smaller than 4, and is consequently zero in all equilibria. The only equilibrium of this network is:

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

We observe that the upper bound is reached for the first two nodes. However, we derived that the last node is always zero in equilibrium. The upper bound for this node is therefore highly inaccurate, which is a shortcoming of the reduction method. The reduction method does not account for the dynamics of each node.

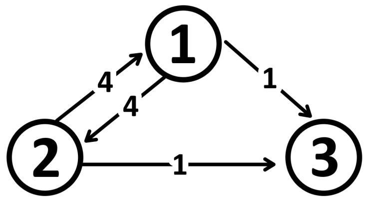

Figure 5: The network of the system we consider in subsection 3.3. The number of each node indicates the index.

4 Sufficient Condition for Cooperative Systems

In this section, we provide a theorem that offers information on whether an equilibrium exists in a N-dimensional box. We will do this by applying Poincaré-Miranda, which states conditions under which multiple functions have a common root. We show that we can relax the conditions of Poincaré-Miranda, resulting in a correlation between the equilibria of the mixed-weight and cooperative system and a framework for the rest of this thesis.

One of the first theorems that a mathematician learns is the Intermediate Value Theorem, which states that a continuous function on the bounds $[a, b]$ attains every value between $f(a)$ and $f(b)$. We can conclude that if $f(a)$ and $f(b)$ have opposite signs, then there exists a $c \in [a, b]$ such that $f(c) = 0$.

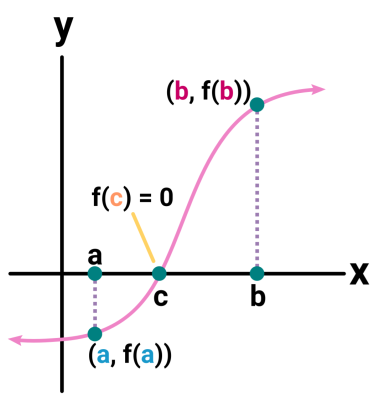

Figure 6: A visual interpretation of the Intermediate Value Theorem. If the function at two points has opposite signs, then there must be a root between the two points. Image taken from the webpage *Intermediate Value Theorem Lesson* by GreeneMath [4].

For our system, the dynamic function is continuous, but we have multiple such functions, one for each node. Therefore, we cannot directly apply the Intermediate Value Theorem to obtain the existence of an equilibrium, due to the higher dimensionality. Fortunately, great minds have come before us and extended the Intermediate Value Theorem to higher dimensions. For example, the result of Poincaré-Miranda, originally conjectured by Henri Poincaré and proven by Carlo Miranda in [5].

Lemma 1. ***Poincaré-Miranda***
Let the two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^N$ *be such that* $a_i \leq b_i$ *for all* $i = 1, \ldots, N$. *Define the N-dimensional box:*

$$\Omega = [a_1, b_1] \times \ldots, \times [a_n, b_n].$$

For all i, *we define the two subsets as:*

$$I_{i,-} = \{\mathbf{x} \in \Omega \mid x_i = a_i\},$$

$$I_{i,+} = \{\mathbf{x} \in \Omega \mid x_i = b_i\}.$$

Suppose that we have continuous functions $f_1, \ldots f_N : \Omega \to \mathbb{R}$. *If for each* i *the function* f_i *satisfies:*

$$\forall \mathbf{x} \in I_{i,-} : f_i(\mathbf{x}) \geq 0 \quad \textit{and} \quad \forall \mathbf{x} \in I_{i,+} : f_i(\mathbf{x}) \leq 0,$$

then there exists a vector $\mathbf{x} \in \Omega$ *such that:*

$$\forall i \in \{1, \ldots, N\}, \quad f_i(\mathbf{x}) = 0.$$

Before we apply this result of Poincaré-Miranda, we would like to give a visual explanation of its conditions. Consider a N-dimensional box. If we fix one coordinate of this N-dimensional box but vary all the other coordinates, we get a face. For each coordinate i, we consider two such faces: one at a_i (the lowest possible value) and one at b_i (the highest possible value). The condition of Poincaré-Miranda states that if on the face with a_i the function f_i is non-negative, and on the face with b_i the function f_i is non-positive, then there exists a point in the box where all $f_i = 0$. The case for $N = 2$ is shown in Figure 7.

We could now, if we find suitable vectors $\mathbf{a}$ and $\mathbf{b}$, conclude the existence of an equilibrium, using Poincaré-Miranda (see Lemma 1). At first sight, it seems that finding the two suitable vectors is no easy task. The conditions imposed on the faces of the N-dimensional box are hard to deduce or calculate. Fortunately, for the systems we examine, there is no need to consider the whole face. We can prove that if at two points the corners of the N-dimensional box satisfy one condition each, then we can also conclude the existence of an equilibrium.

Theorem 4. ***sufficient condition for cooperative systems - SCCS***
Consider a cooperative system with N *nodes that meet Assumptions 1. If there exist two vectors* $\mathbf{l}, \mathbf{h}$ *such that the dynamic function for each node* i *at both vectors is:*

$$F(l_i) + \sum_{j \in S_i^+} w_{ij} G(l_i, l_j) \geq 0,$$

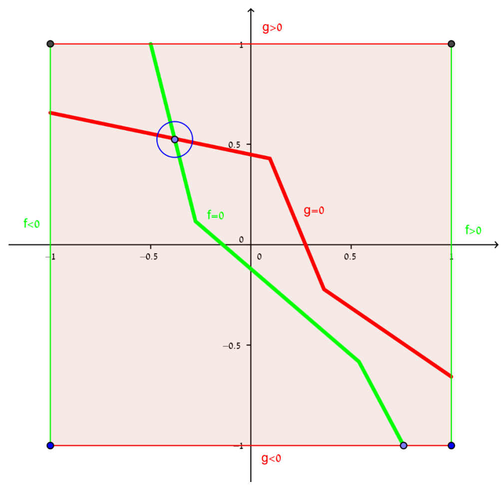

Figure 7: Poincaré-Miranda in two dimensions. The two functions both have a line on which they are zero. The crossing of these two lines is the desired point. Image by Erel Segal-Halevi, via Wikimedia [6].

$$F(h_i) + \sum_{j \in S_i^+} w_{ij} G(h_i, h_j) \leq 0,$$

$$l_i < h_i,$$

then there exists an equilibrium $\mathbf{z}$ *such that:*

$$\mathbf{z} \in [l_1, h_1] \times \cdots \times [l_N, h_N].$$

Proof. We want to apply Poincaré-Miranda (see Lemma 1), therefore, for all $i = 1, \ldots, N$ we define the functions f_i as the dynamic function of node i:

$$f_i(\mathbf{x}) = F(x_i) + \sum_{j \in S_i^+} w_{ij} G(x_i, x_j).$$

It follows from the continuity of $F(x_i)$ and $G(x_i, x_j)$ (see A. 2) that each f_i is continuous. We use the same definition of the faces $I_{i,-}$ and $I_{i,+}$ as in Poincaré-Miranda (see Lemma 1).

Case: for all $\mathbf{x} \in I_{i,-},\ f_i(\mathbf{x}) \geq 0$:

Let $i = 1, \ldots, N$ and take a random vector $\mathbf{a} \in I_{i,-}$. Each entry in this random vector is larger than or equal to the corresponding entry in $\mathbf{l}$, except for the i-th entry, which is always equal. It follows from the monotonicity of $G(x_i, x_j)$ (see A. 5), that:

$$\forall j \in \{1, .., N\}, \quad G(l_i, l_j) \leq G(l_i, a_j) = G(a_i, a_j).$$

To check the sign of $f_i(\mathbf{a})$, we start with the dynamic function of node i at $\mathbf{l}$, which is:

$$\begin{aligned} 0 \le f_i(\mathbf{l}) &= F(l_i) + \sum_{j \in S_i^+} w_{ij}\, G(l_i, l_j) \\ &\le F(l_i) + \sum_{j \in S_i^+} w_{ij}\, G(l_i, a_j) \\ &= F(a_i) + \sum_{j \in S_i^+} w_{ij}\, G(a_i, a_j) = f_i(\mathbf{a}). \end{aligned}$$

Consequently, for a random vector $\mathbf{a} \in I_{i,-}$, we must have $f_i(\mathbf{a}) \ge 0$. We conclude that $\forall \mathbf{x} \in I_{i,-},\ f_i(\mathbf{x}) \ge 0$.

Case: for all $\mathbf{x} \in I_{i,+},\ f_i(\mathbf{x}) \le 0$:

We apply a similar structure for this case; therefore, let $i = 1, \ldots, N$ and take a random vector $\mathbf{b} \in I_{i,+}$. Each entry in this random vector is smaller than or equal to the corresponding entry in $\mathbf{h}$, except for the i-th entry, which is always equal. It follows from the monotonicity of $G(x_i, x_j)$ (see A. 5), that:

$$\forall j \in \{1, .., N\}\,, \quad G(b_i, b_j) \le G(b_i, h_j) = G(h_i, h_j).$$

To check the sign of $f_i(\mathbf{b})$, we start with the dynamic function of node i at $\mathbf{h}$, which is:

$$\begin{aligned} 0 \ge f_i(\mathbf{h}) &= F(h_i) + \sum_{j \in S_i^+} w_{ij}\, G(h_i, h_j) \\ &= F(b_i) + \sum_{j \in S_i^+} w_{ij}\, G(b_i, h_j) \\ &\ge F(b_i) + \sum_{j \in S_i^+} w_{ij}\, G(b_i, b_j) = f_i(\mathbf{b}). \end{aligned}$$

Consequently, for a random vector $\mathbf{b} \in I_{i,+}$, we must have for all $\mathbf{x} \in I_{i,+},\ f_i(\mathbf{x}) \le 0$. By Poincaré-Miranda (see Lemma 1), we conclude that there exists an equilibrium $\mathbf{z}$ to this cooperative system, with:

$$\mathbf{z} \in [l_1, h_1] \times \cdots \times [l_N, h_N].$$

□

Remark 5. We will refer to the vector $\mathbf{l}$ as a **lower vector** and similarly to the vector $\mathbf{h}$ as a **higher vector**.

Instead of directly solving the equilibria, we can try to find the two vectors of the SCCS (see Theorem 4) to approximate the equilibrium. Alternatively, suppose that we only want to know whether a non-zero equilibrium exists. In that case, we can also use the SCCS (see Theorem 4). Finding these two vectors, especially the lower one, can still be challenging. Although one might think that we could use the zero vector, this does not yield any new information since the interval of the equilibrium is closed on the given two vectors. In the next subsection, we provide a method to obtain a lower vector. For the higher vector, we can use the information of section 3. Consider a node i of a cooperative system. If all the neighbours of node i have the same value, we could instead consider only one node that has the same value and a weight equal to the sum of all the previous neighbours, which is equal to k_i^+. By definition, k_i^+ is never greater than $W_{\max}$ (see Theorem 1). We could combine this with Theorem 1, which states that:

$$F(x_{\text{upper}}) + W_{\max} G(x_{\text{upper}}, x_{\text{upper}}) = 0.$$

Since $G(x_i, x_j)$ is never negative, any value that replaces $W_{\max}$, which is strictly smaller, will always result in the dynamic function being negative, which is what we want for our higher vector. The above is visually displayed in Figure 8, and is formalised in the following corollary.

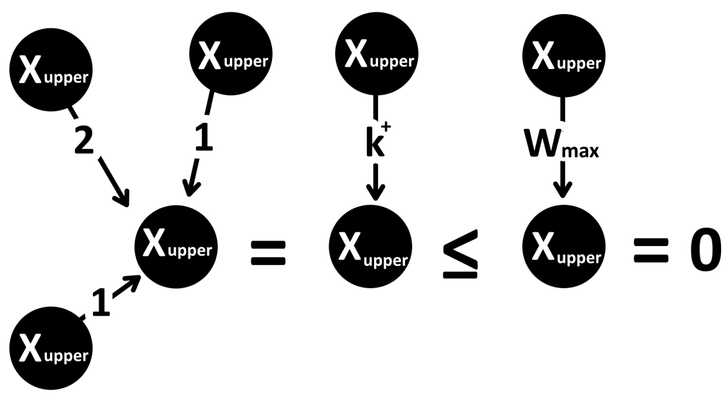

Figure 8: The symbols in each node represent the value of that node. We can determine the sign of the dynamic function of the centre node on the left by comparing it to the equation in Theorem 1. This gives us a higher vector for the SCCS (see Theorem 4).

Corollary 3. *For every cooperative system with N nodes that satisfies Assumptions 1, in the SCCS (see Theorem 4), the higher vector* $\mathbf{h}$ *can always be taken as* $\mathbf{x}_{upper}$*, which is defined by:*

$$\mathbf{x}_{\text{upper}} := [x_{\text{upper}}]^N.$$

Proof. For an arbitrary node i, the dynamic function at $\mathbf{x}_{\text{upper}}$ is:

$$\begin{aligned} \left.\frac{dx_i}{dt}\right|_{\mathbf{x}=\mathbf{x}_{\text{upper}}} &= F(x_{\text{upper}}) && + \sum_{j \in S_i^+} w_{ij}\, G(x_{\text{upper}}, x_{\text{upper}}) \\ &\leq F(x_{\text{upper}}) && + W_{\max} G(x_{\text{upper}}, x_{\text{upper}}) && = 0, \end{aligned}$$

where the equality at the end follows from the definition of the upper bound (see Theorem 1). □

4.1 Weighted k-Core

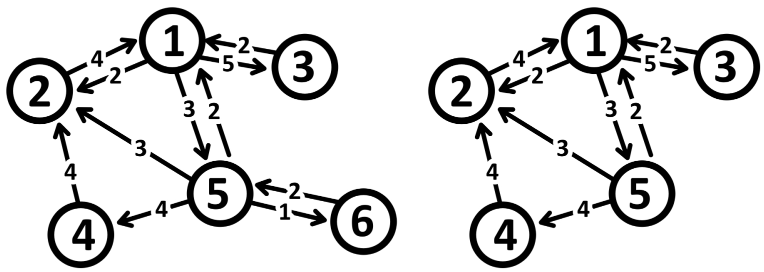

Figure 9: Left: original network. Right: corresponding weighted k-core, where each node has a positive weighted in-degree of at least 3, therefore $k_{\max} = 3$.

In this subsection, we provide a method to obtain a lower vector, which is based on the work of Wu *et al.* [2]. We will consider the subnetwork(s) called the weighted k-core(s). In the weighted k-core, the lowest weighted in-degree of all nodes is maximal, which is often denoted as $k_{\max}$. The value $k_{\max}$ allows us to determine whether the system has a non-zero equilibrium [2]. To find this subnetwork of a network, we calculate the positive weighted in-degree for each node. We denote the smallest weighted in-degree, which is the minimum positive weighted in-degree every node has. Next, we remove the node with the smallest positive weighted in-degree, recalculate the weighted in-degree of the remaining nodes, and compare the smallest to the previous one. We repeat this process, keeping track of each smallest positive weighted in-degree until no nodes are left. The maximum of these minimum weighted in-degrees is defined as $k_{\max}$. A full example on the derivation of the weighted k-cores is given in Appendix Example F.4, of which the result is given in Figure 9. We will use this $k_{\max}$ to find the vector we require in the SCCS (see Theorem 4).

Consider the subnetwork that corresponds to $k_{\max}$. For this subnetwork, it might be the case that some nodes have a weighted in-degree higher than $k_{\max}$. We decrease the weight of the incoming interactions of the nodes with a higher positive weighted in-degree, until they are equal to $k_{\max}$. This results in a semi-symmetric network [3] where every node in the subnetwork has a weighted in-degree of $k_{\max}$. If we assume that all the nodes of this subnetwork have the same value, the dynamic function of each node is similar, and only dependent on one variable:

$$F(x) + G(x, x)k_{\max} = 0.$$

[2]Although used differently, a similar approach was employed by Wu *et al.* [2] to indicate if a non-zero equilibrium exists.

[3]The network is semi-symmetric, as not all the weights are equal, but the weighted in-degrees are.

The solutions of this equation can be used as a lower vector.

Corollary 4. *Consider a subnetwork of a cooperative system with N nodes that meet Assumptions 1. Suppose that each node of the subnetwork has a weighted in-degree of k_{max}. If this subnetwork has a non-zero uniform equilibrium* $\mathbf{v}$*, then this equilibrium* $\mathbf{v}$ *is a lower vector in the SCCS (see Theorem 4).*

Proof. For the proof, we refer to Appendix Proof C.1. □

4.2 Spectral Radius

The main focus of Wu *et al.* [2] is not the search for lower and higher vectors for the SCCS (see Theorem 4); rather, the focus lies on if a non-zero equilibrium exists from a cooperative system, also called the survivability of a system. We can compare the method described in Wu *et al.* [2] with x_{upper}, which we do in this subsection.

The spectral radius, commonly denoted as $\rho(\mathbf{A})$, of the network $\mathbf{A}$, provides critical information if a non-zero equilibrium exists. In Wu *et al.* [2], it is stated that if the only solution to the equation:

$$F(x) + \rho(\mathbf{A})G(x, x) = 0,$$

is $x = 0$, then the only equilibrium is the zero equilibrium [4]. Similarly, if x_{upper} is zero, no non-zero equilibria exist. That is, if the only solution to the equation:

$$F(x) + W_{\max}G(x, x) = 0,$$

is $x = 0$. We want to know which attribute, $\rho(\mathbf{A})$ or x_{upper}, gives the best approximation. We consider a system with functions of the GRN system:

$$F(x_i) = -Bx_i, \quad G(x_i, x_j) = \frac{x_j^h}{1 + x_j^h}.$$

To compare both attributes, we will calculate, depending on h, for which B-values we could still have non-zero equilibria. The same can be done numerically for the system without applying any reduction methods. The lowest B-value, for which the system still has a non-zero equilibrium, is the tipping point. We will do this for two networks, both of which are randomly generated. For the first network, each interaction is generated from a uniform distribution. The second network is made such that a few selected nodes have strong in- and outgoing interactions, and the rest have a weak or non-existent interaction, which can be called a multi-starlike matrix. Both networks are visualised, which can be seen in the top two images of Figure 10.

The results of both attributes are visualised in the lower plots of Figure 10. In both cases, the spectral radius gives a better approximation, although in the

[4]This only holds for certain functions $G(x_i, x_j)$. For details, we refer to Wu *et al.* [2].

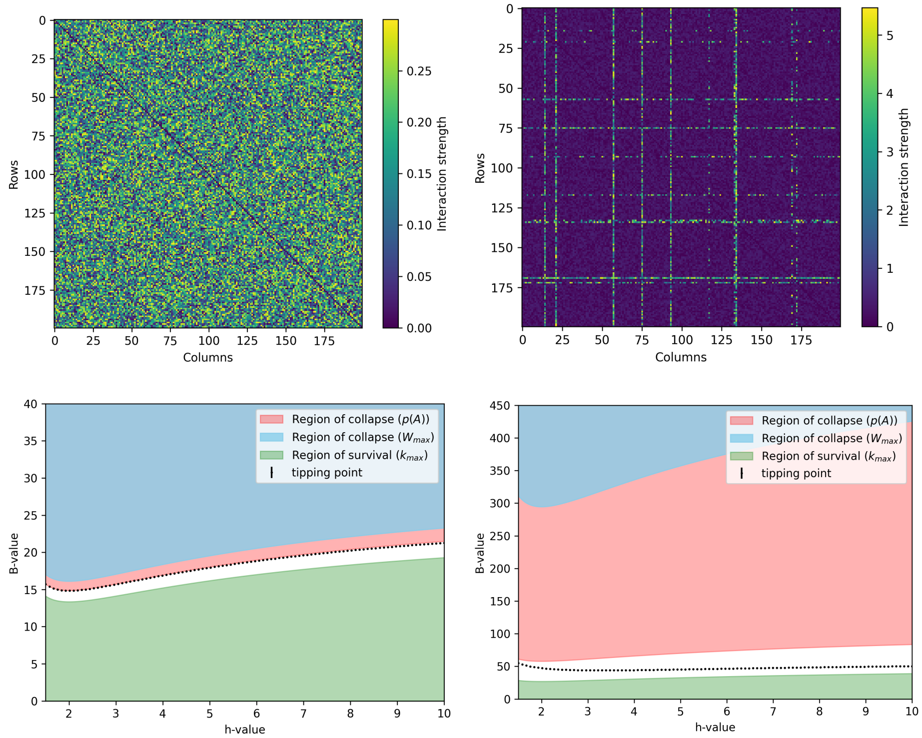

Figure 10: The regions of guaranteed survival and collapse for two randomly generated matrices. The top two images represent the networks, with the left a uniform generated matrix, and on the right a multi-starlike matrix. The region of collapse derived from $W_{\max}$ is close to $\rho(\mathbf{A})$, in the case of the uniform matrix, but not for the multi-starlike matrix.

case of the multi-starlike network, the difference is prominent. The spectral radius gives a better approximation, since:

$$\min_{1\leq i\leq N}\sum_{j=1}^{n} w_{ij} \leq \rho(\mathbf{A}) \leq \max_{1\leq i\leq N}\sum_{j=1}^{n} w_{ij} = W_{\max}.^{5}$$

Since we solve a similar equation for both the spectral radius and $W_{\max}$, the larger the difference between the two, the larger the difference between the regions. For a uniform matrix, the difference is much smaller than for the multi-starlike matrix. The spectral radius as $W_{\max}$ of both matrices is given in Table 1.

[5] This inequality is presented as Exercise 8.2.7 in Matrix Analysis and Applied Linear Algebra by Carl D. Meyer [7], for which we give a formal proof in Appendix Proof C.2

	Uniform	**Multi-starlike**
$\rho(\mathbf{A})$	29.713	115.630
$W_{\max}$	32.198	589.210

Table 1: The values used for the comparison of the regions of collapse.

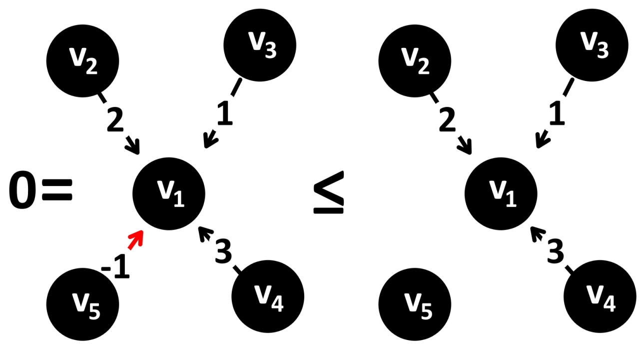

Figure 11: The symbols in each node represent the value of that node. If the dynamic function of a mixed-weight system is at equilibrium, then the dynamic equation of the corresponding cooperative system at the same equilibrium is either zero or positive.

4.3 Cooperative vs Mixed-weight

How do we incorporate the SCCS for our search for a correlation between the equilibria of the mixed-weight and cooperative systems? Let $\mathbf{v}$ be an equilibrium of a mixed-weight system. Consider this vector, but for the dynamic functions of the corresponding cooperative system. The difference is that we no longer have negative interactions; therefore, each dynamic function would be zero or positive (see Figure 11). This is exactly the criteria for a vector we need for a lower vector in the SCCS (see Theorem 4).

Lemma 2. *Consider a system that satisfies Assumptions 1. Every equilibrium* $\mathbf{v}$ *of the mixed-weight system is a lower vector in the SCCS (see Theorem 4).*

Proof. The dynamic function of the mixed-weight system for node i at $\mathbf{v}$ is:

$$\begin{aligned} 0 = \left.\frac{dx_i}{dt}\right|_{\mathbf{x}=\mathbf{v}} &= F(v_i) + \sum_{j \in S_i^+} w_{ij} G(v_i, v_j) + \sum_{j \in S_i^-} w_{ij} G(v_i, v_j) \\ &\leq F(v_i) + \sum_{j \in S_i^+} w_{ij} G(v_i, v_j). \end{aligned}$$

This proves that $\mathbf{v}$ is a suitable lower vector. □

This implies that every equilibrium of the mixed-weight system is upper bounded by an equilibrium of the corresponding cooperative system, since there must be an equilibrium between the equilibrium of the mixed-weight system and $\mathbf{x}_{upper}$.

Corollary 5. *Consider a system with N nodes that satisfies Assumptions 1. For every equilibrium* $\mathbf{v}$ *of the mixed-weight system, there exists an equilibrium* $\mathbf{z}$ *of the corresponding cooperative system such that for each index i:*

$$v_i \leq z_i.$$

Proof. By Lemma 2 we have that $\mathbf{v}$ is a lower vector in the SCCS (see Theorem 4), while we have previously shown in Corollary 3 that we always have a corresponding higher vector. This implies that there exists an equilibrium $\mathbf{z}$ of the cooperative system such that for all i:

$$v_i \leq z_i \leq x_{\text{upper}}.$$

□

Remark 6. If the only equilibrium of the cooperative system is the zero equilibrium, then so will be the case of the corresponding mixed-weight system.

We conclude that the addition of the negative weight will never result in a non-zero equilibrium if the cooperative system on itself does not have a non-zero equilibrium. With this, we can update our visual graphic, which can be seen in Figure 12.

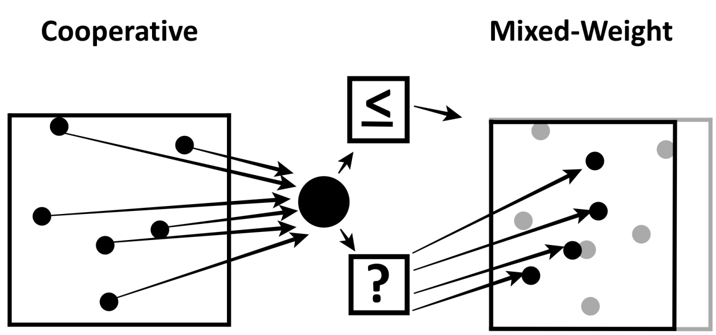

Figure 12: A conceptual illustration of the knowledge obtained. The space of equilibria of the mixed-weight system is bounded by the equilibria of the cooperative network, which follows from Corollary 5. The grey image behind the mixed-weight is the space of the cooperative system.

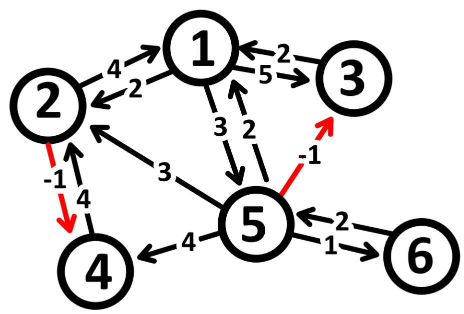

Figure 13: The network of the system we consider in subsection 4.4. The number of each node indicates the index.

4.4 Example: Sufficient Condition for Cooperative Systems

Consider the mixed-weight system:

$$\begin{bmatrix} 0 & 4 & 2 & 0 & 2 & 0 \\ 2 & 0 & 0 & 4 & 3 & 0 \\ 5 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 4 & 0 \\ 3 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$F(x_i) = -\frac{3x_i}{2}, \quad G(x_i, x_j) = \frac{x_j^2}{1 + x_j^2},$$

where the functions are of the gene regulatory network (GRN) system [2]. The network is also visually shown in Figure 13. The corresponding cooperative network, we have previously considered in the subsection 4.1; it is the network on the left of Figure 9. For this cooperative network, we find $k_{\max} = 3$. Solving the equation:

$$-\frac{3x}{2} + k_{\max}\frac{x^2}{1 + x^2} = 0,$$

yields $x = 0$ or $x = 1$. Therefore, a lower vector for the SCCS (see Theorem 4) is:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}^{\intercal}.$$

The last entry is zero, as this node is not in the subnetwork matching the weighted k-core. The upper bound of this network is 5.828, therefore, the higher vector is:

$$\begin{bmatrix} 5.828 & 5.828 & 5.828 & 5.828 & 5.828 & 5.828 \end{bmatrix}^{\intercal}.$$

By the SCCS (see, Theorem 4), there must be an equilibrium between these two vectors, which is the case as the only equilibrium of the cooperative system is:

$$\begin{bmatrix}4.896 & 5.164 & 3.200 & 2.222 & 2.234 & 0.555\end{bmatrix}^\intercal.$$

As this is the only equilibrium of the cooperative system, all equilibria of the mixed-weight system are upper bounded by this equilibrium. There is one equilibrium to the mixed-weight system, namely:

$$\begin{bmatrix}4.835 & 4.849 & 2.641 & 1.581 & 2.232 & 0.555\end{bmatrix}^\intercal.$$

Every element of this equilibrium is indeed upper bounded by the matching element of the equilibrium of the cooperative system.

5 Maximum Equilibria of Cooperative Systems

In this section, we will show the concept of maximum equilibria, which only applies to cooperative systems. Maximum equilibria are equilibria that are element-wise greater than or equal to other equilibria of cooperative systems. These maximum equilibria allow us to group the equilibria of the cooperative network. Consequently, there exists an equilibrium that all the equilibria of the mixed-weight network are upper bounded by.

Before introducing the concept of maximum equilibria, we consider an alternative way to group the equilibria, based on the nodes which have a non-vanishing value. For this, we adopt the common terminology of linear algebra, for example, as used in *Oriented Matroids* by Richter-Gebert and Ziegler [8].

Definition 3. *The support of a vector* $\mathbf{v}$ *is defined as:*

$$\mathit{supp}(\mathbf{v}) := \{i : v_i \neq 0\}\,.$$

Example 1. The support of the vector:

$$\begin{bmatrix}3 & 1 & 0.5 & 0 & 2\end{bmatrix}^{\intercal}$$

is $\{1, 2, 3, 5\}$, 4 is not included since this element is zero.

Consider equilibria that all have the same support. One of these equilibria could be a **maximum equilibrium**. A maximum equilibrium is an equilibrium that has entries that are the maximum of all equilibria on the support.

Definition 4. *For a cooperative network with* N *nodes that meet Assumptions 1, a* ***maximum equilibrium*** $\mathbf{u}$ *is an equilibrium of the cooperative network, if for each equilibrium* $\mathbf{v}$*, with the support of* $\mathbf{v}$ *equal to* $\mathbf{u}$*, the elements of the maximum equilibrium* $\mathbf{u}$ *are higher than or equal to the matching entries of* $\mathbf{v}$*:*

$$\forall i \in \{1, \ldots, N\}\,, \quad u_i \geq v_i.$$

We define the ***order*** *of a* ***maximum equilibrium*** *as:*

$$\mathit{ord}(\mathbf{u}) = \#\mathit{supp}(\mathbf{u}).$$

Remark 7. There is at most one maximum equilibrium on a given support.

Example 2. Consider a cooperative system that has five non-zero equilibria:

$$\mathbf{v}_1 = \begin{bmatrix}1\\2\\0\\4\\3\\5\end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix}2\\3\\0\\5\\4\\5\end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix}0\\4\\0\\6\\2\\1\end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix}0\\2\\0\\4\\0\\1\end{bmatrix}, \quad \mathbf{v}_5 = \begin{bmatrix}0\\1\\0\\3\\0\\5\end{bmatrix}$$

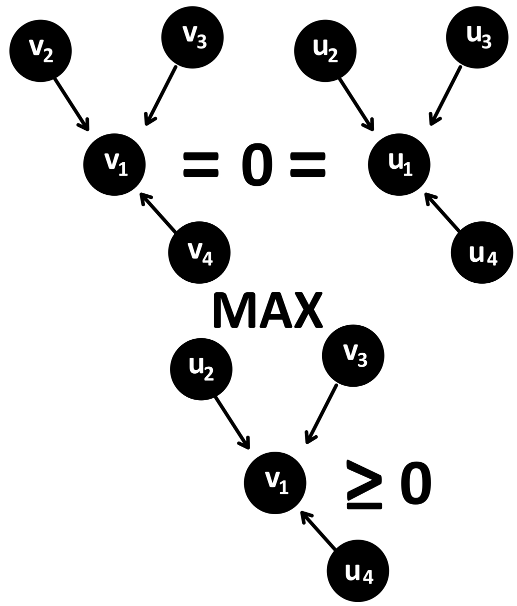

Figure 14: The dynamic equation at equilibria, $\mathbf{v}$ and $\mathbf{u}$, is zero. If we instead consider the dynamic equation, where we take each node's value as the maximum of the corresponding entry in $\mathbf{v}$ and $\mathbf{u}$, the dynamic equation is non-negative.

We have two maximum equilibria: $\mathbf{v}_2$ and $\mathbf{v}_3$. $\mathbf{v}_1$ is not a maximum equilibrium, as $\mathbf{v}_2$ has the same support and at least one entry higher than the corresponding entry in $\mathbf{v}_1$. $\mathbf{v}_3$ is a maximum equilibrium since it is the only equilibrium on its support. $\mathbf{v}_4$ and $\mathbf{v}_5$ are not maximum equilibrium, since both contain an entry higher, that is, higher than the corresponding entry in the other equilibrium.

We will work towards showing that if a system has a non-zero equilibrium, the system also has a non-zero maximum equilibrium [6]. We start by obtaining a new lower vector for the SCCS (see Theorem 4). Suppose that we have multiple equilibria, for example $\mathbf{v}$ and $\mathbf{u}$. The dynamic function for each node at both equilibria is zero. What sign will the dynamic function of each node be if we set the value of the nodes as the maximum of the corresponding entry in $\mathbf{v}$ and $\mathbf{u}$? Take an arbitrary node i of the system, of which the maximum of both equilibria is v_i. If the neighbours have values matching $\mathbf{v}$, node i is at equilibrium. If one of the neighbours, node j, has a value of u_j, then this value must be higher than v_j. Since the interaction function increases in its second argument (see A. 5), it follows that:

$$G\left(v_i, v_j\right) \leq G\left(v_i, u_j\right).$$

[6] the zero equilibrium, which has an empty support, is also a maximum equilibrium, since it is the only equilibrium on its support

Therefore, the dynamic function of this node is non-negative. The same argument can be made for nodes that take the value of the corresponding entry in $\mathbf{u}$. This is also visually shown in Figure 14. We formalise the above in the following lemma.

Lemma 3. *Suppose we have a cooperative system with N nodes that meets Assumptions 1. Let $\mathbf{v}_1, \ldots, \mathbf{v}_M$ be some of the equilibria of the system. We define the vector $\mathbf{v}_{max}$ as taking the element-wise maximum of each of the equilibria:*

$$\mathbf{v}_{max} := \begin{bmatrix} \max\{[\mathbf{v}_1]_1, \ldots, [\mathbf{v}_M]_1\} \\ \max\{[\mathbf{v}_1]_2, \ldots, [\mathbf{v}_M]_2\} \\ \vdots \\ \max\{[\mathbf{v}_1]_N, \ldots, [\mathbf{v}_M]_N\} \end{bmatrix}$$

The vector $\mathbf{v}_{max}$ is an appropriate lower vector in SCCS (see Theorem 4), since for each i, the dynamic function at $\mathbf{v}_{max}$ is:

$$F([\mathbf{v}_{max}]_i) + \sum_{j \in S_i^+} w_{ij} G([\mathbf{v}_{max}]_i, [\mathbf{v}_{max}]_j) \geq 0.$$

Proof. Consider an arbitrary node i. There is an equilibrium $\mathbf{v}_k$, such that for the i-th element, the value $[\mathbf{v}_k]_i$ is the highest over all equilibria:

$$[\mathbf{v}_k]_i = \max\{[\mathbf{v}_1]_i, [\mathbf{v}_2]_i, \ldots, [\mathbf{v}_M]_i\}.$$

To deduce the sign of the dynamic function of node i at $\mathbf{v}_{\max}$ we start at equilibrium $\mathbf{v}_k$:

$$\begin{aligned} 0 &= F([\mathbf{v}_k]_i) + \sum_{j \in S_i^+} w_{ij}\, G([\mathbf{v}_k]_i, [\mathbf{v}_k]_j) \\ &\leq F([\mathbf{v}_k]_i) + \sum_{j \in S_i^+} w_{ij}\, G([\mathbf{v}_k]_i, [\mathbf{v}_{\max}]_j) \\ &= F([\mathbf{v}_{\max}]_i) + \sum_{j \in S_i^+} w_{ij}\, G([\mathbf{v}_{\max}]_i, [\mathbf{v}_{\max}]_j). \end{aligned}$$

□

We combine this lower vector with the SCCS (see Theorem 4). If $\mathbf{v}_{\max}$ is not equal to one of the equilibria used, another equilibrium exists [7]. In the following, we show how this implies that if there is at least one non-zero equilibrium, there must be a non-zero maximum equilibrium.

[7] The bounds on which the equilibrium exists in the SCCC (see Theorem 4) are closed. If $\mathbf{v}_{\max}$ is equal to one of the equilibria, the lower vector itself could be the only equilibrium in the N-dimensional box.

Theorem 5. ***maximum equilibrium with a larger support***
Suppose we have a cooperative system that satisfies Assumptions 1, and let $\mathbf{v}$ *be an equilibrium of the system. Then there exists a maximum equilibrium* $\mathbf{a}$*, with a support that satisfies:*

$$supp(\mathbf{v}) \subseteq supp(\mathbf{a}).$$

Proof. We can distinguish between two cases: either $\mathbf{v}$ is a maximum equilibrium, at which point we would be done. In the other case, consider all the equilibria $\mathbf{u}_1 \dots \mathbf{u}_M$, which have the same support as $\mathbf{v}$. Either one of these equilibria is the maximum equilibrium on the support, or none of them. We consider the case in which neither of them has the maximum equilibrium. Construct the vector $\mathbf{v}_{\max}$, as defined in Lemma 3, from the equilibria $\mathbf{u}_1 \dots \mathbf{u}_M$ and $\mathbf{v}$. Since this vector is a lower vector in Theorem 4, it implies the existence of an equilibrium between $\mathbf{v}_{\max}$ and $\mathbf{x}_{\text{upper}}$. This unknown equilibrium $\mathbf{z}$, must have a support that is strictly larger than that of $\mathbf{v}$, since we assumed that $\mathbf{v}$ and $\mathbf{u}_1 \dots \mathbf{u}_M$ are all the equilibria on the support, but none of them are equal to $\mathbf{v}_{\max}$. For the new equilibrium $\mathbf{z}$, we could apply the same argument as for $\mathbf{v}$. If we keep repeating this argument, we either find a maximum equilibrium, at which point we could stop, or we find an equilibrium that has a support that contains every node. The support can no longer increase; therefore, it must be the case that $\mathbf{v}_{\max}$ on this support does not produce a new equilibrium. This can only be the case if $\mathbf{v}_{\max}$ itself is equal to one of the equilibria, which must be a maximum equilibrium. □

Remark 8. Since $\mathbf{v}_{\max}$ is a lower vector, there must always be an equilibrium $\mathbf{u}$ between $\mathbf{v}_{\max}$ and $\mathbf{x}_{\text{upper}}$. Therefore, instead of considering one equilibrium $\mathbf{v}$ in the above theorem, we could create $\mathbf{v}_{\max}$ of multiple equilibria $\mathbf{v}_1 \dots, \mathbf{v}_M$, which must result in a new unknown equilibrium $\mathbf{u}$. This $\mathbf{u}$ could be used in the above theorem, which results in a maximum equilibrium $\mathbf{z}$, which must satisfy:

$$\bigcup_{i=1}^{M} \text{supp}(\mathbf{v}_i) \subseteq \text{supp}(\mathbf{u}) \subseteq \text{supp}(\mathbf{z}).$$

Example 3. Before we move on to the next subsection, we will give a brief example. In Appendix Example F.2 we considered the following cooperative system:

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix},$$

$$F(x_i) = -4x_i + 4x_i^2 - 4x_i^3, \quad G(x_i, x_j) = 3x_i x_j.$$

Of all equilibria, two are given by:

$$\begin{bmatrix} 1.058 \\ 0 \\ 1.058 \\ 1.415 \end{bmatrix}, \quad \begin{bmatrix} 1.057 \\ 0.707 \\ 0 \\ 0.707 \end{bmatrix}.$$

It follows from Theorem 5 that there should exist a maximum equilibrium which has order 4; an equilibrium with non-vanishing entries. We also found the (maximum) equilibrium:

$$\begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix},$$

which is the maximum equilibrium on the given support.

5.1 Separable Systems

We have shown that a non-zero maximum equilibrium exists if the cooperative system has a non-zero equilibrium. Are there systems that have multiple maximum equilibria, and are there systems that only have one (excluding the zero equilibrium). We first work out a set of networks that could have multiple. Networks that have functions $G(x_i, x_j)$, that are also zero if the first variable is zero, are such networks.

Definition 5. *A network is* ***separable****, if:*

$$\forall x \in \mathbb{R}_{\geq 0}, \quad G(0, x) = 0.$$

Remark 9. If the function $G(x_i, x_j)$ is separable, i.e., it can be written in the form $h(x_i)l(x_j)$ and $h(0) = 0$, the network is also separable. By A. 4 it is already implied that $l(0) = 0$.

Example 4. A system with the function $G_1(x_i, x_j)= \frac{x_j^2 x_i}{x_i x_j + 1}$ is separable, whereas a system with the function $G_2(x_i, x_j)= \frac{x_j^2}{x_i x_j + 1}$ is not. Both functions $G_1(x_i, x_j)$ and $G_2(x_i, x_j)$ are not separable.

The intuition behind a separable network lies in its name. Instead of analysing the whole network, we consider subnetworks by removing nodes. The equilibria of the subnetworks are also equilibria for the original network, where the deleted nodes are set to zero. The key detail is that we set the nodes not included to zero. To clarify why such a node does not disassemble the equilibrium, all interactions of a node in a separable network, where the node has a quantity of zero, will also be zero. Therefore, this node is not influenced by the values of its neighbours. We formalise the above in the following lemma.

Lemma 4. *Consider a separable system that satisfies Assumptions 1. If a subnetwork has an equilibrium, then this is also an equilibrium of the full network if the removed nodes are set to zero.*

Proof. The proof is presented in Appendix Proof D.1. □

We have previously used $\mathbf{x}_{\text{upper}}$ as the higher vector in the SCCS (see Theorem 4), which, although always correct, does not give good constraints on the equilibrium between the lower and higher vectors; the support lies between the support of the lower vector and all nodes. For separable networks, we can modify $\mathbf{x}_{\text{upper}}$ to give us precise control of the support on the resulting equilibrium.

Lemma 5. *Suppose we have separable cooperative system with N nodes that satisfy Assumptions 1, and an arbitrary subset $A \subseteq \{1, \dots, N\}$. We define the partial upper bound vector by:*

$$\mathbf{x}_{upper}^{partial}(A) := (x_1, \dots, x_N), \quad \text{where } x_i = \begin{cases} x_{\text{upper}} & \text{if } i \in A \\ 0 & \text{otherwise} \end{cases}.$$

$\mathbf{x}_{upper}^{partial}(A)$ an appropriate higher vector for the SCCS (see Theorem 4).

Proof. A complete proof is available in Appendix Proof D.2. □

With this new upper vector (which is only applicable for separable cooperative systems), we can do something similar to Theorem 5, except that the maximum equilibrium must have the same support as our starting equilibrium.

Corollary 6. ***maximum equilibrium of the same support***
Consider a separable cooperative system that satisfies Assumptions 1. If there exists an equilibrium $\mathbf{v}$, then there exists a maximum equilibrium $\mathbf{u}$ such that:

$$supp(\mathbf{u}) = supp(\mathbf{v}),$$

Proof. See Appendix Proof D.3 for a detailed proof. □

The maximum equilibria of separable systems are also the upper bounds for the equilibria of the mixed-weight system. This follows because we have previously found that the equilibria of the mixed-weight system are lower vectors, and we can adjust our higher vector for separable systems to our needs.

Corollary 7. *Consider a separable system that satisfies Assumptions 1. An equilibrium $\mathbf{v}$ of the mixed-weight system is upper bounded by the maximum equilibrium $\mathbf{u}$ of the cooperative system, given that:*

$$supp(\mathbf{v}) = supp(\mathbf{u}).$$

Proof. From Lemma 2 we have that $\mathbf{v}$ is a lower vector in the SCCS (see Theorem 4). The higher vector can be taken as the partial upper bound vector in the set $\text{supp}(\mathbf{v})$, defined in Lemma 5. This implies that there exists an equilibrium $\mathbf{z}$ such that for all i:

$$v_i \leq z_i \leq x_{\text{upper}}^{\text{partial}}(\text{supp}[\mathbf{v}])_i.$$

Consequently, by Corollary 6 we have a maximum equilibrium on this support, which proves the statement. □

Example 5. In Appendix Example F.2 we consider the following separable cooperative system:

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix},$$

$$F(x_i) = -4x_i + 4x_i^2 - 4x_i^3, \quad G(x_i, x_j) = 3x_i x_j.$$

In which we find the equilibria:

$$\begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0.707 \\ 1.057 \\ 0.707 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1.058 \\ 1.415 \\ 1.058 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0.707 \\ 1.057 \\ 0.707 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1.058 \\ 1.415 \\ 1.058 \end{bmatrix},$$

$$\begin{bmatrix} 0.707 \\ 0 \\ 0.707 \\ 1.057 \end{bmatrix}, \quad \begin{bmatrix} 1.058 \\ 0 \\ 1.058 \\ 1.415 \end{bmatrix}, \quad \begin{bmatrix} 1.057 \\ 0.707 \\ 0 \\ 0.707 \end{bmatrix}, \quad \begin{bmatrix} 1.415 \\ 1.058 \\ 0 \\ 1.058 \end{bmatrix}.$$

We may conclude from Corollary 6 that on each support, $\{1,2,3\}$, $\{1,2,4\}$, $\{1,3,4\}$, $\{2,3,4\}$, $\{1,2,3,4\}$ and $\{\emptyset\}$, we have a maximum equilibrium, which in this case, in order are:

$$\begin{bmatrix} 1.058 \\ 1.415 \\ 1.058 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1.415 \\ 1.058 \\ 0 \\ 1.058 \end{bmatrix}, \quad \begin{bmatrix} 1.058 \\ 0 \\ 1.058 \\ 1.415 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1.058 \\ 1.415 \\ 1.058 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Separable cooperative systems have conditions that allow multiple maximum equilibria. This condition is that a node that has a quantity of zero is always in equilibrium. We could instead consider systems that are the opposite of this. As a system with functions such that a node with a value of zero can never be at equilibrium if one of its neighbours has a non-zero quantity.

Corollary 8. *Consider a cooperative system with N nodes that satisfies Assumptions 1. Suppose that the system is not separable, the associated matrix is strongly connected and that the function $G(x_i, x_j)$ satisfies:*

$$\forall x \in \mathbb{R}_{>0}, \quad G(0, x) > 0.$$

Then every non-zero equilibrium $\mathbf{v}$ *has a support that contains every node.*

Proof. The proof can be found in Appendix Proof D.4. □

In the upcoming section 6 we consider a system for which Corollary 8 is applicable.

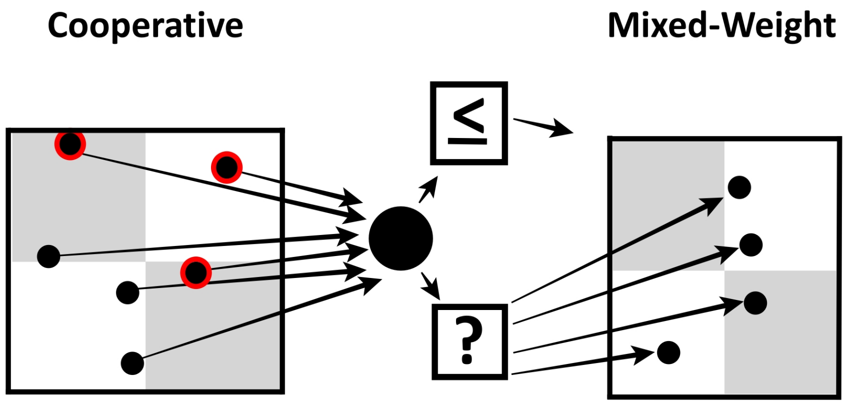

Figure 15: A conceptual illustration of the knowledge obtained. We can group the the equilibria in regions that have the same support. There is at most one maximum equilibria on a given support. The maximum equilibria are the equilibria with a red circle.

5.2 Principal equilibria

Something special happens if we apply the knowledge we have obtained of $\mathbf{v}_{\max}$, to all the equilibria of a cooperative system $\mathbf{v}_1, \ldots, \mathbf{v}_M$. If we create $\mathbf{v}_{\max}$ from the equilibria, we get a lower vector for the SCCS (Theorem 4). Therefore, there must be an equilibrium between $\mathbf{v}_{\max}$ and $\mathbf{x}_{\text{upper}}$. However, since we have already assumed that we know all the equilibria, it can only be that $\mathbf{v}_{\max}$ is itself an equilibrium. This implies that there is one equilibrium in $\mathbf{v}_1, \ldots, \mathbf{v}_M$ equal to $\mathbf{v}_{\max}$. This equilibrium is therefore not only maximal on its support but maximal over all equilibria. This equilibrium we call the **principal equilibrium**.

Definition 6. *Consider a cooperative system that satisfies Assumptions 1. The* ***principal equilibrium*** *is the equilibrium equal to* $\mathbf{v}_{max}$*, where* $\mathbf{v}_{max}$ *is calculated from all the equilibria of a cooperative system.*

Remark 10. The principal equilibrium is also a maximum equilibrium.

There are some direct consequences for the principal equilibrium. One of them is that all its entries must be greater than or equal to the corresponding entries of any other equilibrium. This, in turn, implies that it must have the largest support.

Corollary 9. *Consider a cooperative system that satisfies Assumptions 1. For every equilibrium* $\mathbf{v}$*, the principal equilibrium* $\mathbf{u}$ *is element-wise larger than or* *equal to* $\mathbf{v}$*. That is:*

$$\forall i \in \{1, \ldots, N\}, \quad u_i \geq v_i.$$

Proof. Suppose that $\mathbf{v}_1, \ldots, \mathbf{v}_M$ and $\mathbf{u}$ are all the equilibria of the system, with $\mathbf{u}$ the principal equilibrium. For an arbitrary entry i we have:

$$u_i = \max\{u_i, [\mathbf{v}_1]_i, \ldots, [\mathbf{v}_M]_i\}.$$

□

Corollary 10. *Consider a cooperative system that satisfies Assumptions 1. For an arbitrary equilibrium* $\mathbf{v}$ *and the principal equilibrium* $\mathbf{u}$ *of a cooperative system, we must have:*

$$supp(\mathbf{u}) \subseteq supp(\mathbf{z}).$$

Proof. Consider an element $i \in \text{supp}(\mathbf{v})$. By Corollary 9, $u_i \geq v_i$, from which it follows that $i \in \text{supp}(\mathbf{u})$. □

We can link the above two corollaries to the definition of participatory and partially redundant systems (see Definition 2).

Corollary 11. *Suppose we have a cooperative system with* N *nodes that satisfies Assumptions 1. The following is equivalent:*

1. *The order of the principal equilibrium is equal to* N
2. *the system is participatory*

Proof. See Appendix Proof D.5 for the full proof. □

Corollary 12. *Suppose we have a cooperative system with* N *nodes that satisfies Assumptions 1. The following is equivalent:*

1. *The order of the principal equilibrium is smaller than* N
2. *The system is partially redundant.*

Proof. It follows immediately from taking the negation of the above statements and seeing that this is equal to the two statements in Corollary 11. □

Example 6. In Example 5 we found the maximum equilibria:

$$\begin{bmatrix}1.058\\1.415\\1.058\\0\end{bmatrix}, \quad \begin{bmatrix}1.415\\1.058\\0\\1.058\end{bmatrix}, \quad \begin{bmatrix}1.058\\0\\1.058\\1.415\end{bmatrix}, \quad \begin{bmatrix}0\\1.058\\1.415\\1.058\end{bmatrix}, \quad \begin{bmatrix}2\\2\\2\\2\end{bmatrix}, \quad \begin{bmatrix}0\\0\\0\\0\end{bmatrix}.$$

There is only one equilibrium of this that is the principal equilibrium:

$$\begin{bmatrix}2\\2\\2\\2\end{bmatrix}.$$

The implications of Corollary 9 and Corollary 10 can also be seen in the examples shown in the appendix, namely Appendix Example F.1, Appendix Example F.2, Appendix Example F.5 and Appendix Example F.6.

In Corollary 5, we have shown that an equilibrium of the mixed-weight system is upper bounded by an equilibrium of the cooperative system. However, we did not show which equilibrium in particular. Fortunately, there is no such need, as all the equilibria of the cooperative system are upper bounded by the principal equilibrium.

Corollary 13. *Consider a system that satisfies Assumptions 1. Every equilibrium* $\mathbf{v}$ *of the mixed-weight system is element-wise upper bounded by the principal equilibrium* $\mathbf{u}$ *of the corresponding cooperative system.*

Proof. In Corollary 1, we have proved that every equilibrium $\mathbf{v}$ of the mixed-weight system is upper bounded by an equilibrium of the cooperative system. Since the equilibria of the cooperative system are upper bounded by the principal equilibrium (see Corollary 9), the statement immediately follows. $\square$

Let us now summarise the above theorems and corollaries and relate them to the connection between the equilibria of the cooperative and mixed-weight systems. First, we have seen that there always exists a principal equilibrium, which is element-wise greater than or equal to all other equilibria. This also implies that the equilibria of the mixed-weight system are all element-wise smaller than or equal to the principal equilibrium. We have also seen that there are systems for which we can take subnetworks, of which the equilibria transfer over to the equilibria of the full network. The equilibria of the separable mixed-weight systems are upper bounded by the corresponding maximum equilibria of the cooperative system.

5.3 One Variable Principal Equilibrium Algorithm

We can find the principal equilibrium of a cooperative system by iteratively solving equations dependent on only one variable, which is much easier to solve directly for the equilibrium. Below is the pseudo-code of the algorithm:
Given: The system, and x_{upper}
Result: The principal maximum of the system.

```
Set v_i = x_upper for all i = 1, ..., N.
while at least one element of v changed in the last loop
  loop over i = 1, ..., N :
    calculate the supremum of all the positive values x such that:
      F(x) + sum_{j ∈ S_i^+} w_ij G(x, v_j) = 0
    setv_i equal to the supremum.
```

Set $\mathbf{v}_i = x_{\text{upper}}$ for all $i = 1, \dots, N$.

while at least one element of $\mathbf{v}$ changed in the last loop

  loop over $i = 1, \dots, N$:

    calculate the supremum of all the positive values x such that:

$$F(x) + \sum_{j \in S_i^+} w_{ij} G(x, v_j) = 0$$

    set$\mathbf{v}_i$ equal to the supremum.

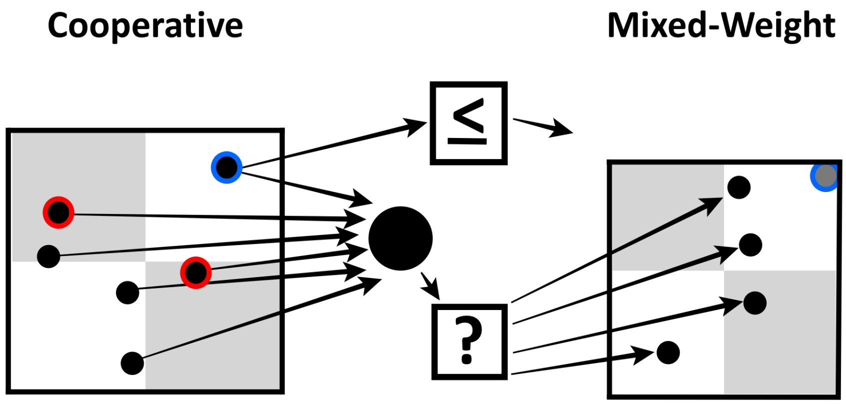

Figure 16: A conceptual illustration of the knowledge obtained. There is one equilibrium of the cooperative system that is element-wise the maximum of all the equilibria, which is called the principal equilibrium. The principal equilibrium also upper bounds the equilibria of the mixed-weight system. The principal equilibrium has a blue circle.

Although we will not put our main focus on the science behind the algorithm, we will give an intuitive explanation of why it works. We start by setting the quantity associated with each node at the maximum value x_{upper}. This maximum value for all nodes is a bit crude, which stems from two things: first, the maximum value is only obtained if all the neighbours also have a maximum value. Secondly, although not independent of the first argument, the maximum value is only obtained if the weighted in-degree of each node is equal to the maximal weighted in-degree. This is certainly true for one node, but depending on the network, it does not have to apply to all other nodes. From this, we may conclude that even if the neighbours of such a node have entries equal to x_{upper}, due to a lower weighted in-degree than the maximum weighted in-degree, this node will have a derivative smaller than zero. Therefore, we can conclude that the current value (x_{upper}) is too large for this node. We can approximate a better upper bound for this node by recalculating the maximum value for which this node has an equilibrium, given that the neighbours have a value x_{upper}. Consequently, we find a new upper bound for this node, smaller than the previous upper bound. We can do this for all nodes with a smaller weighted interaction than the maximum weighted interaction. This will give a better upper bound for each node. But by doing this, we disregarded the assumption that the neighbouring nodes have entries equal to x_{upper}, as we have calculated a better upper bound for the neighbours. Therefore, we can recalculate the upper bound of the quantity of each node with this new information. We can keep repeating this process until the entries no longer change, but what does this imply? This implies that for the current entries, the derivatives of the quantities of the nodes are equal to zero; an equilibrium. Therefore, instead of directly trying to solve the equation governing the dynamics of the complex network, which is multivariable, we iteratively solve equations that depend on only one variable to get closer and closer to the equilibrium.

5.4 Stability of the Maximum Equilibria

The stability of the equilibria of both systems is not the main focus of this thesis; however, it is also, in general, an important piece of information. Hence, in the next section, we briefly provide some information on the stability of different systems.

Theorem 6. *Suppose we have a separable system that satisfies Assumptions 1. Assume that the function $F(x_i)$ is continuously differentiable near $x_i = 0$, and that the function $G(x_i, x_j)$ is twice continuously differentiable near $(x_i, x_j) = (0, 0)$. If*

$$\left.\frac{dF(x_i)}{dx_i}\right|_{x_i=0} < 0,$$

then the zero equilibrium is stable.

Proof. A complete proof is provided in Appendix Proof D.6. The general idea behind the proof is that the Taylor expansion of $G(x_i, x_j)$ around $\mathbf{0}$ will not result in a constant term nor in x_i and x_j, but always in terms of higher order. Therefore, each diagonal element in the Jacobian is dependent on x_i or x_j, resulting in zero at $\mathbf{0}$. The diagonal elements are all negative, which implies that all the eigenvalues are negative. □

Proving the stability of equilibria is in itself a whole activity. Different methods can be used to estimate Jacobian eigenvalues, for example, Gershgorin, but Gershgorin does not always provide conclusive information, as illustrated in Appendix Example F.5.

In Appendix Example F.6, we calculate the equilibria of a very simple system, dependent on a parameter. The stability of the equilibria changes for different values of the parameter. From this example, we also get insight into why some systems do and do not have non-zero equilibria. If a system has non-zero equilibria, and we adjust the parameter, which results in a harsher environment, we reach a tipping point. The tipping point occurs due to bifurcation of the non-zero equilibria.

6 Example: SIS Epidemic Process

In this section, we present a simple example of an SIS epidemic system described in Wu *et al.* [2], which focusses not only on the equilibria, but also on interpreting the various components and results. Given its relevance, we use COVID-19 as a framework to clarify the system.

The nodes of the network are hosts of the disease. The functions of the SIS epidemic process are:

$$F(x_i) = -x_i, \quad G(x_i, x_j) = \beta(1 - x_i)x_j.$$

This system does not meet A. 5, since the function $G(x_i, x_j)$ decreases in x_j if $x_i > 1$. One might think that, as a result, all the above theorems are no longer applicable. Fortunately, this will not be a problem for a cooperative network, as a value above 1 cannot correspond to an equilibrium, nor does it make sense in the system. The quantitative value of each node corresponds to the probability of infection, while the node itself is the host [2]. We see that the organisms themselves fight the disease, represented by $F(x_i)$, which is linear, independent of any parameters. This might seem implausible at first, as one would expect that certain infections are easier to combat. Instead of having an independent parameter for this, we incorporate it into the parameter β. This implies that β is the effectiveness of transmission between hosts, which incorporates effects such as:

1. The method of transmission. One way COVID-19 spreads is in small liquid particles, according to the World Health Organization [9].
2. If a host shows symptoms while carrying the disease. For COVID-19, an infected person can spread the disease without symptoms, making the effectiveness of transmission greater, according to the World Health Orginization [10].

Transmission is modelled through the function $G(x_i, x_j)$. If a host itself has a higher probability of being infected, the influence decreases, reaching zero if the probability is one. Another parameter, similar to β but more personalised to each host, is the weight of the interaction with other hosts. People who live close together are more likely to get infected. However, acquiring a vaccine could lead to a weaker interaction with neighbouring hosts. In addition, wearing a face mask makes one less susceptible to infection. In this thesis, we have always assumed that both the parameters and the weight of the interaction have a constant value over time. Although this simplifies the analysis, it does not fully reflect reality. Different mutations of COVID-19 disease are alpha, beta, and gamma variants, all of which have different transmissibility, which is correlated with our β [11]. People also do not have a fixed distance from one another but move over time, which also makes the weight of the interaction dependent on time. For simplicity, we will consider three cases with constant parameters. Each case represents a different stage of an epidemic. We will do this for a small network, which contradicts a "usual" epidemic, but the results still give us relevant insights.

6.1 Stage 1: Early Infection

For the system we consider below, we will first employ different methods to describe the equilibria, after which we give the numerically calculated equilibrium. We will compare how the numerically calculated equilibrium changes over the three stages. Suppose that we have three people, two people close to each other, and a third who interacts weakly with both. This could result in the following adjacency matrix:

$$\begin{bmatrix} 0 & 3 & 0.3 \\ 2 & 0 & 0.1 \\ 0.2 & 0.1 & 0 \end{bmatrix},$$

with a β value of $\frac{1}{2}$. Rather than directly providing the equilibrium of this network, we will first present some implications derived from the above theorems and corollaries. The upper bound of this network can be calculated by solving:

$$\begin{aligned} -x + \beta \cdot W_{\max}(1-x)x &= 0 \\ x = 0 \quad \text{or} \quad x = 1 - \frac{1}{\beta \cdot W_{\max}}&, \end{aligned}$$

which, for this system, results in an upper bound of 0.394. This implies that even the chance of being sick is slim. Since this system satisfies the requirements of Corollary 8, any non-zero equilibrium must have non-vanishing entries. To approximate the equilibrium, we will use $k_{\max}$, which for this network is 2. The value of $k_{\max}$ equal to 2 corresponds to the subnetwork of node 1 and node 2. To find the lower vector in Theorem 4, we solve the equation:

$$\begin{aligned} -x + \beta \cdot k_{\max}(1-x)x &= 0 \\ x = 0 \quad \text{or} \quad x = 1 - \frac{1}{\beta \cdot k_{\max}}&. \end{aligned}$$

For this stage, it results in $x = 0$, which implies that we cannot obtain a useful approximation from this method. The non-zero equilibrium for this network is:

$$\begin{bmatrix} 0.229 \\ 0.192 \\ 0.058 \end{bmatrix}.$$

It might not come as a surprise that the last entry is significantly smaller than the other two, as the corresponding node has a weak interaction with the other two nodes. In general, someone will become less sick from other people if they have less contact with others.

6.2 Stage 2: Widespread Outbreak

Suppose that we have the same system as above, but now β is increased to 2. This will result in a higher probability of disease for each node. We can again calculate the upper bound, which is now approximately 0.848. This does not mean that the equilibrium is close to this; it simply provides an upper bound. Due to the change in the β value, in this stage, the lower vector obtained from $k_{\max}$ provides useful information. The lower vector is:

$$\begin{bmatrix} 0.75 \\ 0.75 \\ 0 \end{bmatrix}.$$

We observe that the new equilibrium for the first two entries is strictly larger than that in the previous stage. The equilibrium with β equal to 2 is:

$$\begin{bmatrix} 0.828 \\ 0.772 \\ 0.327 \end{bmatrix}.$$

We see that indeed the first two entries are above 0.75, and all entries are below the upper bound of 0.848. The last entry, which was previously well below the other entries, is now much closer. It does not matter whether you have very little contact with other people if the disease is highly contagious. This equilibrium, compared to the previous equilibrium, is higher, as expected.

6.3 Stage 3: Intervention

Suppose that β remains 2, but person 1 and person 2 receive a vaccination that makes them less likely to become infected. This could result in the following adjacency matrix:

$$\begin{bmatrix} 0 & 1 & 0.1 \\ 0.6 & 0 & 0.1 \\ 0.2 & 0.1 & 0 \end{bmatrix}.$$

This results in a lower upper bound, namely 0.545. Compared to the previous equilibrium, vaccination of these two people will result in a lower probability of infection. A lower vector for this network is:

$$\begin{bmatrix} 0.167 \\ 0.167 \\ 0 \end{bmatrix},$$

which is also more promising than the previous lower vector. However, we do see that the difference between the lower and higher vectors is larger than in case 2. If we were to only employ these reduction methods, the range of the first and second nodes is 0.378. This is, compared to the maximum probability of 1, inaccurate. The equilibrium of this network is:

$$\begin{bmatrix} 0.411 \\ 0.336 \\ 0.130 \end{bmatrix},$$

which is still worse than in stage 1, but better than in stage 2. To improve it even further, we must hope that the value β decreases with time due to mutations, or as a result of widespread measures such as handwashing, wearing masks, and maintaining a distance of 1.5-meter.

7 From Cooperative to Mixed-Weight

Finally, we illustrate that it is not straightforward to deduce the equilibria of a mixed-weight system based on the equilibria of the corresponding cooperative system, as the outcome can strongly depend on the magnitude of the negative interactions. However, we can state constraints on the equilibria of the mixed-weight system and give conditions, which state if an equilibrium of the cooperative system is also an equilibrium of the mixed-weight system.

How the equilibria of the cooperative system change due to the added negative interaction is closely related to when a system collapses. How and when a system collapses is part of bifurcation theory. This branch of mathematics studies how changes in parameters influence equilibria and their stability. For simple small systems, we can concretely describe how a change in one of the parameters influences the equilibria, but for the systems we consider, which could have more than 1000 nodes and interact non-linearly, it is much harder, if not impossible. We hope to illustrate this with an example of how a change in the parameters influences the equilibria. Consider the subnetwork of the network used for the system in Appendix Example F.2, with a small modification:

$$\begin{bmatrix} 0 & 1 & -c \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

$$F(x_i) = -4x_i + 4x_i^2 - px_i^3, \quad G(x_i, x_j) = 3x_ix_j.$$

The addition of c could make the system mixed-weight. We find for $p = 4$ and $c = 0$, the two non-zero equilibria are:

$$\begin{bmatrix} 0.707 \\ 1.057 \\ 0.707 \end{bmatrix}, \quad \begin{bmatrix} 1.058 \\ 1.415 \\ 1.058 \end{bmatrix}.$$

We might now wonder what would happen if we increased p or c. In both cases, we get the saddle-node bifurcation, which is visualised in Figure 17.

There always exist parameter values at which the system is at the boundary of losing non-zero equilibria. In this example, the tipping point occurs around $p = 4.12$ for the parameter p and around $c = 0.12$ for the parameter c. To my best knowledge, no theorem exists that characterises when the tipping point occurs for the systems we consider. If such a theorem exists, it would provide information not only on how the equilibria move with a change in the parameter c, but also on when the cooperative system survives.

7.1 Leeway in the Equilibria

Fortunately, we can still impose conditions on the equilibria of the mixed-weight. This will result in equilibria of the cooperative system, which, independent of the negative interaction, extend to equilibria of the mixed-weight network.

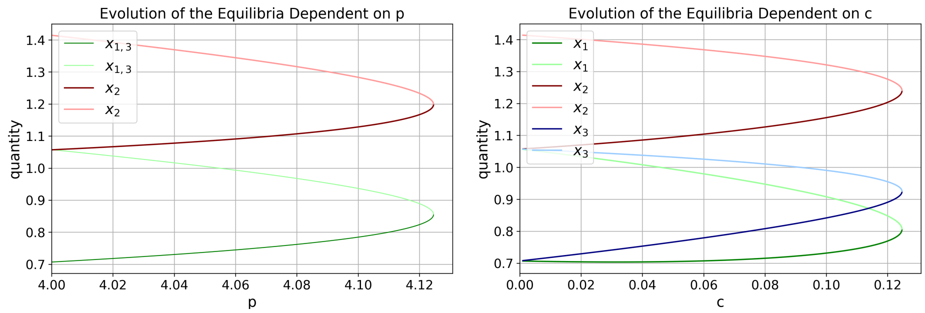

Figure 17: The evolution of the quantities of the nodes of the above system. On the left, p is variable and $c = 0$, and on the right, c is variable and $p = 4$. The two equilibria collide around $p = 4.12$ on the left, around $c = 0.12$ on the right, after which no equilibrium exists for both. The collision of the two equilibria of this system is also referred to as a saddle-node bifurcation.

Theorem 7. *Consider a cooperative system that satisfies Assumptions 1. Let* $\mathbf{v}$ *be the principal equilibrium. We define the* ***leeway*** *of node* i*:*

$$\Delta_i(\mathbf{v}) := \sup_{x \in [0, v_i]} \left[F(x) + \sum_{j \in S_i^+} w_{ij} G(x, v_j) \right].$$

For every equilibrium $\mathbf{z}$ *of the mixed-weight system, the negative interaction received by node* i *must be smaller than or equal to the leeway of node* i*:*

$$-\sum_{j \in S_i^-} w_{ij} G(z_i, z_j) \leq \Delta_i(\mathbf{v}).$$

Proof. We will prove the statement with a contradiction; therefore, suppose that there exists an equilibrium $\mathbf{z}$, such that:

$$-\sum_{j \in S_i^-} w_{ij} G(z_i, z_j) > \Delta_i(\mathbf{v}).$$

The dynamic function of node a at $\mathbf{z}$ is:

$$\begin{aligned} \left.\frac{dx_i}{dt}\right|_{\mathbf{x}=\mathbf{z}} &= F(z_i) + \sum_{j \in S_i^+} w_{ij} G(z_i, z_j) + \sum_{j \in S_i^-} w_{ij} G(z_i, z_j) \\ &< F(z_i) + \sum_{j \in S_i^+} w_{ij} G(z_i, z_j) - \Delta_i(\mathbf{v}) \\ &\leq F(z_i) + \sum_{j \in S_i^+} w_{ij} G(z_i, v_j) - \Delta_i(\mathbf{v}) \qquad \leq 0. \end{aligned}$$

The last inequality follows, since each equilibrium of the mixed-weight system is element-wise upper bounded by the principal equilibrium of the cooperative system (see Corollary 13). At equilibrium $\mathbf{z}$, the dynamic function of node i is negative, which is a contradiction. $\square$

Remark 11. If the leeway of a node i is zero, then for each equilibrium $\mathbf{z}$ of the mixed-weight network, all negative interactions going to node i must be zero.

Example 7. We consider the mixed-weight system we covered in subsection 4.4:

$$\begin{bmatrix} 0 & 4 & 2 & 0 & 2 & 0 \\ 2 & 0 & 0 & 4 & 3 & 0 \\ 5 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 4 & 0 \\ 3 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$F(x_i) = -\frac{3x_i}{2}, \quad G(x_i, x_j) = \frac{x_j^2}{1 + x_j^2}.$$

The principal equilibrium $\mathbf{v}$ is:

$$\begin{bmatrix} 4.896 & 5.164 & 3.200 & 2.222 & 2.234 & 0.555 \end{bmatrix}^\intercal.$$

The only equilibrium $\mathbf{z}$ to the (mixed-weight) system is:

$$\begin{bmatrix} 4.835 & 4.849 & 2.641 & 1.581 & 2.232 & 0.555 \end{bmatrix}^\intercal.$$

Since the function $G(x_i, x_j)$ is not dependent on x_i, the leeway

$$\Delta_i(\mathbf{v}) := \sup_{x \in [0, v_i]} \left[-\frac{3x}{2} + \sum_{j \in S_i^+} w_{ij} \frac{v_j^2}{1 + v_j^2} \right].$$

is always obtained at $x = 0$. Nodes 3 and 4 both have negative interactions. The leeway of node 3, $\Delta_3(\mathbf{v})$, is 4.800. This implies that $-w_{35}G(z_3, z_5) \leq 4.800$, which is the case as $-w_{35}G(z_3, z_5) = 0.833$. For node 4, the leeway, $\Delta_4(\mathbf{v})$, is 3.332. This implies that $-w_{42}G(z_4, z_2) \leq 3.332$, which is the case as $-w_{42}G(z_4, z_2) = 0.964$.

For separable systems, the maximum equilibrium on a support is similar to the principal equilibrium. This implies that we can extend the above theorem to separable systems.

Corollary 14. *Consider a separable cooperative system that satisfies Assumptions 1. Let A be an arbitrary subset of $\{1, \ldots, N\}$ which has an equilibrium $\mathbf{v}$ with:*

$$supp(\mathbf{v}) = A.$$

For every equilibrium $\mathbf{z}$ *of the mixed-weight system, the negative interaction received by node* i *must be smaller than or equal to the leeway of node* i*:*

$$-\sum_{j\in S_i^-} w_{ij} G(z_i, z_j) \leq \Delta_i(\mathbf{v}).$$

Proof. The proof can be found in Appendix Proof E.1. □

Unfortunately, more information on the conditions of the equilibria of the mixed-weight system requires a better understanding of how the negative interactions change the equilibria of the cooperative system.

7.2 The Same Equilibria

Finally, we give conditions that tell us if an equilibrium of the cooperative system is also an equilibrium of the mixed-weight system. This is rather easy, as we only have to consider whether a negative interaction influences the equilibrium equations.

Theorem 8. *Suppose we have a system that satisfies Assumptions 1. If there exists an equilibrium* $\mathbf{v}$ *of the cooperative network, such that the following is true:*

$$\forall\,(i,j) \in A^-, \quad G(v_i, v_j) = 0,$$

then $\mathbf{v}$ *is also an equilibrium of the mixed-weight network.*

Proof. The equilibrium equation of the mixed-weight system for a node i is:

$$\begin{aligned} \left.\frac{dx_i}{dt}\right|_{\mathbf{x}=\mathbf{v}} &= F(v_i) + \sum_{j\in S_i^+} w_{ij} G(v_i, v_j) + \sum_{j\in S_i^-} w_{ij} G(v_i, v_j) \\ &= F(v_i) + \sum_{j\in S_i^+} w_{ij} G(v_i, v_j) + 0 \qquad\qquad = 0. \end{aligned}$$

The sum of all negative interactions is zero, which is implied by our assumption. The last equation is equal to zero, since $\mathbf{v}$ is an equilibrium. □

Remark 12. If $j \neq \text{supp}(\mathbf{v})$, it follows that:

$$\forall i \in \{1, \ldots, N\}, \quad G(v_i, v_j) = 0,$$

which is a direct consequence of A. 3.

Remark 13. If the system is separable and $i \neq \text{supp}(\mathbf{v})$, we deduce that:

$$\forall j \in \{1, \ldots, N\}, \quad G(v_i, v_j) = 0.$$

This follows by the definition of a separable system (see Definition 5).

For the last time, update our visual to include the latest pieces of information, see Figure 18.

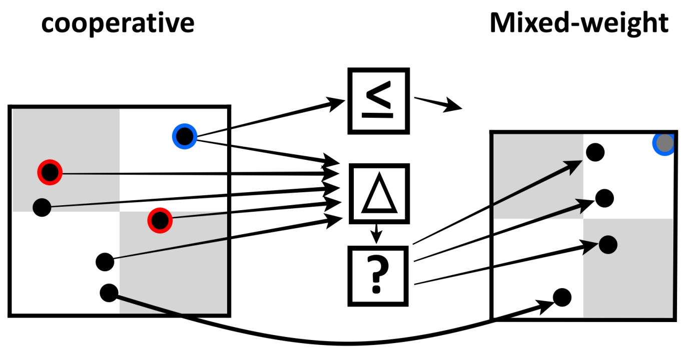

Figure 18: A conceptual illustration of the knowledge obtained. All the equilibria of the mixed-weight system must meet the conditions in Theorem 7. However, no direct translation between the cooperative equilibria and those of the mixed-weight equilibria has become apparent. Some of the equilibria of the cooperative system transfer over to the equilibria of the mixed-weight system.

7.3 Final Example

Finally, we consider an example to illustrate how the theorems and corollaries we have obtained can be applied. Consider the mixed-weight system:

$$\begin{bmatrix} 0 & 12 & 0 & 4 & -3 \\ 6 & 0 & 6 & 0 & 3 \\ 8 & 4 & 0 & -2 & 0 \\ 8 & -1 & 5 & 0 & 2 \\ -6 & 2 & 0 & 1 & 0 \end{bmatrix},$$

$$F(x_i) = -4x_i + 4x_i^2 - 16x_i^3, \quad G(x_i, x_j) = x_i x_j.$$

Although the system is still relatively small and could probably be solved analytically, doing so would contradict what we aim to demonstrate. We visualised the network in Figure 19. We will ignore the trivial zero-equilibrium. We first calculate the upper bound of the entire network. The upper bound x_{upper}, is the supremum over all real positive numbers x such that:

$$-4x + 4x^2 - 16x^3 + W_{\max} x^2 = 0$$

$$x = 0 \quad \text{or} \quad x = \frac{W_{\max} + 4 \pm \sqrt{(W_{\max} + 4)^2 - 256}}{32}.$$

For $W_{\max} = 16$, this implies $x_{\text{upper}} = 1$. Next, we calculate $W_{\min}$, which is the smallest value such that the function:

$$-4x + 4x^2 - 16x^3 + W_{\min} x,$$

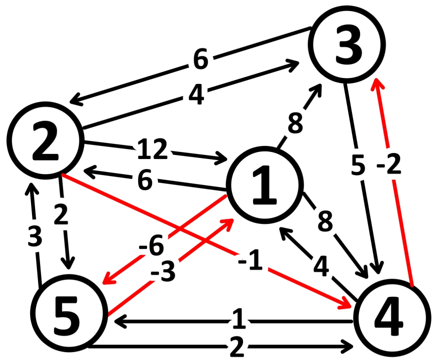

Figure 19: The full network of the system we consider in subsection 7.3.

still has positive roots. This results in:

$$16 + 64(W_{\min} - 4) = 0$$
$$W_{\min} = 3\frac{3}{4}.$$

Node 5 has a smaller positive weighted in-degree than $W_{\min}$, which implies that the quantity of node 5 is zero for all equilibria of positive and mixed-weight systems. This system is partially redundant; therefore, we consider the network:

$$\begin{bmatrix} 0 & 12 & 0 & 4 \\ 6 & 0 & 6 & 0 \\ 8 & 4 & 0 & -2 \\ 8 & -1 & 5 & 0 \end{bmatrix},$$

also shown in Figure 20. This network has the same x_{upper} and $W_{\min}$. Since

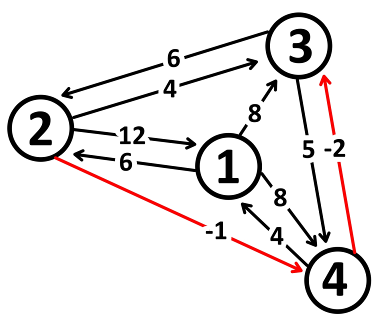

Figure 20: The subnetwork of the system we consider in subsection 7.3.

the system is separable, we could instead consider all the sub-matrices. For all sub-matrices with a $W_{\max} < 12$, there are no equilibria on the support, since:

$$\sqrt{(W_{\max} + 4)^2 - 256},$$

is negative for $W_{\max} < 12$. This implies that x_{upper} is zero. This leaves the sub-matrices:

$$A_{\{1,3,4\}} = \begin{bmatrix} 0 & 0 & 4 \\ 8 & 0 & -2 \\ 8 & 5 & 0 \end{bmatrix}, \quad A_{\{1,2,4\}} = \begin{bmatrix} 0 & 12 & 4 \\ 6 & 0 & 0 \\ 8 & -1 & 0 \end{bmatrix},$$
$$A_{\{1,2,3\}} = \begin{bmatrix} 0 & 12 & -1 \\ 6 & 0 & 6 \\ 8 & 4 & 0 \end{bmatrix}, \quad A_{\{1,2\}} = \begin{bmatrix} 0 & 12 \\ 6 & 0 \end{bmatrix}.$$

At this point, to even reduce the sub-matrices we consider, we calculate the spectral radius of each sub-matrix (only the non-negative entries), which serves a similar role as $W_{\max}$. However, the spectral radius is always smaller than or equal to $W_{\max}$, which we have shown in subsection 4.2. Therefore, instead of first eliminating sub-matrices with a $W_{\max} < 0$, directly eliminating the sub-matrices A with $p(A) < 12$ would have yielded fewer sub-matrices. Calculating the largest eigenvalues, on the other hand, requires more mathematics. The only sub-matrix with a spectral radius larger than or equal to 12 is $A_{\{1,2,3\}}$. We therefore conclude that all the non-zero equilibria either have a support of 1, 2, and 3 or 1, 2, 3, and 4.

$W_{\max}$ on the support of 1, 2, and 3 is 12, which results in $x_{\text{upper}} = 0.5$. Due to symmetry, if nodes 1, 2 and 3 all have the quantity x_{upper}, it is also an equilibrium:

$$\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 0 \end{bmatrix}.$$

Consequently, this is the maximum equilibrium on the support 1,2 and 3. This is also the only equilibrium on the support, which follows from a numerical analysis. This maximum equilibrium also meets the condition to be an equilibrium of the mixed-weight network (see Theorem 8).

Finally, we need to consider the principal equilibrium. The minimum positive weighted in-degree, $k_{\max}$, of the system is 12. This is obtained in the subnetwork with nodes 1, 2, 3 and 4. Solving for:

$$F(x) + G(x, x)k_{\max} = 0.$$

Therefore, the principal equilibrium is lower bounded by the first vector and upper bounded by the second vector:

$$\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

A numerical analysis tells us that the principal equilibrium is:

$$\begin{bmatrix} 0.826 \\ 0.713 \\ 0.722 \\ 0.760 \\ 0 \end{bmatrix}.$$

Every equilibrium of the mixed-weight system is upper bounded by this principal equilibrium. A numerical analysis confirms this, as the equilibria of the mixed-weight system, with the support of nodes 1, 2, 3 and 4 are:

$$\begin{bmatrix} 0.782 \\ 0.662 \\ 0.612 \\ 0.679 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0.449 \\ 0.330 \\ 0.288 \\ 0.369 \\ 0 \end{bmatrix}.$$

8 Conclusion

The analysis of equilibria of complex systems remains a fundamental challenge due to the high dimensionality and non-linear interactions. In this thesis, we approach this problem by reducing the system to an equation that bounds the equilibria of both the cooperative system and the mixed-weight system. This upper bound underlies several Corollaries and Theorems presented throughout the thesis.

We established a sufficient condition for the equilibria of cooperative systems (see Theorem 4), which, given two appropriate vectors, states that an equilibrium exists in a N-dimensional box. The condition shifts the challenge from computing equilibria to searching for two suitable vectors. We have shown that the equilibria of the mixed-weight system are always element-wise upper bounded by an equilibrium of the cooperative system, resulting in a sharper upper bound.

We developed a framework to group the equilibria. An equilibrium could exist which, on its support, is element-wise maximal. This led to the principal equilibrium, an equilibrium which is element-wise always greater than all the other equilibria. Consequently, all the equilibria of the mixed-weight system are upper bounded by the principal equilibrium. For separable systems, we were able to upper bound the equilibria of the mixed-weight system for each support.

Finally, we considered the effect of introducing negative interactions into cooperative systems. Due to the complexity, we were unable to draw general conclusions. However, we derived a constraint on the equilibria of the mixed-weight system. In certain cases, the equilibria of the cooperative system are also equilibria of the mixed-weight system.

In summary, we can upper bound the equilibria of mixed-weight systems by those of cooperative systems. However, we were unable to derive an accurate translation between the systems. Future work could focus on approximation techniques for such a translation.

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Appendix A: Cooperative and competitive systems

There are other frameworks, in addition to the mixed-weight system (described in section 1), that could also describe cooperative and competitive systems. We provide two such systems below.

The **sign-changing system** is unique because it is characterised by very few constraints on the functions that describe the system. It is described by:

$$\begin{aligned}
&\forall\, i \in \{1, \ldots, N\}, && \frac{dx_i}{dt} = F(x_i) + \sum_{j=1}^{N} w_{ij} G(x_i, x_j) \\
&\exists\, x, y \in \mathbb{R}_{\geq 0}, && F(x) > 0 \text{ and } F(y) < 0 \\
&\exists\, x_1, x_2, y_1, y_2 \in \mathbb{R}_{\geq 0}, && G(x_1, x_2) > 0 \text{ and } G(y_1, y_2) < 0 \\
&\forall\, i, j \in \{1, \ldots, N\}, && w_{ij} \geq 0.
\end{aligned}$$

The **dual-interaction system**, compared to sign-changing and mixed-weight systems, has a completely different approach; two networks are laid on top of each other, where one network is for strictly cooperative interactions $G(x_i, x_j)$, and the other for competitive interactions $H(x_i, x_j)$. This system is more sophisticated than the sign-changing system. It can be described by:

$$\begin{aligned}
&\forall i \in \{0, 1, \ldots, N\}, && \frac{dx_i}{dt} = F(x_i) + \sum_{j=1}^{N} w_{ij} G(x_i, x_j) + \sum_{j=1}^{N} v_{ij} H(x_i, x_j) \\
&\forall x \in \mathbb{R}_{\geq 0}, && F(x) \leq 0 \\
&\forall x, y \in \mathbb{R}_{\geq 0}, && G(x, y) \geq 0, \quad H(x, y) \leq 0 \\
&\forall i, j \in \{1, \ldots, N\}, && w_{ij}, v_{ij} \geq 0.
\end{aligned}$$

Appendix B: Upper Bound

The proofs of the theorem and corollaries in section 3, which are not provided in the text.

Proof of Corollary 1:

Proof B.1. Let $\mathbf{v}$ be an arbitrary equilibrium of the system. Then there exists an index k such that:

$$v_k = \max \{v_1, v_2, \ldots, v_N\}.$$

We will show $v_k \leq x_{\text{upper}}$. For node k, the dynamic function at equilibrium $\mathbf{v}$

is:

$$\begin{aligned}
\left.\frac{dx_k}{dt}\right|_{\mathbf{x}=\mathbf{v}} =& F\left(v_k\right) + \sum_{j \in S_k^+} w_{kj} G(v_k, v_j) + \sum_{j \in S_k^-} w_{kj} G(v_k, v_j) \\
\leq& F\left(v_k\right) + \sum_{j \in S_k^+} w_{kj} G(v_k, v_j) \\
\leq& F\left(v_k\right) + \sum_{j \in S_k^+} w_{kj} G(v_k, v_k) \qquad (3) \\
=& F\left(v_k\right) + G(v_k, v_k) \sum_{j \in S_k^+} w_{kj} \\
\leq& F\left(v_k\right) + W_{\max} G(v_k, v_k). \qquad (4)
\end{aligned}$$

In step 3 we reduce the system. The inequality follows as the function $G(x_i, x_j)$ is non-decreasing in its second variable (see A. 5). Inequality 4 follows from the definition of $W_{\max}$ (see, Theorem 1). Inequality 4 follows from the definition of $W_{\max}$. By inequality 4 and the definition of the upper bound (see, Theorem 1) we can conclude that if $v_k > x_{\text{upper}}$ we must have

$$\left.\frac{dx_k}{dt}\right|_{\mathbf{x}=\mathbf{v}} < 0.$$

This would contradict $\mathbf{v}$ being an equilibrium, thus we conclude that for each index i:

$$v_i \leq v_k \leq x_{\text{upper}}.$$

Proof of Corollary 2:

Proof B.2. We consider a system in which only one node is zero for all equilibria. If there were multiples, a recursive argument could be applied. Let node i be a node that vanishes in all equilibria and let $\mathbf{v}$ be an arbitrary equilibrium of the mixed-weigh system or the cooperative system. We only consider the neighbours of node i, as these are the only ones influenced. Let k be an arbitrary node such that:

$$w_{ki} \neq 0.$$

The interaction terms for node k are:

$$\begin{aligned}
&\sum_{j \in S_k^+ \setminus (k,i)} G(v_k, v_j) + \sum_{j \in S_k^- \setminus (k,i)} w_{kj} G(v_k, v_j) + w_{ki} G(v_k, v_i) = \\
&\sum_{j \in S_k^+ \setminus (k,i)} G(v_k, v_j) + \sum_{j \in S_k^- \setminus (k,i)} w_{kj} G(v_k, v_j) + 0.
\end{aligned}$$

Where $G(v_k, v_i) = 0$, which follows from A. 3. We conclude that solving the equilibrium equations without node i results in the same equilibrium equations. Note that the sum over A^- would be zero for cooperative systems.

Proof of Theorem 3:

Proof B.3. We will prove this by contradiction. Suppose that there is an equilibrium $\mathbf{v}$, for which there exists an index i, such that:

$$k_i^+ < W_{\min}, \quad v_i \neq 0.$$

The dynamic function of node i at $\mathbf{v}$ is:

$$\begin{aligned} 0 = \left.\frac{dx_i}{dt}\right|_{\mathbf{x}=\mathbf{v}} &= F(v_i) + \sum_{j \in S_i^+} w_{ij}\, G(v_i, v_j) + \sum_{j \in S_i^-} w_{ij} G(v_i, v_j) \\ &\leq F(v_i) + \sum_{j \in S_i^+} w_{ij}\, G(v_i, v_j) \\ &\leq F(v_i) + \sum_{j \in S_i^+} w_{ij} G(v_i, x_{\text{upper}}) \\ &= F(v_i) + G(v_i, x_{\text{upper}}) k_i^+. \end{aligned}$$

By the continuity of both $F(x_i)$ and $G(x_i, x_j)$ (see A. 2), we may conclude that:

$$\exists x \in [v_i, x_{\text{upper}}] \quad \text{such that } F(x) + G(x, x_{\text{upper}}) k_i^+ = 0,$$

which contradicts the definition of $W_{\min}$. Thus, we proved by contradiction that each node i, which has a positive weighted in-degree smaller than $W_{\min}$, must be zero in all equilibria.

Appendix C: Sufficient Condition for Cooperative Systems

The proofs of the corollary and exercise in section 4, which are not provided in the text.

Proof of Corollary 4:

Proof C.1. Let node i have reduced incoming weights denoted as q_{ij}, where:

$$\sum_{j \in S_i^+} q_{ij} = k_{\max}.$$

The dynamic function of i with these reduced weights at $\mathbf{v}$ is:

$$\begin{aligned} 0 = \left.\frac{dx_i}{dt}\right|_{\mathbf{x}=\mathbf{v}} &= F(v_i) + \sum_{j \in S_i^+} q_{ij} G(v_i, v_j) \\ &= F(v_i) + \sum_{j \in S_i^+} q_{ij} G(v_i, v_i) \\ &= F(v_i) + G(v_i, v_i) k_{\max} \\ &\leq F(v_i) + G(v_i, v_i) \sum_{j \in S_i^+} w_{ij}. \end{aligned}$$

We substitute v_j with v_i, since the equilibrium is uniform. From the last equation, we conclude that we have our desired vector.

Proof of Exercise 8.2.7 in Matrix Analysis and Applied Linear Algebra by Carl D. Meyer [7]:

Proof C.2.

$$\textbf{Case}: \min_{1\le i\le N} \sum_{j=1}^{n} w_{ij} \le \rho(\mathbf{A})$$

The Collatz-Wielandt formula [7] states:

$$\rho(\mathbf{A}) \;=\; \max_{\mathbf{v}\in\Omega} f(\mathbf{v}), \quad f(\mathbf{v}) \;=\; \min_{\substack{1\le i\le N\\ v_i\neq 0}} \frac{[\mathbf{A}\mathbf{v}]_i}{v_i}, \quad \Omega = \{\, \mathbf{v}\in\mathbb{R}^n : \mathbf{v}\ge 0,\ \mathbf{v}\neq 0\}.$$

Consider the vector $\mathbf{1} = (1,\ldots,1)^N \in \Omega$. This vector evaluated in f is:

$$\min_{1\le i\le N} \sum_{j=1}^{n} w_{ij} = \min_{1\le i\le N} \frac{(\mathbf{A}\mathbf{1})_i}{1} = f(\mathbf{1}) \le \max_{\mathbf{v}\in\Omega} f(\mathbf{v}) = \rho(\mathbf{A}).$$

$$\textbf{Case}: \rho(\mathbf{A}) \le \max_{1\le i\le N} \sum_{j=1}^{n} w_{ij}$$

Let $\mathbf{u}$ be the Perron vector, which is a normalised positive eigenvector of A associated to its spectral radius:

$$\mathbf{A}\mathbf{u} = \rho(\mathbf{A})\mathbf{u}, \quad \mathbf{u} > 0, \quad \|\mathbf{u}\|_1 = 1.$$

Let k be the index such that:

$$u_k = \max\{u_1,\ldots,u_N\}.$$

Then

$$(\mathbf{A}\mathbf{u})_k = \rho(\mathbf{A})\, u_k \quad\Longrightarrow\quad \rho(\mathbf{A}) = \frac{(\mathbf{A}\mathbf{u})_k}{u_k}.$$

If we write out $(\mathbf{A}\mathbf{u})_k$, the desired inequality will follow:

$$\rho(\mathbf{A}) = \sum_{j=1}^{n} w_{kj}\frac{u_j}{u_k} \le \sum_{j=1}^{n} w_{kj} \le \max_{1\le i\le N} \sum_{j=1}^{n} w_{ij}.$$

Appendix D: Maximum Equilibria

The proofs of the lemmas, corollaries and theorems in section 5, which are not provided in the text.

Proof of Lemma 4:

Proof D.1. Let $\mathbf{v}$ be an arbitrary equilibrium of a subnetwork, the same as the entire network, except that node i is removed. If we can prove that $\mathbf{v}$ is also an equilibrium of the full network, we can conclude from a recursion argument that it holds for any subnetwork. In comparison to the whole network, we have two kinds of interaction that are not in the subnetwork. Interactions entering or leaving the node i.

For a node $k \in S_i^+$ we have:

$$G(0, x_k) = 0,$$

which is a consequence of the definition of a separable system (see Definition 5). For each node n, with $i \in S_n^+$, we have:

$$G(x_n, 0) = 0,$$

which follows by the constraints on $G(x_i, x_j)$ (see A. 3). We deduce that the addition of these extra interactions will not change the dynamic interaction at $\mathbf{v}$.

Proof of Lemma 5:

Proof D.2. We will distinguish two cases that we need to consider. The dynamic functions of a node that has a quantity of zero or x_{upper}. Let the index k be arbitrary such that $[\mathbf{x}_{\text{upper}}^{\text{partial}}(A)]_k = 0$. The dynamic function of node k is:

$$\left.\frac{dx_k}{dt}\right|_{\mathbf{x}=\mathbf{x}_{\text{upper}}^{\text{partial}}(A)} = F(0) \qquad + \sum_{j \in S_k^+} w_{kj}\, G([0, [\mathbf{x}_{\text{upper}}^{\text{partial}}(A)]_j) = 0$$

Each $G(0, [\mathbf{x}_{\text{upper}}^{\text{partial}}(A)]_j)$ is zero, which is implied by the definition of a separable model (see Definition 5).

Let the index n be arbitrary such that $[\mathbf{x}_{\text{upper}}^{\text{partial}}(A)]_n \neq 0$. The dynamic function of node n is:

$$\begin{aligned}\left.\frac{dx_n}{dt}\right|_{\mathbf{x}=\mathbf{x}_{\text{upper}}^{\text{partial}}(A)} &= F(x_{\text{upper}}) \quad + \sum_{j \in S_n^+} w_{nj}\, G(x_{\text{upper}}, [\mathbf{x}_{\text{upper}}^{\text{partial}}(A)]_j). \\ &\leq F(x_{\text{upper}}) \quad + \sum_{j \in S_n^+} w_{nj}\, G(x_{\text{upper}}, x_{\text{upper}}) \qquad \leq 0.\end{aligned}$$

The last inequality follows from the definition of x_{upper} (see Theorem 2). We conclude that $\mathbf{x}_{\text{upper}}^{\text{partial}}(A)$ is a higher vector in SCCS (see Theorem 4).

Proof of Corollary 6:

Proof D.3. Let $\mathbf{v}_1, \ldots, \mathbf{v}_M$ be the set of equilibria such that they all have the same support:

$$\text{supp}(\mathbf{v}_1) = \cdots = \text{supp}(\mathbf{v}_M).$$

Construct from the set of equilibria the vector $\mathbf{v}_{max}$ of the vector $\mathbf{v}_1, \ldots, \mathbf{v}_M$, as defined in Lemma 3. For the upper vector, take the partial upper bound vector on the set supp($\mathbf{v}_{max}$), as defined in Lemma 5.

It follows from the SCCS (see Theorem 4), that there is an equilibrium between $\mathbf{v}_{\text{max}}$ and $\mathbf{x}_{\text{upper}}^{\text{partial}}(\text{supp}[\mathbf{v}_{max}])$. Since we assumed that the set $\mathbf{v}_1, \ldots, \mathbf{v}_M$ contains all equilibria with this support, it must be that $\mathbf{v}_{max}$ is itself one of them.

Proof of Corollary 8:

Proof D.4. We will prove this by a contradiction. Suppose that there exists an equilibrium, $\mathbf{v}$, which has a support that has fewer than N elements. This implies that there is an index k that satisfies two requirements. First, $v_k = 0$, and second, k also has a neighbour n such that $v_n \neq 0$. k exists since the network is strongly connected. The dynamic function for node k at $\mathbf{v}$ is:

$$\begin{aligned} \left.\frac{dx_k}{dt}\right|_{\mathbf{x}=\mathbf{v}} &= F(0) + \sum_{j \in S_k^+} w_{kj}\, G(0, v_j) \\ &= 0 + \sum_{j \in S_k^+} w_{kj}\, G(0, v_j) \qquad > 0 \end{aligned}$$

which contradicts that $\mathbf{v}$ is an equilibrium; hence, we must have that node k is non-vanishing, from which we conclude that no node has a value equal to zero.

Proof of Corollary 11:

Proof D.5. If the principal equilibrium $\mathbf{v}$ has order N, then each node is in the support of $\mathbf{v}$, which makes the system participatory.

If the system is participatory, for every entry, there is at least one equilibrium such that the entry is non-zero. Denote the equilibrium in which the i-th entry is non-zero as $\mathbf{v}_i$. Creating the vector $\mathbf{v}_{max}$ from all these equilibria is a lower vector, which is proven in Lemma 3. There must be an equilibrium $\mathbf{u}$ for which we have:

$$\#\text{supp}(\mathbf{u}) = N,$$

which is implied by the SCCS (see Theorem 4). Consequently, a maximum equilibrium exists that has a support greater than or equal to the equilibrium $\mathbf{u}$ (see Theorem 5). Since the support of $\mathbf{u}$ contains all nodes, the same must be the case for the maximum equilibrium. There is only one maximum equilibrium that has a support of all nodes, which must be the principal equilibrium.

Proof of Theorem 6:

Proof D.6. We will prove the stability of the zero equilibrium by proving that the eigenvalues of the Jacobian at $\mathbf{0}$ are strictly negative. All off-diagonal elements of the Jacobian are only dependent on the function $G(x_i, x_j)$, since for an arbitrary $k = 1, \ldots, N$, the x_j derivative of the dynamic function of a node $i \neq k$ is:

$$\frac{\partial}{\partial x_k}\left(\frac{dx_i}{dt}\right)\bigg|_{\mathbf{x}=\mathbf{0}} = \frac{dF(x_i)}{dx_i} \cdot \delta_{ik}\bigg|_{\mathbf{x}=\mathbf{0}} + \sum_{j \in S_i^+} w_{ij} \left.\frac{\partial G(v_i, v_j)}{\partial x_k}\right|_{\mathbf{x}=\mathbf{0}} = w_{ik} \left.\frac{\partial G(v_i, v_k)}{\partial x_k}\right|_{\mathbf{x}=\mathbf{0}}.$$

To calculate the last derivative, we use a Taylor expansion of $G(x_i, x_j)$:

$$\begin{aligned} G(x_i, x_j) = G(0,0) &+ \frac{\partial G}{\partial x_i}(0,0)\, x_i + \frac{\partial G}{\partial x_j}(0,0)\, x_j \\ &+ \frac{1}{2}\frac{\partial^2 G}{\partial x_i^2}(0,0)\, x_i^2 + \frac{\partial^2 G}{\partial x_i \partial x_j}(0,0)\, x_i x_j + \frac{1}{2}\frac{\partial^2 G}{\partial x_j^2}(0,0)\, x_j^2 \\ &+ \mathcal{O}(\|(x_i, x_j)\|^3). \end{aligned}$$

- The first term is zero, which is one of the assumptions (see A. 4).
- The second term is zero, which follows since the function $G(x_i, x_j)$ is not intrinsic (see A. 3).
- The third term is zero, which is a consequence of the separability of the system.

All other terms are still dependent on x_k if we take the derivative, which at zero will also vanish. We conclude that the off-diagonal elements are all zero.

For the diagonal elements, we have:

$$\frac{\partial}{\partial x_i}\left(\frac{dx_i}{dt}\right)\bigg|_{\mathbf{x}=\mathbf{0}} = \frac{dF(x_i)}{dx_i} \cdot \delta_{ii}\bigg|_{\mathbf{x}=\mathbf{0}} + \sum_{j \in S_i^+} w_{ij} \left.\frac{\partial G(v_i, v_j)}{\partial x_i}\right|_{\mathbf{x}=\mathbf{0}} = \left.\frac{dF(x_i)}{dx_i}\right|_{\mathbf{x}=\mathbf{0}}.$$

The sum over all interactions is zero, which also follows directly from the Taylor expansion. By assumption, the diagonal elements are negative, which are also the eigenvalues.

Appendix E: From Cooperative to Mixed-Weight

The proof of Corollary 14, which is not provided in the text of section 7:

Proof E.1. We will prove the statement with a contradiction; therefore, suppose that there exists an equilibrium $\mathbf{z}$, such that:

$$-\sum_{j \in S_i^-} w_{ij} G(z_i, z_j) > \Delta_i(\mathbf{v}).$$

The dynamic function of node a at $\mathbf{z}$ is:

$$
\begin{aligned}
\left.\frac{dx_i}{dt}\right|_{\mathbf{x}=\mathbf{z}} &= F(z_i) + \sum_{j\in S_i^+} w_{ij}G(z_i,z_j) + \sum_{j\in S_i^-} w_{ij}G(z_i,z_j) \\
&< F(z_i) + \sum_{j\in S_i^+} w_{ij}G(z_i,z_j) - \Delta_i(\mathbf{v}) \\
&< F(z_i) + \sum_{j\in S_i^+} w_{ij}G(z_i,v_j) - \Delta_i(\mathbf{v}) & \leq 0.
\end{aligned}
$$

The last inequality follows, since each equilibrium of the mixed-weight system is element-wise upper bounded by the corresponding maximum equilibrium of the cooperative system (see Corollary 7). At equilibrium $\mathbf{z}$, the dynamic function of node i is negative, which is a contradiction.

Appendix F: Examples

The collection of examples referred to throughout the text. Some of the examples are thoroughly worked out, whilst others use numerical calculations.

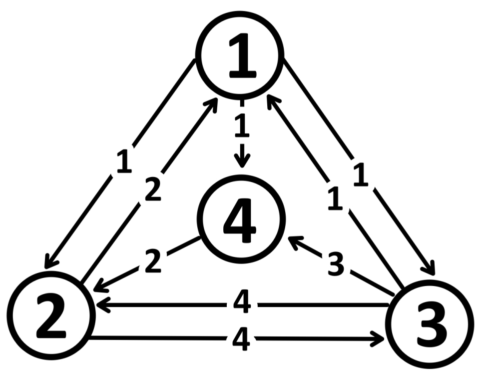

Figure 21: The network of the system in Appendix Example F.1, with the direction and weight of the interaction. This system has no obvious symmetry, which makes it harder to solve analytically. The strong interaction between nodes 2 and 3 could lead to equilibria, in which only these two have non-vanishing entries. The number of each node indicates the index.

Example F.1. An illustration of a cooperative system for which all equilibria are strictly below the upper bound is:

$$\begin{bmatrix} 0 & 2 & 1 & 0 \\ 1 & 0 & 4 & 2 \\ 1 & 4 & 0 & 0 \\ 1 & 0 & 3 & 0 \end{bmatrix},$$

$$F(x_i) = -4x_i + 4x_i^2 - 2x_i^3, \quad G(x_i, x_j) = \frac{x_i x_j}{2}.$$

The network is visualised in Figure 21. For this system, we find an upper bound of 3.106, where, if we numerically calculate the equilibria, we get:

$$\begin{bmatrix} 2.002 \\ 2.758 \\ 2.503 \\ 2.174 \end{bmatrix}, \quad \begin{bmatrix} 0.110 \\ 2.473 \\ 2.225 \\ 1.834 \end{bmatrix}, \quad \begin{bmatrix} 1.872 \\ 2.347 \\ 2.347 \\ 0.000 \end{bmatrix}, \quad \begin{bmatrix} 0.261 \\ 2.061 \\ 2.061 \\ 0.000 \end{bmatrix},$$

$$\begin{bmatrix} 0.000 \\ 2.452 \\ 2.205 \\ 1.809 \end{bmatrix}, \quad \begin{bmatrix} 0.000 \\ 2.085 \\ 2.042 \\ 0.271 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 2 \\ 2 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

All values are strictly below the upper bound. One might ask if this could have been anticipated. Since the positive weighted in-degree for all nodes is not $W_{\max}$, this system will not have an equilibrium that has entries equal to $W_{\max}$.

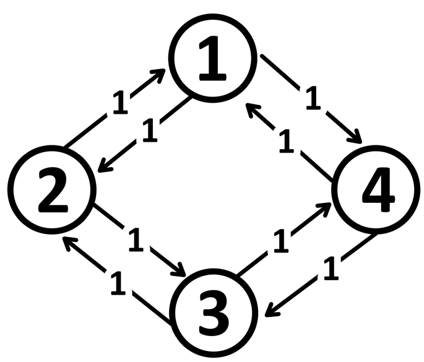

Figure 22: The graph of the network in Appendix Example F.2, with the direction and weight of the interaction. There is a lot of symmetry in this network. The number of each node indicates the index.

Example F.2. An illustration of a system which has an equilibrium in which all components reach the upper bound is the following cooperative system:

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix},$$

$$F(x_i) = -4x_i + 4x_i^2 - 4x_i^3, \quad G(x_i, x_j) = 3x_i x_j.$$

The network is visualised in Figure 22. We can solve this system analytically with basic algebra. To compute the upper bound, we need to find the supremum of all non-negative x that satisfy:

$$\begin{aligned} F(x) + W_{\max} G(x, x) &= -4x + 4x^2 - 4x^3 + 2 \cdot 3x^2 && = 0 \\ & - x(4x^2 - 10x + 4) && = 0. \end{aligned}$$

Solving the equation yields:

$$x = 0 \quad \text{or} \quad x = 2 \quad \text{or} \quad x = \frac{1}{2}.$$

The supremum of these three solutions is 2, which is therefore our upper bound. To find an equilibrium, observe that all nodes are indistinguishable from each other. We can therefore use a symmetry argument to reduce the problem. We proceed as follows: instead of solving the equilibrium equation for each node, we assume that all nodes have the same value x, which gives the equation:

$$-4x + 4x^2 - 4x^3 + 3x^2 + 3x^2 = 0,$$

which is the same as solving for the upper bound. This implies that we have an equilibrium with entries equal to x_{upper}. The corresponding equilibria are:

$$\begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

We have not searched for equilibria that do not have equal entries, which we will do next. Again, we employ a symmetry argument, but instead of assuming that all nodes have the same value, we assume a repeating pattern in which the value of the nodes alternates. Let x_1 and x_2 be the two values that result in the following two equations:

$$-4x_1 + 4x_1^2 - 4x_1^3 + 2 \cdot 3x_1 x_2 = 0 \tag{5}$$

$$-4x_2 + 4x_2^2 - 4x_2^3 + 2 \cdot 3x_2 x_1 = 0. \tag{6}$$

Either x_1 or x_2 is zero; being zero will result in the trivial zero equilibrium. We can isolate x_2 from 5:

$$\frac{2}{3}(1 - x_1 + x_1^2) = x_2.$$

Substituting this expression into equation 6 gives:

$$\begin{aligned}\frac{2}{3}(-1 + x_2 - x_2^2) + x_1 &= 0\\ -\frac{2}{3} + \frac{4}{9}(1 - x_1 + x_1^2) - \frac{8}{27}(1 - x_1 + x_1^2)^2 + x_1 &= 0\\ 8x_1^4 - 16x_1^3 + 12x_1^2 - 31x_1 + 14 &= 0.\end{aligned}$$

Although there are methods to solve this fourth-degree polynomial, we note that if x_2 and x_1 are the same, we obtain the case above, where we found the equilibria $x_{1,2} = 2$ and $x_{1,2} = \frac{1}{2}$. We can factor out these equilibria from the fourth-degree polynomial, which results in the second-degree polynomial:

$$\frac{8x_1^4 - 16x_1^3 + 12x_1^2 - 31x_1 + 14}{(x_1 - 2)(x_1 - \frac{1}{2})} = 8x_1^2 + 4x_1 + 14,$$

Which has no real roots as the discriminant is negative:

$$16 - 4 \cdot 8 \cdot 14 = -432 < 0.$$

We conclude that there are no non-symmetric equilibria. There are still two cases left to consider; the first is if one of the nodes has a quantity equal to zero. Due to the symmetry of the system, it does not matter which node we set to zero. The reduced network is:

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

also shown in Figure 23. This system, as well, has symmetry, but unlike the previous two cases, it is not periodic. Therefore, we need a more advanced method than before. We consider the following equation:

$$-4x_{13} + 4x_{13}^2 - 4x_{13}^3 + 3x_{13}x_2 = 0,$$

where x_{13} denotes possible equilibrium values for the quantities of nodes 1 and 3. Our strategy is to investigate for what x_2 there are multiple possible x_{13} values. If no such x_2 exists, we may conclude that there are no equilibria in which nodes 1 and 3 are unequal. If such x_2 values exist, we investigate further if such values could correspond to an equilibrium. We begin by solving:

$$-4x_{13} + 4x_{13}^2 - 4x_{13}^3 + 3x_{13}x_2 = 0$$

$$-x_{13} = 0 \quad \text{or} \quad 4x_{13}^2 - 4x_{13} + 4 - 3x_2 = 0.$$

Solving the quadratic equation gives:

$$x_{13} = \frac{4 \pm 4\sqrt{3(x_2 - 1)}}{8}.$$

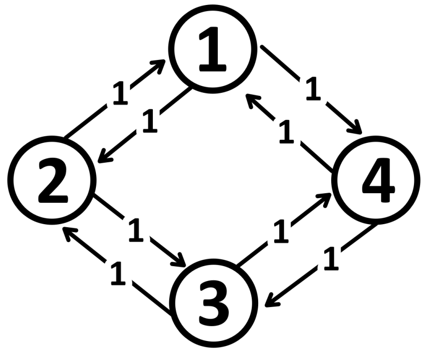

Figure 23: The sub-graphs of the network in Appendix Example F.2, with the direction and weight of the interaction. Both sub-graphs have symmetry in their network, making it easier to analyse. The number of each node indicates the index.

For now, we will completely ignore the $x_{13} = 0$ solution, as this will correspond to another node being zero, which will be covered later. We only have multiple positive values for x_{13} if:

$$\begin{aligned} 0 < 3(x_2 - 1) < 1, \\ 0 < x_2 - 1 < \frac{1}{3}, \\ 1 < x_2 < \frac{4}{3}. \end{aligned}$$

If nodes 1 and 3 have different values, the cooperative term for node 2 will be:

$$\begin{aligned} \sum_{j \in S_2^+} w_{2j}\, G(x_2, x_j) &= 3x_2 \frac{4\ + 4\sqrt{3(x_2-1)}}{8} + 3x_2 \frac{4 - 4\sqrt{3(x_2-1)}}{8} \\ &= 3x_2. \end{aligned}$$

The equilibrium equation for node 2 is therefore:

$$\begin{aligned} -4x_2 + 4x_2^2 - 4x_2^3 + 3x_2 = 0 \\ -x_2 = 0 \quad \text{or} \quad 4x_2^2 - 4x_2 + 1 = 0 \\ x_2 = \frac{1}{2}. \end{aligned}$$

We ignore the zero solution since it corresponds to the trivial equilibrium where all nodes are zero. This found solution contradicts the earlier bounds for x_2, implying no equilibrium exists with nodes 1 and 3 having different values. Consequently, we aim to find equilibria for which nodes 1 and 3 are the same, which result in the following equations:

$$\begin{aligned}-4x_1 + 4x_1^2 - 4x_1^3 + 3x_1x_2 &= 0\\ -4x_2 + 4x_2^2 - 4x_2^3 + 2\cdot 3x_2x_1 &= 0.\end{aligned} \tag{7}$$

These equations are similar to the previously mentioned equations. Isolating x_2 yields:

$$\frac{4}{3}(1 - x_1 + x_1^2) = x_2.$$

Substituting this in the Equation 7 gives:

$$\begin{aligned}\frac{2}{3}(-1 + x_2 - x_2^2) + x_1 &= 0\\ -\frac{2}{3} + \frac{8}{9}(1 - x_1 + x_1^2) - \frac{32}{27}(1 - x_1 + x_1^2)^2 + x_1 &= 0\\ 32x_1^4 - 64x_1^3 + 72x_1^2 - 67x_1 + 26 &= 0,\end{aligned}$$

which is once again a fourth-degree polynomial. Instead of solving it, we give the approximate roots:

$$x_1 = 1.058 \quad \text{or} x_1 = 0.707,$$

which results in the equilibria for the original system to be:

$$\begin{bmatrix}0.707\\1.057\\0.707\\0\end{bmatrix}, \begin{bmatrix}1.058\\1.415\\1.058\\0\end{bmatrix}, \begin{bmatrix}0\\0.707\\1.057\\0.707\end{bmatrix}, \begin{bmatrix}0\\1.058\\1.415\\1.058\end{bmatrix},$$

$$\begin{bmatrix}0.707\\0\\0.707\\1.057\end{bmatrix}, \begin{bmatrix}1.058\\0\\1.058\\1.415\end{bmatrix}, \begin{bmatrix}1.057\\0.707\\0\\0.707\end{bmatrix}, \begin{bmatrix}1.415\\1.058\\0\\1.058\end{bmatrix}.$$

Lastly, only one case is left to consider, a system in which two nodes have value zero. The reduced network is:

$$\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}.$$

The equilibria of this network are not hard to find. We first calculate the upper bound for this adjacency matrix (Figure 23):

$$\begin{aligned}-4x + 4x^2 - 4x^3 + 3x^2 &= 0\\ -x(4x^2 - 7x + 4) &= 0\\ x = 0 \quad \text{or} \quad 4x^2 - 7x + 4 &= 0.\end{aligned}$$

Since the discriminant of the right polynomial is smaller than zero, we only have the zero equilibrium. This network does not result in new equilibria.

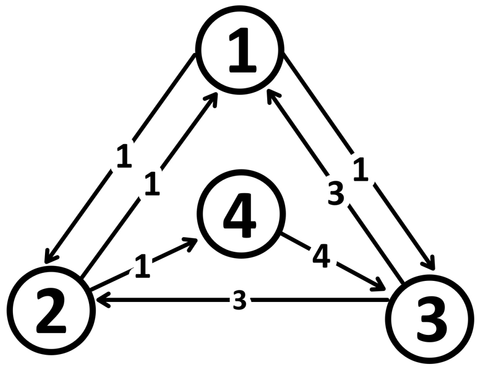

Figure 24: The network of the system in Appendix Example F.3, with the direction and weight of the interaction. The number of each node indicates the index.

Example F.3. Suppose that we have the following cooperative system:

$$\begin{bmatrix} 0 & 1 & 3 & 0 \\ 1 & 0 & 3 & 0 \\ 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

$$F(x_i) = -4x_i + 4x_i^2 - 2x_i^3, \quad G(x_i, x_j) = \frac{x_i x_j}{2}.$$

The network of this system is shown in Figure 24. To calculate W_{min} we solve the equation:

$$4x + 4x^2 - 2x^3 + W_{\text{min}} x_{\text{upper}} x = 0,$$

which only has roots if:

$$W_{\text{min}} \geq \frac{4}{x_{\text{upper}}}.$$

The upper bound of this system is 2.425, so W_{min} is 1.649. Node 4 has a positive weighted in-degree smaller than W_{min}, from which we deduce that its quantity is zero in all equilibria, making the system partially redundant. We reduce the network (as shown in Figure 25) to:

$$\begin{bmatrix} 0 & 1 & 3 \\ 1 & 0 & 3 \\ 1 & 0 & 0 \end{bmatrix},$$

with the same set of functions. This subnetwork has an upper bound of 2, so W_{min} is 2. In this reduced network, node 3 has a positive weighted in-degree below $W_{\min}$. Consequently, we focus on the equilibria of the subnetwork (shown in Figure 25):

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

This subnetwork has an upper bound of zero; hence, the equilibrium of the subnetwork is the zero equilibrium. We deduce that for the original system, the only equilibrium is the zero equilibrium.

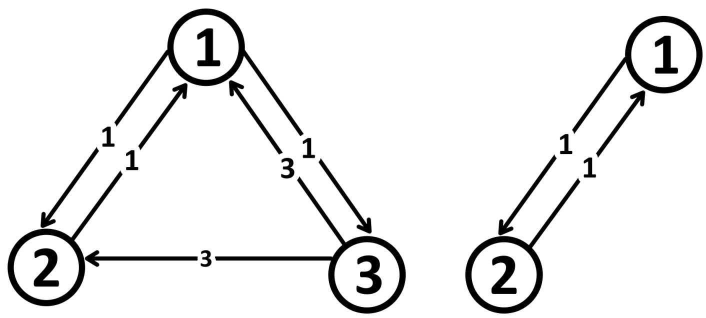

Figure 25: The networks of the subnetworks in Appendix Example F.3, with the direction and weight of the interaction. The number of each node indicate the index.

Example F.4. Consider the network on the top left in Figure 26. We will iteratively find $k_{\max}$ of this network. In the network on the top left, node 6 has a weighted in-degree of 1, coming from node 5, which is the lowest of all the nodes. Removing node 6 results in the network below it. In this network, node 5 has a weighted in-degree of 3, which is the lowest. Deleting node 5 gives the network on the bottom left, in which node 4 has a weighted in-degree of 0, which is the smallest for this network; therefore, we remove node 4. Doing so gives the network on the top right, where node 2 has the smallest weighted in-degree of 2, which is the same as previously found. We finally arrive at the final network in which we once again have that the smallest weighted in-degree is equal to 2, therefore $k_{\max} = 3$.

Example F.5. Consider the system:

$$\begin{bmatrix} 0 & 2 & 2 \\ 1 & 0 & 1 \\ 3 & 2 & 0 \end{bmatrix},$$

$$F(x_i) = -\frac{3x_i}{2}, \quad G(x_i, x_j) = \frac{x_j^2}{1 + x_j^2},$$

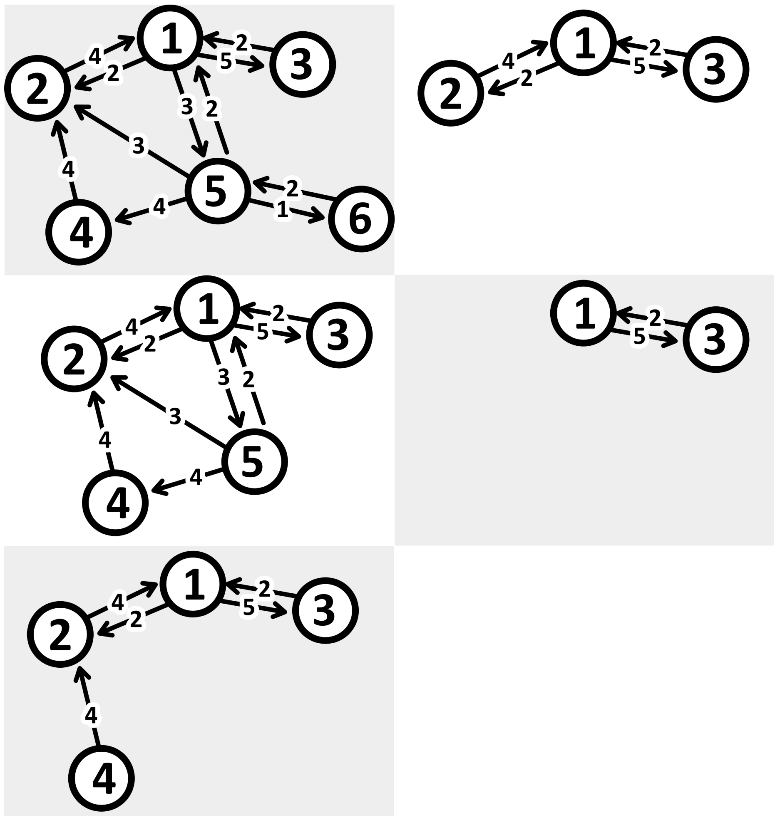

Figure 26: The iterative process of graph to find $k_{\max}$. We start at the top left and work our way down, from which we move to the right. It follows from the procedure that $k_{\max}$ is 3, which is obtained in subnetworks on the middle left. The number of each node indicates the index.

where the functions are of the GRN system [2]. The equilibria for this network are:

$$\begin{bmatrix} 1.825 \\ 1.069 \\ 2.249 \end{bmatrix}, \quad \begin{bmatrix} 0.584 \\ 0.378 \\ 0.675 \end{bmatrix}.$$

The first equilibrium is the maximum equilibrium. To deduce the stability of this equilibrium, we calculate the Jacobian, which is:

$$\begin{bmatrix} -1.5 & 0.931 & 0.245 \\ 0.195 & -1.5 & 0.123 \\ 0.584 & 0.931 & -1.5 \end{bmatrix}.$$

Numerically calculating the eigenvalues of this matrix gives -0.736, -1.901, -1.857. The equilibrium is stable because all the eigenvalues are negative. If we

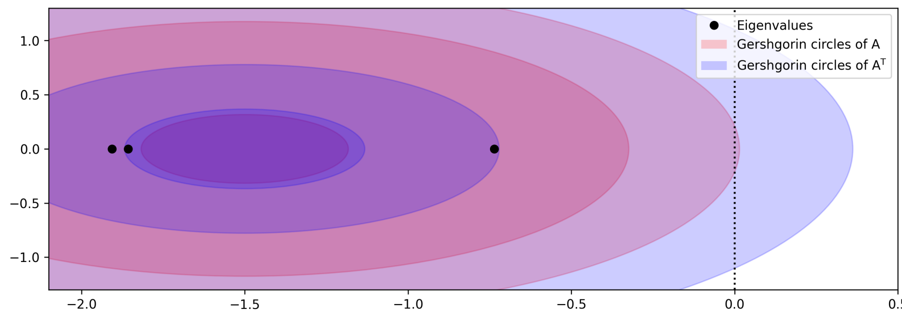

Figure 27: The Gershgorin circles of A in red and A^T in blue, with the eigenvalues of the matrix in example F.5. Both the red and blue circles have a region on the right side of the y-axis, from which we cannot deduce the sign of all the eigenvalues.

apply Gershgorin to this Jacobian, we can not conclude the same. We visualised the results of Gershgorin in Figure 27.

Example F.6. Suppose we have the following complex system:

$$\begin{bmatrix} 0 & 4 \\ 4 & 0 \end{bmatrix},$$

$$F(x_i) = -4x_i + 4x_i^2 - \alpha x_i^3, \quad G(x_i, x_j) = x_i x_j \quad \alpha \in (1, \infty).$$

This system has equilibrium equations:

$$\begin{aligned} -4x_1 + 4x_1^2 - \alpha x_1^3 + 4x_1x_2 = 0 \\ -4x_2 + 4x_2^2 - \alpha x_2^3 + 4x_2x_1 = 0. \end{aligned}$$

Before calculating the equilibria, we will first use the method described in Wu *et al* [2] to find the interval of α for which this system has non-zero equilibria. The so-called difficulty of this system is given as:

$$\lambda^* = 4\sqrt{\alpha} - 4.$$

For this simple network, we see that the largest eigenvalue is equal to $k_{\max}$, which tells us that if the difficulty is higher than both, no non-zero equilibria exist:

$$\begin{aligned} k_{max}(A) = p(A) = & 4 \geq 4\sqrt{\alpha} - 4 = \lambda^* \\ & 2 \geq \sqrt{\alpha}. \end{aligned}$$

As we also require that the function $F(x_i)$ be negative for any positive input, we get $\alpha \in (1, 4]$. We now calculate the equilibria of this system with a variable α. We ignore the zero equilibrium and isolate one of the variables:

$$x_2 = 1 - x_1 + \frac{\alpha}{4}x_1^2,$$

We can substitute this expression in the equilibrium equation of x_2, which results in:

$$\begin{aligned}
-4 + 4x_2 - \alpha x_2^2 + 4x_1 &= 0 \\
-4 + 4(1 - x_1 + \frac{\alpha}{4}x_1^2) - \alpha(1 - x_1 + \frac{\alpha}{4}x_1^2)^2 + 4x_1 &= 0 \\
\alpha^3 x_1^4 - 8\alpha^2 x_1^3 + 8\alpha^2 x_1^2 - 32\alpha x_1 + 16\alpha &= 0,
\end{aligned}$$

which is a fourth-degree polynomial. Although not impossible to solve for analyticity, we can avoid the complexity by first solving for the symmetric case. If both nodes have the same quantitative value, we get the following equation:

$$\begin{aligned}
-4x + 4x^2 - \alpha x^3 + 4x^2 = 0 \\
x = 0 \quad \text{or} \quad \alpha x^2 - 8x + 4 = 0.
\end{aligned}$$

Solving the quadratic equation:

$$x = 0 \quad \text{or} \quad x = \frac{4 \pm 2\sqrt{4-\alpha}}{\alpha}.$$

On the given interval of α, x is always a positive real. Using this, we can simplify the previous polynomial:

$$\frac{\alpha^3 x_1^4 - 8\alpha^2 x_1^3 + 8\alpha^2 x_1^2 - 32\alpha x_1 + 16\alpha}{x^2 + \frac{8}{a}x + \frac{4\alpha}{a^2}} = \alpha^2(4 + \alpha x_1^2),$$

which on the given bounds of α is never zero, thus the only equilibria are the symmetric equilibria. To study the stability of these equilibria, we will calculate the eigenvalues of the Jacobian. The Jacobian is:

$$\mathbf{J}(\mathbf{x}) = \begin{bmatrix} -4 + 8x_1 - 3\alpha x_1^2 + 4x_2 & 4x_1 \\ 4x_2 & -4 + 8x_2 - 3\alpha x_2^2 + 4x_1 \end{bmatrix}.$$

Fortunately, due to the symmetry in the equilibria, we can use that for the following matrix:

$$\begin{bmatrix} a & b \\ b & a \end{bmatrix},$$

the eigenvalues are:

$$\lambda = a \pm b.$$

For the stability of the equilibria, the sign of the eigenvalues is important; therefore, we will calculate the eigenvalues dependent on α, and check what sign they have.

$$\text{Let } \beta = \sqrt{4-\alpha}, \quad \alpha = 4 - \beta^2 \quad \beta \in [0, \sqrt{3}).$$

$$\text{Case 1: } x = \frac{4 - 2\sqrt{4-\alpha}}{\alpha}:$$

$$\begin{aligned}
\lambda_1 &= -4 + 16x - 3\alpha x^2 \\
&= \frac{-4\alpha + 16(4 - 2\sqrt{4-\alpha}) - 3(4 - 2\sqrt{4-\alpha})^2}{\alpha} \\
&= \frac{\beta(16 - 8\beta)}{\alpha}.
\end{aligned}$$

This eigenvalue, on the given bounds, is only zero if $\beta = 0$ and positive on the rest of the interval; therefore, this eigenvalue is positive for all α, except for $\alpha = 4$, where it is zero. For the second eigenvalue, we have:

$$\begin{aligned}
\lambda_2 &= -4 + 8x - 3\alpha x^2 \\
&= \frac{-4\alpha + 8(4 - 2\sqrt{4-\alpha}) - 3(4 - 2\sqrt{4-\alpha})^2}{\alpha} \\
&= \frac{-16 + 4\beta^2 + 32 - 16\beta - 48 + 48\beta - 12\beta^2}{\alpha} \\
&= -8\frac{(\beta - 2)(\beta - 2)}{\alpha}.
\end{aligned}$$

This eigenvalue, on the given bounds, is never zero and thus always negative. Therefore, we conclude that this equilibrium is a saddle for all α values except for $\alpha = 4$.

$$\text{Case 2: } x = \frac{4 + 2\sqrt{4-\alpha}}{\alpha}:$$

$$\begin{aligned}
\lambda_1 &= -4 + 16x - 3\alpha x^2 \\
&= \frac{-4\alpha + 16(4 + 2\sqrt{4-\alpha}) - 3(4 + 2\sqrt{4-\alpha})^2}{\alpha} \\
&= -\frac{\beta(16 + 8\beta)}{\alpha}.
\end{aligned}$$

This eigenvalue is on the bounds zero for $\beta = 0$, and for the rest of the interval it is negative; hence, this eigenvalue is also negative for all α, except $\alpha = 4$. The second eigenvalue is:

$$\begin{aligned}
\lambda_2 &= -4 + 8x - 3\alpha x^2 \\
&= \frac{-4\alpha + 8(4 + 2\sqrt{4-\alpha}) - 3(4 + 2\sqrt{4-\alpha})^2}{\alpha} \\
&= \frac{-16 + 4\beta^2 + 32 + 16\beta - 48 - 48\beta - 12\beta^2}{\alpha} \\
&= -8\frac{(\beta + 2)(\beta + 2)}{\alpha}.
\end{aligned}$$

This eigenvalue, the given bounds, is never zero and, as a result, is always negative. This equilibrium is always stable except when $\alpha = 4$. What happens exactly at $\alpha = 4$? The equilibria:

$$\begin{bmatrix} \frac{4-2\sqrt{4-\alpha}}{\alpha} \\ \frac{4-2\sqrt{4-\alpha}}{\alpha} \end{bmatrix}, \quad \begin{bmatrix} \frac{4+2\sqrt{4-\alpha}}{\alpha} \\ \frac{4+2\sqrt{4-\alpha}}{\alpha} \end{bmatrix},$$

are distinct, except for $\alpha = 4$ where the "two" equilibria merge into one equilibrium, namely:

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The evolution of the equilibria is visualised in Figure 28. The Jacobian for this equilibrium does not yield any information about the stability of this equilibrium. We could instead use a trick to find out if the non-zero equilibrium at $\alpha = 4$ is stable or not. We will look at the change of both variables on the line $x_1 = x_2$. On this line, the change of the quantity of both nodes is the same, and given as:

$$\begin{aligned} \frac{dx_{1,2}}{dt} &= -4x_{1,2} + 4x_{1,2}^2 - 4x_{1,2}^3 + 4x_{1,2}^2 \\ &= -4x_{1,2} + 8x_{1,2}^2 - 4x_{1,2}^3. \end{aligned}$$

Suppose we have a small deviation from the equilibrium at $x_1 = x_2 = 1$. At such a small deviation, the equation becomes:

$$\begin{aligned} \left.\frac{dx_{1,2}}{dt}\right|_{x_{1,2}=1-\epsilon} &= -4(1-\epsilon) + 8(1-\epsilon)^2 - 4(1-\epsilon)^3 \\ &= 4\epsilon - 16\epsilon + 8\epsilon^2 + 12\epsilon - 12\epsilon^2 + \epsilon^3 \\ &\approx -4\epsilon^2, \end{aligned}$$

where the approximation follows for ϵ close enough to zero. We see that no matter if we come from above 1 or below 1, the derivative is always negative. This implies that this equilibrium is not a stable point. This phenomenon, where one saddle point combines with a stable point, is also called saddle-node bifurcation, which is explained by Steven H. Strogatz in the book *Nonlinear Dynamics and Chaos* [12].

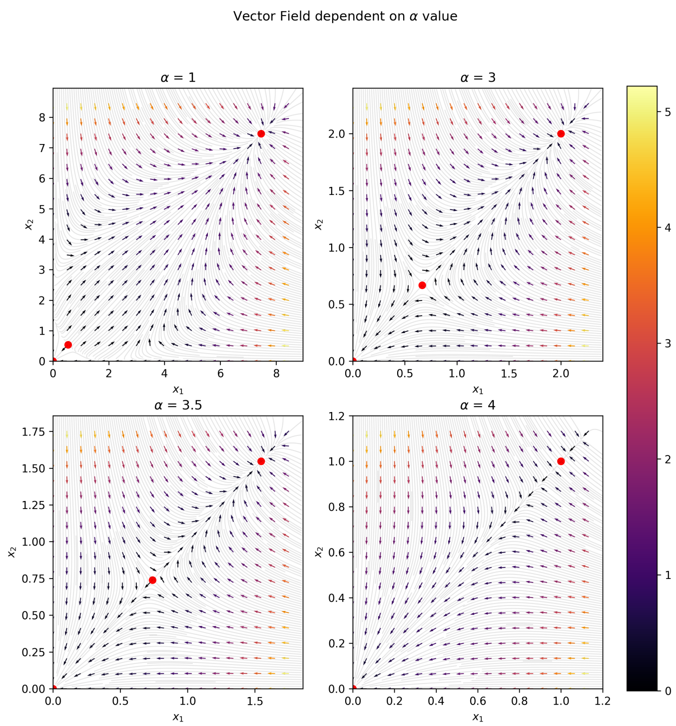

Figure 28: The vector fields plotted for different α values. The red dots are the locations of the equilibria. The colour of the vectors is the magnitude of the vector, according to the colour on the right side. As α increases, the two non-zero equilibria move towards each other. At $\alpha = 4$ "both" non-zero equilibria merge into one equilibrium. This phenomenon, where one saddle point combines with a stable equilibrium, is called a saddle-node bifurcation.