On the Optimal Time-averaged Power Yielding from a Damped Harmonic Oscillator with Variable Damping

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by

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TU Delft

Preface

With this master's thesis, I present my research on the optimal damping value for a damped harmonic oscillator to yield the most time-averaged power .

Readers of this thesis should be familiar with the basics of classical mechanics. Furthermore, knowledge regarding basic numerical methods of approximating functions is advised.

If one is interested in the mathematical foundation behind the analysis, they should read Chapter 2 and Chapter 3, where the model is introduced and solved, respectively. If the reader is interested in the case of no switch at all in damping value, they should first look at Chapter 4. Readers with a special interest in the results of a single switch between damping values are referred to Chapter 5. For more switches, readers should read Chapter 6.

Firstly, I'd like to thank my supervisor, Wim van Horssen, for his support and the time he dedicated to guiding me through this thesis. I especially appreciate the informal and open way of working, which allowed us not only to discuss my thesis but also to exchange thoughts on broader societal topics.

Secondly, I'd like to thank Henk Schuttelaars and Neil Budko for taking a seat in my thesis committee.

Next, I want to thank my family and friends, especially, André, Paula, Charlie, Ellen and Gijs for proofreading and discussing this thesis with me. I also want to thank W.I.S.V. 'Christiaan Huygens', /Pub, Lijst Bèta, VSSD, ASVA and ISO for providing me with a sense of purpose.

Lastly, I'd like to thank a particular Brazilian architect for unintentionally reminding me to take breaks over the past months and showing me that there is more to look forward to.

What is, will be.

Sam de Jong
Delft, August 2025

Abstract

Energy can be harvested from vibrations by using a damped harmonic oscillator with base excitation, providing a sustainable way of yielding energy. By solving the equations of motion for this oscillator and studying the steady state solution, an expression for the time-averaged power is obtained. Different damping values of the oscillator influence how much power is yielded. In this thesis, it is analytically shown that a constant damping value equal to $c_v = \frac{\sqrt{c_m^2 \psi^2 f_s^2 + (f_s^2 - 1)^2}}{\psi f_s}$ yields the most time-averaged power for the case where there is no switch in damping value and when there is a singular arbitrary switch in damping value. It is numerically shown that this damping value also yields the most time-averaged power for multiple switches in the damping value.

Contents

Nomenclature

Symbol	Definition	Unit
F	Force (general)	[N]
F_{spring}	Spring force	[N]
F_{damp}	Damping force	[N]
F_{g}	Gravitational force	[N]
m	Mass	[kg]
p	Momentum	[kg m/s]
v	Velocity	[m/s]
u	Displacement	[m]
k	Spring coefficient	[N/m]
g	Gravitational constant	[m/s^2]
$\hat{z}$	Relative displacement with respect to base excitation	[m]
$\hat{z}_{\text{eq}}$	Equilibrium position of $\hat{z}$	[m]
z	Relative displacement with respect to base excitation and equilibrium position	[m]
z_{hom}	Homogeneous solution for relative displacement z	[m]
z_{part}	Particular solution for relative displacement z	[m]
t	Time	[s]
T_0	Starting time of the model	[s]
T_m	Time of switch of damping value	[s]
T_b	Beginning of time interval of switching damping value	[s]
T_f	End of time interval of switching damping value	[s]
c	Damping coefficient	[Ns/m]
c_m	Parasitic damping	[Ns/m]
c_v	Time-variable damping	[Ns/m]
$c_{v,i}$	Specific value i time-variable damping	[Ns/m]
P_{inst}	Instantaneous power	[J/s]
P_{ave}	Time-averaged power	[J/s]
i	Complex root	[-]
y_s	Base excitation	[m]
y_0	Amplitude of incoming wave of base excitation	[m]
f_s	Frequency of incoming of base excitation	[1/s]
φ	Phase of incoming wave of base excitation	[-]
$\hat{\varphi}$	Shifted phase due to harmonic addition theorem	[-]
ϕ_i	Specific mapping for certain interval i for solution	[-]
ϕ	General mapping for entire interval solution	[-]

List of Figures

List of Tables

1
Introduction

Imagine a world where energy harvesting would take place by cars driving over highways. Imagine that energy can be harvested by people walking across the street. Imagine an electric car being able to fully recharge by driving. All these wonderful applications might be possible due to vibrations. A car driving over the highway causes vibrations in the road, and people walking across the street can cause vibrations in the ground. By creating an oscillator which can harvest energy from these vibrations, a new way of sustainable energy harvesting can be created.

The simplest model of an oscillator is the simple harmonic oscillator. In practice, damping will always exist due to natural causes, so the damped harmonic oscillator is most often used in current research. An alteration to the damped harmonic oscillator model is to include a base excitation, as is done by Stephen [13] and Rao [11]. Both discussed the damped harmonic oscillator with constant damping with base excitation they assumed to be harmonic with frequency ω and amplitude Y. Another alteration could be to assume the mass to be variable. The general behaviour of models with changing mass has already been the topic of extensive research, for example by Awrejcewicz [2]. Specifically oscillator models with varying mass without damping have been discussed by Núñez and Torres [9]. Papers by Horssen, Pischanskyy, and Dubbeldam [5], Horssen and Pischanskyy [4], and Pischanskyy and Horssen [10]. Núñez and Torres [9] assumed a varying mass with constant damping. All of these researches expand on the research by Irschik and Holl [6] who derived the relevant equations of motion.

Power harvesting from these oscillators is a relatively new topic. Though Nikzamir et al. [8] and Scapolan, Tehrani, and Bonisoli [12] have researched how base excitation with time-dependent damping can influence the power that could be gained from such a system, their analysis was purely numerical and not analytical. Also, Di Monaco et al. [3] have researched this topic, by approximating the solution of the damped harmonic oscillator using the harmonic balance method. All these papers assume that the best way to harvest power from the damped harmonic oscillator is by having a variable damping with twice the frequency of the base excitation. With the harvested power from these oscillators, several applications exist. One is discussed by Yang et al. [15], who analysed power harvesting in vehicle suspension applications.

In this thesis, a focus is put on damped harmonic oscillators with time-dependent damping and a base excitation, to compare the findings with research done by Nikzamir et al. [8]. The base excitation is due to external vibrations outside of the model, and the damping is used to yield power from these vibrations. The goal is to analyse which damping value will allow for the most harvesting from the external vibration.

This thesis has two main objectives to achieve the aforementioned goal. The first objective is to get a clear indication of the power that can be harvested from a damped harmonic oscillator with variable damping. This requires a thorough analysis of the oscillator and a way to measure the power over the oscillatory periods.

The second objective is to provide optimal damping for the yielding of power from the system. Optimal damping is herein defined as the damping for which the most power is yielded from the oscillator. This damper value potentially depends on many variables, such as the frequency or phase of the external vibration for the base excitation. The aim is to improve the work done by previous papers ([8], [12] and [3]) by providing an analytical solution and showing that a solution can be found which yields more power and is easier to implement.

The following structure is adhered to in this thesis. First, in Chapter 2, the relevant Equations of Motion are derived for the discussed model. These Equations of Motion are derived stepwise, the notation and assumptions are discussed and it is subsequently rescaled to reduce the number of parameters. Furthermore, in this Chapter, the numerical model behind the damping, the equations for the time-averaged power and finally the parameter bounds are presented. In Chapter 3, the Equations of Motion are solved. The solutions are presented for three cases: constant damping, piecewise changing damping and pseudocontinuous damping. Piecewise changing damping is a stepfunction for the damping value, while pseudocontinuous damping is a damping value which mimicks a continuous damping function by approximating it with several stepwise intervals. Chapter 4 focuses on the power yield for a system where there is no switch in the damping value. In Chapter 5, an analysis of the harvested power from the system is discussed. Here, a proposed method to find the damping value of power-yielding is given for a singular switch. Thereafter, a scenario with multiple switches in damping value is discussed in Chapter 6. Lastly, in Chapter 7 conclusions are drawn and several recommendations for further research into this subject are given.

2

Deriving Relevant Equations and Parameters for the Model

In this Chapter, the relevant equations and parameters for yielding power from a damped harmonic oscillator with base excitation are discussed. In Section 2.1 the equations of motion for the damped harmonic oscillator with base excitation are derived. Then, the model is rescaled for an easier analysis in Section 2.2. Next, the equations regarding the damping value are introduced in Section 2.3. Here, the cases of a piecewise changing damping and pseudocontinuous damping are discussed. Then, an introduction is made to the relevant equations for the power analysis in Section 2.4. Here, the instantaneous power is discussed in Section 2.4.1 and the time-averaged power is discussed in Section 2.4.2. Finally, in Section 2.5, an estimate is given for the bounds of the different parameters in the model for the damped harmonic oscillator.

2.1. Equations of motion

This section derives the equations of motion for the damped harmonic oscillator with a base excitation, which is done in steps for the relevant forces. The model is assumed to be one-dimensional

2.1.1. Newton's second law

At the basis of the model lies the second law of Newton. Though many readers know this formula simply as the sum of all forces on an object equals the object's mass times the object's acceleration, the actual law is slightly different. It states that the sum of all forces on an object equals the change in the momentum of the object, so

$$\sum F = \frac{\mathrm{d}p}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(mv) = \frac{\mathrm{d}}{\mathrm{d}t}\left(m\frac{\mathrm{d}u}{\mathrm{d}t}\right). \tag{2.1}$$

For clarity's sake, F describes a force acting on the object, p describes the object's momentum, m is the object's mass and v is the object's velocity. All variables here can depend on time. Note that the law will be simplified to the familiar version of mass times acceleration when an object has a constant mass.

This law is the foundation for deriving the model used in this thesis.

2.1.2. Simple harmonic oscillator

The simple harmonic oscillator forms the basis of any oscillator. In the simple harmonic oscillator, the only force acting is the spring force. This force is described by Hooke's law, which describes the linear relation between the spring coefficient k and the displacement from equilibrium of the mass u. Hooke's law thus states that

$$F_{\text{spring}} = ku. \tag{2.2}$$

Then, using Newton's second law as stated in Equation (2.1), the equations of motion for the simple harmonic oscillator are obtained as

$$-ku = \frac{\mathrm{d}}{\mathrm{d}t}\left(m\frac{\mathrm{d}u}{\mathrm{d}t}\right), \tag{2.3}$$

where a minus sign is placed before the spring force, as it acts as a restoring force. A diagram of the simple harmonic oscillator is given in Figure 2.1.

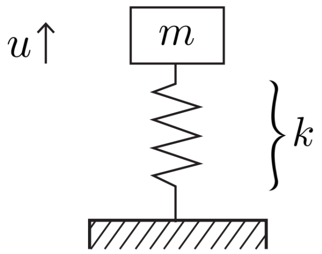


Figure 2.1: A diagram of the simple harmonic oscillator with a mass m and spring coefficient k.

2.1.3. Damped harmonic oscillator

For the damped harmonic oscillator, the previous model for the simple harmonic oscillator is expanded. The damping force is introduced into the model. Damping linearly influences the velocity of the object and is thus denoted by

$$F_{\text{damp}} = c(t)v, \tag{2.4}$$

where $c(t)$ is the, potentially time-dependent, damping coefficient and $\mathbf{v}$ is the velocity of the object. As mentioned previously, velocity is also time-dependent. The damping is assumed to contain a constant, "parasitic" part c_m and a time-variable part $c_v(t)$, so

$$c(t) = c_m + c_v(t). \tag{2.5}$$

The above equation is rewritten with the consideration that $\mathbf{v} = \frac{\mathrm{d}_u}{\mathrm{d}_t}$, giving the equation of motion for the damped harmonic oscillator as:

$$-ku - c(t)\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(m\frac{\mathrm{d}u}{\mathrm{d}t}\right). \tag{2.6}$$

A visualisation of the damped harmonic oscillator is given in Figure 2.2.

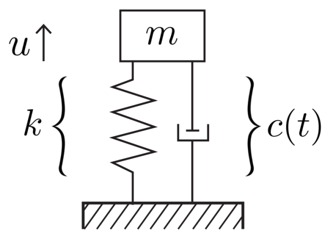


Figure 2.2: A diagram of the damped harmonic oscillator with mass m, spring coefficient k and damping value $c(t)$.

2.1.4. Damped harmonic oscillator with gravity

Objects used in this model will experience a gravitational force, which is defined as

$$F_{\text{g}} = mg, \tag{2.7}$$

with g being equal to the gravitational acceleration. The gravitational force will influence the mass in the damped harmonic oscillator. Thus, the equations of motion will change to

$$-ku - c(t)\frac{\mathrm{d}u}{\mathrm{d}t} - mg = \frac{\mathrm{d}}{\mathrm{d}t}\left(m\frac{\mathrm{d}u}{\mathrm{d}t}\right). \tag{2.8}$$

2.1.5. Damped harmonic oscillator with base excitation

This thesis focuses on energy harvesting from external vibrations. These external vibrations are considered to be base excitations, which means that the base to which the spring is attached, is moved. If it is assumed that this moving of the base is indicated with displacement $y_s(t)$, the governing equations of motion are

$$-c(t)\left(\frac{\mathrm{d}u}{\mathrm{d}t} - \frac{\mathrm{d}y_s}{\mathrm{d}t}\right) - k(u - y_s) - mg = \frac{\mathrm{d}}{\mathrm{d}t}\left(m\frac{\mathrm{d}u}{\mathrm{d}t}\right). \tag{2.9}$$

A visualisation of how the base excitation affects the damped harmonic oscillator is given in Figure 2.3, Figure 2.4 and Figure 2.5. Here, the base excitation travels from left to right.

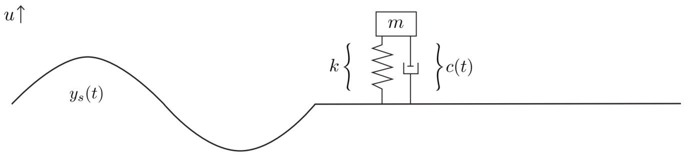


Figure 2.3: Diagram of the damped harmonic oscillator with base excitation, which is not affected yet by the base excitation $y_s(t)$ with mass m, spring coefficient k and damping coefficient $c(t)$.

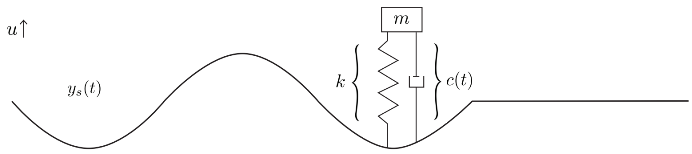


Figure 2.4: Diagram of the damped harmonic oscillator with base excitation, where the oscillator is extended by the base excitation $y_s(t)$ with mass m, spring coefficient k and damping coefficient $c(t)$.

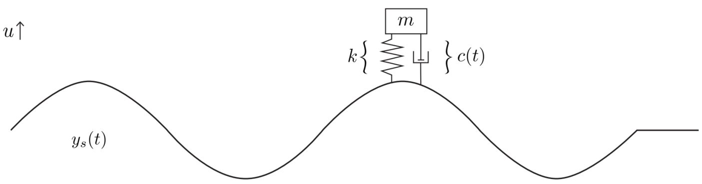


Figure 2.5: Diagram of the damped harmonic oscillator with base excitation, where the oscillator is compressed by the base excitation $y_s(t)$ with mass m, spring coefficient k and damping coefficient $c(t)$.

2.1.6. Notation and assumptions

Equation (2.9) forms the foundation for this thesis. A small change of notation is introduced to simplify the writing of the derivative. This notation is

$$u_t = \frac{\mathrm{d}u}{\mathrm{d}t}, \tag{2.10}$$

where, if a variable already has a subscript, the notation will change to

$$y_{s,t} = \frac{\mathrm{d}y_s}{\mathrm{d}t}. \tag{2.11}$$

This change in notation transforms Equation (2.9) into

$$-c(t)\left(u_t - y_{s,t}\right) - k\left(u - y_s\right) - mg = mu_{tt}. \tag{2.12}$$

Rewriting slightly gives

$$mu_{tt} + c(t)(u_t - y_{s,t}) + k(u - y_s) + mg = 0. \tag{2.13}$$

To simplify the equation, the new variable $\hat{z} = u - y_s$ is introduced. This gives that $u_t = \hat{z}_t + y_{s,t}$ and $u_{tt} = \hat{z}_{tt} + y_{s,tt}$ transforming the equation into

$$m\hat{z}_{tt} + c(t)\hat{z}_t + k\hat{z} + mg = -my_{s,tt}. \tag{2.14}$$

Now, another transformation will be applied with respect to the equilibrium position to remove the influence of gravity on the displacement. The transformation of $z = \hat{z} - \hat{z}_{\text{eq}}$ is made, where $\hat{z}_{\text{eq}}$ represents the equilibrium position of $\hat{z}$. As $\hat{z}_{\text{eq}}$ represents an equilibrium position, the time derivative of this variable will be equal to 0 as the position is constant. Hence, the equation will transform to

$$mz_{tt} + c(t)z_t + kz + k\hat{z}_{\text{eq}} + mg = -my_{s,tt}. \tag{2.15}$$

Note that, at an equilibrium position, the forces are in balance. This implies that

$$\sum F = 0 = F_{\text{spring}} + F_{\text{g}}. \tag{2.16}$$

Specifically,

$$-F_{\text{spring}} = k\hat{z}_{\text{eq}} = -mg, \tag{2.17}$$

thus transforming Equation (2.15) into

$$mz_{tt} + c(t)z_t + kz = -my_{s,tt}. \tag{2.18}$$

Rewriting, with the assumption that $m > 0$, gives

$$z_{tt} + \frac{c(t)}{m}z_t + \frac{k}{m}z = -y_{s,tt}. \tag{2.19}$$

2.1.7. Boundary conditions

The model will need boundary conditions before a solution can be found. Due to the system being moved by a base excitation, the boundary conditions are generally assumed to be

$$z(T_0) = 0, \tag{2.20}$$

and

$$z_t(T_0) = 0, \tag{2.21}$$

where T_0 is the starting time of the model.

2.2. Rescaling

An important step in simplifying the studied equation for the damped harmonic oscillator is to rescale the equation. This way, the number of variables in the equation can be considerably reduced, simplifying the later optimisation process.

To rescale the equation, the time and displacement are scaled with to be determined variables λ and μ, giving

$$t = \frac{1}{\lambda}\bar{t}, z = \mu\bar{z}, \tag{2.22}$$

and so, the derivatives are as

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\mu\bar{z}\right) = \mu\frac{\mathrm{d}\bar{z}}{\mathrm{d}\bar{t}}\frac{\mathrm{d}\bar{t}}{\mathrm{d}t} = \mu\lambda\bar{z}_{\bar{t}}, \tag{2.23}$$

and

$$\frac{\mathrm{d}^2z}{\mathrm{d}t^2} = \mu\lambda^2\bar{z}_{\bar{t}\bar{t}}. \tag{2.24}$$

This transforms Equation (2.19) into

$$\mu\lambda^2\bar{z}_{\bar{t}\bar{t}} + \frac{c(t)}{m}\mu\lambda\bar{z}_{\bar{t}} + \frac{k}{m}\mu\bar{z} = -y_{s,tt}, \tag{2.25}$$

and, so,

$$\bar{z}_{\bar{t}\bar{t}} + \frac{c(t)}{m}\frac{1}{\lambda}\bar{z}_{\bar{t}} + \frac{k}{m}\frac{1}{\lambda^2}\bar{z} = -\frac{1}{\mu\lambda^2}y_{s,tt}. \tag{2.26}$$

Rewriting the right hand side using

$$\frac{\mathrm{d}^2 y_s}{\mathrm{d}t^2} = \frac{\mathrm{d}^2 y_s}{\mathrm{d}\bar{t}^2}\frac{\mathrm{d}^2\bar{t}}{\mathrm{d}t^2} = y_{s,\bar{t}\bar{t}}\lambda^2, \tag{2.27}$$

gives Equation (2.26) to be

$$\bar{z}_{\bar{t}\bar{t}} + \frac{c(t)}{m}\frac{1}{\lambda}\bar{z}_{\bar{t}} + \frac{k}{m}\frac{1}{\lambda^2}\bar{z} = -\frac{1}{\mu}y_{s,\bar{t}\bar{t}}. \tag{2.28}$$

Thus, we define

$$\lambda^2 = \frac{k}{m}, \tag{2.29}$$

and, as $y_{s,\bar{t}\bar{t}} = y_0\cos(f_s\bar{t}\frac{1}{\lambda} + \varphi)$

$$\mu = -y_0, \tag{2.30}$$

so that Equation (2.28) transforms into

$$\bar{z}_{\bar{t}\bar{t}} + \frac{c(t)}{m}\sqrt{\frac{m}{k}}\bar{z}_{\bar{t}} + \bar{z} = \cos(f_s\sqrt{\frac{m}{k}}\bar{t} + \varphi). \tag{2.31}$$

Thus, simplifying further, the rescaled equation is

$$\bar{z}_{\bar{t}\bar{t}} + \frac{c(t)}{\sqrt{mk}}\bar{z}_{\bar{t}} + \bar{z} = \cos(f_s\sqrt{\frac{m}{k}}\bar{t} + \varphi), \tag{2.32}$$

and, defining $\psi = \frac{1}{\sqrt{mk}}$ and $\hat{f}_s = f_s\sqrt{\frac{m}{k}}$, gives

$$\bar{z}_{\bar{t}\bar{t}} + \psi c(t)\bar{z}_{\bar{t}} + \bar{z} = \cos(\hat{f}_s\bar{t} + \varphi), \tag{2.33}$$

where we have that

$$t = \sqrt{\frac{m}{k}}\bar{t}, \tag{2.34}$$

and

$$z = -y_0\bar{z}. \tag{2.35}$$

For simplicity, the notation with bars and hats in Equation (2.33) is dropped for the remainder of this thesis.

2.3. Damping

The behaviour of the damping is crucial to this thesis, as the goal is to research the time-averaged power yielded from the potentially variable damping value. This section describes what is meant by piecewise changing damping and pseudocontinuous damping. Afterwards, the influence of the discontinuity of these variable damping values on the general equations of motion is discussed.

2.3.1. Piecewise Changing Damping

As the name implies, piecewise changing damping is a damping which changes piecewise. We assume that for the piecewise changing damping, it only changes one time. We define this particular moment of switch in damping value to be at a certain time $t = T_m$. This value lies within a certain interval $[T_b, T_f]$, which usually equals one period of the base excitation. Thus,

$$c(t) = \begin{cases} c_1, & \text{for } T_b \leq t < T_m, \\ c_2, & \text{for } T_m \leq t \leq T_f. \end{cases} \tag{2.36}$$

Note that this damping is not continuous, as it jumps at T_m.

2.3.2. Pseudocontinuous Damping

Pseudocontinuous damping mimics the behaviour of a continuous function, but still isn't continuous. It is a combination of multiple piecewise changing dampings at particular intervals. Two examples are given in Figure 2.6 and Figure 2.7.

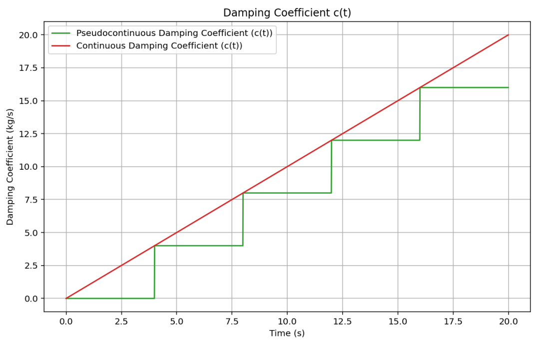


Figure 2.6: An example continuous damping $c(t) = t$ in red with a pseudocontinuous damping in green.

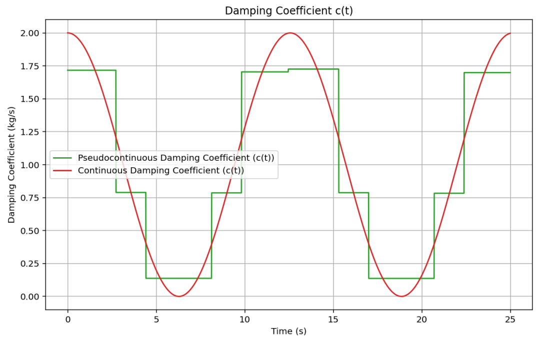


Figure 2.7: Another example continuous damping $c(t) = 1 + \cos(\frac{t}{2})$ in red with a pseudocontinuous damping in green.

Note that in Figure 2.6 the intervals are of the same size and in Figure 2.7 the intervals have different lengths. This is due to the function approximation method that is chosen to approximate the damping.

As mentioned before, for pseudocontinuous damping, multiple piecewise changing dampings are used. Intuitively, this is similar to taking the intervals used in the Riemann sum of a particular function.

Different methods are commonly used for the approximation of functions, for example, the left-hand approximation. Figure 2.6 displays how the left-hand approximation would work. The left-hand rule states that the integral of a particular function $f(x)$ on the interval $[a, b]$ can be approximated by

$$\int_a^b f(x) \approx \sum_{i=1}^{n} f(x_i)\Delta x, \tag{2.37}$$

where n represents the number of intervals chosen and Δx the length of the intervals. Larger n implies a better approximation. Instead of approximating the integral of the particular function, this theory is instead used to approximate the function itself. By using a stepwise function, the approximation is expressed as

$$f(x) \approx \begin{cases} f(x_1), & \text{for } x_1 \leq x < x_2, \\ f(x_2), & \text{for } x_2 \leq x < x_3, \\ \vdots & \\ f(x_j), & \text{for } x_j \leq x \leq x_{j+1}, \end{cases} \tag{2.38}$$

or, generally,

$$f(x) \approx \begin{cases} f(x_i), & \text{for } x_i \leq x < x_{i+1}, \text{ where } i \in 1, ..., j-1, \\ f(x_j), & \text{for } x_j \leq x \leq x_{j+1}. \end{cases} \tag{2.39}$$

Next to the left-hand approximation rule, also the right-hand rule and midpoint rule exist, which, respectively, would yield

$$f(x) = \begin{cases} f(x_{i+1}), & \text{for } x_i \leq x < x_{i+1}, \text{ where } i \in 1, ..., j-1, \\ f(x_{j+1}), & \text{for } x_j \leq x \leq x_{j+1}, \end{cases} \tag{2.40}$$

and

$$f(x) \approx \begin{cases} f(\frac{x_i + x_{i+1}}{2}), & \text{for } x_i \leq x < x_{i+1}, \text{ where } i \in 1, ..., j-1, \\ f(\frac{x_j + x_{j+1}}{2}), & \text{for } x_j \leq x \leq x_{j+1}. \end{cases} \tag{2.41}$$

Clearly, when more intervals are taken, the approximation will be closer to the actual function value. However, using more intervals is computationally more expensive. Hence, an optimum must be found for the minimum intervals and error between the function and the approximation.

The problem with these approximation rules is that they do not take into account the behaviour of the function that is approximated. This can be seen in the following figure.

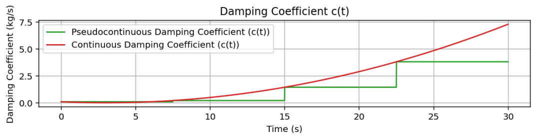


Figure 2.8: The graph $c(t) = \frac{t^2}{100}$ with a left hand approximation for the pseudocontinuous damping.

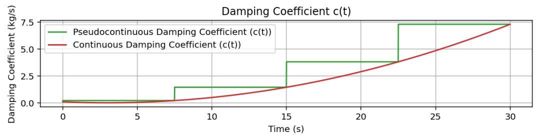


Figure 2.9: The graph $c(t) = \frac{t^2}{100}$ with a right hand approximation for the pseudocontinuous damping.

Here, the intervals are divided equally over the graph with every interval having an equal length, but the approximation would be more accurate if the intervals were distributed unequally and had different interval lengths. Specifically, a distribution of more intervals when the graph becomes steeper would be optimal. To realise the exact distribution of intervals at the best moments to minimise the error between the continuous and pseudocontinuous damping, a numerical implementation is used. This code is given in Appendix A. The code provides the best approximation of a function based on the number of intervals required by computing the difference between different values of a function f, hence effectively judging the steepness of the graph.

2.3.3. Influence of the discontinuous damping

Having a discontinuous damping means that the acceleration of the system will not be continuous, as:

$$\begin{cases} z_t = x, \\ mx_t = -c(t)x - kz + F(t). \end{cases} \tag{2.42}$$

The jump in the damping will make it behave like a Heaviside function. In Figure 2.10 a plot is made of $c(t)$, which illustrates that it acts like a Heaviside function.

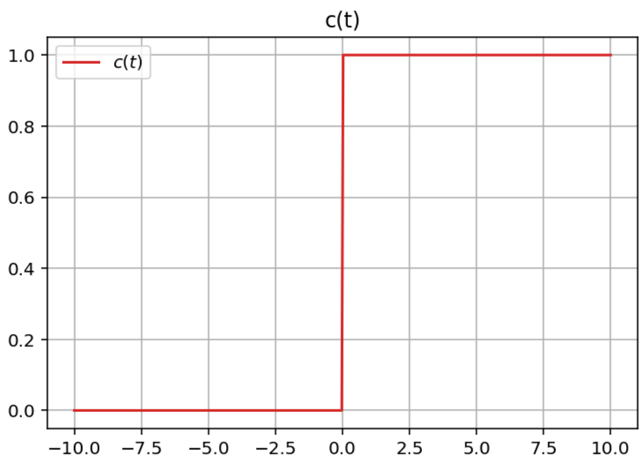


Figure 2.10: A plot of a $c(t)$ where $c_1 = 0$ and $c_2 = 1$, showing how $c(t)$ acts like a Heaviside function.

When a Heaviside function is integrated, it becomes continuous, if the same integration constants are chosen. This is illustrated in Figure 2.11.

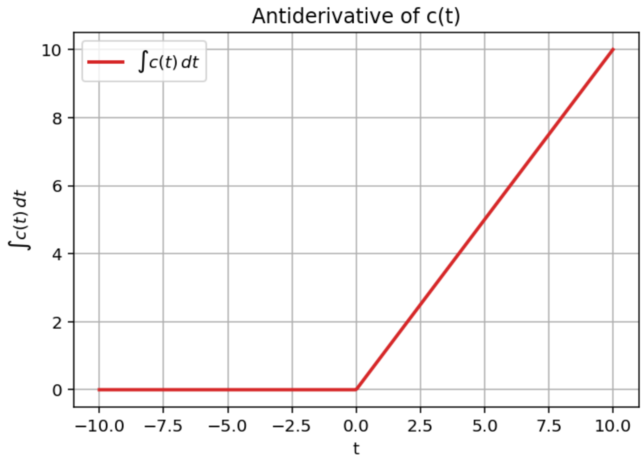


Figure 2.11: A plot of the antiderivative of $c(t)$ where $c_1 = 0$ and $c_2 = 1$, showing how the antiderivative of $c(t)$ is continuous.

Therefore, although the acceleration is discontinuous, the velocity and the displacement are continuous. This is crucial for the influence of the velocity and displacement on the power analysis later on in this thesis. Those being continuous will make the analysis substantially easier.

Before explaining the exact function approximation method that is chosen for this pseudocontinuous damping, it should be noted that, via reasoning similar to that in the previous section, the displacement and velocity are continuous when using pseudocontinuous damping.

2.4. Power analysis

Since one of the two objectives of this thesis is to correctly measure the harvested power from the oscillation, in this section, the measure taken to research how much power is harvested from the damped harmonic oscillator is discussed.

2.4.1. Instantaneous power

Instantaneous power is defined as the power that is obtained by a specific force at a particular moment in time. We have instantaneous power defined as

$$P_{\text{inst}} = F \cdot v. \tag{2.43}$$

Instantaneous power for the damper is therefore

$$P_{\text{inst}} = F_{\text{damp}} \cdot v, \tag{2.44}$$

or, using the previous notations and assumptions

$$P_{\text{inst}} = -c(t) z_t^2. \tag{2.45}$$

As our goal is to determine the power obtained by the non-parasitic damping specifically, above equation should be rewritten to

$$P_{\text{inst}} = -c_v(t) z_t^2. \tag{2.46}$$

Yet, as the damping might experience resonance and therefore power gains at particular moments might not be representative for the whole period of an incoming wave, the time-averaged power is studied.

2.4.2. Time-averaged power

Time-averaged power is defined as, such as the name implies, the average power gain over a certain period in time. Hence,

$$P_{\text{ave}} = \frac{1}{T_e - T_b} \int_{T_b}^{T_e} P_{\text{ins}} \mathrm{d}t, \tag{2.47}$$

where T_b and T_e are the respective begin and end time of the particular time period. This thesis will make use of the time-averaged power for an analysis of the particular energy yield of the system. Specifically, the interval will be taken as one oscillation from the incoming base excitation oscillation. Thus,

$$P_{\text{ave}} = \frac{1}{\frac{2\pi+k2\pi}{f_s} - \frac{k2\pi}{f_s}} \int_{\frac{k2\pi}{f_s}}^{\frac{2\pi+k2\pi}{f_s}} P_{\text{ins}} \mathrm{d}t, \tag{2.48}$$

with k being an integer. For simplicity and due to the periodicity of the goniometric formulas, this form can be assumed to be equal to

$$P_{\text{ave}} = \frac{1}{\frac{2\pi}{f_s}} \int_0^{\frac{2\pi}{f_s}} P_{\text{ins}} \mathrm{d}t. \tag{2.49}$$

and therefore, using the definition of instantaneous power given in Equation (2.46)

$$P_{\text{ave}} = \frac{1}{\frac{2\pi}{f_s}} \int_0^{\frac{2\pi}{f_s}} -c_v(t) z_t^2 \mathrm{d}t. \tag{2.50}$$

2.5. Parameter estimates

This section explains the bounds that exist on the different parameters present in the equations of motion for the damped harmonic oscillator. These bounds are relevant for computing the boundary values, which may influence what solution is optimal for the time-averaged power yield. Bounds which are found in literature are given in Appendix B in Table B.1.

It can be seen that there are several applications of the damped harmonic oscillator for which the parameter values differ significantly. Due to the probability of any application of this thesis first being tested in a lab, parameters close to the precision lab setup are chosen. These values can be found in Table 2.1.

Table 2.1: Parameter values of the damped harmonic oscillator chosen for the damped harmonic oscillator with base excitation.

Parameter value [unit]	Symbol	Value
Mass [kg]	m	1
Spring coefficient [N/m]	k	$4\pi^2$
Parasitic damping [Ns/m]	c_m	0.1
Variable damping [Ns/m]	c_v or $c_{v,i}$	$[-0.15, 0.35]$
Amplitude [m]	y_0	10^{-3}
Frequency [Hz]	f_s	$(0, 2)$
Phase [-]	φ	$[-\pi, \pi)$

3

Damped Harmonic Oscillator with Base Excitation and Variable Damping

The damped harmonic oscillator model forms the basis of the research of this thesis. This chapter will build towards solving the damped harmonic oscillator model with base excitation and variable damping. This is done by first solving the model for a constant damping, as has been done in Section 3.1. Therafter, the model is solved for a piecewise changing damping in Section 3.2. Next, for pseudocontinuous damping, the model is solved in Section 3.3. Each of these sections will discuss how the solution of the damped harmonic oscillator is modelled and show how the different variable parameters influence the behaviour of the oscillator. Lastly, in Section 3.4, concluding remarks are discussed.

3.1. Constant damping

As was concluded in Section 2.1.6 of Chapter 2, the governing equations of motion for the damped harmonic oscillator with base excitation are

$$z_{tt} + \psi c(t) z_t + z = \cos(f_s t + \varphi) \tag{3.1}$$

For this section, the damping is assumed to be constant. Hence, there will be no time-dependency of the damping coefficient, so

$$z_{tt} + \psi c z_t + z = \cos(f_s t + \varphi) \tag{3.2}$$

is the governing equation of motion. To solve this equation, the homogeneous solution and particular solution are derived, as the general solution is the sum of these two solutions, so

$$z(t) = z_{\text{hom}}(t) + z_{\text{part}}(t). \tag{3.3}$$

The homogeneous solution is derived in Section 3.1.1 and the particular solution in Section 3.1.2. Then, the general solution is provided in Section 3.1.3 and visualised in Section 3.1.4. The influence of the different parameters is discussed in Section 3.1.5.

3.1.1. Homogeneous solution

For the homogeneous solution of the governing equation, no base excitation is assumed. This simplifies the equations of motion to

$$z_{\text{hom},tt} + \psi c z_{\text{hom},t} + z_{\text{hom}} = 0. \tag{3.4}$$

Using the ansatz that $z_{\text{hom}} \sim \mathrm{e}^{i\omega t}$, where i is the complex root, and ω is a constant, gives that

$$\mathrm{e}^{i\omega t}\left(-\omega^2 + i\psi c\omega + 1\right) = 0. \tag{3.5}$$

The exponent will not equal 0 and therefore the quadratic formula should equate to 0. Solving this quadratic formula gives that

$$\omega = \frac{1}{2}\left(ci\psi \pm \sqrt{4 - c^2\psi^2}\right). \tag{3.6}$$

Using the superposition principle, the solution of the homogeneous problem is given by

$$z_{\text{hom}}(t) = \mathrm{e}^{\frac{-c\psi t}{2}} \left(d_1 \mathrm{e}^{\frac{it}{2}\sqrt{4-c^2\psi^2}} + d_2 \mathrm{e}^{\frac{-it}{2}\sqrt{4-c^2\psi^2}} \right), \tag{3.7}$$

which could also be rewritten to a form using sines and cosines. This is explored further later in this thesis. For the general solution, the form with exponentials is used.

Rewriting the above form to remove the complex root using the definition that $i^2 = -1$, gives

$$z_{\text{hom}}(t) = \mathrm{e}^{\frac{-c\psi t}{2}} \left(d_1 \mathrm{e}^{\frac{t}{2}\sqrt{c^2\psi^2-4}} + d_2 \mathrm{e}^{\frac{-t}{2}\sqrt{c^2\psi^2-4}} \right), \tag{3.8}$$

which implies that three different solutions exist depending on the value in the square root. These three different solutions are discussed case by case.

$c^2\psi^2 > 4$

The homogenous solution in the case where $c^2\psi^2 > 4$ will be the same as Equation (3.8), as the root is positive in this case. So,

$$z_{\text{hom}}(t) = \mathrm{e}^{\frac{-c\psi t}{2}} \left(d_1 \mathrm{e}^{\frac{t}{2}\sqrt{c^2\psi^2-4}} + d_2 \mathrm{e}^{\frac{-t}{2}\sqrt{c^2\psi^2-4}} \right). \tag{3.9}$$

$c^2\psi^2 = 4$

In the case where $c^2\psi^2 = 4km$, the two solutions of the quadratic formula given in Equation (3.6) are the same, this solution is defined as $z_1(t)$. As there should be two distinct solutions, as it is a quadratic equation, the method of reduction of order is used to find the other solution. So, using a newly time-dependent variable $\sigma(t)$ to get

$$z_{\text{hom}}(t) = z_1(t)\sigma(t), \tag{3.10}$$

which implies that

$$z_{\text{hom},t} = \sigma(t)z_{1,t}(t) + z_1(t)\sigma_t(t), \tag{3.11}$$

and

$$z_{\text{hom},tt} = \sigma(t)z_{1,tt}(t) + 2z_{1,t}(t)\sigma_t(t) + z_1(t)\sigma_{tt}(t). \tag{3.12}$$

Substituting these back into Equation (3.4), gives

$$\sigma(t)z_{1,tt}(t) + 2z_{1,t}(t)\sigma_t(t) + z_1(t)\sigma_{tt}(t) + \psi c\left(\sigma(t)z_{1,t}(t) + z_1(t)\sigma_t(t)\right) + z_1(t)\sigma(t) = 0. \tag{3.13}$$

As z_1 is a solution of Equation (3.4), the equation can be simplified to

$$z_1(t)\sigma_{tt}(t) + 2z_{1,t}(t)\sigma_t(t) + \psi c z_1(t)\sigma_t(t) = 0, \tag{3.14}$$

or, simplified even further,

$$z_1(t)\sigma_{tt}(t) + \sigma_t(t)\left(2z_{1,t}(t) + \psi c z_1(t)\right) = 0. \tag{3.15}$$

This is a first-order linear equation for $\sigma_t(t)$, and its solution is given by

$$\sigma_t(t) = d_3 \mathrm{e}^{-\int\left(2\frac{z_{1,t}(t)}{z_1(t)} + c\psi\right)\mathrm{d}t} = d_3 \mathrm{e}^{-\int c\psi \mathrm{d}t} \mathrm{e}^{-2\int\left(\frac{z_{1,t}(t)}{z_1(t)}\right)\mathrm{d}t} = \frac{d_3 \mathrm{e}^{-\int c\psi \mathrm{d}t}}{z_1^2(t)}. \tag{3.16}$$

Substituting the already-found solution

$$z_1(t) = d_1 \mathrm{e}^{\frac{-c\psi t}{2}}, \tag{3.17}$$

gives

$$\sigma_t(t) = d_3 \frac{1}{d_1^2} \mathrm{e}^{-c\psi t} \mathrm{e}^{-(c\psi t)} = \frac{d_3}{d_1^2}. \tag{3.18}$$

Note that constant d_3 can be freely chosen, as we are only interested in one solution to the differential equation for $\sigma_1(t)$, given in Equation (3.16). Thus, we choose $d_3 = d_1^2$, so that

$$\sigma_t(t) = 1, \tag{3.19}$$

and therefore the solution to the homogeneous problem is

$$z_{\text{hom}}(t) = d_1 \mathrm{e}^{-c\psi t} + d_2 t \mathrm{e}^{-c\psi t}. \tag{3.20}$$

$c^2\psi^2 < 4$

In the case where $c^2\psi^2 < 4$, the square root will be positive in the solution of the quadratic formula, given in Equation (3.6). Therefore, the last step in determining the general homogeneous solution, rewriting the solution to the form given in Equation (3.8), is unnecessary. To remove the complex root in the equation, the exponentials are rewritten to sines and cosines, so

$$\begin{aligned} z_{\text{hom}}(t) = \mathrm{e}^{\frac{-c\psi t}{2}} \Bigg(d_1 \left[\cos\left(t\frac{\sqrt{4-c^2\psi^2}}{2} \right) + i \sin\left(t\frac{\sqrt{4-c^2\psi^2}}{2} \right) \right] \\ + d_2 \left[\cos\left(-t\frac{\sqrt{4-c^2\psi^2}}{2} \right) + i \sin\left(-t\frac{\sqrt{4-c^2\psi^2}}{2} \right) \right] \Bigg). \end{aligned} \tag{3.21}$$

Combining the sines and cosines into their forms gives

$$z_{\text{hom}}(t) = \mathrm{e}^{\frac{-c\psi t}{2}} \left(\cos\left(t\frac{\sqrt{4-c^2\psi^2}}{2} \right) [d_1 + d_2] + i \sin\left(t\frac{\sqrt{4-c^2\psi^2}}{2} \right) [d_1 - d_2] \right). \tag{3.22}$$

Hence, by introducing new constants $d_3 = d_1 + d_2$ and $d_4 = i\,(d_1 - d_2)$, the solution for the case where $c^2\psi^2 < 4$ is

$$z_{\text{hom}}(t) = \mathrm{e}^{\frac{-c\psi t}{2}} \left(d_3 \cos\left(t\frac{\sqrt{4-c^2\psi^2}}{2} \right) + d_4 \sin\left(t\frac{\sqrt{4-c^2\psi^2}}{2} \right) \right). \tag{3.23}$$

3.1.2. Particular Solution

Now that the three different solutions, based on the value of the damping coefficient and variable ψ, have been found, the particular solution of the problem is examined. For the particular solution to the equations of motion with constant damping, the following equation will have to be solved:

$$z_{\text{part},tt} + \psi c(t) z_{\text{part},t} + z_{\text{part}} = \cos(f_s t + \varphi). \tag{3.24}$$

Just as in the solving of the homogeneous problem, the ansatz $z_{\text{part}} \sim \mathrm{e}^{i\omega t}$ is used, which gives

$$\mathrm{e}^{i\omega t}\left(-\omega^2 + i\,\psi\,c\,\omega + 1\right) = F(t), \tag{3.25}$$

which implies that $F(t) \sim \mathrm{e}^{i\omega t}$. Using Fourier series, this exponential function can be created via $F(t) = \sum_n d_n \cos(\theta_n t + \varphi_n)$ for $n \in \mathbb{N}$, where d_n, θ_n and φ_n are constants. This rewriting to a sum of cosines is beneficial, as it can be physically interpreted as a wave with amplitude d_n, frequency θ_n and phase φ_n. So, if a solution is found for

$$z_{n,tt} + \psi c z_{n,t} + z_n = \cos(\theta_n t), \tag{3.26}$$

it can be combined to create

$$\sum_n d_n \left(z_{n,tt} + \psi c z_{n,t} + z_n \right) = \sum_n d_n \cos(\theta_n t), \tag{3.27}$$

which implies that a solution is found for the general equation

$$z_{\text{part},tt} + \psi c z_{\text{part},t} + z_{\text{part}} = \cos(f_s t + \varphi), \tag{3.28}$$

where y_0 represents the amplitude of the incoming wave, f_s represents the frequency and φ the phase. Using Euler's Formula,

$$\mathrm{e}^{-if_s t+\varphi} = \cos(f_s t + \varphi) - i \sin(f_s t + \varphi), \tag{3.29}$$

gives that, if assuming that there exists a $v(t)$ which solves

$$v_{tt} + \psi c v_t + v = \mathrm{e}^{-if_s t+\varphi}, \tag{3.30}$$

and taking $\mathrm{Re}(v) = z_{\text{part}}$ with noting that $\mathrm{Re}(\mathrm{e}^{-if_s t+\varphi}) = \cos(f_s t + \varphi)$, one should have that

$$\mathrm{Re}(v_{tt} + \psi c v_t + v) = \mathrm{Re}(\mathrm{e}^{-if_s t+\varphi}) = \cos(f_s t + \varphi) = z_{\text{part},tt} + \psi c z_{\text{part},t} + z_{\text{part}}. \tag{3.31}$$

This means, that to solve the particular solution, a solution should be found for

$$v_{tt} + \psi c v_t + v = y_0 \mathrm{e}^{-if_s t + \varphi}. \tag{3.32}$$

Again using ansatz that $v(t) = A\mathrm{e}^{-if_s t+\varphi}$,

$$A\mathrm{e}^{-if_s t+\varphi}\left(-f_s^2 - i\psi c f_s + 1\right) = \mathrm{e}^{-if_s t+\varphi}. \tag{3.33}$$

same assumption that $\mathrm{e}^{-if_s t+\varphi} \neq 0$, gives

$$A = \frac{1}{-f_s^2 - i\psi c f_s + 1}, \tag{3.34}$$

which gives then

$$z_{\text{part}}(t) = \text{Re}(v(t)) = \text{Re}\left(\frac{1}{-f_s^2 - i\psi c f_s + 1}\mathrm{e}^{-if_s t+\varphi}\right) \tag{3.35}$$

to remove the complex part of the constant A it is rewritten

$$\frac{1}{-f_s^2 - i\psi c f_s + 1} = \frac{1}{-f_s^2 - i\psi c f_s + 1}\frac{-f_s^2 + i\psi c f_s + 1}{-f_s^2 + i\psi c f_s + 1}, \tag{3.36}$$

$$= \frac{-f_s^2 + i\psi c f_s + 1}{\left(f_s^2 - 1\right)^2 + \left(\psi c f_s\right)^2}, \tag{3.37}$$

$$= \frac{-f_s^2 + 1}{\left(f_s^2 - 1\right)^2 + \left(\psi c f_s\right)^2} + i\frac{\psi c f_s}{\left(f_s^2 - 1\right)^2 + \left(\psi c f_s\right)^2}, \tag{3.38}$$

so,

$$z_{\text{part}}(t) = \text{Re}\left(\left(\frac{-f_s^2 + 1}{\left(f_s^2 - 1\right)^2 + \left(\psi c f_s\right)^2} + i\frac{\psi c f_s}{\left(f_s^2 - 1\right)^2 + \left(\psi c f_s\right)^2}\right)\left(\cos(f_s t + \varphi) - i\sin(f_s t + \varphi)\right)\right). \tag{3.39}$$

This implies that

$$z_{\text{part}}(t) = \frac{-f_s^2 + 1}{\left(f_s^2 - 1\right)^2 + \left(\psi c f_s\right)^2}\cos(f_s t + \varphi) + \frac{\psi c f_s}{\left(f_s^2 - 1\right)^2 + \left(\psi c f_s\right)^2}\sin(f_s t + \varphi), \tag{3.40}$$

or, using the Harmonic Addition Theorem in Appendix C,

$$z_{\text{part}}(t) = \frac{1}{\sqrt{\left(f_s^2 - 1\right)^2 + \left(\psi c f_s\right)^2}}\left(\cos(f_s t + \varphi + \delta)\right). \tag{3.41}$$

where $\delta = \arctan\left(-\frac{\psi c f_s}{-f_s^2 + 1}\right)$, or, assuming that $\hat{\varphi} = \varphi + \delta$,

$$z_{\text{part}}(t) = \frac{1}{\sqrt{\left(f_s^2 - 1\right)^2 + \left(\psi c f_s\right)^2}}\left(\cos(f_s t + \hat{\varphi})\right). \tag{3.42}$$

3.1.3. General Solution

The general solution is, as given in Equation (3.3),

$$z(t) = \frac{1}{\sqrt{\left(f_s^2-1\right)^2+\left(\psi c f_s\right)^2}}\left(\cos(f_s t+\hat{\varphi})\right)+\begin{cases} \mathrm{e}^{\frac{-c\psi t}{2}}\left(d_1\mathrm{e}^{\frac{t}{2}\sqrt{c^2\psi^2-4}}+d_2\mathrm{e}^{\frac{-t}{2}\sqrt{c^2\psi^2-4}}\right) & \text{for } c^2\psi^2>4,\\ d_1\mathrm{e}^{-c\psi t}+d_2 t\mathrm{e}^{-c\psi t} & \text{for } c^2\psi^2=4,\\ \mathrm{e}^{\frac{-c\psi t}{2}}\left(d_1\cos\left(t\frac{\sqrt{4-c^2\psi^2}}{2}\right)+d_2\sin\left(t\frac{\sqrt{4-c^2\psi^2}}{2}\right)\right) & \text{for } c^2\psi^2<4. \end{cases} \tag{3.43}$$

3.1.4. Visualising solution

In Section 3.1.3, the general solution for the damped harmonic oscillator with base excitation with a constant damping has been given. This solution depends on the relationship between the damping, spring coefficient, and mass. The constants of the general solution are still to be determined and can be found by rewriting the solution to

$$\begin{pmatrix} z(t)\\ z_t(t)\end{pmatrix} = C(t)\begin{pmatrix} d_1\\ d_2\end{pmatrix}+\begin{pmatrix} z_{\text{part}}(t)\\ z_{\text{part},t}(t)\end{pmatrix}. \tag{3.44}$$

To determine the constants d_1 and d_2, it can be evaluated at a particular starting time T_0 given the initial displacement $z(T_0)$ and velocity $z_t(T_0)$ to get

$$\begin{pmatrix} z(T_0)\\ z_t(T_0)\end{pmatrix} = C(T_0)\begin{pmatrix} d_1\\ d_2\end{pmatrix}+\begin{pmatrix} z_{\text{part}}(T_0)\\ z_{\text{part},t}(T_0)\end{pmatrix}, \tag{3.45}$$

which can be rewritten to

$$C^{-1}(T_0)\left[\begin{pmatrix} z(T_0)\\ z_t(T_0)\end{pmatrix}-\begin{pmatrix} z_{\text{part}}(T_0)\\ z_{\text{part},t}(T_0)\end{pmatrix}\right]=\begin{pmatrix} d_1\\ d_2\end{pmatrix}, \tag{3.46}$$

giving a final expression which is used for the visualisation of the solution to be

$$\begin{pmatrix} z(t)\\ z_t(t)\end{pmatrix} = C(t)C^{-1}(T_0)\left[\begin{pmatrix} z(T_0)\\ z_t(T_0)\end{pmatrix}-\begin{pmatrix} z_{\text{part}}(T_0)\\ z_{\text{part},t}(T_0)\end{pmatrix}\right]+\begin{pmatrix} z_{\text{part}}(t)\\ z_{\text{part},t}(t)\end{pmatrix}. \tag{3.47}$$

Different parameters influence the behaviour of the damped harmonic oscillator. For example, the damping value or the base excitation. An analysis of the influence of most of these variables is made in the following section.

3.1.5. Influence of different factors

The general solution without rescaling is visualised, as then the physical interpretation of the variables is easier. This means that Equation (2.19) is modelled, of which the solution is

$$z(t) = \frac{y_0}{\sqrt{\left(f_s^2-\frac{k}{m}\right)^2+\left(\frac{c}{m}f_s\right)^2}}\left(\cos(f_s t+\hat{\varphi})\right)+\begin{cases} \mathrm{e}^{\frac{-ct}{2m}}\left(d_1\mathrm{e}^{\frac{t}{2m}\sqrt{c^2-4km}}+d_2\mathrm{e}^{\frac{-t}{2m}\sqrt{c^2-4km}}\right) & \text{for } c^2>4km,\\ d_1\mathrm{e}^{\frac{-ct}{2m}}+d_2 t\mathrm{e}^{\frac{-ct}{2m}} & \text{for } c^2=4km,\\ \mathrm{e}^{\frac{-ct}{2m}}\left(d_1\cos\left(t\frac{\sqrt{4km-c^2}}{2m}\right)+d_2\sin\left(t\frac{\sqrt{4km-c^2}}{2m}\right)\right) & \text{for } c^2<4km. \end{cases} \tag{3.48}$$

It should be noted that only variables y_0, f_s, c and φ are discussed. So, respectively, only the amplitude of the base excitation, the frequency of the base excitation, the damping value and the phase of the base excitation. This spring coefficient and mass are considered fixed.

Influence of Amplitude

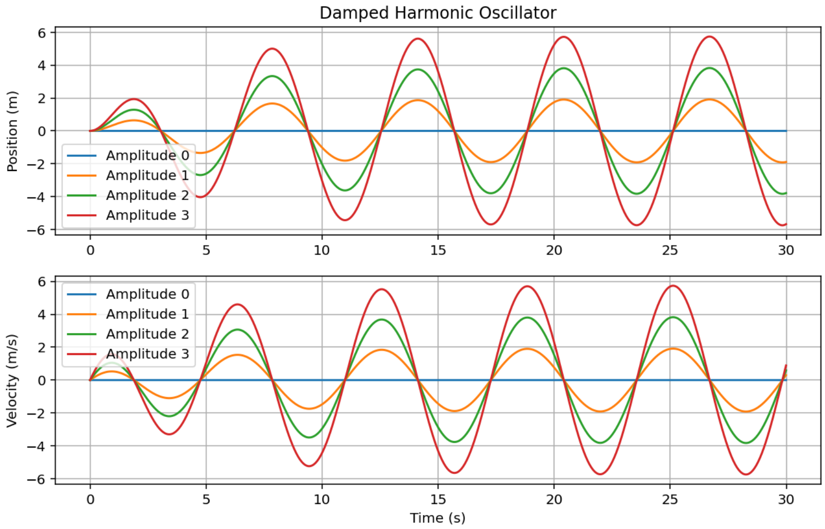


Figure 3.1: The displacement and velocity of the damped harmonic oscillator with values $c = 1, m = 1, k = 2, f_s = 1, \varphi = 0$, initial conditions $z(T_0) = 0, z_t(T_0) = 0$, and changing amplitude from 0 to 3.

The displacement and velocity with different amplitude values are plotted in Figure 3.1. As can be expected, the displacement of the damped harmonic oscillator becomes larger when the incoming base excitation has a higher amplitude. Oscillations do still occur, as the system is underdamped.

Influence of damping

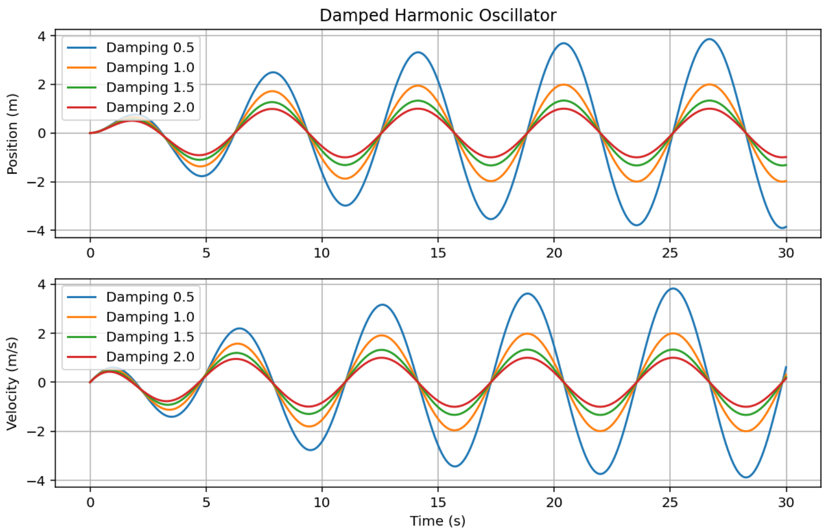


Figure 3.2: The displacement and velocity of the damped harmonic oscillator with values $y_0 = 1, m = 1, k = 2, f_s = 1, \varphi = 0$, initial conditions $z(T_0) = 0, z_t(T_0) = 0$, and changing damping from 0.5 to 2.

The displacement and velocity with different damping values are plotted in Figure 3.2. Again, as can be expected, with higher damping values, the manner of displacement decreases. This is due to the damping having a higher influence on the displacement for higher values. Oscillations do still occur, as the system is underdamped.

Influence of frequency

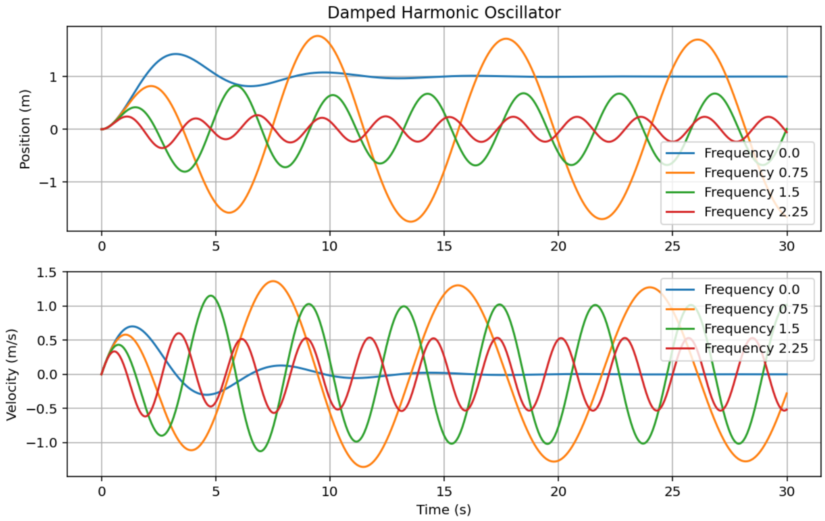


Figure 3.3: The displacement and velocity of the damped harmonic oscillator with values $y_0 = 1, m = 1, k = 2, c = 1, \varphi = 0$, initial conditions $z(T_0) = 0, z_t(T_0) = 0$, and changing frequency from 0 to 4.

The displacement and velocity with different frequencies are plotted in Figure 3.3. It can be seen that when the frequency is equal to 0, the damped harmonic oscillator will come to a standstill. This is due to no external base excitation happening, and therefore, the system will not move. Oscillations do still occur when a frequency is non equal to 0, as the system is underdamped. Frequencies which are close to the eigenfrequency result in bigger oscillations.

Influence of phase

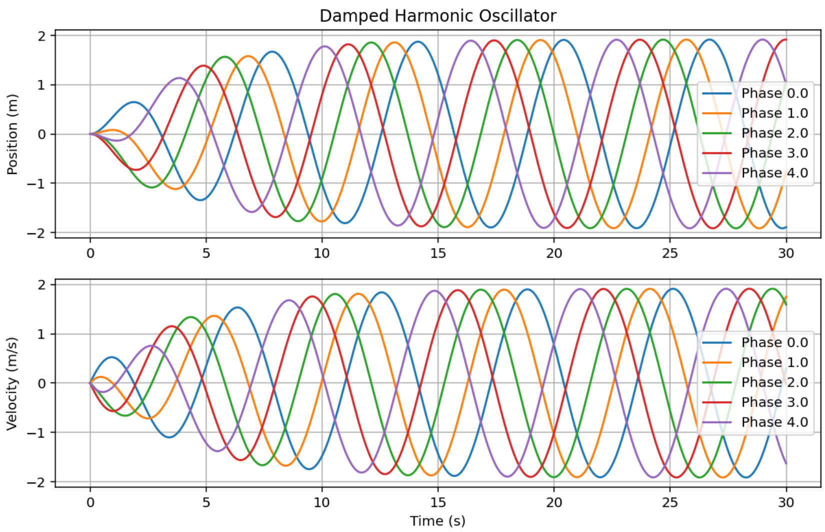


Figure 3.4: The displacement and velocity of the damped harmonic oscillator with values $y_0 = 1, m = 1, k = 2, c = 1, f_s = 1$, initial conditions $z(T_0) = 0, z_t(T_0) = 0$, and changing phase from 0 to 2.25.

The displacement and velocity with different phase values are plotted in Figure 3.4. The phase determines at which level the incoming wave hits the base excitation. Oscillations do still occur, as the system is not overdamped but rather underdamped. Logically, for different values of the phase, the oscillator is excited differently.

3.2. Piecewise changing damping

This section elaborates on the general solution when a piecewise changing damping is assumed. The piecewise changing damping is explained in Section 2.3.1. The general solution for the damped harmonic oscillator with a piecewise changing damping subject to a base excitation is given in 3.2.1. Afterwards, the visualisation of the solution is explained in Section 3.2.2 and the influence of the different parameters in discussed in Section 3.2.3.

3.2.1. General solution

As explained in Section 2.3.3, the displacement and velocity of the damped harmonic oscillator remain continuous, even though the damping value is not. Due to this continuity, the following maps can be created as

$$\begin{pmatrix} z(T_b) \\ z_t(T_b) \end{pmatrix} \xrightarrow[\phi_1(t)]{c(t)=c_1} \begin{pmatrix} z(T_m) \\ z_t(T_m) \end{pmatrix} \xrightarrow[\phi_2(t)]{c(t)=c_2} \begin{pmatrix} z(T_f) \\ z_t(T_f) \end{pmatrix}.$$

Mapping $\phi_1(t)$ and $\phi_2(t)$ are constructed in a similar manner. On the intervals $[T_b, T_m)$ and $[T_m, T_f]$, the damping is constant. This allows the previous solution from Section 3.1 to be used. The map $\phi_1(t)$ can therefore be expressed as

$$\phi_1(t) : \begin{pmatrix} z(T_m) \\ z_t(T_m) \end{pmatrix} = A_1(t) \begin{pmatrix} z(T_b) \\ z_t(T_b) \end{pmatrix} + \begin{pmatrix} z_{\text{part},1}(T_m) \\ z_{\text{part},1,t}(T_m) \end{pmatrix}, \tag{3.49}$$

and, similarly, for map $\phi_2(t)$

$$\phi_2(t) : \begin{pmatrix} z(T_f) \\ z_t(T_f) \end{pmatrix} = A_2(t) \begin{pmatrix} z(T_m) \\ z_t(T_m) \end{pmatrix} + \begin{pmatrix} z_{\text{part},2}(T_f) \\ z_{\text{part},2,t}(T_f) \end{pmatrix}. \tag{3.50}$$

These maps can be combined to create one bigger map, indicated by $\phi(t)$

$$\begin{pmatrix} z(T_b) \\ z_t(T_b) \end{pmatrix} \xrightarrow[\phi_1(t)]{c(t)=c_1} \begin{pmatrix} z(T_m) \\ z_t(T_m) \end{pmatrix} \xrightarrow[\phi_2(t)]{c(t)=c_2} \begin{pmatrix} z(T_f) \\ z_t(T_f) \end{pmatrix}, \qquad \begin{pmatrix} z(T_b) \\ z_t(T_b) \end{pmatrix} \xrightarrow{\phi(t)} \begin{pmatrix} z(T_f) \\ z_t(T_f) \end{pmatrix}.$$

Looking at the construction of the map and Equations (3.49), (3.50), it can be concluded that $\phi(t) = (\phi_2 \circ \phi_1)(t)$. The map $\phi(t)$ can therefore be expressed as

$$\phi(t) : \begin{pmatrix} z(T_f) \\ z_t(T_f) \end{pmatrix} = A_2(t) \left(A_1(t) \begin{pmatrix} z(T_b) \\ z_t(T_b) \end{pmatrix} + \begin{pmatrix} z_{\text{part},1}(T_m) \\ z_{\text{part},1,t}(T_m) \end{pmatrix} \right) + \begin{pmatrix} z_{\text{part},2}(T_f) \\ z_{\text{part},2,t}(T_f) \end{pmatrix}, \tag{3.51}$$

which can be rewritten to

$$\phi(t) : \begin{pmatrix} z(T_f) \\ z_t(T_f) \end{pmatrix} = A_2(t) A_1(t) \begin{pmatrix} z(T_b) \\ z_t(T_b) \end{pmatrix} + A_2 \begin{pmatrix} z_{\text{part},1}(T_m) \\ z_{\text{part},1,t}(T_m) \end{pmatrix} + \begin{pmatrix} z_{\text{part},2}(T_f) \\ z_{\text{part},2,t}(T_f) \end{pmatrix}, \tag{3.52}$$

and thus the solution for the equation of motion with piecewise changing damping can be found.

3.2.2. Visualising the solution

The map $\phi(t)$ forms the equations of motion and can thus be used for the visualisation of the damped harmonic oscillator, if rewritten to determine the displacement and velocity at an arbitrary time t, so

$$\phi(t) : \begin{pmatrix} z(t) \\ z_t(t) \end{pmatrix} = A_2(t) A_1(t) \begin{pmatrix} z(T_b) \\ z_t(T_b) \end{pmatrix} + A_2 \begin{pmatrix} z_{\text{part},1}(T_m) \\ z_{\text{part},1,t}(T_m) \end{pmatrix} + \begin{pmatrix} z_{\text{part},2}(t) \\ z_{\text{part},2,t}(t) \end{pmatrix}. \tag{3.53}$$

3.2.3. Influence of different factors

The influence of the amplitude, frequency and phase is studied. We do not study the influence of the damping value as this is the piecewise changing damping model.

Influence of Amplitude

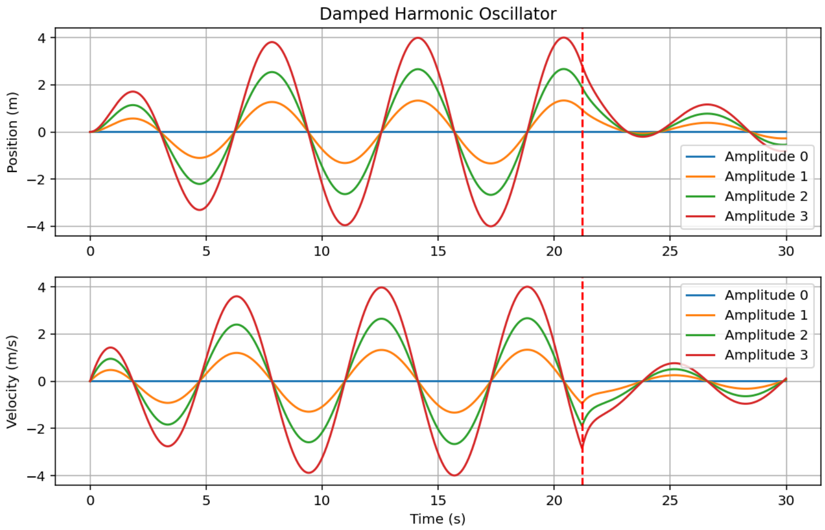


Figure 3.5: The displacement and velocity of the damped harmonic oscillator with values, $m = 1, k = 2, f_s = 1, \varphi = 0$, initial conditions $z(T_0) = 0, z_t(T_0) = 0$, switching damping from $c = 1.5$ to $c = 6.6$ and changing amplitude from 0 to 3.

The displacement and velocity with different amplitude values are plotted in Figure 3.5. After the crossed red line, the damping value has switched. It can immediately be seen that the system changes from an under- to an overdamped damped harmonic oscillator, as the displacement and velocity both die out. Higher amplitude will make the damped harmonic oscillator still move, but not as big as before.

Influence of frequency

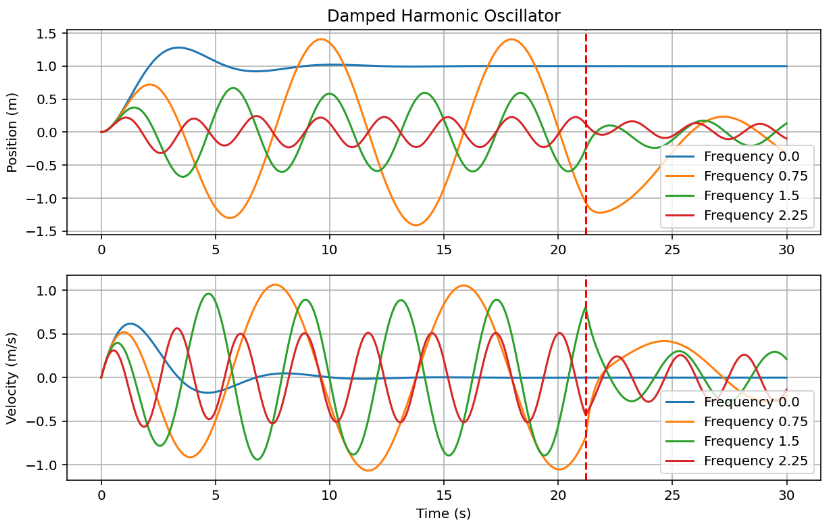


Figure 3.6: The displacement and velocity of the damped harmonic oscillator with values $y_0 = 1, m = 1, k = 2, \varphi = 0$, initial conditions $z(T_0) = 0, z_t(T_0) = 0$, , switching damping from $c = 1.5$ to $c = 6.6$ and changing frequency from 0 to 2.25.

The displacement and velocity with different frequencies are plotted in Figure 3.6. It can be seen that when frequency is equal to 0, the damped harmonic oscillator will come to a standstill. This is due to no external base excitation happening, and therefore the system will not move.

Influence of phase

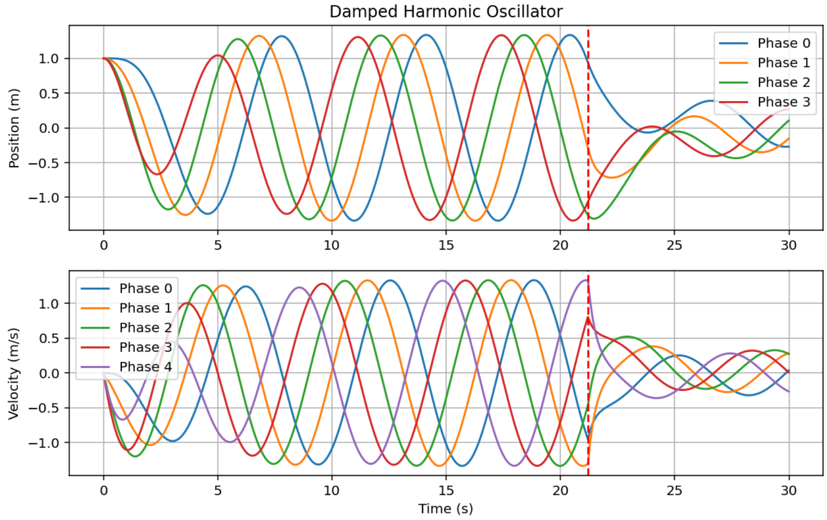


Figure 3.7: The displacement and velocity of the damped harmonic oscillator with values $y_0 = 1, m = 1, k = 2, f_s = 1$, initial conditions $z(T_0) = 0, z_t(T_0) = 0,$, switching damping from $c = 1.5$ to $c = 6.6$ and changing phase from 0 to 3.

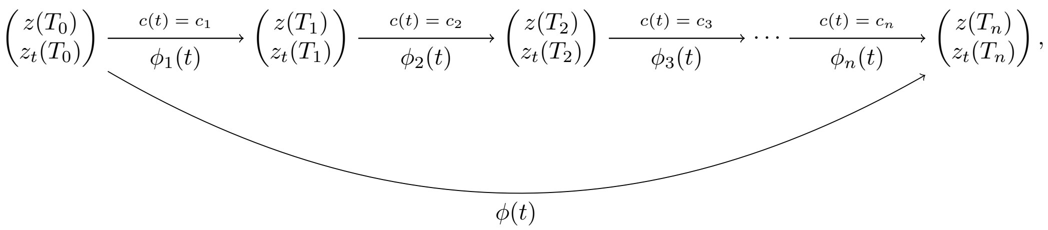
The displacement and velocity with different phase values are plotted in Figure 3.7. Like before, the influence of the phase is limited, as it only shifts the base excitation oscillation.

3.3. Pseudocontinuous damping

In this section, the general solution for the case of pseudocontinuous damping is given in Section 3.3.1. The visualisation is explained in Section 3.3.2 and then, in Section 3.3.3, the influence of the different parameters is discussed.

3.3.1. General solution

Building on the steps taken in the previous section, the map for the pseudocontinuous case will take the following form. Similarly to the previous section, we get a total map $\phi(t)$, which is also indicated below



where

$$\phi_i(t) : \begin{pmatrix} z(T_i) \\ z_t(T_i) \end{pmatrix} = A_i(t) \begin{pmatrix} z(T_{i-1}) \\ z_t(T_{i-1}) \end{pmatrix} + \begin{pmatrix} z_{\text{part},i}(T_i) \\ z_{\text{part},i,t}(T_i) \end{pmatrix}, \tag{3.54}$$

with $i \in 1, 2, ..., n$. Here n denotes the number of switches in damping values. Thus, also

$$\phi_{i+1}(t) : \begin{pmatrix} z(T_{i+1}) \\ z_t(T_{i+1}) \end{pmatrix} = A_{i+1}(t) \left(A_i(t) \begin{pmatrix} z(T_{i-1}) \\ z_t(T_{i-1}) \end{pmatrix} + \begin{pmatrix} z_{\text{part},i}(T_i) \\ z_{\text{part},i,t}(T_i) \end{pmatrix} \right) + \begin{pmatrix} z_{\text{part},i+1}(T_{i+1}) \\ z_{\text{part},i+1,t}(T_{i+1}) \end{pmatrix}, \tag{3.55}$$

and so

$$\phi(t) : \begin{pmatrix} z(T_n) \\ z_t(T_n) \end{pmatrix} = \prod_{j=1}^{n} A_j \begin{pmatrix} z(T_0) \\ z_t(T_0) \end{pmatrix} + \sum_{j=1}^{n} \prod_{k=j}^{n} A_k \begin{pmatrix} z_{\text{part},j}(T_j) \\ z_{\text{part},j,t}(T_j) \end{pmatrix}. \tag{3.56}$$

3.3.2. Visualising the solution

Equation (3.56) is used for the visualisation of the solution of the damped harmonic oscillator model.

3.3.3. Influence of parameters

The influence of the parameters is the same as that of the piecewise changing damping. To illustrate this, the different images are reproduced based on damping values from the approximation of function $c(t) = \frac{(t+8)^3}{5000}$ and four intervals.

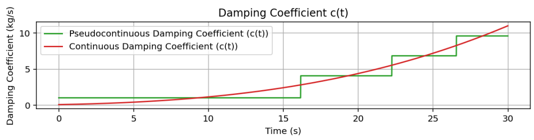


Figure 3.8: An example continuous damping $c(t) = \frac{(t+8)^3}{5000}$ in red with a pseudocontinuous damping in green, giving values $1, 4, 7$ and 9.

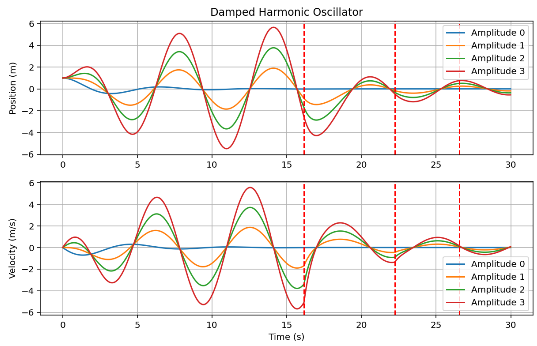


Figure 3.9: The displacement and velocity of the damped harmonic oscillator with values $\varphi = 0, m = 1, k = 2, f_s = 1$, initial conditions $z(T_0) = 0, z_t(T_0) = 0$, damping values from Figure 3.8 changing amplitude from 0 to 3.

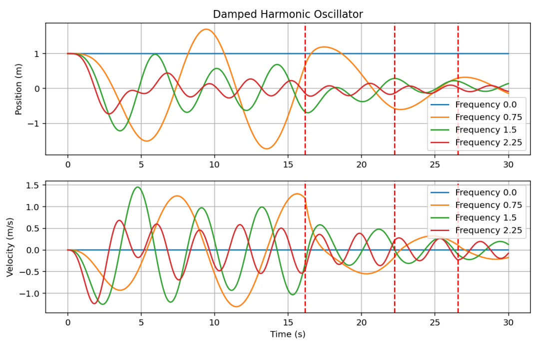


Figure 3.10: The displacement and velocity of the damped harmonic oscillator with values $y_0 = 1, m = 1, k = 2, \varphi = 0$, initial conditions $z(T_0) = 0, z_t(T_0) = 0$, damping values from Figure 3.8 changing frequency from 0 to 2.25.

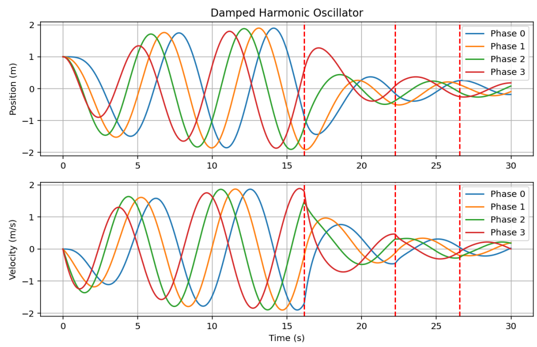


Figure 3.11: The displacement and velocity of the damped harmonic oscillator with values $y_0 = 1, m = 1, k = 2, f_s = 1$, initial conditions $z(T_0) = 0, z_t(T_0) = 0$, damping values from Figure 3.8 changing phase from 0 to 3.

It should be noted that in Figure 3.8, the intervals of the damping value are purposely not equidistant, as then the approximation of the continuous function $c(t) = \frac{(t+8)^3}{5000}$ has the lowest error with using interval values for the damping. This numerical approximation has been discussed in Section 2.3.2.

The influence of the amplitude, frequency and phase, visualised in Figure 3.9, Figure 3.10 and Figure 3.11 respectively, is the same as previously discussed in Section 3.1.5 and Section 3.2.3.

3.4. Concluding the damped harmonic oscillator model

In this section, the conclusion on the modelling of the damped harmonic oscillator with different methods to implement a variable damping, is given. In Section 3.4.1 a table is given which outlines the effect of different parameter values on the damped harmonic oscillator with base excitation. Thereafter, in Section 3.4.2, an explanation is given why only the steady state solution the damped harmonic oscillator with base excitation is studied for the power analysis.

3.4.1. Influence of parameters

An overview of the different influence of the parameters on the damped harmonic oscillator is given in Table 3.1.

Table 3.1: Influence of different parameters on the damped harmonic oscillator.

Parameter	Influence
Amplitude	Higher amplitudes of the base excitation imply a higher amplitude of the oscillator
Damping	Higher damping values reduce the displacement and velocity of the oscillator
Frequency	Frequencies closer to the eigenfrequency imply larger displacement and velocity
Phase	Bigger phase shifts the displacement and velocity with a larger value

The amplitude, frequency, and phase are all related to the base excitation, which excites the damped harmonic oscillator. These terms cannot be influenced in the application of the damped harmonic oscillator to yield power.

3.4.2. Transient solution and steady state solution

The solution of the damped harmonic oscillator typically consists of homogeneous and particular solutions. The piecewise and pseudocontinuous are nothing more than mappings of this solution for a constant damping on different intervals.

It should be noted that the solution consists of a so-called transient and steady state part. The transient solution is the part belonging to the homogeneous equation, while the steady state is related to the particular solution. For time intervals which start close to zero, both terms are required to study the behaviour of the oscillator properly. In simulations where the damped harmonic oscillator is studied far away from zero in time, however, it can be shown that the transient solution will not influence the behaviour of the model. A visualition is given in Figure 3.12.

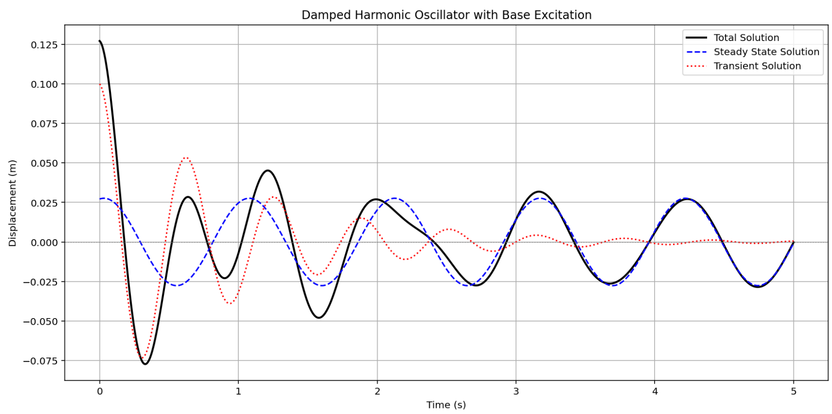


Figure 3.12: Graph of the damped harmonic oscillator with the red line as the transient solution, the blue line as the steady state solution and the black line as the total solution.

So, from the visualisation, we can conclude that after a short number of seconds have passed, only the steady state solution determines the behaviour of the damped harmonic oscillator with base excitation.

This conclusion is logically supported by studying all cases of the homogeneous solutions discussed in Section 3.1.1. All forms of these solutions contain an exponential with a negative power with a factor of time. For these, it holds that

$$\lim_{t\to\infty} \mathrm{e}^{\frac{-ct}{2m}} = 0, \tag{3.57}$$

even when multiplied by the variable t this expression will still converge to zero, so

$$\lim_{t\to\infty} t\mathrm{e}^{\frac{-ct}{2m}} = 0. \tag{3.58}$$

Therefore, we will see that the model will be governed by the dynamics which can be described by

$$z(t) = \frac{1}{\sqrt{\left(f_s^2 - 1\right)^2 + \left(\psi c f_s\right)^2}} \left(\cos(f_s t + \hat{\varphi})\right). \tag{3.59}$$

4

Power Analysis without Damping Switch

This chapter focuses on the power analysis when there is no switch in the damping value for the damped harmonic oscillator. This significantly simplifies the analysis that is done. First, the extrema are found and then they are classified. Thereafter, a visualisation of the optimal solution is given to show it truly yields the most time-averaged power from the damped harmonic oscillator with base excitation.

In the case where there is no switch in the damping value, it implies that

$$c = c_m + c_v, \tag{4.1}$$

and so, using the governing equation of motion given in Equation (3.59), we have as time-averaged power, defined in Equation (2.50),

$$P_{\text{ave}} = \frac{1}{\frac{2\pi}{f_s}} \frac{c_v f_s^2}{2} \frac{1}{(f_s^2 - 1)^2 + (\psi c f_s)^2} \int_0^T 1 - \cos(2 f_s t + 2\hat{\varphi}) \mathrm{d}t, \tag{4.2}$$

where we used the goniometric identity

$$\sin^2(f_s t + \hat{\varphi}) = \frac{1 - \cos(2 f_s t + 2\hat{\varphi})}{2}. \tag{4.3}$$

As mentioned before, a cycle is defined as one oscillation of the base excitation wave. This gives $T = \frac{2\pi}{f_s}$, and so

$$P_{\text{ave}} = \frac{1}{\frac{2\pi}{f_s}} \frac{c_v f_s^2}{2} \frac{1}{(f_s^2 - 1)^2 + (\psi c f_s)^2} \left(\frac{2\pi}{f_s} - \frac{1}{2 f_s} \left(\sin(4\pi + 2\hat{\varphi}) - \sin(2\hat{\varphi}) \right) \right). \tag{4.4}$$

Due to the periodicity of the sine function, Equation (4.4) simplifies to

$$P_{\text{ave}} = \frac{c_v f_s^2}{2} \frac{1}{(f_s^2 - 1)^2 + (\psi c f_s)^2}. \tag{4.5}$$

As the goal is to find the optimal time-averaged power, this equation is now differentiated with respect to variable c_v and put equal to zero to find the extrema. So,

$$\frac{\partial P_{\text{ave}}}{\partial c_v} = \frac{f_s^2}{2} \frac{(f_s^2 - 1)^2 + (c\psi f_s)^2 - 2 c_m c_v \psi^2 f_s^2 - 2 c_v^2 \psi^2 f_s^2}{((f_s^2 - 1)^2 + (c\psi f_s)^2)^2} = 0, \tag{4.6}$$

so,

$$\frac{\partial P_{\text{ave}}}{\partial c_v} = \frac{f_s^2}{2} \frac{(f_s^2 - 1)^2 + \psi^2 f_s^2 (c_m^2 - c_v^2)}{((f_s^2 - 1)^2 + (c\psi f_s)^2)^2} = 0, \tag{4.7}$$

which is simplified further to,

$$(f_s^2-1)^2+\psi^2 f_s^2(c_m^2-c_v^2)=0. \tag{4.8}$$

Solving this equation for variable c_v gives

$$c_{v,\text{opt}}=\pm\frac{\sqrt{c_m^2\psi^2 f_s^2+(f_s^2-1)^2}}{\psi f_s}, \tag{4.9}$$

thus implying there are two extreme values. Here $c_{v,\text{opt}}$ is defined as the optimal value for the variable damping. Specifically, $c_{v,\text{opt},+}$ is the optimal value where the root has a plus sign in front and similarly $c_{v,\text{opt},-}$ is defined.

To check whether these values are a maximum or a minimum, Equation (4.7) is studied further. The second derivative test is used. So, for the second derivative we obtain

$$\frac{\partial P_{\text{ave}}^2}{\partial c_v^2}=\frac{f_s^2}{2}\frac{((f_s^2-1)^2+(c\psi f_s)^2)^2\left(-2c_v c_m\psi^2 f_s^2\right)-\left((f_s^2-1)^2+\psi^2 f_s^2(c_m^2-c_v^2)\right)\left(\frac{\partial((f_s^2-1)^2+(c\psi f_s)^2)^2}{\partial c_v}\right)}{((f_s^2-1)^2+(c\psi f_s)^2)^4}. \tag{4.10}$$

Note that $\left(\frac{\partial((f_s^2-1)^2+(c\psi f_s)^2)^2}{\partial c_v}\right)$ isn't evaluated due to the fact that the $\left((f_s^2-1)^2+\psi^2 f_s^2(c_m^2-c_v^2)\right)$ will be equal to zero at $c_v=c_{v,\text{opt}}$. So,

$$\frac{\partial P_{\text{ave}}^2}{\partial c_v^2}(c_{v,\text{opt}})=\frac{f_s^2}{2}\frac{((f_s^2-1)^2+(c_{\text{opt}}\psi f_s)^2)^2\left(-2c_{v,\text{opt}}c_m\psi^2 f_s^2\right)}{((f_s^2-1)^2+(c_{\text{opt}}\psi f_s)^2)^4}, \tag{4.11}$$

where $c_{\text{opt}}=c_m+c_{v,\text{opt}}$. Note that

$$\text{sgn}\left(\frac{\partial P_{\text{ave}}^2}{\partial c_v^2}(c_{v,\text{opt}})\right)=\text{sgn}\left(-2c_{v,\text{opt}}c_m\psi^2 f_s^2\right), \tag{4.12}$$

thus, since $c_m>0$,

$$\text{sgn}\left(\frac{\partial P_{\text{ave}}^2}{\partial c_v^2}(c_{v,\text{opt}})\right)=-\text{sgn}\left(c_{v,\text{opt}}\right),. \tag{4.13}$$

Therefore we have a maximum when

$$c_{v,\text{opt},+}=\frac{\sqrt{c_m^2\psi^2 f_s^2+(f_s^2-1)^2}}{\psi f_s}, \tag{4.14}$$

and a minimum when

$$c_{v,\text{opt},-}=-\frac{\sqrt{c_m^2\psi^2 f_s^2+(f_s^2-1)^2}}{\psi f_s}. \tag{4.15}$$

There are, however, more possibilities where the time-averaged power yield is more than when we are at a maximum. This is due to the bounds of the damping value, as we might have boundary optimal values which aren't a maximum, but might be higher than the maximum we found. The discussion of the bounds for the variable damping value can be found in Section 2.5. Due to these bounds, we should also test whether $c_v=-0.15$ or $c_v=0.35$ is an optimal value. A visualisation is made to show the time-averaged power.

As the only variable in the optimum damping value is the frequency, a plot is made for $c_v=c_{v,\text{opt},+}$, $c_v=-0.15$ and $c_v=0.35$. This plot can be seen in Figure 4.1.

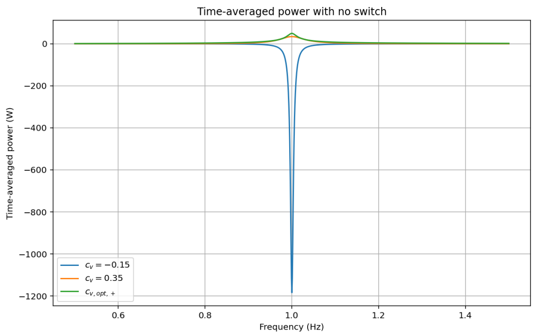


Figure 4.1: Time-averaged power of damped harmonic oscillator with base excitation with no switch in damping value, where damping values on boundary extrema and $c_{v,\text{opt},+}$ are plotted.

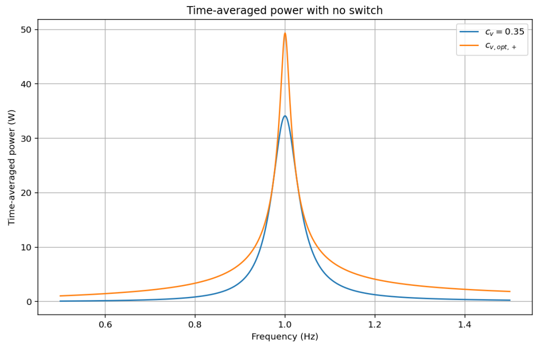


Figure 4.2: Time-averaged power of damped harmonic oscillator with base excitation with no switch in damping value, where damping values $c_v = 0.35$ and $c_v = c_{v,\text{opt},+}$ are plotted.

It can be seen in Figure 4.1, that the damping value $c_v = -0.15$ is not optimal. A closer look at Figure 4.2, where $c_v = -0.15$ is omitted, shows that $c_v = c_{v,\text{opt},+}$ outperforms $c_v = 0.35$. The time-averaged power is plotted for several damping values compared to $c_v = c_{v,\text{opt},+}$ in Figure 4.3.

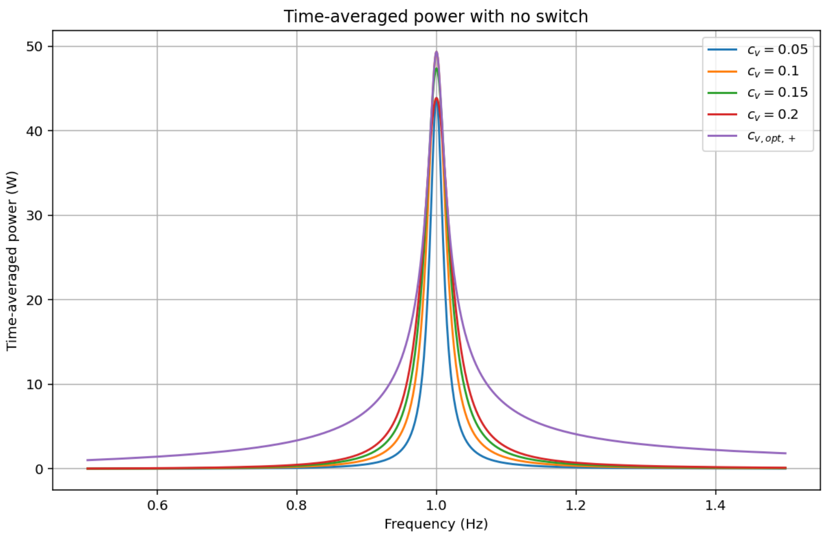


Figure 4.3: Time-averaged power of damped harmonic oscillator with base excitation with no switch in damping value, where damping values $c_v \in \{0.05, 0.1, 0.15, 0.2, c_{v,\text{opt},+}\}$ are plotted.

From these visualisations, it can be concluded that indeed the damping value $c_v = c_{v,\text{opt},+}$ provides the most time-averaged power for the damped harmonic oscillator with base excitation for no switch in damping value.

5

Power Analysis Singular Switch

This chapter will delve into how much power can be yielded from a damped harmonic oscillator with base excitation when a singular switch in damping value is made. This is done for three cases: a halfway switch, a quarter switch and an arbitrary switch. The case of a halfway switch is called a halfway switch due to the switch in damping value happening halfway through the period of the base excitation. The quarter switch is defined similarly. These specific cases are discussed before the general case due to the goniometric identities of sine and cosine at the halfway and quarter switch.

5.1. Power analysis for halfway singular switch

Now, the case of a halfway switch is studied. With a halfway switch, it is meant that the moment the damping value changes, the oscillation of the base excitation is precisely in the middle of the cycle. Hence,

$$c(t) = c_m + c_v(t), \tag{5.1}$$

where,

$$c_v(t) = \begin{cases} c_v = c_{v,1} & t \in [0, \frac{\pi}{f_s}), \\ c_v = c_{v,2} & t \in [\frac{\pi}{f_s}, \frac{2\pi}{f_s}), \end{cases} \tag{5.2}$$

and so, for $i \in \{1, 2\}$

$$c_i = c_m + c_{v,i}. \tag{5.3}$$

Now, generally we had, using the rescaled equation, the expression for the time-averaged power as

$$P_{\text{ave}} = \frac{1}{\frac{2\pi}{f_s}} \frac{c_v f_s^2}{2} \frac{1}{(f_s^2 - 1)^2 + (\psi c f_s)^2} \int_0^T 1 - \cos(2 f_s t + 2\hat{\varphi}) \mathrm{d}t, \tag{5.4}$$

however, as c_v now depends on time, the above equation is now rewritten to

$$P_{\text{ave}} = \frac{1}{\frac{2\pi}{f_s}} \int_0^T \frac{c_v f_s^2}{2} \frac{1}{(f_s^2 - 1)^2 + (\psi c f_s)^2} \left(1 - \cos(2 f_s t + 2\hat{\varphi})\right) \mathrm{d}t. \tag{5.5}$$

Now, as $c_v(t)$ has a switch between variable damping value $c_{v,1}$ and $c_{v,2}$ at a certain time $T_m = \frac{\pi}{f_s}$, which lies between 0 and $\frac{2\pi}{f_s}$ as $T = \frac{2\pi}{f_s}$, the integral can be split into two parts. So,

$$\begin{aligned} P_{\text{ave}} = \frac{1}{\frac{2\pi}{f_s}} \Bigg(& \int_0^{\frac{\pi}{f_s}} \frac{c_{v,1} f_s^2}{2} \frac{1}{(f_s^2 - 1)^2 + (\psi c_1 f_s)^2} \left(1 - \cos(2 f_s t + 2\hat{\varphi}_1)\right) \mathrm{d}t \\ & + \int_{\frac{\pi}{f_s}}^{\frac{2\pi}{f_s}} \frac{c_{v,2} f_s^2}{2} \frac{1}{(f_s^2 - 1)^2 + (\psi c_2 f_s)^2} \left(1 - \cos(2 f_s t + 2\hat{\varphi}_2)\right) \mathrm{d}t \Bigg), \end{aligned} \tag{5.6}$$

which, when working out the integrals, is

$$
\begin{aligned}
P_{\text{ave}} = \frac{1}{\frac{2\pi}{f_s}} \Bigg(& \frac{c_{v,1} f_s^2}{2} \frac{1}{(f_s^2-1)^2 + (\psi c_1 f_s)^2} \left(\frac{\pi}{f_s} - \frac{1}{2f_s} \left(\sin(2\pi + 2\hat{\varphi}_1) - \sin(2\hat{\varphi}_1) \right) \right) \\
& + \frac{c_{v,2} f_s^2}{2} \frac{1}{(f_s^2-1)^2 + (\psi c_2 f_s)^2} \left(\frac{2\pi}{f_s} - \frac{\pi}{f_s} - \frac{1}{2f_s} \left(\sin(4\pi + 2\hat{\varphi}_2) - \sin(2\pi + 2\hat{\varphi}_2) \right) \right) \Bigg),
\end{aligned}
\tag{5.7}
$$

and so, due to the periodicty of the sine function

$$
P_{\text{ave}} = \frac{1}{\frac{2\pi}{f_s}} \left(\frac{c_{v,1} f_s^2}{2} \frac{1}{(f_s^2-1)^2 + (\psi c_1 f_s)^2} \left(\frac{\pi}{f_s} \right) + \frac{c_{v,2} f_s^2}{2} \frac{1}{(f_s^2-1)^2 + (\psi c_2 f_s)^2} \left(\frac{2\pi}{f_s} - \frac{\pi}{f_s} \right) \right), \tag{5.8}
$$

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$$
P_{\text{ave}} = \frac{1}{2} \frac{f_s^2}{2} \left(\frac{c_{v,1}}{(f_s^2-1)^2 + (\psi c_1 f_s)^2} + \frac{c_{v,2}}{(f_s^2-1)^2 + (\psi c_2 f_s)^2} \right). \tag{5.9}
$$

Due to the symmetry of this equation, the optimal solution when optimising for $c_{v,1}$ and $c_{v,2}$ will be the same value, namely, taken from the previous section, with now $i \in \{1, 2\}$

$$
c_{v,i} = \pm \frac{\sqrt{c_m^2 \psi^2 f_s^2 + (f_s^2-1)^2}}{\psi f_s}. \tag{5.10}
$$

These solutions imply that there are four different extreme values. Similar to the previous section, the derivative is studied to determine which of these extreme values are maxima and which are minima. The derivative for $i \in \{1, 2\}$ is

$$
\frac{\partial P_{\text{ave}}}{\partial c_{v,i}} = \frac{1}{2} \frac{f_s^2}{2} \frac{(f_s^2-1)^2 + (c_i \psi f_s)^2 - 2c_m c_{v,i} \psi^2 f_s^2 - 2c_{v,i}^2 \psi^2 f_s^2}{((f_s^2-1)^2 + (c_i \psi f_s)^2)^2}. \tag{5.11}
$$

Note that Equation (5.11) is the same as Equation (4.6). Hence, via similar reasoning as at the end of Chapter 4, it can be concluded that there will be a maximum when $c_{v,i}$ is positive and thus, $c_{v,1} = c_{v,2}$. Therefore, no actual switch is needed, and the situation is reduced to the first case discussed without any switch in Chapter 4.

Similar to Chapter 4, we also have boundary values where the damping value could provide a higher time-averaged power. Thus, we should also test the cases where $c_{v,1} = -0.15$ or $c_{v,1} = 0.35$ and $c_{v,2} = -0.15$ or $c_{v,2} = 0.35$.

So, we create a visualisation for these three damping values, $c_v = c_{v,\text{opt},+}$, $c_{v,1} = -0.15$ and $c_{v,2} = 0.35$ or $c_{v,1} = 0.35$ and $c_{v,2} = -0.15$. We do not have to study the case where the damping value both equal the same boundary damping value, as it has already been shown in Chapter 4 that $c_v = c_{v,\text{opt},+}$ outperforms these damping values. The visualisation can be seen in Figure 5.1.

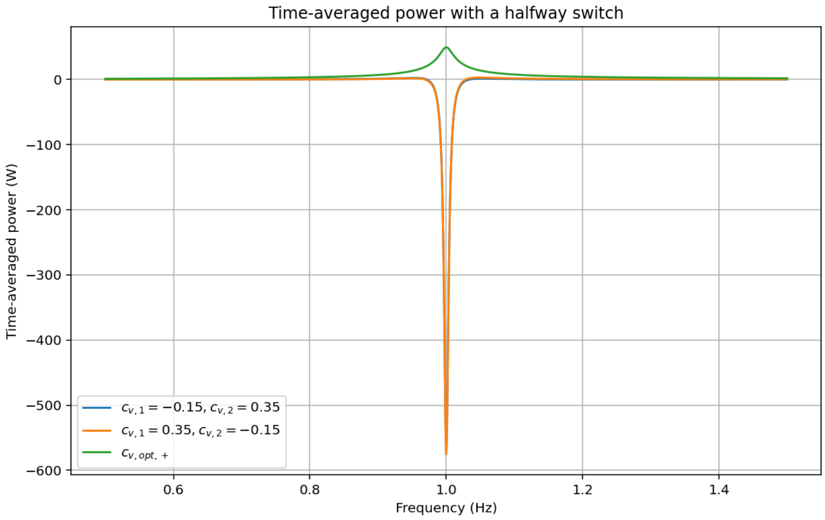

Figure 5.1: Time-averaged power of damped harmonic oscillator with base excitation with a halfway switch in damping value, where damping values $c_{v,1} = 0.1 \pm 0.25$, $c_{v,2} = 0.1 \mp 0.25$ and $c_v = c_{v,\text{opt},+}$ are plotted.

In Figure 5.1 it can be seen that the damping value $c_v = c_{v,\text{opt},+}$ performs better than the boundary extrema. To verify the performance with respect to other damping values, Figure 5.2 is studied.

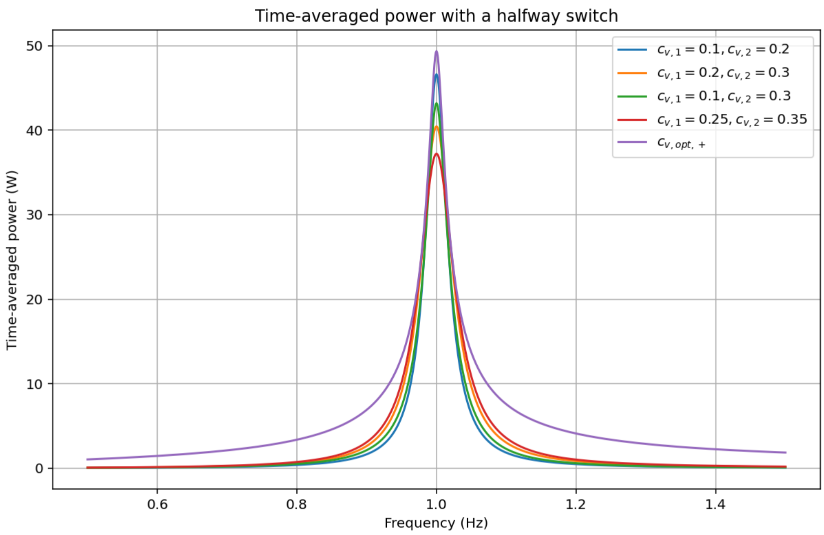

Figure 5.2: Time-averaged power of damped harmonic oscillator with base excitation with a halfway switch in damping value, where damping values with a switch indicated in the legend of the graph and $c_v = c_{v,\text{opt},+}$ are plotted.

We can see in Figure 5.2 that no other damping value combination yields more time-averaged power than $c_v = c_{v,\text{opt},+}$.

Therefore, it can be concluded that when using a halfway switch, the best strategy is to remove the switch and stick to the damping value $c_v = c_{v,\text{opt},+}$.

5.2. Power analysis for a quarter switch

Another special case of a switch is the case of a quarter switch. With a quarter switch, the quarter of the base excitation oscillation is meant for the switch in the damping value. So,

$$c(t) = c_m + c_v(t) \tag{5.12}$$

where,

$$c_v(t) = \begin{cases} c_v = c_{v,1} & t \in [0, \frac{\pi}{2f_s}), \\ c_v = c_{v,2} & t \in [\frac{\pi}{2f_s}, \frac{2\pi}{f_s}), \end{cases} \tag{5.13}$$

and so, for $i \in \{1, 2\}$

$$c_i = c_m + c_{v,i}. \tag{5.14}$$

Now, generally we had, using the rescaled equation, the expression for the time-averaged power as

$$P_{\text{ave}} = \frac{1}{\frac{2\pi}{f_s}} \frac{c_v f_s^2}{2} \frac{1}{(f_s^2-1)^2 + (\psi c f_s)^2} \int_0^T 1 - \cos(2f_s t + 2\hat{\varphi})\text{d}t, \tag{5.15}$$

however, as c_v now depends on time, the above equation is now rewritten to

$$P_{\text{ave}} = \frac{1}{\frac{2\pi}{f_s}} \int_0^T \frac{c_v f_s^2}{2} \frac{1}{(f_s^2-1)^2 + (\psi c f_s)^2} \left(1 - \cos(2f_s t + 2\hat{\varphi})\right) \text{d}t. \tag{5.16}$$

Now, as $c_v(t)$ has a switch between variable damping value $c_{v,1}$ and $c_{v,2}$ on certain time $T_m = \frac{\pi}{f_s}$, which lies between 0 and $\frac{2\pi}{f_s}$ as $T = \frac{2\pi}{f_s}$, the integral can be split into two parts. So,

$$\begin{aligned} P_{\text{ave}} = \frac{1}{\frac{2\pi}{f_s}} \Bigg(&\int_0^{\frac{\pi}{2f_s}} \frac{c_{v,1} f_s^2}{2} \frac{1}{(f_s^2-1)^2 + (\psi c_1 f_s)^2} \left(1 - \cos(2f_s t + 2\hat{\varphi}_1)\right) \text{d}t \\ &+ \int_{\frac{\pi}{2f_s}}^{\frac{2\pi}{f_s}} \frac{c_{v,2} f_s^2}{2} \frac{1}{(f_s^2-1)^2 + (\psi c_2 f_s)^2} \left(1 - \cos(2f_s t + 2\hat{\varphi}_2)\right) \text{d}t \Bigg), \end{aligned} \tag{5.17}$$

which, when working out the integrals, is

$$\begin{aligned} P_{\text{ave}} = \frac{1}{\frac{2\pi}{f_s}} \frac{f_s^2}{2} \Bigg(&\frac{c_{v,1}}{(f_s^2-1)^2 + (\psi c_1 f_s)^2} \left(\frac{\pi}{2f_s} - \frac{1}{2f_s} \left(\sin(\pi + 2\hat{\varphi}_1) - \sin(2\hat{\varphi}_1)\right) \right) \\ &+ \frac{c_{v,2}}{(f_s^2-1)^2 + (\psi c_2 f_s)^2} \left(\frac{2\pi}{f_s} - \frac{\pi}{2f_s} - \frac{1}{2f_s} \left(\sin(4\pi + 2\hat{\varphi}_2) - \sin(\pi + 2\hat{\varphi}_2)\right) \right) \Bigg), \end{aligned} \tag{5.18}$$

and so, due to the periodicty of the sine function and the property of shifting the sine by half a period,

$$\begin{aligned} P_{\text{ave}} = \frac{1}{\frac{2\pi}{f_s}} \frac{f_s^2}{2} \Bigg(&\frac{c_{v,1}}{(f_s^2-1)^2 + (\psi c_1 f_s)^2} \left(\frac{\pi}{2f_s} + \frac{1}{2f_s} 2\sin(2\hat{\varphi}_1) \right) \\ &+ \frac{c_{v,2}}{(f_s^2-1)^2 + (\psi c_2 f_s)^2} \left(\frac{3\pi}{2f_s} - \frac{1}{2f_s} 2\sin(2\hat{\varphi}_2) \right) \Bigg), \end{aligned} \tag{5.19}$$

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$$P_{\text{ave}} = \frac{1}{\frac{2\pi}{f_s}} \frac{f_s}{2} \left(\frac{c_{v,1}}{(f_s^2-1)^2 + (\psi c_1 f_s)^2} \left(\frac{\pi}{2} + \sin(2\hat{\varphi}_1) \right) + \frac{c_{v,2}}{(f_s^2-1)^2 + (\psi c_2 f_s)^2} \left(\frac{3\pi}{2} - \sin(2\hat{\varphi}_2) \right) \right), \tag{5.20}$$

An optimal solution needs to be found on all variables $c_{v,1}, c_{v,2}, f_s$ and φ. The optimal solution for variable φ is studied. Recall that $\hat{\varphi}_i = \varphi + \delta_i$, as discussed in Section 3.1.2. Differentiating with respect to this variable and equalising to zero gives

$$\frac{\partial P_{\text{ave}}}{\partial \varphi} = \frac{1}{\frac{2\pi}{f_s}} \frac{f_s}{2} \left(\frac{c_{v,1}}{(f_s^2-1)^2 + (\psi c_1 f_s)^2} 2\cos(2\hat{\varphi}_1) - \frac{c_{v,2}}{(f_s^2-1)^2 + (\psi c_2 f_s)^2} 2\cos(2\hat{\varphi}_2) \right) = 0, \tag{5.21}$$

where the equality to zero is put to find extrema. Hence, it must hold that

$$\frac{c_{v,1}}{(f_s^2-1)^2+(\psi c_1 f_s)^2} 2\cos(2\hat{\varphi}_1) = \frac{c_{v,2}}{(f_s^2-1)^2+(\psi c_2 f_s)^2} 2\cos(2\hat{\varphi}_2). \tag{5.22}$$

Besides the trivial solution of $c_{v,1} = c_{v,2}$, no other obvious solution can be spotted. Note that this equation is nonsymmetric, nonlinear and transcendental. This equation and its solution are further discussed later in Chapter 6. Instead of solving the equation, we focus on an arbitrary moment of switch in the damping value instead.

5.3. Power analysis for an arbitrary singular switch

Finally, we discuss a switch in damping at a certain time T_m. Note that due to previous two cases discussed, it holds that

$$T_m \in (0, \frac{2\pi}{f_s}) \ \setminus \ \{\frac{\pi}{f_s}\}. \tag{5.23}$$

Note that the values 0 and $\frac{2\pi}{f_s}$ are excluded, as a switch on those moments implies a constant damping for the entire oscillation cycle. Value $\frac{\pi}{f_s}$ is excluded due to it already being analysed in the previous section.

For the damping value this means that

$$c_v(t) = \begin{cases} c_v = c_{v,1} & t \in [0, T_m), \\ c_v = c_{v,2} & t \in [T_m, \frac{2\pi}{f_s}), \end{cases} \tag{5.24}$$

and still for $i \in \{1, 2\}$

$$c_i = c_m + c_{v,i}. \tag{5.25}$$

Now, again, we generally use the expression for the time-averaged power as

$$P_{\text{ave}} = \frac{1}{\frac{2\pi}{f_s}} \frac{c_v f_s^2}{2} \frac{1}{(f_s^2-1)^2 + (\psi c f_s)^2} \int_0^T 1 - \cos(2f_s t + 2\hat{\varphi})\text{d}t. \tag{5.26}$$

However, experiencing a switch between the variable damping value $c_{v,1}$ and $c_{v,2}$ at a certain time T_m which lies between 0 and $\frac{2\pi}{f_s}$, gives

$$\begin{aligned} P_{\text{ave}} = \frac{1}{\frac{2\pi}{f_s}} \Bigg(&\int_0^{T_m} \frac{c_{v,1} f_s^2}{2} \frac{1}{(f_s^2-1)^2 + (\psi c_1 f_s)^2} \left(1 - \cos(2f_s t + 2\hat{\varphi}_1)\right) \text{d}t \\ &\int_{T_m}^{\frac{2\pi}{f_s}} \frac{c_{v,2} f_s^2}{2} \frac{1}{(f_s^2-1)^2 + (\psi c_2 f_s)^2} \left(1 - \cos(2f_s t + 2\hat{\varphi}_2)\right) \text{d}t \Bigg), \end{aligned} \tag{5.27}$$

working this out gives

$$\begin{aligned} P_{\text{ave}} = \frac{1}{\frac{2\pi}{f_s}} \Bigg(&\frac{c_{v,1} f_s^2}{2} \frac{1}{(f_s^2-1)^2 + (\psi c_1 f_s)^2} \left(T_m - \frac{1}{2f_s}\left(\sin(2f_s T_m + 2\hat{\varphi}_1) - \sin(2\hat{\varphi}_1)\right)\right) \\ &+ \frac{c_{v,2} f_s^2}{2} \frac{1}{(f_s^2-1)^2 + (\psi c_2 f_s)^2} \left(\frac{2\pi}{f_s} - T_m - \frac{1}{2f_s}\left(\sin(2\hat{\varphi}_2) - \sin(2f_s T_m + 2\hat{\varphi}_2)\right)\right)\Bigg). \end{aligned} \tag{5.28}$$

Using the goniometric identity

$$\sin(\theta) \pm \sin(\varphi) = 2\sin\left(\frac{\theta \pm \varphi}{2}\right)\cos\left(\frac{\theta \mp \varphi}{2}\right), \tag{5.29}$$

Equation (5.28) is reduced to

$$\begin{aligned} P_{\text{ave}} = \frac{1}{\frac{2\pi}{f_s}} \Bigg(&\frac{c_{v,1} f_s^2}{2} \frac{1}{(f_s^2-1)^2 + (\psi c_1 f_s)^2} \left(T_m - \frac{1}{f_s}\sin(f_s T_m)\cos(f_s T_m + 2\hat{\varphi}_1)\right) \\ &+ \frac{c_{v,2} f_s^2}{2} \frac{1}{(f_s^2-1)^2 + (\psi c_2 f_s)^2} \left(\frac{2\pi}{f_s} - T_m + \frac{1}{f_s}\sin(f_s T_m)\cos(f_s T_m + 2\hat{\varphi}_2)\right)\Bigg). \end{aligned} \tag{5.30}$$

5.3.1. Finding extrema

Before optimising over the existing values in the time-averaged power, we note that besides all optimal values potentially coming out of the extrema, we have boundary values which could also be extrema. Therefore, similarly to Section 5.2, we should also test the cases where $c_{v,1} = -0.15$ or $c_{v,1} = 0.35$ and $c_{v,2} = -0.15$ or $c_{v,2} = 0.35$. The cases where $c_{v,1} = c_{v,2}$ for these boundary values do not need to be studied however, as then we would have no switch in damping value.

An optimal solution needs to be found for all variables $c_{v,1}, c_{v,2}, f_s, T_m$ and φ. First, the optimal solution for variable φ is studied. Differentiating with respect to this variable and equalising to zero gives

$$\begin{aligned}\frac{\partial P_{\text{ave}}}{\partial \varphi} = \frac{1}{\frac{2\pi}{f_s}} \Bigg(& \frac{c_{v,1} f_s^2}{2} \frac{1}{(f_s^2-1)^2 + (\psi c_1 f_s)^2} \frac{2}{f_s} \sin(f_s T_m) \sin(f_s T_m + 2\hat{\varphi}_1) \\ & - \frac{c_{v,2} f_s^2}{2} \frac{1}{(f_s^2-1)^2 + (\psi c_2 f_s)^2} \frac{2}{f_s} \sin(f_s T_m) \sin(f_s T_m + 2\hat{\varphi}_2) \Bigg) = 0,\end{aligned} \tag{5.31}$$

where the equality is put to zero to find extrema. Simplifying,

$$\begin{aligned}\frac{\partial P_{\text{ave}}}{\partial \varphi} = \frac{\sin(f_s T_m)}{\frac{2\pi}{f_s}} \Bigg(& \frac{c_{v,1} f_s^2}{2} \frac{1}{(f_s^2-1)^2 + (\psi c_1 f_s)^2} \frac{2}{f_s} \sin(f_s T_m + 2\hat{\varphi}_1) \\ & - \frac{c_{v,2} f_s^2}{2} \frac{1}{(f_s^2-1)^2 + (\psi c_2 f_s)^2} \frac{2}{f_s} \sin(f_s T_m + 2\hat{\varphi}_2) \Bigg) = 0,\end{aligned} \tag{5.32}$$

implying the two solutions to be

$$\sin(f_s T_m) = 0, \tag{5.33}$$

or,

$$\frac{c_{v,1} f_s^2}{2} \frac{1}{(f_s^2-1)^2 + (\psi c_1 f_s)^2} \frac{2}{f_s} \sin(f_s T_m + 2\hat{\varphi}_1) = \frac{c_{v,2} f_s^2}{2} \frac{1}{(f_s^2-1)^2 + (\psi c_2 f_s)^2} \frac{2}{f_s} \sin(f_s T_m + 2\hat{\varphi}_2). \tag{5.34}$$

Assuming that indeed $\sin(f_s T_m) = 0$ gives

$$f_s T_m = k\pi, \tag{5.35}$$

where k is an integer. And so, also

$$T_m = \frac{k\pi}{f_s}. \tag{5.36}$$

Due to the bounds stated in Equation (5.23), it is implied that $k \in \{0, 1, 2\}$, and so $T_m = 0, T_m = \frac{\pi}{f_s}$ or $T_m = \frac{2\pi}{f_s}$. Therefore, this solution has already been discussed in the previous sections.

The other solution is now studied. So, it is assumed that Equation (5.34) holds. Simplifying this equation gives

$$\frac{c_{v,1}}{(f_s^2-1)^2 + (\psi c_1 f_s)^2} \sin(f_s T_m + 2\hat{\varphi}_1) = \frac{c_{v,2}}{(f_s^2-1)^2 + (\psi c_2 f_s)^2} \sin(f_s T_m + 2\hat{\varphi}_2). \tag{5.37}$$

This equation does not have an obvious general solution. A trivial solution, $c_{v,1} = c_{v,2}$, is quickly spotted, but this solution implies no change in damping value, which is contradictory to the assumption that there should be a switch in damping value. Hence, further work is required to find the general solution to determine the optimal values for the damping.

To find a general solution, the time-averaged power equation is now optimised for the variable T_m. So, we obtain

$$\begin{aligned}\frac{\partial P_{\text{ave}}}{\partial T_m} = \frac{1}{\frac{2\pi}{f_s}} \frac{f_s^2}{2} \Bigg(& \frac{c_{v,1}}{(f_s^2-1)^2 + (\psi c_1 f_s)^2} (1 - \cos(2 f_s T_m + 2\hat{\varphi}_1) \\ & + \frac{c_{v.2}}{(f_s^2-1)^2 + (\psi c_1 f_s)^2} (\cos(2 f_s T_m + 2\hat{\varphi}_2 - 1) \Bigg) = 0,\end{aligned} \tag{5.38}$$

and so,

$$\frac{c_{v,1}}{(f_s^2-1)^2 + (\psi c_1 f_s)^2} (1 - \cos(2 f_s T_m + 2\hat{\varphi}_1) = \frac{c_{v,2}}{(f_s^2-1)^2 + (\psi c_2 f_s)^2} (1 - \cos(2 f_s T_m + 2\hat{\varphi}_2). \tag{5.39}$$

Using the goniometric identity

$$1 - \cos(2\theta) = 2\sin^2(\theta), \tag{5.40}$$

we get

$$\frac{c_{v,1}}{(f_s^2-1)^2+(\psi c_1 f_s)^2}\sin^2(f_sT_m+\hat{\varphi}_1) = \frac{c_{v,2}}{(f_s^2-1)^2+(\psi c_2 f_s)^2}\sin^2(f_sT_m+\hat{\varphi}_2). \tag{5.41}$$

Now, Equation (5.37) and Equation (5.41) are studied. Both are similar, as both equations are oscillations with the same amplitude but different trigonometric functions. In short, we have the following system of equations

$$\begin{cases} \frac{c_{v,1}}{(f_s^2-1)^2+(\psi c_1 f_s)^2}\sin(f_sT_m+2\hat{\varphi}_1) = \frac{c_{v,2}}{(f_s^2-1)^2+(\psi c_2 f_s)^2}\sin(f_sT_m+2\hat{\varphi}_2), \\ \frac{c_{v,1}}{(f_s^2-1)^2+(\psi c_1 f_s)^2}\sin^2(f_sT_m+\hat{\varphi}_1) = \frac{c_{v,2}}{(f_s^2-1)^2+(\psi c_2 f_s)^2}\sin^2(f_sT_m+\hat{\varphi}_2). \end{cases} \tag{5.42}$$

We solve by studying two different cases. We either assume that $\sin(f_sT_m+2\hat{\varphi}_1)=0$ or $\sin(f_sT_m+2\hat{\varphi}_1)\neq 0$.

Case $\sin(f_sT_m+2\hat{\varphi}_1)=0$

This assumption would imply that either $\frac{c_{v,2}}{(f_s^2-1)^2+(\psi c_2 f_s)^2}=0$ or $\sin(f_sT_m+2\hat{\varphi}_2)=0$. Both of these are studied case by case, starting with the first.

Assuming that indeed $\frac{c_{v,2}}{(f_s^2-1)^2+(\psi c_2 f_s)^2}=0$. This implies that either $\frac{c_{v,1}}{(f_s^2-1)^2+(\psi c_1 f_s)^2}=0$ or $\sin^2(f_sT_m+\hat{\varphi}_1)=0$. The first case would imply that $\frac{c_{v,1}}{(f_s^2-1)^2+(\psi c_1 f_s)^2}=\frac{c_{v,2}}{(f_s^2-1)^2+(\psi c_2 f_s)^2}=0$, implying that $c_{v,1}=c_{v,2}=0$. The second case would imply that $f_sT_m+2\hat{\varphi}_1=k_1\pi$ and $f_sT_m+\hat{\varphi}_1=k_2\pi$ with both k_1,k_2 being integers, so

$$\begin{cases} f_sT_m+2\hat{\varphi}_1=k_1\pi, \\ f_sT_m+\hat{\varphi}_1=k_2\pi. \end{cases} \tag{5.43}$$

Substracting the second equation twice from the first equation gives

$$f_sT_m+2\hat{\varphi}_1-2\left(f_sT_m+\hat{\varphi}_1\right)=(k_1-2k_2)\pi, \tag{5.44}$$

so

$$-f_sT_m=(k_1-2k_2)\pi, \tag{5.45}$$

and so

$$f_sT_m=\hat{k}\pi, \tag{5.46}$$

where $\hat{k}=2k_2-k_1$, and so still an integer. Similar to the reasoning regarding the solution form $\sin(f_sT_m)=0$, this gives an already known solution.

Now starting with other assumption, so assuming that $\sin(f_sT_m+2\hat{\varphi}_2)=0$. Now we get both equations that hold which are

$$\begin{cases} f_sT_m+2\hat{\varphi}_2=k_3\pi, \\ f_sT_m+2\hat{\varphi}_1=k_4\pi. \end{cases} \tag{5.47}$$

Using the above equations means that

$$\cos(f_sT_m+2\hat{\varphi}_i)=\pm 1, \tag{5.48}$$

for $i\in\{1,2\}$ and so, using Equation (5.39),

Table 5.1: Outcome of using Equation (5.39).

Assumption on $f_sT_m+2\hat{\varphi}_1$	Assumption on $f_sT_m+2\hat{\varphi}_2$	Result
$\cos(f_sT_m+2\hat{\varphi}_1)=-1$	$\cos(f_sT_m+2\hat{\varphi}_2)=-1$	$c_{v,1}=c_{v,2}$
	$\cos(f_sT_m+2\hat{\varphi}_2)=1$	$c_{v,1}=0$
$\cos(f_sT_m+2\hat{\varphi}_1)=1$	$\cos(f_sT_m+2\hat{\varphi}_2)=-1$	$c_{v,2}=0$
	$\cos(f_sT_m+2\hat{\varphi}_2)=1$	Discussed later.

Note that $c_{v,1} = c_{v,2}$ implies that there is no damping switch. The result $c_{v,2} = 0$ implies that $\frac{c_{v,2}}{(f_s^2-1)^2+(\psi c_2 f_s)^2} = 0$ and thus again, via the same steps in the beginning of this case, implies that $c_{v,1} = c_{v,2}$.

The case where $c_{v,1} = 0$ is studied further. Looking at the time-averaged power, this case yields

$$P_{\text{ave}} = \frac{1}{\frac{2\pi}{f_s}} \frac{c_{v,2} f_s^2}{2} \frac{1}{(f_s^2-1)^2 + (\psi c_2 f_s)^2} \left(\frac{2\pi}{f_s} - T_m + \frac{1}{f_s} \sin(f_s T_m) \right). \tag{5.49}$$

Now, the optimal value $c_{v,2}$ is studied. Hence, we differentiate with respect to this variable and put the result equal to zero to find the extrema.

$$\frac{\partial P_{\text{ave}}}{\partial c_{v,2}} = \frac{1}{\frac{2\pi}{f_s}} \frac{1}{2} \left(\frac{2\pi}{f_s} - T_m + \frac{1}{f_s} \sin(f_s T_m) \right) \frac{(f_s^2-1)^2 + (c_2 \psi f_s)^2 - 2c_m c_{v,2} \psi^2 f_s^2 - 2c_{v,2}^2 \psi^2 f_s^2}{((f_s^2-1)^2 + (c_2 \psi f_s)^2)^2} = 0, \tag{5.50}$$

which implies that either

$$\left(\frac{2\pi}{f_s} - T_m + \frac{1}{f_s} \sin(f_s T_m) \right) = 0, \tag{5.51}$$

which is only true when $f_s T_m = 2\pi$, so it must be the case that

$$\frac{(f_s^2-1)^2 + (c_2 \psi f_s)^2 - 2c_m c_{v,2} \psi^2 f_s^2 - 2c_{v,2}^2 \psi^2 f_s^2}{((f_s^2-1)^2 + (c_2 \psi f_s)^2)^2} = 0. \tag{5.52}$$

And so, the solution is given by

$$c_{v,2} = \pm \frac{\sqrt{c_m^2 \psi^2 f_s^2 + (f_s^2-1)^2}}{\psi f_s}, \tag{5.53}$$

which is similar to the solution found in Chapter 4. Hence, an extremum has been found for $c_{v,1} = 0$ and $c_{v,2} = \pm \frac{\sqrt{c_m^2 \psi^2 f_s^2 + (f_s^2-1)^2}}{\psi f_s}$.

Studying the inconclusive result further, gives, when looking at the time-averaged power

$$\begin{aligned} P_{\text{ave}} = \frac{1}{\frac{2\pi}{f_s}} \Bigg(& \frac{c_{v,1} f_s^2}{2} \frac{1}{(f_s^2-1)^2 + (\psi c_1 f_s)^2} \left(T_m - \frac{1}{f_s} \sin(f_s T_m) \right) \\ & + \frac{c_{v,2} f_s^2}{2} \frac{1}{(f_s^2-1)^2 + (\psi c_2 f_s)^2} \left(\frac{2\pi}{f_s} - T_m + \frac{1}{f_s} \sin(f_s T_m) \right) \Bigg). \end{aligned} \tag{5.54}$$

Now, optimising over the two damping values, giving the equations

$$\frac{\partial P_{\text{ave}}}{\partial c_{v,1}} = \frac{1}{\frac{2\pi}{f_s}} \frac{f_s^2}{2} \left(T_m - \frac{1}{f_s} \sin(f_s T_m) \right) \frac{(f_s^2-1)^2 + (c_1 \psi f_s)^2 - 2c_m c_{v,1} \psi^2 f_s^2 - 2c_{v,1}^2 \psi^2 f_s^2}{((f_s^2-1)^2 + (c_1 \psi f_s)^2)^2} = 0, \tag{5.55}$$

and

$$\frac{\partial P_{\text{ave}}}{\partial c_{v,2}} = \frac{1}{\frac{2\pi}{f_s}} \frac{f_s^2}{2} \left(\frac{2\pi}{f_s} - T_m + \frac{1}{f_s} \sin(f_s T_m) \right) \frac{(f_s^2-1)^2 + (c_2 \psi f_s)^2 - 2c_m c_{v,2} \psi^2 f_s^2 - 2c_{v,2}^2 \psi^2 f_s^2}{((f_s^2-1)^2 + (c_2 \psi f_s)^2)^2} = 0. \tag{5.56}$$

Simplifying gives the following system of equations

$$\begin{cases} \left(T_m - \frac{1}{f_s} \sin(f_s T_m) \right) \frac{(f_s^2-1)^2 + (c_1 \psi f_s)^2 - 2c_m c_{v,1} \psi^2 f_s^2 - 2c_{v,1}^2 \psi^2 f_s^2}{((f_s^2-1)^2 + (c_1 \psi f_s)^2)^2} = 0, \\ \left(\frac{2\pi}{f_s} - T_m + \frac{1}{f_s} \sin(f_s T_m) \right) \frac{(f_s^2-1)^2 + (c_2 \psi f_s)^2 - 2c_m c_{v,2} \psi^2 f_s^2 - 2c_{v,2}^2 \psi^2 f_s^2}{((f_s^2-1)^2 + (c_2 \psi f_s)^2)^2} = 0. \end{cases} \tag{5.57}$$

Note that

$$T_m - \frac{1}{f_s} \sin(f_s T_m) = 0, \tag{5.58}$$

implies that $f_s T_m = 0$ which cannot be true and that

$$\frac{2\pi}{f_s} - T_m + \frac{1}{f_s} \sin(f_s T_m) = 0, \tag{5.59}$$

implies that $f_s T_m = 2\pi$ which also cannot hold. Therefore it must be that

$$\frac{(f_s^2-1)^2+(c_1\psi f_s)^2-2c_m c_{v,1}\psi^2 f_s^2-2c_{v,1}^2\psi^2 f_s^2}{((f_s^2-1)^2+(c_1\psi f_s)^2)^2}=0=\frac{(f_s^2-1)^2+(c_2\psi f_s)^2-2c_m c_{v,2}\psi^2 f_s^2-2c_{v,2}^2\psi^2 f_s^2}{((f_s^2-1)^2+(c_2\psi f_s)^2)^2}. \tag{5.60}$$

which gives solutions where either $c_{v,1}=c_{v,2}$ or $c_{v,1}=-c_{v,2}$. For the case when $c_{v,1}=c_{v,2}$, there is no switch in damping value and thus we refer back to Chapter 4. The case where $c_{v,1}=-c_{v,2}$ gives the following for the time-averaged power that,

$$\begin{aligned}P_{\text{ave}}=\frac{1}{\frac{2\pi}{f_s}}\Bigg(&\frac{c_{v,1}f_s^2}{2}\frac{1}{(f_s^2-1)^2+(\psi c_1 f_s)^2}\left(T_m-\frac{1}{f_s}\sin(f_sT_m)\right)\\&-\frac{c_{v,1}f_s^2}{2}\frac{1}{(f_s^2-1)^2+(\psi\,(c_m-c_{v,1})\,f_s)^2}\left(\frac{2\pi}{f_s}-T_m+\frac{1}{f_s}\sin(f_sT_m)\right)\Bigg).\end{aligned} \tag{5.61}$$

Taking the partial derivative with respect to the first variable damping, gives

$$\begin{aligned}\frac{\partial P_{\text{ave}}}{\partial c_{v,1}}=\frac{1}{\frac{2\pi}{f_s}}\Bigg(&\frac{f_s^2}{2}\left(T_m-\frac{1}{f_s}\sin(f_sT_m)\right)\frac{(f_s^2-1)^2+(\psi f_s)^2\left(c_m^2-c_{v,1}^2\right)}{\left((f_s^2-1)^2+(\psi c_1 f_s)^2\right)^2}\\&-\frac{f_s^2}{2}\left(\frac{2\pi}{f_s}-T_m+\frac{1}{f_s}\sin(f_sT_m)\right)\frac{(f_s^2-1)^2+(\psi f_s)^2\left(c_m^2-c_{v,1}^2\right)}{\left((f_s^2-1)^2+(\psi\,(c_m-c_{v,1})\,f_s)^2\right)^2}\Bigg),\end{aligned} \tag{5.62}$$

which simplifies to

$$\frac{\partial P_{\text{ave}}}{\partial c_{v,1}}=\frac{1}{\frac{2\pi}{f_s}}\frac{f_s^2}{2}\left((f_s^2-1)^2+(\psi f_s)^2\left(c_m^2-c_{v,1}^2\right)\right)\left(\frac{T_m-\frac{1}{f_s}\sin(f_sT_m)}{\left((f_s^2-1)^2+(\psi c_1 f_s)^2\right)^2}-\frac{\frac{2\pi}{f_s}-T_m+\frac{1}{f_s}\sin(f_sT_m)}{\left((f_s^2-1)^2+(\psi\,(c_m-c_{v,1})\,f_s)^2\right)^2}\right), \tag{5.63}$$

which will be equal to zero as the optimum value is sought. As $f_s>0$, Equation (5.63) simplifies to

$$\frac{\partial P_{\text{ave}}}{\partial c_{v,1}}=\left((f_s^2-1)^2+(\psi f_s)^2\left(c_m^2-c_{v,1}^2\right)\right)\left(\frac{T_m-\frac{1}{f_s}\sin(f_sT_m)}{\left((f_s^2-1)^2+(\psi c_1 f_s)^2\right)^2}-\frac{\frac{2\pi}{f_s}-T_m+\frac{1}{f_s}\sin(f_sT_m)}{\left((f_s^2-1)^2+(\psi\,(c_m-c_{v,1})\,f_s)^2\right)^2}\right)=0. \tag{5.64}$$

The case where

$$(f_s^2-1)^2+(\psi f_s)^2\left(c_m^2-c_{v,1}^2\right)=0, \tag{5.65}$$

will imply that

$$c_{v,1}=\pm\frac{\sqrt{c_m^2\psi^2 f_s^2+(f_s^2-1)^2}}{\psi f_s}. \tag{5.66}$$

When

$$\frac{T_m-\frac{1}{f_s}\sin(f_sT_m)}{\left((f_s^2-1)^2+(\psi c_1 f_s)^2\right)^2}-\frac{\frac{2\pi}{f_s}-T_m+\frac{1}{f_s}\sin(f_sT_m)}{\left((f_s^2-1)^2+(\psi\,(c_m-c_{v,1})\,f_s)^2\right)^2}=0, \tag{5.67}$$

we define

$$A=\frac{T_m-\frac{1}{f_s}\sin(f_sT_m)}{\frac{2\pi}{f_s}-T_m+\frac{1}{f_s}\sin(f_sT_m)}, \tag{5.68}$$

and note that

$$A>0, \tag{5.69}$$

as both the numerator and denominator are strictly positive. Then Equation (5.67) will simplify to

$$A\left((f_s^2-1)^2+(\psi\,(c_m-c_{v,1})\,f_s)^2\right)^2=\left((f_s^2-1)^2+(\psi c_1 f_s)^2\right)^2, \tag{5.70}$$

taking the square root on both sides gives

$$\sqrt{A}\left((f_s^2-1)^2+(\psi\,(c_m-c_{v,1})\,f_s)^2\right)=\pm\left((f_s^2-1)^2+(\psi c_1 f_s)^2\right). \tag{5.71}$$

For the positive root and defining

$$\hat{A}=\sqrt{A}-1, \tag{5.72}$$

gives, after simplifying,

$$c_{v,1}^2\psi^2 f_s^2\hat{A} - 2c_{v,1}c_m\psi^2 f_s^2\left(\hat{A}+2\right) + (f_s^2-1)^2\hat{A} + \psi^2 f_s^2\hat{A}c_m^2 = 0. \tag{5.73}$$

This quadratic formula can be solved and will give solutions

$$c_{v,1} = c_m + \frac{2c_m}{\hat{A}} \pm \frac{\sqrt{(2\psi f_s c_m)^2\left(\hat{A}+1\right) - \hat{A}^2\left(f_s^2-1\right)^2}}{\psi f_s \hat{A}}. \tag{5.74}$$

For the negative root and defining

$$\tilde{A} = \sqrt{A} + 1, \tag{5.75}$$

gives, after simplifying,

$$c_{v,1}^2\psi^2 f_s^2\hat{A} - 2c_{v,1}c_m\psi^2 f_s^2\left(\hat{A}+2\right) + (f_s^2-1)^2\hat{A} + \psi^2 f_s^2\hat{A}c_m^2 = 0. \tag{5.76}$$

This quadratic formula can be solved and will give solutions

$$c_{v,1} = c_m - \frac{2c_m}{\tilde{A}} \pm \frac{\sqrt{(2\psi f_s c_m)^2\left(1-\tilde{A}\right) - \tilde{A}^2\left(f_s^2-1\right)^2}}{\psi f_s \tilde{A}}. \tag{5.77}$$

Hence, for the case where $c_{v,1} = -c_{v,2}$, six extrema have been found.

Concluding for the case $\sin(f_sT_m + 2\hat{\varphi}_1) = 0$, the following outcomes are obtained. The first possibility is that the solution has already been discussed in Chapter 4. The second is the case when $c_{v,1} = 0$ with $c_{v,2} = \pm\frac{\sqrt{c_m^2\psi^2 f_s^2 + (f_s^2-1)^2}}{\psi f_s}$, and finally six cases for extrema when $c_{v,1} = -c_{v,2}$.

Case $\sin(f_sT_m + 2\hat{\varphi}_1) \neq 0$

With this assumption, we can write

$$\frac{c_{v,1}}{(f_s^2-1)^2 + (\psi c_1 f_s)^2} = \frac{\frac{c_{v,2}}{(f_s^2-1)^2+(\psi c_2 f_s)^2}\sin(f_sT_m + 2\hat{\varphi}_2)}{\sin(f_sT_m + 2\hat{\varphi}_1)}, \tag{5.78}$$

and therefore

$$\frac{\frac{c_{v,2}}{(f_s^2-1)^2+(\psi c_2 f_s)^2}\sin(f_sT_m + 2\hat{\varphi}_2)}{\sin(f_sT_m + 2\hat{\varphi}_1)}\sin^2(f_sT_m + \hat{\varphi}_1) = \frac{c_{v,2}}{(f_s^2-1)^2 + (\psi c_2 f_s)^2}\sin^2(f_sT_m + \hat{\varphi}_2). \tag{5.79}$$

Thus either $\frac{c_{v,2}}{(f_s^2-1)^2+(\psi c_2 f_s)^2} = 0$, which implies that $c_{v,1} = c_{v,2}$, or

$$\frac{\sin(f_sT_m + 2\hat{\varphi}_2)}{\sin(f_sT_m + 2\hat{\varphi}_1)}\sin^2(f_sT_m + \hat{\varphi}_1) = \sin^2(f_sT_m + \hat{\varphi}_2). \tag{5.80}$$

Grouping the terms with the same damping value, gives

$$\frac{\sin^2(f_sT_m + \hat{\varphi}_1)}{\sin(f_sT_m + 2\hat{\varphi}_1)} = \frac{\sin^2(f_sT_m + \hat{\varphi}_2)}{\sin(f_sT_m + 2\hat{\varphi}_2)}. \tag{5.81}$$

We want to determine when this equality holds. For $i \in \{1, 2\}$, we rewrite the numerator to

$$\sin(f_sT_m + \hat{\varphi}_i) = \sin(f_sT_m + \varphi)\cos(\delta_i) + \cos(f_sT_m + \varphi)\sin(\delta_i). \tag{5.82}$$

Using the identities

$$\sin(\arctan(\theta)) = \frac{\theta}{\sqrt{1+\theta^2}}, \tag{5.83}$$

and,

$$\cos(\arctan(\theta)) = \frac{1}{\sqrt{1+\theta^2}}, \tag{5.84}$$

we obtain

$$\sin(f_sT_m + \hat{\varphi}_i) = \frac{\sin(f_sT_m+\varphi) + x_i\cos(f_sT_m+\varphi)}{\sqrt{1+x_i^2}}, \tag{5.85}$$

where

$$x_i = \frac{-\psi c_i f_s}{1-f_s^2}. \tag{5.86}$$

Rewriting the denominator for the same values of i gives

$$\sin(f_sT_m + 2\hat{\varphi}_i) = \sin(f_sT_m+2\varphi)\cos(2\delta_i) + \cos(f_sT_m+2\varphi)\sin(2\delta_i). \tag{5.87}$$

Using the identities

$$\sin(2\arctan(\theta)) = \frac{2\theta}{1+\theta^2}, \tag{5.88}$$

and,

$$\cos(2\arctan(\theta)) = \frac{1-\theta^2}{1+\theta^2}, \tag{5.89}$$

we obtain

$$\sin(f_sT_m + 2\hat{\varphi}_i) = \frac{\sin(f_sT_m+2\varphi)(1-x_i^2) + \cos(f_sT_m+2\varphi)2x_i}{1+x_i^2}. \tag{5.90}$$

Thus,

$$\frac{\sin^2(f_sT_m+\hat{\varphi}_i)}{\sin(f_sT_m+2\hat{\varphi}_i)} = \frac{(\sin(f_sT_m+\varphi)+x_i\cos(f_sT_m+\varphi))^2}{1+x_i^2}\frac{1+x_i^2}{\sin(f_sT_m+2\varphi)(1-x_i^2)+\cos(f_sT_m+2\varphi)2x_i}, \tag{5.91}$$

which is rewritten to

$$\frac{\sin^2(f_sT_m+\hat{\varphi}_i)}{\sin(f_sT_m+2\hat{\varphi}_i)} = \frac{\sin^2(f_sT_m+\varphi)+2x_i\sin(f_sT_m+\varphi)\cos(f_sT_m+\varphi)+x_i^2\cos^2(f_sT_m+\varphi)}{\sin(f_sT_m+2\varphi)(1-x_i^2)+2x_i\cos(f_sT_m+2\varphi)}. \tag{5.92}$$

Using Equation (5.92) to rewrite Equation (5.81) to get

$$\frac{\sin^2(f_sT_m)+2x_1\sin(f_sT_m)\cos(f_sT_m)+x_1^2\cos^2(f_sT_m)}{\sin(f_sT_m)(1-x_1^2)+2x_1\cos(f_sT_m)} = \frac{\sin^2(f_sT_m)+2x_2\sin(f_sT_m)\cos(f_sT_m)+x_2^2\cos^2(f_sT_m)}{\sin(f_sT_m)(1-x_2^2)+2x_2\cos(f_sT_m)}. \tag{5.93}$$

This equation is further simplified in Appendix D. It is shown that Equation (5.81) reduces to

$$(x_1-x_2)\left(\sin(f_sT_m+2\varphi)(x_1+x_2)+2\sin(f_sT_m+\varphi)\sin(\varphi)-2\cos(f_sT_m+\varphi)x_1x_2\cos(2f_sT_m+3\varphi)\right)=0. \tag{5.94}$$

Assuming that

$$c_{v,1} \neq c_{v,2}, \tag{5.95}$$

as the case that they are equal has already been discussed, simplifies Equation (5.94) to

$$\sin(f_sT_m+2\varphi)(x_1+x_2)+2\sin(f_sT_m+\varphi)\sin(\varphi)-2\cos(f_sT_m+\varphi)x_1x_2\cos(2f_sT_m+3\varphi)=0. \tag{5.96}$$

Expanding x_i for $i \in \{1,2\}$, gives

$$c_2\frac{\psi f_s}{-f_s^2+1}\left(\sin(f_sT_m+2\varphi)+2\cos(f_sT_m+\varphi)\cos(2f_sT_m+3\varphi)\frac{\psi f_s c_1}{-f_s^2+1}\right) = c_1\sin(f_sT_m+2\varphi)+2\sin(f_sT_m+\varphi)\sin(\varphi), \tag{5.97}$$

and, therefore,

$$c_{v,2} = -c_m + \frac{(c_m+c_{v,1})\sin(f_sT_m+2\varphi)+2\sin(f_sT_m+\varphi)\sin(\varphi)}{\frac{\psi f_s}{-f_s^2+1}\left(\sin(f_sT_m+2\varphi)+2\cos(f_sT_m+\varphi)\cos(2f_sT_m+3\varphi)\frac{\psi f_s(c_m+c_{v,1})}{-f_s^2+1}\right)}. \tag{5.98}$$

Hence, we have an expression for the second damping value, based on the first damping value.

5.3.2. Classifying extrema

From the analysis, several extrema for variable damping values $c_{v,1}$ and $c_{v,2}$ were identified. These extrema are noted in Table E.1 in Appendix E.

We note that the first extremum falls under the case where there is no switch. Hence, we refer to Chapter 4, where the case of no switch is discussed.

To determine whether the remaining extrema are a maximum or a minimum, the second derivative test is used. This is done on a per-group basis. For each group, variables f_s, T_m and φ are fixed. As we are dealing with a multivariate function, the determinant of the Hessian matrix is required. The determinant is given by

$$\text{Det}(c_{v,1}, c_{v,2}) = \frac{\partial^2 P_{\text{ave}}}{\partial c_{v,1}^2}\frac{\partial P_{\text{ave}}}{\partial c_{v,2}^2} - \frac{\partial}{\partial c_{v,2}}\left(\frac{\partial P_{\text{ave}}}{\partial c_{v,1}}\right)\frac{\partial}{\partial c_{v,1}}\left(\frac{\partial P_{\text{ave}}}{\partial c_{v,2}}\right), \tag{5.99}$$

which will be used in later calculations.

Case $f_sT_m + 2\hat{\varphi}_1 = \pi + k_1 2\pi$ and $f_sT_m + 2\hat{\varphi}_2 = k_2 2\pi$

We first look at the cases where the assumptions of $f_sT_m + 2\hat{\varphi}_1 = \pi + k_1 2\pi$ and also $f_sT_m + 2\hat{\varphi}_2 = k_2 2\pi$ are made. The time-averaged power for these assumptions is equal to

$$\begin{aligned} P_{\text{ave}} = \frac{1}{\frac{2\pi}{f_s}}\Bigg(&\frac{c_{v,1}f_s^2}{2}\frac{1}{(f_s^2-1)^2+(\psi c_1 f_s)^2}\left(T_m + \frac{1}{f_s}\sin(f_sT_m)\right) \\ &+\frac{c_{v,2}f_s^2}{2}\frac{1}{(f_s^2-1)^2+(\psi c_2 f_s)^2}\left(\frac{2\pi}{f_s} - T_m + \frac{1}{f_s}\sin(f_sT_m)\right)\Bigg). \end{aligned} \tag{5.100}$$

Now, the derivative with respect to $c_{v,2}$ is taken two times. Note that the first derivative has already been given in Equation (5.50) and is similar to the first derivative in the case of no switch, given in Equation (4.7). It only differs by a term which doesn't contain the damping value $c_{v,2}$, thus the second derivative will look similar. So,

$$\begin{aligned} \frac{\partial P_{\text{ave}}^2}{\partial c_{v,2}^2} = &\frac{1}{\frac{2\pi}{f_s}}\frac{1}{2}\left(\frac{2\pi}{f_s} - T_m + \frac{1}{f_s}\sin(f_sT_m)\right) \\ &\frac{((f_s^2-1)^2+(c_2\psi f_s)^2)^2\left(-2c_{v,2}c_m\psi^2 f_s^2\right) - \left((f_s^2-1)^2+\psi^2 f_s^2(c_m^2-c_{v,2}^2)\right)\left(\frac{\partial((f_s^2-1)^2+(c_2\psi f_s)^2)^2}{\partial c_{v,2}}\right)}{((f_s^2-1)^2+(c_2\psi f_s)^2)^4}. \end{aligned} \tag{5.101}$$

Similar to Chapter 4, $\frac{\partial((f_s^2-1)^2+(c_2\psi f_s)^2)^2}{\partial c_{v,2}}$ drops out when filling in either value of $c_{v,2}$ for this case.

Now, for $c_{v,1}$. Note that the first-order partial derivative has been given in Equation (5.55). The second-order partial derivative is given by

$$\begin{aligned} \frac{\partial P_{\text{ave}}^2}{\partial c_{v,1}^2} = &\frac{1}{\frac{2\pi}{f_s}}\frac{1}{2}\left(T_m + \frac{1}{f_s}\sin(f_sT_m)\right) \\ &\frac{((f_s^2-1)^2+(c_1\psi f_s)^2)^2\left(-2c_{v,1}c_m\psi^2 f_s^2\right) - \left((f_s^2-1)^2+\psi^2 f_s^2(c_m^2-c_{v,1}^2)\right)\left(\frac{\partial((f_s^2-1)^2+(c_1\psi f_s)^2)^2}{\partial c_{v,1}}\right)}{((f_s^2-1)^2+(c_1\psi f_s)^2)^4}, \end{aligned} \tag{5.102}$$

where, when evaluating at the value $c_{v,1} = 0$, we get

$$\frac{\partial P_{\text{ave}}^2}{\partial c_{v,1}^2}(0) = \frac{1}{\frac{2\pi}{f_s}}\frac{1}{2}\left(T_m + \frac{1}{f_s}\sin(f_sT_m)\right)\frac{-\left((f_s^2-1)^2+\psi^2 f_s^2 c_m^2\right)\left(\frac{\partial((f_s^2-1)^2+(c_1\psi f_s)^2)^2}{\partial c_{v,1}}\right)}{((f_s^2-1)^2+(c_m\psi f_s)^2)^4}. \tag{5.103}$$

This simplifies to

$$\frac{\partial P_{\text{ave}}^2}{\partial c_{v,1}^2}(0) = \frac{1}{\frac{2\pi}{f_s}}\frac{1}{2}\left(T_m + \frac{1}{f_s}\sin(f_sT_m)\right)\frac{-\left(\frac{\partial((f_s^2-1)^2+(c_1\psi f_s)^2)^2}{\partial c_{v,1}}\right)}{((f_s^2-1)^2+(c_m\psi f_s)^2)^3}, \tag{5.104}$$

and when working out the partial derivative

$$\frac{\partial((f_s^2-1)^2+(c_1\psi f_s)^2)^2}{\partial c_{v,1}} = 4c_1\psi^2 f_s^2\left(\left(f_s^2-1\right)^2+(c_1\psi f_s)^2\right), \tag{5.105}$$

we get

$$\frac{\partial P^2_{\text{ave}}}{\partial c^2_{v,1}}(0) = \frac{1}{\frac{2\pi}{f_s}}\frac{1}{2}\left(T_m+\frac{1}{f_s}\sin(f_sT_m)\right)\frac{-\left(4c_m\psi^2f_s^2\left(\left(f_s^2-1\right)^2+(c_m\psi f_s)^2\right)\right)}{((f_s^2-1)^2+(c_m\psi f_s)^2)^3}, \tag{5.106}$$

For the mixed partial derivative for $c_{v,1}$ and $c_{v,2}$, it is noted that in Equation (5.55) and Equation (5.50) contain no damping value $c_{v,2}$ and $c_{v,1}$ respectively. Hence

$$\frac{\partial}{\partial c_{v,2}}\left(\frac{\partial P_{\text{ave}}}{\partial c_{v,1}}\right) = 0, \tag{5.107}$$

and

$$\frac{\partial}{\partial c_{v,1}}\left(\frac{\partial P_{\text{ave}}}{\partial c_{v,2}}\right) = 0. \tag{5.108}$$

Now, an analysis is made of the determinant given in Equation (5.99).

Note that

$$\text{sgn}\left(\frac{\partial P^2_{\text{ave}}}{\partial c^2_{v,2}}\right) = \text{sgn}\left(\frac{2\pi}{f_s}-T_m+\frac{1}{f_s}\sin(f_sT_m)\right)\text{sgn}\left(-2c_{v,2}c_m\psi^2f_s^2\right), \tag{5.109}$$

where

$$\text{sgn}\left(\frac{2\pi}{f_s}-T_m+\frac{1}{f_s}\sin(f_sT_m)\right) > 0, \tag{5.110}$$

as, with $x = f_sT_m$,

$$f(x) = 2\pi - x + \sin(x) > 0, \tag{5.111}$$

for $x \in (0, 2\pi)$, since $f(0) = 2\pi$ and $f(2\pi) = 0$. Thus

$$\text{sgn}\left(\frac{\partial P^2_{\text{ave}}}{\partial c^2_{v,2}}\right) = \text{sgn}\left(-2c_{v,2}c_m\psi^2f_s^2\right), \tag{5.112}$$

but, as also $c_m > 0$,

$$\text{sgn}\left(\frac{\partial P^2_{\text{ave}}}{\partial c^2_{v,2}}\right) = -\text{sgn}(c_{v,2}). \tag{5.113}$$

Now, for

$$\frac{\partial P^2_{\text{ave}}}{\partial c^2_{v,1}}(0) < 0. \tag{5.114}$$

as all terms, except for the minus, in Equation (5.106) are positive.

To classify what type of extrema we have, the determinant is studied. Note that for this case, we have that

$$\text{Det}(c_{v,1},c_{v,2}) = \underbrace{\frac{\partial^2 P_{\text{ave}}}{\partial c^2_{v,1}}}_{<0}\underbrace{\frac{\partial P_{\text{ave}}}{\partial c^2_{v,2}}}_{\text{sgn}=\pm1} - \underbrace{\frac{\partial}{\partial c_{v,2}}\left(\frac{\partial P_{\text{ave}}}{\partial c_{v,1}}\right)}_{=0}\underbrace{\frac{\partial}{\partial c_{v,1}}\left(\frac{\partial P_{\text{ave}}}{\partial c_{v,2}}\right)}_{=0}, \tag{5.115}$$

and thus, when sgn$(c_{v,2}) > 0$ we have a saddle point and for sgn$(c_{v,2}) < 0$ we have a minimum.

Case $f_sT_m + 2\hat{\varphi}_1 = k_1 2\pi$ and $f_sT_m + 2\hat{\varphi}_2 = k_2 2\pi$

Now, assuming that both $f_sT_m + 2\hat{\varphi}_1 = k_1 2\pi$ and $f_sT_m + 2\hat{\varphi}_2 = k_2 2\pi$. Then,

$$P_{\text{ave}} = \frac{1}{\frac{2\pi}{f_s}} \left(\frac{c_{v,1} f_s^2}{2} \frac{1}{(f_s^2 - 1)^2 + (\psi c_1 f_s)^2} \left(T_m - \frac{1}{f_s} \sin(f_s T_m) \right) \right. \\ \left. + \frac{c_{v,2} f_s^2}{2} \frac{1}{(f_s^2 - 1)^2 + (\psi c_2 f_s)^2} \left(\frac{2\pi}{f_s} - T_m + \frac{1}{f_s} \sin(f_s T_m) \right) \right). \tag{5.116}$$

Note that the term which is multiplied with $c_{v,2}$ is no different than from previous case which is studied. Hence, again

$$\frac{\partial P_{\text{ave}}^2}{\partial c_{v,2}^2} = \frac{1}{\frac{2\pi}{f_s}} \frac{1}{2} \left(\frac{2\pi}{f_s} - T_m + \frac{1}{f_s} \sin(f_s T_m) \right) \\ \frac{((f_s^2 - 1)^2 + (c_2 \psi f_s)^2)^2 \left(-2 c_{v,2} c_m \psi^2 f_s^2 \right) - \left((f_s^2 - 1)^2 + \psi^2 f_s^2 (c_m^2 - c_{v,2}^2) \right) \left(\frac{\partial ((f_s^2 - 1)^2 + (c_2 \psi f_s)^2)^2}{\partial c_{v,2}} \right)}{((f_s^2 - 1)^2 + (c_2 \psi f_s)^2)^4}. \tag{5.117}$$

Now, for $c_{v,1}$. Note that the first order partial derivative has been given in Equation (5.55). The second-order partial derivative is given by

$$\frac{\partial P_{\text{ave}}^2}{\partial c_{v,1}^2} = \frac{1}{\frac{2\pi}{f_s}} \frac{1}{2} \left(T_m - \frac{1}{f_s} \sin(f_s T_m) \right) \\ \frac{((f_s^2 - 1)^2 + (c_1 \psi f_s)^2)^2 \left(-2 c_{v,1} c_m \psi^2 f_s^2 \right) - \left((f_s^2 - 1)^2 + \psi^2 f_s^2 (c_m^2 - c_{v,1}^2) \right) \left(\frac{\partial ((f_s^2 - 1)^2 + (c_1 \psi f_s)^2)^2}{\partial c_{v,1}} \right)}{((f_s^2 - 1)^2 + (c_1 \psi f_s)^2)^4}. \tag{5.118}$$

Note that for both second-order partial derivatives, it is the case that

$$c_m^2 - c_{v,i}^2 = 0, \tag{5.119}$$

for $i \in \{1, 2\}$, as

$$c_{v,1} = -c_{v,2}. \tag{5.120}$$

Thus the derivatives simplify to

$$\frac{\partial P_{\text{ave}}^2}{\partial c_{v,1}^2} = \frac{1}{\frac{2\pi}{f_s}} \frac{1}{2} \left(T_m - \frac{1}{f_s} \sin(f_s T_m) \right) \frac{((f_s^2 - 1)^2 + (c_1 \psi f_s)^2)^2 \left(-2 c_{v,1} c_m \psi^2 f_s^2 \right)}{((f_s^2 - 1)^2 + (c_1 \psi f_s)^2)^4}, \tag{5.121}$$

and

$$\frac{\partial P_{\text{ave}}^2}{\partial c_{v,2}^2} = \frac{1}{\frac{2\pi}{f_s}} \frac{1}{2} \left(\frac{2\pi}{f_s} - T_m + \frac{1}{f_s} \sin(f_s T_m) \right) \frac{((f_s^2 - 1)^2 + (c_2 \psi f_s)^2)^2 \left(-2 c_{v,2} c_m \psi^2 f_s^2 \right)}{((f_s^2 - 1)^2 + (c_2 \psi f_s)^2)^4}. \tag{5.122}$$

Note that, like before,

$$\text{sgn}\left(\frac{\partial P_{\text{ave}}^2}{\partial c_{v,i}^2} \right) = -\text{sgn}\left(c_{v,i} \right), \tag{5.123}$$

for $i \in \{1, 2\}$.

For the mixed partial derivative for $c_{v,1}$ and $c_{v,2}$, it is again noted that in Equation (5.55) and Equation (5.50) contain no damping value $c_{v,2}$ and $c_{v,1}$ respectively. Hence

$$\frac{\partial}{\partial c_{v,2}} \left(\frac{\partial P_{\text{ave}}}{\partial c_{v,1}} \right) = 0, \tag{5.124}$$

and

$$\frac{\partial}{\partial c_{v,1}} \left(\frac{\partial P_{\text{ave}}}{\partial c_{v,2}} \right) = 0. \tag{5.125}$$

To classify what type of extrema we have, the determinant is studied. Note that for this case, we have that

$$\text{Det}(c_{v,1}, c_{v,2}) = \underbrace{\frac{\partial^2 P_{\text{ave}}}{\partial c_{v,1}^2}}_{\text{sgn}=\pm 1} \underbrace{\frac{\partial P_{\text{ave}}}{\partial c_{v,2}^2}}_{\text{sgn}=\pm 1} - \underbrace{\frac{\partial}{\partial c_{v,2}}\left(\frac{\partial P_{\text{ave}}}{\partial c_{v,1}}\right)}_{=0} \underbrace{\frac{\partial}{\partial c_{v,1}}\left(\frac{\partial P_{\text{ave}}}{\partial c_{v,2}}\right)}_{=0}, \tag{5.126}$$

and the signs are given in Table 5.2.

Table 5.2: Outcome of extrema for damping values $c_{v,1}$ and $c_{v,2}$.

Value of $c_{v,1}$	sgn $(c_{v,1})$	Value of $c_{v,2}$	sgn $(c_{v,2})$
$\frac{\sqrt{c_m^2\psi^2 f_s^2+(f_s^2-1)^2}}{\psi f_s}$	$+1$	$-\frac{\sqrt{c_m^2\psi^2 f_s^2+(f_s^2-1)^2}}{\psi f_s}$	-1
$-\frac{\sqrt{c_m^2\psi^2 f_s^2+(f_s^2-1)^2}}{\psi f_s}$	-1	$\frac{\sqrt{c_m^2\psi^2 f_s^2+(f_s^2-1)^2}}{\psi f_s}$	$+1$
$c_m + \frac{2c_m}{\hat{A}} + \frac{\sqrt{(2\psi f_s c_m)^2(\hat{A}+1)-\hat{A}^2(f_s^2-1)^2}}{\psi f_s \hat{A}}$	$\pm$	$-c_m - \frac{2c_m}{\hat{A}} - \frac{\sqrt{(2\psi f_s c_m)^2(\hat{A}+1)-\hat{A}^2(f_s^2-1)^2}}{\psi f_s \hat{A}}$	$\mp$
$c_m + \frac{2c_m}{\hat{A}} - \frac{\sqrt{(2\psi f_s c_m)^2(\hat{A}+1)-\hat{A}^2(f_s^2-1)^2}}{\psi f_s \hat{A}}$	$\pm$	$-c_m - \frac{2c_m}{\hat{A}} + \frac{\sqrt{(2\psi f_s c_m)^2(\hat{A}+1)-\hat{A}^2(f_s^2-1)^2}}{\psi f_s \hat{A}}$	$\mp$
$c_m - \frac{2c_m}{\tilde{A}} + \frac{\sqrt{(2\psi f_s c_m)^2(1-\tilde{A})-\tilde{A}^2(f_s^2-1)^2}}{\psi f_s \tilde{A}}$	$\pm$	$-c_m + \frac{2c_m}{\tilde{A}} - \frac{\sqrt{(2\psi f_s c_m)^2(1-\tilde{A})-\tilde{A}^2(f_s^2-1)^2}}{\psi f_s \tilde{A}}$	$\mp$
$c_m - \frac{2c_m}{\tilde{A}} - \frac{\sqrt{(2\psi f_s c_m)^2(1-\tilde{A})-\tilde{A}^2(f_s^2-1)^2}}{\psi f_s \tilde{A}}$	$\pm$	$-c_m + \frac{2c_m}{\tilde{A}} + \frac{\sqrt{(2\psi f_s c_m)^2(1-\tilde{A})-\tilde{A}^2(f_s^2-1)^2}}{\psi f_s \tilde{A}}$	$\mp$

The signs of the first two cases are trivial and thus denoted in the table. The signs of the other cases are more difficult to determine, due to the extensive number of terms in the fraction and due to the sign switch of $\hat{A}$ and $\tilde{A}$. This difficulty is indicated in the figures below.

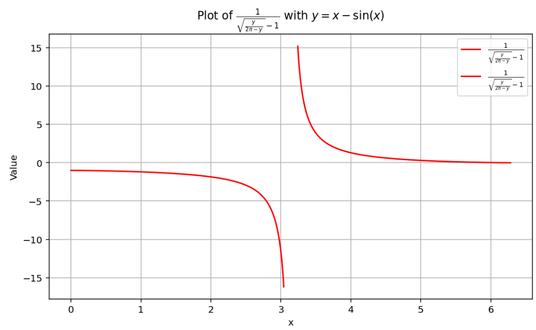

Figure 5.3: Graph of the function $\frac{1}{\hat{A}}$ for $x \in [1, \pi - \frac{1}{10}]$ and for $x \in [\pi + \frac{1}{10}, 2pi]$.

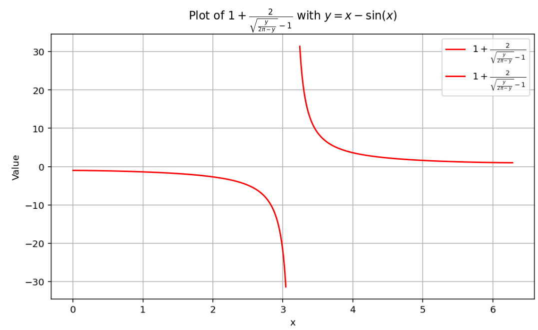

Figure 5.4: Graph of the function $1 + \frac{2}{\hat{A}}$ for for $x \in [1, \pi - \frac{1}{10}]$ and for $x \in [\pi + \frac{1}{10}, 2pi]$.

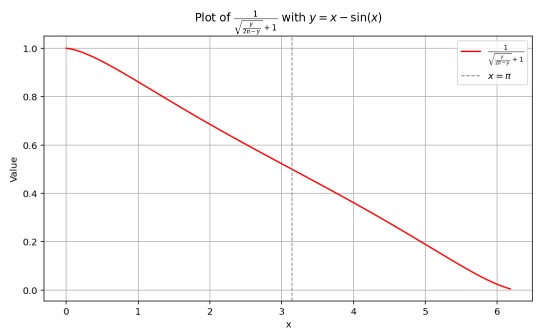

Figure 5.5: Graph of the function $\frac{1}{\tilde{A}}$ for $x \in [0, 2\pi)$ with a grey dotted line indicating where $x = \pi$.

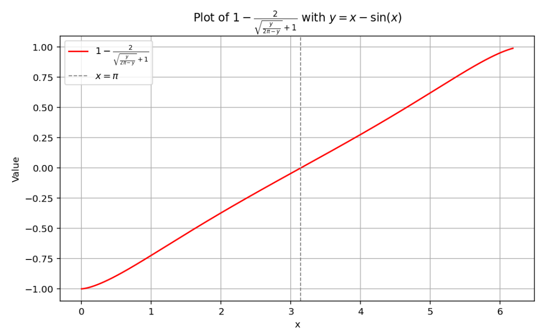

Figure 5.6: Graph of the function $\frac{1}{\tilde{A}}$ for $x \in [0, 2\pi)$ with a grey dotted line indicating where $x = \pi$.

Note that the interval isn't chosen for x-value π as then we reach the asymptote for $\frac{1}{\hat{A}}$. The grey dotted line in Figure 5.5 and Figure 5.6 indicates the point where the graph crosses the x-axis.

The exact sign of $c_{v,1}$ and $c_{v,2}$ doesn't need to be determined, however. Due to the shown result that

$$\text{sgn}\left(\frac{\partial P_{\text{ave}}^2}{\partial c_{v,i}^2}\right) = -\text{sgn}\left(c_{v,i}\right), \tag{5.127}$$

and the given fact for these six cases from Table 5.2, that

$$\text{sgn}\left(c_{v,1}\right) = -\text{sgn}\left(c_{v,2}\right), \tag{5.128}$$

and the result that

$$c_{v,i} \neq 0, \tag{5.129}$$

as then $c_{v,1} = c_{v,2}$, and we would have no switch and thus we can refer back to Chapter 4, we can conclude that

$$\text{sgn}\left(\frac{\partial P_{\text{ave}}^2}{\partial c_{v,1}^2}\right) = -\text{sgn}\left(\frac{\partial P_{\text{ave}}^2}{\partial c_{v,2}^2}\right), \tag{5.130}$$

and therefore all extrema from Table 5.2 are saddle points.

Case $f_sT_m + 2\hat{\varphi}_1 \neq k_1\pi$

For this case, the damping variable of $c_{v,1}$ has been left free. This means that the sign of the determinant can not be determined and therefore no classification can be given to this extremum. Hence, a visualisation will show us the behaviour of this extremum.

5.3.3. Visualising extrema

For the different extrema, a visualisation is created to show which damping values yield the most time-averaged power. These visualations are given in Appendix F. Note that there are no visualisations given of the case where the damping value contains the value $\tilde{A}$, as those values do not exist for the bounds picked of the parameters.

Due to the moment of switch in damping value being at an arbitrary moment, this variable is crucial for the correct visualisation of the power analysis. Per case, a different method is used to determine this moment of switch.

Damping value equal to optimal damping value

When one of the damping values is equal to the optimal damping value, we have the assumption that $\sin\left(f_sT_m + 2\hat{\varphi}_1\right) = 0$. This implies that $f_sT_m = k_1\pi - 2\varphi - 2\delta_1$. As $f_sT_m \in (0, 2\pi) \setminus \{\pi\}$, the moment of switch should be calculated modulo 2π. The necessity of this module calculation is highlighted in Figure 5.7.

The modulo calculation is visualised in Figure 5.8. It can be seen that there is a moment where the constant k_1 switches in value.

Damping values are negative reciprocal and dependent on moment of switch

When the damping value is dependent on the moment of switch, but also a negative reciprocal, the time-averaged power is equal to

$$\begin{aligned} P_{\text{ave}} = \frac{1}{\frac{2\pi}{f_s}} \Bigg(& \frac{c_{v,1}f_s^2}{2} \frac{1}{(f_s^2-1)^2 + (\psi c_1 f_s)^2} \left(T_m - \frac{1}{f_s}\sin(f_sT_m)\right) \\ & + \frac{c_{v,2}f_s^2}{2} \frac{1}{(f_s^2-1)^2 + (\psi c_2 f_s)^2} \left(\frac{2\pi}{f_s} - T_m + \frac{1}{f_s}\sin(f_sT_m)\right)\Bigg), \end{aligned} \tag{5.131}$$

meaning that the two variables are the frequency and the moment of switch. Hence, for these damping values, the extrema can be visualised by a 3D graph of frequency versus moment of switch versus the time-averaged power. Thus, no module calculation needs to be made.

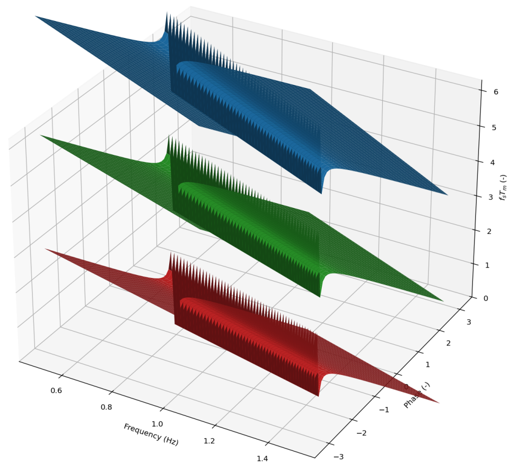

Figure 5.7: Moment of switch between damping values for different frequency and phase values where the blue graph indicates $k_1 = 2$, the green line has $k_1 = 1$ and the red line is $k_1 = 0$ for $c_{v,1} = 0$.

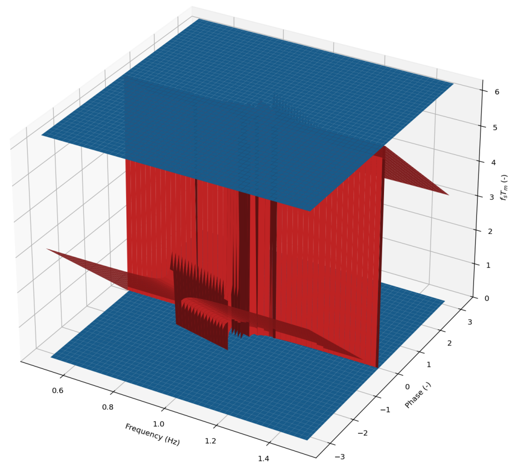

Figure 5.8: Moment of switch between damping values for different frequency and phase values where the blue graph indicates the bound of 0 and 2π and the red is the moment of switch for $c_{v,1} = 0$.

Boundary extrema

On the boundary values of the damping, we have the frequency, phase and moment of switch as unknown variables. As these cannot be determined from the model, these all need to be plotted. Therefore, a numerical implementation is created which plots the maximum value of the time-averaged power per different phase value in the 3D graph. This code can be seen in Appendix G.

Free first damping value

For the case where $c_{v,1}$ is assumed to be a free variable, the time-averaged power is dependent on four variables. Thus, a visualisation is created based on a similar way of work for the boundary extrema, but then the maximum value is taken per phase and first damping value. This code can be seen in Appendix H.

5.4. Conclusion on power analysis singular switch

Previous sections found and classified all extrema of the damped harmonic oscillator subject to base excitation with an arbitrary switch in damping value. An overview of these extrema and their type is given in Table 5.3.

From the visualisations in Appendix F, it can be concluded that the best solution is that where the damping value does not change. So, the optimal power yielding is when we have the situation discussed in Chapter 4 and optimal damping value

$$c_{v,1} = c_{v,2} = c_{v,\text{opt},+}. \tag{5.132}$$

Table 5.3: Types of extrema for damping values $c_{v,1}$ and $c_{v,2}$.

Value of $c_{v,1}$	Value of $c_{v,2}$	Type
$c_{v,\text{opt}}$	$c_{v,\text{opt}}$	Maximum
0	$\frac{\sqrt{c_m^2\psi^2 f_s^2+(f_s^2-1)^2}}{\psi f_s}$	Saddle point
0	$-\frac{\sqrt{c_m^2\psi^2 f_s^2+(f_s^2-1)^2}}{\psi f_s}$	Minium
$\frac{\sqrt{c_m^2\psi^2 f_s^2+(f_s^2-1)^2}}{\psi f_s}$	$-\frac{\sqrt{c_m^2\psi^2 f_s^2+(f_s^2-1)^2}}{\psi f_s}$	Saddle point
$-\frac{\sqrt{c_m^2\psi^2 f_s^2+(f_s^2-1)^2}}{\psi f_s}$	$\frac{\sqrt{c_m^2\psi^2 f_s^2+(f_s^2-1)^2}}{\psi f_s}$	Saddle point
$c_m+\frac{2c_m}{\hat{A}}+\frac{\sqrt{(2\psi f_s c_m)^2\left(\hat{A}+1\right)-\hat{A}^2(f_s^2-1)^2}}{\psi f_s\hat{A}}$	$-c_m-\frac{2c_m}{\hat{A}}-\frac{\sqrt{(2\psi f_s c_m)^2\left(\hat{A}+1\right)-\hat{A}^2(f_s^2-1)^2}}{\psi f_s\hat{A}}$	Saddle point
$c_m+\frac{2c_m}{\hat{A}}-\frac{\sqrt{(2\psi f_s c_m)^2\left(\hat{A}+1\right)-\hat{A}^2(f_s^2-1)^2}}{\psi f_s\hat{A}}$	$-c_m-\frac{2c_m}{\hat{A}}+\frac{\sqrt{(2\psi f_s c_m)^2\left(\hat{A}+1\right)-\hat{A}^2(f_s^2-1)^2}}{\psi f_s\hat{A}}$	Saddle point
$c_m-\frac{2c_m}{\tilde{A}}+\frac{\sqrt{(2\psi f_s c_m)^2\left(1-\tilde{A}\right)-\tilde{A}^2(f_s^2-1)^2}}{\psi f_s\tilde{A}}$	$-c_m+\frac{2c_m}{\tilde{A}}-\frac{\sqrt{(2\psi f_s c_m)^2\left(1-\tilde{A}\right)-\tilde{A}^2(f_s^2-1)^2}}{\psi f_s\tilde{A}}$	Saddle point
$c_m-\frac{2c_m}{\tilde{A}}-\frac{\sqrt{(2\psi f_s c_m)^2\left(1-\tilde{A}\right)-\tilde{A}^2(f_s^2-1)^2}}{\psi f_s\tilde{A}}$	$-c_m+\frac{2c_m}{\tilde{A}}+\frac{\sqrt{(2\psi f_s c_m)^2\left(1-\tilde{A}\right)-\tilde{A}^2(f_s^2-1)^2}}{\psi f_s\tilde{A}}$	Saddle point
$c_{v,1}$	$-c_m+\frac{(c_m+c_{v,1})\sin(f_sT_m+2\varphi)+2\sin(f_sT_m+\varphi)\sin(\varphi)}{\frac{\psi f_s}{-f_s^2+1}\left(\sin(f_sT_m+2\varphi)+2\cos(f_sT_m+\varphi)\cos(2f_sT_m+3\varphi)\frac{\psi f_s(c_m+c_{v,1})}{-f_s^2+1}\right)}$	-
-0.15	0.35	Boundary Extrema
0.35	-0.15	Boundary Extrema

6 Energy Analysis Multiple Switch

6.1. Power analysis according to paper

For the case where there is more than one switch, first, the case is studied similarly to Nikzamir et al. [8]. In this paper, switches are defined as

$$T_0 = 0, T_1 = \frac{\pi}{2f_s}, T_2 = \frac{\pi}{f_s}, T_3 = \frac{3\pi}{2f_s}, T_4 = \frac{2\pi}{f_s}. \tag{6.1}$$

However, the switches aren't used to implement four different damping values, but rather to switch between two values. Hence

$$c_v(t) = \begin{cases} c_v = c_{v,1} & t \in [T_0, T_1) \cup [T_2, T_3), \\ c_v = c_{v,2} & t \in [T_1, T_2) \cup [T_3, T_4), \end{cases} \tag{6.2}$$

and still for $i \in \{1, 2\}$

$$c_i = c_m + c_{v,i}. \tag{6.3}$$

An attentive reader might note this expression for damping values to be an approximation of the cosine function with a frequency of twice the base excitation frequency.

Now, again, we generally use the expression for the time-averaged power as

$$P_{\text{ave}} = \frac{1}{\frac{2\pi}{f_s}} \frac{c_v f_s^2}{2} \frac{1}{(f_s^2 - 1)^2 + (\psi c f_s)^2} \int_0^T 1 - \cos(2f_s t + 2\hat{\varphi}) \mathrm{d}t. \tag{6.4}$$

However, experiencing a switch between variable damping value $c_{v,1}$ and $c_{v,2}$ on certain time T_m which lies between 0 and $\frac{2\pi}{f_s}$, gives

$$\begin{aligned} P_{\text{ave}} = \frac{1}{\frac{2\pi}{f_s}} \Bigg(& \int_{T_0}^{T_1} \frac{c_{v,1} f_s^2}{2} \frac{1}{(f_s^2 - 1)^2 + (\psi c_1 f_s)^2} \left(1 - \cos(2f_s t + 2\hat{\varphi}_1)\right) \mathrm{d}t \\ & \int_{T_1}^{T_2} \frac{c_{v,2} f_s^2}{2} \frac{1}{(f_s^2 - 1)^2 + (\psi c_2 f_s)^2} \left(1 - \cos(2f_s t + 2\hat{\varphi}_2)\right) \mathrm{d}t \\ & \int_{T_2}^{T_3} \frac{c_{v,1} f_s^2}{2} \frac{1}{(f_s^2 - 1)^2 + (\psi c_1 f_s)^2} \left(1 - \cos(2f_s t + 2\hat{\varphi}_1)\right) \mathrm{d}t \\ & \int_{T_3}^{T_4} \frac{c_{v,2} f_s^2}{2} \frac{1}{(f_s^2 - 1)^2 + (\psi c_2 f_s)^2} \left(1 - \cos(2f_s t + 2\hat{\varphi}_2)\right) \mathrm{d}t \Bigg), \end{aligned} \tag{6.5}$$

working this out gives

$$
\begin{aligned}
P_{\text{ave}} = \frac{1}{\frac{2\pi}{f_s}} \Bigg(&\int_0^{\frac{\pi}{2f_s}} \frac{c_{v,1} f_s^2}{2} \frac{1}{(f_s^2-1)^2 + (\psi c_1 f_s)^2} \left(1 - \cos(2 f_s t + 2\hat{\varphi}_1)\right) \mathrm{d}t \\
&\int_{\frac{\pi}{2f_s}}^{\frac{\pi}{f_s}} \frac{c_{v,2} f_s^2}{2} \frac{1}{(f_s^2-1)^2 + (\psi c_2 f_s)^2} \left(1 - \cos(2 f_s t + 2\hat{\varphi}_2)\right) \mathrm{d}t \\
&\int_{\frac{\pi}{f_s}}^{\frac{3\pi}{2f_s}} \frac{c_{v,1} f_s^2}{2} \frac{1}{(f_s^2-1)^2 + (\psi c_1 f_s)^2} \left(1 - \cos(2 f_s t + 2\hat{\varphi}_1)\right) \mathrm{d}t \\
&\int_{\frac{3\pi}{2f_s}}^{\frac{2\pi}{f_s}} \frac{c_{v,2} f_s^2}{2} \frac{1}{(f_s^2-1)^2 + (\psi c_2 f_s)^2} \left(1 - \cos(2 f_s t + 2\hat{\varphi}_2)\right) \mathrm{d}t \Bigg),
\end{aligned} \tag{6.6}
$$

and so

$$
\begin{aligned}
P_{\text{ave}} = \frac{1}{\frac{2\pi}{f_s}} \frac{f_s^2}{2} \Bigg(&\frac{c_{v,1}}{(f_s^2-1)^2 + (\psi c_1 f_s)^2} \left(\frac{\pi}{f_s} - \frac{1}{2f_s} \left(2\sin(\pi + 2\hat{\varphi}_1) - 2\sin(2\hat{\varphi}_1)\right) \right) \\
&\frac{c_{v,2}}{(f_s^2-1)^2 + (\psi c_2 f_s)^2} \left(\frac{\pi}{f_s} - \frac{1}{2f_s} \left(2\sin(2\hat{\varphi}_2) - 2\sin(\pi + 2\hat{\varphi}_2)\right) \right) \Bigg).
\end{aligned} \tag{6.7}
$$

Using the goniometric identity with the π shift gives

$$
P_{\text{ave}} = \frac{1}{\frac{2\pi}{f_s}} \frac{f_s^2}{2} \left(\frac{c_{v,1}}{(f_s^2-1)^2 + (\psi c_1 f_s)^2} \left(\frac{\pi}{f_s} + \frac{1}{2f_s} 4\sin(2\hat{\varphi}_1) \right) + \frac{c_{v,2}}{(f_s^2-1)^2 + (\psi c_2 f_s)^2} \left(\frac{\pi}{f_s} - \frac{1}{2f_s} 4\sin(2\hat{\varphi}_2) \right) \right), \tag{6.8}
$$

and so, simplified,

$$
P_{\text{ave}} = \frac{1}{\frac{2\pi}{f_s}} \frac{f_s}{2} \left(\frac{c_{v,1}}{(f_s^2-1)^2 + (\psi c_1 f_s)^2} \left(\pi + 2\sin(2\hat{\varphi}_1)\right) + \frac{c_{v,2}}{(f_s^2-1)^2 + (\psi c_2 f_s)^2} \left(\pi - 2\sin(2\hat{\varphi}_2)\right) \right). \tag{6.9}
$$

An optimal solution needs to be found on all variables $c_{v,1}, c_{v,2}, f_s$ and φ. For reasons that become clear later, first an optimal solution for variable φ is studied. Differentiating with respect to this variable and equalising to zero gives

$$
\frac{\partial P_{\text{ave}}}{\partial \varphi} = \frac{1}{\frac{2\pi}{f_s}} \frac{f_2}{2} \left(\frac{c_{v,1}}{(f_s^2-1)^2 + (c_1 \psi f_s)^2} 4\cos(2\hat{\varphi}_1) - \frac{c_{v,2}}{(f_s^2-1)^2 + (c_2 \psi f_s)^2} 4\cos(2\hat{\varphi}_2) \right) = 0. \tag{6.10}
$$

Thus, it must hold that

$$
\frac{c_{v,1}}{(f_s^2-1)^2 + (c_1 \psi f_s)^2} 4\cos(2\hat{\varphi}_1) = \frac{c_{v,2}}{(f_s^2-1)^2 + (c_2 \psi f_s)^2} 4\cos(2\hat{\varphi}_2), \tag{6.11}
$$

and so, simplifying further,

$$
\frac{c_{v,1}}{(f_s^2-1)^2 + (c_1 \psi f_s)^2} \cos(2\hat{\varphi}_1) = \frac{c_{v,2}}{(f_s^2-1)^2 + (c_2 \psi f_s)^2} \cos(2\hat{\varphi}_2). \tag{6.12}
$$

Note that this equation is the same nonsymmetric, nonlinear, transcendental equation as discussed in Section 5.2. Hence, the same trivial solution of $c_{v,1} = c_{v,2}$ holds.

Note that Nikzamir et al. [8] study the solution $c_{v,2} = -c_{v,1}$. However, this solution is not a general solution to Equation (6.12), as

$$
c_1^2 = c_m^2 + 2 c_m c_{v,1} + c_{v,1}^2, \tag{6.13}
$$

and

$$
c_2^2 = c_m^2 + 2 c_m c_{v,2} + c_{v,2}^2 = c_m^2 - 2 c_m c_{v,1} + c_{v,1}^2, \tag{6.14}
$$

which implies that $c_{v,1} = 0$ must hold, thus implying that indeed $c_{v,1} = c_{v,2}$. Therefore, the optimal solution that Nikzamir et al. [8] propose is suboptimal.

To determine the optimal solution to this problem, we study the general case of multiple switches. This is done in the next section.

6.2. General multiple switches

Now, we discuss the general case of multiple switches in the damping value at general times. This coincides with the pseudocontinuous damping as discussed earlier. The general approach means that

$$c_v(t) = \begin{cases} c_v = c_{v,i} & t \in [T_{i-1}, T_i), \end{cases} \tag{6.15}$$

for $i \in \{1, ..., n\}$ with $T_0 = 0$ and $T_n = \frac{2\pi}{f_s}$. For the same values of i, it again holds that

$$c_i = c_m + c_{v,i}. \tag{6.16}$$

Discussing this general form for the damping values will give the possibility to fully implement the pseudocontinuous damping in the model.

The general expression for time-averaged power is used. Substituting the different integrals that appear due to the different switches in damping, it gives

$$P_{\text{ave}} = \frac{1}{\frac{2\pi}{f_s}} \sum_{i=1}^{n} \frac{c_{v,i}}{(f_s^2-1)^2 + (c_i \psi f_s)^2} \int_{T_{i-1}}^{T_i} 1 - \cos(2f_s t + 2\hat{\varphi})\text{d}t. \tag{6.17}$$

Evaluating the integrals in this expression for the time-averaged power will give the following

$$P_{\text{ave}} = \frac{1}{\frac{2\pi}{f_s}} \frac{f_s^2}{2} \sum_{i=1}^{n} \frac{c_{v,i}}{(f_s^2-1)^2 + (c_i \psi f_s)^2} \left(T_i - T_{i-1} - \frac{1}{2f_s} \left(\sin(2f_s T_i + 2\hat{\varphi}_i) - \sin(2f_s T_{i-1} + 2\hat{\varphi}_i) \right) \right), \tag{6.18}$$

so, optimising over φ gives

$$\frac{\partial P_{\text{ave}}}{\partial \varphi} = \frac{1}{\frac{2\pi}{f_s}} \frac{f_s^2}{2} \sum_{i=1}^{n} \frac{-c_{v,i}}{(f_s^2-1)^2 + (c_i \psi f_s)^2} \frac{1}{2f_s} \left(2\cos(2f_s T_i + 2\hat{\varphi}_i) - 2\cos(2f_s T_{i-1} + 2\hat{\varphi}_i) \right) = 0, \tag{6.19}$$

and

$$\frac{\partial P_{\text{ave}}}{\partial \varphi} = \frac{1}{\frac{2\pi}{f_s}} \frac{f_s}{2} \sum_{i=1}^{n} \frac{c_{v,i}}{(f_s^2-1)^2 + (c_i \psi f_s)^2} \left(\cos(2f_s T_{i-1} + 2\hat{\varphi}_i) - \cos(2f_s T_i + 2\hat{\varphi}_i) \right) = 0. \tag{6.20}$$

Hence,

$$\sum_{i=1}^{n} \frac{c_{v,i}}{(f_s^2-1)^2 + (c_i \psi f_s)^2} \left(\cos(2f_s T_{i-1} + 2\hat{\varphi}_i) - \cos(2f_s T_i + 2\hat{\varphi}_i) \right) = 0, \tag{6.21}$$

where it should be noted that for $n = 2$ this equation reduces to the solved equation of Section 5.2 and previous Section 6.1.

We are curious to see if the trivial solution, so $\sum_{i=1}^{n} c_{v,i} = nc_{v,i}$ with $i \in \{1, ..., n\}$, would still solve Equation (6.21). Indeed, this holds, as for the same damping value for the entire duration of a period of the base excitation, the equation will reduce to

$$\frac{c_{v,i}}{(f_s^2-1)^2 + (c_i \psi f_s)^2} \left(\cos(2\pi + 2\hat{\varphi}_i) - \cos(2\hat{\varphi}_i) \right) = 0. \tag{6.22}$$

The sum disappears due to the fact that there are no switches between damping values as all damping values are equal. Moreover, the equality holds due to the periodicity of the cosine function.

Equation (6.21) appears to also be subject to the telescopic sum construction. However, due to the different damping values, these terms will not equal each other. Hence, the telescopic sum construction can only be used if

$$\frac{c_{v,j}}{(f_s^2-1)^2 + (c_j \psi f_s)^2} \cos(2f_s T_{i-1} + 2\hat{\varphi}_j) = \frac{c_{v,i}}{(f_s^2-1)^2 + (c_i \psi f_s)^2} \cos(2f_s T_i + 2\hat{\varphi}_i) = 0. \tag{6.23}$$

Another question that arises is whether there are any other specific cases for which a solution to the above equation can be found. This thesis will not study this equation further analytically, but rather poses a numerical simulation.

The code for this numerical simulation can be found in Appendix I. It can be seen that the code optimises the time-averaged power for an input number of switches and the moment of switches. For each iteration of determining the time-averaged power, the formula recalculates for which damping values the most time-averaged power was yielded. In Figure 6.1, the time-averaged power is given for two switches and Figure 6.2 for three switches in the damping value.

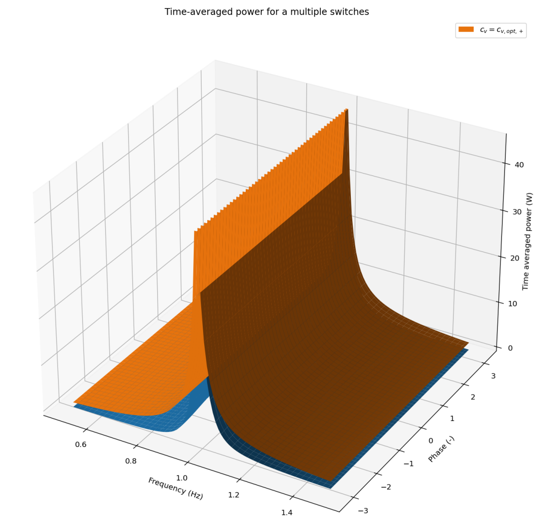

Figure 6.1: Time-averaged power for two switches at $T_1 = \frac{2\pi}{3f_s}$ and $T_2 = \frac{4\pi}{3f_s}$ compared to the time-averaged power for $c_v = c_{v,\text{opt},+}$.

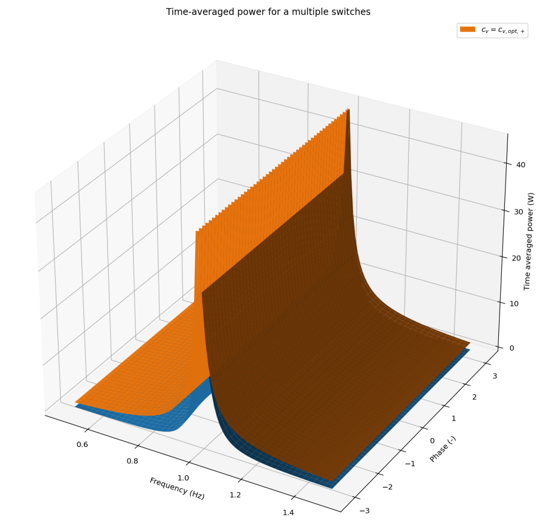

Figure 6.2: Time-averaged power for three switches at $T_1 = \frac{\pi}{2f_s}$, $T_2 = \frac{\pi}{f_s}$ and $T_3 = \frac{3\pi}{2f_s}$ compared to the time-averaged power for $c_v = c_{v,\text{opt},+}$.

These simulations are also done for four switches and five switches, which can be seen in Figure 6.3 and Figure 6.4 respectively.

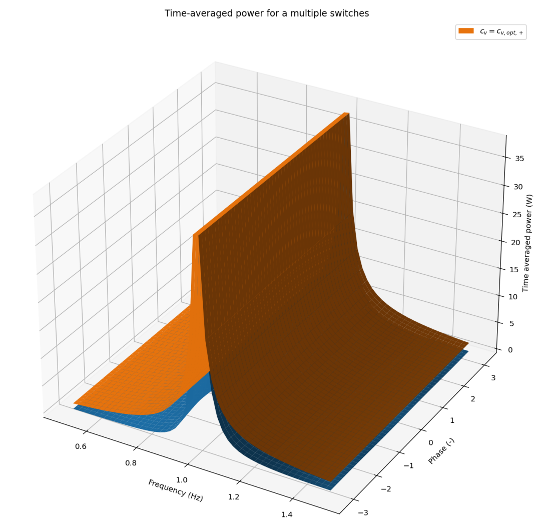

Figure 6.3: Time-averaged power for four switches at $T_1 = \frac{2\pi}{5f_s}$, $T_2 = \frac{4\pi}{5f_s}$, $T_3 = \frac{6\pi}{5f_s}$ and $T_4 = \frac{8\pi}{5f_s}$ compared to the time-averaged power for $c_v = c_{v,\text{opt},+}$.

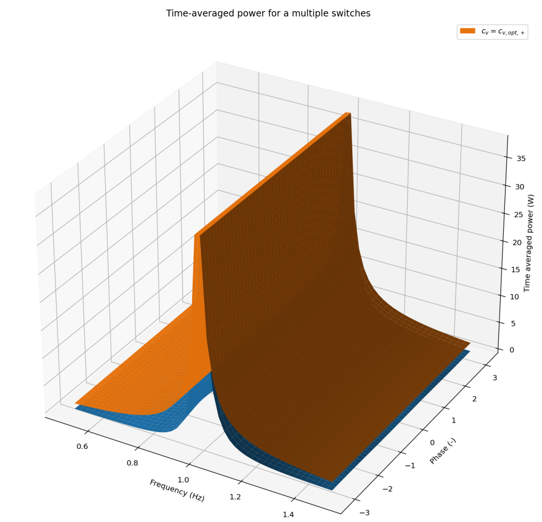

Figure 6.4: Time-averaged power for five switches at $T_1 = \frac{2\pi}{6f_s}$, $T_2 = \frac{4\pi}{6f_s}$, $T_3 = \frac{6\pi}{5f_s}$, $T_4 = \frac{8\pi}{6f_s}$ and $T_5 = \frac{10\pi}{6f_s}$ compared to the time-averaged power for $c_v = c_{v,\text{opt},+}$.

It can be seen that, again, damping value $c_v = c_{v,\text{opt},+}$ yields the most time-averaged power.

7
Conclusion

In this chapter, a summary and several recommendations are given. In Section 7.1, the summary is discussed, and in Section 7.2, the recommendations are discussed.

7.1. Summary

This thesis aimed to get a clear indication of the power that can be harvested from a damped harmonic oscillator with a variable damping and to optimise the yielded power.

The analysis of the power harvesting showed that the time-averaged power is the physically appropriate choice, due to the periodic nature of the base excitation acting on the damped harmonic oscillator. The time-averaged power was studied with the variable damping value and the velocity of the steady state part of the damped harmonic oscillator model. This completes the first objective of this thesis.

Next, several cases of variable damping values and their effects on the yielded time-averaged power were studied. It has been analytically shown that for no switch in damping value the strategy that yields as much time-averaged power as possible is to use $c_v = c_{v,\text{opt},+}$. For the case of a singular switch in damping value, it has been shown that the strategy to yield the most time-averaged power is to remove the switch and stick to damping value $c_v = c_{v,\text{opt},+}$. For the case of multiple switches, previous work of Nikzamir et al. [8], Scapolan, Tehrani, and Bonisoli [12] and Di Monaco et al. [3] has been expanded to a solution which yields more power and is easier to implement, completing the second objective of this thesis. It has been numerically shown that for multiple switches the solution where $c_v = c_{v,\text{opt},+}$ remains the best option to yield the most time-averaged power.

7.2. Recommendations

This section discusses the recommendations for future research regarding this thesis. These are given in separate subsections.

7.2.1. Improved analytical analysis of single switch

In the analysis of the single switch of the damped harmonic oscillator, the case where $f_s T_m + 2\hat{\varphi}_1 \neq k_1\pi$ did not yield any value for $c_{v,1}$. This value could still be sought after if the value of $c_{v,2}$ is substituted into the expression of P_ave and then optimised for $c_{v,1}$. Due to the time constraint of this thesis, unfortunately, this was not done. Future research could explore this path to fully conclude the analytical work of this thesis, allowing the extremum to be classified.

7.2.2. Research influence of scaled variables and parasitic damping

Currently, the value ψ is left outside of the scope of this research. The influence of this value could impact the amount of power that is harvested from the oscillations. Determining the ideal value of the variable ψ will provide an optimal ratio between the mass and spring coefficient of the oscillator and is therefore useful for the application of this thesis. This variable can be researched by treating it as the other variables, such as the frequency or phase of the base excitation, in this thesis. Hence, effectively creating the function for the time averaged power to be

$$P_{\text{ave}}(f_s, c_{v,i}, c_m, \psi, \varphi, T_i). \tag{7.1}$$

The amplitude of the base excitation also influences the time-averaged power, via the displacement. Future research should rescale this variable back and then study influence the amplitude of the base excitation.

The parasitic damping, c_m, is a variable that influences the time-averaged power yield, but as it doesn't yield any power in itself, it should be aimed for to be as close to zero. Thus, it is not a variable that should optimised for in itself.

7.2.3. Including transient solution in oscillator model

In this thesis, only the steady state solution of the damped harmonic oscillator was studied, as the influence of the transient response was considered minimal. However, for a more accurate representation of the oscillator's behaviour, the transient solution should also be included in the power analysis. Incorporating this component may offer new insights into the optimal damping value during the initial phase of power transfer from base excitation. The conclusions drawn in this thesis could then be applied after the transient solution is not significant anymore.

7.2.4. Improved research behind multiple switches

An analytical examination was only done in the specific case of a single switch to demonstrate improved solutions over those currently proposed in the field. Thereafter, a numerical approach was used to study the case of multiple switches. In this numerical approach, the moment of switch is considered input in the simulation. A better numerical simulation would also leave this variable up to the numerical implementation to be optimally chosen.

Next, future work could improve the research behind multiple switches by exploring how the numerical conclusions given can be shown analytically. This can be done by expanding the number of variables which are studied in the optimisation process. Note that this also makes the classification of the extrema more difficult as the number of variables increases.

7.2.5. Multidimensional oscillator model

Further research could expand on the current model by allowing two of the three-dimensional movements. Currently, only up and down movement is considered, while in practice also side-to-side or back-and-forth movement will also occur. A diagram of the oscillator is given in Figure 7.1

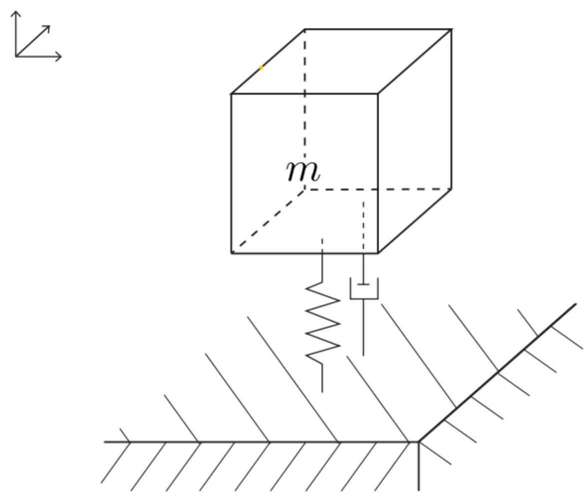

Figure 7.1: Diagram of a three-dimensional damped harmonic oscillator.

The equations will transform into a three-dimensional form for the damped harmonic oscillator.

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A

Code for Pseudocontinuous Damping

```python
import numpy as np

def adaptive_fixed_intervals(f, a, b, size, num_intervals):
    """
    Divide the function into a fixed number of adaptive intervals based on variability.

    Parameters:
        f: function to be approximated
        a: begin of interval where f is approximated
        b: end of interval where f is approximated
        size: the sample size of f
        num_intervals: number of intervals

    Returns:
        stepwise_x: array of x values of interval begin- and endpoints
        stepwise_f: arrays of f values belonging to x values of stepwise_x
    """

    # Creating the interval
    x = np.linspace(a, b, size)

    # Calculate variability (absolute difference between successive y values)
    variability = np.abs(np.diff(f))

    # Normalize variability to calculate relative weights
    weights = variability / np.sum(variability)

    # Determine cumulative weights to split into intervals
    cumulative_weights = np.cumsum(weights)
    interval_bounds = np.linspace(0, 1, num_intervals + 1)  # Divide cumulative weights into intervals

    stepwise_x = []
    stepwise_f = []

    start_idx = 0

    if num_intervals == 1:
        avg_y = np.mean(f)
        return np.array([x[0], x[-1]]), np.array([avg_y, avg_y])
    else:
        for i in range(num_intervals):
            # Find the end index based on cumulative weights
            end_idx = np.searchsorted(cumulative_weights, interval_bounds[i + 1],side='right')

            # Ensure the end index stays within bounds
            end_idx = min(end_idx, len(x) - 1)

            # Get the x and y values in this interval
```

```
            x_interval = x[start_idx:end_idx + 1]
            f_interval = f[start_idx:end_idx + 1]

            if len(x_interval) > 0:
                if i == num_intervals -1:
                    avg_f = np.mean(f_interval)  # Average y value in the interval
                    stepwise_x.extend([stepwise_x[-1], b])
                    stepwise_f.extend([avg_f, avg_f])
                else:
                    if start_idx > 1:
                        avg_f = np.mean(f_interval)  # Average y value in the interval
                        stepwise_x.extend([stepwise_x[-1], x_interval[-1]])
                        stepwise_f.extend([avg_f, avg_f])
                    else:
                        avg_f = np.mean(f_interval)  # Average y value in the interval
                        stepwise_x.extend([x_interval[0], x_interval[-1]])
                        stepwise_f.extend([avg_f, avg_f])


            start_idx = end_idx + 1  # Move to the next interval

        return np.array(stepwise_x), np.array(stepwise_f)
```

B

Table with parameter values for the damped harmonic oscillator

Table B.1: Parameter values for different applications of the damped harmonic oscillator.

System Type	Mass m (kg)	Spring Constant k (N/m)	Damping Coefficient c (kg/s)
Micro-mechanical systems (MEMS) [14]	$1.44 \cdot 10^{-9}$	6057	8.8
Precision lab setups (e.g. exploring viscous damping) [7]	0.12	1.39	0.25
Vehicle suspension [1]	35	160000	980

C

Harmonic Addition Theorem

The harmonic addition theorem states that

$$f(\theta) = a\cos(\theta) + b\sin(\theta), \tag{C.1}$$

can be rewritten to a form like

$$g(\theta) = c\cos(\theta + \delta). \tag{C.2}$$

This can be shown by first rewriting function $g(\theta)$ using the angle addition identity for the cosine function, giving

$$g(\theta) = c\cos(\theta)\cos(\delta) - c\sin(\theta)\sin(\delta), \tag{C.3}$$

which means that, if $f(\theta) = g(\theta)$, it should be true that

$$a = c\cos(\delta) \text{ and } b = -c\sin(\delta). \tag{C.4}$$

Therefore, using the identity that

$$\cos^2(\phi) + \sin^2(\phi) = 1, \tag{C.5}$$

we get that

$$a^2 + b^2 = c^2\cos^2(\delta) + c^2\sin^2(\delta), \tag{C.6}$$

which implies that

$$c = \pm\sqrt{a^2 + b^2}. \tag{C.7}$$

Also, assuming that $a \neq 0$, it should hold that

$$\frac{b}{a} = \frac{-c\sin(\delta)}{c\cos(\delta)} \tag{C.8}$$

and therefore

$$-\frac{b}{a} = \tan(\delta), \tag{C.9}$$

and thus

$$\delta = \arctan(-\frac{b}{a}). \tag{C.10}$$

Now, note that $\delta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, and therefore $\cos(\delta) > 0$. This implies, due to the fact that it should hold that $a = c\cos(\delta)$, that it should also hold that

$$\text{sgn}(a) = \text{sgn}(c). \tag{C.11}$$

Hence, concluding that indeed

$$f(\theta) = a\cos(\theta) + b\sin(\theta) \tag{C.12}$$

equals

$$g(\theta) = c\cos(\theta + \delta) \tag{C.13}$$

when

$$c = \mathsf{sgn}(a)\sqrt{a^2 + b^2} \tag{C.14}$$

and, assuming that $a \neq 0$,

$$\delta = \arctan(-\frac{b}{a}). \tag{C.15}$$

D

Simplifying Equation (5.93)

The aim is to simplify

$$\frac{\sin^2(f_sT_m)+2x_1\sin(f_sT_m)\cos(f_sT_m)+x_1^2\cos^2(f_sT_m)}{\sin(f_sT_m)(1-x_1^2)+2x_1\cos(f_sT_m)}=\frac{\sin^2(f_sT_m)+2x_2\sin(f_sT_m)\cos(f_sT_m)+x_2^2\cos^2(f_sT_m)}{\sin(f_sT_m)(1-x_2^2)+2x_2\cos(f_sT_m)}. \tag{D.1}$$

For simplicity, we write

$$s_1=\sin(f_sT_m+\varphi), s_2=\sin(f_sT_m+2\varphi) \tag{D.2}$$

and

$$c_1=\cos(f_sT_m+\varphi), c_2=\cos(f_sT_m+2\varphi) \tag{D.3}$$

which gives

$$\left(s_1^2+2s_1c_1x_1+x_1^2c_1^2\right)\left(s_2(1-x_2^2)+2x_2c_2\right)=\left(s_1^2+2s_1c_1x_2+x_2^2c_1^2\right)\left(s_2(1-x_1^2)+2x_1c_2\right), \tag{D.4}$$

and so

$$s_1^2s_2(1-x_2^2-1+x_1^2)+2s_1^2c_2(x_2-x_1)+2s_1c_1s_2(x_1-x_2^2x_1-x_2+x_2x_1^2)+c_1^2s_2(x_1^2-x_1^2x_2^2-x_2^2+x_2^2x_1^2)+2c_1^2c_2(x_2x_1^2-x_1x_2^2)=0. \tag{D.5}$$

Rewriting

$$(x_1^2-x_2^2)(s_1^2s_2-c_1^2s_2)+(x_2-x_1)(2s_1^2c_2)+2s_1c_1s_2(x_1-x_2)+2s_1c_1s_2(x_2x_1^2-x_2^2x_1)+2c_1c_2(x_2x_1^2-x_1x_2^2=0, \tag{D.6}$$

and thus

$$(x_1-x_2)\left((x_1+x_2)(s_2(s_1^2+c_1^2))+2s_1(c_1s_2-s_1c_2)+2x_1x_2c_1(s_1s_2-c_1c_2)\right)=0. \tag{D.7}$$

Using goniometric identities

$$\sin^2(\theta)+\cos^2(\theta)=1, \tag{D.8}$$

and

$$\sin(\theta-\phi)=\sin(\theta)\cos(\phi)-\cos(\theta)\sin(\phi), \tag{D.9}$$

and finally,

$$\cos(\theta+\phi)=\cos(\theta)\cos(\phi)-\sin(\theta)\sin(\phi), \tag{D.10}$$

we get that

$$(x_1-x_2)\left(s_2(x_1+x_2)+2s_1\sin(\varphi)-2c_1x_1x_2\cos(2f_sT_m+3\varphi)\right)=0. \tag{D.11}$$

E

Table with extrema values for arbitrary singular switch

Table E.1: Outcome of extrema for damping values $c_{v,1}$ and $c_{v,2}$.

Assumptions	Value of $c_{v,1}$	Value of $c_{v,2}$
$f_sT_m + 2\hat{\varphi}_1 = \pi + k_1 2\pi$ and $f_sT_m + 2\hat{\varphi}_2 = \pi + k_2 2\pi$	$c_{v,\text{opt}}$	$c_{v,\text{opt}}$
$f_sT_m + 2\hat{\varphi}_1 = \pi + k_1 2\pi$ and $f_sT_m + 2\hat{\varphi}_2 = k_2 2\pi$	0	$\frac{\sqrt{c_m^2\psi^2 f_s^2+(f_s^2-1)^2}}{\psi f_s}$
$f_sT_m + 2\hat{\varphi}_1 = \pi + k_1 2\pi$ and $f_sT_m + 2\hat{\varphi}_2 = k_2 2\pi$	0	$-\frac{\sqrt{c_m^2\psi^2 f_s^2+(f_s^2-1)^2}}{\psi f_s}$
$f_sT_m + 2\hat{\varphi}_1 = k_1 2\pi$ and $f_sT_m + 2\hat{\varphi}_2 = k_2 2\pi$	$\frac{\sqrt{c_m^2\psi^2 f_s^2+(f_s^2-1)^2}}{\psi f_s}$	$-\frac{\sqrt{c_m^2\psi^2 f_s^2+(f_s^2-1)^2}}{\psi f_s}$
$f_sT_m + 2\hat{\varphi}_1 = k_1 2\pi$ and $f_sT_m + 2\hat{\varphi}_2 = k_2 2\pi$	$-\frac{\sqrt{c_m^2\psi^2 f_s^2+(f_s^2-1)^2}}{\psi f_s}$	$\frac{\sqrt{c_m^2\psi^2 f_s^2+(f_s^2-1)^2}}{\psi f_s}$
$f_sT_m + 2\hat{\varphi}_1 = k_1 2\pi$ and $f_sT_m + 2\hat{\varphi}_2 = k_2 2\pi$	$c_m + \frac{2c_m}{\hat{A}} + \frac{\sqrt{(2\psi f_s c_m)^2\left(\hat{A}+1\right)-\hat{A}^2(f_s^2-1)^2}}{\psi f_s \hat{A}}$	$-c_m - \frac{2c_m}{\hat{A}} - \frac{\sqrt{(2\psi f_s c_m)^2\left(\hat{A}+1\right)-\hat{A}^2(f_s^2-1)^2}}{\psi f_s \hat{A}}$
$f_sT_m + 2\hat{\varphi}_1 = k_1 2\pi$ and $f_sT_m + 2\hat{\varphi}_2 = k_2 2\pi$	$c_m + \frac{2c_m}{\hat{A}} - \frac{\sqrt{(2\psi f_s c_m)^2\left(\hat{A}+1\right)-\hat{A}^2(f_s^2-1)^2}}{\psi f_s \hat{A}}$	$-c_m - \frac{2c_m}{\hat{A}} + \frac{\sqrt{(2\psi f_s c_m)^2\left(\hat{A}+1\right)-\hat{A}^2(f_s^2-1)^2}}{\psi f_s \hat{A}}$
$f_sT_m + 2\hat{\varphi}_1 = k_1 2\pi$ and $f_sT_m + 2\hat{\varphi}_2 = k_2 2\pi$	$c_m - \frac{2c_m}{\tilde{A}} + \frac{\sqrt{(2\psi f_s c_m)^2\left(1-\tilde{A}\right)-\tilde{A}^2(f_s^2-1)^2}}{\psi f_s \tilde{A}}$	$-c_m + \frac{2c_m}{\tilde{A}} - \frac{\sqrt{(2\psi f_s c_m)^2\left(1-\tilde{A}\right)-\tilde{A}^2(f_s^2-1)^2}}{\psi f_s \tilde{A}}$
$f_sT_m + 2\hat{\varphi}_1 = k_1 2\pi$ and $f_sT_m + 2\hat{\varphi}_2 = k_2 2\pi$	$c_m - \frac{2c_m}{\tilde{A}} - \frac{\sqrt{(2\psi f_s c_m)^2\left(1-\tilde{A}\right)-\tilde{A}^2(f_s^2-1)^2}}{\psi f_s \tilde{A}}$	$-c_m + \frac{2c_m}{\tilde{A}} + \frac{\sqrt{(2\psi f_s c_m)^2\left(1-\tilde{A}\right)-\tilde{A}^2(f_s^2-1)^2}}{\psi f_s \tilde{A}}$
$f_sT_m + 2\hat{\varphi}_1 \neq k_1\pi$	$c_{v,1}$	$-c_m + \frac{(c_m+c_{v,1})\sin(f_sT_m+2\varphi)+2\sin(f_sT_m+\varphi)\sin(\varphi)}{\frac{\psi f_s}{-f_s^2+1}\left(\sin(f_sT_m+2\varphi)+2\cos(f_sT_m+\varphi)\cos(2f_sT_m+3\varphi)\frac{\psi f_s(c_m+c_{v,1}}{-f_s^2+1}\right)}$
None	-0.15	0.35
None	0.35	-0.15

F

Visualisation of extrema of singular switch

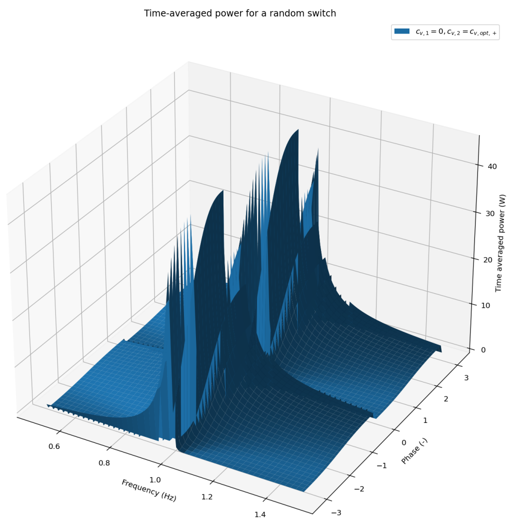

Figure F.1: The graph of the time averaged power with $c_{v,1} = 0$ and $c_{v,2} = c_{v,\text{opt},+}$.

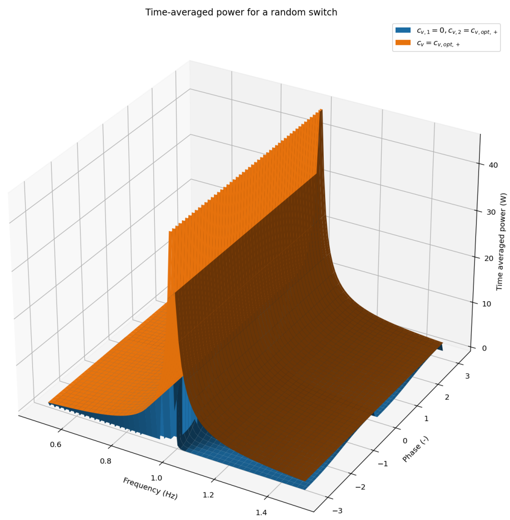

Figure F.2: The graph of the time averaged power with in blue $c_{v,1} = 0$ and $c_{v,2} = c_{v,\text{opt},+}$ and in orange $c_v = c_{v,\text{opt},+}$.

Note that in Figure F.14, the blue appears to performs better than the orange, however, this is only when $T_m = \frac{2\pi}{f_s}$, hence a value which does not fall in the bounds allowed for the moment of switch.

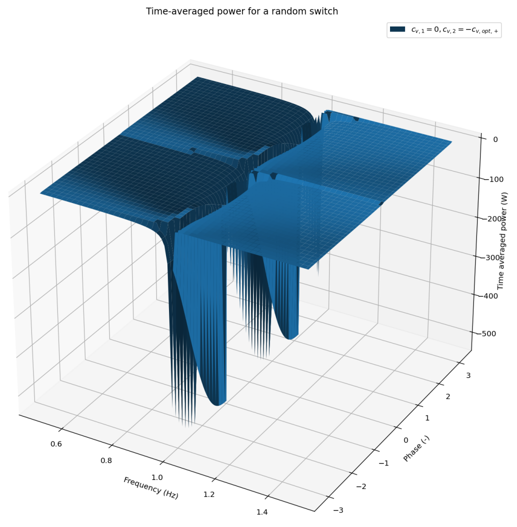

Figure F.3: The graph of the time averaged power with $c_{v,1} = 0$ and $c_{v,2} = c_{v,\text{opt},-}$.

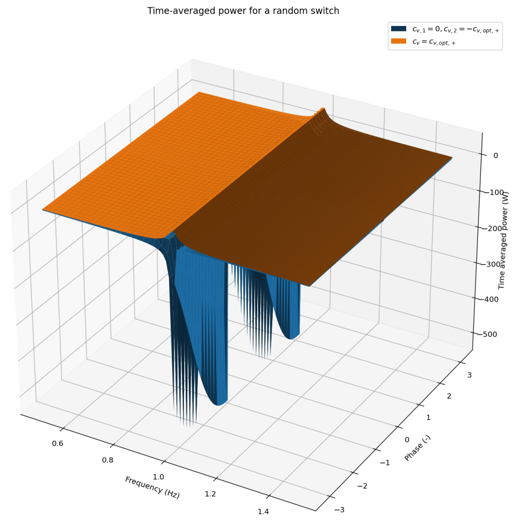

Figure F.4: The graph of the time averaged power with in blue $c_{v,1} = 0$ and $c_{v,2} = c_{v,\text{opt},-}$ and in orange $c_v = c_{v,\text{opt},+}$.

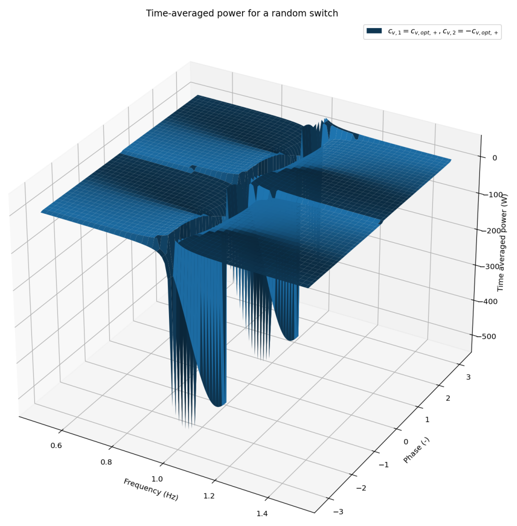

Figure F.5: The graph of the time averaged power with $c_{v,1} = c_{v,\text{opt},+}$ and $c_{v,2} = c_{v,\text{opt},-}$.

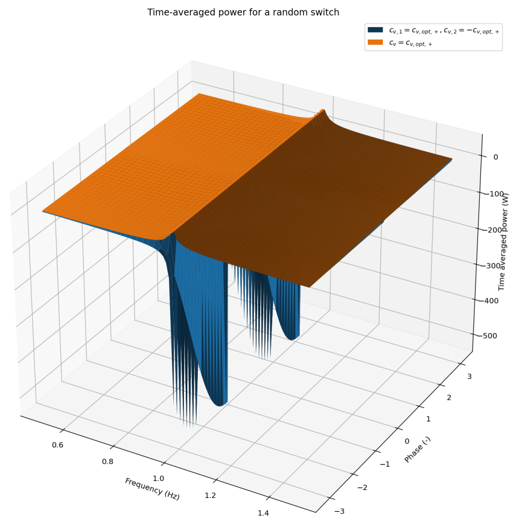

Figure F.6: The graph of the time averaged power with in blue $c_{v,1} = c_{v,\text{opt},+}$ and $c_{v,2} = c_{v,\text{opt},-}$ and in orange $c_v = c_{v,\text{opt},+}$.

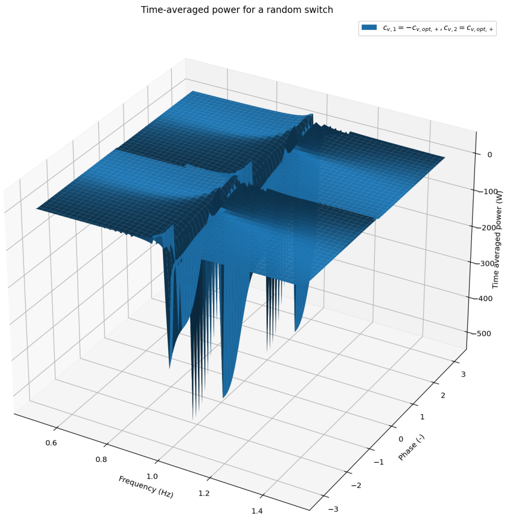

Figure F.7: The graph of the time averaged power with $c_{v,1} = c_{v,\text{opt},-}$ and $c_{v,2} = c_{v,\text{opt},+}$.

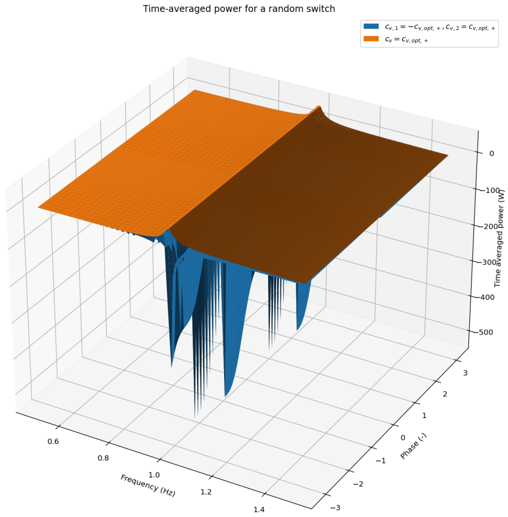

Figure F.8: The graph of the time averaged power with in blue $c_{v,1} = c_{v,\text{opt},-}$ and $c_{v,2} = c_{v,\text{opt},+}$ and in orange $c_v = c_{v,\text{opt},+}$.

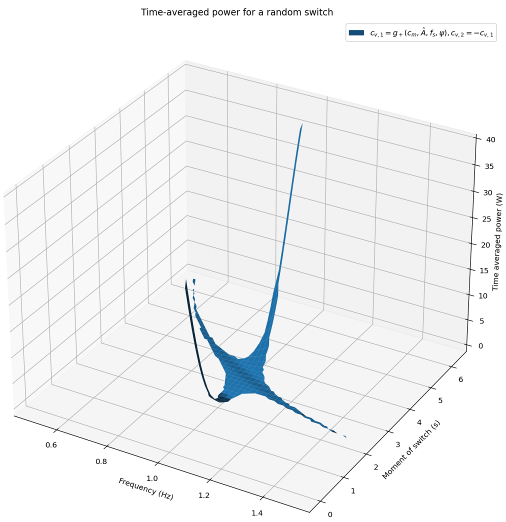

Figure F.9: The graph of the time averaged power with $c_{v,1} = c_m + \frac{2c_m}{\hat{A}} + \frac{\sqrt{(2\psi f_s c_m)^2(\hat{A}+1) - \hat{A}^2(f_s^2-1)^2}}{\psi f_s \hat{A}}$ and $c_{v,2} = -c_{v,1}$.

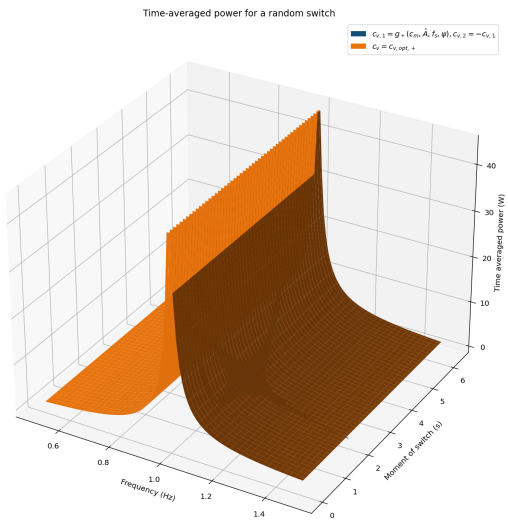

Figure F.10: The graph of the time averaged power with $c_{v,1} = c_m + \frac{2c_m}{\hat{A}} + \frac{\sqrt{(2\psi f_s c_m)^2(\hat{A}+1) - \hat{A}^2(f_s^2-1)^2}}{\psi f_s \hat{A}}$ and $c_{v,2} = -c_{v,1}$ and in orange $c_v = c_{v,\text{opt},+}$.

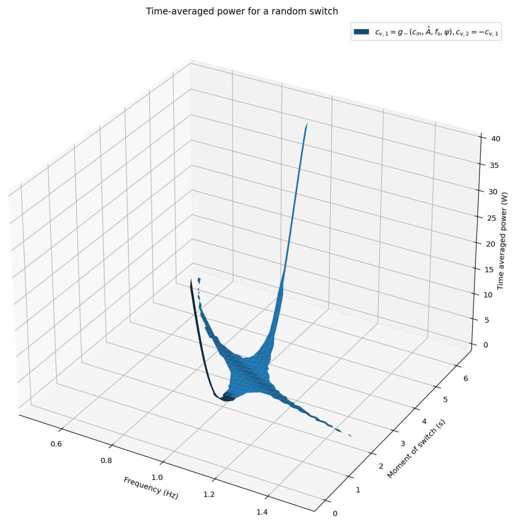

Figure F.11: The graph of the time averaged power with $c_{v,1} = c_m + \frac{2c_m}{\hat{A}} - \frac{\sqrt{(2\psi f_s c_m)^2(\hat{A}+1)-\hat{A}^2(f_s^2-1)^2}}{\psi f_s \hat{A}}$ and $c_{v,2} = -c_{v,1}$.

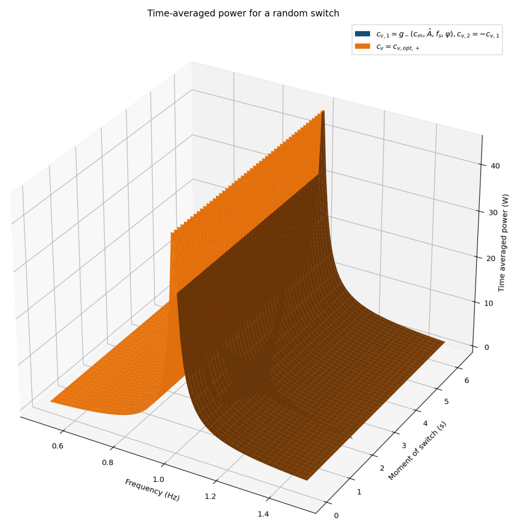

Figure F.12: The graph of the time averaged power with $c_{v,1} = c_m + \frac{2c_m}{\hat{A}} - \frac{\sqrt{(2\psi f_s c_m)^2(\hat{A}+1)-\hat{A}^2(f_s^2-1)^2}}{\psi f_s \hat{A}}$ and $c_{v,2} = -c_{v,1}$ and in orange $c_v = c_{v,\text{opt},+}$.

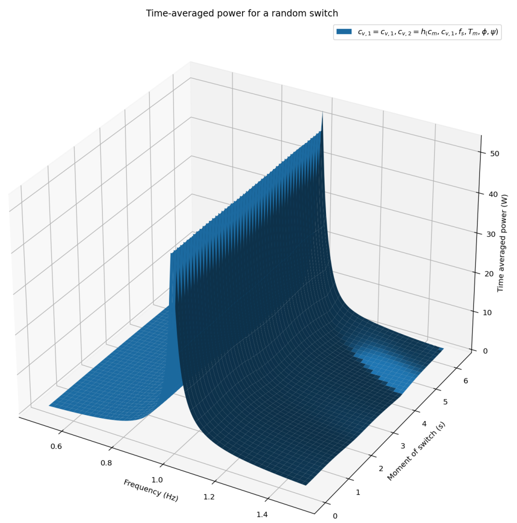

Figure F.13: The graph of the time averaged power with $c_{v,1} = c_{v,1}$ and $c_{v,2} = -c_m + \frac{(c_m+c_{v,1})\sin(f_s T_m+2\varphi)+2\sin(f_s T_m+\varphi)\sin(\varphi)}{\frac{\psi f_s}{-f_s^2+1}\left(\sin(f_s T_m+2\varphi)+2\cos(f_s T_m+\varphi)\cos(2 f_s T_m+3\varphi)\frac{\psi f_s(c_m+c_{v,1})}{-f_s^2+1}\right)}$.

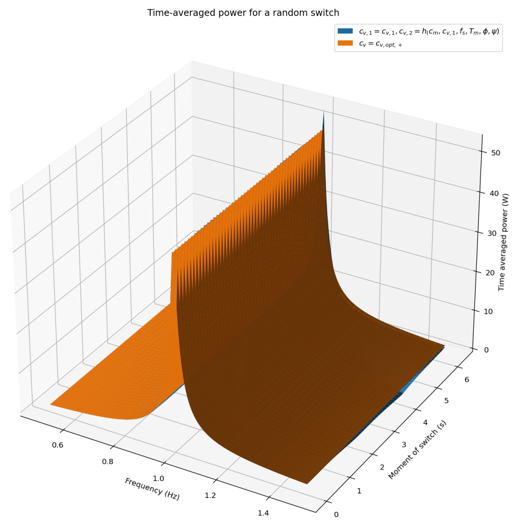

Figure F.14: The graph of the time averaged power with $c_{v,1} = c_{v,1}$ and $c_{v,2} = -c_m + \frac{(c_m+c_{v,1})\sin(f_s T_m+2\varphi)+2\sin(f_s T_m+\varphi)\sin(\varphi)}{\frac{\psi f_s}{-f_s^2+1}\left(\sin(f_s T_m+2\varphi)+2\cos(f_s T_m+\varphi)\cos(2 f_s T_m+3\varphi)\frac{\psi f_s(c_m+c_{v,1})}{-f_s^2+1}\right)}$ and in orange $c_v = c_{v,\text{opt},+}$.

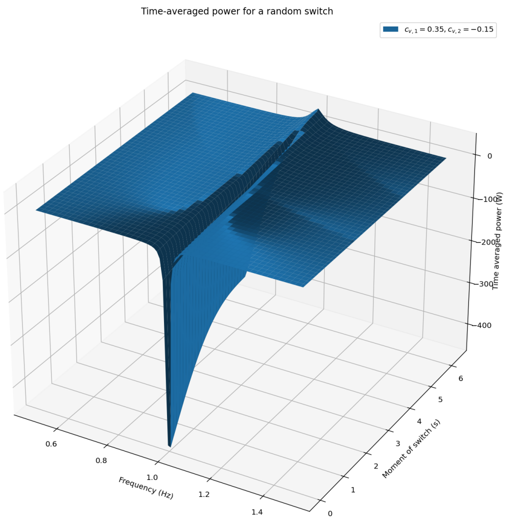

Figure F.15: The graph of the time averaged power with $c_{v,1} = -0.15$ and $c_{v,2} = 0.35$.

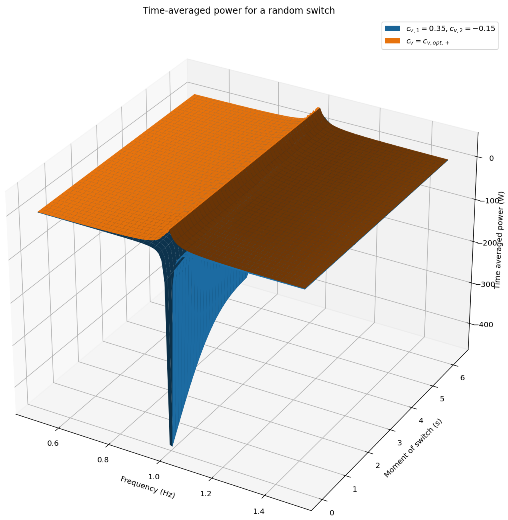

Figure F.16: The graph of the time averaged power with in blue $c_{v,1} = -0.15$ and $c_{v,2} = 0.35$ and in orange $c_v = c_{v,\text{opt},+}$.

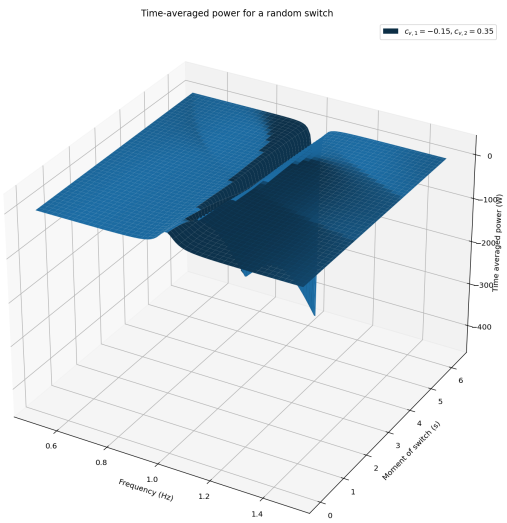

Figure F.17: The graph of the time averaged power with $c_{v,1} = 0.35$ and $c_{v,2} = -0.15$.

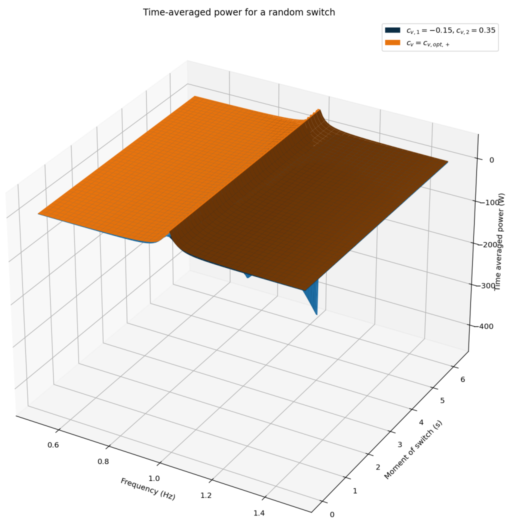

Figure F.18: The graph of the time averaged power with in blue $c_{v,1} = 0.35$ and $c_{v,2} = -0.15$ and in orange $c_v = c_{v,\text{opt},+}$.

G

Code for visualising the boundary extrema

```
import matplotlib.pyplot as plt
import numpy as np

# Constant Variables
mass = 1.0                    # Mass of oscillator (kg)
k = 4 * np.pi ** 2            # Spring constant (N/m)

# Initial conditions
u0 = 0                        # Initial position (m)
v0 = 0                        # Initial velocity (m/s)

# Time
t_start = 3000                # Starting time (s)
t_end = 3020                  # End time (s)
dt = 0.01                     # Time step (s)

time = np.arange(t_start, t_end, dt)

y0_1 = 1                      # Amplitude of base excitation (m)
fs_1 = 1                      # Frequency of base excitation (Hz)

no_damping_array = np.array([0.1])

# Scaled variables
psi = 1/(np.sqrt(k * mass))
c_m = no_damping_array[0]

def opt_damping(fs):
    return (np.sqrt( (c_m * psi * fs) ** 2 + (fs ** 2 - 1) ** 2 )) / (psi * fs)

def averaged_power_zero_switch(fs, damping):
    return damping * 0.5 * fs ** 2 * 1 / ((fs ** 2 - 1) ** 2 + ((c_m + damping)* psi * fs) **
        2)

def averaged_power_one_switch(fs, phase, damping_one, damping_two, T_m):
    delta_one = np.arctan( (-psi * fs * (c_m + damping_one) ) / (-fs ** 2 + 1) )
    delta_two = np.arctan( (-psi * fs * (c_m + damping_two) ) / (-fs ** 2 + 1) )
    phi_hat_one = phase + delta_one
    phi_hat_two = phase + delta_two
    return 1 / (2 * np.pi / fs) * (damping_one * fs ** 2 * 0.5 * 1 / ((fs ** 2 - 1) ** 2 + ((
        c_m + damping_one)* psi * fs) ** 2) * (T_m - 1 / (2 * fs) * (np.sin(2 * fs * T_m + 2
        * phi_hat_one) - np.sin(2 * phi_hat_one) ) ) + damping_two * fs ** 2 * 0.5 * 1 / ((fs
         ** 2 - 1) ** 2 + ((c_m + damping_two)* psi * fs) ** 2) * (2*np.pi/fs - T_m - 1 / (2
        * fs) * (np.sin(2 * phi_hat_two) - np.sin(2 * fs * T_m + 2 * phi_hat_two) ) ) )

### 3D Time Avg Power
fig = plt.figure(figsize=(20, 10))
```

```python
ax = fig.add_subplot(111,projection='3d')
freq_vals = np.linspace(0.5, 1.5, 100)
Tm_vals = np.linspace(0 , 2*np.pi, 100)
freq, Tm = np.meshgrid(freq_vals, Tm_vals)

phase_list = np.linspace(-np.pi, np.pi, 100)

max_time_avg_power = None

for phase in phase_list:
    time_avg_power_random_switch = averaged_power_one_switch(freq, phase, -0.15, 0.35, Tm)
    if max_time_avg_power is None:
        max_time_avg_power = time_avg_power_random_switch
    else:
        max_time_avg_power = np.maximum(max_time_avg_power, time_avg_power_random_switch)


ax.plot_surface(freq, Tm, max_time_avg_power, label='$c_{v,1} = = -0.15, c_{v,2} = 0.35$')

time_avg_power_zero_switch = averaged_power_zero_switch(freq, opt_damping(freq))
ax.plot_surface(freq, Tm, time_avg_power_zero_switch, label='$c_{v} = c_{v,opt,+}$')
ax.set_xlabel("Frequency (Hz)")
ax.set_ylabel("Moment of switch (s)")
ax.set_zlabel("Time averaged power (W)", labelpad=1)
plt.tight_layout()
plt.legend()
plt.title('Time-averaged power for a random switch')
plt.show()
```

H

Code for visualising the free first damping value extrema

```
import matplotlib.pyplot as plt
import numpy as np

# Constant Variables
mass = 1.0                  # Mass of oscillator (kg)
k = 4 * np.pi ** 2          # Spring constant (N/m)

# Initial conditions
u0 = 0                      # Initial position (m)
v0 = 0                      # Initial velocity (m/s)

# Time
t_start = 3000              # Starting time (s)
t_end = 3020                # End time (s)
dt = 0.01                   # Time step (s)

time = np.arange(t_start, t_end, dt)

y0_1 = 1                    # Amplitude of base excitation (m)
fs_1 = 1                    # Frequency of base excitation (Hz)

no_damping_array = np.array([0.1])

# Scaled variables
psi = 1/(np.sqrt(k * mass))
c_m = no_damping_array[0]

def opt_damping(fs):
    return (np.sqrt( (c_m * psi * fs) ** 2 + (fs ** 2 - 1) ** 2 )) / (psi * fs)

def averaged_power_zero_switch(fs, damping):
    return damping * 0.5 * fs ** 2 * 1 / ((fs ** 2 - 1) ** 2 + ((c_m + damping)* psi * fs) **
         2)

def averaged_power_one_switch(fs, phase, damping_one, damping_two, T_m):
    delta_one = np.arctan( (-psi * fs * (c_m + damping_one) ) / (-fs ** 2 + 1) )
    delta_two = np.arctan( (-psi * fs * (c_m + damping_two) ) / (-fs ** 2 + 1) )
    phi_hat_one = phase + delta_one
    phi_hat_two = phase + delta_two
    return 1 / (2 * np.pi / fs) * (damping_one * fs ** 2 * 0.5 * 1 / ((fs ** 2 - 1) ** 2 + ((
        c_m + damping_one)* psi * fs) ** 2) * (T_m - 1 / (2 * fs) * (np.sin(2 * fs * T_m + 2
        * phi_hat_one) - np.sin(2 * phi_hat_one) ) ) + damping_two * fs ** 2 * 0.5 * 1 / ((fs
         ** 2 - 1) ** 2 + ((c_m + damping_two)* psi * fs) ** 2) * (2*np.pi/fs - T_m - 1 / (2
        * fs) * (np.sin(2 * phi_hat_two) - np.sin(2 * fs * T_m + 2 * phi_hat_two) ) ) )

def damping_value_two(fs,  phase, damping_one, T_m):
```

```
    return -c_m + ( (c_m + damping_one) * np.sin(fs * T_m + 2 * phase) + 2 * np.sin(fs * T_m + phase) * np.sin(phase) ) / ( ((psi * fs) / (fs ** 2 + 1)) * (np.sin(fs * T_m + 2 * phase) + 2 * np.cos(fs * T_m + phase) * np.cos(2 * fs * T_m + 3 * phase) * ( (psi * fs * (c_m + damping_one)) / (fs ** 2 + 1) ) ) )


### 3D Time Avg Power
fig = plt.figure(figsize=(20, 10))
ax = fig.add_subplot(111,projection='3d')
freq_vals = np.linspace(0.5, 1.5, 200)
Tm_vals = np.linspace(0 , 2*np.pi, 200)
freq, Tm = np.meshgrid(freq_vals, Tm_vals)

phase_list = np.linspace(-np.pi, np.pi, 200)
damping_value_one_list = np.linspace(-0.15, 0.35, 200)

max_time_avg_power = None

for damping_value_one in damping_value_one_list:
    for phase in phase_list:
        time_avg_power_random_switch = averaged_power_one_switch(freq, phase,
            damping_value_one, damping_value_two(freq, phase, damping_value_one, Tm), Tm)
        if max_time_avg_power is None:
            max_time_avg_power = time_avg_power_random_switch
        else:
            max_time_avg_power = np.maximum(max_time_avg_power, time_avg_power_random_switch)


ax.plot_surface(freq, Tm, max_time_avg_power, label='$c_{v,1} = ' + str(damping_value_one) +
    ', c_{v,2} = 0.35$')

time_avg_power_zero_switch = averaged_power_zero_switch(freq, opt_damping(freq))
ax.plot_surface(freq, Tm, time_avg_power_zero_switch, label='$c_{v} = c_{v,opt,+}$')
ax.set_xlabel("Frequency (Hz)")
ax.set_ylabel("Moment of switch (s)")
ax.set_zlabel("Time averaged power (W)", labelpad=1)
plt.tight_layout()
plt.legend()
plt.title('Time-averaged power for a random switch')
plt.show()
```

I

Code for visualising multiple switches in damping value

```python

import matplotlib.pyplot as plt
import numpy as np
import itertools

# Constant Variables
mass = 1.0                  # Mass of oscillator (kg)
k = 4 * np.pi ** 2          # Spring constant (N/m)

# Initial conditions
u0 = 0                      # Initial position (m)
v0 = 0                      # Initial velocity (m/s)

# Time
t_start = 3000              # Starting time (s)
t_end = 3020                # End time (s)
dt = 0.01                   # Time step (s)

time = np.arange(t_start, t_end, dt)

y0_1 = 1                    # Amplitude of base excitation (m)
fs_1 = 1                    # Frequency of base excitation (Hz)

no_damping_array = np.array([0.1])

# Scaled variables
psi = 1/(np.sqrt(k * mass))
c_m = no_damping_array[0]

def opt_damping(fs):
    return (np.sqrt( (c_m * psi * fs) ** 2 + (fs ** 2 - 1) ** 2 )) / (psi * fs)

def averaged_power_zero_switch(fs, damping):
    return damping * 0.5 * fs ** 2 * 1 / ((fs ** 2 - 1) ** 2 + ((c_m + damping)* psi * fs) **
        2)

def compute_max_time_avg_power(fs, phase, damping_linspace, moment_of_switches_list, x):
    if len(moment_of_switches_list) - 2 != x:
        return print("Length of moment of switches not equal to amount of switches!")
    else:
        max_time_avg_power = -np.inf
        initial = 1 / (2 * np.pi / fs) * (fs ** 2) / 2

        damping_values = np.empty((detail, detail) + (x,))

        for combo in itertools.product(damping_linspace, repeat=x):
            summations = []
```

```
            for i, k in enumerate(combo):
                # compute varphi_hat directly from this k
                varphi_hat = phase + np.arctan((-psi * fs * (c_m + k)) / (-fs**2 + 1))

                # compute the summation for this interval
                summation = (k / ((fs**2 - 1)**2 + (psi * fs * (c_m + k))**2)) * (
                    moment_of_switches_list[i+1] - moment_of_switches_list[i]- (1/(2*fs)) * (
                    np.sin(2*fs*moment_of_switches_list[i+1] + 2*varphi_hat)- np.sin(2*fs*
                    moment_of_switches_list[i] + 2*varphi_hat) ) )
                summations.append(summation)

            # # total time-avg power for this combo
            time_avg_power = initial * sum(summations)

            max_time_avg_power = np.maximum(max_time_avg_power, time_avg_power)

            new_max = np.maximum(max_time_avg_power, time_avg_power)

            mask = new_max > max_time_avg_power

            # Store tuple for updated elements
            for dim in range(x):
                damping_values[..., dim][mask] = combo[dim]

        return max_time_avg_power, damping_values

### 3D Time Avg Power
fig = plt.figure(figsize=(20, 10))
ax = fig.add_subplot(111,projection='3d')

detail = 100

freq_vals = np.linspace(0.5, 1.5, detail)
phase_vals = np.linspace(-np.pi, np.pi, detail)
freq, phase = np.meshgrid(freq_vals, phase_vals)

trial_switches = 3

trial_moments = [0,
                 np.pi/2, np.pi, 3*np.pi/2,
                 2 * np.pi]

boundary_damping_left = -0.15
boundary_damping_right = 0.35
damping = np.linspace(boundary_damping_left, boundary_damping_right, detail)

max_time_avg_power, damping_values = compute_max_time_avg_power(freq, phase, damping,
    trial_moments, trial_switches)
ax.plot_surface(freq, phase, max_time_avg_power, label='$c_{v,1} = , c_{v,2} = 0.35$')

time_avg_power_zero_switch = averaged_power_zero_switch(freq, opt_damping(freq))
ax.plot_surface(freq, phase, time_avg_power_zero_switch, label='$c_{v} = c_{v,opt,+}$')

ax.set_xlabel("Frequency (Hz)")
ax.set_ylabel("Phase (-)")
ax.set_zlabel("Time averaged power (W)", labelpad=1)
plt.tight_layout()
plt.legend()
plt.title('Time-averaged power for a random switch')
plt.show()

i, j = 5, 3  # pick some index in your meshgrid

print("Max value:", max_time_avg_power[i, j])
print("Best damping tuple:", tuple(damping_values[i, j]))
print("Optimal damping value", opt_damping(freq[i,j]))
print("Timeavgpower zero switch", time_avg_power_zero_switch[i,j])
```