Cohomological local-to-global principles and integration in finite- and infinite-dimensional Lie theory

Miaskiwskyi, L.T.

DOI
10.4233/uuid:60c65f23-8c3c-4050-85e4-39ef675e2168

Publication date
2022

Document Version
Final published version

Citation (APA)
Miaskiwskyi, L. T. (2022). *Cohomological local-to-global principles and integration in finite- and infinite-dimensional Lie theory*. [Dissertation (TU Delft), Delft University of Technology]. https://doi.org/10.4233/uuid:60c65f23-8c3c-4050-85e4-39ef675e2168

Cohomological local-to-global principles and integration in finite- and infinite-dimensional Lie theory

Cohomological local-to-global principles and integration in finite- and infinite-dimensional Lie theory

Proefschrift

ter verkrijging van de graad van doctor
aan de Technische Universiteit Delft,
op gezag van de Prof.dr.ir. T.H.J.J. van der Hagen,
voorzitter van het College voor Promoties,
in het openbaar te verdedigen op 13 october 2022 om 12:30 uur

door

Lukas Miaskiwskyi

geboren te Neumarkt i.d.OPf., Duitsland.

Dit proefschrift is goedgekeurd door de promotoren.

Samenstelling promotiecommissie:

Rector Magnificus,	voorzitter
Prof. dr. J. M. A. M. van Neerven	Technische Universiteit Delft, promotor
Dr. B. Janssens	Technische Universiteit Delft, copromotor
Onafhankelijke leden:	
Prof. dr. F. H. J. Redig	Technische Universiteit Delft
Prof. dr. K.-H. Neeb	Friedrich-Alexander-Universität Erlangen-Nürnberg, Duitsland
Prof. dr. S. Waldmann	Julius-Maximilians-Universität Würzburg
Prof. dr. M. Crainic	Universiteit Utrecht
Dr. C. Esposito	Università degli studi de Salerno, Italië
Prof. dr. D. C. Gijswijt	Technische Universiteit Delft, reservelid

Het onderzoek voor dit proefschrift is gefinancierd door de Nederlandse Organisatie voor Wetenschappelijk Onderzoek (NWO).

TU Delft Delft University of Technology

NWO Nederlandse Organisatie voor Wetenschappelijk Onderzoek

Keywords: Lie theory, Lie algebra cohomology, Gelfand-Fuks cohomology, factorization algebras, Loday-Quillen-Tsygan theorem, formal vector fields, diagonal cohomology, topological tensor products, Fréchet algebras, LF-spaces, Slodowy slices, orbital limits, Wilson loops, Brownian motion, Haar measure, cosheaf theory, Hochschild-Serre spectral sequence

Printed by: Ipskamp Printing

Front & Back: Cover art a collaboration by the author, Elisabeth Lempa, and Markus Delitz

ISBN 978-94-6421-832-9

An electronic version of this dissertation is available at
http://repository.tudelft.nl/.

Früher war alles einfach, so einfach wie die Buchstaben in einem Lesebuch.
Jetzt ist nichts mehr einfach, nicht einmal mehr die Buchstaben.

Hermann Hesse, Siddhartha

Contents

ACKNOWLEDGMENTS

These last weeks have been filled with job applications, moving arrangements, eternal proofreading, and the occasional, mild nervous breakdown, so, to be frank, it is difficult to feel thankful for anything except that all of this will soon be over. Hence, I want to apologize right at the start if I forget anyone or if my sincere appreciation is dampened or not carried over by the words I choose here. To be clear: Everyone who opens this thesis is, in doubt, sincerely loved and appreciated in some way or another – and even many a one who never does. Probably more than you would imagine, too. You know how it is with male socialization and communicating affection. Thank you for being here.

To begin, Bas, thank you for sharing your knowledge with me and for letting me come to this wonderful country in the first place. Through your guidance, I learned a wonderful many things about mathematics, some useful, some beautiful, some nerve-racking, and probably enough to serve me a lifetime. I know it all was a challenge for you, too, with relentless teaching, a global pandemic, and small humans at home. De eerste pannenkoek mislukt altijd, but I would like to think your first PhD student turned out quite alright. Take good care of Milan, too!

I also want to thank the countless PhD candidate colleagues, whose position and fate I have been sharing throughout the last two unusually strange years, and the two usually strange years before that. Work as a PhD student is often quite lonely, but commiseration keeps the spirits high. So thank you, in somewhat chronological order: Ivan, Elek, Nick, Zoe, Emiel, Mario, Milan, Gerrit, Max, Bálint, Carel, and Matthijs #1 and #2. Some of you I got to meet as wise guiding figures to teach me all about the complex machine that is the PhD grinding mill. Some of you I first met as students in Functional Analysis, and I hope I could be your wise guiding figure for a while. And some of you I just think of as friends. I hope all of you achieve your goals, mathematical and non-mathematical, and I look forward to seeing you again after all is said and done.

I also thank my colleagues higher up the academic ladder, such as Tobias, Amudhan, Jordy, Kristin, Wolter, Martijn, Mark, and Jan. All of you helped me out, taught me or organized something valuable, or simply provided me with perspectives which enriched me as a mathematician, or just as a person. With some of you, I have ongoing collaborations, which I am very happy worked out the way they did, and I will do my best to get them published with you, despite being a mathematician only in spirit from April onwards. After this, the joys of academic publishing remain in your tenurable hands.

At this possible end of my mathematical journey, it would be amiss not to thank my academic roots back in Würzburg. I still recognize the environment at Stefan Waldmann's Chair X during my time there as a uniquely honest and inspiring one, and I hope the attitudes that made it so are spread far and wide across global mathematics. Even though the pandemic has made our encounters sparse, I thank you for contributing to this, Stefan, Knut, Chiara, Philipp, Thomas, Marvin, Andreas, and Matthias.

Nun habe ich vier Jahre lang das perfekteste Englisch der Welt gesprochen, gekritzelt und geschrieben, und vermutlich werde ich das noch eine Weile lang machen müssen. Für meine nichtakademischen Danksagungen legen wir eine kurze Pause ein.

An meine Lieben in Würzburg, Lukas, Stefan, Tessi, Jonathan, Martin. Ich bin froh, dass wir nach dem Ende unserer Unizeit noch in gutem Kontakt bleiben konnten, auch, wenn ich weiß, dass das schwierig ist – wir leben alle ganz unterschiedliche Leben an ganz unterschiedlichen Orten. Unsere Gespräche werden vermutlich seltener werden, aber ich hoffe, ihr wisst, dass euer Einfluss auf mich nicht zu überschätzen ist. Es war eine wunderschöne Zeit mit euch, und ich denke oft dahin zurück. Ich hoffe, es geht euch allen gut. Das ist alles, was ich mir für euch wünsche. Ich hab' euch sehr gern!

An Elisabeth, Niklas, Tobias, Markus, Jonathan und Menki richte ich formell mein Beileid, gute Ernte, und meine nie enttäuschten Grüße aus. Fast alles, was an meinem Leben gut ist, ist vermutlich irgendwie auf euch zurückzuführen. Ich glaube, ich verstehe euch oft nicht so sehr, wie ich es mir wünsche, weil ihr in meinem Leben einzigartig seid. Nach diesen Danksagungen werde ich, versprochen, wieder der alte Helger sein mit dem undankbaren, arroganten Studentenlächeln, weil ich diese Rolle spiele und nicht wüsste, wer ich ohne sie wäre. Aber darunter seid ihr ein wichtiger Teil von mir. Ich hoffe, wir gehen unsere Wege noch lange gemeinsam.

An meine Familie, Christine & Erich, Caroline, Sabine, und Ewald – und eure mittlerweile unzähligen Anhängsel in Form von Partnern und Kindern. Es ist lange her, dass ich damals ausgezogen bin, und vermutlich bin ich nicht mehr so ganz die Person, die ihr damals kanntet — vermutlich ist das auch gut so! Aber bis auf die eine oder andere Macke bin ich froh über und stolz auf die Person, die ich jetzt bin. Ich hoffe auch, dass ihr ein bisschen stolz sein könnt auf euren Sohn bzw. euren kleinen Bruder. Ich glaube, ich habe mittlerweile genug Dinge falsch gemacht, um zu wissen, wie man Dinge richtig macht. Ich hab' euch lieb, und so anstrengend die acht Stunden Zugfahrt auch sind, ich gehe sie immer gerne ein für euch. Seid nett zueinander!

Ja lopulta: Moi Jere! Tiedän, ettet pidä siitä, kun olen tunteellinen, mutta tämän kerran täytyy olla. Het waren vier lange jaren voor ons allebei: we zijn beide gegroeid en veranderd – en ik ben ongelooflijk blij over de mens die ik nu ben, de mens die jij bent en de mensen die wij samen zijn geworden. Für uns ist ein neuer Lebensabschnitt begonnen, und, ganz ehrlich, ich hab' ein bisschen Angst, aber ich freue mich auch wahnsinnig darauf. Minä rakastan sinua ja kiitos kaikesta.

Summary

In this thesis, we study local-to-global principles for continuous Chevalley-Eilenberg cohomology of certain, infinite-dimensional Lie algebras of geometric origin (Part I), and integral formulas for finite-dimensional Lie groups (Part II).

In Chapter 2, we begin with a detailed exposition of certain well-known results on continuous cohomology of the Lie algebra of vector fields $\mathfrak{X}(M)$ on a smooth manifold M, or *Gelfand-Fuks cohomology* of M. We present a novel construction of cohomological spectral sequences by Gelfand and Fuks based on a local-to-global principle closely related to the modern theory of factorization algebras.

For a large class of smooth manifolds, e.g. M compact and orientable, these spectral sequences converge towards so-called *diagonal cohomology* $\Delta_k H^\bullet(\mathfrak{X}(M))$ for $k \in \mathbb{N}$, and their second pages can be fully specified in terms of known and easily calculable data, namely continuous Chevalley-Eilenberg cohomology of formal vector fields and relative homology of Cartesian powers M^k with respect to their fat diagonals M^k_{k-1}.

In Chapter 3, we apply the local-to-global principle prepared in the previous chapter to continuous homology of gauge algebras, i.e. Lie algebras given by the sections $\Gamma(\operatorname{Ad} P \to M)$ of the adjoint bundle associated to a principal fiber bundle $P \to M$. Due to isomorphisms of the form

$$\mathfrak{gl}_n(C^\infty(M)) \cong C^\infty(M, \mathfrak{gl}_n(\mathbb{R})), \quad n \in \mathbb{N}$$

it will be helpful to begin by formulating a topological version of the Loday-Quillen-Tsygan (LQT) theorem, which says that for an arbitrary unital algebra A one has

$$H_\bullet(\mathfrak{gl}(A)) \cong \Lambda^\bullet H^\lambda_{\bullet-1}(A),$$

where the left-hand side homology is algebraic Chevalley-Eilenberg homology and the right-hand side homology is cyclic homology. We show that a topological/bornological analog of this result holds for a certain class of Fréchet and LF-algebras A, including $A = C^\infty(M)$ for smooth manifolds M and $A = C^\infty_c(\mathbb{R}^n)$. The same spectral sequence approach as in Chapter 2 yields, when $P \to M$ is a principal $GL_t(\mathbb{R})$-bundle, a spectral sequence converging to diagonal homology of $\Gamma(\operatorname{Ad} P \to M)$. A certain stable part of its second page can be specified in terms of Čech homology of (topological tensor products of) certain cosheaves given by associating to $U \subset M$ the quotient of k-forms by exact k-forms on U. This whole chapter requires a careful analysis of functional-analytic details, since many homology spaces in question are infinite-dimensional Fréchet or LF-spaces.

In Chapter 4, let G be a real, semisimple Lie group G. We study the ratios

$$\delta_F(U) := \frac{\mu\left(\bigcap_{g\in F}\mathrm{Ad}_g U\right)}{\mu(U)}, \quad \delta_F(\mathscr{U}) := \liminf_{U\in\mathscr{U}} \delta_F(U)$$

where $F \subset G$ arbitrary, μ denotes the Haar measure, Ad denotes the adjoint action, $U \subset G$ is a relatively compact neighborhood of the identity with nonzero Haar volume, and $\mathscr{U}$ is a neighborhood basis of the identity in G. Lastly, we set δ_F as the supremum over $\delta_F(\mathscr{U})$ for all such $\mathscr{U}$. If $B_\rho^G := \{g \in G : \|\mathrm{Ad}_g\| \le \rho\}$ for some $\rho > 0$ and $\|\cdot\|$ the operator norm, then we show, by constructing a specific neighborhood basis of the identity, that

$$\delta_{B_\rho^G} \ge \rho^{-d/2},$$

where d is the maximal dimension of a nilpotent, adjoint orbit in the Lie algebra $\mathfrak{g} := \mathrm{Lie}(G)$. This is done through the means of certain orbital limit formulas due to Barbasch, Harris, and Vogan, and Lie-theoretic integration formulas due to Harish-Chandra and Varadarajan. Our result is useful for a quantitative analysis of the coefficients of non-commutative de Leeuw inequalities in Harmonic Analysis.

Lastly, in Chapter 5, we explore for compact Lie groups G with bi-invariant Riemannian metric $\langle\cdot,\cdot\rangle$ the consequences of simple integration by parts formulas of the form

$$\int_G \Delta\rho(g)_{ij}\rho(g)_{kl}\,\mathrm{d}g = -\int_G \langle d\rho(g)_{ij}, d\rho(g)_{kl}\rangle\,\mathrm{d}g,$$

where $\mathrm{d}g$ denotes its Haar measure, Δ denotes the Laplacian associated to the Riemannian metric, $\rho : G \to V$ is a finite-dimensional representation of G and $\rho(g)_{ij}$ are its matrix entries with respect to some basis. Through the Peter-Weyl theorem and the study of Casimir operators associated to ρ, this equation can provide nontrivial insights into integrals of matrix coefficients on G, which we identify as the underlying principle for many results in the literature, such as expectation values of Wilson loops in lattice gauge theory, of Riemannian Brownian motions on Lie groups in stochastic calculus, and for the calculation of *Weingarten functions*, i.e. expectation values of matrix coefficients in the Haar measure. We pay special attention in particular to the latter question, which, using this integration by parts formula, can be rephrased as a question about tensor invariants $\left(V^{\otimes k}\otimes(V^*)^{\otimes l}\right)^G$, a well-studied problem for many representations ρ.

Samenvatting

In dit proefschrift bestuderen we lokaal-naar-globaal-principes voor continue Chevalley-Eilenberg cohomologie van bepaalde, oneindig-dimensionele Lie-algebra's met een geometrische oorsprong (Deel I), en integratieformules voor eindig-dimensionale Lie-groepen (Deel II).

In Hoofdstuk 2 beginnen we met een gedetailleerde uiteenzetting van bepaalde bekende resultaten over de continue cohomologie van de Lie-algebra van vectorvelden $\mathfrak{X}(M)$ op een gladde variëteit M, ook wel de *Gelfand-Fuks cohomologie* van M genoemd. We presenteren een nieuwe constructie van cohomologische spectraalrijen, oorspronkelijk van Gelfand en Fuks, gebaseerd op een lokaal-naar-globaal principe dat verwant is aan de moderne theorie van factorisatie-algebra's. Voor een grote klasse gladde variëteiten, zoals M compact en oriënteerbaar, convergeren deze spectraalrijen naar de zogenaamde *diagonale cohomologie* $\Delta_k H^\bullet(\mathfrak{X}(M))$ voor $k \in \mathbb{N}$ en kunnen hun tweede pagina's volledig worden gespecificeerd door bekende en makkelijk te bepalen data, namelijk de continue Chevalley-Eilenberg-cohomologie van formele vectorvelden en relatieve homologie van cartesische machten M^k met betrekking tot hun dikke diagonalen M^k_{k-1}.

In Hoofdstuk 3 passen we het lokaal-naar-globaal-principe, dat in het vorige hoofdstuk is voorbereid, toe op continue homologie van ijkalgebra's, Lie-algebra's gegeven door snedes $\Gamma(\mathrm{Ad}\,P \to M)$ van de geadjungeerde bundel associeerd aan een hoofdvezelbundel $P \to M$. Dankzij isomorfismen van de vorm

$$\mathfrak{gl}_n(C^\infty(M)) \cong C^\infty(M, \mathfrak{gl}_n(\mathbb{R})), \quad n \in \mathbb{N}$$

is het om te beginnen nuttig om een topologische versie van de Loday-Quillen-Tsygan (LQT) stelling te formuleren, die zegt dat voor een algebra A met eenheid er geldt dat

$$H_\bullet(\mathfrak{gl}(A)) \cong \Lambda^\bullet H^\lambda_{\bullet-1}(A),$$

waarbij de homologie aan de linkerkant de algebraïsche Chevalley-Eilenberg-homologie is en die aan de recheterkant de algebraïsche cyclische homologie. We laten zien dat een topologische/bornologische versie van dit resultaat geldt voor een bepaalde klasse van Fréchet- en LF-algebra's A, waaronder $A = C^\infty(M)$ voor een gladde variëteit M en $A = C^\infty_c(\mathbb{R}^n)$. Wanneer $P \to M$ een hoofdvezelbundel met vezel $GL_t(\mathbb{R})$ is, leidt dezelfde benadering als in Hoofdstuk 2 tot een spectraalrij die convergeert naar diagonale homologie van $\Gamma(\mathrm{Ad}\,P \to M)$. Een stabiel deel van de tweede pagina kan worden gespecificeerd in termen van Čech-homologie van (topologische tensorproducten van)

bepaalde co-schoven, gegeven door de afbeelding die het quotiënt van k-vormen op U door exacte k-vormen op U toekent aan de open verzameling $U \subset M$. Omdat veel homologieruimten in kwestie oneindig-dimensionale Fréchet- of LF-ruimten zijn, vereist dit hele hoofdstuk een zorgvuldige analyse van functioneel-analytische details.
Zij nu G een reële, halfenkelvoudige Lie-groep in Hoofdstuk 4. We bestuderen de ratio's

$$\delta_F(U) := \frac{\mu\left(\bigcap_{g\in F} \mathrm{Ad}_g U\right)}{\mu(U)}, \quad \delta_F(\mathscr{U}) := \liminf_{U\in\mathscr{U}} \delta_F(U)$$

waarbij $F \subset G$ arbitrair is, μ de Haar-maat, Ad de geadjungeerde actie, $U \subset G$ een relatief compacte omgeving van de identiteit is met Haar-maat ongelijk aan nul, en $\mathscr{U}$ een omgevingsbasis van de identiteit in G. Tot slot definiëren we δ_F als het supremum van $\delta_F(\mathscr{U})$ voor alle dergelijke $\mathscr{U}$. Zij $B^G_\rho := \{g \in G : \|\mathrm{Ad}_g\| \le \rho\}$ voor $\rho > 0$ en $\|\cdot\|$ de operatornorm. Door een specifieke omgevingsbasis van de identiteit te construeren, laten we zien dat

$$\delta_{B^G_\rho} \ge \rho^{-d/2},$$

waarbij d de maximale dimensie is van nilpotente geadjungeerde banen in de Lie-algebra $\mathfrak{g} := \mathrm{Lie}(G)$. Dit wordt gedaan door middel van bepaalde orbitale limietformules van Barbasch, Harris en Vogan, en Lie-theoretische integratieformules van Harish-Chandra en Varadarajan. Ons resultaat is nuttig voor een kwantitatieve analyse van de coëfficiënten van niet-commutatieve de Leeuw ongelijkheden in harmonische analyse.
Ten slotte onderzoeken we in Hoofdstuk 5 voor een compacte Lie-groep G met bi-invariante Riemann-metriek $\langle\cdot,\cdot\rangle$ de gevolgen van eenvoudige, partiële-integratie-formules van de vorm

$$\int_G \Delta\rho(g)_{ij}\rho(g)_{kl}\,\mathrm{d}g = -\int_G \langle d\rho(g)_{ij}, d\rho(g)_{kl}\rangle\,\mathrm{d}g,$$

waarbij $\mathrm{d}g$ de Haar-maat is, Δ de Laplace-operator associeerd aan de Riemann-metriek, $\rho : G \to V$ een eindig-dimensionele representatie van G, en $\rho(g)_{ij}$ de matrixcoëfficiënten in een bepaalde basis. Vanwege de Peter-Weyl stelling en door Casimir-operatoren geassocieerd aan ρ te bestuderen leidt deze vergelijking tot niet-triviale inzichten over integralen van matrixcoëfficiënten op G. Deze zien we als het onderliggende principe voor veel bekende resultaten in de literatuur, zoals als verwachtingswaarden van Wilson-loops in roosterijktheorie, Riemanniaanse Brownse bewegingen op Lie-groepen in stochastische calculus, en voor de berekening van *Weingarten-functies*, de verwachtingswaarden van matrixcoëfficiënten met betrekking tot de Haar-maat. We besteden in het bijzonder aandacht aan de laatste vraag, die met behulp van partiële integratie kan worden hergeformuleerd als een vraag over tensorinvarianten van de vorm $\left(V^{\otimes k} \otimes (V^*)^{\otimes l}\right)^G$; een goed bestudeerd probleem voor vele representaties ρ.

Parts of the translation were provided by Milan Niestijl & Bas Janssens.

1

Introduction

Understanding any mathematical problem or physical system generally relies on understanding its underlying symmetries. Of particular interest are the symmetry groups governing the solutions of (partial) differential equations, which are of continuous type and also called *Lie groups*. This continuity allows one to construct an infinitesimal counterpart to Lie groups, their *Lie algebras*, which are often easier to analyze and whose properties in many ways mirror properties of the corresponding Lie group. In this thesis, we study methods to analyze Lie groups and Lie algebras from representation-theoretic, measure-theoretic, and cohomological perspectives. The methods required to conduct such analyses differ wildly depending on whether these objects are of infinite- or finite-dimensional type, and since we study both situations, this document is divided into two largely independent parts.

1.1. Part I: Local-to-global principles for cohomology of infinite-dimensional Lie algebras

In the first part, we study certain geometric, infinite-dimensional Lie algebras, more specifically, the Lie algebra of vector fields $\mathfrak{X}(M)$ for a smooth manifold M, and the so-called *gauge algebra* $\Gamma(\operatorname{Ad} P \to M)$ associated to a principal fiber bundle $P \to M$. Here, the aspect that we are most interested in is the calculation of the *continuous Chevalley–Eilenberg cohomology* $H^\bullet(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$, a naturally assigned, graded vector space containing certain invariants associated to $\mathfrak{g}$. In particular, degree 2 cohomology of a Lie algebra $\mathfrak{g}$ classifies *central extensions* of $\mathfrak{g}$, which are exact sequences of Lie algebras

1

of the form

$$0 \to \mathbb{R} \to \hat{\mathfrak{g}} \to \mathfrak{g} \to 0. \tag{1.1.1}$$

These, in turn, characterize projective representations of $\mathfrak{g}$ and are hence of interest to formulate quantizations of physical systems admitting $\mathfrak{g}$-symmetries. The Lie algebra of vector fields occurs, in this sense, in conformal field theory. On the circle $M = S^1$, this Lie algebra admits an (up to isomorphism) unique nontrivial central extension called the *Virasoro algebra* [Vir70].

1.1.1. AN EXPOSITION TO GELFAND–FUKS COHOMOLOGY

A study of continuous Chevalley–Eilenberg cohomology of vector fields, also called *Gelfand–Fuks cohomology*, has already been conducted in the 1970s, giving rise to a large body of literature [GF70a, GF69, GF70b, BS77]. However, many of these results are not easily accessible to non-experts today as they rely on many (at the time) novel techniques. The ideas discussed in these papers appear more relevant than ever in the new millenium, with many new results appearing about Chevalley–Eilenberg cohomology of similar Lie algebras, such as symplectic and Hamiltonian vector fields on a smooth manifold [JV16, JV18]. Furthermore, in the last decade, *factorization algebras* have not only been shown to offer a promising view on conformal field theory [CG17], but they are also a perfectly suited structure with which to study cochains and cohomology over topological spaces, closely mirroring some local-to-global techniques used in the calculation of Gelfand–Fuks cohomology [HK18].

Hence, in Chapter 2, which is based on the preprint [Mia22b], we give a detailed construction of some of the main results in the study of Gelfand–Fuks cohomology. We view this setting as a prototype to calculate continuous Chevalley–Eilenberg cohomology for similar infinite-dimensional Lie algebras of geometric origin. While the results in this chapter are essentially all captured in the literature [Fuk86, Bot73], we do modernize some of the language and provide novel proof techniques which easily generalize to different settings, such as the continuous cohomology of gauge algebras that we study in Chapter 3.

The traditional approach, which we will follow, is to first study the case of *infinitesimal* Gelfand–Fuks cohomology, which is represented by continuous Chevalley–Eilenberg cohomology of the Lie algebra of formal vector fields

$$W_n := \left\{ \sum_{i=1}^{n} f_i \partial_i : \quad f_1, \dots, f_n \in \mathbb{R}[\![x_1, \dots, x_n]\!] \right\}, \quad [f\partial_i, g\partial_j] := f \cdot \frac{\partial g}{\partial x_i}\partial_j - g \cdot \frac{\partial g}{\partial x_j}\partial_i. \tag{1.1.2}$$

On this level, the continuity requirement is equivalent to cochains vanishing on formal vector fields of sufficiently high degree, and the calculation of continuous cohomology is almost purely algebraic. One identifies the reductive Lie subalgebra

$$\mathfrak{gl}_n(\mathbb{R}) \cong \left\{ \sum_{i,j=1}^{n} a_{ij} x_i \partial_j : a_{ij} \in \mathbb{R} \right\} \subset W_n \tag{1.1.3}$$

and uses this to construct a Hochschild–Serre spectral sequence for continuous Chevalley–Eilenberg cohomology of W_n. All pages of this spectral sequence can be determined completely, through an analysis of $\mathfrak{gl}_n(\mathbb{R})$-tensor invariants $\left(V^{\otimes k} \otimes (V^*)^{\otimes l}\right)^{\mathfrak{gl}_n(\mathbb{R})}$ where $V = \mathbb{R}^n$, an application of the Bott transgression theorem, and using the functoriality of the Hochschild–Serre spectral sequence. The most important information is summarized in the following:

Theorem A (Gelfand, Fuks 1970)**.** *The space $H^k(W_n)$ is trivial when $1 \leq k \leq 2n$ or $k > n^2 + 2n$. The wedge product of two cohomology classes of positive degree in $H^\bullet(W_n)$ is zero.*

Building on this, one studies the *local* Gelfand–Fuks cohomology, represented by the continuous Chevalley–Eilenberg cohomology of vector fields $\mathfrak{X}(\mathbb{R}^n)$ on Euclidean space. Here, we have at our disposal the Euler vector field

$$E(x) := \sum_{i=1}^{n} x_i \partial_i, \quad x \in \mathbb{R}^n, \tag{1.1.4}$$

which generates a family of scaling diffeomorphisms $\{T_t\}_{t>0}$. Reminiscent of Taylor expansion, one can study the behavior of limits of vector fields

$$\lim_{t \to 0} t^{-k} T_t^* X, \quad X \in \mathfrak{X}(\mathbb{R}^n), k \in \mathbb{N}, \tag{1.1.5}$$

giving rise to a filtration of cochain complexes on the Gelfand–Fuks cochains. The same filtration exists on W_n with the formal analog of the Euler vector field, and through the limits (1.1.5) a correspondence between them can be established. To prepare our local-to-global principle, we show that the same analysis holds for Gelfand–Fuks cohomology of finite disjoint unions $\mathbb{R}^n \sqcup \cdots \sqcup \mathbb{R}^n$, and we conclude:

Theorem B (Bott, 1973)**.** *Let $M := \bigsqcup_{i=1}^r \mathbb{R}^n$ be a disjoint collection of copies of $\mathbb{R}^n$. Then every choice of order on the copies of $\mathbb{R}^n$ induces an algebra isomorphism*

$$H^\bullet(\mathfrak{X}(M)) \cong H^\bullet(W_n)^{\otimes^r}. \tag{1.1.6}$$

We want to once again emphasize that, up to this point, our proof methods are largely an elaboration, modernization, and occasionally slight correction of the ones available in the literature. In the last part of Chapter 2, however, we use novel tools closely related to the theory of factorization algebras[1] to construct the spectral sequences of Gelfand

[1] Note that we do not describe the Chevalley–Eilenberg cochains as factorization algebras themselves here. While this setting is natural in light of [CG17], we have not been able to use it to simplify our constructions.

and Fuks that calculate the continuous cohomology of vector fields $\mathfrak{X}(M)$ on a smooth manifold, using a local-to-global principle.

While the results here are not new, this provides an approach that, in principle, can be used as a local-to-global principle for continuous Chevalley–Eilenberg cohomology of section spaces of more general Lie algebroids, an example of which we study in Chapter 3. The rough idea is to consider the following double complex:

$$\begin{tikzcd}
\cdots & \cdots & \\
\bigoplus_i C^3(\mathfrak{X}(U_i)) \arrow[u] & \bigoplus_{i,j} C^3(\mathfrak{X}(U_{ij})) \arrow[u] \arrow[l] & \cdots \arrow[l] \\
\bigoplus_i C^2(\mathfrak{X}(U_i)) \arrow[u] & \bigoplus_{i,j} C^2(\Gamma(\mathfrak{X}(U_{ij})) \arrow[u] \arrow[l] & \cdots \arrow[l] \\
\bigoplus_i C^1(\mathfrak{X}(U_i)) \arrow[u] & \bigoplus_{i,j} C^1(\Gamma(\mathfrak{X}(U_{ij})) \arrow[u] \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

Here, the rows consist of cosheaf-theoretic Čech complexes associated to the precosheaf structures on the continuous cochain spaces

$$U \mapsto C^k(\mathfrak{X}(U)), \quad U \subset M \text{ open}, \, k \geq 1 \tag{1.1.7}$$

with respect to a fixed cover $\mathscr{U}$ of M. The column differentials are given by the direct sum of Chevalley–Eilenberg differentials associated to the Lie algebras $\mathfrak{X}(U)$ for all $U \in \mathscr{U}$. The idea is a standard spectral sequence argument for double complexes: If the rows were well-behaved, flabby cosheaves, the spectral sequence arising from filtering this complex by columns collapses on the second page into a single column given by the Chevalley–Eilenberg cohomology of the global sections of the cosheaf. On the other hand, the cohomology in the columns is simply Gelfand–Fuks cohomology on the elements of the cover, which, ideally, have a more simple geometry than M, like the local case of $M = \mathbb{R}^n$. This gives rise to a spectral sequence whose entries are, ideally, built from local Gelfand–Fuks cohomology, and which converges to global Gelfand–Fuks cohomology.

There are many details swept under the rug in the above: One significant problem that arises in this approach is the fact that the assignment $U \mapsto C^k(\mathfrak{X}(U))$ does *not* define a cosheaf over M for $k > 1$, so the Čech complexes may be badly behaved in general; However, if $\boxtimes$ denotes the external tensor product of vector bundles, we can link $C^k(\mathfrak{X}(U))$ to the cosheaf given by the distributions of section spaces

$$U \mapsto \Gamma\left(TM^{\boxtimes k}|_U \to U\right)^*, \quad U \subset M^k \text{ open}. \tag{1.1.8}$$

This is done through the isomorphism

$$C^k(\mathfrak{X}(M)) \cong \left(\Gamma\left(TM^{\boxtimes k} \to M^k\right)_k^{\Sigma}\right)^*, \tag{1.1.9}$$

where Σ_k denotes the symmetric group in k elements, acting on the sections of the external tensor product through antisymmetric permutation of the factors. The correct open cover to work with in this setting are *k-good covers*, a generalization of the notion of good covers from the theory of factorization algebras due to [BdBW13], useful in the setting when a single cover is needed to capture information about the Cartesian powers $M, M^2, \ldots, M^k$. This controls the cochain spaces $C^r(\mathfrak{X}(U))$ up to degree $\leq k$. By replacing the higher degree cochain spaces by *k-diagonal cochains* $\Delta_k C^r(\mathfrak{X}(U))$, defined as the cochains which under the isomorphism (1.1.9) arise from distributions whose support lies in a certain diagonal subspace $M_k^r \subset M^r$, all precosheaves in the double complex can be related to cosheaves over the Cartesian powers and diagonals $M, \ldots, M^k, M_k^{k+1}, M_k^{k+2}, \ldots$, which can be fully controlled by k-good covers.
We call this double complex the *Čech–Bott–Segal* double complex, honoring similar but different localization strategies carried out by Bott and Segal in [BS77]. Now, after an analysis of the precosheaf structure associated to the assignments $U \mapsto H^\bullet(\mathfrak{X}(U))$ for U open disks in M, one can apply standard double complex techniques, we show that this reproduces well-known Gelfand–Fuks spectral sequences:

Theorem C (Gelfand, Fuks 1969)**.** *Let M be an orientable manifold which admits a finite, k-good open cover (e.g. if M is compact). There exists a cohomological spectral sequence $\{E_r^{\bullet,\bullet}, d_r\}$ which converges to reduced k-diagonal cohomology $\Delta_k \tilde{H}^\bullet(\mathfrak{X}(M))$, and the entries $E_2^{p,q}$ of its second page are, for $q \geq 1$, of the following form:*

$$\begin{aligned} E_2^{p,q} &\cong H_{-p}(M) \otimes H^q(W_n) \\ &\oplus \bigoplus_{\substack{q_1+q_2=q \\ q_i>0}} \left(H_{-p}(M^2, M_1^2) \otimes H^{q_1}(W_n) \otimes H^{q_2}(W_n)\right)^{\Sigma_2} \\ &\oplus \quad \ldots \\ &\oplus \bigoplus_{\substack{q_1+\cdots+q_k=q \\ q_i>0}} \left(H_{-p}(M^k, M_{k-1}^k) \otimes H^{q_1}(W_n) \otimes \cdots \otimes H^{q_k}(W_n)\right)^{\Sigma_k}. \end{aligned} \tag{1.1.10}$$

Here, a permutation $\sigma \in \Sigma_r$ acts by simultaneous permutation of the Cartesian factors of M^k and the tensor factors $H^q(W_n)$.

1.1.2. BORNOLOGICAL LODAY–QUILLEN–TSYGAN THEOREMS AND CONTINUOUS COHOMOLOGY OF GAUGE ALGEBRAS

The second kind of infinite-dimensional Lie algebras we study are *gauge algebras*, spaces of sections $\Gamma(\mathrm{Ad}\, P \to M)$ of the adjoint bundle associated to a principal fiber bundle

1

$P \to M$. The case of degree 2, which is relevant in the unitary representation theory of gauge algebras, has been fully calculated in [JW13], in the case when the fibers K of P are semisimple Lie groups with finitely generated $\pi_0(K)$, and M is compact and connected. Closely related to our investigation in Chapter 3, which is based on the preprint [Mia22a], they used a sheaf-theoretic local-to-global principle to calculate the continuous cohomology in this degree. More precisely, denote by K the typical fiber Lie group of P, by $\mathfrak{k} := \mathrm{Lie}(K)$ its Lie algebra, and by $V(\mathrm{Ad}\,P) \to M$ the vector bundle arising from applying the functor

$$T_eK = \mathfrak{k} \mapsto (S^2\mathfrak{k})_{\mathfrak{k}} := \frac{S^2\mathfrak{k}}{\mathfrak{k}\cdot(S^2\mathfrak{k})}. \tag{1.1.11}$$

to every fiber of the adjoint bundle $\mathrm{Ad}\,P = P \times_{\mathrm{Ad}} \mathfrak{k} \to M$. Choose further a Lie connection ∇ on $\mathrm{Ad}\,P$.

In the local case, when the base M is just Euclidean space, one finds the following using results from [Mai02]:

$$H^2(\Gamma(\mathrm{Ad}\,P \to \mathbb{R}^n)) \cong \left(\frac{\Omega^2_c(\mathbb{R}^n, V(\mathrm{Ad}\,P))}{d_\nabla\Omega^1_c(\mathbb{R}^n, V(\mathrm{Ad}\,P)}\right)^*, \tag{1.1.12}$$

where $\Omega^\bullet_c(\mathbb{R}^n, V(\mathrm{Ad}\,P))$ denotes the compactly supported differential forms on $\mathbb{R}^n$ with values in the vector bundle $V(\mathrm{Ad}\,P)$, and d_∇ denotes the connection induced by ∇. The dual is the continuous dual with respect to the standard LF-topology on the quotient space. Through degree-specific calculations, they show that for a fixed principal bundle $P \to M$, both the assignment of open sets $U \subset M$ to $H^2(\Gamma(\mathrm{Ad}\,P|_U \to U))$ an and the assignment of open sets $U \subset M$ to $\left(\frac{\Omega^2_c(U,V(\mathrm{Ad}\,P))}{d_\nabla\Omega^1_c(U,V(\mathrm{Ad}\,P)}\right)^*$ define not only presheaves over M, but sheaves. Sheaf theory provides simple arguments to then prove that local isomorphisms lift to global ones, with which they show that under the given assumptions on P and M, we have

$$H^2(\Gamma(\mathrm{Ad}\,P \to M)) \cong \left(\frac{\Omega^2_c(M, V(\mathrm{Ad}\,P))}{d_\nabla\Omega^1_c(M, V(\mathrm{Ad}\,P)}\right)^*. \tag{1.1.13}$$

The obvious question, especially in conjunction with our work in Chapter 2, is if a similar local-to-global principle is applicable to $H^k(\Gamma(\mathrm{Ad}\,P \to M))$ for $k > 2$.

In Chapter 3, we will study this question in the dual setting of continuous Chevalley–Eilenberg *homology* $H_\bullet(\Gamma(\mathrm{Ad}\,P \to M))$, which is defined in terms of certain topological tensor products. We do this because in the analysis of $\Gamma(\mathrm{Ad}\,P \to M)$, nontrivial functional-analytic arguments take place at the intersection of homological algebra and functional analysis, and we find it helpful to separate topological phenomena of multilinearity from phenomena of duality.

We explore the *Loday–Quillen–Tsygan (LQT) theorem* [LQ84, Tsy83], which says that if A is a unital algebra and $\mathfrak{gl}(A) := \varinjlim \mathfrak{gl}_n(A)$, then in *algebraic* Chevalley–Eilenberg cohomology, we have

$$H_\bullet(\mathfrak{gl}(A)) \cong \Lambda^\bullet H^\lambda_{\bullet-1}(A) \text{ and } H_k(\mathfrak{gl}_n(A)) \cong H_k(\mathfrak{gl}(A)) \text{ if } n \geq k, \tag{1.1.14}$$

where $H^\lambda_\bullet(A)$ denotes the cyclic homology of the algebra A. Similar theorems are available in the literature for the direct limit Lie algebras associated to other classical Lie algebras, such as $\mathfrak{sl}(A)$, $\mathfrak{sp}(A)$ and $\mathfrak{so}(A)$ [Lod92].

The reason this is interesting to us for the study of gauge algebra homology lies in isomorphisms of the form

$$\mathfrak{gl}_n(C^\infty(M)) \cong C^\infty(M, \mathfrak{gl}_n(\mathbb{R})) \cong \Gamma(M \times \mathfrak{gl}_n(\mathbb{R}) \to M), \tag{1.1.15}$$

so that in the case $A = C^\infty(M)$, the Lie algebras $\mathfrak{gl}(A)$, $\mathfrak{sl}(A)$ and so on are closely related to gauge algebras associated to trivial principal fiber bundles whose typical fibers have the Lie algebras $\mathfrak{gl}_n(\mathbb{R})$, $\mathfrak{sl}_n(\mathbb{R})$. What, however, is *not* available in the literature at this point, is a version of the LQT theorem for *continuous* homology. The study of this is one of the aims of Chapter 3, where we begin by studying the local case where M is Euclidean space and the Lie algebra in question is either $\mathfrak{gl}(C^\infty(M)) = \varinjlim \mathfrak{gl}_n(C^\infty(M))$ or its compactly supported analog $\mathfrak{gl}(C^\infty(M)) = \varinjlim \mathfrak{gl}_n(C^\infty_c(M))$, both equipped with their natural LF-topologies arising from the direct limits. We find that the most natural way to adapt the proof of the algebraic LQT theorem is to consider *bornological* Chevalley–Eilenberg homology, which is the homology of the usual Chevalley–Eilenberg complex with all involved tensor products replaced by completed bornological tensor products $\widehat{\otimes}$ [KM97]. The usefulness of this tensor product is its compatibility with inductive limits and the fact that it reproduces Grothendieck's inductive tensor product [Gro95] and the projective tensor product [Sch71] on LF-spaces and Fréchet spaces, respectively.

Adapting the proof of the algebraic LQT theorem to the bornological setting is then fairly easy whenever the algebra A is unital and Fréchet, and the most difficult part is a necessary application of a Künneth formula

$$H_\bullet(C_\bullet \widehat{\otimes} D_\bullet) \cong H_\bullet(C_\bullet) \widehat{\otimes} H_\bullet(D_\bullet) \tag{1.1.16}$$

for chain complexes $C_\bullet, D_\bullet$ of topological vector spaces. Such Künneth formulas exist for chain complexes of Fréchet spaces, but require the differentials of both chain complexes to be toplogical homomorphisms, which for linear, continuous maps of Fréchet spaces is equivalent to their range being closed in the codomain. With this, we can adapt the LQT theorem as follows:

Theorem D. *Let A be a nuclear unital Fréchet algebra, and assume that the differential of the bornological cyclic complex $C^{\lambda,\mathrm{born}}_\bullet(A)$ has closed range. Then we have, for all $r, n \in \mathbb{N}$ with $r + 1 \leq n$*

$$H^{\mathrm{born}}_r(\mathfrak{gl}_n(A)) \cong \left(\hat{\Lambda}^\bullet H^{\lambda,\mathrm{born}}_{\bullet-1}(A)\right)_r, \tag{1.1.17}$$

and

$$H^{\mathrm{born}}_\bullet(\mathfrak{gl}(A)) \cong \hat{\Lambda}^\bullet H^{\lambda,\mathrm{born}}_{\bullet-1}(A). \tag{1.1.18}$$

1

We are also interested in the case of certain non-unital algebras, such as $C_c^\infty(\mathbb{R}^n)$ with its LF-topology arising from

$$C_c^\infty(\mathbb{R}^n) = \varinjlim C_{D_n}^\infty(\mathbb{R}^n) \tag{1.1.19}$$

where $\{D_n\}_{n\geq 1}$ is any compact exhaustion of $\mathbb{R}^n$ and $C_{D_n}^\infty(\mathbb{R}^n)$ denotes the smooth functions on $\mathbb{R}^n$ whose support is contained in D_n. Algebraic LQT theorems exist for non-unital algebras [Han88, Cor05], in the case they are *H-unital*, defined as their bar complex being acyclic. We show that an analogous statement holds for bornological LQT theorems, even if the necessary proof is a lot more involved. The reason for this is that in the unital case, one can identify the reductive subalgebra $\mathfrak{gl}_n(\mathbb{K}) \subset \mathfrak{gl}_n(A)$. The existence of such a subalgebra is the used to show that the homology of the Chevalley–Eilenberg complex is equal to the homology of the $\mathfrak{gl}_n(\mathbb{K})$-invariant subcomplex, heavily simplifying the process. In the non-unital case, a $\mathfrak{gl}_n(\mathbb{K})$-action on the complex still exists, but one needs to instead decompose it into isotypic types and show that the non-invariant components nontrivially involve the bar complex of the underlying algebra. Adapting this to the bornological setting, one finds the following:

Theorem E. *Let A be a nuclear Fréchet algebra, and assume that the differential of the bornological cyclic complex $C_\bullet^{\lambda,\mathrm{born}}(A)$ has closed range. Additionally, assume that A is* bornologically H-unital, *i.e. the bornological bar complex $C^{\mathrm{bar,born}}(A)$ is acyclic.*
Then we have, for all $r, n \in \mathbb{N}$ with $2r+1 \leq n$:

$$H_r^{\mathrm{born}}(\mathfrak{gl}_n(A)) \cong \left(\hat{\Lambda}^\bullet H_{\bullet-1}^{\lambda,\mathrm{born}}(A)\right)_r, \tag{1.1.20}$$

and

$$H_\bullet^{\mathrm{born}}(\mathfrak{gl}(A)) \cong \hat{\Lambda}^\bullet H_{\bullet-1}^{\lambda,\mathrm{born}}(A) \tag{1.1.21}$$

Finally, using a local-to-global principle as in Chapter 2, we can reconstruct from the above results in spectral sequences analogous to the Gelfand–Fuks spectral sequences which provide insight into higher order bornological/continuous homology of gauge algebras whose fibers are classical, simple Lie algebras. However, there are some problems that do not show up in the Gelfand–Fuks setting which we are not able to resolve at the current point in time. Specifically, since the local homology spaces are infinite-dimensional, we end up having to calculate Čech homology of completed tensor products of cosheaves with respect to a fixed cover. If Z^1, Z^2 are two cosheaves of Fréchet spaces on M, and $Z^1 \hat{\otimes} Z^2$ is a certain bornological tensor product cosheaf over M^2, we can use the previous Künneth methods to deduce statements of the form

$$\check{H}_\bullet(\mathcal{U}\times\mathcal{V}, Z^1 \hat{\otimes} Z^2) \cong \check{H}_\bullet(\mathcal{U}, Z^1) \hat{\otimes} \check{H}_\bullet(\mathcal{V}, Z^2), \tag{1.1.22}$$

for open covers $\mathcal{U},\mathcal{V}$ of M and their product cover $\mathcal{U}\times\mathcal{V} = \{U\times V : U\in\mathcal{U}, V\in\mathcal{V}\}$. However, since such product covers are not in general cofinal in the family of open covers

over topological spaces, this argument is insufficient to calculate the Čech homology of the manifold itself, and thus some arguments stay unavailable to us.
Nevertheless, in a similar way to Chapter 2, we arrive at the following spectral sequence:

Theorem F. *Let M be a manifold of finite dimension n and $P \to M$ a principal $GL_t(\mathbb{R})$-bundle. Let $\mathrm{Ad}(P) \to M$ be the associated adjoint bundle, and $q := \lfloor \frac{t-1}{2} \rfloor$. Denote by Z^k for $k \geq 0$ the cosheaves over M defined by*

$$Z^k(U) := \frac{\Omega_c^k(U)}{d\Omega_c^{k-1}(U)}. \tag{1.1.23}$$

Set also for all $k \geq 1$:

$$\xi_n(k) := \min\{k, n + (k - n \mod 2)\}. \tag{1.1.24}$$

Then there is a homological first-quadrant spectral sequence $\{E^\bullet_{r,s}\}_{r,s\geq 0}$ with

$$E^k_{r,s} \implies H^{\mathrm{born}}_{r+s}(\Gamma_c(\mathrm{Ad}(P))). \tag{1.1.25}$$

For $r \geq 0$ and $1 \leq s \leq q$, the second page $E^2_{r,s}$ equals

$$E^2_{r,s} = \bigoplus_{k\geq 1} \left(\bigoplus_{s_1+\dots+s_k=s} \check{H}_r \left(\mathscr{U}^k, Z^{\xi_n(s_1-1)} \widehat{\otimes} \cdots \widehat{\otimes} Z^{\xi_n(s_k-1)} \right) \right)_{\Sigma_k} \tag{1.1.26}$$

1.2. PART II: INTEGRATION FORMULAS ON FINITE-DIMENSIONAL LIE GROUPS

In the second part, we restrict our attention to finite-dimensional Lie groups and Lie algebras, more specifically, integration and measure theoretic methods thereon. Since every finite-dimensional Lie group can be equipped with a canonical measure, the *Haar measure*, Lie theory offers a large playground of well-behaved measure spaces to conduct differential geometry and functional analysis on.

1.2.1. VOLUMES OF IDENTITY-NEIGHBORHOODS IN REAL REDUCTIVE LIE GROUPS

In Chapter 4, which is based on the preprint [CJKUM22][2], we begin this investigation by studying how the adjoint action transforms the volume of small neighborhoods of the identity in a real, semisimple Lie group G. More precisely, we consider ratios of the form

$$\delta_F(U) := \frac{\mu\left(\bigcap_{g\in F} \mathrm{Ad}_g U\right)}{\mu(U)}, \tag{1.2.1}$$

[2]This preprint is a joint effort together with Caspers, Krishnaswamy-Usha, and Janssens; the content given within this thesis is a contribution by Janssens and the author.

1

where $F \subset G$ is some subset, μ denotes the Haar measure, Ad denotes the adjoint action, and $U \subset G$ is a relatively compact neighborhood of the identity with nonzero Haar volume. For a neighborhood basis $\mathscr{U}$ of the identity in G we set $\delta_F(\mathscr{U}) := \liminf_{U \in \mathscr{U}} \delta_F(U)$, and δ_F denotes the supremum of $\delta_F(\mathscr{U})$ where $\mathscr{U}$ varies over all symmetric neighborhood bases of the identity in G.
The main theorem in Chapter 4 is a nontrivial lower bound on $\delta_{B_\rho^G}$, where

$$B_\rho^G := \{g \in G : \| \mathrm{Ad}_g \| \leq \rho\} \tag{1.2.2}$$

is a neighborhood of the identity in G, where the norm is the operator norm induced by a Hilbert space structure on $\mathfrak{g}$ based on an invariant bilinear form and a Cartan involution:

Theorem G. *If d is the maximal dimension of a nilpotent, adjoint orbit in the Lie algebra $\mathfrak{g} := \mathrm{Lie}(G)$, then*

$$\delta_{B_\rho^G} \geq \rho^{-d/2}. \tag{1.2.3}$$

We derive this method by constructing a specific neighborhood basis $\{V_{\epsilon,R} \subset \mathfrak{g}\}_{\epsilon,R>0}$ of zero in the Lie algebra which exponentiates to a symmetric neighborhood of the identity in the Lie group G. The reason for the occurrence of nilpotent orbits is that if $B_\epsilon(0)$ is the nilpotent cone $\mathscr{N} \subset \mathfrak{g}$ lies at the center of every open, invariant neighborhood of zero, in the sense that if $B_\epsilon(0)$ is the ϵ-ball around zero with respect to the Hilbert space structure on $\mathfrak{g}$, then

$$\bigcap_{\epsilon>0} \mathrm{Ad}_G B_\epsilon(0) = \mathscr{N}. \tag{1.2.4}$$

Theorem G then arises by employing certain orbital limit formulas due to Barbasch, Harris, and Vogan [BV80, Har12] and by employing analytic tools and integration formulas on Lie algebras due to Harish-Chandra and Varadarajan [HC57, HC65, Var77].
Not only does Theorem G provide geometric insight into the scaling behavior of neighborhood systems of G under the adjoint action, but it is also of interest in the setting of harmonic analysis, noncommutative L^p-spaces and Fourier multipliers on Lie groups.
We will explain in the beginning of Chapter 4 very briefly the ideas on how to make this connection and direct the interested reader towards [CJKUM22] for details.

1.2.2. EXPECTATION VALUES OF POLYNOMIALS AND MOMENTS ON COMPACT LIE GROUPS

Finally, in Chapter 5, which is based on the preprint [DM22] together with Tobias Diez, we propose a general framework in which to work with expectation values of polynomials of matrix coefficients based on representation theory and a simple integration by parts formula. Such polynomials have ubiquitous applications and take a core role in lattice gauge theory (LGT) [Wei78], which is the original motivation to our study. Specifically, we consider expectation values of *Wilson loops*. If we fix a compact Lie group

G and a finite-dimensional representation (ρ, V) of G, then a Wilson loop is, briefly, a function on finitely many copies of a Lie group G of the form

$$W : G \times \cdots \times G \to \mathbb{C}, \quad W(g_1, \ldots, g_r) := \mathrm{tr}(\rho(g_{i_1})^{\pm 1} \ldots \rho(g_{i_s})^{\pm 1}), \tag{1.2.5}$$

with certain choices of a subset $\{i_1, \ldots, i_s\} \subset \{1, \ldots, r\}$ and signs ± 1 in the exponents. We will largely restrict ourselves to *single-argument* Wilson loops, where we keep all but one argument fixed:

$$W : G \to \mathbb{C}, \quad W(g) := \mathrm{tr}(c_1 \rho(g)^{\pm 1} c_2 \rho(g)^{\pm 1} \ldots c_s \rho(g)^{\pm 1}), \quad c_1, \ldots, c_s \in G. \tag{1.2.6}$$

In LGT, an important problem is the calculation of expectation values of products of Wilson loops under a certain measure called the *Wilson action*. Performing analytic calculations in this setting, however, is one of the most difficult challenges of LGT, and as such, a lot of effort is spent on the development of techniques to simplify this problem. In [Cha19], a so-called master loop equation is developed for $G = \mathrm{SO}(N)$. This equation represents the expectation value of a product of Wilson loops as the expectation value of a product of different Wilson loops. This is done in order to carrying out the *'t Hooft limit* $N \to \infty$ [tH74], in which the master loop equation simplifies and allows an inductive calculation of the expectation values. They arrive at this equation through an application of what they call *Stein's method* and long explicit calculations with matrix coefficients in $\mathrm{SO}(N)$.
In Chapter 5, we show that there is a simple, representation-theoretic principle underlying this master loop equation. For G an arbitrary compact Lie group with bi-invariant Riemannian metric $\langle \cdot, \cdot \rangle$, an elementary application of integration by parts gives rise to formulas of the type

$$\int_G \Delta\rho(g)_{ij} \rho(g)_{kl} \, \mathrm{d}g = -\int_G \langle d\rho(g)_{ij}, d\rho(g)_{kl} \rangle \, \mathrm{d}g, \tag{1.2.7}$$

where $\mathrm{d}g$ denotes the Haar measure and Δ is the Laplacian associated to the Riemannian metric. If the representation is irreducible, the Peter–Weyl theorem tells us that the matrix coefficients are eigenfunctions of the Laplacian, hence the left-hand side of (1.2.7) is directly related to the desired expectation value of polynomials of matrix coefficients, whereas the right-hand side can be expanded further using the *completeness relation*, represented by tensor K on the Lie algebra $\mathfrak{g} := \mathrm{Lie}(G)$ of the form

$$K_{ijkl} := \sum_a \rho(\xi)^a_{ij} \rho(\xi)^a_{kl}, \tag{1.2.8}$$

with respect to an orthonormal basis $\{\xi_r : 1, \ldots, \dim \mathfrak{g}\}$ of $\mathfrak{g} \cong T_{\mathrm{id}} G$. This operator is closely related to the quadratic Casimir invariant associated to the representation ρ and its tensor square $\rho^{\otimes 2}$. As such, it is proportional to the identity on irreducible subrepresentations and thus easily determined.

1

It defines two operations on Wilson loops, the *merging* of Wilson loops $\mathcal{M}(W_{l_1}, W_{l_2})$ in which two copies of G in different loops are contracted with one another, and the *twisting* of a Wilson loop $\mathcal{T}(W)$, which is essentially a merging of two copies of G within the same loop. We delay the precise formulas to Chapter 5 and present the key theorem of the chapter.

Theorem H. *Let G be a compact Lie group equipped with a probability density ν, $\rho : G \to V$ an irreducible, finite-dimensional representation and $W_{l_1}, \dots, W_{l_q} : G \to \mathbb{C}$ a collection of single-argument Wilson loops. Let $\lambda \in \mathbb{C}$ be the eigenvalue of the Casimir $\rho(C)$, and denote the number of factors of g or g^{-1} in the canonical representation of the Wilson loop W_{l_r} by n_r. Then we have*

$$\begin{aligned}\lambda \sum_{r=1}^{q} n_r \cdot \int_G W_{l_1} \cdots W_{l_q} \, \nu \, \mathrm{d}g = \\ & - 2 \sum_{\substack{r,s=1 \\ r<s}}^{q} \int_G \mathcal{M}(W_{l_r}, W_{l_s}) \cdot W_{l_1} \cdots \widehat{W}_{l_r} \cdots \widehat{W}_{l_s} \cdots W_{l_q} \, \nu \, \mathrm{d}g \\ & - \sum_{r=1}^{q} \int_G \mathcal{T}(W_{l_r}) \cdot W_{l_1} \cdots \widehat{W}_{l_r} \cdots W_{l_q} \, \nu \, \mathrm{d}g \\ & + \int_G W_{l_1} \cdots W_{l_q} \, \Delta\nu \, \mathrm{d}g. \end{aligned} \tag{1.2.9}$$

The proof of this is a simple twofold application of integration by parts and an appeal to the Peter–Weyl theorem, but nonetheless this equation has far-reaching implications. For one, this principle essentially replicates the master loop formula for the defining representation of $G = \mathrm{SO}(N)$ of the previously mentioned [Cha19] and for the defining representation of $G = \mathrm{SU}(N)$ from the related preprint [Jaf16] through a simple proof which is, besides determining the tensor K, almost free of explicit calculations. We explicitly show how to construct such master loop equations for the classical examples of compact Lie groups $G = \mathrm{SO}(N)$, $\mathrm{Sp}(N)$, $\mathrm{SU}(N)$, $\mathrm{U}(N)$, and also the exceptional Lie group G_2. Beyond this, our method has far-reaching generelizations with respect to other measures. We generalize results of Lévy and Dahlqvist about the large-time behavior of the expectation value of moments of Riemannian Brownian motion [Lé08, Dah17], and we provide a method for calculating such moments in the Haar measure, also called *Weingarten functions* [Col03, CS06], based purely on an understanding of the tensor invariants $(V^{\otimes n} \otimes (V^*)^{\otimes n'})^G$, which are well-explored for many Lie groups. We apply this last method explicitly to $G = G_2$ to construct some novel integral formulas.

REFERENCES

[BdBW13] Pedro Boavida de Brito and Michael Weiss. Manifold calculus and homotopy sheaves. *Homology Homotopy Appl.*, 15(2):361–383, 2013.

[Bot73] R. Bott. Lectures on Gelfand-Fuks cohomology and characteristic classes. *Notes by M. Mostow and J. Perchik, New Mexico State University*, 1973.

[BS77] R. Bott and G. Segal. The cohomology of the vector fields on a manifold. *Topology*, 16(4):285–298, 1977.

[BV80] Dan Barbasch and David A. Vogan, Jr. The local structure of characters. *J. Functional Analysis*, 37(1):27–55, 1980.

[CG17] Kevin Costello and Owen Gwilliam. *Factorization algebras in quantum field theory. Vol. 1*, volume 31 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2017.

[Cha19] Sourav Chatterjee. Rigorous solution of strongly coupled $SO(N)$ lattice gauge theory in the large N limit. *Comm. Math. Phys.*, 366(1):203–268, 2019.

[CJKUM22] Martijn Caspers, Bas Janssens, Amudhan Krishnaswamy-Usha, and Lukas Miaskiwskyi. Local and multilinear noncommutative de Leeuw theorems. *arXiv preprint arXiv:2201.10400*, 2022.

[Col03] Benoit Collins. Moments and cumulants of polynomial random variables on unitary groups, the Itzykson-Zuber integral, and free probability. *Int. Math. Res. Not.*, (17):953–982, 2003.

[Cor05] G. Cortiñas. Cyclic homology of h-unital (pro-) algebras, Lie algebra homology of matrices, and a paper of Hanlon's. *arXiv preprint math/0504148*, 2005.

[CS06] Benoît Collins and Piotr Śniady. Integration with respect to the Haar measure on unitary, orthogonal and symplectic group. *Comm. Math. Phys.*, 264(3):773–795, 2006.

[Dah17] Antoine Dahlqvist. Integration formulas for Brownian motion on classical compact Lie groups. *Ann. Inst. Henri Poincaré Probab. Stat.*, 53(4):1971–1990, 2017.

[DM22] Tobias Diez and Lukas Miaskiwskyi. Expectation values of polynomials and moments on general compact lie groups. *arXiv preprint arXiv:2203.11607*, 2022.

[Fuk86] D. B. Fuks. *Cohomology of infinite-dimensional Lie algebras*. Contemporary Soviet Mathematics. Consultants Bureau, New York, 1986. Translated from the Russian by A. B. Sosinskiĭ.

1

1

[GF69] I. M. Gel'fand and D. B. Fuks. Cohomologies of the Lie algebra of tangent vector fields of a smooth manifold. *Funkcional. Anal. i Priložen.*, 3(3):32–52, 1969.

[GF70a] I. M. Gel'fand and D. B. Fuks. Cohomologies of the Lie algebra of formal vector fields. *Izv. Akad. Nauk SSSR Ser. Mat.*, 34:322–337, 1970.

[GF70b] I. M. Gel'fand and D. B. Fuks. Cohomologies of the Lie algebra of tangent vector fields of a smooth manifold. II. *Funkcional. Anal. i Priložen.*, 4(2):23–31, 1970.

[Gro95] A. Grothendieck. Produits tensoriels topologiques et espaces nucléaires. In *Séminaire Bourbaki, Vol. 2,* pages Exp. No. 69, 193–200. Soc. Math. France, Paris, 1995.

[Han88] Phil Hanlon. On the complete $\mathrm{GL}(n,\mathbf{C})$-decomposition of the stable cohomology of $\mathrm{gl}_n(A)$. *Trans. Amer. Math. Soc.*, 308(1):209–225, 1988.

[Har12] Benjamin Harris. Tempered representations and nilpotent orbits. *Represent. Theory*, 16:610–619, 2012.

[HC57] Harish-Chandra. Fourier transforms on a semisimple Lie algebra. II. *Amer. J. Math.*, 79:653–686, 1957.

[HC65] Harish-Chandra. Invariant eigendistributions on a semisimple Lie group. *Trans. Amer. Math. Soc.*, 119:457–508, 1965.

[HK18] B. Hennion and M. Kapranov. Gelfand-Fuchs cohomology in algebraic geometry and factorization algebras. *arXiv preprint arXiv:1811.05032*, 2018.

[Jaf16] Jafar Jafarov. Wilson loop expectations in $SU(N)$ lattice gauge theory. *arXiv preprint arXiv:1610.03821*, 2016.

[JV16] Bas Janssens and Cornelia Vizman. Universal central extension of the Lie algebra of Hamiltonian vector fields. *Int. Math. Res. Not. IMRN*, (16):4996–5047, 2016.

[JV18] Bas Janssens and Cornelia Vizman. Integrability of central extensions of the Poisson Lie algebra via prequantization. *J. Symplectic Geom.*, 16(5):1351–1375, 2018.

[JW13] Bas Janssens and Christoph Wockel. Universal central extensions of gauge algebras and groups. *J. Reine Angew. Math.*, 682:129–139, 2013.

[KM97] Andreas Kriegl and Peter W. Michor. *The convenient setting of global analysis*, volume 53 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1997.

[Lod92] Jean-Louis Loday. *Cyclic homology*, volume 301 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1992. Appendix E by María O. Ronco.

[LQ84] Jean-Louis Loday and Daniel Quillen. Cyclic homology and the Lie algebra homology of matrices. *Comment. Math. Helv.*, 59(4):569–591, 1984.

[Lé08] Thierry Lévy. Schur-Weyl duality and the heat kernel measure on the unitary group. *Adv. Math.*, 218(2):537–575, 2008.

[Mai02] Peter Maier. Central extensions of topological current algebras. In *Geometry and analysis on finite- and infinite-dimensional Lie groups (Będlewo, 2000)*, volume 55 of *Banach Center Publ.*, pages 61–76. Polish Acad. Sci. Inst. Math., Warsaw, 2002.

[Mia22a] Lukas Miaskiwskyi. Continuous cohomology of gauge algebras and bornological Loday-Quillen-Tsygan theorems. *arXiv preprint arXiv:2206.08879*, 2022.

[Mia22b] Lukas Miaskiwskyi. A local-to-global analysis of Gelfand-Fuks cohomology. *arXiv preprint arXiv:2202.09666*, 2022.

[Sch71] Helmut H. Schaefer. *Topological vector spaces*. Graduate Texts in Mathematics, Vol. 3. Springer-Verlag, New York-Berlin, 1971. Third printing corrected.

[tH74] G. 't Hooft. A planar diagram theory for strong interactions. *Nuclear Physics B*, 72(3):461–473, April 1974.

[Tsy83] B. L. Tsygan. Homology of matrix Lie algebras over rings and the Hochschild homology. *Uspekhi Mat. Nauk*, 38(2(230)):217–218, 1983.

[Var77] V. S. Varadarajan. *Harmonic analysis on real reductive groups*. Lecture Notes in Mathematics, Vol. 576. Springer-Verlag, Berlin-New York, 1977.

[Vir70] Miguel Angel Virasoro. Subsidiary conditions and ghosts in dual-resonance models. *Physical Review D*, 1(10):2933, 1970.

[Wei78] Don Weingarten. Asymptotic behavior of group integrals in the limit of infinite rank. *J. Mathematical Phys.*, 19(5):999–1001, 1978.

I

Local-to-global principles for cohomology of infinite-dimensional Lie algebras

2

An Exposition to Gelfand–Fuks Cohomology

In this chapter, we lay out a detailed, easily accessible exposition on the continuous Chevalley-Eilenberg cohomology of formal vector fields, of vector fields on Euclidean space, and the construction of the Gelfand–Fuks spectral sequence that describes this cohomology for vector fields on a more general class of smooth manifolds. While the results in this section are well known, we employ a novel proof technique to construct the Gelfand–Fuks spectral sequence for diagonal cohomology that is not captured in the literature, using a generalization of good covers of a manifold from the theory of factorization algebras.

This chapter is based on the preprint [Mia22].

2.1. INTRODUCTION

Beginning around fifty years ago, a plethora of literature has been created to understand the continuous Chevalley–Eilenberg cohomology $\mathfrak{X}(M)$ of the vector fields on a smooth manifold M. This cohomology carries the name *Gelfand–Fuks cohomology*, in reference to the authors who opened the investigation of this subject with a series of highly novel papers [GF69, GF70b, GF70a]. Initially, it was hoped that this cohomology might contain invariants for the smooth structure of M, hence be a potential tool for a classification of smooth structures on a given topological manifold.

Unfortunately, these hopes were denied by a paper by Bott and Segal, which showed that the Gelfand–Fuks cohomology was isomorphic to the singular cohomology of a mapping space that can be functorially constructed from M, and from which no new invariants arise [BS77]. Regardless, these explorations brought with them a lot of applications, for example in the theory of foliations [Fuk73] or for the construction of the Virasoro algebra [Vir70]. Further, many related open problems are still being pursued, like the continuous cohomology of the Lie algebra of symplectic, Hamiltonian or divergence-free vector fields on symplectic/Riemannian manifolds [JV16, JV18].

The goal of this document is two-fold: For one, we want to present a novel proof technique, using so-called k*-good covers* from the theory of functor calculus and factorization algebras (cf. [BdBW13]), to construct well-known spectral sequences which calculate the Gelfand–Fuks cohomology of certain smooth manifolds, cf. [Fuk86, Theorem 2.4.1a, 2.4.1.b]. This approach is inspired by the treatment of Gelfand–Fuks cohomology in the framework of factorization algebras in a preprint by Kapranov and Hennion [HK18]. The idea is to use a local-to-global analysis to reconstruct the spectral sequence for the Gelfand–Fuks cohomology of the manifold from the Gelfand–Fuks cohomology on the local level. The use of k-good covers solves the problem that this reconstruction necessarily compares data between different Cartesian powers $M, M^2, M^3, \ldots$ of a manifold M. This is a natural problem to encounter here, since cochains for Gelfand–Fuks cohomology are maps on multiple copies of $\mathfrak{X}(M)$. An advantage of this approach over previous proofs is that it is easily generalizable to other situations: we show in Section 3 that the Chevalley–Eilenberg cohomology of *gauge algebras* can be treated similarly.

The other goal of this document is to lay out a streamlined, detailed, and relatively elementary path to the fundamental results of Gelfand–Fuks cohomology, accessible to any researcher with a solid understanding of homological algebra, sheaf theory and differential geometry. To this end, we largely follow the general strategies in [Fuk86, Bot73], filling in nontrivial details that have been left to the reader in the original literature, modernizing some of the language used, and replacing some of the arguments with ones which the author perceives as clearer. We make no claim to exhaustiveness: we restrict ourselves to Gelfand–Fuks cohomology with trivial coefficients, and direct the reader to [Tsu81] for an overview of the study of other coefficient modules.

We begin in Section 2.2 with a study of continuous cohomology of the Lie algebra of formal vector fields, i.e. vector fields whose coefficient functions are formal power series. They represent the infinitesimal counterpart of $\mathfrak{X}(M)$ and their cohomology is calculated using a spectral sequence over which one can get full control. This section is largely a review of [GF70a]. In Section 2.3, we tie the cohomology of formal vector fields to the Gelfand–Fuks cohomology of Euclidean space, which may itself be understood as the local counterpart to Gelfand–Fuks cohomology. This section is a review of [Bot73]. The Section 2.4, we examine the transformation behavior of Gelfand–Fuks cohomology on Euclidean space under diffeomorphisms. The proofs and formulations are original, though the results are implicitly used in the literature. This prepares the local-to-global analysis of the Gelfand–Fuks cohomology on an arbitrary smooth manifold, which we carry out in Section 2.5. We give a variation of the well-known spectral sequences that calculate Gelfand–Fuks cohomology for a class of orientable, smooth manifolds. For the sake of completion, we explain how it allows a full calculation of the Gelfand–Fuks cohomology of the circle S^1 and may be used to make certain general statements about finite-dimensionality of the Gelfand–Fuks cohomology.

2.2. THE LIE ALGEBRA OF FORMAL VECTOR FIELDS

In this section, we mainly elaborate on the methods given in [Fuk86, Chapter 2.2] and [GF70a] to analyze the Lie algebra of formal vector fields, an infinitesimal version of the Lie algebra of vector fields on a smooth manifold. We also use methods from [GS73, Corollary 1] to analyze stable Chevalley–Eilenberg cohomology of this space.

2.2.1. DEFINITION AND FIRST PROPERTIES

Definition 2.2.1 (Formal vector fields)**.** Let $n \in \mathbb{N}$. We define the *Lie algebra of formal vector fields* W_n to be equal to the topological Lie algebra

$$W_n := \mathbb{R}[\![x_1,\dots,x_n]\!] \otimes \mathbb{R}^n. \tag{2.2.1}$$

Its topology is induced by the projective limit topology of $\mathbb{R}[\![x_1,\dots,x_n]\!]$ and its Lie bracket is given by

$$[f\partial_i, g\partial_j] := f\frac{\partial g}{\partial x_i}\cdot\partial_j - g\frac{\partial f}{\partial x_j}\cdot\partial_i, \quad f,g \in \mathbb{R}[\![x_1,\dots,x_n]\!]. \tag{2.2.2}$$

Remark 2.2.2. There is a more geometric definition of W_n which we will use in Section 2.3, as the space of infinity-jets $J^\infty_p\mathfrak{X}(\mathbb{R}^n)$ of vector fields at an arbitrary point $p \in \mathbb{R}^n$. Any choice of local frame around p induces a continuous Lie algebra isomorphism

$$J^\infty_p\mathfrak{X}(\mathbb{R}^n) \cong W_n. \tag{2.2.3}$$

We first examine the structure of W_n.

2

Definition 2.2.3. The element $E := \sum_{i=1}^{n} x_i \partial_i \in \mathfrak{g}_0$ is the *Euler vector field* in W_n. The eigenspaces

$$\mathfrak{g}_k = \{X \in W_n : [E, X] = k \cdot X\} \tag{2.2.4}$$

give $W_n = \widehat{\bigoplus}_{k \geq \mathbb{Z}} \mathfrak{g}_k$ as the completion of a graded Lie algebra. Elements of $\mathfrak{g}_k$ are called *homogeneous (of degree k)*.

More explicitly, we find for all $k \in \mathbb{Z}$

$$\mathfrak{g}_k = \left\{ \sum_{i=1}^{n} p_i \partial_i \in W_n : p_i \text{ homogeneous polynomials of degree } k+1 \right\}. \tag{2.2.5}$$

In particular, $\mathfrak{g}_k = 0$ if $k < -1$, and in low orders, we have the following Lie algebra isomorphisms:

$$\mathfrak{g}_{-1} = \operatorname{span}\{\partial_i : i = 1, \ldots, n\} \cong \mathbb{R}^n, \tag{2.2.6}$$

$$\mathfrak{g}_0 = \operatorname{span}\left\{x_i \partial_j : i, j = 1, \ldots, n\right\} \cong \mathfrak{gl}_n(\mathbb{R}). \tag{2.2.7}$$

Definition 2.2.4. Let $\mathfrak{g}$ be a topological Lie algebra. Its *continuous Chevalley–Eilenberg cohomology* is the cohomology of the cochain complex

$$C^\bullet(\mathfrak{g}) := \bigoplus_{k \geq 0} C^k(\mathfrak{g}). \tag{2.2.8}$$

By $C^k(\mathfrak{g})$ we denote the space of multilinear, skew-symmetric, continuous maps

$$c : \mathfrak{g}^k \to \mathbb{R}, \tag{2.2.9}$$

and the differential $d : C^\bullet(\mathfrak{g}) \to C^{\bullet+1}(\mathfrak{g})$ of the complex is[1]

$$dc(X_1, \ldots, X_{k+1}) = \sum_{1 \leq i < j \leq k+1} (-1)^{i+j-1} c([X_i, X_j], X_1, \ldots, X_{k+1}), \tag{2.2.10}$$

for all $X_1, \ldots, X_{k+1} \in \mathfrak{g}$.
The space $C^\bullet(\mathfrak{g})$ assumes the structure of a differential graded algebra with the wedge product

$$\begin{aligned}(c_1 \wedge c_2)(X_1, \ldots, X_{k+l}) :=& \\ \frac{1}{k!l!} \sum_{\sigma \in \Sigma_{k+l}} &\operatorname{sign}(\sigma) c_1(X_{\sigma(1)}, \ldots, X_{\sigma(k)}) c_2(X_{\sigma(k+1)}, \ldots, X_{\sigma(k+l)})\end{aligned} \tag{2.2.11}$$

for $c_1 \in C^k(\mathfrak{g}), c_2 \in C^l(\mathfrak{g}), X_1, \ldots, X_{k+l} \in \mathfrak{g}$.

[1]The definition of this differential is not uniform throughout the literature and may deviate from our choice by a global sign. The one we give here is found, for example, in [Fuk86].

Remark 2.2.5. If $\mathfrak{g}$ is finite-dimensional, the continuity assumption for cochains in $C^\bullet(\mathfrak{g})$ is redundant. If $\mathfrak{g} = W_n$ with its projective topology, then the continuity assumption on $c \in C^\bullet(W_n)$ just means that there is a $k \in \mathbb{Z}$ so that $c(X, \cdot, \dots, \cdot) = 0$ for all X with $\deg X > k$. In particular, c is only nonzero on a finite-dimensional subspace of $\Lambda^\bullet W_n$.

Recall the following:

Definition 2.2.6. Let $\mathfrak{g}$ be a Lie algebra, $Y \in \mathfrak{g}$ and $c \in C^k(\mathfrak{g})$ for some $k \geq 0$.

i) Denote the natural Lie algebra action of an element Y on $C^k(\mathfrak{g})$ by $Y \cdot c$; the formula is given for $Y, X_1, \dots, X_k \in \mathfrak{g}$ by

$$(Y \cdot c)(X_1, \dots, X_k) := -\sum_{i=1}^{k} c(X_1, \dots, [Y, X_i], \dots, X_k), \tag{2.2.12}$$

and $Y \cdot c = 0$ if $c \in C^0(\mathfrak{g})$.

ii) Denote by $Y \lrcorner c \in C^{k-1}(\mathfrak{g})$ the *interior product of c with Y*, which is defined via

$$(Y \lrcorner c)(X_1, \dots, X_{k-1}) = c(Y, X_1, \dots, X_{k-1}) \tag{2.2.13}$$

and $Y \lrcorner c = 0$ if $c \in C^0(\mathfrak{g})$.

A straightforward calculation yields the following homotopy relation:

Lemma 2.2.7. *Let $\mathfrak{g}$ be a Lie algebra, $c \in C^\bullet(\mathfrak{g})$ and $Y \in \mathfrak{g}$. Then we have the following chain homotopy formula:*

$$d(Y \lrcorner c) + Y \lrcorner dc = -Y \cdot c \tag{2.2.14}$$

A well-known corollary of the previous statement is:

Corollary 2.2.8. *The action of a Lie algebra $\mathfrak{g}$ on its cochains $C^\bullet(\mathfrak{g})$ commutes with the Chevalley–Eilenberg differential, and the induced action on $H^\bullet(\mathfrak{g})$ is trivial.*

Using the grading of W_n induced by the Euler vector field E, we can also define a grading of the cochains:

Definition 2.2.9. Let $r \in \mathbb{Z}$ and $k \geq 0$ an integer. Define

$$C^k_{(r)}(W_n) := \{c \in C^k(W_n) : E \cdot c = -r \cdot c\}. \tag{2.2.15}$$

Remark 2.2.10. More explicitly, $c \in C^k_{(r)}(W_n)$ if and only if we have for all homogeneous formal vector fields $X_1, \dots, X_k \in W_n$

$$\sum_{i=1}^{k} \deg X_i \neq r \implies c(X_1, \dots, X_k) = 0. \tag{2.2.16}$$

2

Proposition 2.2.11 ([Fuk86], Section 1.5 and 2.2)**.** *The spaces $C^\bullet_{(r)}(W_n)$ fulfill the following properties:*

i) We have $C^\bullet(W_n) = \bigoplus_{r\in\mathbb{Z}} C^\bullet_{(r)}(W_n)$.

ii) For all $r \in \mathbb{Z}$, the spaces $C^\bullet_{(r)}(W_n)$ are subcomplexes of $C^\bullet(W_n)$.

iii) For all $r, s \in \mathbb{Z}$ we have

$$C^\bullet_{(r)}(W_n) \wedge C^\bullet_{(s)}(W_n) \subset C^\bullet_{(r+s)}(W_n). \tag{2.2.17}$$

iv) If $r < -k$, then $C^k_{(r)}(W_n) = 0$.

v) The inclusion $C^\bullet_{(0)}(W_n) \subset C^\bullet(W_n)$ induces an algebra isomorphism

$$H^\bullet(C^\bullet_{(0)}(W_n)) \cong H^\bullet(W_n). \tag{2.2.18}$$

Proof. *i)* The direct sum decomposition follows since every $c \in C^k(W_n)$ is zero on homogeneous vector fields of sufficiently high degree, and its evaluation on any collection $X_1, \dots, X_k \in W_n$ can be uniquely decomposed into summands of homogeneous vector fields.
ii) This follows since the action of E commutes with the Lie algebra differential by Lemma 2.2.7.
iii) This is due to

$$E \cdot (c_1 \wedge c_2) = (E \cdot c_1) \wedge c_2 + c_1 \wedge (E \cdot c_2) \quad \forall c_1, c_2 \in C^\bullet(W_n). \tag{2.2.19}$$

iv) Due to the pidgeonhole principle, any collection of k elements in W_n whose degrees sum up to a value smaller than $-k$ must have an element with degree smaller -1. Such an element is necessarily zero, which shows the statement.
v) Lemma 2.2.7 shows that for all $c \in C^\bullet_{(r)}(W_n)$ we have

$$d(E \lrcorner c) + E \lrcorner (dc) = -r \cdot c. \tag{2.2.20}$$

Thus, for $r \neq 0$, the map $-\frac{1}{r}(E \lrcorner \cdot)$ defines a chain homotopy between the identity and zero for the cochain complex $C^\bullet_{(r)}(W_n)$, and hence $H^\bullet(C^\bullet_{(r)}(W_n)) = 0$. We conclude that all cohomology classes of $C^\bullet(W_n)$ admit a representative fully contained in $C^\bullet_{(0)}(W_n)$. This shows that the inclusion induces an isomorphism of vector spaces. By statement iii) $C^\bullet_{(0)}(W_n)$ is a subalgebra of $C^\bullet(W_n)$ with respect to the wedge product, hence the inclusion induces an algebra isomorphism on cohomology. □

In the following we will write $H^\bullet_{(r)}(W_n) := H^\bullet(C^k_{(r)}(W_n))$.

2.2.2. STABLE COHOMOLOGY OF FORMAL VECTOR FIELDS

We first focus on certain low-dimensional cohomology, the so-called *stable cohomology* of W_n, due to Guillemin and Shnider. They prove in [GS73, Corollary 1] that $H^k(W_n)$ is trivial in dimension $k = 1,\dots,n$. Note that their paper makes much more general statements, in particular about stable cohomology of formal Lie algebras corresponding to other classical vector field Lie algebras, e.g. formal Hamiltonian and divergence-free vector fields.

Definition 2.2.12. Define for all $r \in \mathbb{Z}$,

$$\partial C^\bullet_{(r)}(W_n) := \{\partial_i \cdot c \in C^\bullet_{(r+1)}(W_n) : c \in C^\bullet_{(r)}(W_n)\}, \tag{2.2.21}$$

and

$$\partial C^\bullet(W_n) := \bigoplus_{r\in\mathbb{Z}} \partial C^\bullet_{(r)}(W_n). \tag{2.2.22}$$

Recall that $\partial_i \cdot c$ denotes the action of $\partial_i \in \mathfrak{g}_{-1}$ on the cochain c (see Definition 2.2.6).

By Corollary 2.2.8, the Lie algebra action of $\mathfrak{g}$ on $C^\bullet(W_n)$ commutes with the Chevalley-Eilenberg differential, and thus:

Lemma 2.2.13. *For all $r \in \mathbb{Z}$, the space $\partial C^\bullet_{(r)}(W_n)$ is a subcomplex of $C^\bullet_{(r+1)}(W_n)$.*

We need one more preparing definition, since the degree zero component of cochain complexes is often troublesome.

Definition 2.2.14 (Reduced Complex)**.** If $C^\bullet$ is a cochain complex, define the *reduced complex* $\tilde{C}^\bullet$ as

$$\tilde{C}^0 = 0, \quad \tilde{C}^k := C^k \quad \forall k \geq 1, \tag{2.2.23}$$

equipped with the inherited differential from $C^\bullet$.
We denote by $\tilde{H}^\bullet$ the cohomology of the reduced complex.

The aim of this section is the construction of a Koszul complex relating the complexes $C^\bullet_{(r)}(W_n)$ for different values of r.

Proposition 2.2.15. *There exists an exact sequence of cochain complexes*

$$\begin{aligned} 0 \to \tilde{C}^\bullet_{(0)}(W_n) \otimes \Lambda^n \mathfrak{g}_{-1} \to \tilde{C}^\bullet_{(1)}(W_n) \otimes \Lambda^{n-1} \mathfrak{g}_{-1} \to \dots \\ \to \tilde{C}^\bullet_{(n)}(W_n) \otimes \Lambda^0 \mathfrak{g}_{-1} \to \tilde{C}^\bullet_{(n)}(W_n)/\partial \tilde{C}^\bullet_{(n-1)}(W_n) \to 0, \end{aligned} \tag{2.2.24}$$

where the differentials in every term are induced by the Chevalley–Eilenberg differential of $C^\bullet(W_n)$.

2

Proof. Write $V := \mathfrak{g}_{-1}$. Denote by $\vee$ the product of symmetric tensors, and define for all $r \le n-1$ the map

$$\begin{aligned} \sigma_r : S^\bullet V \otimes \Lambda^r V &\to S^\bullet V \otimes \Lambda^{r-1} V, \\ u \otimes (\partial_{i_1} \wedge \cdots \wedge \partial_{i_r}) &\mapsto \sum_{j=1}^{r} (-1)^j (\partial_{i_j} \vee u) \otimes (\partial_{i_1} \wedge \cdots \widehat{\partial_{i_j}} \cdots \wedge \partial_{i_r}) \quad \forall u \in S^\bullet V. \end{aligned} \tag{2.2.25}$$

These maps give rise to the well-known, acyclic Koszul complex

$$\begin{aligned} 0 \to S^\bullet V \otimes \Lambda^n V \to S^\bullet V \otimes \Lambda^{n-1} V \to \dots \\ \to S^\bullet V \otimes \Lambda^1 V \to S^\bullet V \otimes \Lambda^0 V \to \mathbb{R} \to 0. \end{aligned} \tag{2.2.26}$$

Taking the tensor product of the above exact sequence with $S^\bullet V$, and using the canonical isomorphism $S^\bullet(V^2) \cong S^\bullet(V) \otimes S^\bullet(V)$, we get the exact sequence

$$\begin{aligned} 0 \to S^\bullet V^2 \otimes \Lambda^n V \to S^\bullet V^2 \otimes \Lambda^{n-1} V \to \dots \\ \to S^\bullet V^2 \otimes \Lambda^0 V \to S^\bullet V \to 0, \end{aligned} \tag{2.2.27}$$

and inductively, if we denote by $V_\Delta \subset V^k$ the diagonal subspace, then we get an exact sequence

$$\begin{aligned} 0 \to S^\bullet V^k \otimes \Lambda^n V \to S^\bullet V^k \otimes \Lambda^{n-1} V \to \dots \\ \to S^\bullet V^k \otimes \Lambda^0 V \to S^\bullet(V^k / V_\Delta) \to 0. \end{aligned} \tag{2.2.28}$$

Let $k \ge 1$ and denote by Σ_k the permutation group in k elements. The tensor product $S^\bullet(V^k) \otimes (V^*)^{\otimes k}$ admits a Σ_k action by signed, simultaneous permutation of the tensor factors in $(V^*)^{\otimes k}$ and the direct summands in V^k. Taking invariants with respect to this action, we find

$$\left(S^\bullet(V^k) \otimes (V^*)^{\otimes k}\right)^{\Sigma_k} \cong C^k(W_n). \tag{2.2.29}$$

Thus, taking the tensor product of the complex (2.2.28) with $(V^*)^{\otimes k}$ and taking invariants under a finite group are exact functors of $\mathbb{R}$-vector spaces, and as such we get for every $k \ge 1$ an exact sequence

$$\begin{aligned} 0 \to C^k(W_n) \otimes \Lambda^n V \to C^k(W_n) \otimes \Lambda^{n-1} V \to \dots \\ \to C^k(W_n) \otimes \Lambda^0 V \to C^k(W_n)/\partial \tilde{C}^k(W_n) \to 0. \end{aligned} \tag{2.2.30}$$

The last nontrivial map in (2.2.30) is the canonical quotient projection, the others are:

$$\begin{aligned} C^k(W_n) \otimes \Lambda^r V &\to C^k(W_n) \otimes \Lambda^{r-1} V, \\ c \otimes (\partial_{i_1} \wedge \cdots \wedge \partial_{i_r}) &\mapsto \sum_{j=1}^{r} (-1)^j (\partial_{i_j} \cdot c) \otimes (\partial_{i_1} \wedge \cdots \widehat{\partial_{i_j}} \cdots \wedge \partial_{i_r}) \quad \forall c \in \tilde{C}^\bullet(W_n). \end{aligned} \tag{2.2.31}$$

By Corollary 2.2.8, the differential of $C^\bullet(W_n)$ commutes with the action of $S^\bullet V$, and by Lemma 2.2.13, the differential commutes with the projection

$$C^\bullet(W_n) \to C^\bullet(W_n)/\partial C^\bullet(W_n). \tag{2.2.32}$$

Hence, by taking the direct sum of the complexes (2.2.30) for all $k \geq 1$, we receive the following exact sequence of chain complexes

$$\begin{aligned} 0 \to \tilde{C}^\bullet(W_n) \otimes \Lambda^n V \to \tilde{C}^\bullet(W_n) \otimes \Lambda^{n-1} V \to \dots \\ \to \tilde{C}^\bullet(W_n) \otimes \Lambda^0 V \to \tilde{C}^\bullet(W_n)/\partial \tilde{C}^\bullet(W_n) \to 0. \end{aligned} \tag{2.2.33}$$

With respect to the grading $\tilde{C}^\bullet(W_n) = \bigoplus_r \tilde{C}^\bullet_{(r)}(W_n)$ the maps (2.2.31) restrict to maps on the graded components

$$C^\bullet_{(k)}(W_n) \otimes \Lambda^r(V) \to C^\bullet_{(k+1)}(W_n) \otimes \Lambda^{r-1}(V) \quad \forall k, r \in \mathbb{Z}, \tag{2.2.34}$$

and the canonical quotient projection $C^\bullet(W_n) \to C^\bullet(W_n)/\partial C^\bullet(W_n)$ restricts to

$$C^\bullet_{(k)}(W_n) \to C^\bullet_{(k)}(W_n)/\partial C^\bullet_{(k-1)}(W_n) \quad \forall k \in \mathbb{Z}. \tag{2.2.35}$$

Considering the graded component of (2.2.33) whose leftmost term is $C^\bullet_{(0)}(W_n) \otimes \Lambda^n V$ yields the desired statement. □

Remark 2.2.16. The construction of this sequence in [GS73, Theorem 1] is carried out differently. A detailed proof of their construction would require a study of Hopf algebra theory, which we do not want to carry out here: they implicitly use that if M is a free module over a Hopf algebra H, then so is $\Lambda^k_{\mathbb{K}} M$ with its induced, diagonal action for all $k > 0$. This can, for example, be shown with the methods of [DNR01, Chapter 7.2].

Proposition 2.2.17 ([GS73], Corollary 1)**.** *We have $H^k(W_n) = 0$ if $k = 1, \dots, n$.*

Proof. Consider the exact sequence (2.2.24). We have established that $\tilde{C}^\bullet_{(r)}(W_n)$ is an acyclic subcomplex whenever $r \neq 0$, and as such, all terms in the exact sequence are acyclic except for the leftmost and rightmost nontrivial ones. Now we can combine this with the exactness of (2.2.24) and either apply a straightforward diagram chase, or by view this sequence of cochain complexes as a double complex and comparing the associated spectral sequences. Both methods allow one to deduce the following isomorphisms:

$$H^k\left(\tilde{C}^\bullet_{(0)}(W_n)\right) \cong H^k\left(\tilde{C}^\bullet_{(0)}(W_n) \otimes \Lambda^n \mathfrak{g}_{-1}\right) \cong H^{k-n}\left(\tilde{C}^\bullet_{(n)}(W_n)/\partial \tilde{C}^\bullet_{(n-1)}(W_n)\right). \tag{2.2.36}$$

But, as a relative complex, the complex on the right-hand side is zero in all degrees smaller than 1, hence so is its cohomology. Hence, for all $k = 1, \dots, n$ we have

$$H^k(W_n) \cong H^k_{(0)}(W_n) = \tilde{H}^k_{(0)}(W_n) = 0. \tag{2.2.37}$$

□

2

2.2.3. A SPECTRAL SEQUENCE FOR THE COHOMOLOGY OF FORMAL VECTOR FIELDS

One can do even better than Proposition 2.2.17: We will formulate a spectral sequence due to Gelfand and Fuks [GF70a] which calculates the cohomology of W_n, and fully specify its differentials. In other words, we will be able to calculate the dimension of $H^\bullet(W_n)$ in every degree and for every $n \in \mathbb{N}$. The information from the previous section about low degree cohomology will aid us for the analysis of this spectral sequence. To this end, we begin with a short recollection of some representation and cohomology theory of $\mathfrak{gl}_n(\mathbb{R})$. We largely follow the proof in [Fuk86], with certain adaptations that will be indicated.

Theorem 2.2.18 ([KMS93], Theorem 24.4)**.** *Write $V = \mathbb{R}^n$, and consider V and V^* as $\mathfrak{gl}_n(\mathbb{R})$-modules with the defining representation and the dual thereof.*
For every $\sigma \in \Sigma_r$, define

$$\Psi_\sigma : V^{\otimes r} \otimes (V^*)^{\otimes r} \to \mathbb{R},$$
$$v_1 \otimes \cdots \otimes v_r \otimes \alpha_1 \otimes \cdots \otimes \alpha_r \mapsto \alpha_1(v_{\sigma(1)}) \ldots \alpha_r(v_{\sigma(r)}) \quad \forall \alpha_i \in V^*, v_i \in V.$$

Then $\{\Psi_\sigma\}_{\sigma\in\Sigma_r}$ is a spanning set for

$$\mathrm{Hom}_{\mathbb{R}}\left(V^{\otimes r} \otimes (V^*)^{\otimes r}, \mathbb{R}\right)^{\mathfrak{gl}_n(\mathbb{R})} \cong \left((V^*)^{\otimes r} \otimes V^{\otimes r}\right)^{\mathfrak{gl}_n(\mathbb{R})} \tag{2.2.38}$$

For $r \leq n$, the set $\{\Psi_\sigma\}$ is linearly independent. Further, if $r \neq s$, then

$$\mathrm{Hom}_{\mathbb{R}}\left(V^{\otimes r} \otimes (V^*)^{\otimes s}, \mathbb{R}\right)^{\mathfrak{gl}_n(\mathbb{R})} = 0. \tag{2.2.39}$$

Theorem 2.2.19 ([Fuk86], Theorem 2.1.1)**.** *The cohomology ring $H^\bullet(\mathfrak{gl}_n(\mathbb{R}))$ is isomorphic to the exterior algebra*

$$\Lambda^\bullet[\phi_1, \ldots, \phi_{2n-1}], \tag{2.2.40}$$

where the ϕ_i are generators in degree i. The inclusion $\mathfrak{gl}(n-1,\mathbb{R}) \to \mathfrak{gl}(n,\mathbb{R})$ induces a morphism

$$H^q(\mathfrak{gl}(n,\mathbb{R})) \to H^q(\mathfrak{gl}(n-1,\mathbb{R})) \tag{2.2.41}$$

which is an isomorphism for $q \leq 2n-3$.

Remark 2.2.20. Note that our reference states the above theorem in an erroneous way: They state the map induced by the inclusion has a one-dimensional kernel for $q = n$, which, for example, cannot be true when $n = 2$, since the second cohomology vanishes for all $\mathfrak{gl}(n,\mathbb{R})$. They also write that the inclusion only induces an isomorphism in degree $< n$, but their spectral sequence argument actually shows the above, stronger property (see also [Hat02, Corollary 4D.3].

Lemma 2.2.21. *Let $n \in \mathbb{N}$ be arbitrary, and consider the Hochschild–Serre spectral sequence $\{E_r^{p,q}, d_r\}$ of the Lie-algebra-subalgebra pair $\mathfrak{g}_0 \subset W_n$ in continuous cohomology (cf. Appendix B).*
We have for all $p, q \geq 0$

$$E_2^{p,q} = \begin{cases} H^q(\mathfrak{g}_0) \otimes (\Lambda^r \mathfrak{g}_{-1} \otimes \Lambda^r \mathfrak{g}_1)^{\mathfrak{g}_0} & \text{if } p \text{ even and } p = 2r, \\ 0 & \text{if } p \text{ odd.} \end{cases} \tag{2.2.42}$$

Proof. By definition of the Hochschild–Serre spectral sequence, the first page takes the following form:

$$\begin{aligned} E_1^{p,q} &= H^q\left(\mathfrak{g}_0; \Lambda^p (W_n/\mathfrak{g}_0)^*\right) = H^q\left(\mathfrak{g}_0; \Lambda^p \left(\bigoplus_{j\neq 0} \mathfrak{g}_j\right)^*\right) \\ &= \bigoplus_{p_{-1}+p_1+p_2+\cdots=p} H^q\left(\mathfrak{g}_0; \bigotimes_{j\neq 0} \Lambda^{p_j} \mathfrak{g}_j^*\right) \end{aligned} \tag{2.2.43}$$

Note that, as Lie algebras, $\mathfrak{g}_0 \subset W_n$ is isomorphic to $\mathfrak{gl}_n(\mathbb{R}) = \mathfrak{gl}(V)$ via

$$\sum_{i,j=1}^{n} a_{ij} x_i \partial_j \mapsto (a_{ij})_{1\leq i,j\leq n}. \tag{2.2.44}$$

By Weyl's complete reducibility theorem [Hal15, Theorem 10.9], all coefficient modules in (2.2.43) are completely reducible. Together with reducibility of the Lie algebra $\mathfrak{gl}_n(\mathbb{R})$ and [HS53, Theorem 10], we may reduce the coefficient space in the above cohomologies to the $\mathfrak{gl}_n(\mathbb{R})$-invariants. Hence,

$$\begin{aligned} E_1^{p,q} &= \bigoplus_{p_{-1}+p_1+p_2+\cdots=p} H^q\left(\mathfrak{g}_0; \left(\bigotimes_{j\neq 0} \Lambda^{p_j} \mathfrak{g}_j^*\right)^{\mathfrak{gl}(V)}\right) \\ &= H^q(\mathfrak{g}_0) \otimes \left(\bigoplus_{p_{-1}+p_1+p_2+\cdots=p} \left(\bigotimes_{j\neq 0} \Lambda^{p_j} \mathfrak{g}_j^*\right)^{\mathfrak{gl}(V)}\right). \end{aligned} \tag{2.2.45}$$

By definition, $\mathfrak{g}_j^* = (S^{j+1}V) \otimes V^*$ contains $j+1$ tensor factors that transform *covariantly* (i.e. copies of V) under the $\mathfrak{gl}(V)$ action, and one tensor factor that transforms *contravariantly* (i.e. a copy of V^*), hence $\Lambda^{p_j} \mathfrak{g}_j^*$ contains $j \cdot p_j$ covariant and p_j contravariant tensor factors. Hence

$$\bigotimes_{j\neq 0} \Lambda^{p_j} \mathfrak{g}_j^* \subset (V^*)^{\sum_{j\neq 0} p_j} \otimes V^{\sum_{j\neq 0}(j+1)p_j} \tag{2.2.46}$$

The last part of Theorem 2.2.18 then implies that the space of $\mathfrak{gl}(V)$-invariants of the space $\bigotimes_{j\neq 0} \Lambda^{p_j} \mathfrak{g}_j^*$ is only nonzero if

$$\sum_{j\neq 0} p_j = \sum_{j\neq 0} (j+1) p_j, \text{ or equivalently } p_{-1} = \sum_{j=1}^{\infty} j \cdot p_j \tag{2.2.47}$$

Simultaneously, again from Theorem 2.2.18, we know that $\mathfrak{gl}_n(\mathbb{R})$-invariants in a tensor module

$$V^{\otimes r} \otimes (V^*)^{\otimes r} \cong \mathrm{Hom}((V^*)^{\otimes r} \otimes V^{\otimes r}, \mathbb{R}) \tag{2.2.48}$$

can be described as the linear combinations of the functionals which contract all covariant indices with permutations of the contravariant indices. Correspondingly, the $\mathfrak{gl}(V)$ invariants in the subspaces $\bigotimes_{j\neq 0} \Lambda^{p_j} \mathfrak{g}_j^*$ are given by subjecting these functionals to the required (skew-)symmetrizations. Hence if

$$p_{-1} > \sum_{j=1}^{\infty} p_j, \tag{2.2.49}$$

then by the pidgeonhole principle, any invariant tensor contracts at least two contravariant factors belonging to $\Lambda^{p_{-1}}\mathfrak{g}_{-1}$ with two covariant factors, both belonging to a single copy of $\mathfrak{g}_j$ within $\Lambda^{p_j}\mathfrak{g}_j = \mathfrak{g}_j \wedge \cdots \wedge \mathfrak{g}_j$ for some $j \geq 1$. However, in such a contraction the contravariant factors would behave skew-symmetrically and the covariant ones symmetrically under permutation, hence their contraction is zero. Thus, we get the additional requirement

$$p_{-1} \leq \sum_{j=1}^{\infty} p_j \tag{2.2.50}$$

Combining (2.2.47) and (2.2.50) we end up with $p_k = 0$ for $k \geq 2$ and $p_{-1} = p_1 =: r$. This implies that $p = 2r$ is even whenever there are nontrivial invariants, and thus every other column in the page $E_1^{p,q}$ vanishes. Hence all differentials on the first page are trivial, and $E_1^{p,q} = E_2^{p,q}$. This concludes the proof. □

Since $H^q(\mathfrak{g}_0)$, $\Lambda^r\mathfrak{g}_{-1}$, and $\Lambda^r\mathfrak{g}_1$ are nonzero only for finitely many $q, r \geq 0$, and all involved spaces are finite-dimensional in every degree, we have:

Corollary 2.2.22. *For all $q \geq 0$, the continuous cohomology $H^q(W_n)$ is finite-dimensional, and $H^q(W_n) \neq 0$ only in finitely many degrees.*

Let us further analyze the invariant space $(\Lambda^r\mathfrak{g}_{-1} \otimes \Lambda^r\mathfrak{g}_1)^{\mathfrak{g}_0}$.

Lemma 2.2.23. *Let $\{E_2^{\bullet,\bullet}, d_r\}$ be the spectral sequence from Lemma 2.2.21 of the Lie-algebra-subalgebra pair $\mathfrak{g}_0 \subset W_n$.*
For all $r = 1, \dots, n$, there exist multiplicative generators $\Psi_{2r} \in E_2^{2r,0}$ so that

$$E_2^{\bullet,0} = \mathbb{R}[\Psi_2, \Psi_4 \dots, \Psi_{2n}]/\langle \Psi_{i_1} \dots \Psi_{i_k} : i_1 + \cdots + i_k > 2n\rangle. \tag{2.2.51}$$

Proof. By Lemma 2.2.21 we have

$$E_2^{\bullet,0} = \bigoplus_{r\geq 0} (\Lambda^r\mathfrak{g}_{-1} \otimes \Lambda^r\mathfrak{g}_1)^{\mathfrak{g}_0}. \tag{2.2.52}$$

By Theorem 2.2.18 the elements in the invariant space $(\Lambda^r\mathfrak{g}_{-1}\otimes\Lambda^r\mathfrak{g}_1)^{\mathfrak{g}_0}$ arise by taking, for every permutation $\sigma\in\Sigma_r$, the functional

$$\bigotimes^r V\times\bigotimes^r\left(V^*\otimes V^*\otimes V\right)\to\mathbb{R}, \tag{2.2.53}$$

with

$$\begin{aligned}(\alpha_1^1\otimes\ldots\otimes\alpha_r^1,(\beta_1^1\otimes\beta_1^2\otimes\alpha_{r+1}^2)\otimes\cdots\otimes(\beta_r^1\otimes\beta_r^2\otimes\alpha_{2r}^2))&\\ \mapsto\beta_1^1(\alpha_1^1)\ldots\beta_r^1(\alpha_r^1)\cdot\beta_1^2(\alpha_{\sigma(1)}^2)\ldots\beta_r^2(\alpha_{\sigma(r)}^2),\quad\forall\alpha_i\in V,\beta_i^1,\beta_i^2\in V^*,&\end{aligned} \tag{2.2.54}$$

skew-symmetrizing over the first r and the last r arguments, and symmetrizing over the exchange $\beta_i^1\leftrightarrow\beta_i^2$. Denote the arising functional by $\Psi_\sigma\in(\Lambda^r\mathfrak{g}_{-1}\otimes\Lambda^r\mathfrak{g}_1)^{\mathfrak{gl}(V)}$. From the skew-symmetry of the functionals in the first r and last r arguments, one deduces that

$$\Psi_\sigma=\Psi_{\tau\sigma\tau^{-1}}\quad\forall\sigma,\tau\in\Sigma_k. \tag{2.2.55}$$

Another straightforward calculation shows that for $\sigma\in\Sigma_r$ and $\tau\in\Sigma_l$ we have

$$\Psi_\sigma\wedge\Psi_\tau=\Psi_{\sigma\tau}\in\left(\Lambda^{r+l}\mathfrak{g}_{-1}\otimes\Lambda^{r+l}\mathfrak{g}_1\right)^{\mathfrak{gl}(V)}. \tag{2.2.56}$$

Because $\mathfrak{g}_{-1}$ is n-dimensional, $(\Lambda^r\mathfrak{g}_{-1}\otimes\Lambda^r\mathfrak{g}_1)^{\mathfrak{gl}(V)}=0$ if $r>n$, so in particular

$$\Psi_\sigma\wedge\Psi_\tau=0\text{ if }\sigma\in\Sigma_r,\tau\in\Sigma_l,r+l>n. \tag{2.2.57}$$

Denote by $\Psi_r\in(\Lambda^r\mathfrak{g}_{-1}\otimes\Lambda^r\mathfrak{g}_1)^{\mathfrak{gl}(V)}$ the Ψ_σ corresponding to an r-cycle $\sigma\in\Sigma_r$. This is well-defined, since by (2.2.55) the functional Ψ_σ only depends on the conjugacy class of σ, and all r-cycles are conjugate to one another.
Since every permutation can be uniquely (up to ordering) decomposed into a composition of cycles, (2.2.56) shows that $(\Lambda^r\mathfrak{g}_{-1}\otimes\Lambda^r\mathfrak{g}_1)^{\mathfrak{gl}(V)}$ is multiplicatively generated by the $\Psi_{2r}\in(\Lambda^r\mathfrak{g}_{-1}\otimes\Lambda^r\mathfrak{g}_1)^{\mathfrak{gl}(V)}$ for every $r=1,\ldots,n$. Hence every element in $E_2^{\bullet,0}$ is given as a unique (up to ordering) product of elements in $\{\Psi_2,\ldots,\Psi_{2n}\}$. Since all Ψ_σ for $\sigma\in\Sigma_r$ and $r\le n$ are nonzero, the only relation between these generators is that products $\Psi_{i_1}\ldots\Psi_{i_k}$ are zero if $i_1+\cdots+i_k>2n$, and the lemma is proven. □

To understand this spectral sequence further, we will need the Borel transgression theorem. To formulate it, let us first define some terminology.

Definition 2.2.24. Let $\{E_r^{p,q},d_r\}_{r\ge0}$ be a cohomological first-quadrant spectral sequence. Denote by $\kappa_r^{r+1}:\ker d_r\to E_{r+1}^{\bullet,\bullet}$ the natural quotient map from cocycles of the r-th page differential d_r to the $r+1$-th page, and

$$\kappa_r^s=\kappa_{s-1}^s\circ\cdots\circ\kappa_r^{r+1}\quad\forall s>r, \tag{2.2.58}$$

where the domain of κ_r^s is defined inductively as all the $c\in E_{r+1}^{\bullet,\bullet}$ in the domain of κ_r^{s-1} so that $\kappa_r^{s-1}c\in\ker d_{s-1}$. We call an element $c\in E_2^{p,0}$ *transgressive* if, for all r with $2\le r\le p$, we have that c is in the domain of κ_2^r.

Intuitively, the transgressive elements in $E_2^{p,0}$ are the ones which "survive" until the very last moment: Only the differential $d_{p+1} : E_{p+1}^{p,0} \to E_{p+1}^{0,p+1}$, also called the *transgression*, can map it to something nontrivial. By abuse of notation, we often denote an element in the domain of κ_r^s by the same symbol as its image under κ_r^s in the higher page $E_s^{\bullet,\bullet}$. The Borel transgression theorem was originally proven in [Bor53], but we cite a slightly stronger version from [MT91, Theorem 2.9].

Theorem 2.2.25 (Borel transgression theorem)**.** *Consider two finite-dimensional, graded vector spaces $B^\bullet := \bigoplus_{p\in\mathbb{N}_0} B^p$ and $F^\bullet := \bigoplus_{q\in\mathbb{N}_0} F^q$. Assume there are elements $x_i \in F^\bullet$ of odd degree such that*

$$\Lambda^\bullet[x_1,\ldots,x_l] \to F^\bullet \tag{2.2.59}$$

is bijective in degrees $\le N$ and injective in degree $N+1$. Let further $\{E_r^{p,q}, d_r\}_{r\ge 0}$ be a cohomological spectral sequence whose second page has the form

$$E_2^{p,q} = B^p \otimes F^q, \tag{2.2.60}$$

and which converges towards a graded vector space $H^\bullet$ with $H^k = 0$ if $0 < k \le N+2$. Then we can choose the generators x_i to be transgressive, and if $y_1,\ldots,y_l \in B^\bullet$ denotes a collection of elements with

$$d_{\deg x_i+1} x_i = y_i \quad i = 1,\ldots,l, \tag{2.2.61}$$

then the map

$$\mathbb{R}[y_1,\ldots,y_l] \to B^\bullet \tag{2.2.62}$$

is bijective for degrees $\le N$ and injective for degree $N+1$.

Using this, we can fully describe the desired spectral sequence:

Theorem 2.2.26 ([Fuk86], Theorem 2.2.4)**.** *Let $n \in \mathbb{N}$ be arbitrary, and consider the Hochschild-Serre spectral sequence $\{E_r^{p,q}, d_r\}$ of the pair $\mathfrak{g}_0 \subset W_n$ in continuous cohomology. Its second page takes the form*

$$\begin{aligned} E_2^{0,\bullet} &= \Lambda^\bullet[\phi_1,\phi_3,\ldots,\phi_{2n-1}], && (2.2.63)\\ E_2^{\bullet,0} &= \mathbb{R}[\Psi_2,\Psi_4\ldots,\Psi_{2n}]/\langle \Psi_{i_1}\ldots\Psi_{i_k} : i_1+\cdots+i_k > 2n\rangle, && (2.2.64)\\ E_2^{p,q} &= E_2^{p,0}\otimes E_2^{0,q}, && (2.2.65) \end{aligned}$$

and all differentials of the spectral sequence are fully specified on the generators by

$$d_{i+1}\phi_i = \Psi_{i+1} \quad i \in \{1,3,\ldots,2n-1\}. \tag{2.2.66}$$

Proof. The form of the second page follows from Theorem 2.2.19, Lemma 2.2.21, and Lemma 2.2.23. It remains to show the statement about the differentials. Consider the Hochschild–Serre spectral sequence for the Lie-algebra-subalgebra pair $(W_{3n}, \mathfrak{gl}_{3n}(\mathbb{R}))$. By Proposition 2.2.17, we know that $H^k(W_{3n}) = 0$ in degrees $k = 1, \ldots, 3n$, and up to degree $2n$, the zeroth column of the spectral sequence is equal to $\Lambda^\bullet[\phi_1, \phi_3, \ldots, \phi_{2n-1}]$, the ϕ_i being the generators of $H^\bullet(\mathfrak{gl}_{3n}(\mathbb{R}))$. Hence we can apply the Borel transgression theorem 2.2.25 with $N = 2n-1$, implying that generators $\tilde{\phi}_1, \ldots, \tilde{\phi}_{2n-1}$ of the zeroth column can be chosen so that

$$d_{2i}\tilde{\phi}_i = \Psi_{i+1}, \quad i \in \{1, 3, \ldots, 2n-1\}. \tag{2.2.67}$$

Now the inclusion $W_n \to W_{3n}$ induces a morphism from the Hochschild–Serre spectral sequence for the Lie-algebra-subalgebra pair $(W_{3n}, \mathfrak{gl}_{3n}(\mathbb{R}))$ to the spectral sequence for $(W_n, \mathfrak{gl}_n(\mathbb{R}))$. Under this morphism, the generators Ψ_i for W_{3n} restrict to equivalent ones for W_n by the explicit formula for them given in Lemma 2.2.23.
By functoriality of the Hochschild–Serre spectral sequence, the generators $\tilde{\phi}_i$ must then restrict to something nonzero in the space $E_{2i}^{0,2i-1}$ of the spectral sequence for W_n. By an inductive argument, this vector space is one-dimensional and generated by the generator ϕ_i, so the image of $\tilde{\phi}_i$ under this morphism must be a nonzero multiple of ϕ_i. Hence, up to a nonzero constant, we have in the spectral sequence for W_n that

$$d_{2i}\phi_i = \Psi_{i+1}, \quad i \in \{1, 3, \ldots, 2n-1\}. \tag{2.2.68}$$

This proves that all generators $\phi_1, \ldots, \phi_{2n-1}$ in the spectral sequence for W_n map as desired. Since the differential of the Hochschild–Serre spectral sequence is multiplicative and all pages are generated by the ϕ_i and the Ψ_i, this fully specifies the differential on every page. □

Remark 2.2.27. In [Fuk86], the above argument is carried out with the Lie-algebra--subalgebra pair $(W_{2n}, \mathfrak{gl}_{2n}(\mathbb{R}))$ rather than $(W_{3n}, \mathfrak{gl}_{3n}(\mathbb{R}))$, which would not fulfill the requirements of the version of the Borel transgression theorem we use here.

This allows one to fully calculate the dimensions of $H^\bullet(W_n)$ in all degrees and even offers some insight into the behavior of representatives of the cohomology classes. We are going to sumarize the most important properties of $H^\bullet(W_n)$ in the following corollary:

Corollary 2.2.28. *The space $H^k(W_n)$ is trivial when $1 \leq k \leq 2n$ or $k > n^2 + 2n$. The wedge product of two cohomology classes of positive degree in $H^\bullet(W_n)$ is zero.*

Proof. Any element in $E_2^{p,q}$ with $(p,q) \neq (0,0)$ is a linear combination of terms of the form

$$\phi_{i_1} \ldots \phi_{i_s} \Psi_{j_1}^{m_1} \ldots \Psi_{j_t}^{m_t}, \tag{2.2.69}$$

2

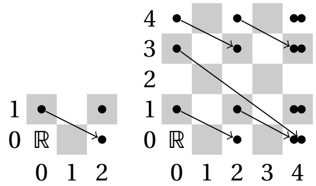

Figure 2.1: The spectral sequences for W_1 and W_2, with nonvanishing differentials indicated. Every dot represents one basis element of the term in the given position. The cohomology of W_1 is only nontrivial in degree 0 and 3, whereas the cohomology of W_2 is nontrivial in degree $0, 5, 7$ and 8, degree 5 and 8 having multiplicity 2.

where we have ordered the groups of indices so that $i_1 < \cdots < i_s$ and $j_1 < \cdots < j_t$. We further have

$$i_1 + \cdots + i_s = q, \quad m_1 j_1 + \cdots + m_t j_t = p, \tag{2.2.70}$$

and s and t are possibly zero, but not both at the same time. Theorem 2.2.26 shows:

i) If $s = 0$ or $i_1 > j_1$, then $\phi_{j_1-1}\phi_{i_1}\dots\phi_{i_s}\Psi_{j_1}^{m_1-1}\dots\Psi_{j_t}^{m_t}$ maps to the term (2.2.69) under the differential $d_{2(j_1-1)}$.

ii) If $t = 0$ or $i_1 < j_1$, then the term (2.2.69) maps to $\phi_{i_2}\dots\phi_{i_s}\Psi_{i_1+1}\Psi_{j_1}^{m_1}\dots\Psi_{j_t}^{m_t}$ under the differential d_{2i_1}.

Since necessarily i_1 is odd and j_1 is even, one of i) or ii) must hold. Let us show that under the assumptions of ii), and if $p \le n$ or $p + q \le 2n$, then the product $\Psi_{i_1+1}\Psi_{j_1}^{m_1}\dots\Psi_{j_t}^{m_t}$ is nonzero. This is equivalent to showing

$$i_1 + 1 + m_1 j_1 + \cdots + m_t j_t \le 2n. \tag{2.2.71}$$

If $t = 0$, then this is trivial, so assume from here on $t > 0$ and $i_1 < j_1$. If $p \le n$, then we know that

$$i_1 < j_1 \le m_1 j_1 + \cdots + m_t j_t = p, \tag{2.2.72}$$

thus

$$i_1 + 1 + m_1 j_1 + \cdots + m_t j_t \le 2p \le 2n. \tag{2.2.73}$$

On the other hand, assume $p + q \le 2n$. Note that $s = 1$ is never the case, since i_1 is always odd and q is always even. Hence assume $s > 1$, so that $i_1 + 1 \le i_1 + \cdots + i_s$. But then

$$i_1 + 1 + m_1 j_1 + \cdots + m_t j_t \le i_1 + \cdots + i_s + m_1 j_1 + \cdots + m_t j_t = q + p \le 2n. \tag{2.2.74}$$

With this, we have shown that $E_\infty^{p,q}$ is zero when $0 < p+q \le 2n$, implying that $H^k(W_n)$ vanishes in degree $0 < k \le 2n$. Further, we have shown that $E_\infty^{p,q}$ is zero when $p \le n$, so two cohomology classes of positive degree in $H^\bullet(W_n)$ correspond to two equivalence classes in some spaces $E_\infty^{p,q}, E_\infty^{p',q'}$ with $p, p' > n$. By multiplicativity of the Hochschild–Serre spectral sequence, their product must then correspond to an equivalence class in $E_\infty^{p+p',q+q'}$, which must be zero because $p + p' > 2n$. This proves the corollary. □

2.3. GELFAND–FUKS COHOMOLOGY ON EUCLIDEAN SPACE

In this section, we calculate the Gelfand–Fuks cohomology $H^\bullet(\mathfrak{X}(M))$ for $M = \mathbb{R}^n$. We follow an elaborate outline by Bott, see [Bot73]. This approach will allow us to easily extend our proof to the Gelfand–Fuks cohomology of finite disjoint unions $\mathbb{R}^n \sqcup \cdots \sqcup \mathbb{R}^n$ and also to certain *diagonal* cohomologies thereof, a concept which we introduce in Section 2.5.

2.3.1. DEFINITIONS AND CALCULATION

The Lie algebra of smooth vector fields $\mathfrak{X}(\mathbb{R}^n)$ on Euclidean space is a locally convex Lie algebra with respect to the standard Fréchet topology. We are interested in its continuous Chevalley–Eilenberg cohomology with respect to this topology. We can express vector fields in the canonical coordinates

$$\mathfrak{X}(\mathbb{R}^n) = \left\{ \sum_{i=1}^n f_i \partial_i : f_i \in C^\infty(\mathbb{R}^n) \right\} \tag{2.3.1}$$

Let us again identify some structures:

Definition 2.3.1. Let $t > 0$. We define the family of scaling operators $\{T_t\}_{t>0}$ as

$$T_t : \mathbb{R}^n \to \mathbb{R}^n, \quad x \mapsto tx. \tag{2.3.2}$$

We adopt the notation that if $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is a local diffeomorphism, then we denote by ϕ^* its pullback on vector fields $\mathfrak{X}(\mathbb{R}^n)$

$$\phi^*(X) := (d\phi)^{-1}(X \circ \phi). \tag{2.3.3}$$

We overload our notation and also write ϕ^* for the pullback on cochains $C^k(\mathfrak{X}(\mathbb{R}^n))$ for all $k \ge 0$:

$$(\phi^* c)(X_1, \ldots, X_k) := c(\phi^* X_1, \ldots, \phi^* X_k), \quad \forall k \ge 0. \tag{2.3.4}$$

Note that for $c \in C^0(\mathfrak{X}(\mathbb{R}^n))$ this amounts to $\phi^* c = c$.
For the scaling operators $\{T_t\}_{t>0}$, this translates to

$$T_t^* X = \frac{1}{t}(X \circ T_t) \quad \forall X \in \mathfrak{X}(\mathbb{R}^n),$$
$$(T_t^* c)(X_1, \ldots, X_k) := \frac{1}{t^k} c(X_1 \circ T_t, \ldots, X_k \circ T_t) \quad \forall c \in C^k(\mathfrak{X}(\mathbb{R}^n)).$$

2

Definition 2.3.2. We define the subspace $\mathfrak{X}_{\mathrm{pol}}(\mathbb{R}^n) \subset \mathfrak{X}(\mathbb{R}^n)$ of *polynomial vector fields*

$$\mathfrak{X}_{\mathrm{pol}}(\mathbb{R}^n) := \left\{ \sum_{i=1}^{n} f_i \partial_i : f_i \in \mathbb{R}[x_1, \ldots, x_n] \right\}. \tag{2.3.5}$$

It admits the structure of a graded Lie algebra

$$\mathfrak{X}_{\mathrm{pol}}(\mathbb{R}^n) = \bigoplus_{k \in \mathbb{Z}} P_k, \quad P_k := \{X \in \mathfrak{X}_{\mathrm{pol}}(\mathbb{R}^n) : T_t^* X = t^k X \quad \forall t > 0\}. \tag{2.3.6}$$

Elements of P_k are called *homogeneous* vector fields (of degree k).

Remark 2.3.3. Compare this to the grading $W_n = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_k$ on formal vector fields. The sets P_k are the images of the natural embeddings $\mathfrak{g}_k \to \mathfrak{X}(\mathbb{R}^n)$ that we get from considering finite formal vector fields in W_n as polynomial vector fields on $\mathbb{R}^n$.

Definition 2.3.4. For all $k \in \mathbb{Z}, q \geq 0$, define

$$F^k C^q(\mathfrak{X}(\mathbb{R}^n)) := \left\{ c \in C^q(\mathfrak{X}(\mathbb{R}^n)) : \lim_{t \to 0} t^{-k} T_t^* c(X_1, \ldots, X_q) \text{ exists } \forall X_i \in \mathfrak{X}(\mathbb{R}^n) \right\}. \tag{2.3.7}$$

Lemma 2.3.5. *The spaces $F^k C^\bullet(\mathfrak{X}(\mathbb{R}^n))$ for $k \in \mathbb{Z}$ fulfill the following properties:*

i) For every $k \in \mathbb{Z}$, the space $F^k C^\bullet(\mathfrak{X}(\mathbb{R}^n))$ is a subcomplex of $C^\bullet(\mathfrak{X}(\mathbb{R}^n))$.

ii) We have a descending chain

$$\cdots \subset F^k C^\bullet(\mathfrak{X}(\mathbb{R}^n)) \subset F^{k-1} C^\bullet(\mathfrak{X}(\mathbb{R}^n)) \subset F^{k-2} C^\bullet(\mathfrak{X}(\mathbb{R}^n)) \subset \ldots \tag{2.3.8}$$

iii) For all $k, l \in \mathbb{Z}$ we have

$$F^k C^\bullet(\mathfrak{X}(\mathbb{R}^n)) \wedge F^l C^\bullet(\mathfrak{X}(\mathbb{R}^n)) \subset F^{k+l} C^\bullet(\mathfrak{X}(\mathbb{R}^n)). \tag{2.3.9}$$

iv) For $k \leq -n$ we have $F^k C^\bullet(\mathfrak{X}(\mathbb{R}^n)) = C^\bullet(\mathfrak{X}(\mathbb{R}^n))$.

Summarizing, $\{F^k C^\bullet(\mathfrak{X}(\mathbb{R}^n))\}_{k \in \mathbb{Z}}$ constitutes a descending filtration of the Chevalley-Eilenberg complex $C^\bullet(\mathfrak{X}(\mathbb{R}^n))$ which is bounded from above.

Proof. *i)* Pulling back vector fields along local diffeomorphisms is a Lie algebra homomorphism, so

$$T_t^*[X, Y] = [T_t^* X, T_t^* Y] \quad \forall X, Y \in \mathfrak{X}(\mathbb{R}^n). \tag{2.3.10}$$

Hence, the pullback on cochains T_t^* commutes with the Lie algebra differential, so if the appropriate limits exist for a cochain c, they also do for dc. Thus the $F^k C^\bullet(\mathfrak{X}(\mathbb{R}^n))$ are indeed subcomplexes.
ii) This follows since if the limit $\lim_{t \to 0} t^{-k} f(t)$ exists for some function $t \mapsto f(t)$, so

does $\lim_{t\to 0} t^{-(k-1)} f(t) = 0$.
iii) The compatibility with the wedge product follows from

$$T_t^*(c_1 \wedge c_2) = T_t^* c_1 \wedge T_t^* c_2 \quad \forall c_1, c_2 \in C^\bullet(\mathfrak{X}(\mathbb{R}^n)). \tag{2.3.11}$$

iv) In degree zero the statement is clear, since

$$F^k C^0(\mathfrak{X}(\mathbb{R}^n)) = C^0(\mathfrak{X}(\mathbb{R}^n)) \quad \forall k \le 0. \tag{2.3.12}$$

Assume now $q > 0$. Fix $c \in C^q(\mathfrak{X}(\mathbb{R}^n))$ and $X_1, \dots, X_q \in \mathfrak{X}(\mathbb{R}^n)$. Applying the Hadamard lemma to every one of the X_i shows that there are vector fields $X_k^{(i)}$ for all $k = 1, \dots, q$ and $i = 1, \dots, n$ so that

$$X_k(x) = X_k(0) + \sum_{i=1}^n x_i X_k^{(i)}(x), \quad \forall x \in \mathbb{R}^n. \tag{2.3.13}$$

Then

$$T_t^* X_k(x) = \frac{1}{t} X_k(0) + \sum_{i=1}^n x_i X_k^{(i)}(tx) \quad \forall x \in \mathbb{R}^n. \tag{2.3.14}$$

Hence we can rewrite

$$T_t^* c(X_1, \dots, X_q) = c\left(\frac{1}{t} X_1(0) + \sum_{i=1}^n x_i X_1^{(i)}(tx), \dots, \frac{1}{t} X_q(0) + \sum_{i=1}^n x_i X_q^{(i)}(tx)\right). \tag{2.3.15}$$

Decomposing this expression using multilinearity of c, we find that all the terms whose order in t is lower than $-n$ have to vanish, since any set $\{X_{i_1}(0), \dots, X_{i_{n+1}}(0)\}$ is necessarily linearly dependent and c is skew-symmetric. Note also that on any compact set in $\mathbb{R}^n$, the vector fields $x \mapsto x_i X_k^{(i)}(tx)$ converge uniformly to the vector field $x \mapsto x_i X_k^{(i)}(0)$ for $t \to 0$, and the same holds for their derivatives. Combining the two previous facts, the continuity of c lets us conclude that the limit $\lim_{t\to 0} \frac{1}{t^{-n}} T_t^* c(X_1, \dots, X_q)$ exists. This proves the statement. □

The analysis at the end of the previous proof motivates a different characterization of the filtration:

Lemma 2.3.6. *Let $q > 0$. A cochain $c \in C^q(\mathfrak{X}(\mathbb{R}^n))$ lies in $F^k C^q(\mathfrak{X}(\mathbb{R}^n))$ if and only if for all homogeneous vector fields $X_1, \dots, X_q \in \mathfrak{X}_{\mathrm{pol}}(\mathbb{R}^n)$ we have*

$$\sum_{i=1}^q \deg X_i < k \implies c(X_1, \dots, X_k) = 0. \tag{2.3.16}$$

Proof. $\Rightarrow$: Let $c \in F^k C^q(\mathfrak{X}(\mathbb{R}^n))$, and let $X_1, \dots, X_q \in \mathfrak{X}_{\mathrm{pol}}(\mathbb{R}^n)$ be any homogeneous polynomial vector fields. Then

$$t^{-k} T_t^* c(X_1, \dots, X_q) = t^{\left(\sum_{i=1}^q \deg X_i\right) - k} c(X_1, \dots, X_q). \tag{2.3.17}$$

2

If $\sum_{i=1}^{q} \deg X_i < k$, this can only converge to a finite value in the limit $t \to 0$ if c vanishes on $(X_1, \dots, X_q)$. This proves the first implication.
$\Leftarrow$: Let $c \in C^q(\mathfrak{X}(\mathbb{R}^n))$ be so that it vanishes on all homogeneous vector fields whose degree adds to a value smaller k. Set $r := \max\{q+k, 1\}$. Given any vector fields $X_1, \dots, X_q \in \mathfrak{X}(\mathbb{R}^n)$, apply the Hadamard lemma to each of them r times to write, in multiindex notation,

$$X_k(x) = Y_k(x) + \sum_{\substack{\vec{\alpha} \in \mathbb{N}_0^n \\ |\vec{\alpha}| = r+1}} x_1^{\alpha_1} \dots x_n^{\alpha_n} Z_k^{\vec{\alpha}}(x) =: Y_k(x) + Z_k(x), \tag{2.3.18}$$

where Y_k is a polynomial vector field whose homogeneous components are of degree $\le r-1$, and $Z_k^{\vec{\alpha}} \in \mathfrak{X}(\mathbb{R}^n)$. Using multilinearity of c, decompose $T_t^* c(X_1, \dots, X_q)$ into summands of the form $\frac{1}{t^k} T_t^* c$ with all arguments being some Y_k or some Z_k for $k = 1, \dots, q$. The limits $\lim_{t \to 0} t T_t^* Y_k$ and $\lim_{t \to 0} t^{-r} T_t^* Z_k$ in $\mathfrak{X}(\mathbb{R}^n)$ exist for all $k = 1, \dots, q$. Thus any summand in the decomposition of $\frac{1}{t^k} \left(T_t^* c\right)(X_1, \dots, X_q)$ is of the following form for a certain $s \ge 0$ and certain $i_1, \dots, i_s, j_1, \dots, j_{q-s} \in \{1, \dots, q\}$,:

$$\begin{aligned} &\frac{1}{t^k} \left(T_t^* c\right)(Z_{i_1}, \dots, Z_{i_s}, Y_{j_1}, \dots, Y_{j_{q-s}}) \\ &\qquad = t^{rs-(q-s)-k} (T_t^* c)(t^{-r} Z_{i_1}, \dots, t^{-r} Z_{i_s}, t Y_{j_1}, \dots, t Y_{j_{q-s}}). \end{aligned} \tag{2.3.19}$$

If $s \ge 1$, then we have due to $r \ge q + k$,

$$rs - (q-s) - k \ge s \ge 1, \tag{2.3.20}$$

and the limit $t \to 0$ exists.
If $s = 0$, then the summand is of the form $\frac{1}{t^k} T_t^* c(Y_1, \dots, Y_q)$. The Y_k are polynomial vector fields, so we may use multilinearity to decompose this term so that we get terms of $\frac{1}{t^k} T_t^* c$ whose arguments are homogeneous polynomial vector fields. In every such summand, $\frac{1}{t^k} T_t^* c$ can be replaced by $t^{\Sigma - k} c$, where Σ is the sum of the degrees of inserted homogeneous vector fields. By assumption on c, every summand where $\Sigma < k$ must vanish. This implies that as $t \to 0$, the term $\frac{1}{t^k} T_t^* c(Y_1, \dots, Y_k)$ converges to a finite value. This concludes the proof. $\square$

2.3.2. GELFAND–FUKS COHOMOLOGY OF EUCLIDEAN SPACE

Let us now connect the Lie algebra $\mathfrak{X}(\mathbb{R}^n)$ to the Lie algebra of formal vector fields W_n. To this end, fix, for the rest of the section some local frame of vector fields around $0 \in \mathbb{R}^n$ to induce an isomorphism $J_0^\infty \mathfrak{X}(\mathbb{R}^n) \xrightarrow{\sim} W_n$, cf. Remark 2.2.2, so that every elements of W_n can be written as the infinity-jet $j_0^\infty X$ at zero of some $X \in \mathfrak{X}(\mathbb{R}^n)$.

Definition 2.3.7. If $X \in \mathfrak{X}(\mathbb{R}^n)$, denote by $\tilde{X}^{(r)}$ the polynomial vector field corresponding to the r-jet of X at zero. Define then for all $k \in \mathbb{Z}$ the maps

$$\begin{aligned} \gamma_k : F^k C^\bullet(\mathfrak{X}(\mathbb{R}^n)) &\to C^\bullet_{(k)}(W_n), \\ (\gamma_k c)(j_0^\infty X_1, \dots, j_0^\infty X_q) &:= \lim_{r\to\infty} \lim_{t\to 0} t^{-k} (T_t^* c')(\tilde{X}_1^{(r)}, \dots, \tilde{X}_q^{(r)}), \end{aligned} \tag{2.3.21}$$

$$\beta_k : C^\bullet_{(k)}(W_n) \to F^k C^\bullet(\mathfrak{X}(\mathbb{R}^n)), \quad (\beta_k c')(Y_1, \dots, Y_k) := c'\left(j_0^\infty Y_1, \dots, j_0^\infty Y_k\right). \tag{2.3.22}$$

for all $X_1, \dots, X_q \in W_n$, $Y_1, \dots, Y_q \in \mathfrak{X}(\mathbb{R}^n)$, $c \in F^k C^q(\mathfrak{X}(\mathbb{R}^n))$ and $c' \in C^q_{(k)}(W_n)$.

Lemma 2.3.8. *The maps β_k and γ_k are well-defined chain maps with $\gamma_k \circ \beta_k = \mathrm{id}$.*

Proof. Note first that γ_k is well-defined: By definition of the filtration, for every $c \in F^k C^\bullet(\mathfrak{X}(\mathbb{R}^n))$ the pointwise limit $\lim_{t\to 0} t^{-k} T_t^* c$ exists. Further, the sequence

$$\left(\lim_{t\to 0} t^{-k} (T_t^* c)(\tilde{X}_1^{(r)}, \dots, \tilde{X}_q^{(r)})\right)_{r\in\mathbb{N}} \tag{2.3.23}$$

is eventually constant in r, since the cochain $\lim_{t\to 0} T_t^* c$ vanishes on homogeneous vector fields whose sum of degrees is larger than k, cf. (2.3.17). Further, if $c \in F^k C^q(\mathfrak{X}(\mathbb{R}^n))$, then by Lemma 2.3.6, c vanishes on polynomial vector fields the sum of whose degrees is smaller than k. Hence $\gamma_k c \in C^q_{(k)}(W_n)$.

Analogously, if $c \in C^k_{(r)}(W_n)$, then $\beta_k c$ vanishes on polynomial vector fields whose sum of degrees is smaller than k, hence Lemma 2.3.6 implies $\beta_k c \in F^k C^q(\mathfrak{X}(\mathbb{R}^n))$.

The identification of a finite formal vector field X_n with its Taylor polynomial in $\mathfrak{X}_{\mathrm{pol}}(\mathbb{R}^n)$ is a Lie algebra morphism, and so is the pullback of a vector field by the diffeomorphism T_t. Hence γ_k is a chain map.

The map β_k is a chain map since taking the infinite jet of a vector field at zero is a Lie algebra morphism $\mathfrak{X}(\mathbb{R}^n) \to W_n$.

It remains to check the composition of the two maps. Let us prove

$$(\gamma_k \beta_k c)(j_0^\infty X_1, \dots, j_0^\infty X_q) = c(j_0^\infty X_1, \dots, j_0^\infty X_q) \tag{2.3.24}$$

for all $c \in C^q_{(k)}(W_n)$ and homogeneous formal vector fields $j_0^\infty X_1, \dots, j_0^\infty X_q \in W_n$ whose degree sums up to k. This suffices, since we have $c, \gamma_k \beta_k c \in C^q_{(k)}(W_n)$, so both cochains vanish on all other homogeneous vector fields.

For a homogeneous vector field X and sufficiently large r, we have

$$j_0^\infty \tilde{X}^{(r)} = j_0^\infty X \text{ and } \deg \tilde{X}^{(r)} = \deg j_0^\infty X. \tag{2.3.25}$$

Hence, since if $j_0^\infty X_1, \dots, j_0^\infty X_q \in W_n$ are homogeneous formal vector fields whose degree sums up to k, then

$$\begin{aligned} (\gamma_k \beta_k c)(j_0^\infty X_1, \dots, j_0^\infty X_q) &= \lim_{r\to\infty} c(j_0^\infty \tilde{X}_1^{(r)}, \dots, j_0^\infty \tilde{X}_q^{(r)}) \\ &= c(j_0^\infty X_1, \dots, j_0^\infty X_q). \end{aligned} \tag{2.3.26}$$

This concludes the proof. □

Corollary 2.3.9. *For every $k \in \mathbb{Z}$,*

$$0 \to F^{k+1}C^\bullet(\mathfrak{X}(\mathbb{R}^n)) \hookrightarrow F^k C^\bullet(\mathfrak{X}(\mathbb{R}^n)) \xrightarrow{\gamma_k} C^\bullet_{(k)}(W_n) \to 0 \tag{2.3.27}$$

is a split exact sequence of cochain complexes.

Proof. Because $\{F^k C^\bullet(\mathfrak{X}(\mathbb{R}^n))\}_{k\in\mathbb{Z}}$ constitutes a filtration of complexes for $C^\bullet(\mathfrak{X}(\mathbb{R}^n))$, we have that the inclusion $F^{k+1}C^\bullet(\mathfrak{X}(\mathbb{R}^n)) \to F^k C^\bullet(\mathfrak{X}(\mathbb{R}^n))$ is a chain map. The map γ_k is a chain map due to Lemma 2.3.8. Hence, the sequence is a sequence of chain complexes. The injectivity of the first map is clear, and the second map is surjective, since it is split by β_k. Exactness at the middle term follows from the characterization of the filtration in Lemma 2.3.6. This concludes the proof. □

For the following lemma, recall the action of a Lie algebra on its Chevalley–Eilenberg cochains and the interior product $\lrcorner$ defined in Definition 2.2.6.

Lemma 2.3.10. *Let M be a smooth manifold, let $\{\phi_t\}_{t>0}$ a one-parameter semigroup of diffeomorphisms on M, and $\{X_t \in \mathfrak{X}(M)\}_{t>0}$ its time-dependent generator. Then, for all $t_0, t_1 > 0$ and $c \in C^\bullet(\mathfrak{X}(M))$ we have*

$$\phi_{t_1}^* c - \phi_{t_0}^* c = K_{t_1,t_0} dc + dK_{t_1,t_0} c, \tag{2.3.28}$$

where $K_{t_1,t_0} := -\int_{t_0}^{t_1} \phi_t^*(X_t \lrcorner c)dt$.

Proof. By definition of the generator $\{X_t\}_{t\in\mathbb{R}}$, we have for every $Y \in \mathfrak{X}(\mathbb{R}^n)$

$$\frac{d}{dt}\phi_t^* Y = \phi_t^*[X_t, Y] \quad \forall t \in \mathbb{R}. \tag{2.3.29}$$

By standard differentiation rules and the homotopy formula from Lemma 2.2.7 we have for $c \in C^q(\mathfrak{X}(M))$ and $Y_1, \ldots, Y_q \in \mathfrak{X}(\mathbb{R}^n)$:

$$\begin{aligned} \frac{d}{dt}\phi_t^* c(X_1,\ldots,X_q) &= \sum_{k=1}^{q} (\phi_t^* c)(X_1,\ldots,[X_t,X_k],\ldots,X_q) \\ &= -(X_t \cdot \phi_t^* c)(X_1,\ldots,X_q) = -d(X_t \lrcorner \phi_t^* c) - X_t \lrcorner (\phi_t^* dc). \end{aligned} \tag{2.3.30}$$

An application of the fundamental theorem of calculus now gives the desired statement. □

Corollary 2.3.11. *The complex $F^1 C^\bullet(\mathfrak{X}(\mathbb{R}^n))$ is acyclic.*

Proof. Consider for every $t_0, t_1 > 0$ the operator $K_{t_0,t_1} : C^\bullet(\mathfrak{X}(\mathbb{R}^n)) \to C^\bullet(\mathfrak{X}(\mathbb{R}^n))$ from Lemma 2.3.10 associated to the one-parameter semigroup $\{T_t\}_{t>0}$. By definition, for $c \in F^1 C^k(\mathfrak{X}(\mathbb{R}^n))$ and all all $X_1, \ldots, X_k \in \mathfrak{X}(\mathbb{R}^n)$, we have

$$\lim_{t\to 0}(T_t^* c)(X_1,\ldots,X_k) = 0. \tag{2.3.31}$$

But then

$$\begin{aligned} c(X_1,\dots,X_k) &= \lim_{t\to 0}(T_1^* c - T_t^* c)(X_1,\dots,X_k) \\ &= \lim_{t\to 0}\left(K_{1,t} dc + dK_{1,t} c\right)(X_1,\dots,X_k) =: (Kdc + dKc)(X_1,\dots,X_k). \end{aligned} \tag{2.3.32}$$

In the above, we defined the operator $K : C^\bullet(\mathfrak{X}(\mathbb{R}^n) \to C^\bullet(\mathfrak{X}(\mathbb{R}^n))$ as the pointwise limit of the operators $K_{1,t}$:

$$Kc = \lim_{t\to 0} K_{1,t} c = \int_0^1 T_t^*(X_t \lrcorner c)\,\mathrm{d}t, \tag{2.3.33}$$

where X_t is the generator of the semigroup of diffeomorphisms $\{T_t\}_{t>0}$. Hence K is a chain homotopy between the identity map and zero on $F^1 C^\bullet(\mathfrak{X}(\mathbb{R}^n))$, which proves the statement. □

Finally we can state a variation of Lemma 1 in Section 2.4.B. of [Fuk86]:

Theorem 2.3.12. *The inclusion*

$$F^0 C^\bullet(\mathfrak{X}(\mathbb{R}^n)) \hookrightarrow C^\bullet(\mathfrak{X}(\mathbb{R}^n)) \tag{2.3.34}$$

and the maps

$$\gamma_0 : F^0 C^q(\mathfrak{X}(\mathbb{R}^n)) \to C^q_{(0)}(W_n), \quad \beta_0 : C^q_{(0)}(W_n) \to F^0 C^q(\mathfrak{X}(\mathbb{R}^n)) \tag{2.3.35}$$

from Definition 2.3.7 are quasi-isomorphisms and unital algebra morphisms. In particular, there is an isomorphism of algebras

$$H^\bullet(\mathfrak{X}(\mathbb{R}^n)) \cong H^\bullet(W_n), \tag{2.3.36}$$

and the wedge product of two elements of positive degree in $H^\bullet(\mathfrak{X}(\mathbb{R}^n))$ is zero.

Proof. The compatibility of the maps with the algebra structure is immediate from the multiplicativity of the filtration $F^k C^q(\mathfrak{X}(\mathbb{R}^n))$ and the formulas for γ_0, β_0. By the split exact sequence of Corollary 2.3.9, we have for every $k \in \mathbb{Z}$ an isomorphism of cochain complexes

$$F^k C^\bullet(\mathfrak{X}(\mathbb{R}^n)) \cong F^{k+1} C^\bullet(\mathfrak{X}(\mathbb{R}^n)) \oplus C^\bullet_{(k)}(W_n). \tag{2.3.37}$$

Inserting $k = 0$ into (2.3.37), the acyclicity of $F^1 C^\bullet(\mathfrak{X}(\mathbb{R}^n))$ by Corollary 2.3.11 shows that we have algebra isomorphisms

$$H^\bullet(F^0 \tilde{C}^\bullet(\mathfrak{X}(\mathbb{R}^n))) \cong H^\bullet_{(0)}(W_n) \overset{\text{Prop. 2.2.11}}{\cong} H^\bullet(W_n), \tag{2.3.38}$$

hence γ_0 and its splitting β_0 must be quasi-isomorphisms.
Let further $k = -1, \dots, -n$ in (2.3.37). By Proposition 2.2.11, the complexes $\tilde{C}^\bullet_{(k)}(W_n)$ are acyclic and we have isomorphisms

$$\begin{aligned} H^\bullet(F^0 C^\bullet(\mathfrak{X}(\mathbb{R}^n))) &\cong H^\bullet(F^{-1} C^\bullet(\mathfrak{X}(\mathbb{R}^n))) \\ &\cong \dots \\ &\cong H^\bullet(F^{-n} C^\bullet(\mathfrak{X}(\mathbb{R}^n))) \\ &\overset{\text{Lem. 2.3.5}}{=} H^\bullet(C^\bullet(\mathfrak{X}(\mathbb{R}^n))) = H^\bullet(\mathfrak{X}(\mathbb{R}^n)). \end{aligned} \tag{2.3.39}$$

Hence the inclusion $F^0 C^\bullet(\mathfrak{X}(\mathbb{R}^n)) \hookrightarrow C^\bullet(\mathfrak{X}(\mathbb{R}^n))$ is a quasi-isomorphism.
The product of positive degree elements in $H^\bullet(\mathfrak{X}(\mathbb{R}^n))$ is zero since this is true for $H^\bullet(W_n)$ by Corollary 2.2.28. □

We end this section by an extension of our proof of $H^\bullet(\mathfrak{X}(\mathbb{R}^n)) \cong H^\bullet(W_n)$ to the setting of Gelfand–Fuks cohomology of a disjoint union of finitely many copies of Euclidean space.

Remark 2.3.13. The isomorphism of topological Lie algebras

$$\mathfrak{X}(\mathbb{R}^n \sqcup \mathbb{R}^n) \cong \mathfrak{X}(\mathbb{R}^n) \oplus \mathfrak{X}(\mathbb{R}^n) \tag{2.3.40}$$

insinuates that the Gelfand–Fuks cohomology of such a disjoint union may be calculated by the use of a *Künneth formula*: For finite-dimensional Lie algebras $\mathfrak{g}, \mathfrak{h}$ over $\mathbb{R}$, the Künneth theorem implies

$$H^\bullet(\mathfrak{g} \oplus \mathfrak{h}) \cong H^\bullet(\mathfrak{g}) \otimes H^\bullet(\mathfrak{h}). \tag{2.3.41}$$

And indeed, such Künneth theorems in Lie algebra cohomology are well known in the purely algebraic setting, but extending them to *continuous* Lie algebra cohomology relies on nontrivial topological assumptions in order to deal with the arising topological tensor products and their completion. For such formulas, the reader may, for example, consult [GLW05]. We avoid this approach here.

Proposition 2.3.14. *Let $M := \bigsqcup_{i=1}^r \mathbb{R}^n$ be a disjoint collection of copies of $\mathbb{R}^n$. Then every choice of order on the copies of $\mathbb{R}^n$ induces an algebra isomorphism*

$$H^\bullet(\mathfrak{X}(M)) \cong H^\bullet(\mathfrak{X}(\mathbb{R}^n))^{\otimes^r}. \tag{2.3.42}$$

Proof. We mimic the proof for the $r = 1$ situation, but we expand the scaling of $\mathbb{R}^n$ to the same scaling in every copy of $\mathbb{R}^n$:

$$T_t : \bigsqcup_{i=1}^r \mathbb{R}^n \to \bigsqcup_{i=1}^r \mathbb{R}^n, \quad x \mapsto tx. \tag{2.3.43}$$

The definition of the corresponding spaces $F^kC^q(\mathfrak{X}(M))$ is identical to in the $r=1$ case, and by the same proofs, they constitute a descending, filtration that is bounded from below with

$$F^kC^\bullet(\mathfrak{X}(M)) = C^\bullet(\mathfrak{X}(M)) \quad \forall k \leq -rn. \tag{2.3.44}$$

In analogy to Definition 2.3.7, we can define a map $\gamma_k^{(r)}$, which, together with some choice of ordering on the components of M gives rise to an exact sequence for every $k \in \mathbb{Z}$:

$$\begin{aligned} 0 \to F^{k+1}C^\bullet(\mathfrak{X}(\mathbb{R}^n)) \to F^kC^\bullet(\mathfrak{X}(\mathbb{R}^n)) \\ \xrightarrow{\gamma_k^{(r)}} \bigoplus_{k_1+\cdots+k_r=k} C^\bullet_{(k_1)}(W_n) \otimes \cdots \otimes C^\bullet_{(k_r)}(W_n) \to 0. \end{aligned} \tag{2.3.45}$$

For the tensor product complex on the right-hand side, we can use the Künneth theorem to calculate its cohomology, and due to acyclicity of $C^\bullet_{(k)}(W_n)$ for $k \neq 0$, the only one of the complexes with nontrivial cohomology is the one with the condition $k_1 + \cdots + k_r = 0$. By the same steps as in Corollary 2.3.11 and Theorem 2.3.12 we arrive at the desired isomorphism of vector spaces. This isomorphism respects the wedge product, as we see with the arising quasi-isomorphism

$$\beta_0^{(r)} : C^\bullet_{(0)}(W_n)^{\otimes^r} \to C^\bullet(\mathfrak{X}(M)), \quad c_1 \otimes \cdots \otimes c_r \mapsto \beta_0^1 c_1 \wedge \cdots \wedge \beta_0^r c_r, \tag{2.3.46}$$

where β_0^k maps formal cochains exactly like the map β_0 from the Definition 2.3.7, but all jets of vector fields are evaluated at the zero in the k-th copy of $\mathbb{R}^n$. Because the β_0 in the $r=1$ case respect the wedge product, so does $\beta_0^{(r)}$. □

The formula for the quasi-isomorphism $\beta_0^{(r)}$ from the previous proof implies:

Corollary 2.3.15. *Let $B_1, \ldots, B_r \subset \mathbb{R}^n$ be pairwise disjoint sets diffeomorphic to $\mathbb{R}^n$. Assume their union is contained in another set $C \subset \mathbb{R}^n$ diffeomorphic to $\mathbb{R}^n$. The extension map*

$$[\iota^C_{B_1\cup\cdots\cup B_r}] : \bigotimes_{i=1}^r H^\bullet(\mathfrak{X}(B_i)) \cong H^\bullet\,(\mathfrak{X}\,(B_1 \cup \cdots \cup B_r)) \to H^\bullet(\mathfrak{X}(C)) \tag{2.3.47}$$

is given by

$$[c_1] \otimes \cdots \otimes [c_r] \mapsto [\iota^C_{B_i} c_1 \wedge \cdots \wedge \iota^C_{B_r} c_r]. \tag{2.3.48}$$

2.4. COSHEAF-THEORETIC ASPECTS OF GELFAND-FUKS COHOMOLOGY

The previous section concludes the analysis of the cohomology of $\mathfrak{X}(\mathbb{R}^n)$. This constitutes an important building block to understand the Gelfand–Fuks cohomology for

2

arbitrary smooth manifolds M. We will explore some properties related to *extension* of the cochains from a smaller to a larger open set of M. The following presentation of these results, especially the framing in terms of cosheaf-theoretic data, is a novel contribution to the literature, though the results themselves are implicitly used in [GF69, GF70b, Fuk86]. We also remark a close similarity of these methods to the standard constructions in the theory of factorization algebras [CG17].

Recall that we identify $J_0^\infty \mathfrak{X}(\mathbb{R}^n) \cong W_n$ by a choice of local frame of vector fields around 0. The group of local diffeomorphisms $\phi : \mathbb{R}^n \to \mathbb{R}^n$ that fix zero admits a right action on infinity-jets $j_0^\infty X \in J_0^\infty \mathfrak{X}(\mathbb{R}^n)$ via the pullback of vector fields:

$$\phi^* j_0^\infty X := j_0^\infty(\phi^* X). \tag{2.4.1}$$

This action factors through to an action of the group of infinity-jets of diffeomorphisms that fix zero, denoted $J_0^\infty \operatorname{Diff}(\mathbb{R}^n)$. Hence, $J_0^\infty \operatorname{Diff}(\mathbb{R}^n)$ acts on W_n and by pullback on the complex $C^\bullet(W_n)$, and we write for all $\phi \in \operatorname{Diff}(\mathbb{R}^n)$, all $c \in C^k(W_n)$ and $X_1, \ldots, X_k \in \mathfrak{X}(\mathbb{R}^n)$:

$$((j_0^\infty \phi)^* c)(j_0^\infty X_1, \ldots, j_0^\infty X_k) := c(j_0^\infty(\phi^* X_1), \ldots, j_0^\infty(\phi^* X_k)). \tag{2.4.2}$$

Lemma 2.4.1. *Let $\phi : \mathbb{R}^n \to \mathbb{R}^n$ be a local diffeomorphism.*

i) The induced map $[\phi^*] : H^\bullet(\mathfrak{X}(\mathbb{R}^n)) \to H^\bullet(\mathfrak{X}(\mathbb{R}^n))$ is a unital algebra isomorphism.

ii) Assume ϕ is additionally orientation-preserving with respect to some fixed orientation on $\mathbb{R}^n$. Then $[\phi^*] = \mathrm{id}$.

Proof. Lemma 2.3.10 shows that the maps induced on $C^\bullet(\mathfrak{X}(\mathbb{R}^n))$ by translations

$$\tau_a : \mathbb{R}^n \to \mathbb{R}^n, \quad x \mapsto x + a, \quad a \in \mathbb{R}^n \tag{2.4.3}$$

are homotopic to the identity $\mathrm{id} = \tau_0^*$. Hence on cohomology τ_a acts as the identity for all $a \in \mathbb{R}^n$. Since all τ_a are orientation-preserving, we may without loss of generality assume that, in all that follows, ϕ fixes zero.

i) Recall the map $\beta_0 : C^\bullet(W_n) \to C^\bullet(\mathfrak{X}(\mathbb{R}^n))$ from Definition 2.3.7. By Theorem 2.3.12, β_0 is a quasi-isomorphism, so every cohomology class in $H^\bullet(\mathfrak{X}(\mathbb{R}^n))$ has a representative of the form $\beta_0 c$ for some $c \in C^\bullet(W_n)$. By definition of β_0, it intertwines the action of the group of local diffeomorphisms that fix zero on $C^\bullet(\mathfrak{X}(\mathbb{R}^n)$ with the action of $J_0^\infty \operatorname{Diff}(\mathbb{R}^n)$ on $C^\bullet(W_n)$, meaning

$$\phi^*(\beta_0)c = \beta_0((j_0^\infty \phi)^* c). \tag{2.4.4}$$

Since ϕ is a local diffeomorphism, it admits a local inverse around 0, so the action of $j_0^\infty \phi$ on c is invertible. Thus $\phi^* : H^\bullet(\mathfrak{X}(\mathbb{R}^n)) \to H^\bullet(\mathfrak{X}(\mathbb{R}^n))$ is an isomorphism of vector spaces.

The conditions $\phi^*(c_1 \wedge c_2) = \phi^* c_1 \wedge \phi^* c_2$ for all $c_1, c_2 \in C^\bullet(\mathfrak{X}(\mathbb{R}^n))$ and $\phi^*(1) = 1$ follow

directly from the definition of the pullback of local diffeomorphisms. Hence ϕ induces a unital algebra isomorphism.
ii) Fix an arbitrary cocycle $c \in C^\bullet(W_n)$. Since c is a continuous cochain, it is zero on formal vector fields of sufficiently high degree. Hence, for some $N \in \mathbb{N}$, it factors through to a multilinear map

$$c_N : \left(\frac{W_n}{\bigoplus_{r=N+1}^\infty \mathfrak{g}_r}\right)^k \to \mathbb{R}. \tag{2.4.5}$$

The action of $J_0^\infty \operatorname{Diff}(\mathbb{R}^n)$ on W_n descends to an action of the Lie group $J_0^N \operatorname{Diff}(\mathbb{R}^n)$ on the quotient $\frac{W_n}{\bigoplus_{r=N+1}^\infty \mathfrak{g}_r}$, so that so that for all $X_1, \dots, X_k \in \mathfrak{X}(\mathbb{R}^n)$

$$(j_0^\infty \phi^* c)(j_0^\infty X_1, \dots, j_0^\infty X_k) = (j_0^N \phi^* c_N)(j_0^N X_1, \dots, j_0^N X_k), \tag{2.4.6}$$

where the action on the right-hand side is defined analogously to the action of the infinity-jets. The Lie algebra of $J_0^N \operatorname{Diff}(\mathbb{R}^n)$ is given by

$$J_0^N \mathfrak{X}(\mathbb{R}^n) \cong \frac{\bigoplus_{r=0}^\infty \mathfrak{g}_r}{\bigoplus_{r=N+1}^\infty \mathfrak{g}_r} \subset \frac{W_n}{\bigoplus_{r=N+1}^\infty \mathfrak{g}_r}. \tag{2.4.7}$$

Since ϕ is orientation-preserving, its N-jet at zero lies in the identity component of the Lie group $J_0^N \operatorname{Diff}(\mathbb{R}^n)$. This component is generated by the image of the exponential map, there exist vector fields $Y_1, \dots, Y_r \in \mathfrak{X}(\mathbb{R}^n)$ so that for all $X_1, \dots, X_k \in \mathfrak{X}(\mathbb{R}^n)$

$$\begin{aligned}(j_0^\infty \phi^* c)(j_0^\infty X_1, \dots, j_0^\infty X_k) &= (j_0^N \phi^* c_N)(j_0^N X_1, \dots, j_0^N X_k) \\ &= (\exp(j_0^N Y_1)^* \cdots \exp(j_0^N Y_r)^* c_N)(j_0^N X_1, \dots, j_0^N X_k) \\ &= (\exp(j_0^\infty Y_1)^* \cdots \exp(j_0^\infty Y_r)^* c)(j_0^\infty X_1, \dots, j_0^\infty X_k).\end{aligned} \tag{2.4.8}$$

The action of $J_0^\infty \mathfrak{X}(\mathbb{R}^n) \cong W_n$ on $H^\bullet(W_n)$ is trivial by Corollary 2.2.8, and as a consequence

$$[\phi^*]([\beta_0 c]) = [\beta_0(\exp(j_0^\infty Y_1)^* \cdots \exp(j_0^\infty Y_r)^* c)] = [\beta_0 c]. \tag{2.4.9}$$

Hence $[\phi^*] = \mathrm{id}$ as a map on cohomology, and the statement is shown. □

We will now make use of the language of *cosheaves* to describe the extension of Gelfand-Fuks cochains between open sets of a smooth manifold. While sheaf theory is well known, the dual concept of cosheaves is less commonly considered. For self-containedness, we direct the reader to Appendix A, or [Bre97] for a more detailed study of both sheaf and cosheaf theory.

Definition 2.4.2. Let M be a smooth manifold, and $U \subset V$ open subsets of M.

i) Define the *extension of cochains*

$$\iota_U^V : C^\bullet(\mathfrak{X}(U)) \to C^\bullet(\mathfrak{X}(V)) \tag{2.4.10}$$

on cochains of degree $k > 0$ as

$$(\iota_U^V c)(X_1, \dots, X_k) := c(X_1\big|_U, \dots, X_k\big|_U) \tag{2.4.11}$$

for all $c \in C^k(\mathfrak{X}(U))$, $X_1, \dots, X_k \in \mathfrak{X}(V)$.
On cochains of degree 0, we set $\iota_U^V : C^0(\mathfrak{X}(U)) \to C^0(\mathfrak{X}(V))$ to be the identity.

ii) The extension of cochains induces an *extension of cohomology classes*

$$\iota_U^V : H^\bullet(\mathfrak{X}(U)) \to H^\bullet(\mathfrak{X}(V)) \tag{2.4.12}$$

which we will denote with the same symbol ι_U^V by an abuse of notation.

iii) We define a precosheaf $\mathscr{H}^\bullet$ of algebras over M, assigning to an open set $U \subset M$ the algebra $H^\bullet(\mathfrak{X}(U))$, and to an inclusion of open sets $U \subset V$ the extension map $\iota_U^V : H^\bullet(\mathfrak{X}(U)) \to H^\bullet(\mathfrak{X}(V))$.

An alternative perspective on the extension maps ι_U^V is that they are the pullback along the inclusion map $U \to V$, as defined in (2.3.4). This map is a local diffeomorphism, hence this pullback is well-defined.
For the next corollary, recall that a C-valued precosheaf on M is simply a covariant functor from the category Open(M) of open sets of M to the category C.

Definition 2.4.3. Let A be a associative $\mathbb{R}$- algebra and X a locally connected topological space. We define the *constant precosheaf of algebras* associated to A as the assignment

$$U \mapsto A^{\otimes \pi_0(U)}. \tag{2.4.13}$$

Here, if $\pi_0(U) = \infty$, we set $A^{\otimes \pi_0(U)}$ to be the infinite coproduct of algebras, i.e.

$$A^{\otimes \pi_0(U)} := \varinjlim A_1 \otimes \cdots \otimes A_n, \tag{2.4.14}$$

and the inclusion maps $A_1 \otimes \cdots \otimes A_n \to A_1 \otimes \cdots \otimes A_m$ for $n \leq m$ of this colimit are given by

$$a_1 \otimes \cdots \otimes a_n \mapsto a_1 \otimes \cdots \otimes a_n \otimes 1 \otimes \cdots \otimes 1. \tag{2.4.15}$$

The extension maps of the precosheaf are given by taking products along connected components.

Remark 2.4.4. This is the categorical generalization of a constant cosheaf from the category of vector spaces, replacing the coproduct $\oplus$ of vector spaces by the coproduct $\otimes$ of algebras. However, the category of $\mathbb{R}$-algebras is not even preadditive. Hence, we cannot define exact sequences, and hence cosheaves, in the way we did in Appendix A. This is why we refrain from calling this construction a cosheaf.

Corollary 2.4.5. *Let M be a smooth, orientable manifold of dimension n. Consider the full subcategory of* $\mathrm{Open}(M)$ *defined by finite unions of pairwise disjoint sets diffeomorphic to* $\mathbb{R}^n$*:*

$$\mathscr{B}_M := \{U_1 \cup \cdots \cup U_k : k \in \mathbb{N}, U_i \cong \mathbb{R}^n, U_i \cap U_j = \emptyset \text{ for } i \neq j\}. \tag{2.4.16}$$

The functor associated to the restriction of the precosheaf $\mathscr{H}^\bullet$ to this subcategory is naturally isomorphic to the analogous restriction of the functor associated to the constant cosheaf of algebras $U \mapsto H^\bullet(\mathfrak{X}(\mathbb{R}^n))^{\otimes \pi_0(U)}$.

Proof. Fix some orientation on $\mathbb{R}^n$. Using orientability of M, we can construct for every connected $U \in \mathscr{B}_M$ a diffeomorphism $\phi_U : U \to \mathbb{R}^n$ so that if $V \in \mathscr{B}_M$ is any other connected set with $U \cap V \neq \emptyset$, the transition map $(\phi_U \circ \phi_V^{-1})|_{\phi_V(V\cap U)}$ is orientation-preserving. Fix now some set $U \in \mathscr{B}_M$ with connected components $U_1, \ldots, U_k$. In Proposition 2.3.14 we defined an isomorphism

$$H^\bullet(\mathfrak{X}(U_1)) \otimes \cdots \otimes H^\bullet(\mathfrak{X}(U_k)) \cong H^\bullet(\mathfrak{X}(U)). \tag{2.4.17}$$

Thus, under the identification (2.4.17) we can express every element in $H^\bullet(\mathfrak{X}(U))$ as a span of elements of the form

$$[\phi_{U_1}^* c_1] \otimes \cdots \otimes [\phi_{U_k}^* c_k], \quad c_1, \ldots, c_k \in C^\bullet(\mathfrak{X}(\mathbb{R}^n)). \tag{2.4.18}$$

Let us show that the collection of the maps

$$[\phi_{U_1}^*] \otimes \cdots \otimes [\phi_{U_k}^*] : H^\bullet(\mathfrak{X}(\mathbb{R}^n))^{\pi_0(U)} \to H^\bullet(\mathfrak{X}(U)) \tag{2.4.19}$$

for all $U \in \mathscr{B}$ induce the desired natural isomorphism of functors. By Lemma 2.4.1, every tensor factor is an isomorphism of vector spaces, and by an explicit calculation with the wedge product, the maps are compatible with the algebra structures. It remains to show that the precosheaf extension maps are respected.

Let $V \in \mathscr{B}_M$ contain U. Assume first that two connected components of U, say U_1, U_2 lie in a single connected component of V, say V_1. By Corollary 2.2.28 the wedge product of $H^\bullet(W_n)$ is zero in nonzero degree, so the extension map

$$H^{q_1}(\mathfrak{X}(U_1)) \otimes H^{q_2}(\mathfrak{X}(U_2)) \to H^{q_1+q_2}(\mathfrak{X}(V_1)) \tag{2.4.20}$$

is zero if q_1 and q_2 are simultaneously nonzero.

If one of the q_1, q_2 is equal to zero, say $q_1 = 0$, then the extension map $H^{q_1}(\mathfrak{X}(U_1)) \otimes H^{q_2}(\mathfrak{X}(U_2)) \to H^{q_1+q_2}(\mathfrak{X}(V_1))$ is simply the isomorphism

$$\mathbb{R} \otimes H^{q_2}(\mathfrak{X}(U_2)) \xrightarrow{\sim} H^{q_2}(\mathfrak{X}(U_2)), \quad 1 \otimes [c] \mapsto [c]. \tag{2.4.21}$$

For the constant precosheaf of algebras $U \mapsto H^\bullet(\mathfrak{X}(\mathbb{R}^n))^{\otimes \pi_0(U)}$ analogous properties hold, and as a consequence the maps (2.4.19) commute with inclusions where two

2

connected components of U are contained in a single connected component of V. Hence it remains to consider the case in which every connected component V_i of V contains at most one connected component of U, so by reordering we may assume $U_i \subset V_i$ for all $i = 1,\dots,k$. It suffices to show that $(\phi_{V_i}^{-1})^* \circ \iota_{U_i}^{V_i} \circ \phi_{U_i}^*$ induces the identity map on $H^\bullet(\mathfrak{X}(\mathbb{R}^n))$ for all $i = 1,\dots,k$. As maps on cochains $C^\bullet(\mathfrak{X}(\mathbb{R}^n))$ we have the identity

$$(\phi_{V_i}^{-1})^* \circ \iota_{U_i}^{V_i} \circ \phi_{U_i}^* = \left(\phi_{V_i}^{-1}|_{U_i} \circ \phi_{U_i}\right)^*. \tag{2.4.22}$$

Now $\phi_{V_i}^{-1}|_{U_i} \circ \phi_{U_i}$ is an orientation-preserving local diffeomorphism, so by Lemma 2.4.1 the map $(\phi_{V_i}^{-1})^* \circ \iota_{U_i}^{V_i} \circ \phi_{U_i}^*$ equals the identity on cohomology and $[\iota_{U_i}^{V_i} \circ \phi_{U_i}] = [\phi_{V_i}]$. Hence, the statement is shown. $\square$

Remark 2.4.6. In the non-orientable case, the previous proof shows that the restriction of $\mathscr{H}^\bullet$ is naturally isomorphic to the restriction of a *locally constant cosheaf* $U \mapsto S(U)$, i.e. every point $x \in M$ has an open neighborhood U_x so that the restriction of S to U_x is a constant cosheaf.

2.5. Gelfand–Fuks cohomology for smooth manifolds

In this section, we construct a spectral sequence due to Gelfand and Fuks that calculates the continuous Lie algebra cohomology for smooth manifolds, following a local-to-global principle using sheaf theoretic ideas. The spectral sequences were originally constructed in [GF69], by an involved global analysis of the cochain spaces $C^\bullet(\mathfrak{X}(M))$ in terms of explicit distributions.

The proposed local-to-global principle has originally been outlined in [Bot73] and [Bot75], and, according to the last reference, was initially suggested by Segal. However, in these latter two references, there are some unaddressed subtleties: It is (indirectly) claimed that the assignment of open sets U to $C^\bullet(\mathfrak{X}(U))$ is a *cosheaf* of graded vector spaces, i.e. its Čech homology (see Appendix A) vanishes with respect to every good cover $\mathscr{U}$ of M. We show in Proposition 2.5.1 that this is false. We present a novel proof that works around this problem by using so-called k-*good* covers, an adaptation to the concept of a good cover originating from [BdBW13]. This was inspired by the recent preprint [HK18], treating Gelfand–Fuks cohomology in the setting of factorization algebras.

However, we want to emphasize that this subtlety does not influence the validity of the final results of Bott and Segal. The mistake is not repeated in [BS77] and [Fuk86], where similar, but more sophisticated Čech-theoretic methods are used.

Regardless, our proof gives a more elementary way to calculate Gelfand–Fuks spectral sequences for k-*diagonal cohomology*, an approximation of Gelfand–Fuks cohomology which we will introduce in the following section. The expressions for the spectral sequences have been given in [Fuk86] without an explicit proof for $k \neq 1$; the proof in [Bot73] is only a sketch, with previously mentioned issues, and the proof in [GF69] uses explicit, distribution-theoretic formulas which may be difficult for non-experts to retrace.

The construction we propose in the following section is inspired by a preprint by Kapranov and Hennion [HK18], in which they use the theory of factorization algebras to prove that Gelfand–Fuks cohomology can be identified with singular cohomology of a certain section space. We borrow from this theory a notion of generalized good covers which makes it possible to construct the desired spectral sequences from a local-to-global principle. This strategy is easily generalizable to construct local-to-global spectral sequences in other cohomology theories, as we will show in Chapter 3 with the example of continuous Chevalley–Eilenberg cohomology of gauge algebras.

2.5.1. DIAGONAL FILTRATION

Fix a smooth manifold M of dimension n. The previously established precosheaf structure (see Definition 2.4.2) of the cochains $C^\bullet(\mathfrak{X}(M))$ does *not* extend to a cosheaf structure.

Proposition 2.5.1. *Let M be a smooth manifold. If $k>1$ and* $\dim M > 0$*, then the precosheaf $U \mapsto C^k(\mathfrak{X}(U))$ from Definition 2.4.2 is not a cosheaf.*

Proof. Because $\dim M > 0$, there are smooth, nonzero $X_1, \dots, X_k \in \mathfrak{X}(M)$ whose supports are pairwise disjoint, and some $c \in C^k(\mathfrak{X}(M))$ with $c(X_1, \dots, X_k) \neq 0$. Then the sets

$$U_i := M \setminus \left(\bigcup_{j \neq i} \operatorname{supp} X_j \right), \quad i = 1, \dots, k \tag{2.5.1}$$

define an open cover $\{U_i\}_{i=1,\dots,k}$ of M. If the assignment of an open set $U \subset M$ to cochains $C^k(\mathfrak{X}(U))$ was a cosheaf, then there would exist $c_i \in C^\bullet(\mathfrak{X}(U_i))$ for $i = 1, \dots, k$ with

$$c = \sum_{i=1}^{k} \iota_{U_i}^M c_i. \tag{2.5.2}$$

But then, because $X_i|_{U_j} = 0$ for $i \neq j$, it follows that

$$0 \neq c(X_1, \dots, X_k) = \sum_{i=1}^{k} c_i(X_1|_{U_i}, \dots, X_k|_{U_i}) = 0. \tag{2.5.3}$$

A clear contradiction, hence, the precosheaf $U \mapsto C^k(\mathfrak{X}(U))$ is not a cosheaf for $k > 1$. □

Hence, as we increase the number of arguments in our cochains, we may get *locality* or *diagonality problems* as in the above proof. It will be valuable to replace these spaces by certain diagonal replacements:

Definition 2.5.2. Let U be an open subset of M.

i) Define the graded vector space $B^\bullet(\mathfrak{X}(U)) := \bigoplus_{q\geq 0} B^q(\mathfrak{X}(U))$, where

$$B^q(\mathfrak{X}(U)) := \{c : \mathfrak{X}(U)^q \to \mathbb{R} \mid c \text{ multilinear and jointly continuous}\}. \tag{2.5.4}$$

ii) Let $\{X_1,\dots,X_q\} \subset \mathfrak{X}(U)$ be a finite collection of vector fields and $k \geq 1$ arbitrary. We say that this collection *has the property* Δ_k if for every finite set $\Gamma \subset U$ of k arbitrary points, there is an X_i that vanishes in a neighborhood of Γ.

iii) For $q, k \geq 1$ integers, we define the k*-diagonal distributions* as those $c \in B^q(\mathfrak{X}(U))$ with

$$\{X_1,\dots,X_q\} \text{ has property } \Delta_k \implies c(X_1,\dots,X_q) = 0. \tag{2.5.5}$$

Their collection is denoted $\Delta_k B^q(\mathfrak{X}(U))$.
If $q = 0$, set $\Delta_k B^0(\mathfrak{X}(U)) = B^0(\mathfrak{X}(U))$ for all $k \geq 1$.

iv) Define the k*-diagonal cochains* $\Delta_k C^q(\mathfrak{X}(U)) \subset C^q(\mathfrak{X}(M))$ as the skew-symmetric cochains which are contained in $\Delta_k B^q(\mathfrak{X}(U))$.

Proposition 2.5.3. *For all $q \geq 0$ and all open $U \subset M$, we have the ascending chain*

$$\begin{aligned} 0 =: \Delta_0 C^q(\mathfrak{X}(U)) \subset \Delta_1 C^q(\mathfrak{X}(U)) \subset \dots \\ \subset \Delta_{q-1} C^q(\mathfrak{X}(U)) \subset \Delta_q C^q(\mathfrak{X}(U)) = C^k(\mathfrak{X}(U)). \end{aligned} \tag{2.5.6}$$

Further, the $\Delta_k C^\bullet(\mathfrak{X}(U))$ constitute a multiplicative filtration of the complex $C^\bullet(\mathfrak{X}(U))$.

Proof. The property Δ_k for a set $\{X_1,\dots,X_q\}$ implies the property Δ_{k-1}. This proves the ascending chain of inclusions.
Further, a set $\{X_1,\dots,X_k\}$ of k vector fields can only have the property Δ_k if one of the X_i is zero everywhere. Hence $\Delta_k C^k(\mathfrak{X}(M)) = C^k(\mathfrak{X}(M))$. Further, notice that if $\{X_1,\dots,X_{q+1}\}$ has the property Δ_k, so does $\{[X_1,X_2],X_3,\dots,X_q\}$. From this it follows that

$$d(\Delta_k C^q(\mathfrak{X}(M)) \subseteq \Delta_k C^{q+1}\mathfrak{X}(M). \tag{2.5.7}$$

Lastly, if $\{X_1,\dots,X_{q+r}\}$ has the property Δ_{k+l}, then the set $\{X_1,\dots,X_q\}$ must have the property Δ_k or the set $\{X_{q+1},\dots,X_{q+r}\}$ must have the property Δ_l. Hence

$$\Delta_k C^\bullet(\mathfrak{X}(M)) \wedge \Delta_l C^\bullet(\mathfrak{X}(M)) \subseteq \Delta_{k+l} C^\bullet(\mathfrak{X}(M)). \tag{2.5.8}$$

□

Example 2.5.4. A set $\{X_1,\dots,X_k\} \subset \mathfrak{X}(M)$ has the property Δ_1 if and only if

$$\bigcap_{i=1}^k \operatorname{supp} X_i = \emptyset. \tag{2.5.9}$$

Note that $\Delta_q C^q(\mathfrak{X}(U)) = C^q(\mathfrak{X}(U))$, hence

$$\Delta_k H^q(\mathfrak{X}(U)) = H^q(\mathfrak{X}(U)) \quad \forall k \geq q+1. \tag{2.5.10}$$

To put this in terms of more sheaflike data, let us view these cochains through a different lens.

Definition 2.5.5. Given $q \geq 1$ and the canonical projections $\mathsf{pr}_1, \dots, \mathsf{pr}_q : M^q \to M$, consider the vector bundle

$$\boxtimes^q TM := \bigotimes_{i=1}^{q} \mathsf{pr}_i^* TM \to M^q. \tag{2.5.11}$$

Equipping the space of sections $\mathfrak{X}(M)$ with its standard Fréchet topology, it is a well-known consequence of the Schwartz kernel theorem that there is a natural vector space isomorphism

$$B^q(\mathfrak{X}(M)) \cong \Gamma(\boxtimes^q TM)^*, \tag{2.5.12}$$

the star denoting the continuous dual with respect to the Fréchet topology (cf. [Hö03] for the distributional statement). This isomorphism is dual to the map

$$\begin{aligned} \mathfrak{X}(M) \otimes \cdots \otimes \mathfrak{X}(M) \to \Gamma(\boxtimes^q TM), \quad (X_1, \dots, X_q) \mapsto X_1 \boxtimes \cdots \boxtimes X_q, \\ (X_1 \boxtimes \cdots \boxtimes X_q)(x_1, \dots, x_q) := X_1(x_1) \otimes \cdots \otimes X_q(x_q) \quad \forall x_1, \dots, x_q \in M. \end{aligned} \tag{2.5.13}$$

Definition 2.5.6. Let X be a topological space and $k, q \in \mathbb{N}$. The *k-th diagonal* of X in X^q is the subspace

$$X_k^q := \{(x_1, \dots, x_q) \in X^q : |\{x_1, \dots, x_q\}| \leq k\}. \tag{2.5.14}$$

Special examples are the *thin diagonal* M_1^q and *fat diagonal* M_{q-1}^q, which take the following form:

$$\begin{aligned} M_1^q &:= \{(x, \dots, x) \in M^q\}, \\ M_{q-1}^q &= \{(x_1, \dots, x_q) \in M^q \mid \exists i, j : i \neq j \text{ and } x_i = x_j\}. \end{aligned}$$

Clearly $M_1^q \subset M_2^q \subset \cdots \subset M_q^q = M^q$. Note that a set of vector fields $\{X_1, \dots, X_q\}$ has the property Δ_k if and only if the support of $X_1 \boxtimes \cdots \boxtimes X_q$ is contained in $M^q \setminus M_k^q$. This proves the following:

Lemma 2.5.7. *An element $c \in B^q(\mathfrak{X}(M))$ is k-diagonal if and only if the support of its image under the Schwartz kernel map in $\mathrm{Hom}_{\mathbb{R}}(\Gamma(\boxtimes^q TM), \mathbb{R})$ is contained in M_k^q.*

With this perspective, we can deduce:

Lemma 2.5.8. *For $U \subset M^q$, the assignments*

$$M^q \supset U \mapsto \mathscr{B}^q(U) := \mathrm{Hom}_{\mathbb{R}}(\Gamma(\boxtimes^q TM)\big|_U, \mathbb{R}), \tag{2.5.15}$$

$$M^q_k \supset U \mapsto \mathscr{B}^q_k(U) := \left\{c \in \mathrm{Hom}_{\mathbb{R}}(\Gamma(\boxtimes^q TM), \mathbb{R}) : \operatorname{supp} c \subset U\right\}, \tag{2.5.16}$$

constitute flabby cosheaves on M^q and M^q_k, respectively, where the extension maps are induced by the restriction maps of the section spaces.

Proof. The sheaves of distributions

$$M^q \supset U \mapsto \mathscr{D}^q(U) := \mathrm{Hom}_{\mathbb{R}}\left(\Gamma_c(\boxtimes^q TM)\big|_U, \mathbb{R}\right), \tag{2.5.17}$$

$$M^q_k \supset U \mapsto \mathscr{D}^q_k(U) := \left\{c \in \mathrm{Hom}_{\mathbb{R}}(\Gamma_c(\boxtimes^q TM), \mathbb{R}) : \operatorname{supp} c \subset U\right\} \tag{2.5.18}$$

are shown to be soft using standard partition of unity arguments. Alternatively, this follows since the first sheaf is a module over a soft sheaf of rings, namely the sheaf of smooth functions on M^q, and the second one is a restriction of the first sheaf to a closed subspace, hence soft (see [Bre97, Chapter II, Theorems 9.2 & 9.16]). But $\mathscr{B}^q$ is exactly the precosheaf of compactly supported sections of the sheaf $\mathscr{D}^q$, and analogously for $\mathscr{B}^q_k$ and $\mathscr{D}^q_k$. By Proposition A.5, this implies that these precosheaves are flabby cosheaves. □

2.5.2. GENERALIZED GOOD COVERS

We have seen that the precosheaf $U \mapsto C^k(\mathfrak{X}(U))$ for open sets U of a smooth manifold M does not define a cosheaf for $k \geq 2$. However, Lemma 2.5.8 gives some hope that we can meaningfully study them over the Cartesian power M^k. As such, we will need methods to compare different Cartesian powers $M, M^2, M^3, \ldots$ of M. One such tool we can use is the notion of a k-good cover in the sense of [BdBW13, Definition 2.9]:

Definition 2.5.9. Let $k \geq 1$. An open cover $\mathscr{U}$ of M is k-*good* if:

i) Given k points $x_1, \ldots, x_k \in M$, there is a $U \in \mathscr{U}$ with $x_1, \ldots, x_k \in U$.

ii) Intersections of elements of $\mathscr{U}$ are diffeomorphic to a disjoint union of at most k copies of $\mathbb{R}^n$.

For $k = 1$ this agrees with the usual notion of a good cover.

Remark 2.5.10. The k-good covers are, in a sense, finite approximations to so-called *Weiss covers*, which have property i) of the previous definition with no restriction on the number k, but without any replacement for property ii), so the sets in the cover may, a priori, be homologically wild. Weiss covers are heavily used in the theory of *factorization algebras*, which have strong ties to our setting, see [CG17, HK18, Gin15].

Definition 2.5.11. Let X be a topological space, $\mathscr{U}$ an open cover of X, and $k, q \geq 1$ integers. Define

$$\mathscr{U}^q := \{V \subset M^q : V \text{ a connected component of } U^q \text{ for some } U \in \mathscr{U}\},$$
$$\mathscr{U}^q_k := \{V \cap M^q_k : V \in \mathscr{U}^q\}.$$

Property i) of a k-good cover $\mathscr{U}$ implies that the sets $\mathscr{U}^1, \mathscr{U}^2, \ldots, \mathscr{U}^k$ are open covers of $M, M^2, \ldots, M^k$, making k-good covers useful tools in comparing data between Cartesian powers of M.
If $\mathscr{U}$ is a k-good cover, then the equation

$$(U \cap V)^q = U^q \cap V^q \tag{2.5.19}$$

shows that all intersections of the open cover $\mathscr{U}^q$ are diffeomorphic to finite disjoint unions of at most k-copies of $\mathbb{R}^{qn}$. To show a similar result for the open cover $\mathscr{U}^q_k$ of M^q_k, let us prepare an auxiliary statement:

Lemma 2.5.12. *Let X be a topological space and $q, k \geq 1$ integers. If X has finitely many connected components and all of them are contractible, then the same holds for X^q_k.*

Proof. Let $X_1, \ldots, X_s$ be the connected components of X. Every connected component C of X^q is then of the form

$$C = X_{i_1} \times \cdots \times X_{i_q} \tag{2.5.20}$$

for some $i_1, \ldots, i_q \in \{1, \ldots, s\}$, not necessarily all different. By assumption, the $X_1, \ldots, X_s$ are contractible, hence, for all $j = 1, \ldots, s$, there are deformation retracts $F_j : X_j \times [0,1] \to X_j$ of X_j onto a point. Then the map

$$F : C \times [0,1] \to C, \quad (x_1, \ldots, x_q, t) \mapsto (F_{i_1}(x_1, t), \ldots, F_{i_q}(x_q, t)) \tag{2.5.21}$$

is a deformation retract of C onto a point.
If $C \cap X^q_k$ is nonempty, this map restricts to a deformation retract of $C \cap X^q_k$ to a point. Hence, if C is a connected component of U^q, and $C \cap X^q_k \neq \emptyset$, then $C \cap X^q_k$ is contractible. But the connected components of X^q_k are exactly the nonempty sets $C \cap X^q_k$ for connected components C of X^q. Hence all connected components of X^q_k are contractible. Lastly, since X^q only has finitely many connected components, so does X^q_k. This finishes the proof. □

Lemma 2.5.13. *Let X be a locally connected topological space, $q, k \geq 1$ integers, and $\mathscr{U}$ a k-good open cover of X. Then $\mathscr{U}^q_k$ is an open cover of X^q_k. Further, all nonempty, finite intersections of elements in $\mathscr{U}^q_k$ have finitely many connected components, and all these connected components are contractible.*

2

Proof. If $(x_1,\dots,x_q) \in X^q_k$, then $|\{x_1,\dots,x_q\}| \le k$. But since $\mathscr{U}$ is k-good, there is some $U \in \mathscr{U}$ containing all $x_1,\dots,x_k$. Hence $x \in U^q \cap X^q_k$. Since x was arbitrary, this shows that $\mathscr{U}^q_k$ is an open cover of X^q_k.
Let us now show that if $V, V' \in \mathscr{U}^q_k$ have nonempty intersection, then $V \cap V'$ is a finite disjoint union of contractible sets. By definition there exist $U, U' \in \mathscr{U}$ so that V and V' are connected components of U^q_k and U'^q_k, respectively. Since $\mathscr{U}$ is a k-good cover, $U \cap U'$ has only finitely many connected components, and all these connected components are contractible. Thus by Lemma 2.5.12 the same holds for $(U \cap U')^q_k = U^q_k \cap (U')^q_k$.
Since X is locally connected, so are X^q_k and its open subsets U^q_k and $(U')^q_k$. In a locally connected space, all connected components are closed and open, hence V and V' are closed and open in U^q_k and $(U')^q_k$, respectively. Thus $V \cap V'$ is closed and open in $U^q_k \cap (U')^q_k$, and hence must be a union of connected components of $U^q_k \cap (U')^q_k$. But we have seen that there are only finitely many connected components of $U^q_k \cap (U')^q_k$, and that they are all contractible. This shows that the same holds for $V \cap V'$.
By induction this extends to arbitrary finite intersections of sets $V_1,\dots,V_q \in \mathscr{U}^q_k$, and the lemma is shown. □

The first part of the following theorem is Proposition 2.10 in [BdBW13]:

Theorem 2.5.14. *For every smooth manifold M, a k-good open cover exists. Further, if M is compact, then M admits finite k-good open covers.*

Proof. The existence of k-good open covers is shown in [BdBW13]. If M is compact, choose any k-good cover $\mathscr{U}$, then $\mathscr{U}^k$ is a cover of M^k, and since M^k is compact, there is a finite subcover $\tilde{\mathscr{U}} \subset \mathscr{U}$ so that $\tilde{\mathscr{U}}^k$ is a cover of M^k. Hence the set $\tilde{\mathscr{U}}$ fulfills property i) of being a k-good cover, and as a subset of a k-good cover, it also fulfills property ii). □

2.5.3. THE ČECH–BOTT–SEGAL DOUBLE COMPLEX

Finally, let us define a double complex which intertwines Čech complexes with the Chevalley-Eilenberg complex structure. Analyzing this double complex will provide us with spectral sequences that calculate the Gelfand–Fuks cohomology of M. We name this double complex after Bott and Segal, in reference to their spirtually similar local-to-global analysis in [BS77], though we emphasize that our double complex differs from theirs.

Definition 2.5.15. Let $\mathscr{U} := \{U_i\}_{i\in I}$ be an open cover of a smooth manifold M, and $k \ge 1$. For $U_{i_1},\dots,U_{i_q} \in \mathscr{U}$, set

$$U_{i_1\dots i_q} := U_{i_1} \cap \cdots \cap U_{i_q}. \tag{2.5.22}$$

We define the *k-th Čech–Bott–Segal (CBS) double complex* for the cover $\mathscr{U}$ as the following double complex:

$$\begin{tikzcd}
\cdots & \cdots & \\
\bigoplus_i \Delta_k C^3(\mathfrak{X}(U_i)) \arrow[u] & \bigoplus_{i,j} \Delta_k C^3(\mathfrak{X}(U_{ij})) \arrow[l] \arrow[u] & \cdots \arrow[l] \\
\bigoplus_i \Delta_k C^2(\mathfrak{X}(U_i)) \arrow[u] & \bigoplus_{i,j} \Delta_k C^2(\Gamma(\mathfrak{X}(U_{ij})) \arrow[l] \arrow[u] & \cdots \arrow[l] \\
\bigoplus_i \Delta_k C^1(\mathfrak{X}(U_i)) \arrow[u] & \bigoplus_{i,j} \Delta_k C^1(\Gamma(\mathfrak{X}(U_{ij})) \arrow[l] \arrow[u] & \cdots \arrow[l]
\end{tikzcd}$$

The horizontal maps are given by the Čech differentials associated to the precosheaf structure, the vertical maps by the direct sum of Chevalley–Eilenberg differentials for the complexes $C^\bullet\left(\mathfrak{X}\left(U_{i_1\dots i_p}\right)\right)$. The grading is defined so that $\bigoplus_{i_1,\dots,i_p} \Delta_k C^q(\mathfrak{X}(U_{i_1\dots i_p}))$ lies in degree (p,q).
The *k-th skew-symmetrized CBS double complex* is the CBS double complex with all horizontal Čech complexes replaced by their skew-symmetrized versions, see Remark A.3.

Remark 2.5.16. Many remarks on the form of this double complex are in order:

i) Note that the zeroth row of the CBS double complex is zero, and not, as one might expect, the Čech complex associated to $\Delta_k C^0(\mathfrak{X}(M))$. Since the zeroth degree is connected to the rest of the complex by a zero differential, we do not lose any information by leaving it out.

ii) To deduce the ring structure on Gelfand–Fuks cohomology, it would be helpful if we could define a product structure on the CBS double complex in a way which extends the wedge product of cochains. At this point in time, it is unclear to the author how to accomplish this.

iii) The CBS double complex is not a first-quadrant double complex: It mixes a cohomological and a homological differential. A priori, this means there is an ambiguity in defining the associated total complex, given by the choice of taking either direct sums or direct products on the relevant diagonals, since there may now be infinitely many nonzero terms on each such diagonal. The usual convergence theorems for the spectral sequences arising from horizontal and vertical filtration will, in general, not apply.

Especially part iii) of the previous remark poses a significant problem. We borrow an argument from [BS77] to circument this:

Lemma 2.5.17. *Let $k \in \mathbb{N}$ and $\mathcal{U}$ a finite open cover of a smooth manifold M. The k-th skew-symmetrized CBS double complex associated to $\mathcal{U}$ has only finitely many nonzero columns. In particular, it is bounded as a double complex.*

2

Proof. By finiteness of $\mathscr{U}$, there is a largest n such that there is a nonempty intersection $U_1 \cap \cdots \cap U_n$ with $U_i \neq U_j$ for $i \neq j$. Hence all columns in degree $> n$ vanish in the skew-symmetrized double complex. This concludes the proof. □

We begin with horizontal cohomology.

Proposition 2.5.18. *Let $k, q \geq 1$ be integers and $\mathscr{U} = \{U_i\}_{i \in I}$ a k-good cover of a smooth manifold M. The Čech complex*

$$\bigoplus_i \Delta_k B^q(\mathfrak{X}(U_i)) \leftarrow \bigoplus_{i,j} \Delta_k B^q(\mathfrak{X}(U_{ij})) \leftarrow \dots \tag{2.5.23}$$

is isomorphic to the Čech complex associated to the flabby cosheaf $\mathscr{B}_k^q$ on M_k^q defined in Lemma 2.5.8) with respect to the open cover $\{U_k^q \subset M_k^q : U \in \mathscr{U}\}$ of M_k^q.
The same statement holds for the skew-symmetrized Čech complex, and the isomorphism is equivariant under the natural permutation action of the symmetric group Σ_q.

Proof. Note first that $\mathscr{U}_k^q$ is a cover of M_k^q by Lemma 2.5.13. The (restriction of the) Schwartz kernel maps (2.5.12) give us a family of isomorphisms $\{\phi_U : U \subset M \text{ open}\}$ as in Lemma 2.5.7, so that for all open $U \subset V$ the following diagram commutes:

$$\begin{tikzcd}
\Delta_k B^q(\mathfrak{X}(U)) \arrow[r] \arrow[d, "\phi_U"] & \Delta_k B^q(\mathfrak{X}(V)) \arrow[d, "\phi_V"] \\
\mathscr{B}_k^q(U_k^q) \arrow[r] & \mathscr{B}_k^q(V_k^q)
\end{tikzcd}$$

Hence, we have isomorphisms on the precosheaf data; this lifts to an isomorphism of the two Čech complexes. This argument is independent of the choice of the standard or the skew-symmetrized Čech complex. Since the sets U_k^q are invariant under the natural Σ_q-action on M^q, both of the terms $\Delta_k B^q(\mathfrak{X}(U))$ and $\mathscr{B}_k^q(U_k^q)$ admit a Σ_k-action by permutation of vector fields. The Schwartz kernel map is equivariant with respect to this permutation, as one finds from the explicit formula of its dual map 2.5.13. □

Theorem 2.5.19. *Let $q, k \geq 1$ be integers, and consider the k-th (skew-symmetrized) CBS double complex for a k-good cover $\mathscr{U}$ of M. The cohomology of the q-th row is equal to $\Delta_k C^q(\mathfrak{X}(M))$ in degree zero, and trivial in all other degrees.*

Proof. By Proposition 2.5.18, the Čech complex in this row, associated to the cover $\mathscr{U}$ and the presheaf $U \mapsto \Delta_k B^q(\mathfrak{X}(U))$ over M, has the same homology as the Čech complex of the flabby cosheaf $U \mapsto \mathscr{B}_k^q(U)$ over M_k^q with respect to the cover $\mathscr{U}_k^q$. Flabby cosheaves have trivial Čech homology independent of the chosen cover by Proposition A.6, hence the homology is equal to $\Delta_k B^q(\mathfrak{X}(M))$ in zeroth degree and zero in higher degree.
The isomorphism identifying the two complexes is equivariant with respect to the Σ_q-action on both spaces. The functor taking the complexes to its Σ_q-invariants is exact,

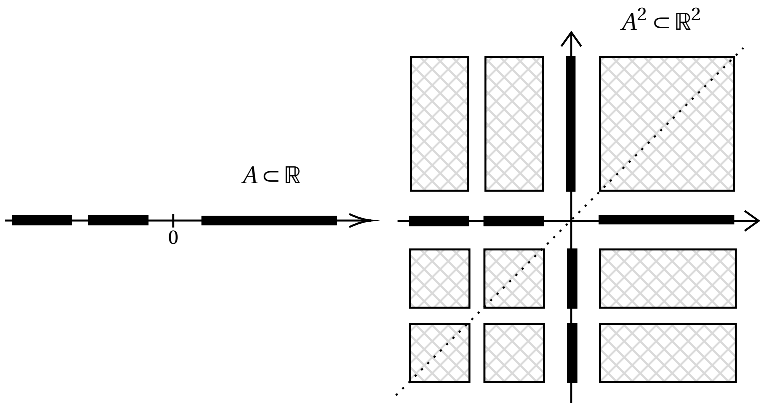

Figure 2.2: An illustration of Lemma 2.5.22: the set A has three connected components in $\mathbb{R}$, so its square $A^2 \subset \mathbb{R}^2$ has $9 = 3^2$, all arising by taking products of connected components of A. The products of a connected component of A with itself are exactly the connected components of A^2 which intersect the diagonal in $\mathbb{R}^2$.

as it arises from the action of a finite group in characteristic zero. Hence, the skew-symmetrized complex also has trivial cohomology in nonzero degree, and in degree zero $\left(\Delta_k B^q(\mathfrak{X}(M))\right)^{\Sigma_q} = \Delta_k C^q(\mathfrak{X}(M))$. Since the skew-symmetrized complex is exactly the q-th row of the skew-symmetrized CBS complex, this concludes the proof. □

Corollary 2.5.20. *Let $k \geq 1$ and assume there exists a finite, k-good cover $\mathcal{U}$ of M, e.g. when M is compact. The spectral sequences $\{E_r^{p,q}, d_r\}$ associated to the skew-symmetrized k-th CBS double complex for $\mathcal{U}$ by filtering horizontally or vertically converges to $\Delta_k \tilde{H}^\bullet(\mathfrak{X}(M))$. The tilde denotes reduced cohomology, cf. Definition 2.2.14.*

Proof. By Theorem 2.5.19, filtering by rows makes the spectral sequence collapse on the second page, with the indicated limit term $\Delta_k \tilde{H}^\bullet(\mathfrak{X}(M))$. Due to finiteness of $\mathcal{U}$ and Lemma 2.5.17, the skew-symmetrized double complex has bounded rows, and for such double complexes both filtrations yield spectral sequences which converge to the same cohomology, see [CE56, Chapter XV]. This shows the statement. □

2.5.4. SPECTRAL SEQUENCES FOR DIAGONAL COHOMOLOGY

To arrive at Corollary 2.5.20 we filtered the CBS double complex by rows, so now, let us study its filtration by rows. The cohomology among the arising vertical complexes amounts to calculating k-diagonal Lie algebra cohomology of $\mathfrak{X}(U)$, where the sets U are finite disjoint unions of $\mathbb{R}^n$. We begin by showing that our methods of Section 2.3 allow us to calculate k-diagonal cohomology for such U.

Proposition 2.5.21. *Let $1 \leq r \leq k$ and $U = \bigsqcup_{i=1}^r \mathbb{R}^n$. The inclusion*

$$\Delta_k C^\bullet(\mathfrak{X}(U)) \subset C^\bullet(\mathfrak{X}(U)) \tag{2.5.24}$$

2

induces an isomorphism

$$\Delta_k H^\bullet(\mathfrak{X}(U)) \cong H^\bullet(\mathfrak{X}(U)). \tag{2.5.25}$$

Proof. The construction in the proof of Proposition 2.3.14 restricts without change to the diagonally filtered complex. Specifically, the filtration $F^q C^\bullet(\mathfrak{X}(\mathbb{R}^n))$ restricts to a filtration $F^q(\Delta_k C^\bullet)(\mathfrak{X}(\mathbb{R}^n))$, it is straightforward to check that the exact sequence

$$\begin{aligned}0 \to F^{q+1} C^\bullet(\mathfrak{X}(\mathbb{R}^n)) \to F^q C^\bullet(\mathfrak{X}(\mathbb{R}^n)) \\ \to \bigoplus_{k_1+\cdots+k_r=q} C^\bullet_{(k_1)}(W_n) \otimes \cdots \otimes C^\bullet_{(k_r)}(W_n) \to 0\end{aligned} \tag{2.5.26}$$

restricts to an exact sequence

$$\begin{aligned}0 \to F^{q+1}(\Delta_k C^\bullet)(\mathfrak{X}(\mathbb{R}^n)) \to F^q(\Delta_k C^\bullet)(\mathfrak{X}(\mathbb{R}^n)) \\ \to \bigoplus_{k_1+\cdots+k_r=q} C^\bullet_{(k_1)}(W_n) \otimes \cdots \otimes C^\bullet_{(k_r)}(W_n) \to 0,\end{aligned} \tag{2.5.27}$$

and the image of the splitting $\beta_0^{(r)} : C^\bullet_{(0)}(W_n)^{\otimes r} \to C^\bullet(\mathfrak{X}(M))$ from the proof of Proposition 2.3.14 is contained in $\Delta_k C^\bullet(\mathfrak{X}(M))$. Hence,

$$\Delta_k H^\bullet(\mathfrak{X}(U)) \cong H^\bullet(W_n)^{\otimes^k} \cong H^\bullet(\mathfrak{X}(U)), \tag{2.5.28}$$

so that all nontrivial cohomology classes in $H^\bullet(\mathfrak{X}(U))$ have representatives contained in $\Delta_k H^\bullet(\mathfrak{X}(U))$. This shows that the inclusion of complexes induces an isomorphism and the proposition is shown. □

Consider now the CBS double complex for some k-good cover $\mathscr{U}$, and the spectral sequence with respect to the filtration by columns. Every intersection in the cover $\mathscr{U}$ is diffeomorphic to a disjoint union of at most k copies of $\mathbb{R}^n$. Hence, Proposition 2.5.21 applies in every column and we can replace diagonal cohomology with standard Lie algebra cohomology. Hence, the first page of the spectral sequence assumes the form in Figure 2.3. We state the following simple lemma without proof (cf. Figure 2.2):

Lemma 2.5.22. *Let X be a topological space. The connected components of X^q that do not intersect X^q_{q-1} are exactly the Cartesian products of q pairwise different connected components of X.*

The following proposition makes use of *relative* Čech homology of a pair of topological spaces (X, A) with respect to a cover $\mathscr{U}$ of X, cf. [ES52, Chapter IX]. In this situation, the space

$$\mathscr{U}_A := \{U \cap A : U \in \mathscr{U}\} \tag{2.5.29}$$

$$
\begin{array}{ccccc}
\cdots & & \cdots & & \\
\bigoplus_i H^3(\mathfrak{X}(U_i)) & \longleftarrow & \bigoplus_{i,j} H^3(\mathfrak{X}(U_{ij})) & \longleftarrow & \cdots \\
\bigoplus_i H^2(\mathfrak{X}(U_i)) & \longleftarrow & \bigoplus_{i,j} H^2(\Gamma(\mathfrak{X}(U_{ij})) & \longleftarrow & \cdots \\
\bigoplus_i H^1(\mathfrak{X}(U_i)) & \longleftarrow & \bigoplus_{i,j} H^1(\Gamma(\mathfrak{X}(U_{ij})) & \longleftarrow & \cdots
\end{array}
$$

Figure 2.3: The first page of the spectral sequence associated to the CBS double complex by beginning with taking the cohomology along the vertical, Chevalley–Eilenberg differential.

is a cover of A, and the relative complex is the quotient by the inclusion of Čech complexes $\check{C}(\mathscr{U}_A) \to \check{C}(\mathscr{U}_X)$. We denote the relative complex by $\check{C}_\bullet(\mathscr{U}_X, \mathscr{U}_A)$ and its homology by $\check{H}_\bullet(\mathscr{U}_X, \mathscr{U}_A)$. If $\mathscr{U}$ and $\mathscr{U}_A$ are good covers of X and A, respectively, it is well known that the Čech homologies are isomorphic to singular homology[2]

$$\check{H}^\bullet(\mathscr{U}_X) \cong H^\bullet(X), \quad \check{H}^\bullet(\mathscr{U}_A) \cong H^\bullet(A). \tag{2.5.30}$$

Hence, by an argument on long exact sequences in homology, one has an isomorphism to relative singular homology.

$$\check{H}_\bullet(\mathscr{U}_M, \mathscr{U}_A) \cong \check{H}_\bullet(M, A) \tag{2.5.31}$$

Proposition 2.5.23. *Let $q, k \geq 1$ be integers, M be a smooth, orientable manifold, and $\mathscr{U}$ a k-good cover of M. Denote by $\{E_r^{p,q}, d_r\}$ the spectral sequence associated to the k-th CBS double complex with respect to $\mathscr{U}$, arising from the horizontal filtration by columns. The q-th row of the first page $E_1^{\bullet,\bullet}$ is naturally isomorphic to a direct sum of relative Čech complexes with respect to covers $(\mathscr{U}^r, \mathscr{U}^r_{r-1})$ of (M^r, M^r_{r-1}) for $r = 1, \dots k$ (cf. Definition 2.5.11):*

$$
\begin{aligned}
E_1^{\bullet,q} \cong\ & \check{C}_\bullet(\mathscr{U}_M) \otimes H^q(W_n) \\
& \oplus \left(\bigoplus_{\substack{q_1+q_2=q \\ q_1,q_2>0}} \check{C}_\bullet(\mathscr{U}^2_{M^2}, \mathscr{U}^2_{M^2_1}) \otimes H^{q_1}(W_n) \otimes H^{q_2}(W_n) \right)^{\Sigma_2} \\
& \oplus \ \dots \\
& \oplus \left(\bigoplus_{\substack{q_1+\cdots+q_k=q \\ q_1,\dots,q_k>0}} \check{C}_\bullet(\mathscr{U}^k_{M^k}, \mathscr{U}^k_{M^k_{k-1}}) \otimes H^{q_1}(W_n) \otimes \cdots \otimes H^{q_k}(W_n) \right)^{\Sigma_k} .
\end{aligned}
\tag{2.5.32}
$$

[2] see [GR09, Chapter VI.D, Theorem 4] for a proof in Čech cohomology. This easily dualizes to our setting.

2

Here, the symmetric groups $\Sigma_2,\dots,\Sigma_k$ act by simultaneous, skew-symmetric permutation of the Cartesian factors $U_1\times\cdots\times U_k$ of any set in the covers $(\mathscr{U}^k,\mathscr{U}^k_{k-1})$ and the tensor factors of $H^{q_1}(W_n)\otimes\cdots\otimes H^{q_k}(W_n)$.
The same statement holds for the skew-symmetrized CBS complex, when the Čech complexes in (2.5.32) are replaced by their skew-symmetrized versions.

Proof. Set $n:=\dim M$. Since $\mathscr{U}$ is k-good, all intersections of elements in $\mathscr{U}$ are diffeomorphic to a finite disjoint union of open balls in $\mathbb{R}^n$. Hence we can apply Corollary 2.4.5 to find that the Čech complex in q-th row of $E_1^{\bullet,\bullet}$ is naturally isomorphic to the Čech complex of the q-th degree component of the constant precosheaf of algebras $U\mapsto H^\bullet(\mathfrak{X}(\mathbb{R}^n))^{\pi_0(U)}$, i.e.

$$U\mapsto \bigoplus_{q_1+\cdots+q_{\pi_0(U)}=q} H^{q_1}(\mathfrak{X}(\mathbb{R}^n))\otimes\cdots\otimes H^{q_{\pi_0(U)}}(\mathfrak{X}(\mathbb{R}^n)). \tag{2.5.33}$$

Denote this precosheaf by H^q_M. The multiplication of $H^\bullet(\mathfrak{X}(\mathbb{R}^n))$ is trivial on two elements of positive degree. Hence, if $H^q_M(U)\to H^q_M(V)$ is an extension map associated to an inclusion $U\subset V$, then the associated map

$$H^{q_1}(\mathfrak{X}(\mathbb{R}^n))\otimes\cdots\otimes H^{q_{\pi_0(U)}}(\mathfrak{X}(\mathbb{R}^n))\to H^{q_1'}(\mathfrak{X}(\mathbb{R}^n))\otimes\cdots\otimes H^{q'_{\pi_0(V)}}(\mathfrak{X}(\mathbb{R}^n)) \tag{2.5.34}$$

is only nonzero if

$$|\{r:q_r>0, r=1,\dots,\pi_0(U)\}|=|\{r:q_r'>0,r=1,\dots,\pi_0(V)\}|. \tag{2.5.35}$$

Hence the Čech complex associated to (2.5.33) decomposes into a direct sum of complexes $C_1,C_2,\dots$, where C_r is defined as the subcomplex on which the number of tensor factors of nonzero degree in every term equals r. Since $\mathscr{U}$ is a k-good cover, $C_r=0$ if $r>k$, so

$$\check{C}(\mathscr{U}_M,H^q_M)=C_1\oplus\cdots\oplus C_k. \tag{2.5.36}$$

We are done if we can show that we have the following isomorphism of chain complexes:

$$C_r\cong\left(\bigoplus_{\substack{q_1+\cdots+q_r=q\\ q_1,\dots,q_r>0}}\check{C}_\bullet(\mathscr{U}^r_{M^r},\mathscr{U}^r_{M^r_{r-1}})\otimes H^{q_1}(W_n)\otimes\cdots\otimes H^{q_r}(W_n)\right)^{\Sigma_r}. \tag{2.5.37}$$

Let now U be an intersection of elements of $\mathscr{U}$, and write $U_1,\dots,U_s\subset U$ for the connected components of U, respectively. If $s\geq r$, then there is a nontrivial direct summand in the complex C_r associated to U, specifically

$$\bigoplus_{\substack{q_1+\cdots+q_s=q\\ |\{i:q_i>0,i=1,\dots,s\}|=r}} H^{q_1}(\mathfrak{X}(\mathbb{R}^n))\otimes\cdots\otimes H^{q_s}(\mathfrak{X}(\mathbb{R}^n)) \tag{2.5.38}$$

$$=\bigoplus_{\substack{q_1+\cdots+q_r=q\\ q_1,\dots,q_r>0}} H^{q_1}(\mathfrak{X}(\mathbb{R}^n))\otimes\cdots\otimes H^{q_r}(\mathfrak{X}(\mathbb{R}^n)). \tag{2.5.39}$$

Every term $H^{q_1}(\mathfrak{X}(\mathbb{R}^n)) \otimes \cdots \otimes H^{q_r}(\mathfrak{X}(\mathbb{R}^n))$ in the latter direct sum is associated to a a subset of r pairwise different connected components $U_{i_1}, \ldots, U_{i_r}$ of U. The product $U_{i_{\sigma(1)}} \times \cdots \times U_{i_{\sigma(r)}}$ is a connected component of the set U^r, and U is an intersection of elements in $\mathscr{U}$. Hence $U_{i_{\sigma(1)}} \times \cdots \times U_{i_{\sigma(r)}}$ is an intersection of elements of the cover $\mathscr{U}^r$, and we can write $c_{i_1 \ldots i_r} \in \check{C}_\bullet(\mathscr{U}^r_{M^r})$ for the Čech simplex associated to it.
Now we can identify the term $H^{q_1}(\mathfrak{X}(\mathbb{R}^n)) \otimes \cdots \otimes H^{q_r}(\mathfrak{X}(\mathbb{R}^n))$ with a direct summand of the right-hand side of (2.5.37):

$$
\begin{aligned}
&H^{q_1}(\mathfrak{X}(\mathbb{R}^n)) \otimes \cdots \otimes H^{q_r}(\mathfrak{X}(\mathbb{R}^n)) \\
&\cong \mathbb{R} c_{i_1 \ldots i_r} \otimes H^{q_1}(\mathfrak{X}(\mathbb{R}^n)) \otimes \cdots \otimes H^{q_r}(\mathfrak{X}(\mathbb{R}^n)) \\
&\cong \left(\bigoplus_{\sigma \in \Sigma_r} \mathbb{R} c_{i_{\sigma(1)} \ldots i_{\sigma(r)}} \otimes H^{q_{\sigma(1)}}(\mathfrak{X}(\mathbb{R}^n)) \otimes \cdots \otimes H^{q_{\sigma(r)}}(\mathfrak{X}(\mathbb{R}^n)) \right)^{\Sigma_r}.
\end{aligned}
$$

Lemma 2.5.22 shows that this isomorphism induces the isomorphism (2.5.37):
Firstly, the lemma implies that the Cartesian product of the pairwise different connected components $U_{i_1}, \ldots, U_{i_r}$ does not intersect the diagonal M^r_{r-1}. Hence the Čech simplices $c_{i_1 \ldots i_r}$ do not vanish in the relative Čech complex.
Secondly, the lemma implies that every direct summand in the relative Čech is associated to a product of r pairwise different connected components. Thus, this construction exhausts the right-hand side of (2.5.37).
Hence the isomorphism (2.5.37) holds on a level of vector spaces, and it is straightforward to see that this identification respects extension maps. Hence it is even an isomorphism of chain complexes, and the statement is shown. □

By the argument before Proposition 2.5.23, the cohomology of the relative Čech complexes in the previous proposition is isomorphic to relative singular homology of the associated spaces. Together with Corollary 2.5.20 and a degree reflection $p \mapsto -p$ to bring the spectral sequence into a cohomological form, we end up with the following corollary:

Corollary 2.5.24. *Let M be an orientable manifold which admits a finite, k-good open cover (e.g. if M is compact). There exists a cohomological spectral sequence $\{E_r^{\bullet,\bullet}, d_r\}$ which converges to reduced k-diagonal cohomology $\Delta_k \tilde{H}^\bullet(\mathfrak{X}(M))$, and the entries $E_2^{p,q}$ of its second page are, for $q \geq 1$, of the following form:*

$$
\begin{aligned}
E_2^{p,q} &\cong H_{-p}(M) \otimes H^q(W_n) \\
&\oplus \bigoplus_{\substack{q_1+q_2=q \\ q_i>0}} \left(H_{-p}(M^2, M^2_1) \otimes H^{q_1}(W_n) \otimes H^{q_2}(W_n) \right)^{\Sigma_2} \\
&\oplus \quad \ldots \\
&\oplus \bigoplus_{\substack{q_1+\cdots+q_k=q \\ q_i>0}} \left(H_{-p}(M^k, M^k_{k-1}) \otimes H^{q_1}(W_n) \otimes \cdots \otimes H^{q_k}(W_n) \right)^{\Sigma_k}.
\end{aligned}
\tag{2.5.40}
$$

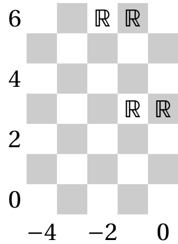

Figure 2.4: The spectral sequence for 2-diagonal Lie algebra cohomology for $\mathfrak{X}(S^1)$. For the k-diagonal spectral sequences for $k \geq 2$, this pattern continues into the upper-left direction.

Here, a permutation $\sigma \in \Sigma_r$ acts by simultaneous permutation of the Cartesian factors of M^k and the tensor factors $H^q(W_n)$.

Remark 2.5.25. These spectral sequences differ from the ones stated in [Fuk86], but only insofar as [Fuk86] considers the quotient complexes $\Delta_k C^\bullet(\mathfrak{X}(\mathbb{R}^n)/\Delta_{k-1}C^\bullet(\mathfrak{X}(\mathbb{R}^n)$ rather than the diagonal complexes themselves. This essentially gives one spectral sequence for every row in (2.5.40).

For $k \geq q+1$, we have $\Delta_k H^q(\mathfrak{X}(M)) = H^q(\mathfrak{X}(M))$ so in principle, these spectral sequences can be used to calculate the full Lie algebra cohomology of $\mathfrak{X}(M)$, degree by degree. In particular, since we know that the nontrivial cohomology of W_n is contained within the degrees $q = 2n+1, \ldots, 2n+n^2$ and the relative cohomology of (M^k, M^k_{k-1}) in degrees $\leq nk$, we have the following:

Corollary 2.5.26. *For all smooth manifolds M that admit finite k-good open covers for all $k \in \mathbb{N}$, and all $n \geq 0$, the Gelfand–Fuks cohomology $H^n(\mathfrak{X}(M))$ is finite-dimensional. Further, if $1 \leq k \leq \dim M$ then $H^k(\mathfrak{X}(M)) = 0$.*

Example 2.5.27. We sketch here how one now can calculate the well-known Gelfand–Fuks cohomology of $M = S^1$. By using excision and Poincaré duality[3], we can find

$$H_r\left((S^1)^k, (S^1)^k_{k-1}\right) = \tilde{H}_r\left((S^1)^k \setminus (S^1)^k_{k-1}\right) = \begin{cases} \mathbb{R}^{(k-1)!} & \text{if } r = k, k-1, \\ 0 & \text{else,} \end{cases} \tag{2.5.41}$$

where the copies of $\mathbb{R}$ in $\mathbb{R}^{(k-1)!}$ are enumerated by permutations of the $(k-1)$-th symmetric group, and the invariant space under the action of the k-th symmetric group Σ_k is one-dimensional.[4] Using this, we find that in the spectral sequence for k-diagonal cohomology, there is only ever at most a single nontrivial term on every diagonal $p+q = \text{const}$,

[3] We thank Moishe Kohan for communicating to us a proof idea for this.

[4] Note that in [GF69] it is incorrectly claimed that the nontrivial degrees of $H_\bullet\left((S^1)^k, (S^1)^k_{k-1}\right)$ are $k(k-1)/2$-dimensional. Already $(S^1)^3 \setminus (S^1)^3_2$ has only 2, not 3 connected components.

and those only exist on the diagonals $p+q=0,2,3,5,6,8,9,\dots$. From lacunary arguments, one concludes that all differentials beyond the second page must be trivial.
By retracing the construction of the spectral sequence, one can verify that the following cocycles $c_2, c_3 \in C^\bullet(\mathfrak{X}(S^1))$ indeed are representatives of the nontrivial cohomology classes in degree 2 and 3: For $x_0 \in S^1$ an arbitrary point, ∂_ϕ the standard coordinate vector field on S^1, and $f, g \in C^\infty(S^1)$, we set

$$c_2(f\partial_\phi, g\partial_\phi) = \int_{S^1} \big(f(\phi)g'(\phi) - f'(\phi)g(\phi)\big)\,\mathrm{d}\phi, \tag{2.5.42}$$

$$c_3(f\partial_\phi, g\partial_\phi, h\partial_\phi) = \det\begin{pmatrix} f(x_0) & g(x_0) & h(x_0) \\ f'(x_0) & g'(x_0) & h'(x_0) \\ f''(x_0) & g''(x_0) & h''(x_0)\end{pmatrix}. \tag{2.5.43}$$

We note that higher cohomology classes arise as the wedge products of these diagonal generators, but due to the lack of an obvious product on our spectral sequence, this is difficult to see directly. In any case, as a ring, we have

$$H^\bullet(\mathfrak{X}(S^1)) \cong S^\bullet(\mathbb{R}c_2) \otimes \Lambda^\bullet(\mathbb{R}c_3). \tag{2.5.44}$$

References

[BdBW13] Pedro Boavida de Brito and Michael Weiss. Manifold calculus and homotopy sheaves. *Homology Homotopy Appl.*, 15(2):361–383, 2013.

[Bor53] Armand Borel. Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts. *Ann. of Math. (2)*, 57:115–207, 1953.

[Bot73] R. Bott. Lectures on Gelfand-Fuks cohomology and characteristic classes. *Notes by M. Mostow and J. Perchik, New Mexico State University*, 1973.

[Bot75] Raoul Bott. On the Gelfand-Fuks cohomology. In *Differential geometry (Proc. Sympos. Pure Math., Vol. XXVII, Part 1, Stanford Univ., Stanford, Calif., 1973)*, pages 357–364, 1975.

[Bre97] Glen E. Bredon. *Sheaf theory*, volume 170 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1997.

[BS77] R. Bott and G. Segal. The cohomology of the vector fields on a manifold. *Topology*, 16(4):285–298, 1977.

[CE56] Henri Cartan and Samuel Eilenberg. *Homological algebra*. Princeton University Press, Princeton, N. J., 1956.

2

[CG17] Kevin Costello and Owen Gwilliam. *Factorization algebras in quantum field theory. Vol. 1*, volume 31 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2017.

[DNR01] Sorin Dăscălescu, Constantin Năstăsescu, and Şerban Raianu. *Hopf algebras*, volume 235 of *Monographs and Textbooks in Pure and Applied Mathematics*. Marcel Dekker, Inc., New York, 2001. An introduction.

[ES52] Samuel Eilenberg and Norman Steenrod. *Foundations of algebraic topology*. Princeton University Press, Princeton, New Jersey, 1952.

[Fuk73] D. B. Fuks. Characteristic classes of foliations. *Uspehi Mat. Nauk*, 28(2(170)):3–17, 1973.

[Fuk86] D. B. Fuks. *Cohomology of infinite-dimensional Lie algebras*. Contemporary Soviet Mathematics. Consultants Bureau, New York, 1986. Translated from the Russian by A. B. Sosinskiĭ.

[GF69] I. M. Gel'fand and D. B. Fuks. Cohomologies of the Lie algebra of tangent vector fields of a smooth manifold. *Funkcional. Anal. i Priložen.*, 3(3):32–52, 1969.

[GF70a] I. M. Gel'fand and D. B. Fuks. Cohomologies of the Lie algebra of formal vector fields. *Izv. Akad. Nauk SSSR Ser. Mat.*, 34:322–337, 1970.

[GF70b] I. M. Gel'fand and D. B. Fuks. Cohomologies of the Lie algebra of tangent vector fields of a smooth manifold. II. *Funkcional. Anal. i Priložen.*, 4(2):23–31, 1970.

[Gin15] Grégory Ginot. Notes on factorization algebras, factorization homology and applications. In *Mathematical aspects of quantum field theories*, Math. Phys. Stud., pages 429–552. Springer, Cham, 2015.

[GLW05] F. Gourdeau, Z. A. Lykova, and M. C. White. A Künneth formula in topological homology and its applications to the simplicial cohomology of $l^1(\mathbb{Z}_+^k)$. *Studia Math.*, 166(1):29–54, 2005.

[GR09] Robert Clifford Gunning and Hugo Rossi. *Analytic functions of several complex variables*, volume 368. American Mathematical Soc., 2009.

[GS73] Victor Guillemin and Steven Shnider. Some stable results on the cohomology of the classical infinite-dimensional Lie algebras. *Trans. Amer. Math. Soc.*, 179:275–280, 1973.

2

[Hö03] Lars Hörmander. *The analysis of linear partial differential operators. I*. Classics in Mathematics. Springer-Verlag, Berlin, 2003. Distribution theory and Fourier analysis, Reprint of the second (1990) edition [Springer, Berlin; MR1065993 (91m:35001a)].

[Hal15] Brian Hall. *Lie groups, Lie algebras, and representations*, volume 222 of *Graduate Texts in Mathematics*. Springer, Cham, second edition, 2015. An elementary introduction.

[Hat02] Allen Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002.

[HK18] B. Hennion and M. Kapranov. Gelfand-Fuchs cohomology in algebraic geometry and factorization algebras. *arXiv preprint arXiv:1811.05032*, 2018.

[HS53] G. Hochschild and J.-P. Serre. Cohomology of Lie algebras. *Ann. of Math. (2)*, 57:591–603, 1953.

[JV16] Bas Janssens and Cornelia Vizman. Universal central extension of the Lie algebra of Hamiltonian vector fields. *Int. Math. Res. Not. IMRN*, (16):4996–5047, 2016.

[JV18] Bas Janssens and Cornelia Vizman. Integrability of central extensions of the Poisson Lie algebra via prequantization. *J. Symplectic Geom.*, 16(5):1351–1375, 2018.

[KMS93] Ivan Kolář, Peter W. Michor, and Jan Slovák. *Natural operations in differential geometry*. Springer-Verlag, Berlin, 1993.

[Mia22] Lukas Miaskiwskyi. A local-to-global analysis of Gelfand-Fuks cohomology. *arXiv preprint arXiv:2202.09666*, 2022.

[MT91] Mamoru Mimura and Hirosi Toda. *Topology of Lie groups. I, II*, volume 91 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1991. Translated from the 1978 Japanese edition by the authors.

[Tsu81] Toru Tsujishita. *Continuous cohomology of the Lie algebra of vector fields*, volume 34. 1981.

[Vir70] Miguel Angel Virasoro. Subsidiary conditions and ghosts in dual-resonance models. *Physical Review D*, 1(10):2933, 1970.

3

Continuous cohomology of gauge algebras and bornological Loday–Quillen–Tsygan theorems

In this chapter, we investigate the well-known Loday–Quillen–Tsygan theorem, which calculates the Lie algebra homology of the general linear algebra gl(A) for an associative algebra A in terms of cyclic homology, and extend the proof to bornological Lie algebra homology of Fréchet and LF-algebras. For Fréchet spaces, this equals the usual continuous Lie algebra homology and is hence closely tied to the dual continuous cohomology. To this end we prepare several statements about homological algebra of topological vector spaces, and discuss when the differential of the bornological Hochschild and cyclic complex are topological homomorphisms in the setting of Fréchet algebras. We apply the results to the algebras of smooth functions on a smooth manifold and compactly supported smooth functions on Euclidean space, and construct from a local-to-global principle a Gelfand–Fuks-like spectral sequence which calculates the stable part of bornological Lie algebra homology of non-trivial gauge algebras. This complements results by Maier, Janssens and Wockel.

This chapter is based on the preprint [Mia22].

3.1. INTRODUCTION

Beyond the previously discussed Lie algebra of vector fields, there are more infinite-dimensional symmetries with great significance to physics, namely the infinite-dimensional symmetry group of gauge transformations and its corresponding infinite-dimensional Lie algebra. This Lie algebra is modeled as the sections of a Lie algebra bundle $\mathcal{K} \to M$, and we call section spaces of this shape *gauge algebras*. When the fibers of the bundle are semisimple finite-dimensional Lie algebras, the second degree cohomology is fully understood, see [Mai02] for the globally trivial case and [JW13] for the general case of nontrivial gauge algebras. However, the methods within these papers are very specifically suited to degree 2, which raises the question of how one might calculate higher degree cohomology.

Independently, roughly 40 years ago, Loday, Quillen and Tsygan fully described the algebraic Lie algebra homology of $\mathfrak{gl}(A) = \varinjlim \left(\mathfrak{gl}_n(\mathbb{K}) \otimes A\right)$ for arbitrary unital algebras A and fields $\mathbb{K}$ with $\mathbb{Q} \subset \mathbb{K}$ in terms of cyclic homology $H^{\lambda}_{\bullet}(A)$, see [LQ84], [Tsy83]. Their proof lays the groundwork for results about the homology of many so-called *current algebras*, Lie algebras of the shape $\mathfrak{g} \otimes A$, where $\mathfrak{g}$ is another Lie algebra and A is an associative algebra. In particular, when $\mathfrak{g}$ equals any of the classical simple Lie algebras, their method allows one to extract quite a lot of information.

Now, if $\mathfrak{g}$ is finite-dimensional, the current algebra $\mathfrak{g} \otimes C^{\infty}(M) \cong C^{\infty}(M, \mathfrak{g})$ represents exactly the gauge algebra of a globally trivial Lie algebra bundle with fibers equal to $\mathfrak{g}$, establishing a connection between the work of Loday, Quillen and Tsygan, and the study of gauge algebras. However, since one is in general not only interested in the algebraic Lie algebra cohomology of gauge algebras, but their continuous counterpart, one may ask the question if the proof of the Loday–Quillen–Tsygan (LQT) result holds when the involved homology theories are modified to take topological data into account. This would provide a unified way to calculate continuous (co-)homology of locally trivial gauge algebras with many different fiber Lie algebras, providing information in more than just low degree.

The goal of this chapter is to explore this question and answer it in the affirmative for *bornological* Lie algebra homology. On the $\mathfrak{gl}(A) = \varinjlim \mathfrak{gl}_n(A)$, this is essentially Lie algebra homology defined in terms of Grothendieck's completed inductive topological tensor product, rather than the more standard projective tensor product. Due to the fact that this tensor product is compatible with the ubiquitous direct limit arguments in the proof of the LQT theorem, this appears to be the most natural framing for a topological LQT theorem.

Note that in [Fei88], a result for the continuous cohomology of $\mathfrak{gl}(C^{\infty}(M))$ for closed manifolds M is stated, but lacking a full proof. They claim the jointly continuous cohomology is freely generated by the continuous cyclic cohomology of the algebra; in contrast, our result is that the bornological homology is equal to the *topological completion* of this freely generated space. We cannot disprove their claim outright due

to the difference in continuity notions. However, it seems likely that the statement in joint continuity should also include such a completion, as the generators contain infinite-dimensional components.
We begin by laying out the foundation of this study: we recall in Section 3.2 and 3.3 the definition and important properties of topological vector spaces, e.g. Fréchet and LF-spaces, their associated topological tensor products, and bornological homology theories for associative algebras. In particular, we prepare statements about the Fréchet algebra $A = C^\infty(M)$ and the LF-algebra $A = C_c^\infty(\mathbb{R}^n)$ and prove that the Hochschild and cyclic differentials are topological homomorphisms in these cases.

In Section 3.4, the algebraic LQT-Theorem is extended to bornological Lie algebra homology and (homologically) unital Fréchet algebras in Theorems 3.4.8 and 3.4.10. In essence, this requires tracking through the algebraic proofs and making sure that all algebraic isomorphisms lift to topological isomorphisms in the respective topologies and on the completions of the tensor products. As an application, in Corollary 3.4.11 we fully state the bornological Lie algebra homology of $\mathfrak{gl}(C^\infty(M)) = \varinjlim \mathfrak{gl}_n(C^\infty(M))$ and $\mathfrak{gl}(C_c^\infty(\mathbb{R}^n)) = \varinjlim \mathfrak{gl}_n(C_c^\infty(\mathbb{R}^n))$, both spaces equipped with their respective direct limit topologies.
Lastly, in Section 3.5 we globalize our results to approximate the bornological Lie algebra homology of gauge algebras $\Gamma(\operatorname{Ad} P \to M)$ for principal bundles $P \to M$. We restrict the calculations to when the fiber Lie algebra is $\mathfrak{gl}_n(\mathbb{K})$, but the general method is easily transferrable to other classical, simple Lie algebras. We construct in Theorem 3.5.15 a spectral sequence which calculates this homology in stable degree. This is parallel to our local-to-global construction in Chapter 2 for Gelfand–Fuks cohomology. Unfortunately, the entries of the second page of the spectral sequence can be specified only in terms of a certain Čech homology of product cosheaves, which we are unable to calculate and can only conjecture. This is due to the lack of a Künneth theorem in the cosheaf-theoretical setting. Assuming this conjecture, however, this spectral sequence yields a unified approach to compute low-dimensional bornological cohomology of a large class of gauge algebras. An example of such results is given in Corollary 3.5.17.
A related approach is given in [GW21], which does not consider continuous, but *local* Loday–Quillen–Tsygan Theorems, in the language of factorization algebras.

3.2. Topological vector spaces, bornologies, and tensor products

3.2.1. Preliminaries and definitions

We want to begin by collecting some definitions and results regarding topological vector spaces and their tensor products. For a more detailed discussion, we direct the reader to [Trè67], [Sch71], [MV97]. Fix, once and for all, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. All vector spaces and algebras in the following will be over $\mathbb{K}$ unless specified otherwise.

Definition 3.2.1. Let V be a vector space.

i) We call V a *topological vector space (TVS)* if it is equipped with equipped with a topology in which addition $V \times V \to V$ and scalar multiplication $V \times V \to V$ are continuous with respect to the according (product) topologies.

ii) We call V *locally convex vector space (LCTVS)* if it is a TVS which is Hausdorff and in which every point has a neighborhood basis consisting of convex sets.

iii) We call V a *Fréchet space* if it is a metrizable and complete LCTVS.

iv) We call V an *LF-space* if it is the inductive limit of a countable direct system of Fréchet spaces $(V_n)_{n\in\mathbb{N}}$, equipped with the inductive limit topology.

v) An LF-space $V = \varinjlim V_n$ is a *strict* LF-space if the maps $V_n \to V_m$ in the direct system (for $m \geq n$) are topological embeddings.

vi) We call V a *bornological algebra* if it is a LCTVS with a bounded multiplication map $\mu : V \times V \to V$ that makes V into an associative $\mathbb{K}$-algebra.

vii) We call V a *bornological Lie algebra* if it is a LCTVS with a bounded map $[\cdot,\cdot]$: $V \times V \to V$ that makes V into a $\mathbb{K}$-Lie algebra.

Remark 3.2.2. In the above context, a map $V \times V \to V$ is bounded if it maps products of bounded sets to bounded sets, see [Mey99] for details.

Remark 3.2.3. A reader intimidated by the word "bornological" showing up here may console themselves as follows: if V is a Fréchet space, then bounded multilinear maps are exactly the continuous multilinear maps. If V is an LF-space, the bounded multilinear maps are exactly the separately continuous multilinear maps, see [Mey99, Section 2.1.2] for further details. The correspondence will become even clearer in Proposition 3.2.8.

Example 3.2.4. Let M be a smooth manifold. Our main Fréchet space of interest will be the space of smooth functions on M, denoted $C^\infty(M)$ with its standard topology. This topology is sequential, and a sequence $(f_n \in C^\infty(M))_{n\in\mathbb{N}}$ converges to $f \in C^\infty(M)$ if and only if all (locally defined) derivatives of the f_n uniformly converge to the derivatives of f on all compact sets K contained within charts. The standard pointwise multiplication is easily shown to be continuous (hence bounded) with respect to this topology, making it into a Fréchet algebra.
These considerations are straightforwardly extended to the space of sections $\Gamma(E)$ of a finite-dimensional vector bundle $E \to M$, giving it a Fréchet structure.

Example 3.2.5. An important LF-space for us will be $\mathfrak{gl}(\mathbb{K}) := \varinjlim \mathfrak{gl}_n(\mathbb{K})$. Since all $\mathfrak{gl}_n(\mathbb{K})$ are finite-dimensional, they admit canonical Fréchet space structures, and they make up a direct system via the inclusions $\mathfrak{gl}_n(\mathbb{K}) \to \mathfrak{gl}_m(\mathbb{K})$ for $n \leq m$. The matrix multiplication is compatible with these inclusion maps. Note that if $y \in \mathfrak{gl}_m(\mathbb{K}) \subset \mathfrak{gl}(\mathbb{K})$, we have that

$$m_{n,y} : \mathfrak{gl}_n(\mathbb{K}) \to \mathfrak{gl}(\mathbb{K}), \quad x \mapsto x \cdot y$$

is continuous for all $n \in \mathbb{N}$, as a composition of continuous maps

$$\mathfrak{gl}_n(\mathbb{K}) \to \mathfrak{gl}_{n+m}(\mathbb{K}) \hookrightarrow \mathfrak{gl}(\mathbb{K}).$$

Hence, by [Trè67, Proposition 13.1] the induced multiplication maps $m_{\infty,y} : \mathfrak{gl}(\mathbb{K}) \to \mathfrak{gl}(\mathbb{K})$ are continuous for all $y \in \mathfrak{gl}(\mathbb{K})$. This shows that the multiplication map $\mathfrak{gl}(\mathbb{K}) \times \mathfrak{gl}_n(\mathbb{K}) \to \mathfrak{gl}(\mathbb{K})$ and, as a consequence, the arising commutator Lie bracket on $\mathfrak{gl}(\mathbb{K})$ are separately continuous (hence bounded) with respect to the LF-topology.

3

3.2.2. TOPOLOGICAL TENSOR PRODUCTS

In contrast to the finite-dimensional case, tensor products become very delicate in infinite dimensions. Certainly one always has the algebraic tensor product $\otimes = \otimes_{\mathbb{K}}$, but there are multiple non-equivalent ways to equip this with a topology. For our purposes we will recall two such notions.

Definition 3.2.6. Let V, W be two LCTVS and consider the canonical map

$$\phi : V \times W \to V \otimes W. \tag{3.2.1}$$

i) The *projective tensor product* $V \otimes_\pi W$ denotes the vector space $V \otimes W$ equipped with the strongest locally convex topology such that ϕ is continuous.

ii) The *inductive tensor product* $V \otimes_\iota W$ denotes the vector space $V \otimes W$ equipped with the strongest locally convex topology such that ϕ is separately continuous.

iii) The *bornological tensor product* $V \otimes_\beta W$ denotes the vector space $V \otimes W$ equipped with the strongest locally convex topology such that ϕ is *bounded*, meaning if $B \subset V \times W$ is bounded, then $\phi(B) \subset V \otimes_\beta W$ is, too.

Remark 3.2.7. The tensor products here should not be confused with the *injective* tensor product, generally denoted by $\otimes_\epsilon$.

See [Sch71, Chapter 3.6], [Gro95, Chapter 1], [KM97] for proofs of the existence of these tensor product topologies and additional details. In particular, it may be interesting to the reader that all presented tensor products fulfill the expected universal properties with respect to jointly continuous, separately continuous, or bounded bilinear maps.

In our setting, it suffices to work only with $\otimes_\beta$, and we denote by $V \widehat{\otimes} W := \overline{V \otimes_\beta W}$ the completion of the bornological tensor product. We set no notation for the completion with respect to other tensor products.

3

Topological tensor products appear to have somewhat of a bad reputation, and in complete generality, they may well deserve it. However, in our setting, the categorical properties of the tensor products are fairly pleasant, especially the bornological tensor product:

Proposition 3.2.8. *Let V, W, U be LCTVS.*

i) [Trè67, Chapter 34.2, Chapter 43] [KM97, Chapter 5] If V and W are Fréchet, then all separately continuous bilinear maps $V \times W \to U$ are continuous,

$$V \otimes_\pi W \cong V \otimes_\iota W \cong V \otimes_\beta W, \tag{3.2.2}$$

and $V \hat{\otimes} W$ is Fréchet.

ii) [Mey99, Appendix A.1.4] If V and W are nuclear strict LF-spaces

$$V = \varinjlim V_i, \quad W = \varinjlim W_j, \tag{3.2.3}$$

then $V \hat{\otimes} W$ is a nuclear strict LF-space, and

$$V \otimes_\iota W \cong V \otimes_\beta W, \quad V \hat{\otimes} W = \varinjlim (V_i \hat{\otimes} W_j). \tag{3.2.4}$$

iii) [Mey99, Proposition 2.25] We canonically have

$$(U \hat{\otimes} V) \hat{\otimes} W \cong U \hat{\otimes} (V \hat{\otimes} W), \quad U \hat{\otimes} V \cong V \hat{\otimes} U. \tag{3.2.5}$$

Remark 3.2.9. Note that commutativity follows quite easily using the topological isomorphism $X \times Y \cong Y \times X$. However, the associativity is not generally quite so obvious: For general topological tensor products, the natural vector space isomorphisms between $X \otimes (Y \otimes Z)$ and $(X \otimes Y) \otimes Z$ might not necessarily be continuous, see [Glö04].

Lastly, we have an exactness property of $\hat{\otimes}$. The following is a consequence of [EP96, Theorem A1.6]:

Proposition 3.2.10. *Let U, V, W and H be nuclear Fréchet spaces, and*

$$0 \to U \xrightarrow{f} V \xrightarrow{g} W \to 0 \tag{3.2.6}$$

an exact sequence, in the sense that f and g are continuous and linear, f is injective, g is surjective, and $\operatorname{im} f = \ker g$. Then

$$0 \to U \hat{\otimes} H \xrightarrow{f \hat{\otimes} \mathrm{id}} V \hat{\otimes} H \xrightarrow{g \hat{\otimes} \mathrm{id}} W \hat{\otimes} H \to 0 \tag{3.2.7}$$

is exact in the same sense.

3.3. BORNOLOGICAL HOCHSCHILD AND CYCLIC HOMOLOGY

3.3.1. PRELIMINARIES AND DEFINITIONS

In this section, we recall how to modify the algebraic notions of Hochschild/cyclic homology to include topological information. For a succinct presentation of the algebraic picture of Hochschild and cyclic homology and certain topological modifications, we cite [Lod92], [Kha13], [Con85]. A related discussion of the topological modifications also takes place in [BL01]. We lay no claim to originality within this section, with the exception of investigating the property of differentials to be topological morphisms; we will explain this below.

3

Definition 3.3.1. Let A be a bornological algebra. The *bornological Hochschild complex* of A with coefficients in itself is given by

$$HC_\bullet^{\mathrm{born}}(A) := \bigoplus_{k\geq 0} HC_k(A), \quad HC_k(A) := A^{\widehat{\otimes}^{k+1}}. \tag{3.3.1}$$

where the differential is induced by the *Hochschild differential*, so

$$\begin{aligned} b\colon HC_k^{\mathrm{born}}(A) &\to HC_{k-1}^{\mathrm{born}}(A), \\ b(a_0\otimes\cdots\otimes a_n) &:= \sum_{i=0}^{n-1}(-1)^i a_0\otimes\cdots\otimes a_i a_{i+1}\otimes\cdots\otimes a_n \\ &\quad + (-1)^{n+1} a_n a_0\otimes a_1\otimes\cdots\otimes a_{n-1}. \end{aligned} \tag{3.3.2}$$

The homology of this complex is called the *bornological Hochschild homology* of A and denoted $HH_\bullet^{\mathrm{born}}(A)$.

Remark 3.3.2. A reader who is not familiar with bornology may instead be interested in replacing $\otimes_\beta$ with, say, the completion of $\otimes_\pi$ or $\otimes_\iota$ rather than $\widehat{\otimes} = \overline{\otimes_\beta}$, to get something which could reasonably be called *(jointly) continuous* or *separately* continuous homology. Due to Proposition 3.2.8, this is possible whenever A is, respectively, Fréchet or a strict LF-space. These cases will be studied further in Section 3.4.

Remark 3.3.3. Note that for nonunital algebras A, the above definition is the bornological version of what, in [Lod92], is called the *naive* Hochschild homology. This does not necessarily agree with the "correct" version of Hochschild homology. However, in our applications, all algebras will be *(bornologically) H-unital*, a term we define later on, which suffices for both notions of Hochschild homology to coincide.

Definition 3.3.4. Let A be a bornological algebra. The *bornological Connes complex* of A is given by

$$C_\bullet^{\lambda,\mathrm{born}}(A) := \bigoplus_{n\geq 0} C_n^{\lambda,\mathrm{born}}(A), \quad C_n^{\lambda,\mathrm{born}}(A) := HC_n^{\mathrm{born}}(A,A)_{\mathbb{Z}/(n+1)\mathbb{Z}}, \tag{3.3.3}$$

where the action of the generator $\tau \in \mathbb{Z}/(n+1)\mathbb{Z}$ on $HC_n^{\text{born}}(A)$ is given by cyclic permutation, meaning

$$\tau \cdot (a_0 \otimes \cdots \otimes a_n) := (-1)^n a_n \otimes a_0 \otimes \cdots \otimes a_{n-1} \quad \forall a_0, \ldots, a_n \in A. \tag{3.3.4}$$

The differential of this complex is induced by the Hochschild differential, which factors through to this complex. The homology of this complex is called *bornological cyclic homology*.

Like in the algebraic setting, we show the compatibility with an alternative definition of cyclic homology, in terms of the following double complex, see [Lod92, Chapter 2]:

Definition 3.3.5. Let A be a bornological algebra. The *bornological cyclic double complex* $CC_{\bullet,\bullet}^{\text{born}}(A)$ of A is given by

$$\begin{tikzcd}
\cdots \arrow[d, "b"] & \cdots \arrow[d, "-b'"] & \cdots \arrow[d, "b"] & \cdots \arrow[d, "-b'"] & \cdots \\
A^{\widehat{\otimes}3} \arrow[d, "b"] & A^{\widehat{\otimes}3} \arrow[l, "1-\tau"'] \arrow[d, "-b'"] & A^{\widehat{\otimes}3} \arrow[l, "N"'] \arrow[d, "b"] & A^{\widehat{\otimes}3} \arrow[l, "1-\tau"'] \arrow[d, "-b'"] & \cdots \arrow[l, "N"'] \\
A^{\widehat{\otimes}2} \arrow[d, "b"] & A^{\widehat{\otimes}2} \arrow[l, "1-\tau"'] \arrow[d, "-b'"] & A^{\widehat{\otimes}2} \arrow[l, "N"'] \arrow[d, "b"] & A^{\widehat{\otimes}2} \arrow[l, "1-\tau"'] \arrow[d, "-b'"] & \cdots \arrow[l, "N"'] \\
A & A \arrow[l, "1-\tau"'] & A \arrow[l, "N"'] & A \arrow[l, "1-\tau"'] & \cdots \arrow[l, "N"']
\end{tikzcd} \tag{3.3.5}$$

Here:

- b is the Hochschild differential as in Definition 3.3.1,
- $b' : A^{\widehat{\otimes}n+1} \to A^{\widehat{\otimes}n}$ is the *bar differential*, which is defined as the Hochschild differential without the last summand, so

$$b'(a_0 \otimes \cdots \otimes a_n) := \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n, \tag{3.3.6}$$

- τ is the cyclic permutation as in Definition 3.3.4,
- $N : A^{\widehat{\otimes}n} \to A^{\widehat{\otimes}n}$ is the *norm operator*, defined as

$$N := 1 + \tau + \tau^2 + \cdots + \tau^{n-1}. \tag{3.3.7}$$

All squares anticommute, and we denote the total complex by $CC_\bullet^{\text{born}}(A)$, also called the *bornological cyclic complex* of A.

Analogously, one defines the *cohomological* bornological Hochschild, Connes, and cyclic double complexes, by taking the continuous dual of all spaces in the above. We respectively denote them by

$$HC_{\text{born}}^\bullet(A, A), \quad C_{\lambda,\text{born}}^\bullet(A), \quad CC_{\text{born}}^{\bullet,\bullet}(A). \tag{3.3.8}$$

Proposition 3.3.6. *Let A be a bornological algebra over $\mathbb{R}$. The canonical quotient*

$$CC_\bullet^{\mathrm{born}}(A) \to C_\bullet^{\lambda,\mathrm{born}}(A) \tag{3.3.9}$$

onto the zeroth column and the canonical inclusion

$$C^\bullet_{\lambda,\mathrm{born}}(A) \to CC^\bullet_{\mathrm{born}}(A) \tag{3.3.10}$$

into the zeroth column are quasi-isomorphisms, *i.e. chain maps which reduce to isomorphisms of vector spaces on homology.*

Proof. Both maps are straightforwardly chain maps, so it remains to show that they induce isomorphisms on (co-)homology.
We begin with a sketch of the algebraic, homological case, see [Lod92, Theorem 2.1.5]. There, one constructs homotopy operators h', h on every row of the algebraic double complex $CC_{\bullet,\bullet}(A)$, fulfilling

$$h'N + (1-\tau)h = \mathrm{id} = Nh' + h(1-\tau), \tag{3.3.11}$$

showing that, all rows are acyclic in nonzero degree, and degree zero homology of the n-th row equals the algebraic cyclic complex $C_n^\lambda(A)$. This shows the algebraic case. Now, all involved operators in [Lod92, Theorem 2.1.5] are easily seen to be continuous and hence extend to the respective topological completions, hence the homotopy argument extends to $CC_\bullet^{\mathrm{born}}(A)$ and $C^\bullet_{\lambda,\mathrm{born}}(A)$.
The cohomological statement follows identically by dualizing the above homotopy operators. □

If A is additionally unital, there is another complex which calculates cyclic homology:

Definition 3.3.7. Let A be a unital bornological algebra. The *bornological* (b,B)*-double complex* $B_{\bullet,\bullet}^{\mathrm{born}}(A)$ of A is defined as

$$\begin{tikzcd}
\cdots \arrow[d, "b"] & \cdots \arrow[d, "b"] & \cdots \arrow[d, "b"] \\
A^{\hat{\otimes}3} \arrow[d, "b"] & A^{\hat{\otimes}2} \arrow[l, "B"'] \arrow[d, "b"] & A \arrow[l, "B"'] \\
A^{\hat{\otimes}2} \arrow[d, "b"] & A \arrow[l, "B"'] & \\
A & &
\end{tikzcd} \tag{3.3.12}$$

Here b is the Hochschild differential and B is the *Connes operator*, defined as

$$B = (1-\tau)sN, \tag{3.3.13}$$

3

with τ the cyclic permutation, N the norm operator (see Definition 3.3.5) and s the *extra degeneracy*, defined as

$$s : A^{\widehat{\otimes}^n} \to A^{\widehat{\otimes}^{n+1}}, \quad a_1 \otimes \cdots \otimes a_n \mapsto 1 \otimes a_1 \otimes \cdots \otimes a_n. \tag{3.3.14}$$

All squares anticommute, and we denote the total complex by $B_\bullet^{\text{born}}(A)$.

Proposition 3.3.8. *Let A be a unital bornological algebra, then $B_\bullet^{\text{born}}(A)$ and $CC_\bullet^{\text{born}}(A)$ are continuously homotopy equivalent.*

Proof. Similar to the proof of Proposition 3.3.6, one builds homotopy operators by compositions of the continuous operators $N, s, (1-\tau)$, thus the well-known algebraic homotopy equivalence lifts to a topological one. See [Kha13, Proposition 3.8.1] for details. □

3.3.2. CLOSED RANGE THEOREMS

In the following we will need to understand whether the range of the differential of the cyclic complex is a topologically closed subspace of the complex. This condition makes the complex more well-behaved, in the situation of Fréchet spaces turning all differentials into topological homomorphisms, and allowing for the use of a Künneth formula. In this subsection, we will only consider Fréchet spaces, so we may always think of $\widehat{\otimes}$ as the closure of the projective tensor product due to Proposition 3.2.8. The following statement due to Serre is occasionally helpful:

Theorem 3.3.9. *[Ser55, Lemma 2] Let $d : A \to B$ be a continuous, linear map of Fréchet spaces. If $d(A)$ is cofinite-dimensional in B, then $d(A)$ is closed and complemented.*

We cite the following Theorem from [GLW05, Corollary 5.3], a rudimentary version of which was already given in [Gro].

Theorem 3.3.10. *Let A, B be chain complexes of nuclear Fréchet spaces, bounded from below in the sense that*

$$A_n = B_n = 0 \text{ if } n < 0. \tag{3.3.15}$$

If the differentials of both complexes have closed range, we have an isomorphism of TVS

$$H_\bullet(A \widehat{\otimes} B) \cong H_\bullet(A) \widehat{\otimes} H_\bullet(B). \tag{3.3.16}$$

Note that in the above situation, $H_\bullet(A \widehat{\otimes} B)$ is again canonically Fréchet, and thus the differential of $A \widehat{\otimes} B$ has closed range. Hence we can iterate this formula:

Corollary 3.3.11. *If $A_1, \ldots, A_n$ are finitely many complexes of nuclear Fréchet spaces, and all differentials have closed range, we have*

$$H_\bullet\left(A_1 \widehat{\otimes} \cdots \widehat{\otimes} A_n\right) \cong H_\bullet(A_1) \widehat{\otimes} \cdots \widehat{\otimes} H_\bullet(A_n). \tag{3.3.17}$$

However, there does not seem to be a simple argument for why the Hochschild/cyclic differentials should have closed range in large generality. In the cases that interest us, we can make use of the following easy proposition:

Proposition 3.3.12. *Let $(C_\bullet, d_C), (D_\bullet, d_D)$ be complexes of Hausdorff TVS, and let there be a continuous quasi-isomorphism $\phi : C_\bullet \to D_\bullet$.*

i) If d_D has closed range, then d_C has closed range.

ii) If $C_\bullet$ and $D_\bullet$ admit a continuous homotopy equivalence, then d_C has closed range if and only if d_D does.

Proof. *i)* Since the range of d_D is closed and fully contained in $\ker d_D$, the projection

$$\pi : \ker d_D \to \frac{\ker d_D}{\operatorname{Im} d_D[-1]} = H_\bullet(D_\bullet) \tag{3.3.18}$$

is a continuous, linear map of TVS. Denote by $\tilde{\phi}$ the continuous, linear map arising from the composition

$$\ker d_C \xrightarrow{\phi} \ker d_D \xrightarrow{\pi} H_\bullet(D_\bullet). \tag{3.3.19}$$

Since ϕ reduces to an isomorphism on homology and $\operatorname{Im} d_C \subset \ker \phi$, we have that the image of d_C is equal to $\ker \tilde{\phi}$, hence $\operatorname{Im} d_C$ is the kernel of a continuous, linear map. Since all involved spaces are Hausdorff, this kernel is closed, and we are done.
ii) This follows from i) since a continuous homotopy equivalence of chain complexes induces continuous quasi-isomorphisms in both directions. □

Proposition 3.3.13. *[MV97, Theorem 26.3] If $\phi : E \to F$ is a continuous linear map of Fréchet spaces, its range is closed if and only if its transpose $\phi^* : F^* \to E^*$ has closed range, where we denote by E^* and F^* the respective strong duals of E and F.*

Corollary 3.3.14. *Let A be a Fréchet algebra.*

i) The differential of the Connes complex $C_\bullet^{\lambda,\mathrm{born}}(A)$ has closed range if and only if the differential of $CC_\bullet^{\mathrm{born}}(A)$ has closed range.

ii) If A is unital, then all differentials of the complexes

$$C_\bullet^{\lambda,\mathrm{born}}(A), \quad CC_\bullet^{\mathrm{born}}(A), \quad B_\bullet^{\mathrm{born}}(A) \tag{3.3.20}$$

have closed range if and only if a single one of them does.

Proof. **i)** $\Longrightarrow$: If $C_\bullet^{\lambda,\mathrm{born}}(A)$ has closed range, then the quasi-isomorphism from Proposition 3.3.6 together with Proposition 3.3.12 shows that $CC_\bullet^{\mathrm{born}}(A)$ has closed range as well.
$\Longleftarrow$: If $CC_\bullet^{\mathrm{born}}(A)$ has closed range, then by Proposition 3.3.13 so does the cohomological

complex $CC^{\bullet}_{\mathrm{born}}(A)$ since all spaces in the homological total complex are Fréchet and all differentials of the cohomological complex are induced by dualization. Again Propositions 3.3.6 and 3.3.12 show that the cohomological complex $C^{\bullet}_{\lambda,\mathrm{born}}(A)$ has closed range as well. By Proposition 3.3.13 this extends to $C^{\lambda,\mathrm{born}}_{\bullet}(A)$. The equivalence is shown.
ii) This follows from i) and the continuous homotopy equivalence in Proposition 3.3.8.

□

3

3.3.3. THE CASE OF SMOOTH FUNCTIONS

The most important algebra in our context is $A = C^\infty(M)$, the algebra of smooth functions on a smooth manifold M with its standard Fréchet structure, see for example [Trè67, Chapter 10.1] for details. The following theorem on its bornological/continuous Hochschild homology with coefficients in itself its well-established, see [Con85] for the cohomological proof when M is compact, and [Pfl98], [Tel98] for extensions to the noncompact case.

Theorem 3.3.15. *Let M be a smooth manifold (possibly with boundary). Equipping the Fréchet space of forms $\Omega^\bullet(M)$ with the zero differential, the map*

$$\begin{aligned} HC^{\mathrm{born}}_{\bullet}(C^\infty(M), C^\infty(M)) &\to \Omega^\bullet(M), \\ f_0 \otimes \cdots \otimes f_n &\mapsto f_0 d_{\mathrm{dR}} f_1 \wedge \cdots \wedge d_{\mathrm{dR}} f_n \quad \forall f_i \in C^\infty(M) \end{aligned} \tag{3.3.21}$$

is a quasi-isomorphism.

The calculation of bornological cyclic homology of $C^\infty(M)$ builds on the above fact and is also well-established. We will state it here together with the additional information that the differential has closed range.

Theorem 3.3.16. *Let $A = C^\infty(M)$ with its standard Fréchet space structure. Then*

$$H^{\lambda,\mathrm{born}}_n(C^\infty(M)) = \frac{\Omega^n(M)}{d_{\mathrm{dR}}\Omega^{n-1}(M)} \oplus H^{n-2}_{\mathrm{dR}}(M) \oplus H^{n-4}_{\mathrm{dR}}(M) \oplus \ldots \tag{3.3.22}$$

and the differential of the Connes complex $C^{\lambda,\mathrm{born}}_{\bullet}(C^\infty(M))$ has closed range.

Proof. The first statement is proven, for example, in [Kha13, Example 3.10.1], and we sketch the proof. The isomorphism from Theorem 3.3.15 induces continuous chain map from the bornological (b, B)-double complex of $C^\infty(M)$ to the following complex of Fréchet spaces:

$$
\begin{tikzcd}
\cdots \arrow[d, "0"] & \cdots \arrow[d, "0"] & \cdots \arrow[d, "0"] \\
\Omega^2(M) \arrow[d, "0"] & \Omega^1(M) \arrow[l, "d_{\mathrm{dR}}"'] \arrow[d, "0"] & \Omega^0(M) \arrow[l, "d_{\mathrm{dR}}"'] \\
\Omega^1(M) \arrow[d, "0"] & \Omega^0(M) \arrow[l, "d_{\mathrm{dR}}"'] & \\
\Omega^0(M) & &
\end{tikzcd}
\tag{3.3.23}
$$

3

This map is compatible with the differentials, and if we filter both double complexes by their columns and consider the induced spectral sequences, the map descends to an isomorphism on the first page. Thus, by the spectral sequence comparison theorem (see [Wei94, Theorem 5.2.12]), the original map is a quasi-isomorphism of the total complexes. Then the calculation of bornological cyclic homology of $C^\infty(M)$ follows from the easy calculation of the total homology of (3.3.23). [Pal72, Proposition 5.4] states that the de Rham differential associated to a smooth manifold has closed range, hence also the total differential of the double complex (3.3.23)). Thus, by Proposition 3.3.12, so do the differentials of the total complex of the bornological (b, B)-double complex, and by Corollary 3.3.14 also the differential of $C^\lambda_\bullet(A)$. This proves the statement. □

3.3.4. THE CASE OF COMPACTLY SUPPORTED FUNCTIONS

In preparation for results about compactly supported gauge algebras, we want to establish some properties of algebras related to compactly supported functions. Specifically, recall that one generally defines the topology of $C_c^\infty(\mathbb{R}^n)$ as a direct limit: For every compact $K \subset \mathbb{R}^n$, define the Fréchet subalgebra

$$C_K^\infty(\mathbb{R}^n) := \{f \in C^\infty(\mathbb{R}^n) : \operatorname{supp} f \subset K\} \subset C^\infty(\mathbb{R}^n). \tag{3.3.24}$$

Denote by $\bar{D}_r(0) \subset \mathbb{R}^n$ the closed disk of radius r around 0. Then the inclusions $\bar{D}_r(0) \subset \bar{D}_{r'}(0)$ for $r' > r$ induce a direct system $\{C^\infty_{\bar{D}_r(0)}(\mathbb{R}^n)\}_{r>0}$ with

$$C_c^\infty(\mathbb{R}^n) = \varinjlim C^\infty_{\bar{D}_r(0)}(\mathbb{R}^n). \tag{3.3.25}$$

All spaces within this direct system are Fréchet spaces, so if one equips $C_c^\infty(\mathbb{R}^n)$ with the inductive limit topology, it is a strict LF-space. We will avoid working with this LF-space directly and just work with the underlying Fréchet spaces, since homology and $\widehat{\otimes}$ commute with direct limits on strict LF-spaces. However, we want to remark that the LF-algebra $C_c^\infty(\mathbb{R}^n)$ itself also has good properties with respect to bornological homology theories, see [Mey10]. From here on, set $D := \bar{D}_1(0) \subset \mathbb{R}^n$. We first cite the following important factorization result from [Voi84, Theorem 3.4]:

Theorem 3.3.17. *The Fréchet algebra $C^\infty_D(\mathbb{R}^n)$ fulfills the* bounded strong factorization property: *For every bounded set $B \subset C^\infty_D(\mathbb{R}^n)$, there is a $z \in C^\infty_D(\mathbb{R}^n)$, a continuous linear operator $T : C^\infty_D(\mathbb{R}^n) \to C^\infty_D(\mathbb{R}^n)$ and a sequence $\{\phi_n\}_{n\geq 1} \subset C^\infty_D(\mathbb{R}^n)$ with*

$$z \cdot T(x) = x, \qquad T(x) = \lim_{n\to\infty} \phi_n \cdot x \quad \forall x \in B. \tag{3.3.26}$$

Remark 3.3.18. Note that in [Voi84, Definition 1.1], the definition of the strong factorization property is stated slightly different, and weaker in one particularly important aspect: They require that for every $x \in B$, the element $T(x)$ must be contained in $\overline{C^\infty_D(\mathbb{R}^n) \cdot x}$, the closed ideal generated by x. This, too, would imply $T(x) = \lim_{n\to\infty} \phi_n \cdot x$ for some ϕ_n, but the choice of ϕ_n may depend on x. The stronger statement that the ϕ_n can be chosen independent of x is also true, as seen in the proof [Voi84, Proposition 2.7] which is used to prove the above theorem. This goes unstated in our cited material, but is important not only for our application, but also for other users of this material such as [Ewa04, Proposition 3.4], [Gon92, Theorem 7], [Wod89, Theorem 6.1].

Corollary 3.3.19. *For all $f \in C^\infty_{D^k}((\mathbb{R}^n)^k)$, there are $g \in C^\infty_D(\mathbb{R}^n)$, $h \in C^\infty_{D^k}((\mathbb{R}^n)^k)$ with*

$$f(x_1,\ldots,x_k) = g(x_1)\cdot h(x_1,\ldots,x_k) \quad \forall x_i \in D, \tag{3.3.27}$$

so that if f in the above is a cycle in bar homology, so is h.

Proof. Since $C^\infty_{D^k}((\mathbb{R}^n)^k) \cong (C^\infty_D(\mathbb{R}^n))^{\widehat{\otimes}k}$, [Trè67, Theorem 45.1] implies that we can represent every $f \in C^\infty_{D^k}((\mathbb{R}^n)^k))$ as a certain series, i.e. there are null sequences

$$\{f_1^n\}_{n\geq 1},\ldots,\{f_k^n\}_{n\geq 1} \subset C^\infty_D(\mathbb{R}^n) \tag{3.3.28}$$

and a sequence of complex numbers $\{\lambda_n\}_{n\geq 1}$ with $\sum_{n=1}^\infty |\lambda_n| < 1$, so that

$$f = \sum_{n=1}^{\infty} \lambda_n f_1^n \otimes \cdots \otimes f_k^n, \tag{3.3.29}$$

where the sum is *absolutely convergent* as a series of Fréchet space elements, meaning that for a generating sequence of seminorms $\{p_i\}_{i\geq 0}$ of the topology of $C^\infty_D(\mathbb{R}^n)$, the following real-valued series converges for all $i_1,\ldots,i_k \geq 0$:

$$\sum_{n=1}^{\infty} \lambda_n \cdot p_{i_1}(f_1^n)\ldots p_{i_k}(f_k^n). \tag{3.3.30}$$

Since the set $\{f_1^n\}_{n\geq 1}$ is convergent, it is bounded, and we can apply Theorem 3.3.17 to get a continuous operator $T : C^\infty_D(\mathbb{R}^n) \to C^\infty_D(\mathbb{R}^n)$, a function $g \in C^\infty_D(\mathbb{R}^n)$ and a sequence $\{\phi_m\}_{m\geq 1} \in C^\infty_D(\mathbb{R}^n)$ with

$$f_1^n = g \cdot T f_1^n, \quad T f_1^n = \lim_{m\to\infty} \phi_m \cdot f_1^n \quad \forall n \in \mathbb{N}. \tag{3.3.31}$$

By continuity of T the sequence

$$h := \sum_{n=1}^{\infty} \lambda_n (T f_1^n) \otimes \cdots \otimes f_k^n \tag{3.3.32}$$

is also absolutely convergent and thus defines an element in $C^\infty_{D^k}((\mathbb{R}^n)^k)$. Thus

$$f(x_1, \ldots, x_n) = g(x_1) \cdot h(x_1, \ldots, x_n) \quad \forall x_i \in \mathbb{R}^n. \tag{3.3.33}$$

Additionally, if f is a cycle in bar homology, then, since the bar differential b' is continuous and $C^\infty_D(\mathbb{R}^n)$-linear with respect to multiplication in the first tensor argument, we have

$$\lim_{m\to\infty} b'((\phi_m f_1^n) \otimes \cdots \otimes f_k^n) = \lim_{m\to\infty} \phi_m \cdot b'(f_1^n \otimes \cdots \otimes f_k^n). \tag{3.3.34}$$

But then $b'(h) = \lim_{m\to\infty} \phi_m \cdot b'(f)$, so if f was a cycle in bar homology, so is h. This concludes the proof. $\square$

Corollary 3.3.20. *The bornological bar complex of $C^\infty_D(\mathbb{R}^n)$ is acyclic.*

Proof. In the notation of Corollary 3.3.19, every bar chain $f \in C_k^{\mathrm{bar}}(C^\infty_D(\mathbb{R}^n))$ fulfills $f = b'(g \otimes h) + g \otimes b'(h)$, with $b'(f) = 0$ implying $b'(h) = 0$. This shows the statement. $\square$

Proposition 3.3.21. *In the notation of Appendix D, consider the space*

$$\Omega^\bullet_{\mathrm{flat}}(D, \partial D) := \{\omega \in \Omega^\bullet(D) : (j^\infty \omega)\big|_{\partial D} = 0\} \subset \Omega^\bullet(D). \tag{3.3.35}$$

If $\Omega^\bullet_{\mathrm{flat}}(D, \partial D)$ is equipped with the zero differential, then the quasi-isomorphism on the bornological Hochschild complex

$$HC_\bullet^{\mathrm{born}}(C^\infty(D)) \to \Omega^\bullet(D) \tag{3.3.36}$$

from Theorem 3.3.15 restricts to a quasi-isomorphism

$$HC_\bullet^{\mathrm{born}}(C^\infty_D(\mathbb{R}^n)) \to \Omega^\bullet_{\mathrm{flat}}(D, \partial D). \tag{3.3.37}$$

Proof. In the sense of [BL09, Section 2], the bornological Hochschild complex of the subalgebra $C^\infty_D(\mathbb{R}^n) \subset C^\infty(D \setminus \partial D)$ is *locally isomorphic* to the bornological Hochschild complex of $C^\infty(D \setminus \partial D)$. But then [BL09, Proposition 2.2] shows that the bornological Hochschild homology of $C^\infty_D(\mathbb{R}^n)$ is, via the above morphism, isomorphic to the subalgebra of differential forms of $\Omega^\bullet(D \setminus \partial D)$ generated by the functions in $C^\infty_D(\mathbb{R}^n)$. But this is exactly the space $\Omega^\bullet_{\mathrm{flat}}(D, \partial D)$. $\square$

We investigate this flat de Rham complex more in Appendix D. From the results there, we conclude the following:

Theorem 3.3.22. *The differential of the bornological cyclic complex of $C^\infty_D(\mathbb{R}^n)$ has closed range and*

$$H_k^{\lambda,\mathrm{born}}(C^\infty_D(\mathbb{R}^n)) = \frac{\Omega^k_{\mathrm{flat}}(D,\partial D)}{d_{\mathrm{dR}}\Omega^{k-1}_{\mathrm{flat}}(D,\partial D)} \oplus H^{k-2}_{\mathrm{sg}}(D,\partial D) \oplus H^{k-4}_{\mathrm{sg}}(D,\partial D) \oplus \ldots \tag{3.3.38}$$

where the relative homology denotes relative singular homology.

Proof. By Lemma D.1, the differential of $\Omega^\bullet_{\mathrm{flat}}(D,\partial D)$ has closed range, thus the quotient map

$$\Omega^\bullet_{\mathrm{flat}}(D,\partial D) \to \frac{\Omega^\bullet_{\mathrm{flat}}(D,\partial D)}{d\Omega^\bullet_{\mathrm{flat}}(D,\partial D)[-1]} \tag{3.3.39}$$

is a continuous map of Fréchet spaces.
The composition of the quasi-isomorphism $HC^{\mathrm{born}}_\bullet(C^\infty_D(\mathbb{R}^n)) \to \Omega^\bullet_{\mathrm{flat}}(D,\partial D)$ from Proposition 3.3.21 with this quotient map gives us a continuous morphism from the double complex $CC^{\mathrm{born}}_{\bullet,\bullet}(C^\infty_D(\mathbb{R}^n))$ to the following:

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \iddots \\
\frac{\Omega^2_{\mathrm{flat}}(D,\partial D)}{d_{\mathrm{dR}}\Omega^1_{\mathrm{flat}}(D,\partial D)} \arrow[d] & 0 \arrow[l] \arrow[d] & H^2_{\mathrm{sg}}(D,\partial D) \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & H^2_{\mathrm{sg}}(D,\partial D) \arrow[l] \arrow[d] & \ldots \arrow[l] \\
\frac{\Omega^1_{\mathrm{flat}}(D,\partial D)}{d_{\mathrm{dR}}\Omega^0_{\mathrm{flat}}(D,\partial D)} \arrow[d] & 0 \arrow[l] \arrow[d] & H^1_{\mathrm{sg}}(D,\partial D) \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & H^1_{\mathrm{sg}}(D,\partial D) \arrow[l] \arrow[d] & \ldots \arrow[l] \\
\Omega^0_{\mathrm{flat}}(D,\partial D) & 0 \arrow[l] & H^0_{\mathrm{sg}}(D,\partial D) \arrow[l] & 0 \arrow[l] & H^0_{\mathrm{sg}}(D,\partial D) \arrow[l] & \ldots \arrow[l]
\end{tikzcd} \tag{3.3.40}$$

All differentials in (3.3.40) are set to zero. By Lemma D.1, $H^k_{\mathrm{sg}}(D,\partial D)$ is either zero or equal to $\frac{\Omega^n_{\mathrm{flat}}(D,\partial D)}{d_{\mathrm{dR}}\Omega^{n-1}_{\mathrm{flat}}(D,\partial D)}$ for all $k \geq 0$, so the zero map and the quotient map (3.3.39) suffice to construct the indicated map of double complexes. Calculate the spectral sequence of $C^{\mathrm{born}}_{\bullet,\bullet}(C^\infty_D(\mathbb{R}^n))$ arising from filtration by columns. By Proposition 3.3.21 and Corollary 3.3.20 the first page $E_1^{\bullet,\bullet}$ contains $\Omega^\bullet_{\mathrm{flat}}(D,\partial D)$ in even-numbered columns and 0 in odd-numbered columns, so we have $E_1^{\bullet,\bullet} = E_2^{\bullet,\bullet}$.
By unravelling the connecting homomorphisms using the homotopy equation in the proof of Corollary 3.3.20 and the construction of the elements in the equation from Corollary 3.3.19, one can explicitly spell out the differential on the second page, showing that the nontrivial differentials $\Omega^\bullet_{\mathrm{flat}}(D,\partial D) \to \Omega^{\bullet+1}_{\mathrm{flat}}(D,\partial D)$ are equal to the de Rham differential. The cohomology of this flat, relative de Rham complex is equal to relative singular homology, see Lemma D.1. Hence the third page of the spectral sequence is exactly equal to the double complex (3.3.40). Hence the map of double complexes we

constructed is an isomorphism between the third pages of the spectral sequences. By the spectral sequence comparison theorem [Wei94, Theorem 5.2.12], this implies that the original map of double complexes is a quasi-isomorphism of the total complexes. This calculates the bornological cyclic homology as stated. Lastly, since the total differential of the double complex (3.3.40) is zero, it is closed, and hence the differential of the total complex $CC_\bullet^{\mathrm{born}}(C_D^\infty(\mathbb{R}^n))$ has closed range by Proposition 3.3.12. This concludes the proof. □

3

Corollary 3.3.23. *We have for all $k \geq 0$*

$$H_\bullet^{\lambda,\mathrm{born}}(C_c^\infty(\mathbb{R}^n)) \cong \frac{\Omega_c^k(\mathbb{R}^n)}{d_{\mathrm{dR}}\Omega_c^{k-1}(\mathbb{R}^n)} \oplus H_{\mathrm{dR},c}^{k-2}(\mathbb{R}^n) \oplus H_{\mathrm{dR},c}^{k-4}(\mathbb{R}^n) \oplus \dots \tag{3.3.41}$$

where $H_{\mathrm{dR},c}^\bullet(\mathbb{R}^n)$ denotes compactly supported de Rham cohomology of $\mathbb{R}^n$.

Proof. This follows from Theorem 3.3.22 and the direct limit $C_c^\infty(\mathbb{R}^n) = \varinjlim C_{\bar{D}_r(0)}^\infty(\mathbb{R}^n)$, since homology, the cyclic action, and the bornological tensor product commute with strict direct limits, and

$$\frac{\Omega_c^k(\mathbb{R}^n)}{d_{\mathrm{dR}}\Omega_c^{k-1}(\mathbb{R}^n)} = \varinjlim \frac{\Omega_{\mathrm{flat}}^k(\bar{D}_r, \partial\bar{D}_r(0))}{d_{\mathrm{dR}}\Omega_{\mathrm{flat}}^{k-1}(\bar{D}_r(0), \partial\bar{D}_r(0))}. \tag{3.3.42}$$

□

Remark 3.3.24. In preparation for what follows, we want to note that the above methods generalize to finite, topological direct sums of copies of $C_D^\infty(\mathbb{R}^n)$, meaning

$$H_\bullet^{\mathrm{bar,born}}\left(\bigoplus_{i=1}^r C_D^\infty(\mathbb{R}^n)\right) = 0, \quad H_k^{\lambda,\mathrm{born}}\left(\bigoplus_{i=1}^r C_D^\infty(\mathbb{R}^n)\right) \cong \bigoplus_{i=1}^r H_k^\lambda\left(C_D^\infty(\mathbb{R}^n)\right). \tag{3.3.43}$$

3.4. LODAY–QUILLEN–TSYGAN THEOREMS FOR FRÉCHET ALGEBRAS

In this section, we want to extend the classical Loday–Quillen–Tsygan theorem to the setting of certain bornological algebras. For a detailed exposition of the algebraic Loday–Quillen–Tsygan theorem, we direct the reader to [Lod92, Chapter 9 & 10], or, for an abridged version of the relevant details, Appendix C.

3.4.1. A GENERAL TOPOLOGICAL LQT-THEOREM

Definition 3.4.1. Let A be a bornological algebra and $n \in \mathbb{N}$. Consider the bornological Lie algebra

$$\mathfrak{gl}_n(A) := \mathfrak{gl}_n(\mathbb{K}) \hat{\otimes} A \tag{3.4.1}$$

3

whose Lie bracket is induced by the products on $\mathfrak{gl}_n(\mathbb{K})$ and A in the following way:

$$[g \otimes a, h \otimes b] := (gh) \otimes (ab) - (hg) \otimes (ba) \quad \forall g, h \in \mathfrak{gl}(\mathbb{K}),\ a, b \in A. \tag{3.4.2}$$

For $m \geq n$, the inclusions $\mathfrak{gl}_n(A) \to \mathfrak{gl}_m(A)$ define a direct system of Lie algebras, and we can define

$$\mathfrak{gl}(A) := \mathfrak{gl}_\infty(A) := \varinjlim \mathfrak{gl}_n(A) \tag{3.4.3}$$

Remark 3.4.2. Note that all $\mathfrak{gl}_n(\mathbb{K})$ for $1 \leq n < \infty$ are finite-dimensional. Hence if A is a complete bornological algebra, we can equivalently write

$$\mathfrak{gl}_n(\mathbb{K}) \mathbin{\widehat{\otimes}} A = \mathfrak{gl}_n(\mathbb{K}) \otimes_\beta A = \mathfrak{gl}_n(\mathbb{K}) \otimes A. \tag{3.4.4}$$

If A is Fréchet, then all $\mathfrak{gl}_n(\mathbb{K}) \otimes A$ are Fréchet, so $\mathfrak{gl}(A)$ is a strict LF-space. In this case, since the bornological tensor product on strict LF-spaces is compatible with inductive limits, we have

$$\mathfrak{gl}(A) = \varinjlim (\mathfrak{gl}_n(\mathbb{K}) \mathbin{\widehat{\otimes}} A) \cong \mathfrak{gl}(\mathbb{K}) \mathbin{\widehat{\otimes}} A. \tag{3.4.5}$$

Definition 3.4.3. Let $\mathfrak{g}$ be a bornological Lie algebra. We define *bornological Lie algebra homology* of this $\mathfrak{g}$ to be the homology of the *bornological Chevalley–Eilenberg chain complex*

$$C_\bullet^{\mathrm{born}}(\mathfrak{g}) := \left(\mathbb{K} \xleftarrow{0} \hat{\Lambda}^1\mathfrak{g} \xleftarrow{d} \hat{\Lambda}^2\mathfrak{g} \xleftarrow{d} \dots\right), \tag{3.4.6}$$

where $\Lambda^k\mathfrak{g}$ denotes the coinvariants of $\otimes^k\mathfrak{g}$ with respect to the action of the symmetric group Σ_k by antisymmetrization, and the hat denotes completion in the bornological tensor product. The differential is the extension of the Chevalley–Eilenberg differential to the completion:

$$d(g_1 \wedge \cdots \wedge g_n) := \sum_{i<j} (-1)^{i+j-1} [g_i, g_j] \wedge g_1 \wedge \cdots \hat{g}_i \cdots \hat{g}_j \cdots \wedge g_n \quad \forall g_i \in \mathfrak{g}. \tag{3.4.7}$$

Proposition 3.4.4. *For every $n \in \mathbb{N}$ and bornological algebra A, the space $\mathfrak{gl}_n(A)$ admits an action by $\mathfrak{gl}_n(\mathbb{K})$. If A is unital, then the reduction to coinvariants*

$$C_\bullet^{\mathrm{born}}(\mathfrak{gl}_n(A)) \to C_\bullet^{\mathrm{born}}(\mathfrak{gl}_n(A))_{\mathfrak{gl}_n(\mathbb{K})} \tag{3.4.8}$$

is a quasi-isomorphism.

Proof. Clearly, $\mathfrak{gl}_n(\mathbb{K})$ acts on $\mathfrak{gl}_n(A) \cong \mathfrak{gl}_n(\mathbb{K}) \otimes A$ by a tensor product of the adjoint action and the trivial action. Unitality of A implies that $\mathfrak{gl}_n(\mathbb{K}) \subset \mathfrak{gl}_n(A)$ exists as a subalgebra, and hence $\mathfrak{gl}_n(\mathbb{K})$ acts on the $\mathfrak{gl}_n(A)$-cochains. By Proposition 3.2.8, the completed bornological tensor product is associative and commutative, and if one of its factors is finite-dimensional, it agrees with the algebraic tensor product as a vector space. Hence:

$$C_k^{\mathrm{born}}(\mathfrak{gl}_n(A)) \cong \left(\mathfrak{gl}_n(\mathbb{K})^{\otimes^k} \otimes A^{\widehat{\otimes}^k}\right)_{\Sigma_k}. \tag{3.4.9}$$

We know that since $\mathfrak{gl}_n(\mathbb{K})$ acts trivially on A, and the finite-dimensional tensor module $\mathfrak{gl}_n(\mathbb{K})^{\otimes^k}$ is completely reducible, hence $C_k^{\mathrm{born}}(\mathfrak{gl}_n(A))$ is completely reducible. From here on, the proof is essentially identical to the proof in algebraic setting, see [Lod92, Proposition 10.1.18]. Complete reducibility gives us

3

$$C_\bullet^{\mathrm{born}}(\mathfrak{gl}_n(A)) = C_\bullet^{\mathrm{born}}(\mathfrak{gl}_n(A))_{\mathfrak{gl}_n(\mathbb{K})} \oplus L_\bullet, \tag{3.4.10}$$

where $L_\bullet$ is a direct sum of simple modules, all of which $\mathfrak{gl}_n(\mathbb{K})$ acts nontrivially on. Since the $\mathfrak{gl}_n(\mathbb{K})$-action commutes with the Lie algebra differential, this is even a decomposition into subcomplexes. We can further decompose $L_\bullet = Z_\bullet \oplus K_\bullet$, where $Z_\bullet = L_\bullet \cap \ker d$, and $K_\bullet$ is a module-theoretic complement of $Z_\bullet$ in $L_\bullet$ so that both $Z_\bullet, K_\bullet$ are direct sums of simple modules. Clearly, $H_\bullet(L_\bullet) = H_\bullet(Z_\bullet)$. A simple $\mathfrak{gl}_n(\mathbb{K})$-module $M \subset Z_\bullet$ is generated by any of its elements that $\mathfrak{gl}_n(\mathbb{K})$ acts nontrivially on. As a consequence, every element of $L_\bullet$ is of the form $X \cdot c \in Z_\bullet$ for some cochain $c \in Z_\bullet$ and $X \in \mathfrak{gl}(\mathbb{K})$. We further have, for all $X \in \mathfrak{gl}_n(\mathbb{K})$ and $c \in C_\bullet(\mathfrak{gl}_n(A))$ the homotopy equation

$$X \cdot c = d(X \wedge c) + X \wedge dc. \tag{3.4.11}$$

Hence every element of $Z_\bullet$ is of the form $X \cdot c = d(X \wedge c)$, so a boundary. This shows that $Z_\bullet$ and $L_\bullet$ are acyclic, and the proof is done. □

Corollary 3.4.5. *If A is a unital Fréchet algebra, then*

$$H_\bullet^{\mathrm{born}}(\mathfrak{gl}(A)) \cong H_\bullet\left(\bigoplus_k \left(\mathbb{K}[\Sigma_k] \otimes A^{\widehat{\otimes}^k}\right)_{\Sigma_k}\right). \tag{3.4.12}$$

Proof. By Proposition 3.2.8, the tensor product $\widehat{\otimes}$ commutes with direct limits on LF-spaces, so

$$\begin{aligned} C_k^{\mathrm{born}}(\mathfrak{gl}(A)) &= \left(\mathfrak{gl}(\mathbb{K})^{\otimes_\beta^k} \otimes_\beta A^{\widehat{\otimes}^k}\right)_{\Sigma_k} \\ &= \varinjlim \left(\mathfrak{gl}_n(\mathbb{K})^{\otimes^k} \otimes A^{\widehat{\otimes}^k}\right)_{\Sigma_k} = \varinjlim C_k^{\mathrm{born}}(\mathfrak{gl}_n(A)). \end{aligned} \tag{3.4.13}$$

By Proposition 3.4.4, the reduction to coinvariants

$$C_\bullet^{\mathrm{born}}(\mathfrak{gl}_n(A)) \to C_\bullet^{\mathrm{born}}(\mathfrak{gl}_n(A))_{\mathfrak{gl}_n(\mathbb{K})} \tag{3.4.14}$$

is a quasi-isomorphism for all $n \in \mathbb{N}$. From the case $A = \mathbb{K}$ considered in Propostion C.2, we get, for $n \geq k$, the isomorphism

$$C_\bullet^{\mathrm{born}}(\mathfrak{gl}_n(A))_{\mathfrak{gl}_n(\mathbb{K})} \cong \left(\left(\mathfrak{gl}_n(\mathbb{K})^{\otimes^k}\right)_{\mathfrak{gl}_n(\mathbb{K})} \otimes A^{\widehat{\otimes}^k}\right)_{\Sigma_k} \cong \left(\mathbb{K}[\Sigma_k] \otimes A^{\widehat{\otimes}^k}\right)_{\Sigma_k}. \tag{3.4.15}$$

3

Finally, since homology commutes with direct limits, and as every graded component of $C_\bullet^{\text{born}}(\mathfrak{gl}_n(A))_{\mathfrak{gl}_n(\mathbb{K})}$ becomes constant at some point in the direct limit, we get the following chain of isomorphisms:

$$\begin{aligned} H_\bullet^{\text{born}}(\mathfrak{gl}(A)) &\cong \varinjlim H_\bullet^{\text{born}}(\mathfrak{gl}_n(A)) \\ &\cong \varinjlim H_\bullet(C_\bullet^{\text{born}}(\mathfrak{gl}_n(A))_{\mathfrak{gl}_n(\mathbb{K})}) \cong H_\bullet\left(\bigoplus_k \left(\mathbb{K}[\Sigma_k] \otimes A^{\widehat{\otimes}^k}\right)_{\Sigma_k}\right). \end{aligned} \tag{3.4.16}$$

□

Hence we may work with the latter complex instead, which is in some sense simpler: Since all Σ_k are finite groups, it is a complex of Fréchet spaces. A drawback is that the differential on this subcomplex is more difficult to describe. Regardless, we have the following:

Proposition 3.4.6. *Let A be a Fréchet algebra. The isomorphism of chain complexes*

$$\theta : \Lambda^\bullet C_{\bullet-1}^\lambda(A) \to \bigoplus_{k\in\mathbb{N}} \left(\mathbb{K}[\Sigma_k] \otimes A^{\otimes^k}\right)_{\Sigma_k} \tag{3.4.17}$$

from Proposition C.3 extends to a continuous isomorphism of chain complexes

$$\hat{\theta} : \hat{\Lambda}^\bullet C_{\bullet-1}^{\lambda,\text{born}}(A) \to \bigoplus_{k\in\mathbb{N}} \left(\mathbb{K}[\Sigma_k] \otimes A^{\widehat{\otimes}^k}\right)_{\Sigma_k}. \tag{3.4.18}$$

Proof. By definition of θ, we can decompose

$$\Lambda^\bullet C_{\bullet-1}^\lambda(A) = \bigoplus_{k\geq 1} \bigoplus_{[\sigma]\subset\Sigma_k} Z_{[\sigma]}, \tag{3.4.19}$$

where the direct sum over $[\sigma]$ is carried out over all conjugacy classes $[\sigma] \in \Sigma_k$, and $Z_{[\sigma]}$ the span of all elements $[u_1] \wedge \cdots \wedge [u_r] \in \Lambda^\bullet C_{\bullet-1}^\lambda(A)$ with

$$\exists \tau \in [\sigma] : \theta([u_1] \wedge \cdots \wedge [u_r]) = [\tau \otimes (u_1 \otimes \cdots \otimes u_r)]. \tag{3.4.20}$$

For each $k \geq 1$ and conjugacy class $[\sigma] \subset \Sigma_k$, choose a representative σ in cycle decomposition

$$\sigma = (1 \cdots k_1) \circ (k_1 + 1 \cdots k_2) \circ \cdots \circ (k_{r-1} + 1 \cdots k_r). \tag{3.4.21}$$

Then, on $Z_{[\sigma]}$, the map θ arises from the continuous map

$$A^{\otimes^k} \to \mathbb{K}[\Sigma_k] \otimes A^{\otimes^k}, \quad a_1 \otimes \cdots \otimes a_k \mapsto \sigma \otimes a_1 \otimes \cdots \otimes a_k, \tag{3.4.22}$$

by composition with the quotient map

$$\mathbb{K}[\Sigma_k] \otimes A^{\otimes^k} \to \left(\mathbb{K}[\Sigma_k] \otimes A^{\otimes^k}\right)_{\Sigma_k}, \tag{3.4.23}$$

factoring through the kernel, and then restricting to the direct summand $Z_{[\sigma]}$ in the kernel. These actions leave continuity invariant, so θ, when restricted to the direct summand $Z_{[\sigma]}$, inherits continuity. But then θ itself is continuous. Also, for every conjugacy class $[\sigma] \subset \Sigma_k$, the restriction $\theta|_{Z_{[\sigma]}} : Z_{[\sigma]} \to \theta(Z_{[\sigma]})$ admits a continuous inverse, induced in the same way by a map

$$\mathbb{K}[[\sigma]] \otimes A^{\otimes^k} \to A^{\otimes^k}, \quad (\tau^{-1}\sigma\tau) \otimes a_1 \otimes \cdots \otimes a_k \mapsto a_{\tau(1)} \otimes \cdots \otimes a_{\tau(k)}. \tag{3.4.24}$$

The above assignment defines a continuous map due to the finiteness of the conjugacy class $[\sigma]$. Since both θ and its inverse are continuous, they extend to continuous maps between the completions of their respective domains and codomains, and these extensions still compose to the identity. Thus, these extensions are isomorphisms of chain complexes, proving the statement. □

Corollary 3.4.7. *Let A be a unital Fréchet algebra. There is a continuous quasi-isomorphism*

$$C^{\text{born}}_{\bullet}(\mathfrak{gl}(A)) \to \hat{\Lambda}^{\bullet} C^{\lambda,\text{born}}_{\bullet-1}(A). \tag{3.4.25}$$

Corollary 3.3.11 then finally implies:

Theorem 3.4.8. *Let A be a nuclear unital Fréchet algebra, and assume that the differential of the bornological cyclic complex $C^{\lambda,\text{born}}_{\bullet}(A)$ has closed range. Then we have, for all $r, n \in \mathbb{N}$ with $r+1 \le n$*

$$H^{\text{born}}_r(\mathfrak{gl}_n(A)) \cong \left(\hat{\Lambda}^{\bullet} H^{\lambda,\text{born}}_{\bullet-1}(A)\right)_r, \tag{3.4.26}$$

and

$$H^{\text{born}}_{\bullet}(\mathfrak{gl}(A)) \cong \hat{\Lambda}^{\bullet} H^{\lambda,\text{born}}_{\bullet-1}(A). \tag{3.4.27}$$

Remark 3.4.9. Note that the only place where it mattered that we used $\mathfrak{gl}(A)$ rather than any of the other limits of classical simple Lie algebras $\mathfrak{sl}(A)$, $\mathfrak{sp}(A)$ or $\mathfrak{so}(A)$ was in the explicit form of the coinvariants for the tensor modules $\mathfrak{gl}(\mathbb{K})^{\otimes^k}$. We will not explicitly present this, but we do want to remark that one can gain statements analogous to Theorem 3.4.8 for these other Lie algebras with very little modification, save that one occasionally may need to replace cyclic homology with the closely related *dihedral homology*, which we have not defined here. We direct the reader to [Lod92, Chapter 9 & 10] for a detailed discussion of the algebraic setting.

Lastly, these results extend to certain non-unital algebras as well. Specifically, in the algebraic setting one can weaken the assumption of unitality to *H-unitality*, a property which is defined as the acyclicity of the algebraic bar complex of A, see [Han88] for a proof in the finite-dimensional setting and the preprint [Cor05] (supplementing the publication [Cn06]) for an explicit generalization to infinite-dimensional algebras. We delay the lengthy proof to Section 3.7:

3

Theorem 3.4.10. *Let A be a nuclear Fréchet algebra, and assume that the differential of the bornological cyclic complex $C_\bullet^{\lambda,\mathrm{born}}(A)$ has closed range. Additionally, assume that A is* bornologically H-unital, *i.e. the bornological bar complex $C^{\mathrm{bar},\mathrm{born}}(A)$ is acyclic. Then we have, for all $r, n \in \mathbb{N}$ with $2r+1 \leq n$:*

$$H_r^{\mathrm{born}}(\mathfrak{gl}_n(A)) \cong \left(\hat{\Lambda}^\bullet H_{\bullet-1}^{\lambda,\mathrm{born}}(A)\right)_r, \tag{3.4.28}$$

and

$$H_\bullet^{\mathrm{born}}(\mathfrak{gl}(A)) \cong \hat{\Lambda}^\bullet H_{\bullet-1}^{\lambda,\mathrm{born}}(A) \tag{3.4.29}$$

We give a rough sketch of the proof here: The complex $C_\bullet^{\mathrm{born}}(\mathfrak{gl}_n(A))$ can be decomposed into isotypic components with respect to the action of $\mathfrak{gl}_n(\mathbb{K})$. In the unital case, we have seen that only the invariant component contributes to homology. In a similar vein to the proof of Proposition 3.4.6, one constructs a morphism of every component to a certain complex involving combinations of the bar complex and the Connes complex of A, see the construction of ϕ in [Cor05, Theorem 3.1]. As one takes the direct limit $n \to \infty$, these morphisms become stable isomorphisms. In every isotypic component in which the bornological bar complex appears nontrivially, the acyclicity of the bornological bar complex forces the whole component to become acyclic; this leaves only the invariant component to contribute to homology, and this component is related to the cyclic complex exactly as in the unital setting.

3.4.2. APPLICATION TO FRÉCHET ALGEBRAS OF SMOOTH FUNCTIONS

We now apply the results of the previous section to the case when A equals some spaces of smooth functions.

Corollary 3.4.11. *Let M be a smooth manifold, we have*

$$H_\bullet^{\mathrm{born}}(\mathfrak{gl}(C^\infty(M))) \cong \hat{\Lambda}^\bullet H_\bullet^{\lambda,\mathrm{born}}(C^\infty(M)), \tag{3.4.30}$$

$$H_\bullet^{\mathrm{born}}(\mathfrak{gl}(C_c^\infty(\mathbb{R}^n))) \cong \hat{\Lambda}^\bullet H_\bullet^{\lambda,\mathrm{born}}(C_c^\infty(\mathbb{R}^n)). \tag{3.4.31}$$

Proof. We have established in Theorem 3.3.16 that the closed-range assumption of Theorem 3.4.8 holds for the nuclear Fréchet algebra $C^\infty(M)$, so the first isomorphism is shown. Further, we have shown in Corollary 3.3.20 and Theorem 3.3.22 that $C_{\bar{D}_r(0)}^\infty(\mathbb{R}^n)$ is bornologically H-unital for arbitrary $r > 0$ and fulfills the closed-range assumption of Theorem 3.4.10. This shows

$$H_\bullet^{\mathrm{born}}(\mathfrak{gl}(C_{\bar{D}_r(0)}^\infty(\mathbb{R}^n))) = \hat{\Lambda}^\bullet H_{\bullet-1}^{\lambda,\mathrm{born}}(C_{\bar{D}_r(0)}^\infty(\mathbb{R}^n)). \tag{3.4.32}$$

Now, homology and $\hat{\otimes}$ commute with direct limits, and so we have

$$\begin{aligned} H_\bullet^{\mathrm{born}}(\mathfrak{gl}(C_c^\infty(\mathbb{R}^n))) &= \varinjlim H_\bullet^{\mathrm{born}}(\mathfrak{gl}(C_{\bar{D}_r(0)}^\infty(\mathbb{R}^n))) \\ &= \varinjlim \hat{\Lambda}^\bullet H_{\bullet-1}^{\lambda,\mathrm{born}}(C_{\bar{D}_r(0)}^\infty(\mathbb{R}^n)) = \hat{\Lambda}^\bullet H_\bullet^{\lambda,\mathrm{born}}(C_c^\infty(\mathbb{R}^n)). \end{aligned} \tag{3.4.33}$$

□

3.5. BORNOLOGICAL HOMOLOGY OF NONTRIVIAL GAUGE ALGEBRAS

Now, let M be a finite-dimensional, smooth manifold, H a finite-dimensional Lie group with associated Lie algebra $\mathfrak{h}$, and $P \to M$ a principal H-bundle. Consider the *adjoint bundle* $\operatorname{Ad} P := P \times_{\operatorname{Ad}} \mathfrak{h}$. We would like to understand the bornological Lie algebra homology of the compactly supported gauge algebra $\Gamma_c(\operatorname{Ad} P)$. For small open sets $U \subset M$, sections of $\operatorname{Ad} P|_U \to U$ can be identified with $\mathfrak{h} \otimes C^\infty(U)$. As a consequence, for simple classical Lie algebras $\mathfrak{h}$, the methods of the previous section allow a calculation of the stable part of bornological homology $H_\bullet^{\mathrm{born}}(\mathfrak{h} \otimes C_c^\infty(U))$. By repeating the local-to-global methods from Chapter 2, we will be able to deduce information about $H_\bullet^{\mathrm{born}}(\Gamma_c(\operatorname{Ad} P))$ from this local data. As a model of the more general case, we will only consider the case $\mathfrak{h} = \mathfrak{gl}_n(\mathbb{K})$ for some $1 \leq n < \infty$, as this connects directly to the methods from Section 3.4. We emphasize again that these methods are not exclusive to Gelfand–Fuks cohomology or gauga algebras: In more generality, they can be applied to calculate bornological Lie algebra homology of $\Gamma_c(A)$ for any Lie algebroid $A \to M$, assuming that the bornological Lie algebra homology of $\Gamma_c(A|_U)$ can be calculated whenever U is a small disk over which A trivializes.

3.5.1. COSHEAVES OF LIE ALGEBRA CHAINS

Fix in this subsection a smooth, locally trivial Lie algebra bundle $\mathcal{K} \to M$ with finite-dimensional fiber $\mathfrak{gl}_n(\mathbb{K})$ for some $1 \leq n < \infty$. Whenever we speak of (pre-)cosheaves in the following, we think of them as valued in the category of abelian groups as in Appendix A, but this is largely unimportant. We first introduce some notation: Define for a compact $K \subset M$ the closed Fréchet subspace

$$\Gamma_K(\mathcal{K}) := \{s \in \Gamma(\mathcal{K}) : \operatorname{supp} s \subset K\} \subset \Gamma(\mathcal{K}). \tag{3.5.1}$$

Then, given any compact exhaustion $\{K_n\}$ of M, the topology on the compactly supported section space arises from the direct limit topology $\Gamma_c(\mathcal{K}) = \varinjlim \Gamma_{K_n}(\mathcal{K})$. It is well known that if we have two vector bundles $A, B \to M$, then

$$\Gamma(A) \mathbin{\widehat{\otimes}} \Gamma(B) \cong \Gamma(A \boxtimes B), \tag{3.5.2}$$

where $A \boxtimes B := \mathrm{pr}_1^* A \otimes \mathrm{pr}_2^* B \to M \times M$ denotes the exterior tensor product of vector bundles, see for example [Trè67, Theorem 51.6] for the statement in trivial fibers. Then, using Proposition 3.2.8:

$$\Gamma_c(\mathcal{K})^{\widehat{\otimes} k} \cong \varinjlim \Gamma_{K_n}(\mathcal{K})^{\widehat{\otimes} k} \cong \varinjlim \Gamma_{K_n \times \cdots \times K_n}(\mathcal{K}^{\boxtimes k}) \cong \Gamma_c(\mathcal{K}^{\boxtimes k}). \tag{3.5.3}$$

This justifies the following definition:

Definition 3.5.1. We define, for every $k \geq 1$ the precosheaf $B_k(\mathcal{K}, \cdot)$ over M^k, assigning to an open set $U \subset M^k$ the set

$$B_k(\mathcal{K}, U) := \Gamma_c(\mathcal{K}^{\boxtimes k}|_U). \tag{3.5.4}$$

The precosheaf map $\Gamma_c(\mathcal{K}^{\boxtimes k}|_U) \to \Gamma_c(\mathcal{K}^{\boxtimes k}|_V)$ associated to the inclusion $U \subset V$ is defined via extension by zero.

Remark 3.5.2. From this definition and the previous isomorphism, we get the bornological Lie algebra complex via restricting to global sections and Σ_k-coinvariants, i.e. for all $k \geq 1$ we have

$$C_k^{\mathrm{born}}(\Gamma_c(\mathcal{K})) \cong (B_k(\mathcal{K}, M^k))_{\Sigma_k}. \tag{3.5.5}$$

Note in particular that we set the zero degree part to zero, so we think of B_k as a reduced complex.

Since the precosheaf $B_k(\mathcal{K}, \cdot)$ arises from the compactly supported sections of a soft sheaf (given by the sections of a smooth vector bundle), Proposition A.5 implies:

Lemma 3.5.3. *For every $k \geq 1$, the precosheaf $B_k(\mathcal{K}, \cdot)$ over M^k is a flabby cosheaf.*

3.5.2. Cosheaves of compactly supported differential forms

Fix a smooth manifold M of dimension n, and some $k \in \mathbb{N}_0$ with $0 \leq k \leq n$.

Definition 3.5.4. Define the precosheaves Ω_c^k and Z^k, respectively given by assigning to an open $U \subset M$ the compactly supported k-forms $\Omega_c^k(U)$ and

$$Z^k(U) := \frac{\Omega_c^k(U)}{d_{\mathrm{dR}}\Omega_c^{k-1}(U)}. \tag{3.5.6}$$

The extension maps of Ω_c^k are induced by the extension of compactly supported forms by zero. They induce the extension maps on the quotient Z^k.

We find:

Lemma 3.5.5. *The precosheaves Ω_c^k and Z^k over M are cosheaves. Further, Ω_c^k is flabby and Z^k admits the flabby coresolution*

$$0 \to \Omega_c^0 \to \Omega_c^1 \to \cdots \to \Omega_c^k \to Z^k \to 0, \tag{3.5.7}$$

where the last nontrivial map is the canonical quotient map, and the other nontrivial maps are given by the de Rham differential.

Proof. The Ω_c^k are flabby cosheaves by Proposition A.5, since they arise as the precosheaf of compactly supported sections of the soft sheaf of differential forms on M. The de

Rham differential induces a cosheaf morphism $\Omega_c^{k-1} \to \Omega_c^k$ whose cokernel precosheaf equals exactly Z^k, and cokernels precosheaves of cosheaf morphisms are automatically cosheaves [Bre97, Chapter VI, Proposition 1.2]. The Poincaré lemma for $\Omega_c^\bullet(\mathbb{R}^n)$ implies that the sequence (3.5.7) is locally exact, and hence it is a flabby coresolution for Z^k. The statement is proven. □

The coresolution shows, together with Proposition A.8:

Corollary 3.5.6. *The Čech homology of Z^k equals*

$$\check{H}_r(M, Z^k) = \begin{cases} Z^k(M) & \text{if } r = 0, \\ H^{k-r}_{\mathrm{dR},c}(M) & \text{if } r > 0, \end{cases} \tag{3.5.8}$$

where $H^\bullet_{\mathrm{dR},c}(M)$ denotes compactly supported de Rham cohomology of M.

We will be working with certain products of the above cosheaves over the cartesian products $M^1, M^2, M^3, \ldots$. Let us formalize what we mean by this:

Lemma 3.5.7. *Let $l \in \mathbb{N}$ and $0 \le k_1, \ldots, k_l \le n$. There is a cosheaf $Z^{k_1} \hat{\otimes} \cdots \hat{\otimes} Z^{k_l}$ over M^l with the property that for all open $U_1, \ldots, U_l \subset M$ we have*

$$(Z^{k_1} \hat{\otimes} \cdots \hat{\otimes} Z^{k_l})(U_1 \times \cdots \times U_l) := Z^{k_1}(U_1) \hat{\otimes} \cdots \hat{\otimes} Z^{k_l}(U_l), \tag{3.5.9}$$

and the cosheaf map associated to an inclusion $U_1 \times \cdots \times U_l \subset V_1 \times \cdots \times V_l$ equals the tensor product of the extension maps of the $Z^{k_1}, \ldots, Z^{k_l}$.

Proof. For simplicity, we treat the case $l = 2$, from which the general case easily follows. Note also that this proof relies on the concept of a *cosheaf on a topological base*, which we elaborate on in Appendix A.
Consider the topological base $\mathscr{B} := \{U \times V : U, V \subset M \text{ open}\}$ of M^2, and the precosheaf $\Omega_c^{k_1} \hat{\otimes} \Omega_c^{k_2}$ on $\mathscr{B}$ given by

$$(\Omega_c^{k_1} \hat{\otimes} \Omega_c^{k_2})(U \times V) := \Omega_c^{k_1}(U) \hat{\otimes} \Omega_c^{k_2}(V) \quad \forall U \times V \in \mathscr{B}. \tag{3.5.10}$$

Similar to (3.5.2), we have for all open $U, V \subset M$ the isomorphism

$$\Omega_c^{k_1}(U) \hat{\otimes} \Omega_c^{k_2}(V) \cong \Gamma_c\Big(\big(\Lambda^{k_1} T^*M \boxtimes \Lambda^{k_2} T^*M\big)\big|_{U \times V}\Big), \tag{3.5.11}$$

which extends to an isomorphism of precosheaves on the base $\mathscr{B}$. The right-hand side defines a cosheaf on M^2 as the cosheaf of compactly supported sections of a vector bundle. Hence, it restricts to a cosheaf on the base $\mathscr{B}$, and thus $U \times V \mapsto \Omega_c^{k_1}(U) \hat{\otimes} \Omega^{k_2}(V)$ defines a cosheaf on the base $\mathscr{B}$. Consider the morphism of cosheaves on $\mathscr{B}$ given by

$$d_{\mathrm{dR}} \hat{\otimes} \mathrm{id} : \Omega_c^{k_1-1} \hat{\otimes} \Omega_c^{k_2} \to \Omega_c^{k_1} \hat{\otimes} \Omega_c^{k_2}. \tag{3.5.12}$$

3

If $\Omega^\bullet_K(U)$ denotes the Fréchet space of differential forms on an open set U whose support is contained in a compact set K, and $\{K_n\}_{n\geq 1}$ and $\{L_n\}_{n\geq 1}$ denote compact exhaustions of open sets $U, V \subset M$, then the image of the morphism $d \,\hat{\otimes}\, \mathrm{id}$ at the open set $U \times V$ is

$$\varinjlim d_{\mathrm{dR}} \Omega^{k_1-1}_{K_n}(U) \,\hat{\otimes}\, \Omega^{k_2}_{L_n}(V) = d_{\mathrm{dR}} \Omega^{k_1-1}_c(U) \,\hat{\otimes}\, \Omega^{k_2}_c(V). \tag{3.5.13}$$

The range of the de Rham differential $d : \Omega^k_{K_n}(U) \to \Omega^{k+1}_{K_n}(U)$ is closed (c.f. Lemma D.1), so using Proposition 3.2.10 we deduce

$$\begin{aligned} \frac{\Omega^{k_1}_c(U) \,\hat{\otimes}\, \Omega^{k_2}_c(V)}{d_{\mathrm{dR}}\Omega^{k_1-1}_c(U) \,\hat{\otimes}\, \Omega^{k_2}_c(V)} &\cong \varinjlim \frac{\Omega^{k_1}_{K_n}(U) \,\hat{\otimes}\, \Omega^{k_2}_{L_n}(V)}{d_{\mathrm{dR}}\Omega^{k_1-1}_{K_n}(U) \,\hat{\otimes}\, \Omega^{k_2}_{L_n}(V)} \\ &\cong \varinjlim \frac{\Omega^{k_1}_{K_n}(U)}{d_{\mathrm{dR}}\Omega^{k_1-1}_{K_n}(U)} \,\hat{\otimes}\, \Omega^{k_2}_{L_n}(V) \cong Z^{k_1}_c(U) \,\hat{\otimes}\, \Omega^{k_2}_c(V). \end{aligned} \tag{3.5.14}$$

By Proposition A.11 the cokernel of $d_{\mathrm{dR}} \,\hat{\otimes}\, \mathrm{id}$ defines a cosheaf on the base $\mathscr{B}$, and, by the previous calculations, this cosheaf on $\mathscr{B}$ assumes on $U \times V \in \mathscr{B}$ the shape $Z^{k_1}(U) \,\hat{\otimes}\, \Omega^{k_2}(V)$; we denote this cosheaf by $Z^{k_1} \,\hat{\otimes}\, \Omega^{k_2}$. Analogously by considering the cokernel of

$$Z^{k_1} \,\hat{\otimes}\, \Omega^{k_2-1}_c \overset{\mathrm{id}\hat{\otimes} d_{\mathrm{dR}}}{\longrightarrow} Z^{k_1} \,\hat{\otimes}\, \Omega^{k_2}_c \tag{3.5.15}$$

we find the cosheaf $Z^{k_1} \,\hat{\otimes}\, Z^{k_2}$ on the base $\mathscr{B}$, which admits for all $U \times V \in \mathscr{B}$ the desired local form

$$(Z^{k_1} \,\hat{\otimes}\, Z^{k_2})(U \times V) = Z^{k_1}(U) \,\hat{\otimes}\, Z^{k_2}(V). \tag{3.5.16}$$

Now the statement follows since a cosheaf on $\mathscr{B}$ extends uniquely to a cosheaf on M by Theorem A.10. □

Unfortunately, we are currently not able to confidently state the Čech homology of these product cosheaves. Let us remark the difficulties. Consider open covers $\mathscr{U}, \mathscr{V}$ of M, and the arising product cover $\mathscr{U} \times \mathscr{V} := \{U \times V : U \in \mathscr{U}, V \in \mathscr{V}\}$ of M^2. Then one can deduce

$$\check{H}_\bullet(\mathscr{U} \times \mathscr{V}, Z^{k_1} \,\hat{\otimes}\, Z^{k_2}) \cong \check{H}_\bullet(\mathscr{U}, Z^{k_1}) \,\hat{\otimes}\, \check{H}_\bullet(\mathscr{V}, Z^{k_2}), \tag{3.5.17}$$

using that the respective Čech complex for $Z^{k_1} \,\hat{\otimes}\, Z^{k_2}$ factorizes, together with a direct limit argument and the Künneth formula from Theorem 3.3.11. However, product covers $\mathscr{U} \times \mathscr{V}$ do *not* generally constitute a cofinal subsystem in the directed system of open covers of M^2, and as a consequence, the Čech homology can not be deduced from the associated inverse system. Without this cofinality, it is unclear to the author how to arrive at a Künneth theorem in the topological setting, such as [Cas73] and [Kau67] – see also [Kau68], where the non-cofinality of product covers is acknowledged. We state the likely Künneth formula for our setting as a conjecture.

Conjecture 3.5.8. If $\mathscr{U}$ is a cover of M^l so that any intersection of elements in U are diffeomorphic to a finite union of Euclidean spaces, then

$$\check{H}_\bullet(\mathscr{U}, Z^{k_1} \hat{\otimes} \cdots \hat{\otimes} Z^{k_l}) \cong \check{H}_\bullet(M, Z^{k_1}) \hat{\otimes} \cdots \hat{\otimes} \check{H}_\bullet(M, Z^{k_l}). \tag{3.5.18}$$

3.5.3. A GLOBALIZING ČECH DOUBLE COMPLEX

Let us use the tools developed so far to examine homology of nontrivial $\mathfrak{gl}_t(\mathbb{K})$-bundles. The idea of using local-to-global principles for the cohomology of section spaces originates from Bott, Segal, Gelfand and Fuks, who used a similar strategy to describe the continuous Lie algebra cohomology of vector fields, see [BS77] and [GF69]. We transfer this idea to our current setting, using methods closely related to the theory of factorization algebras, see [CG17].

3

Definition 3.5.9. Let $q \geq 1$ and $\mathscr{U} := \{U_\alpha\}$ an open cover of M. The *Čech-gauge double complex associated to* $\mathscr{U}$ is the following double complex

$$\begin{tikzcd} \vdots \arrow[d] & \vdots \arrow[d] & \\ \bigoplus_\alpha C_2(\mathscr{K}, U_\alpha) \arrow[d] & \bigoplus_{\alpha,\beta} C_2(\mathscr{K}, U_\alpha \cap U_\beta) \arrow[l] \arrow[d] & \dots \arrow[l] \\ \bigoplus_\alpha C_1(\mathscr{K}, U_\alpha) & \bigoplus_{\alpha,\beta} C_1(\mathscr{K}, U_\alpha \cap U_\beta) \arrow[l] & \dots \arrow[l] \end{tikzcd} \tag{3.5.19}$$

The horizontal differentials are given by the Čech differentials associated to the precosheaves $C_k(\mathscr{K}, \cdot)$, and the vertical differentials arise from the Chevalley–Eilenberg differential.

Remark 3.5.10. Once again, we disregard the zeroth homology groups on purpose, since they do not behave quite as neatly in the cosheaf-theoretic sense, and are only connected to the complex by a zero differential.

As in Chapter 2, it will prove insightful to calculate the spectral sequences associated to this double complex in certain cases. However, since this is a first quadrant double complex, we do not need to be as careful regarding convergence phenomena. This, in particular, allows us to skip defining diagonal cochains and k-good covers as in Definition 2.5.9.

For the rest of the chapter, equip the manifold M with an arbitrary Riemannian metric.

Definition 3.5.11. We say that $U \subset M$ is *strongly convex* if all two points $x, y \in U$ are connected by a unique, minimizing geodesic contained in U. The *geodesic Weiss cover* $\mathscr{U}$ associated to M is given by

$$\mathscr{U} := \{B_1 \cup \cdots \cup B_r : r \in \mathbb{N},\ B_i \cap B_j = \emptyset \text{ for } i \neq j,\ B_i \subset M \text{ open \& geodesically convex}\}.$$

Remark 3.5.12. The geodesic Weiss cover is indeed a Weiss cover in the language of factorization algebras, cf. Remark 2.5.10 or [CG17]. Note further that that if $n = \dim M$, then nonempty strongly convex sets in M are well known to be diffeomorphic to $\mathbb{R}^n$, and intersections of strongly convex sets remain strongly convex. Hence, this cover has the property that all nonempty, finite intersections of elements of $\mathscr{U}$ are diffeomorphic to a finite, disjoint union of copies of $\mathbb{R}^n$.

3

Proposition 3.5.13. *For every $r \geq 1$, the r-th row of the Čech-gauge double complex associated to the geodesic Weiss cover $\mathscr{U}$ are acyclic in nonzero degree, and their zeroth homology is $C_r^{\mathrm{born}}(\Gamma_c(\mathscr{K}))$.*

Proof. Recall from Remark 3.5.2

$$C_r(\mathscr{K}, U) = \big(B_r(\mathscr{K}, U^r)\big)_{\Sigma_r}. \tag{3.5.20}$$

As a consequence, the r-th row of the double complex is equal to the Σ_r-coinvariants associated to the Čech complex of the cosheaf $B_r(\mathscr{K}, \cdot)$ with respect to the cover $\mathscr{U}^r$ of M^r. Taking coinvariants with respect to the finite group Σ_r is an exact functor, hence the calculation of the homologies of the rows reduces to calculating the Čech homology of the cosheaf $B_r(\mathscr{K}, \cdot)$ and taking coinvariants afterwards. But this cosheaf is flabby and thus has trivial Čech homology, see Proposition A.6. This proves the statement. □

Corollary 3.5.14. *The total homology of the Čech-gauge double complex associated to the geodesic Weiss cover is equal to $H_\bullet^{\mathrm{born}}(\Gamma_c(\mathscr{K}))$.*

Hence the spectral sequence associated to the horizontal filtration will give us information about the desired global homology, assuming we understand the homology of the restricted algebras on all $U \in \mathscr{U}$, and we understand the Čech homology associated to the precosheaves of the homology groups $U \mapsto H_\bullet(C_\bullet(\mathscr{K}, U))$ associated with the geodesic Weiss cover $\mathscr{U}$.

For a simple presentation of the following spectral sequence, let us introduce a piece of notation. Fix some $n \in \mathbb{N}$. Then we set, for all $k \in \mathbb{N}_0$:

$$\xi_n(k) := \min\{k, n + (k - n \mod 2)\}, \tag{3.5.21}$$

in other words, the sequence $\{\xi_n(k)\}_{k\geq 0}$ assumes the shape

$$0, 1, 2, \ldots, n-1, n, n+1, n, n+1, n, \ldots \tag{3.5.22}$$

Then, due to the periodic nature of cyclic homology, we can rephrase Corollary 3.3.23 in the following way:

$$H_k^{\lambda,\mathrm{born}}(C^\infty(\mathbb{R}^n)) \cong \frac{\Omega_c^{\xi_n(k)}(\mathbb{R}^n)}{d_{\mathrm{dR}}\Omega_c^{\xi_n(k)-1}(\mathbb{R}^n)} = Z^{\xi_n(k)}(\mathbb{R}^n) \quad \forall k \geq 0. \tag{3.5.23}$$

Theorem 3.5.15. *Let M be a manifold of finite dimension n and $P \to M$ a principal fiber bundle with generic fiber $GL_t(\mathbb{R})$, and set $q := \lfloor \frac{t-1}{2} \rfloor$. Denote by Z^k for $k \geq 0$ the cosheaves over M from Definition 3.5.4 for $k \geq 0$. Then there is a homological first-quadrant spectral sequence $\{E^\bullet_{r,s}\}_{r,s\geq 0}$ with*

$$E^k_{r,s} \implies H^{\mathrm{born}}_{r+s}(\Gamma_c(\mathrm{Ad}(P))). \tag{3.5.24}$$

For $r \geq 0$ and $1 \leq s \leq q$, we can express the second page $E^2_{r,s}$ with the notation (3.5.21):

$$E^2_{r,s} = \bigoplus_{k\geq 1} \left(\bigoplus_{s_1+\cdots+s_k=s} \check{H}_r \left(\mathscr{U}^k, Z^{\xi_n(s_1-1)} \widehat{\otimes} \cdots \widehat{\otimes} Z^{\xi_n(s_k-1)} \right) \right)_{\Sigma_k} \tag{3.5.25}$$

Remark 3.5.16. Assuming Conjecture 3.5.8, we find that, graded by its diagonals, the second page $\{E^2_{r,s}\}_{r,s\geq 0}$ looks exactly like the compactly supported analog of the continuous homology $H_\bullet(\mathfrak{gl}(C^\infty(M)) = \hat{\Lambda}^\bullet H^{\lambda,\mathrm{born}}_{\bullet-1}(C^\infty(M))$ in total degree $\leq q$, all instances of $C^\infty(M)$ and $\Omega^\bullet(M)$ replaced by $C^\infty_c(M)$ and $\Omega^\bullet_c(M)$, respectively. Thus, while it is not instantly recognizable, the second page of this spectral sequence is a double grading of what one would expect bornological Lie algebra homology of $\mathfrak{gl}_n(C^\infty_c(M))$ to be, intertwining the homological grading with a certain grading that encodes the "locality" of the data.

Proof of Theorem 3.5.15. We consider the Čech-gauge double complex associated to the geodesic Weiss cover $\mathscr{U}$, and consider the spectral sequence arising by filtering along the columns. By standard arguments, this first quadrant spectral sequence converges towards the total homology of the double complex, which is equal to $H^{\mathrm{born}}_\bullet(\Gamma_c(\mathrm{Ad}(P)))$ due to Corollary 3.5.14. It remains to describe the second page. The differential on the zeroth page is simply given by taking the homology in vertical direction. If $s \leq q$ and $U \in \mathscr{U}$, then

$$H^{\mathrm{born}}_s(\mathfrak{gl}_t(C^\infty_c(U))) \cong H^{\mathrm{born}}_s(\mathfrak{gl}(C^\infty_c(U))). \tag{3.5.26}$$

If further $U \cong \bigsqcup_{i=1}^r \mathbb{R}^n$, then

$$\Gamma_c(\mathscr{K}\big|_U) \cong \bigoplus_{i=1}^r \mathfrak{gl}(C^\infty_c(\mathbb{R}^n)). \tag{3.5.27}$$

With Remark 3.3.24 and (3.5.23) we get

$$H^{\mathrm{born}}_\bullet(\mathfrak{gl}(C^\infty_c(U))) \cong \hat{\Lambda}^\bullet \left(\bigoplus_{i=1}^r H^{\lambda,\mathrm{born}}_{\bullet-1}(C^\infty_c(\mathbb{R}^n)) \right) \cong \hat{\Lambda}^\bullet \left(Z^{\xi_n(\bullet-1)}(U) \right). \tag{3.5.28}$$

In degrees, this translates to:

$$H^{\mathrm{born}}_s(\Gamma_c(\mathscr{K}\big|_U)) \cong \bigoplus_{k\geq 1} \left(\bigoplus_{s_1+\cdots+s_k=s} Z^{\xi_n(s_1-1)}(U) \widehat{\otimes} \cdots \widehat{\otimes} Z^{\xi_n(s_k-1)}(U) \right)_{\Sigma_k}. \tag{3.5.29}$$

3

This calculates the first page of the spectral sequence. Now, recall that the horizontal differential of the double complex arises from a Čech differential. To understand how this differential acts on the first page, it suffices to understand, for open sets $U \subset V \subset \mathbb{R}^n$ with $U \cong V \cong \mathbb{R}^n$, the composition

$$\begin{aligned} H_k^{\text{born}}(\mathfrak{gl}_t(C_c^\infty(U))) \cong H_k^{\text{born}}(\Gamma_c(\mathcal{K}|_U)) \\ \to H_k^{\text{born}}(\Gamma_c(\mathcal{K}|_V)) \cong H_k^{\text{born}}(\mathfrak{gl}_t(C_c^\infty(V))), \end{aligned} \tag{3.5.30}$$

where the middle map is induced by the extension map

$$C_k^{\text{born}}(\Gamma_c(\mathcal{K}|_U)) \to C_k^{\text{born}}(\Gamma_c(\mathcal{K}|_V)). \tag{3.5.31}$$

The differential of the first page is then given as a the usual Čech-theoretic, antisymmetric linear combination of such maps. Under the identification of $H_k^{\text{born}}(\mathfrak{gl}(C_c^\infty(U)))$ with antisymmetrized tensor products of terms $Z^k(U)$, the map (3.5.30) is induced by extensions $\iota_U^V : Z^k(U) \to Z^k(V)$, up to an action by the transition function $g_{UV} : U \cap V \to \mathrm{GL}_t(\mathbb{R})$ arising from the choice of local trivializations on U and V. Now recall that on $\mathrm{Ad}(P)$, the transition functions act by the adjoint action of $\mathrm{GL}_t(\mathbb{K})$. However, in the calculation of $H_k^\iota(\mathfrak{gl}_t(C_c^\infty(U)))$ we have seen that we may reduce to the $\mathfrak{gl}_t(\mathbb{K})$ tensor invariants, which are equal to the $\mathrm{GL}_t(\mathbb{K})$ tensor invariants by [Lod92, Lemma 9.2.5]. Thus the transition functions act trivially on this space. Hence, under the isomorphism (3.5.29), the map (3.5.30) can be identified with the appropriate direct sum of extension maps for the product cosheaves $Z^{s_1-1} \hat{\otimes} \cdots \hat{\otimes} Z^{s_k-1}$. As a consequence, the r-th row on the first page of the spectral sequence can be identified with the Σ_r-coinvariants of the direct sum of Čech complexes of product cosheaves with respect to the cover $\mathcal{U}^r$. This concludes the proof. □

Writing out the spectral sequence in low degrees, the previous theorem gives us the following corollary:

Corollary 3.5.17. *Let $P \to M$ be a principal $\mathrm{GL}_t(\mathbb{K})$-bundle and* $\dim M \geq 1$, *and assume that Conjecture 3.5.8 holds.*

i) If $t \geq 3$, then

$$H_1^{\text{born}}(\Gamma_c(\mathrm{Ad}(P))) \cong \Omega_c^0(M). \tag{3.5.32}$$

ii) If $t \geq 5$, then

$$H_2^{\text{born}}(\Gamma_c(\mathrm{Ad}(P))) \cong \left(\Omega_c^0(M)^{\hat{\otimes}2}\right)_{\Sigma_2} \oplus \frac{\Omega_c^1(M)}{d_{\mathrm{dR}}\Omega_c^0(M)}. \tag{3.5.33}$$

iii) If $t \geq 7$, then

$$\begin{aligned} &H_3^{\text{born}}(\Gamma_c(\mathrm{Ad}(P)) \cong \\ &\left(\Omega_c^0(M)^{\hat{\otimes}3}\right)_{\Sigma_3} \oplus \left(\frac{\Omega_c^1(M)}{d_{\mathrm{dR}}\Omega_c^0(M)} \hat{\otimes} \Omega_c^0(M)\right) \oplus \frac{\Omega_c^2(M)}{d_{\mathrm{dR}}\Omega_c^1(M)} \oplus H_c^0(M). \end{aligned} \tag{3.5.34}$$

We emphasize once again that, using the analogs of the LQT theorems for G any of the other classical Lie groups (cf. Remark 3.4.9), our methods can straightforwardly be adapted to construct spectral sequences for principal G-bundles.

3.6. COMMENTS AND FURTHER OUTLOOK ON THE SPECTRAL SEQUENCE

We want to dedicate this section to a large amount of comments that can be made about Theorem 3.5.15, its assumptions and possible outlooks for future generalizations.
Firstly, the only reason we needed to restrict to bundles $\mathrm{Ad}(P) \to M$ was to make sure that the transition functions of the bundle can be chosen to act by inner automorphisms of the Lie algebra $\mathfrak{gl}_t(\mathbb{K})$. In a more general Lie algebra bundle, *outer automorphisms* may show up, i.e. automorphisms which are not in the image of $\mathrm{Ad}: GL_t(\mathbb{K}) \to \mathrm{Aut}(\mathfrak{gl}_t(\mathbb{K}))$. In this case, one may need to twist the cosheaf structure of the Z^k by a locally constant system depending on the bundle. Hence, by including this data, it should be a straightforward exercise to generalize Theorem 3.5.15 to more general $\mathfrak{gl}_t(\mathbb{K})$-bundles.
Secondly, in contrast to Theorems 3.4.8 and 3.4.10, one cannot phrase Theorem 3.5.15 in terms of Lie algebra bundles with infinite-dimensional fiber $\mathfrak{gl}(\mathbb{K})$, since the local sections in this situation use the wrong tensor product: [Trè67, Theorem 44.1] implies

$$C_c^\infty(U, \mathfrak{gl}(\mathbb{K})) \cong \overline{C_c^\infty(U) \otimes_\pi \mathfrak{gl}(\mathbb{K})}, \tag{3.6.1}$$

which is not necessarily equal to $C_c^\infty(U) \widehat{\otimes} \mathfrak{gl}(\mathbb{K}) = \mathfrak{gl}(C_c^\infty(U))$, as in general $\otimes_\beta \neq \otimes_\pi$ on LF-spaces.
Thirdly, the reader may be tempted to consider the non-compactly supported analog of Theorem 3.5.15. Constructing this would, in some ways, be more standard, as this would rather use the *sheaf* theory associated to the sheaf of sections of $\mathrm{Ad}(P) \to M$, rather than the cosheaf theory of its compactly supported counterpart. Another advantage would be that the stabilization

$$H_k^{\mathrm{born}}(\mathfrak{gl}(C^\infty(\mathbb{R}^n))) \cong H_k^{\mathrm{born}}(\mathfrak{gl}_n(C^\infty(\mathbb{R}^n))) \tag{3.6.2}$$

occurs already when $k+1 \leq n$, which is strictly stronger than the stabilization for $C_c^\infty(\mathbb{R}^n)$ in Theorem 3.4.10, occurring only when $2k+1 \leq n$. However, the crucial difference is that the arising double complex would not be of first-quadrant anymore, as it arises as a mixture of a cohomological Čech complex with a homological Lie algebra complex. Such double complexes are in many ways more inconvenient: For one, it is not immediately clear whether the spectral sequences associated to the horizontal and vertical filtration converge to the same infinity-page, and even if they do, the relevant diagonals of the spectral sequence may hit infinitely many nonzero terms.
In particular, knowing only a part of the second page $E_2^{p,q}$, as it is the case in Theorem 3.5.15, is then a lot less helpful, as it does not significantly restrict the size of

any total homology group. The relevant diagonals of the double complex cross an unbounded, unidentified part of the spectral sequence. For further material on similar calculations with non-first-quadrant complexes, we direct the reader to [BS77]. Furthermore, one may be interested to construct explicit cochains that generate the terms of the spectral sequence. To this end, it may be helpful to compare this to [GM92, Chapter 3]; there, continuous cochains for $\mathfrak{gl}_n(\mathbb{K})$-Lie algebra bundles are constructed from elements in $\left(\frac{\Omega^\bullet(M)}{d\Omega^{\bullet-1}(M)}\right)^*$. When the bundle equals the endomorphism bundle $\mathrm{End}(E)$ of some vector bundle $E \to M$, these cochains equal the antisymmetrizations of the cyclic cochains constructed in [Qui88, Chapter 7].
Finally, we want to remark that to remove the condition $s \leq q$ in the description of $E^2_{r,s}$ in Theorem 3.5.15, one requires a full description of the unstable bornological homology groups of the Lie algebras $\mathfrak{gl}_n(C^\infty_c(\mathbb{R}^n))$. Already in the algebraic setting, this appears to be highly nontrivial: Conjecture 10.3.9 in [Lod92] attempts to give a description for Lie algebra homology of $\mathfrak{gl}_n(A)$ when A is commutative and unital, and it is stated that their conjecture implies a certain case of the Macdonald conjectures. In [Tel02], the conjecture is verified for many special cases, but it is also shown that, in full generality, it does not hold. To our knowledge, a satisfying, general description of these unstable homology groups is an open problem.

3.7. Proof of Theorem 3.4.10

The strategy and proof of Theorem 3.4.10 takes some preparation. The material and notation of this section originates from [Han88], and we also refer to the related preprint [Cor05]. There, the algebraic analog of Theorem 3.4.10 was proven, and our contribution will be the extension of the results to nuclear Fréchet algebras. We closely follow the outline of the proof of [Cor05] and make remarks to how this extends to our setting at the appropriate places.

Definition 3.7.1. Let $m, n \geq 1$.

i) We say that $\alpha := (\alpha_1, \dots, \alpha_l) \in \mathbb{Z}^l_{>0}$ is a *partition of m of length $l = l(\alpha)$* if $\sum_{i=1}^l \alpha_i = m$ and $\alpha_1 \geq \cdots \geq \alpha_l$.
Additionally, we define $\emptyset$ to be a partition of 0 of length 0.
The set of partitions of m is denoted by $P(m)$.

ii) Let α, β be two partitions of m, and $l(\alpha) + l(\beta) \leq n$, then we set

$$[\alpha, \beta]_n := (\alpha_1, \dots, \alpha_{l(\alpha)}, 0, \dots, 0, -\beta_{l(\beta)}, \dots, -\beta_1) \in \mathbb{Z}^n. \tag{3.7.1}$$

iii) Let V be a $\mathfrak{gl}_n(\mathbb{K})$-representation and $\mu \in \mathbb{Z}^n$, then we define the *highest weight module V_μ* via

$$M_\mu(V) := \{v \in V : e_{ij} \cdot v = 0,\ e_{kk} \cdot v = \mu_k \cdot v \quad \forall k, \forall i < j\}, \tag{3.7.2}$$

$$V_\mu := U(\mathfrak{gl}_n(\mathbb{K})) \cdot M_\mu(V). \tag{3.7.3}$$

Here, $e_{ij} \in \mathfrak{gl}_n(\mathbb{K})$ denotes the elementary matrix with a one in the (i,j)-th entry and zeroes everywhere else.

Lemma 3.7.2. *Let $n,k \geq 1$, and A be any bornological algebra. Then:*

$$C_k^{\text{born}}(\mathfrak{gl}_n(A)) = \bigoplus_{m\geq 0} \bigoplus_{\substack{\alpha,\beta\in P(m)\\ l(\alpha)+l(\beta)\leq n}} C_k^{\text{born}}(\mathfrak{gl}_n(A))_{[\alpha,\beta]_n}. \tag{3.7.4}$$

Proof. Consider $\mathfrak{gl}_n(\mathbb{K})$ and its tensor products $\mathfrak{gl}_n(\mathbb{K})^{\otimes k}$ as a $\mathfrak{gl}_n(\mathbb{K})$-representation in the natural way, via the adjoint action and tensor products thereof. Since $\mathfrak{gl}_n(\mathbb{K})$ is a reductive Lie algebra, its adjoint representation is completely reducible, and the decomposition of the following finite-dimensional tensor modules is standard (for a detailed discussion, see [Han88, p.211f.])

$$\mathfrak{gl}_n(\mathbb{K})^{\otimes k} = \bigoplus_{m\geq 0} \bigoplus_{\substack{\alpha,\beta\in P(m)\\ l(\alpha)+l(\beta)\leq n}} \left(\mathfrak{gl}_n(\mathbb{K})^{\otimes k}\right)_{[\alpha,\beta]_n}. \tag{3.7.5}$$

Now, since $\mathfrak{gl}_n(\mathbb{K})$ acts trivially on A, this extends to a decomposition for the $\mathfrak{gl}_n(\mathbb{K})$-module $\left(\mathfrak{gl}_n(\mathbb{K})^{\otimes k} \otimes A^{\otimes^k}\right)_{\Sigma_k}$. Lastly, since completion commutes with topological direct sums, this extends to $C_k^{\text{born}}(\mathfrak{gl}_n(A))$ and the statement is shown. □

The action of a Lie algebra respects associated Chevalley–Eilenberg differentials, so we have the subcomplexes

$$C_\bullet^{\text{born}}(\mathfrak{gl}_n(A))_{[\alpha,\beta]_n} \subset C_\bullet^{\text{born}}(\mathfrak{gl}_n(A)), \tag{3.7.6}$$

$$M_{[\alpha,\beta]_n}(C_\bullet^{\text{born}}(\mathfrak{gl}_n(A))) \subset M_{[\alpha,\beta]_n}(C_\bullet^{\text{born}}(\mathfrak{gl}_n(A))). \tag{3.7.7}$$

By reducing to the finite-dimensional highest weight theory as in Lemma 3.7.2, one shows:

Lemma 3.7.3. *Let A be any bornological algebra. For every $n \geq 1, m \geq 0$ and $\alpha,\beta \in P(m)$ with $l(\alpha)+l(\beta) \leq n$, we have*

$$H_\bullet(C_\bullet^{\text{born}}(\mathfrak{gl}_n(A))_{[\alpha,\beta]_n}) \cong U(\mathfrak{gl}_n(\mathbb{K})) \cdot H_\bullet(M_{[\alpha,\beta]_n}(C_\bullet^{\text{born}}(\mathfrak{gl}_n(A)))). \tag{3.7.8}$$

Some last definitions before we get to the relevant theorems: Given a partition $\alpha \in P(m)$, we denote by V^α the well-known *Specht Σ_m-module* associated to α, see [Han88, p.216] for a detailed definition. Lastly, given a chain complex $C_\bullet$, denote by $T^m(C_\bullet) := (C_\bullet)^{\otimes^m}$ the chain complex given by the m-th algebraic tensor power of $C_\bullet$. The following will be our main proposition:

Proposition 3.7.4. *Assume A is a nuclear Fréchet algebra. For every $n,m \geq 1$ and $\alpha,\beta \in P(m)$ with $l(\alpha)+l(\beta) \leq n$, there is a chain morphism*

$$\begin{aligned} &M_{[\alpha,\beta]_n}(C_\bullet^{\text{born}}(\mathfrak{gl}_n(A))) \to \\ &\qquad \hat{\Lambda}^\bullet(C_{\bullet-1}^{\lambda,\text{born}}(A)) \otimes \left(\hat{\tilde{T}}^m(C_\bullet^{\text{bar},\text{born}}(A)) \otimes V^\alpha \otimes V^\beta\right)_{\Sigma_m}, \end{aligned} \tag{3.7.9}$$

which is an isomorphism of TVS in degree $\le \frac{n}{2}$*. Here,* $\hat{T}$*,* $\hat{\tilde{T}}$ *and* $\hat{\Lambda}$ *denote the completions of the (reduced/exterior) tensor algebra in the bornological tensor product topology, and the codomain is equipped with the product differential arising from the cyclic and bar differentials.*

Proof. Assume first that A is an arbitrary TVS, not necessarily with an algebra structure. Set

3

$$S(A) := T^\bullet(\tilde{T}^\bullet(A)) \otimes (\tilde{T}^m(C_\bullet^{\mathrm{bar}}(A)) \otimes V^\alpha \otimes V^\beta). \tag{3.7.10}$$

We first construct a map of graded vector spaces

$$\tilde{\psi}_A : S(A) \to M_{[\alpha,\beta]_n} T^\bullet(\mathfrak{gl}_n(A)) \tag{3.7.11}$$

as follows: Denote by $e_{ij} \in \mathfrak{gl}_n(\mathbb{K})$ the elementary matrix with a one in the (i,j)-th entry and zeroes everywhere else. Set then for $c = a_1 \otimes \cdots \otimes a_p \in A^{\otimes p}$ and $1 \le r, s \le n$:

$$\begin{aligned} \zeta_{rs}(c) := \sum_{i_2,\dots,i_p} &(e_{ri_2} \otimes a_1) \otimes (e_{i_2 i_3} \otimes a_2) \otimes \dots \\ &\cdots \otimes (e_{i_{p-1} i_p} \otimes a_{p-1}) \otimes (e_{i_p s} \otimes a_p) \in (\mathfrak{gl}_n(A))^{\otimes p}. \end{aligned} \tag{3.7.12}$$

With this we define

$$\tilde{\theta} : T^\bullet(\tilde{T}^\bullet(A)) \to T^\bullet(\mathfrak{gl}_n(A)) \tag{3.7.13}$$

as the graded algebra map induced by, for $a_1 \otimes \cdots \otimes a_p \in T^1(T^p(A))$:

$$\begin{aligned} \tilde{\theta}(a_1 \otimes \cdots \otimes a_p) &:= \sum_{1 \le k \le n} \zeta_{kk}(a_1 \otimes \cdots \otimes a_p) \\ &= \sum_{i_1,\dots,i_p} (e_{i_1 i_2} \otimes a_1) \otimes \cdots \otimes (e_{i_n i_1} \otimes a_n). \end{aligned} \tag{3.7.14}$$

Further, recall that for some partition γ of a number m, the Specht module V^γ is defined as generated by equivalence classes of standard Young tableaux of shape γ in a certain way [Han88, p.216]. Two standard Young tableaux are considered equivalent if their rows contain the same numbers. Let x be such a Young diagram of length $\le n$ and $1 \le i \le m$, then we set $\rho_i(x)$ to be the row of x containing the number i. We define:

$$\begin{aligned} &\tilde{\epsilon} : \tilde{T}^m(C_\bullet^{\mathrm{bar}}(A)) \otimes V^\alpha \otimes V^\beta \to T^\bullet(\mathfrak{gl}_n(A)), \\ &\quad (c_1 \otimes \cdots \otimes c_m) \otimes x \otimes y \mapsto \zeta_{\rho_1(x), n+1-\rho_1(y)}(c_1) \otimes \cdots \otimes \zeta_{\rho_m(x), n+1-\rho_m(y)}(c_m) \end{aligned} \tag{3.7.15}$$

Then, finally, we define

$$\tilde{\psi}_A : S(A) \to M_{[\alpha,\beta]_n} T^\bullet(\mathfrak{gl}_n(A)) \tag{3.7.16}$$

as the tensor product of $\tilde{\theta}$ and $\tilde{\epsilon}$. [Han88, Theorem 3.4] shows that this indeed maps into the highest weight module $M_{[\alpha,\beta]_n} T^\bullet(\mathfrak{gl}_n(A))$. Note also that $\tilde{\psi}_A$ intertwines the actions

$$\mathbb{Z}/k_1\mathbb{Z} \circlearrowright T^{k_1}(A), \quad \Sigma_{k_2} \circlearrowright T^{k_2}(\tilde{T}^\bullet(A)), \quad \Sigma_m \circlearrowright (\tilde{T}^m(C^{\mathrm{bar}}_\bullet(A)) \otimes V^\alpha \otimes V^\beta) \tag{3.7.17}$$

on the domain with corresponding permutations of Σ_m on the codomain, and the invariants of both spaces with respect to these actions are

$$R^\Sigma(A) := \Lambda^\bullet(C^\lambda_{\bullet-1}(A)) \otimes \left(\tilde{T}^m(C^{\mathrm{bar}}_\bullet(A)) \otimes V^\alpha \otimes V^\beta\right)^{\Sigma_m}, \tag{3.7.18}$$

$$R_\Sigma(A) := \Lambda^\bullet(C^\lambda_{\bullet-1}(A)) \otimes \left(\tilde{T}^m(C^{\mathrm{bar}}_\bullet(A)) \otimes V^\alpha \otimes V^\beta\right)_{\Sigma_m}, \tag{3.7.19}$$

and $M_{[\alpha,\beta]_n} C_\bullet(\mathfrak{gl}(A))$. Denote by $\psi_A : R^\Sigma(A) \to C^\bullet(\mathfrak{gl}(A))$ the arising map on invariants. We show now:

Lemma 3.7.5. *Let A be a complete TVS. Consider $R^\Sigma(A)$ and $M_{[\alpha,\beta]_n}(C_\bullet(\mathfrak{gl}_n(A)))$ as TVS with the topology induced by the projective tensor product. Then ψ_A extends to a morphism of topological graded vector spaces on the completions*

$$\overline{R^\Sigma(A)} \to M_{[\alpha,\beta]_n}(C^{\mathrm{born}}_\bullet(\mathfrak{gl}_n(A))), \tag{3.7.20}$$

which is an isomorphism in degree $\leq \frac{n}{2}$.

Proof. [Han88, Theorem 3.6] shows that for every finite-dimensional A, the map ψ_A is an isomorphism of vector spaces in degree $\leq \frac{n}{2}$. Maschke's theorem implies that in characteristic zero and for a finite group G, we can always assign to every equivariant morphism $f : U \to V$ between G-modules an equivariant morphism $h : V \to U$, with the property that if f reduces to an isomorphism on invariants $f^G : U^G \to V^G$, then h^G inverts f^G. Hence in the case $A = \mathbb{K}$ there is a linear map

$$\tilde{\kappa}_\mathbb{K} : T^\bullet(\mathfrak{gl}_n(\mathbb{K})) \to S(\mathbb{K}), \tag{3.7.21}$$

intertwining the actions (3.7.17), whose reduction to invariants $\kappa_\mathbb{K} : C_\bullet(\mathfrak{gl}_n(\mathbb{K})) \to R(\mathbb{K})$ inverts $\psi_\mathbb{K}$ in degree $\leq \frac{n}{2}$. Note that the domain and codomain of $\tilde{\kappa}_\mathbb{K}$ have finite-dimensional graded components and hence $\tilde{\kappa}_\mathbb{K}$ and $\kappa_\mathbb{K}$ are automatically continuous. In the case of A being an arbitrary TVS, we can use canonical isomorphisms to identify

$$S(A) \cong S(\mathbb{K}) \otimes T^\bullet(A), \quad T^\bullet(\mathfrak{gl}_n(A)) \cong T^\bullet(\mathfrak{gl}_n(\mathbb{K})) \otimes T^\bullet(A). \tag{3.7.22}$$

These isomorphisms identify $\tilde{\psi}_A$ with $\tilde{\psi}_\mathbb{K} \otimes \mathrm{id}_{T^\bullet(A)}$. Under these identifications, consider $\tilde{\kappa}_A := \tilde{\kappa}_\mathbb{K} \otimes \mathrm{id}_{T^\bullet(A)}$. The equivariance of $\tilde{\kappa}_A$ and $\tilde{\psi}_A$ under the actions (3.7.17) show that the reduction to invariants $\kappa_A : C_\bullet(\mathfrak{gl}_n(A)) \to R(A)$ is a two-sided inverse to ψ_A in degree $\leq \frac{n}{2}$, just as in the case $A = \mathbb{K}$. This map is continuous as the tensor product of continuous maps. Hence ψ_A has a continuous inverse, and thus lifts to a continuous morphism of graded TVS on the topological closures, and that this lift is an isomorphism in degree $\leq \frac{n}{2}$. The lemma is shown. □

3

Now, assume A is a nuclear Fréchet algebra. We use the previously constructed map ψ_A, which only induced a morphism of TVS, to induce a morphism of chain complexes. Denote by A^* the strongly continuous dual of A and by $A^\vee$ the algebraic dual. Define the following maps for $a \in A, \beta \in A^*, g, h \in \mathfrak{gl}(\mathbb{K})$:

$$\mathrm{ev}: A \to (A^\vee)^\vee, \qquad \mathrm{ev}(a)(\beta) := \beta(a), \tag{3.7.23}$$

$$\nu: \mathfrak{gl}_n(A) \to \mathfrak{gl}_n(A^\vee)^\vee, \qquad \nu(g \otimes a)(h \otimes \beta) := \mathrm{tr}(g^T \cdot h) \cdot \mathrm{ev}(a)(\beta), \tag{3.7.24}$$

$$K: C^\lambda_\bullet(A) \to C^\lambda_\bullet(A), \qquad K([a_1 \otimes \cdots \otimes a_n]) := n \cdot [a_1 \otimes \cdots \otimes a_n]. \tag{3.7.25}$$

In [Cor05, Theorem 3.1], a continuous graded chain map ϕ is defined which makes the following diagram commute:

$$\begin{tikzcd}
M_{[\alpha,\beta]_n} C_\bullet(\mathfrak{gl}_n(A)) \arrow[r, "\phi"] \arrow[d] & R_\Sigma(A) \arrow[d] \\
M_{[\alpha,\beta]_n} C_\bullet(\mathfrak{gl}_n(A^\vee))^\vee \arrow[r, "(\psi_{A^\vee} \circ (K \otimes 1))^T"] & \left(R^\Sigma(A^\vee)\right)^\vee
\end{tikzcd}$$

Here, the left vertical arrow is induced by ν, the right vertical arrow by ev. Since nuclear Fréchet algebras are automatically reflexive, ev and ν are topological isomorphisms when all algebraic duals are replaced with strongly continuous duals and $C_\bullet$ with $C^{\mathrm{born}}_\bullet$. We have also proven in Lemma 3.7.5 that ψ_{A^*} extends to an isomorphism of TVS in degree $\leq \frac{n}{2}$ on the completion, and so does its transpose $\psi^T_{A^*}$. Hence, if in the above diagram the algebraic duals are replaced with continuous duals and $C_\bullet$ with $C^{\mathrm{born}}_\bullet$, the bottom arrow, too, is a topological isomorphism. Hence all arrows except ϕ in the above diagram induce topological isomorphisms in degree $\leq \frac{n}{2}$ under the given replacements. Hence ϕ does, too. Hence the extension of ϕ to the closures is a morphism of chain complexes and an isomorphism in degree $\leq \frac{n}{2}$. This concludes the statement. $\square$

Proof of Theorem 3.4.10. Due to the Lemmata 3.7.2 and 3.7.3, we have

$$\begin{aligned} H^{\mathrm{born}}_\bullet(\mathfrak{gl}_n(A)) \cong{}& H_\bullet\big(C^{\mathrm{born}}_\bullet(\mathfrak{gl}_n(A))_{[\emptyset,\emptyset]_n}\big) \\ &\oplus \bigoplus_{m \geq 1} \bigoplus_{\substack{\alpha,\beta \in P(m) \\ l(\alpha)+l(\beta) \leq n}} U(\mathfrak{gl}_n(\mathbb{K})) \cdot H_\bullet\left(M_{[\alpha,\beta]_n}(C^{\mathrm{born}}_\bullet(\mathfrak{gl}_n(A)))\right). \end{aligned} \tag{3.7.26}$$

If A is bornologically H-unital, then its bornological bar complex is acyclic, and thus its differential has closed range. By assumption on A, the differential of the cyclic complex has closed range, we can calculate the homology of the codomain of (3.7.9) via the Künneth isomorphism from Corollary 3.3.11. But acyclicity of the bar complex then implies acyclicity of this product complex. Since (3.7.9) is a chain isomorphism in degree $r \leq \frac{n}{2}$, it induces isomorphisms of homology groups in degree $2r+1 \leq n$. Thus, if $2r+1 \leq n$, we have

$$H^{\mathrm{born}}_r(\mathfrak{gl}_n(A)) \cong H_r\left(C^{\mathrm{born}}_\bullet(\mathfrak{gl}_n(A))_{[\emptyset,\emptyset]_n}\right) = H_r\left(C^{\mathrm{born}}_\bullet(\mathfrak{gl}_n(A))_{\mathfrak{gl}_n(\mathbb{K})}\right), \tag{3.7.27}$$

and the calculation of the homology of the invariant complex $C_\bullet^{\mathrm{born}}(\mathfrak{gl}_n(A))_{\mathfrak{gl}_n(\mathbb{K})}$ is carried out exactly as in the unital case in Section 3.4. In particular, this homology stabilizes so that $H_r^\iota(\mathfrak{gl}(A)) \cong H_r^\iota(\mathfrak{gl}_n(A)) \cong H_r^{\mathrm{born}}(\mathfrak{gl}_n(A))$ when $2r+1 \leq n$. Hence the statement is shown. □

REFERENCES

[BL01] Jacek Brodzki and Zinaida A. Lykova. Excision in cyclic type homology of Fréchet algebras. *Bull. London Math. Soc.*, 33(3):283–291, 2001.

[BL09] J.-P. Brasselet and A. Legrand. Teleman localization of Hochschild homology in a singular setting. *Russ. J. Math. Phys.*, 16(3):391–403, 2009.

[Bre97] Glen E. Bredon. *Sheaf theory*, volume 170 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1997.

[BS77] R. Bott and G. Segal. The cohomology of the vector fields on a manifold. *Topology*, 16(4):285–298, 1977.

[Cas73] Antonio Cassa. Formule di Künneth per la coomologia a valori in un fascio. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3)*, 27:905–931 (1974), 1973.

[CG17] Kevin Costello and Owen Gwilliam. *Factorization algebras in quantum field theory. Vol. 1*, volume 31 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2017.

[Cn06] Guillermo Cortiñas. The obstruction to excision in K-theory and in cyclic homology. *Invent. Math.*, 164(1):143–173, 2006.

[Con85] Alain Connes. Noncommutative differential geometry. *Inst. Hautes Études Sci. Publ. Math.*, (62):257–360, 1985.

[Cor05] G. Cortiñas. Cyclic homology of H-unital (pro-) algebras, Lie algebra homology of matrices, and a paper of Hanlon's. *arXiv preprint math/0504148*, 2005.

[EP96] Jörg Eschmeier and Mihai Putinar. *Spectral decompositions and analytic sheaves*, volume 10 of *London Mathematical Society Monographs. New Series*. The Clarendon Press, Oxford University Press, New York, 1996. Oxford Science Publications.

[Ewa04] Christian-Oliver Ewald. Hochschild- and cyclic-homology of LCNT-spaces. *Comm. Math. Phys.*, 250(1):195–213, 2004.

[Fei88] B. L. Feigin. On the cohomology of the Lie algebra of vector fields and of the current algebra. volume 7, pages 49–62. 1988. Selected translations.

3

3

[GF69] I. M. Gel'fand and D. B. Fuks. Cohomologies of the Lie algebra of tangent vector fields of a smooth manifold. *Funkcional. Anal. i Priložen.*, 3(3):32–52, 1969.

[Glö04] Helge Glöckner. Tensor products in the category of topological vector spaces are not associative. *Comment. Math. Univ. Carolin.*, 45(4):607–614, 2004.

[GLW05] F. Gourdeau, Z. A. Lykova, and M. C. White. A Künneth formula in topological homology and its applications to the simplicial cohomology of $l^1(\mathbb{Z}_+^k)$. *Studia Math.*, 166(1):29–54, 2005.

[GM92] I. M. Gel'fand and O. Mathieu. On the cohomology of the Lie algebra of Hamiltonian vector fields. *J. Funct. Anal.*, 108(2):347–360, 1992.

[Gon92] Dong Geng Gong. Excision of equivariant cyclic cohomology of topological algebras. *Michigan Math. J.*, 39(3):455–473, 1992.

[Gro] A. Grothendieck. Opérations algébriques sur les distributions à valeur vectorielle. théorème de künneth. *Séminaire Schwartz*, 1:1–6.

[Gro95] A. Grothendieck. Produits tensoriels topologiques et espaces nucléaires. In *Séminaire Bourbaki, Vol. 2*, pages Exp. No. 69, 193–200. Soc. Math. France, Paris, 1995.

[GW21] Owen Gwilliam and Brian R. Williams. Higher Kac-Moody algebras and symmetries of holomorphic field theories. *Adv. Theor. Math. Phys.*, 25(1):129–239, 2021.

[Han88] Phil Hanlon. On the complete $\mathrm{GL}(n,\mathbf{C})$-decomposition of the stable cohomology of $\mathrm{gl}_n(A)$. *Trans. Amer. Math. Soc.*, 308(1):209–225, 1988.

[JW13] Bas Janssens and Christoph Wockel. Universal central extensions of gauge algebras and groups. *J. Reine Angew. Math.*, 682:129–139, 2013.

[Kau67] Ludger Kaup. Eine Künnethformel für Fréchetgarben. *Math. Z.*, 97:158–168, 1967.

[Kau68] Ludger Kaup. Das topologische Tensorprodukt kohärenter analytischer Garben. *Math. Z.*, 106:273–292, 1968.

[Kha13] Masoud Khalkhali. *Basic noncommutative geometry*. EMS Series of Lectures in Mathematics. European Mathematical Society (EMS), Zürich, second edition, 2013.

[KM97] Andreas Kriegl and Peter W. Michor. *The convenient setting of global analysis*, volume 53 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1997.

[Lod92] Jean-Louis Loday. *Cyclic homology*, volume 301 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1992. Appendix E by María O. Ronco.

[LQ84] Jean-Louis Loday and Daniel Quillen. Cyclic homology and the Lie algebra homology of matrices. *Comment. Math. Helv.*, 59(4):569–591, 1984.

[Mai02] Peter Maier. Central extensions of topological current algebras. In *Geometry and analysis on finite- and infinite-dimensional Lie groups (Będlewo, 2000)*, volume 55 of *Banach Center Publ.*, pages 61–76. Polish Acad. Sci. Inst. Math., Warsaw, 2002.

[Mey99] Ralf Meyer. *Analytic cyclic cohomology*. PhD thesis, Westfälische Wilhelms-Universität Münster, 1999.

[Mey10] Ralf Meyer. Excision in Hochschild and cyclic homology without continuous linear sections. *J. Homotopy Relat. Struct.*, 5(1):269–303, 2010.

[Mia22] Lukas Miaskiwskyi. Continuous cohomology of gauge algebras and bornological Loday-Quillen-Tsygan theorems. *arXiv preprint arXiv:2206.08879*, 2022.

[MV97] Reinhold Meise and Dietmar Vogt. *Introduction to functional analysis*, volume 2 of *Oxford Graduate Texts in Mathematics*. The Clarendon Press, Oxford University Press, New York, 1997. Translated from the German by M. S. Ramanujan and revised by the authors.

[Pal72] V. P. Palamodov. On a Stein manifold the Dolbeault complex splits in positive dimensions. *Mat. Sb. (N.S.)*, 88(130):287–315, 1972.

[Pfl98] Markus J. Pflaum. On continuous Hochschild homology and cohomology groups. *Lett. Math. Phys.*, 44(1):43–51, 1998.

[Qui88] Daniel Quillen. Algebra cochains and cyclic cohomology. *Inst. Hautes Études Sci. Publ. Math.*, (68):139–174 (1989), 1988.

[Sch71] Helmut H. Schaefer. *Topological vector spaces*. Graduate Texts in Mathematics, Vol. 3. Springer-Verlag, New York-Berlin, 1971. Third printing corrected.

[Ser55] Jean-Pierre Serre. Un théorème de dualité. *Comment. Math. Helv.*, 29:9–26, 1955.

[Tel98] Nicolae Teleman. Microlocalisation de l'homologie de Hochschild. *C. R. Acad. Sci. Paris Sér. I Math.*, 326(11):1261–1264, 1998.

[Tel02] Constantin Teleman. Some Hodge theory from Lie algebras. In *Motives, polylogarithms and Hodge theory, Part II (Irvine, CA, 1998)*, volume 3 of *Int. Press Lect. Ser.*, pages 731–744. Int. Press, Somerville, MA, 2002.

3

[Trè67] François Trèves. *Topological vector spaces, distributions and kernels*. Academic Press, New York-London, 1967.

[Tsy83] B. L. Tsygan. Homology of matrix Lie algebras over rings and the Hochschild homology. *Uspekhi Mat. Nauk*, 38(2(230)):217–218, 1983.

[Voi84] Jürgen Voigt. Factorization in some Fréchet algebras of differentiable functions. *Studia Math.*, 77(4):333–348, 1984.

[Wei94] Charles A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.

[Wod89] Mariusz Wodzicki. Excision in cyclic homology and in rational algebraic K-theory. *Ann. of Math. (2)*, 129(3):591–639, 1989.

II

Integration formulas in finite-dimensional Lie theory

4

VOLUMES OF IDENTITY-NEIGHBORHOODS IN REAL REDUCTIVE LIE GROUPS

Bas JANSSENS, Lukas MIASKIWSKYI

We study ratios $\delta_F(U) := \frac{\mu\left(\bigcap_{g\in F}\mathrm{Ad}_g U\right)}{\mu(U)}$ of Haar measures of identity neighborhoods in real reductive Lie groups, measuring their volume changes under the adjoint action. We study adjoint orbits within both the Lie group and its associated Lie algebra, orbital limits due to Barbasch, Harris, and Vogan, and use integration formulas by Harish-Chandra and Varadarajan. Using this, we conclude that there exists a neighborhood basis $\mathscr{U}$ so that if $B^G_\rho = \{g \in G : \|\mathrm{Ad}_g\| \leq \rho\}$, then $\limsup_{U\in\mathscr{U}} \delta_{B^G_\rho}(U) \geq \rho^{-d/2}$, where d is the maximal dimension of a nilpotent orbit in $\mathfrak{g}$. This gives quantitative insight into norm coefficients for noncommutative de Leeuw inequalities in Harmonic Analysis as studied by, for example, Caspers, Parcet, Perrin, and Ricard.

This chapter is an excerpt of the preprint [CJKUM22].

4

4.1. INTRODUCTION

In this section, we study for a real, reductive Lie group G, ratios of the form

$$\delta_F(U) := \frac{\mu\left(\bigcap_{g\in F} \mathrm{Ad}_g U\right)}{\mu(U)}, \tag{4.1.1}$$

where $F \subset G$ is some subset, μ denotes the Haar measure, Ad denotes the adjoint action, and $U \subset G$ is a relatively compact neighborhood of the identity with nonzero Haar volume. For a neighborhood basis $\mathscr{U}$ of the identity in G we set $\delta_F(\mathscr{U}) := \liminf_{U\in\mathscr{U}} \delta_F(U)$, and δ_F denotes the supremum of $\delta_F(\mathscr{U})$ where $\mathscr{U}$ varies over all symmetric neighborhood bases of the identity in G.

The study in this section is, perhaps surprisingly so, based in noncommutative harmonic analysis and the so-called *de Leeuw theorems*. Let us lay out the essential ideas that guide us there. For now, let G be a locally compact, unimodular group, and denote by $\lambda : L_2(G) \to L_2(G)$ its left-regular representation. One defines the non-commutative L_p-spaces $L_p(\hat{G})$ as a collection of elements of the group von Neumann algebra $\mathscr{L}(G)$ with a certain finite p-trace. The notation reflects the fact that if G abelian, this is isomorphic to the L_p-space of the Pontryagin dual of G. We set for $\lambda(f) := \int_G f(s)\lambda(s)\,\mathrm{d}s$ for $f \in L_p(G)$, and denote by $\star$ the convolution product of compactly supported functions $C_c(G)$.

One says that a bounded, continuous map $m \in C_b(G)$ is a *p-multiplier* if there exists a bounded operator $T_m : L_p(\hat{G}) \to L_p(\hat{G})$ with

$$T_m(\lambda(f)) = \lambda(mf) \quad \forall f \in C_c(G) \star C_c(G). \tag{4.1.2}$$

In [CJKUM22], a variation of a non-commutative de Leeuw restriction theorem is proven, c.f. [CPPR15], the study of which originates from [dL65]: If $m \in C_b(G)$ is a p-multiplier, $\Gamma < G$ is a discrete subgroup, and $1 \le p < \infty$, then

$$c(\mathrm{supp}(m|_\Gamma))\, \| T_{m|_\Gamma} : L_p(\hat{\Gamma}) \to L_p(\hat{\Gamma}) \| \le \| T_m : L_p(\hat{G}) \to L_p(\hat{G}) \|, \tag{4.1.3}$$

where $c(\mathrm{supp}(m|_\Gamma)) := \inf\left\{ \sqrt{\delta_F} | F \subset \mathrm{supp}(m|_\Gamma) \text{ finite} \right\}$. For the example $G = \mathrm{SL}(n,\mathbb{R})$ and $\Gamma = \mathrm{SL}(n,\mathbb{Z})$, the existence of nontrivial multipliers for $n > 3$ is still an open problem. To this end, inequalities on δ_F can be used to gather information about the factor c in inequality (4.1.3), and this is valuable to provide quantitative insight into the necessary behavior of multipliers on Γ.

While we do not pick up on this background again, this explains the interest in being able to quantify δ_F. Our main theorem will be a nontrivial inequality for $\delta_{B_\rho^G}$, where

$$B_\rho^G := \{ g \in G : \| \mathrm{Ad}_g \| \le \rho \}, \quad \rho > 0. \tag{4.1.4}$$

To be precise, if d is the maximal dimension of a nilpotent, adjoint orbit in the Lie algebra $\mathfrak{g} := \mathrm{Lie}(G)$, then we prove that

$$\delta_{B_\rho^G} \ge \rho^{-d/2}, \quad \forall \rho > 0. \tag{4.1.5}$$

Our result is based on the construction of a well-behaved neighborhood basis of the identity in G and a careful analysis of the geometry of G and its associated Lie algebra $\mathfrak{g}$, based on orbital limit theorems due to Barbasch, Harris, and Vogan [BV80, Har12] and Lie theoretic integration formulas due to Harish-Chandra and Varadarajan [HC57, HC65, Var77].

We begin in Section 4.2 by recalling some essentials on Lie theory of real reductive Lie groups and transferring the problem from a study of the Lie-group-theoretic quantity δ_F to the study of the Lie-algebraic quantity

$$\lim_{\epsilon\to 0}\frac{\Lambda(V^{\mathfrak{g}}_{\epsilon,\rho R})}{\Lambda(V^{\mathfrak{g}}_{\epsilon,R})}, \tag{4.1.6}$$

where Λ denotes the Lebesgue measure and $\{V^{\mathfrak{g}}_{\epsilon,\rho}\}_{\epsilon,\rho>0}$ is a certain family of zero-neighborhoods in $\mathfrak{g}$, whose exponentials constitute the desired neighborhood basis in G. In particular we show that the Haar measure on G can, in an infinitesimal sense, be replaced by the Lebesgue measure gained from the vector space structure on $\mathfrak{g}$, and that the analysis only depends on the maximal semisimple ideal $\mathfrak{g}_0 \subset \mathfrak{g}$. This makes it possible to replace the original question with a question in the tightly controllable setting of semisimple Lie algebras. Here, the $V_{\epsilon,\rho}$ take the shape

$$V^{\mathfrak{g}}_{\epsilon,R} = \mathrm{Ad}_G(B^{\mathfrak{g}}_{\epsilon}) \cap B^{\mathfrak{g}}_{R}, \tag{4.1.7}$$

where $B^{\mathfrak{g}}_R$ denotes the unit ball with respect to a certain Hilbert space structure on $\mathfrak{g}$.
In Section 4.3 we recall a standard toolbox for the structure of semisimple Lie algebras, and expand on the earlier announced orbital limit theorem due to [BV80, Har12]. This will show that the cone of nilpotent elements $\mathcal{N} \subset \mathfrak{g}$ is directly related to invariant zero neighborhoods in $\mathfrak{g}$, and that the dimensionality of the former influences the scaling behavior of the latter. This will be an essential component of the calculation of (4.1.6). Using analytic results due to Harish-Chandra and Varadarajan [HC57, Var77], we show that their orbital limits do not only converge in the pointwise sense, but uniformly over a certain domain of orbits.
Lastly, in Section 4.4 we combine the results of the previous section with a Fubini-like Lie algebraic integral formula due to Harish-Chandra and Varadarajan, which expresses integrals on $\mathfrak{g}$ as iterated integrals over its adjoint orbits $O_h \subset \mathfrak{g}$ and quotient spaces G/H, where H denotes the respective stabilizer of the orbit O_h [HC65, Var77]. This will make the measures in (4.1.6) explicit enough that, together with the orbital limit formulas, we can calculate the limit exactly.

4.2. PRELIMINARIES

Let G be a real reductive Lie group with Lie algebra $\mathfrak{g}$, Cartan involution θ, maximal compact subgroup K, and invariant bilinear form B, cf. [Kna96, Chapter VII.2]. The inner product

$$B_\theta(x,y) := -B(x,\theta y) \tag{4.2.1}$$

on $\mathfrak{g}$ endows $\mathrm{End}(\mathfrak{g})$ with the operator norm $A \mapsto \|A\|$. For $\rho \geq 1$, we denote by

$$B_\rho^G := \{g \in G;\ \|\mathrm{Ad}_g\| \leq \rho\} \tag{4.2.2}$$

the preimage under the adjoint representation $\mathrm{Ad}\colon G \to \mathrm{End}(\mathfrak{g})$ of the closed ball of radius ρ around the origin. The aim of this section is to prove the following lower bound on $\delta_{B_\rho^G}$ in terms of the radius.

4

Theorem 4.2.1. *If d is the maximal dimension of a nilpotent orbit of G in $\mathfrak{g}$, then*

$$\delta_{B_\rho^G} \geq \rho^{-d/2}. \tag{4.2.3}$$

Remark 4.2.2. As the adjoint orbit O_X of a nilpotent element $X \in \mathfrak{g}$ is a symplectic manifold, the coefficient $d/2$ in (4.2.3) is an integer. Since $T_X O_X = \mathfrak{g}/\mathfrak{g}_X$ is the quotient of $\mathfrak{g}$ by the centralizer $\mathfrak{g}_X$ of X, the maximal dimension d can be expressed as

$$d = \dim(\mathfrak{g}) - \min_{X\in\mathcal{N}} \dim(\mathfrak{g}_X), \tag{4.2.4}$$

where $\mathcal{N} \subseteq \mathfrak{g}$ is the nilpotent cone of $\mathfrak{g}$. In particular, $d = 0$ if $\mathfrak{g}$ is compact or abelian, $d = \dim(\mathfrak{g}) - \mathrm{rank}(\mathfrak{g})$ if $\mathfrak{g}$ is split (or quasisplit [Rot72, Theorem 5.1]), and $d = 2(\dim_{\mathbb{C}}(\mathfrak{g}) - \mathrm{rank}_{\mathbb{C}}(\mathfrak{g}))$ if $\mathfrak{g}$ is complex. In particular, $d = n(n-1)$ for $\mathrm{SL}(n,\mathbb{R})$ and $\mathrm{GL}(n,\mathbb{R})$, and $d = 2n(n-1)$ for $\mathrm{SL}(n,\mathbb{C})$ and $\mathrm{GL}(n,\mathbb{C})$.

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$, let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$, and let A be the corresponding analytic subgroup. If $g = k_1 a k_2$ is the KAK-decomposition of $g \in G$ (see [Kna96, §VII.3]), then $\|\mathrm{Ad}_g\| = \|\mathrm{Ad}_a\|$ since both B and θ are invariant under $\mathrm{Ad}(K)$. Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\lambda\in\Sigma} \mathfrak{g}_\lambda$ be the restricted root space decomposition, where the sum runs over the set $\Sigma \subseteq \mathfrak{a}^*$ of restricted roots. Since A is simply connected, the set B_ρ^G can be equivalently described as

$$B_\rho^G = K\rho^{\mathscr{P}}K := K\exp\big(\log(\rho)\mathscr{P}\big)K, \tag{4.2.5}$$

where $\mathscr{P} \subseteq \mathfrak{a}$ is the polygon $\mathscr{P} = \{h \in \mathfrak{a};\ \alpha(h) \leq 1\ \forall\, \alpha \in \Sigma\}$. From this description, the following result easily follows.

Proposition 4.2.3. *The sets B_ρ^G are invariant under inversion, and under left and right multiplication by K. Furthermore, $\bigcup_{\rho>1} B_\rho^G = G$ and $\bigcap_{\rho>1} B_\rho^G = K$.*

Proof. Invariance under left and right multiplication by K is clear from (4.2.5), and invariance under inversion follows from the fact that $\Sigma = -\Sigma$. The formula for the union is obvious, and the formula for the intersection follows from $\bigcap_{\rho>1} B_\rho^G = B_1^G$, and the fact that $\|\mathrm{Ad}_{\exp(\pm h)}\| = \|\exp(\mathrm{ad}_{\pm h})\| = 1$ for $h \in \mathfrak{a}$ if and only if $h = 0$. □

4.2.1. A NEIGHBORHOOD BASIS

The reductive Lie algebra $\mathfrak{g}$ decomposes as the direct sum $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{z}$ of the maximal semisimple ideal $\mathfrak{g}_0 = [\mathfrak{g}, \mathfrak{g}]$ and the center $\mathfrak{z}$. The former admits the Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$, with $\mathfrak{k}_0 = \mathfrak{k} \cap \mathfrak{g}_0$ and $\mathfrak{p}_0 = \mathfrak{p} \cap \mathfrak{g}_0$.

Let $B_r^{\mathfrak{z}} \subseteq \mathfrak{z}$ and $B_r^{\mathfrak{g}_0} \subseteq \mathfrak{g}_0$ be the open balls of radius r with respect to the inner product B_θ in $\mathfrak{z}$ and $\mathfrak{g}_0$, respectively. Both are K-invariant, and the global Cartan decomposition $G = K\exp(\mathfrak{p})$ implies that $B_r^{\mathfrak{z}} \subseteq \mathfrak{z}$ is invariant under all of G.

We are interested in the neighborhood basis

$$V^{\mathfrak{g}}_{\epsilon,R,r} := \left(\mathrm{Ad}_G(B_\varepsilon^{\mathfrak{g}_0}) \cap B_R^{\mathfrak{g}_0}\right) \times B_r^{\mathfrak{z}}, \tag{4.2.6}$$

obtained by intersecting the G-invariant neighborhoods

$$U_\varepsilon := \mathrm{Ad}_G(B_\varepsilon^{\mathfrak{g}_0}) \times \mathfrak{z} \tag{4.2.7}$$

with the bounded sets $B_R^{\mathfrak{g}_0} \times B_r^{\mathfrak{z}}$. It will be convenient to write $V^{\mathfrak{g}}_{\epsilon,R,r} = V^{\mathfrak{g}_0}_{\varepsilon,R} \times B_r^{\mathfrak{z}}$, where $V^{\mathfrak{g}_0}_{\varepsilon,R}$ is the bounded open subset of $\mathfrak{g}_0$ defined by

$$V^{\mathfrak{g}_0}_{\varepsilon,R} := \mathrm{Ad}_G(B_\varepsilon^{\mathfrak{g}_0}) \cap B_R^{\mathfrak{g}_0}. \tag{4.2.8}$$

Remark 4.2.4 (K-invariance). Since $B_\varepsilon^{\mathfrak{g}_0}$ is K-invariant, the global Cartan decomposition $G = K\exp(\mathfrak{p})$ yields $\mathrm{Ad}_G(B_\varepsilon^{\mathfrak{g}_0}) = \mathrm{Ad}_{\exp(\mathfrak{p})}(B_\varepsilon^{\mathfrak{g}_0})$. Further, since $\mathfrak{z}$ acts trivially on $\mathfrak{g}_0$, we have $\mathrm{Ad}_G(B_\varepsilon^{\mathfrak{g}_0}) = \mathrm{Ad}_{\exp(\mathfrak{p}_0)}(B_\varepsilon^{\mathfrak{g}_0})$ for $\mathfrak{p}_0 = \mathfrak{p} \cap \mathfrak{g}_0$. It follows that $V^{\mathfrak{g}}_{\epsilon,R,r} = V^{\mathfrak{g}_0}_{\epsilon,R} \times B_r^{\mathfrak{z}}$ is the product of the K-invariant set $V^{\mathfrak{g}_0}_{\epsilon,R} \subseteq \mathfrak{g}_0$ that depends only on the restriction of B_θ to $\mathfrak{g}_0$, and the G-invariant set $B_r^{\mathfrak{z}} \subseteq \mathfrak{z}$ that depends only on the restriction of B_θ to $\mathfrak{z}$. In particular the sets $V^{\mathfrak{g}}_{\epsilon,R,r} \subseteq \mathfrak{g}$ are K-invariant, and they depend only on the triple $(\mathfrak{g}, \theta, B)$, not on the Lie group G.

The lower bound (4.2.3) will be established by calculating

$$\delta^0_{B_\rho^G} := \limsup_{(\epsilon,R,r)\to 0} \frac{\mu\left(\bigcap_{g\in B_\rho^G} \mathrm{Ad}_{g^{-1}} \exp(V^{\mathfrak{g}}_{\epsilon,R,r})\right)}{\mu(\exp(V^{\mathfrak{g}}_{\epsilon,R,r}))}, \tag{4.2.9}$$

where μ is a Haar measure on G. Since $V^{\mathfrak{g}}_{\epsilon,R,r} = -V^{\mathfrak{g}}_{\epsilon,R,r}$, the sets $\exp(V^{\mathfrak{g}}_{\epsilon,R,r})$ constitute a symmetric neighborhood basis of the identity. It follows that $\delta_{B_\rho^G} \geq \delta^0_{B_\rho^G}$, so in order to establish (4.2.3), it suffices to prove that $\delta^0_{B_\rho^G} \geq \rho^{-d/2}$.

4.2.2. RELATION BETWEEN HAAR MEASURE AND LEBESGUE MEASURE.

The first step is to reformulate this in terms of the Lebesgue measure on the Lie algebra $\mathfrak{g}$.

Let Vol_G be a left invariant volume form on G, so that integrating against Vol_G corresponds to a left Haar measure μ on G. Let $\mathrm{Vol}_{\mathfrak{g}}$ be a constant volume form on $\mathfrak{g}$, corresponding to a Lebesgue measure Λ on $\mathfrak{g}$. We normalize these volume forms in such a way that $\mathrm{Vol}_{\mathfrak{g}}$ agrees with $\exp^* \mathrm{Vol}_G$ at the origin in $\mathfrak{g}$. Then $\exp^* \mathrm{Vol}_G = \nu \mathrm{Vol}_{\mathfrak{g}}$, where the density ν of $\exp^* \mathrm{Vol}_G$ with respect to the Lebesgue measure satisfies $\nu(0) = 1$. We show that ν can be chosen arbitrarily close to 1 on $U_\varepsilon \subseteq \mathfrak{g}$ for small ε.

Proposition 4.2.5. *The density is given by $\nu(x) = \det(\Phi_x)$, where $\Phi\colon \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ is the left logarithmic derivative of the exponential map. The function $\nu\colon \mathfrak{g} \to \mathbb{R}$ is smooth, G-invariant, and equal to 1 on $\mathfrak{z} \subseteq \mathfrak{g}$. Moreover, there exists a constant $c_{\mathfrak{g}_0} > 0$ (depending only on $\mathfrak{g}_0$) such that for ε sufficiently small, $\|\nu - 1\|_\infty \leq c_{\mathfrak{g}_0}\, \varepsilon$ uniformly on U_ε.*

4

Proof. Let $\Phi\colon \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ be the left logarithmic derivative of the exponential map, defined by $\Phi_x(y) := (D_{\exp(x)} L_{\exp(x)^{-1}}) \circ (D_x \exp)(y)$, where $L_g\colon G \to G$ denotes left multiplication by g. Then for all $x, y_1, \ldots, y_n \in \mathfrak{g}$,

$$\begin{aligned}
(\exp^* \mathrm{Vol}_G)_x(y_1, \ldots, y_n) &= (\mathrm{Vol}_G)_{\exp(x)}(D_x \exp(y_1), \ldots, D_x \exp(y_n)) \\
&= (\mathrm{Vol}_G)_{\exp(x)}(D_{\mathbf{1}} L_{\exp(x)} \Phi_x(y_1), \ldots, D_{\mathbf{1}} L_{\exp(x)} \Phi_x(y_n)) \\
&= (\mathrm{Vol}_G)_{\mathbf{1}}(\Phi_x(y_1), \ldots, \Phi_x(y_n)) \\
&= \det(\Phi_x)(\mathrm{Vol}_{\mathfrak{g}})_0(y_1, \ldots, y_n),
\end{aligned} \tag{4.2.10}$$

where the last two steps use that Vol_G is left invariant, and that $\exp^* \mathrm{Vol}_G$ agrees with $\mathrm{Vol}_{\mathfrak{g}}$ at the origin. It follows that $\nu(x) = \det(\Phi_x)$. Since $\exp\colon \mathfrak{g} \to G$ is equivariant with respect to the adjoint action on $\mathfrak{g}$ and the conjugate action on G, its logarithmic derivative satisfies $\Phi_{\mathrm{Ad}_g(x)} = \mathrm{Ad}_g \circ \Phi_x \circ \mathrm{Ad}_{g^{-1}}$. In particular, $\nu(x) = \det(\Phi_x)$ is invariant under the adjoint action.

In fact, the left logarithmic derivative $\Phi_x\colon \mathfrak{g} \to \mathfrak{g}$ of the exponential map is explicitly given (cf. [Far08, Theorem 2.1.4]) by the convergent power series

$$\Phi_x = \frac{\mathrm{Id} - \exp(-\mathrm{ad}_x)}{\mathrm{ad}_x} = \mathrm{Id} - \frac{1}{2!}\,\mathrm{ad}_x + \frac{1}{3!}(\mathrm{ad}_x)^2 + \ldots \tag{4.2.11}$$

Note that the determinant of the real linear map $\Phi_x\colon \mathfrak{g} \to \mathfrak{g}$ is equal to the determinant over $\mathbb{C}$ of the complexification $\Phi_x^{\mathbb{C}}\colon \mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}$. In terms of the Jordan–Chevalley decomposition $\mathrm{ad}_x = \mathrm{ad}_{x_s} + \mathrm{ad}_{x_n}$ into a semisimple and a nilpotent element of $\mathfrak{g}_{\mathbb{C}}$, we thus find

$$\nu(x) = \left|\det\nolimits_{\mathbb{C}}\left(\frac{\mathrm{Id} - \exp(-\mathrm{ad}_{x_s})}{\mathrm{ad}_{x_s}}\right)\right| = \prod_{i=1}^{\dim \mathfrak{g}} \left|\frac{1 - e^{-\mu_i}}{\mu_i}\right|, \tag{4.2.12}$$

where μ_i are the eigenvalues of ad_x as a complex linear transformation of $\mathfrak{g}_{\mathbb{C}}$.

In particular, $\nu(x)$ depends only on the second factor in $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{g}_0$, and we can write $\nu(x) = \nu_0(\mathrm{ad}_x)$ for a smooth function $\nu_0\colon \mathfrak{g}_0 \to \mathbb{R}$ with $\nu_0(0) = 1$. It follows that for any $c_{\mathfrak{g}_0} > \|\nabla \nu_0(0)\|$, there exists an $\varepsilon_0 > 0$ such that $\|\nu_0 - 1\| \leq c_{\mathfrak{g}_0} \varepsilon$ uniformly on $B_\varepsilon^{\mathfrak{g}_0}$ for all $\varepsilon < \varepsilon_0$. The uniform estimate for ν on U_ε now follows from G-invariance. $\square$

For r and R smaller than the injectivity radius of the exponential $\exp\colon \mathfrak{g} \to G$, we can therefore relate the Haar measure of $\exp(V^{\mathfrak{g}}_{\varepsilon,R,r})$ to the Lebesgue measure of $V^{\mathfrak{g}}_{\varepsilon,R,r}$,

$$(1 - c_{\mathfrak{g}_0}\,\varepsilon)\Lambda(V^{\mathfrak{g}}_{\varepsilon,R,r}) < \mu\big(\exp(V^{\mathfrak{g}}_{\varepsilon,R,r})\big) < (1 + c_{\mathfrak{g}_0}\,\varepsilon)\Lambda(V^{\mathfrak{g}}_{\varepsilon,R,r}). \tag{4.2.13}$$

Since exp is equivariant under the adjoint action on $\mathfrak{g}$ and G, this allows us to express $\delta^0_{B^G_\rho}$ in terms of the Lebesgue measure on $\mathfrak{g}$,

$$\delta^0_{B^G_\rho} = \limsup_{(\varepsilon,R,r)\to 0} \frac{\Lambda\Big(\bigcap_{g\in B^G_\rho} \mathrm{Ad}_{g^{-1}}(V^{\mathfrak{g}}_{\varepsilon,R,r})\Big)}{\Lambda(V^{\mathfrak{g}}_{\varepsilon,R,r})}. \tag{4.2.14}$$

Proposition 4.2.6. *The number $\delta^0_{B^G_\rho}$ depends on G only through the maximal semisimple ideal $\mathfrak{g}_0 = [\mathfrak{g},\mathfrak{g}]$, and the restriction of B_θ to $\mathfrak{g}_0$.*

Proof. By Remark 4.2.4 the sets $V^{\mathfrak{g}}_{\varepsilon,R,r}$ are K-invariant. If $g = k\exp(p)$ is the global Cartan decomposition of $g \in G = K\exp(\mathfrak{p})$, we thus have

$$\mathrm{Ad}_{g^{-1}}(V^{\mathfrak{g}}_{\varepsilon,R,r}) = \mathrm{Ad}_{\exp(-p)}(V^{\mathfrak{g}}_{\varepsilon,R,r}). \tag{4.2.15}$$

Further, since $\mathfrak{z}$ acts trivially on $\mathfrak{g}$, the decomposition $p = p_0 + p_{\mathfrak{z}}$ of $p \in \mathfrak{p}$ with respect to $\mathfrak{p} = \mathfrak{p}_0 \oplus (\mathfrak{z} \cap \mathfrak{p})$ yields

$$\mathrm{Ad}_{g^{-1}}(V^{\mathfrak{g}}_{\varepsilon,R,r}) = \mathrm{Ad}_{\exp(-p_0)}(V^{\mathfrak{g}_0}_{\varepsilon,R}) \times B^{\mathfrak{z}}_r. \tag{4.2.16}$$

Since B^G_ρ is K-invariant (Proposition 4.2.3), we have $k\exp(p) \in B^G_\rho$ if and only if $\exp(p) \in B^G_\rho$, which is the case if and only if $\exp(p_0) \in B^G_\rho$. But since $\mathrm{Ad}_{\exp(p_0)}$ acts by the identity on second factor of $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{z}$, we have $\|\mathrm{Ad}^{\mathfrak{g}}_{\exp(p_0)}\| \le \rho$ for the adjoint action on $\mathfrak{g}$ if and only $\|\mathrm{Ad}^{\mathfrak{g}_0}_{\exp(p_0)}\| \le \rho$ for the adjoint action on $\mathfrak{g}_0$. If G_0 denotes the connected adjoint group of the semisimple Lie algebra $\mathfrak{g}_0$, we thus have

$$\begin{aligned}\frac{\Lambda^{\mathfrak{g}}\Big(\bigcap_{g\in B^G_\rho} \mathrm{Ad}_{g^{-1}}(V^{\mathfrak{g}}_{\varepsilon,R,r})\Big)}{\Lambda^{\mathfrak{g}}(V^{\mathfrak{g}}_{\varepsilon,R,r})} &= \frac{\Lambda^{\mathfrak{g}}\Big(\bigcap_{g_0\in B^{G_0}_\rho} \mathrm{Ad}_{g_0^{-1}}(V^{\mathfrak{g}_0}_{\varepsilon,R}) \times B^{\mathfrak{z}}_r\Big)}{\Lambda^{\mathfrak{g}}(V^{\mathfrak{g}_0}_{\varepsilon,R} \times B^{\mathfrak{z}}_r)} \\ &= \frac{\Lambda^{\mathfrak{g}_0}\Big(\bigcap_{g_0\in B^{G_0}_\rho} \mathrm{Ad}_{g_0^{-1}}(V^{\mathfrak{g}_0}_{\varepsilon,R})\Big)}{\Lambda^{\mathfrak{g}_0}(V^{\mathfrak{g}_0}_{\varepsilon,R})},\end{aligned} \tag{4.2.17}$$

where $\Lambda^{\mathfrak{g}}$ and $\Lambda^{\mathfrak{g}_0}$ denote the Lebesgue measure on $\mathfrak{g}$ and $\mathfrak{g}_0$, respectively. □

4.2.3. Proof of Theorem 4.2.1

By Proposition 4.2.6, it suffices to prove Theorem 4.2.1 for the case where G is the adjoint group of the semisimple Lie algebra $\mathfrak{g}_0$. The proof hinges on the following lemma.

Lemma 4.2.7 (Key Lemma)**.** *Let G be a connected, real reductive Lie group with semisimple Lie algebra $\mathfrak{g}$. Then for all $R>0$ and all $\rho>1$, we have*

$$\lim_{\epsilon\to 0}\frac{\Lambda(V^{\mathfrak{g}}_{\epsilon,\rho R})}{\Lambda(V^{\mathfrak{g}}_{\epsilon,R})}=\rho^{d/2}. \tag{4.2.18}$$

The proof of this lemma requires a rather detailed discussion of limits of orbital integrals, and will be deferred to Section 4.4. Assuming Lemma 4.2.7, the proof of Theorem 4.2.1 is quite straightforward.

Proof of Theorem 4.2.1, assuming Lemma 4.2.7. In view of (4.2.4), the maximal dimension d of the nilpotent orbits is the same in $\mathfrak{g}$ and $\mathfrak{g}_0$. By Proposition 4.2.6, we may therefore assume without loss of generality that G is a connected, real reductive Lie group with semisimple Lie algebra $\mathfrak{g}$.

Since $\|\mathrm{Ad}_g\|\le\rho$, we have $\mathrm{Ad}_{g^{-1}}(B^{\mathfrak{g}}_R)\supseteq B^{\mathfrak{g}}_{R/\rho}$. Since $\mathrm{Ad}_G(B^{\mathfrak{g}}_\varepsilon)$ is Ad_G-invariant, we find for $V_{\varepsilon,R}=\mathrm{Ad}_G(B^{\mathfrak{g}}_\varepsilon)\cap B^{\mathfrak{g}}_R$ that

$$\mathrm{Ad}_{g^{-1}}(V_{\varepsilon,R})=\mathrm{Ad}_G(B^{\mathfrak{g}}_\varepsilon)\cap\mathrm{Ad}_{g^{-1}}B^{\mathfrak{g}}_R\supseteq V_{\varepsilon,R/\rho}. \tag{4.2.19}$$

From (4.2.14) (without r because $\mathfrak{z}=\{0\}$) we thus find

$$\begin{aligned}\delta^0_{B^G_\rho}=\limsup_{\varepsilon,R\to 0}\frac{\Lambda\left(\bigcap_{g\in B^G_\rho}\mathrm{Ad}_{g^{-1}}(V^{\mathfrak{g}}_{\varepsilon,R})\right)}{\Lambda(V^{\mathfrak{g}}_{\varepsilon,R})}&\ge\limsup_{\varepsilon,R\to 0}\frac{\Lambda(V^{\mathfrak{g}}_{\varepsilon,R/\rho})}{\Lambda(V^{\mathfrak{g}}_{\varepsilon,R})}\\&\ge\lim_{R\to 0}\lim_{\epsilon\to 0}\frac{\Lambda(V^{\mathfrak{g}}_{\varepsilon,R/\rho})}{\Lambda(V^{\mathfrak{g}}_{\varepsilon,R})}=\rho^{-d/2}.\end{aligned} \tag{4.2.20}$$

□

The strategy to prove Lemma 4.2.7 is as follows. Since the closure of an adjoint orbit O_x through $x\in\mathfrak{g}$ contains 0 if and only if x is nilpotent, the set $\bigcap_{\varepsilon>0}V_{\varepsilon,R}$ is the intersection of the nilpotent cone $\mathcal{N}$ with the unit ball $B^{\mathfrak{g}}_R(0)$. The union of the nilpotent orbits O_X of maximal dimension is a dense open subset of the nilpotent cone. Using results of Harish-Chandra and Barbasch–Vogan on limiting orbit integrals, we will show that as ε approaches 0, the volume of $V_{\varepsilon,R}$ with respect to the Lebesgue measure on $\mathfrak{g}$ scales with R in the same way as the Liouville volume of the symplectic manifold $O_X\cap B^{\mathfrak{g}}_R(0)$. Since the Kostant–Kirillov–Souriau symplectic form $\omega^{\mathrm{KKS}}_{O_X}$ on the cone O_X scales as R under dilation, the corresponding Liouville volume form $\mathrm{Vol}^{\mathrm{KKS}}_{O_X}$ scales as $R^{d/2}$, yielding the factor $\rho^{d/2}$ in Lemma 4.2.7.

4.3. LIMITS OF ORBITAL MEASURES

In the remainder of this section, we focus on the proof of Lemma 4.2.7. From now on, we assume that the Lie algebra $\mathfrak{g}$ is semisimple, and that the invariant bilinear form B is the Killing form κ. We will generally denote generic elements of $\mathfrak{g}$ by x, y, z, elements of a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ by h, and elements of the nilpotent cone $\mathcal{N} \subset \mathfrak{g}$ by X, Y, Z. For a subset $A \subseteq \mathfrak{g}$, we denote the centralizer and the normalizer in $\mathfrak{g}$ by $Z_{\mathfrak{g}}(A)$ and $N_{\mathfrak{g}}(A)$ respectively. On the group level, we similarly define

$$Z_G(A) := \{g \in G : \mathrm{Ad}_g\, y = y \quad \forall y \in A\}, \quad N_G(A) := \{g \in G : \mathrm{Ad}_g\, A \subset A\}. \tag{4.3.1}$$

4.3.1. REGULAR ELEMENTS OF LIE ALGEBRAS

We recall the notion of *regularity* in $\mathfrak{g}$. For $x \in \mathfrak{g}$ consider the characteristic polynomial

$$\det(\mathrm{ad}\, x - t) =: \sum_{k \geq 0} a_k(x) t^k, \quad t \in \mathbb{R}. \tag{4.3.2}$$

4

Let $k(x)$ be the minimal index so that $a_{k(x)}(x) \neq 0$. If $k(x) = \min_{y \in \mathfrak{g}} k(y)$, we say x is *regular*. If $A \subset \mathfrak{g}$ is any subset of $\mathfrak{g}$, we define A_{reg} to be the set of regular elements in A. If $\mathfrak{h}$ is a Cartan subalgebra (CSA) and C is a connected component of $\mathfrak{h}_{\mathrm{reg}}$, we call C an *open Weyl chamber* of $\mathfrak{h}$.

Remark 4.3.1. Our notion of regularity is different from a common definition where x is regular if its centralizer has minimal dimension among the centralizers of all $x' \in \mathfrak{g}$. Our current definition is used, for example, in [Bou05].

We will use the following standard properties of regular elements:

Lemma 4.3.2. *Let $\mathfrak{g}$ be a real Lie algebra.*

i) Regular elements in $\mathfrak{g}$ lie in a unique CSA given by their centralizer.

ii) If a single element in an adjoint orbit $O_x \subset \mathfrak{g}$ is regular, then all elements are.

iii) The set of regular elements is dense and open in $\mathfrak{g}$, and its complement in $\mathfrak{g}$ is a Lebesgue null set.

Proof. i) The semisimplicity statement is [Bou05, Chapter VIII.4, Cor 2], and the statement about the CSA is [Bou05, Theorem VII.3.1].
ii) This follows since the characteristic polynomial is invariant under the adjoint action.
iii) From [Bou05, Chapter VIII.2] we know that the set of regular elements is Zariski-open, which implies that it is dense and open in the standard topology. The non-regular elements, as a complement of a Zariski-open set, are intersections of closed submanifolds of lower dimension, hence they constitute a Lebesgue null set by Sard's Theorem. □

4.3.2. MEASURES AND DISTRIBUTIONS ON ORBITS

Let $x \in \mathfrak{g}$, and let O_x be the adjoint orbit through $x \in \mathfrak{g}$. Since O_x can be identified with a coadjoint orbit via the invariant bilinear form, it comes equipped with a canonical symplectic form. The *Kostant–Kirillov–Souriau (KKS) form* $\omega^{\mathrm{KKS}}_{O_x} \in \Omega^2(O_x)$ is given by

$$\left(\omega^{\mathrm{KKS}}_{O_x}\right)_{x'} (\mathrm{ad}_{x'}\, y, \mathrm{ad}_{x'}\, z) = \kappa(x', [y, z]), \quad \forall x' \in O_x,\ y, z \in \mathfrak{g}. \tag{4.3.3}$$

This induces a volume form called the Liouville form

$$\mathrm{Vol}_{O_x} := \frac{1}{k!}(\wedge^k \omega^{\mathrm{KKS}}_{O_x}), \tag{4.3.4}$$

where $k = \dim O_x/2$. By [RR72, Theorem 2] the assignment of Borel sets $A \subset \mathfrak{g}$ to

$$\mu_{O_x}(A) := \int_{O_x \cap A} \mathrm{Vol}_{O_x} \tag{4.3.5}$$

defines a Radon measure μ_{O_x} on $\mathfrak{g}$. In particular, it is finite on compact subsets of $\mathfrak{g}$. On nilpotent orbits, these measures are homogeneous:

Lemma 4.3.3. *Let $A \subset \mathfrak{g}$ be a Borel set, let $X \in \mathfrak{g}$ a nilpotent element, and let $k = \dim O_X/2$. Then for all $\rho > 0$, we have*

$$\mu_{O_X}(\rho \cdot A) = \rho^k \mu_{O_X}(A). \tag{4.3.6}$$

Proof. By the Jacobson–Morozov theorem, the nilpotent orbit O_X is a *cone*, i.e. $\rho \cdot O_X = O_X$ for all $\rho > 0$. Thus we have

$$(\rho \cdot A) \cap O_X = \rho \cdot (A \cap O_X). \tag{4.3.7}$$

Denote by $m_\rho : O_X \to O_X$ the multiplication by ρ. By definition of the KKS form we have for all $X' \in O_X$ and $y, z \in \mathfrak{g}$:

$$\begin{aligned}(m_\rho^* \omega^{\mathrm{KKS}}_{O_X})_{X'}(\mathrm{ad}_{X'} y, \mathrm{ad}_{X'} z) &= (\omega^{\mathrm{KKS}}_{O_X})_{\rho X'}(\mathrm{ad}_{\rho X'} y, \mathrm{ad}_{\rho X'} z)\\ &= \kappa(\rho X', [y,z])\\ &= \rho \cdot \kappa(X', [y,z]),\end{aligned} \tag{4.3.8}$$

hence

$$m_\rho^* \omega^{\mathrm{KKS}}_{O_X} = \rho \cdot \omega^{\mathrm{KKS}}_{O_X}. \tag{4.3.9}$$

From (4.3.4) we then find

$$m_\rho^* \mathrm{Vol}_{O_X} = \rho^k \cdot \mathrm{Vol}_{O_X}, \tag{4.3.10}$$

so that

$$\mu_{O_X}(\rho \cdot A) = \int_{m_\rho(A \cap O_X)} \mathrm{Vol}_{O_X} = \int_{A \cap O_X} m_\rho^* \mathrm{Vol}_{O_X} = \rho^k \mu_{O_X}(A) \tag{4.3.11}$$

as required. □

The measure μ_{O_x} on the adjoint orbit O_x through $x \in \mathfrak{g}$ yields the distribution $\mathscr{D}_{O_x}$ on $\mathfrak{g}$ defined by

$$\mathscr{D}_{O_x} : C_c^\infty(\mathfrak{g}) \to \mathbb{R}, \quad \mathscr{D}_{O_x}(f) := \frac{1}{(2\pi)^{2k}} \int_{O_x} f|_{O_x} \mathrm{Vol}_{O_x}, \tag{4.3.12}$$

again with $k := \dim(O_x)/2$. The 2π-normalization factor ensures that the orbital distributions $\mathscr{D}_{O_x}$ coincide with the ones in [Har12], where this normalization occurs in the volume form Vol_{O_x}. Let $\mathfrak{h} \subset \mathfrak{g}$ be any θ-invariant Cartan subalgebra, and $H := Z_G(\mathfrak{h})$ the associated Cartan subgroup. Since G and H are unimodular, we can fix a G-invariant volume form $\mathsf{Vol}_{G/H}$ on the quotient G/H. Note that $\mathsf{Vol}_{G/H}$ is unique up to a nonzero scalar. Let $h \in \mathfrak{g}$ be any element with centralizer H. Then the orbit map $g \mapsto \mathrm{Ad}_g(h)$ descends to a diffeomorphism $\iota\colon G/H \xrightarrow{\sim} O_h$, and the pullback $\iota^*\mathsf{Vol}_{O_h}$ of the KKS volume form on O_h defines yet another invariant volume form on G/H. The two invariant volume forms agree up to a scalar which depends only on h,

$$\iota^*\mathsf{Vol}_{O_h} = \pi(h)\mathsf{Vol}_{G/H}, \tag{4.3.13}$$

4

yielding a function $\pi : \mathfrak{h} \to \mathbb{R}$.

Proposition 4.3.4. *In the above setting, there is some $c > 0$ depending only on the choice of $\mathsf{Vol}_{G/H}$, so that for all $h \in \mathfrak{h}$,*

$$|\pi(h)| = c \cdot \prod_{\alpha\in\Delta_+} |\alpha(h)|. \tag{4.3.14}$$

Here, $\Delta^+ \subset \Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ is a choice of positive roots for the complexified Lie algebra $\mathfrak{g}_\mathbb{C}$ with respect to the CSA $\mathfrak{h}_\mathbb{C}$.

Proof. It suffices to consider the volume forms $\iota^*\mathsf{Vol}_{O_h}$ and $\mathsf{Vol}_{G/H}$ at a single point of G/H, hence we restrict to $T_{[e]}(G/H) \cong \mathfrak{g}/\mathfrak{h}$. We identify $\mathfrak{g}/\mathfrak{h} \cong \mathfrak{h}^\perp$, where $\mathfrak{h}^\perp$ denotes the orthogonal complement of $\mathfrak{h} \subset \mathfrak{g}$ with respect to the inner product κ_θ. There is some scalar $\tilde{c} \neq 0$ so that $\tilde{c} \cdot (\mathsf{Vol}_{G/H})_{[e]} = \mathsf{Vol}_{\mathfrak{h}^\perp}$, where $\mathsf{Vol}_{\mathfrak{h}^\perp}$ is the volume form on $\mathfrak{h}^\perp$ associated to the inner product κ_θ and some choice of orientation on $\mathfrak{h}^\perp$. Consider the map $\theta \circ \mathrm{ad}_h \colon \mathfrak{g} \to \mathfrak{g}$ for $h \in \mathfrak{h}$. Since it preserves $\mathfrak{h}$ and is skew-symmetric with respect to the inner product κ_θ, it restricts to a skew-symmetric endomorphism of $\mathfrak{h}^\perp$. The pullback of the KKS form at $[e] \in G/H$ is given, for $x, y \in \mathfrak{h}^\perp$, by

$$\iota^*\omega^{\mathrm{KKS}}_{O_h}(x, y) = \kappa(h, [x, y]) = \kappa_\theta(x, \theta \circ \mathrm{ad}_h(y)). \tag{4.3.15}$$

Recall that if (V, B) is an oriented inner product space of even dimension $2k$, then the Pfaffian of a skew-symmetric linear map $A\colon V \to V$ is defined by

$$\frac{1}{k!}\left(\wedge^k \omega_A\right) = \mathrm{Pf}(A)\mathsf{Vol}, \tag{4.3.16}$$

where $\mathsf{Vol} \in \wedge^{2k} V^*$ is the volume form associated to the inner product B on the oriented vector space V, and $\omega_A(v, w) = B(v, Aw)$ is the 2-form associated to A with respect to the inner product B. With $V = \mathfrak{h}^\perp$ and $A = \theta \circ \mathrm{ad}_h$, this yields

$$\iota^*\mathsf{Vol}_{O_h} = \frac{1}{k!} \wedge^k (\iota^*\omega^{\mathrm{KKS}}_{O_h})_{[e]} = \mathrm{Pf}\Big((\theta \circ \mathrm{ad}_h)\,|_{\mathfrak{h}^\perp}\Big)\mathsf{Vol}_{\mathfrak{h}^\perp}. \tag{4.3.17}$$

Recall that the Pfaffian is related to the determinant by

$$\mathrm{Pf}\left((\theta\circ \mathrm{ad}_h)\big|_{\mathfrak{h}^\perp}\right)^2 = \det\left((\theta\circ \mathrm{ad}_h)\big|_{\mathfrak{h}^\perp}\right) = \pm\det\left(\mathrm{ad}_h\big|_{\mathfrak{h}^\perp}\right). \tag{4.3.18}$$

Since the determinant is the product of the eigenvalues over $\mathbb{C}$, we can determine $|\pi(h)|$ from the eigenvalues of ad_h on the complexification $(\mathfrak{h}^\perp)_\mathbb{C} \cong \mathfrak{g}_\mathbb{C}/\mathfrak{h}_\mathbb{C}$. In view of the root space decomposition $\mathfrak{g}_\mathbb{C}/\mathfrak{h}_\mathbb{C} \cong \bigoplus_{\alpha\in\Delta_+}\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$, the eigenvalues of the complex linear map ad_h are $\pm\alpha(h)$. This proves the statement with $c = |\tilde{c}|$. □

4.3.3. Slodowy Slices and Pointwise Orbital Limits

Let $\mathcal{N}\subset\mathfrak{g}$ be the nilpotent cone. For any nonzero $X\in\mathcal{N}$, there exists an $\mathfrak{sl}_2$-triple $\{X, Y, H\}$ containing X as the nilpositive element by the Jacobson–Morosov Theorem. We denote by

$$S_X := X + Z_\mathfrak{g}(Y) \tag{4.3.19}$$

the corresponding *Slodowy slice* through X, cf. [Slo80, Chapter 7.4]. It is transversal to the orbit O_X due to the decomposition

$$\mathfrak{g} = \mathrm{ad}_X\,\mathfrak{g} \oplus Z_\mathfrak{g}(Y), \tag{4.3.20}$$

cf. [Bou05, Chapter VIII.2]. It is indeed transversal to *all* orbits O_x with $x\in S_X$, and in particular, the set $G\cdot S_X$ is an open neighborhood of the orbit O_X, cf. [Slo80, Chapter 7.4]. We recall from [Har12, Chapter 2] the construction of a canonical measure $m_{x,X}$ on the intersection $S_X\cap O_x$: we can consider the composition

$$\mathrm{ad}_x\,\mathfrak{g} \hookrightarrow \mathfrak{g} \to \mathrm{ad}_X\,\mathfrak{g}, \tag{4.3.21}$$

where the first map is the natural embedding and the second map the projection of the direct sum (4.3.20) onto the first direct summand. Since the Slodowy slice intersects O_x transversally, the composition of these two maps is surjective. Using $T_xO_x \cong \mathrm{ad}_x\,\mathfrak{g}$ and $T_XO_X \cong \mathrm{Ad}_X\,\mathfrak{g}$, this surjective map induces the following exact sequence:

$$0 \to T_x(O_x\cap S_X) \to T_xO_x \to T_XO_X \to 0. \tag{4.3.22}$$

We obtain a canonical volume form on $O_x\cap S_X$ as the quotient of the KKS volume forms on O_x and O_X, which in turn gives rise to the measure $m_{x,X}$. In [Har12, Chapter 2], the following limit of orbits is defined for all $x\in\mathfrak{g}$:

$$\mathcal{N}_x := \mathcal{N}\cap\overline{\bigcup_{\epsilon>0} O_{\epsilon x}}. \tag{4.3.23}$$

One may think of this set as the limit of the orbits $O_{\epsilon x}$ as ϵ approches zero, hence as the orbits approach the nilpotent cone. Let us first show that every nilpotent orbit arises, in this sense, as a limit of regular orbits:

Lemma 4.3.5. *Every nilpotent orbit O_X lies in the set $\mathcal{N}_x$ for some regular $x \in \mathfrak{g}$.*

Proof. The idea of this proof is essentially due to [BV80, p.48]. If X is nilpotent, choose an $\mathfrak{sl}_2$-triple $\{X, Y, H\}$ with X as the nilpositive and H as the semisimple element. Consider the associated Slodowy slice $S_X = X + Z_{\mathfrak{g}}(Y)$. Since $G \cdot S_X$ is an open neighborhood of O_X, and since the set of regular elements is dense in $\mathfrak{g}$, there exists a regular element $x \in G \cdot S_X$. In other words, there exist $g \in G$ and $V \in Z_{\mathfrak{g}}(Y)$ with

$$\mathrm{Ad}_g x = X + V. \tag{4.3.24}$$

The centralizer $Z_{\mathfrak{g}}(Y)$ is stable under ad_H, hence we can decompose it into the eigenspaces of ad_H on this space. Due to the structure theory of finite-dimensional $\mathfrak{sl}_2$-modules (cf. [Bou05, Chapter VIII.2]), the eigenvalues λ on these eigenspaces are all nonpositive:

$$Z_{\mathfrak{g}}(Y) = \bigoplus_{\lambda \le 0} (Z_{\mathfrak{g}}(Y))_\lambda, \quad V = \sum_{\lambda \le 0} V_\lambda. \tag{4.3.25}$$

Consider then the element $g_t := \exp\left(-\frac{1}{2}\log(t) H\right) \in G$. Then we have

$$\begin{aligned} \mathrm{Ad}_{g_t g}\, tx &= t\,\mathrm{Ad}_{g_t} X + t\,\mathrm{Ad}_{g_t} V & (4.3.26)\\ &= t \exp(-\log(t)) X + \sum_{\lambda \le 0} t \exp\left(-\frac{\lambda}{2}\log(t)\right) V_\lambda & (4.3.27)\\ &= X + \sum_{\lambda \le 0} t^{1-\lambda/2} V_\lambda \overset{t \to 0}{\to} X. & (4.3.28) \end{aligned}$$

But this means that $X \in \mathcal{N} \cap \overline{\bigcup_{t>0} \mathrm{Ad}_G(tx)} = \mathcal{N}_x$. ☐

Remark 4.3.6. A closer inspection of such orbital limits is given in [FM21]. Their definition of $\mathcal{N}_x$ coincides with the one given here by [FM21, Remark 3.6].

We will need the following asymptotic expression for the orbital integrals. It is proven in [Har12, Corollary 2.3], relying on [BV80, Theorem 3.2].

Definition 4.3.7. For $x \in \mathfrak{g}$, we define

$$m(x) := \min_{O_X \subset \mathcal{N}_x} \tfrac{1}{2} (\dim O_x - \dim O_X), \tag{4.3.29}$$

where the minimum is taken over all adjoint orbits O_X contained in $\mathcal{N}_x$.

Theorem 4.3.8. *Let $\mathfrak{g}$ be a real, reductive Lie algebra, and let $x \in \mathfrak{g}$. Then for all $f \in C_c^\infty(\mathfrak{g})$ we have*

$$\lim_{\epsilon \to 0} \epsilon^{-m(x)} \mathscr{D}_{O_{\epsilon x}}(f) = \sum_{O_X} \mathsf{Vol}(S_X \cap O_h) \mathscr{D}_{O_X}(f), \tag{4.3.30}$$

where the sum is taken over the set of nilpotent orbits $O_X \subset \mathcal{N}_x$ which are of maximal dimension among all orbits contained in $\mathcal{N}_x$, and the volume of $S_X \cap O_x$ is calculated with respect to the measure $m_{x,X}$.

Remark 4.3.9. If C is an open Weyl chamber of some CSA $\mathfrak{h}$, one actually has $\mathcal{N}_h = \mathcal{N}_{h'}$ for all $h, h' \in C$ by [Har12, Corollary 2.4] (originally attributed to [BV80]). In particular, the number m in Theorem 4.3.8 is equal for all h in one such open Weyl chamber. In this case we also write $m(C)$ for the m associated to any $h \in C$.

4.3.4. Uniform Orbital Limits

Using results of Harish-Chandra and Varadarajan, we will show that the convergence in Theorem 4.3.8 is uniform on certain subsets of $\mathfrak{h}$. For an open subset $C \subseteq V$ of a vector space V, we denote by $C^k(\overline{C}, \mathbb{R})$ the space of functions $u\colon C \to \mathbb{R}$ that are k times continuously differentiable on C, and whose derivatives of order at most k extend continuously to the closure. The following result follows from [HC57, Theorem 3] (see also [Var77, Theorem I.3.23]).

Theorem 4.3.10. *Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra, $C \subset \mathfrak{h}$ an open Weyl chamber, and let $f \in C_c^\infty(\mathfrak{g}, \mathbb{R})$. Then the function $u_f : C \to \mathbb{R}$ defined by*

$$u_f(h) := \mathscr{D}_{O_h}(f) \tag{4.3.31}$$

is in $C^\infty(\overline{C}, \mathbb{R})$.

Remark 4.3.11. In fact, u_f extends to a Schwartz function on the connected component of the non-zero sets of the singular imaginary roots.

Remark 4.3.12. The definition of the invariant integral in [HC57] is, up to a positive scalar, equivalent to ours by Proposition 4.3.4.

Proposition 4.3.13. *Let C be an open cone in V and $u \in C^{m+1}(\overline{C}, \mathbb{R})$ with $D^{m-1}u(0) = 0$. Then for any compact $K \subseteq \overline{C}$,*

$$\lim_{\varepsilon \to 0^+} \varepsilon^{-m} u(\varepsilon v) = \frac{1}{m!} \partial_v^m u(0) \tag{4.3.32}$$

uniformly for $v \in K$.

Proof. Let $p(t) := u(tv)$. Then since $p \in C^{m+1}([0,1], \mathbb{R})$, Taylor's Theorem yields $p(\varepsilon) = \frac{1}{m!}\varepsilon^m \partial_v^m u(0) + R$, where $R = \frac{1}{(m+1)!}\varepsilon^{m+1}\partial_v^{m+1} u(\theta v)$ for some $\theta \in [0, \varepsilon]$. Since $(v, w) \mapsto \partial_v^{m+1} u(w)$ is uniformly bounded on $K \times K$, the result follows. □

Fix again an open Weyl chamber C of some Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. From Theorem 4.3.8, Theorem 4.3.10, and the fact that integrals over compact domains commute with uniform limits, it immediately follows that

$$\lim_{\epsilon \to 0} \int_{K \cap C} \epsilon^{-m} \mathscr{D}_{O_{\epsilon h}}(f)\, w(h)\, \mathrm{d}\Lambda_{\mathfrak{h}}(h) = \sum_{O_X} \int_{K \cap C} \mathrm{Vol}(S_X \cap O_h) \mathscr{D}_{O_X}(f)\, w(h)\, \mathrm{d}\Lambda_{\mathfrak{h}}(h), \tag{4.3.33}$$

for all compact $K \subset \mathfrak{h}$, all $f \in C_c^\infty(\mathfrak{g})$ and all continuous functions $w\colon \mathfrak{h} \to \mathbb{R}$. In particular, the integral on the right-hand side is well-defined. A similar statement holds with $\mathscr{D}_{O_X}(f)$ replaced by $\mu_{O_X}(B_R(0))$.

Corollary 4.3.14. *Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra, $K \subset \mathfrak{h}$ a compact subset, and $C \subset \mathfrak{h}$ an open Weyl chamber. Let $w\colon \mathfrak{h} \to \mathbb{R}$ be a continuous function, and let $R > 0$. Then*

$$\begin{aligned}\lim_{\epsilon\to 0} \int_{K\cap C} & \epsilon^{-m} \mu_{O_{\epsilon h}}(B_R(0))\, w(h)\, \mathrm{d}\Lambda_{\mathfrak{h}}(h) \\ &= \sum_{O_X} \int_{K\cap C} \mathrm{Vol}(S_X \cap O_h) \mu_{O_X}(B_R(0))\, w(h)\, \mathrm{d}\Lambda_{\mathfrak{h}}(h),\end{aligned} \tag{4.3.34}$$

with the sum over the O_X as in Theorem 4.3.8, and the right-hand side is integrable.

Proof. The following proof is essentially taken from [Kal17, Lemma 4.1]. Choose sequences of functions $\{f_n \in C_c^\infty(\mathfrak{g})\}_{n\geq 1}, \{g_n \in C_c^\infty(\mathfrak{g})\}_{n\geq 1}$ with monotone, pointwise convergence

$$f_n \nearrow 1_{B_R(0)}, \quad g_n \searrow 1_{\overline{B_R(0)}}. \tag{4.3.35}$$

Let $k = \dim O_X/2$ for any of the orbits O_X in the sum. Since w can be written as the difference of two nonnegative functions, we may assume without loss of generality that w is nonnegative. Then, for all $n \in \mathbb{N}$, we have

$$\begin{aligned}&(2\pi)^{2k} \sum_{O_X} \int_{K\cap C} \mathrm{Vol}(S_X \cap O_h) \mathscr{D}_{O_X}(f_n)\, w(h)\, \mathrm{d}\Lambda_{\mathfrak{h}}(h) \\ &\leq \liminf_{\epsilon\to 0} \int_{K\cap C} \epsilon^{-m} \mu_{O_{\epsilon h}}(B_R(0))\, w(h)\, \mathrm{d}\Lambda_{\mathfrak{h}}(h) \\ &\leq \limsup_{\epsilon\to 0} \int_{K\cap C} \epsilon^{-m} \mu_{O_{\epsilon h}}(\overline{B_R}(0))\, w(h)\, \mathrm{d}\Lambda_{\mathfrak{h}}(h) \\ &\leq (2\pi)^{2k} \sum_{O_X} \int_{K\cap C} \mathrm{Vol}(S_X \cap O_h) \mathscr{D}_{O_X}(g_n)\, w(h)\, \mathrm{d}\Lambda_{\mathfrak{h}}(h).\end{aligned} \tag{4.3.36}$$

Note that $\mu_{O_X}(B_R(0)) = \mu_{O_X}(\overline{B_R(0)})$ for all nilpotent elements X. Indeed, since O_X is a cone, the boundary $\partial B_R(0)$ intersects O_X transversally so that the intersection $O_X \cap \partial B_R(0) \subset O_X$ is either empty (when $X = 0$) or a submanifold of codimension at least one (when $X \neq 0$). Its Liouville measure is thus zero by Sard's Theorem. Finally, the statement follows by taking the monotone limits $n \to \infty$ in (4.3.36). □

4.4. PROOF OF THE KEY LEMMA 4.2.7

In all that follows, fix a maximal set of mutually nonconjugate, θ-stable CSAs $\mathfrak{h}_1,\dots,\mathfrak{h}_n$ of $\mathfrak{g}$, with associated Cartan subgroups $H_1,\dots,H_n \subset G$ defined by $H_i := Z_G(\mathfrak{h}_i)$. We denote by $W_j := N_G(\mathfrak{h}_j)/H_j$ the Weyl group associated to $\mathfrak{h}_j$. We will need the following Lie algebraic version of the well-known Harish-Chandra integral formula [HC65, Lemma 41], see e.g. [Var77, Part I, Section 3, Lemma 2].

Lemma 4.4.1. *Let $\mathfrak{g}$ be a reductive Lie algebra with connected adjoint group G, and let $f \in L^1(\mathfrak{g}, \Lambda_\mathfrak{g})$ be an integrable function on $\mathfrak{g}$. Then, for $j = 1, \dots, n$, there are G-invariant volume forms Vol_{G/H_j} on G/H_j, so that*

$$\int_\mathfrak{g} f(x)\,\mathrm{d}\Lambda_\mathfrak{g}(x) = \sum_{j=1}^n \frac{1}{|W_j|} \int_{\mathfrak{h}_j} \left(\int_{G/H_j} f(\mathrm{Ad}_g h) \mathsf{Vol}_{G/H_j}([g]) \right) |\pi_j(h)|^2 \,\mathrm{d}\Lambda_{\mathfrak{h}_j}(h). \quad (4.4.1)$$

Here, $|\pi_j|$ is a product of positive roots of $(\mathfrak{g}_\mathbb{C}, (\mathfrak{h}_j)_\mathbb{C})$ as in Proposition 4.3.4.

Lemma 4.4.2. *Let $\mathfrak{h} \subset \mathfrak{g}$ be a CSA, and define*

$$V_\epsilon^\mathfrak{h} := \mathrm{Ad}_G B_\epsilon(0) \cap \mathfrak{h}. \quad (4.4.2)$$

4

Then for all $\epsilon, R > 0$ we have

$$(V_{\epsilon,R})_{\mathrm{reg}} = \bigsqcup_{j=1}^n \{\mathrm{Ad}_g h : h \in (V_\epsilon^{\mathfrak{h}_j})_{\mathrm{reg}}, \mathrm{Ad}_g(h) \in B_R(0)\}. \quad (4.4.3)$$

Proof. To see that the right-hand side is indeed a disjoint union, suppose that $\mathrm{Ad}_g h = \mathrm{Ad}_{g'} h'$ for regular elements $h \in \mathfrak{h}_i$, $h' \in \mathfrak{h}_j$ and $g, g' \in G$. Since h and h' are regular, $\mathfrak{h}_i = Z_\mathfrak{g}(h)$ and $\mathfrak{h}_j = Z_\mathfrak{g}(h')$. Since h is conjugate to h', $\mathfrak{h}_i$ is conjugate to $\mathfrak{h}_j$. As the various CSAs are mutually nonconjugate, we conclude that $\mathfrak{h}_i = \mathfrak{h}_j$.
$\subset$: Let $\mathrm{Ad}_g x \in (V_{\epsilon,R})_{\mathrm{reg}}$ with $g \in G$ and $x \in B_\epsilon(0)$. Since x is regular, it lies in a unique CSA, and is conjugate to an element $h \in \mathfrak{h}_j$ for some j, i.e.

$$\exists g' \in G : \mathrm{Ad}_{g'} x = h \in \mathfrak{h}_j. \quad (4.4.4)$$

By Lemma 4.3.2, the orbit of a regular element consists only of regular elements. Hence we have $h \in (V_\epsilon^{\mathfrak{h}_j})_{\mathrm{reg}}$ and $\mathrm{Ad}_{g(g')^{-1}} h = \mathrm{Ad}_g x \in B_R(0)$. Hence $\mathrm{Ad}_g x$ lies in the right-hand side.
$\supset$: Let $\mathrm{Ad}_g h$ lie in the right-hand side. Because $h \in \mathrm{Ad}_G B_\epsilon(0)$, we can write $h = \mathrm{Ad}_{g'} x$ for some $g' \in G$ and $x \in B_\epsilon(0)$. But then $\mathrm{Ad}_g h = \mathrm{Ad}_{gg'} x \in \mathrm{Ad}_G B_\epsilon(0)$, and since h was regular, so is every element in its orbit, and we have $\mathrm{Ad}_g h \in (V_{\epsilon,R})_{\mathrm{reg}}$. $\square$

Lemma 4.4.3. *Let $\mathfrak{h}$ be a θ-invariant CSA. For all $\epsilon > 0$, the sets $V_\epsilon^\mathfrak{h} = \mathrm{Ad}_G B_\epsilon(0) \cap \mathfrak{h}$ are bounded.*

Proof. With respect to the inner product $\kappa_\theta : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$, we have

$$\begin{aligned} \kappa_\theta(\mathrm{ad}_x y, z) &= -\kappa([x,y], \theta z) = \kappa(y, [x, \theta z]) \\ &= \kappa(y, \theta([\theta x, z])) = -\kappa_\theta(y, \mathrm{ad}_{\theta x} z). \end{aligned} \quad (4.4.5)$$

It follows that $\mathrm{ad}_h^* = -\mathrm{ad}_{\theta h}$. Since $[h, \theta h] = 0$, the operator $\mathrm{ad}_h : \mathfrak{g} \to \mathfrak{g}$ is normal, and its operator norm $\|\mathrm{ad}_h\|$ with respect to the inner product κ_θ satisfies

$$\|\mathrm{ad}_h\| = \max_{\lambda \in \mathrm{Spec}(\mathrm{ad}_h)} |\lambda|. \quad (4.4.6)$$

Let $g \in G$ such that $\mathrm{Ad}_g(h) \in B_\varepsilon(0)$. Since $\mathrm{ad}_{\mathrm{Ad}_g h}$ is conjugate to ad_h, it is a normal operator with the same eigenvalues, so in particular $\|\mathrm{ad}_h\| = \|\mathrm{ad}_{\mathrm{Ad}_g h}\|$. So since the operator norm is bounded on $B_\varepsilon(0)$, it is bounded on $\mathrm{Ad}_G(B_\varepsilon(0)) \cap \mathfrak{h}$ as well, and the latter is a bounded subset of $\mathfrak{h}$. □

Recall that $V_{\epsilon,R}$ differs from $(V_{\epsilon,R})_{\mathrm{reg}}$ only in a Lebesgue null set. Also, we can write every $(\mathfrak{h}_j)_{\mathrm{reg}}$ as the union of its open Weyl chambers $C_{j,r}$. Hence, we can use Lemma 4.4.1 and the decomposition (4.4.3) to find

$$\begin{aligned}\Lambda(V_{\epsilon,R}) &= \sum_{j=1}^{n} \frac{1}{|W_j|} \int_{(V_\epsilon^{\mathfrak{h}_j})_{\mathrm{reg}}} \left(\int_{\{[g]\in G/H_j : \|\mathrm{Ad}_g h\| \le R\}} \mathsf{Vol}_{G/H_j} \right) |\pi_j(h)|^2 \, \mathrm{d}\Lambda_{\mathfrak{h}_j}(h) \\ &= \sum_{j=1}^{n} \sum_{C_{j,r}\subset \mathfrak{h}_j} \frac{1}{|W_j|} \int_{V_\epsilon^{\mathfrak{h}_j} \cap C_{j,r}} \left(\int_{\{[g]\in G/H_j : \|\mathrm{Ad}_g h\| \le R\}} \mathsf{Vol}_{G/H_j} \right) |\pi_j(h)|^2 \, \mathrm{d}\Lambda_{\mathfrak{h}_j}(h).\end{aligned} \tag{4.4.7}$$

To simplify, we will fix a single CSA $\mathfrak{h}_j$ and a single open Weyl chamber $C_{j,r}$, and suppress the indices:

$$H := H_j, \quad \pi := \pi_j, \quad \mathfrak{h} := \mathfrak{h}_j, \quad \mathsf{Vol}_{G/H} := \mathsf{Vol}_{G/H_j}, \quad C := C_{j,r}. \tag{4.4.8}$$

For now, let us look at the single summand of (4.4.7) corresponding to $\mathfrak{h}$ and C.

Lemma 4.4.4. *Fix the notation as in (4.4.8), let $m := m(h)$ as in Definition 4.3.7 for an arbitrary $h \in C$, and $R > 0$ arbitrary. Then there is some $c \neq 0$ which does not depend on R so that*

$$\begin{aligned}\lim_{\epsilon\to 0^+} \epsilon^{-m-\dim\mathfrak{h}-|\Delta^+|} \int_{V_\epsilon^{\mathfrak{h}} \cap C} \left(\int_{\{[g]\in G/H : \|\mathrm{Ad}_g h\| \le R\}} \mathsf{Vol}_{G/H} \right) |\pi(h)|^2 \, \mathrm{d}\Lambda_{\mathfrak{h}}(h) \\ = c \sum_{O_X} \int_{V_1^{\mathfrak{h}} \cap C} \mathsf{Vol}(S_X \cap O_h) \mu_{O_X}(B_R(0)) \pi(h) \, \mathrm{d}\Lambda_{\mathfrak{h}}(h),\end{aligned} \tag{4.4.9}$$

where the sum is taken over all nilpotent orbits O_X contained in $\mathcal{N}_h$ for an arbitrary $h \in C$, cf. Remark 4.3.9.

Proof. Recall the notation μ_{O_h} for the measure defined in Section 4.3, and that $\pi(h)$ was defined in Equation (4.3.13) as the volume density function of the orbit-stabilizer-diffeomorphism $G/H \to O_h$ with respect to a fixed invariant volume form on G/H and the KKS volume form on O_h. Using this property of $\pi(h)$, we find that there exists some nonzero scalar c depending only on the choice of invariant measure $\mathsf{Vol}_{G/H}$ on G/H

with:

$$\begin{aligned}
&\int_{V_\epsilon^{\mathfrak{h}}\cap C}\left(\int_{\{[g]\in G/H:\|\mathrm{Ad}_g h\|\le R\}}\mathsf{Vol}_{G/H}\right)|\pi(h)|^2\,\mathrm{d}\Lambda_{\mathfrak{h}}(h)\\
&\qquad\qquad = c\cdot\int_{V_\epsilon^{\mathfrak{h}}\cap C}\left(\int_{B_R(0)\cap O_h}\mathsf{Vol}_{O_h}\right)\pi(h)\,\mathrm{d}\Lambda_{\mathfrak{h}}(h)\\
&\qquad\qquad = c\cdot\int_{V_\epsilon^{\mathfrak{h}}\cap C}\mu_{O_h}(B_R(0))\pi(h)\,\mathrm{d}\Lambda_{\mathfrak{h}}(h)\\
&\qquad\qquad = \epsilon^{\dim\mathfrak{h}+|\Delta^+|}c\cdot\int_{V_1^{\mathfrak{h}}\cap C}\mu_{O_{\epsilon h}}(B_R(0))\pi(h)\,\mathrm{d}\Lambda_{\mathfrak{h}}(h).
\end{aligned}\tag{4.4.10}$$

In the last step, we used that $V_\epsilon^{\mathfrak{h}}=\epsilon V_1^{\mathfrak{h}}$, that $\mathrm{d}\Lambda_{\mathfrak{h}}(\epsilon h)=\epsilon^{\dim\mathfrak{h}}d\Lambda_{\mathfrak{h}}(h)$, and that $\pi(\epsilon h)=\epsilon^{|\Delta_+|}\pi(h)$. By Corollary 4.3.14 and compactness of $V_1^{\mathfrak{h}}\cap C$ (Lemma 4.4.3), we have:

$$\begin{aligned}
&\lim_{\epsilon\to 0}\int_{V_1^{\mathfrak{h}}\cap C}\epsilon^{-m}\mu_{O_{\epsilon h}}(B_R(0))\pi(h)\,\mathrm{d}\Lambda_{\mathfrak{h}}(h)\\
&\qquad\qquad = \sum_{O_X}\int_{V_1^{\mathfrak{h}}\cap C}\mathsf{Vol}(S_X\cap O_h)\mu_{O_X}(B_R(0))\pi(h)\,\mathrm{d}\Lambda_{\mathfrak{h}}(h).
\end{aligned}\tag{4.4.11}$$

This shows the statement. □

Finally, we use this to prove the key lemma:

Proof of key Lemma 4.2.7. Let $O_X\subset\mathfrak{g}$ be a nilpotent orbit of dimension d. Lemma 4.3.5 and Remark 4.3.9 imply that there is some CSA $\mathfrak{h}\subset\mathfrak{g}$ and some open Weyl chamber $C\subset\mathfrak{h}_{\mathrm{reg}}$ so that $O_X\subset\mathcal{N}_h$ for all $h\in C$. Since O_X is of maximal dimension, the number $m:=m(x)$ from Definition 4.3.7 is minimal among $m(x')$ for all $x'\in\mathfrak{g}_{\mathrm{reg}}$. Then, by (4.4.7) and Lemma 4.4.4, there are numbers $c_{j,r}\neq 0$, independent of R, such that

$$\begin{aligned}
&\lim_{\epsilon\to 0^+}\epsilon^{-m-\dim\mathfrak{h}-|\Delta^+|}\Lambda(V_{\epsilon,R})\\
&\quad=\sum_{j=1}^n\sum_{C_{j,r}\subset\mathfrak{h}_j}\sum_{O_X}\frac{1}{|W_j|}c_{j,r}\cdot\int_{V_1^{\mathfrak{h}}\cap C}\mathsf{Vol}(S_X\cap O_h)\mu_{O_X}(B_R(0))\pi_j(h)\,\mathrm{d}\Lambda_{\mathfrak{h}_j}(h),
\end{aligned}\tag{4.4.12}$$

where the sum over the O_X is carried out over all nilpotent orbits $O_X\subset\mathcal{N}_h$ of dimension d. Note that this sum is independent of the choice of $h\in C_{j,r}$ for fixed j and r, cf. Remark 4.3.9.
All summands in the above sum are positive as they arise from volumes of subsets of $\mathfrak{g}$, and thus the total sum is nonzero. Lastly, the sum is homogeneous of degree $d/2$ in R by Lemma 4.3.3. Hence we may consider their quotient and conclude the proof:

$$\lim_{\epsilon\to 0^+}\frac{\Lambda(V_{\epsilon,\rho R})}{\Lambda(V_{\epsilon,R})}=\frac{\lim_{\epsilon\to 0^+}\epsilon^{-m-\dim\mathfrak{h}-|\Delta^+|}\Lambda(V_{\epsilon,\rho R})}{\lim_{\epsilon\to 0^+}\epsilon^{-m-\dim\mathfrak{h}-|\Delta^+|}\Lambda(V_{\epsilon,R})}=\rho^{d/2}.\tag{4.4.13}$$

□

REFERENCES

[Bou05] Nicolas Bourbaki. *Lie groups and Lie algebras. Chapters 7–9*. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 2005. Translated from the 1975 and 1982 French originals by Andrew Pressley.

[BV80] Dan Barbasch and David A. Vogan, Jr. The local structure of characters. *J. Functional Analysis*, 37(1):27–55, 1980.

[CJKUM22] Martijn Caspers, Bas Janssens, Amudhan Krishnaswamy-Usha, and Lukas Miaskiwskyi. Local and multilinear noncommutative de Leeuw theorems. *arXiv preprint arXiv:2201.10400*, 2022.

[CPPR15] Martijn Caspers, Javier Parcet, Mathilde Perrin, and Éric Ricard. Noncommutative de Leeuw theorems. *Forum Math. Sigma*, 3:Paper No. e21, 59, 2015.

[dL65] Karel de Leeuw. On L_p multipliers. *Ann. of Math. (2)*, 81:364–379, 1965.

[Far08] Jacques Faraut. *Analysis on Lie groups*, volume 110 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2008. An introduction.

[FM21] Lucas Fresse and Salah Mehdi. Approximation of nilpotent orbits for simple lie groups, 2021.

[Har12] Benjamin Harris. Tempered representations and nilpotent orbits. *Represent. Theory*, 16:610–619, 2012.

[HC57] Harish-Chandra. Fourier transforms on a semisimple Lie algebra. II. *Amer. J. Math.*, 79:653–686, 1957.

[HC65] Harish-Chandra. Invariant eigendistributions on a semisimple Lie group. *Trans. Amer. Math. Soc.*, 119:457–508, 1965.

[Kal17] Olav Kallenberg. *Random measures, theory and applications*, volume 77 of *Probability Theory and Stochastic Modelling*. Springer, Cham, 2017.

[Kna96] Anthony W. Knapp. *Lie groups beyond an introduction*, volume 140. Birkhäuser, 1996.

[Rot72] L. Preiss Rothschild. Orbits in a real reductive Lie algebra. *Trans. Amer. Math. Soc.*, 168:403–421, 1972.

[RR72] R. Ranga Rao. Orbital integrals in reductive groups. *Ann. of Math. (2)*, 96:505–510, 1972.

[Slo80] Peter Slodowy. Simple singularities and simple algebraic groups, 1980.

[Var77] V. S. Varadarajan. *Harmonic analysis on real reductive groups.* Lecture Notes in Mathematics, Vol. 576. Springer-Verlag, Berlin-New York, 1977.

4

5

Expectation values of polynomials and moments on compact Lie groups

Tobias Diez, Lukas Miaskiwskyi

We develop a powerful framework to calculate expectation values of polynomials and moments on compact Lie groups based on elementary representation-theoretic arguments and an integration by parts formula. In the setting of lattice gauge theory, we generalize expectation value formulas for products of Wilson loops by Chatterjee and Jafarov to arbitrary compact Lie groups, and study explicit examples for many classical compact Lie groups and the exceptional Lie group G_2*. Extending classical results by Collins and Lévy, we use our framework to derive expectation value formulas of polynomials of matrix coefficients under the Haar measure, Brownian motion, and the Wilson action. In particular, we construct Weingarten functions for general compact Lie groups by studying the underlying tensor invariants, and apply this to* $\mathrm{SU}(N)$ *and* G_2*.*

This chapter is based on the preprint [DM22].

5.1. INTRODUCTION

Integration over Lie groups plays a central role in many areas of mathematics and theoretical physics. It lies at the core of random matrix theory and has become an important tool to describe a wide range of physical systems including lattice gauge theory [Wei78], quantum chaotic systems [CHJLY17], many-body quantum systems [GMGW98], quantum information theory [CN16] and matrix models for quantum gravity and Yang–Mills theory in two dimensions [DFGZJ95, Xu97]. In this chapter, we develop a general framework to calculate expectation values of polynomials of group elements and their inverses on a compact Lie group G of the form

$$\int_G \mathrm{tr}_\rho(c_1 g^{\pm 1} \cdots c_n g^{\pm 1})\, \nu(g)\, \mathrm{d}g, \qquad c_i \in G, \tag{5.1.1}$$

where $\mathrm{d}g$ denotes the normalized Haar measure, ν is a probability density and the trace is taken in a given representation ρ of G. Expanding the integrand, the problem reduces to a computation of the expectation value of the so-called moments

5

$$\int_G g_{i_1 i'_1} \cdots g_{i_p i'_p}\, g^{-1}_{j_1 j'_1} \cdots g^{-1}_{j_q j'_q}\, \nu(g)\, \mathrm{d}g, \tag{5.1.2}$$

where $g_{ij} = \rho(g)_{ij}$ are the matrix entries of $g \in G$ in the representation ρ.

Given the numerous applications, these integrals are well-studied in the literature. For matrices drawn randomly from the Haar distribution ($\nu = 1$), the calculation of the moments has been initiated by theoretical physicist [Wei78] motivated by problems in lattice gauge theory. [Col03] developed a rigorous mathematical framework for computing moments for the unitary group, which has been extended to the orthogonal and symplectic group by [CS06]. In the unitary case, the developed Weingarten calculus expresses the integral (5.1.2) as a sum over so-called Weingarten functions, which are functions defined on the symmetric group. This approach makes heavy use of representation theory in the form of Schur–Weyl duality. Recently, the Weingarten calculus has been rephrased in terms of Jucys–Murphy elements [Nov10, ZJ10, MN13]. A special but important case is the computation of joint moments of traces of powers of group elements, that is, essentially integrals of the form (5.1.1) with all coefficients c_i set to the identity element. For the unitary group, this has been extensively studied in [DS94, DE01] where it was used to obtain central limit theorems of eigenvalue distributions; see also [PV04, HR03] for analogous results for other groups.

Lattice gauge theories are a natural area where integrals over Lie groups play a central role. They were originally introduced by Wilson as discrete approximations to quantum Yang–Mills theory and evolved to become one of the most promising approaches to study non-perturbative effects in QCD such as quark confinement. The most important gauge-invariant observables in lattice theory are Wilson loops (traced holonomies) whose long range decay serves as an indicator for the confining behavior. In the Euclidean formulation of the theory, the expectation value of a Wilson loop is an integral of the form (5.1.1) with the probability density ν being a Boltzmann weight relative to the Yang–Mills action. However, explicitly computing expectation values of and correlations between Wilson loops is a difficult, if not impossible, challenge. One thus usually resorts to numerical methods such as classical Monte Carlo simulation to approximate such integrals.

Indeed, an analytical understanding of the Wilson loop expectation values in the continuum and infinite volume limit is an essential ingredient to solve the Yang–Mills mass gap problem, which is one of the seven Millennium Problems posed by the Clay Mathematics Institute. [tH74] realized that, when the rank N of the gauge group tends to infinity, the theory simplifies in many ways and can be solved analytically in certain cases. In particular, the Wilson loops then satisfy the Makeenko–Migdal equations and factorize, i.e., the expectation value of a product of Wilson loops equals the product of the expectation values of the individual Wilson loops. Recently, [Cha19] established in the large N-limit of $\mathrm{SO}(N)$ lattice gauge theory an asymptotic formula for expectation values of products of Wilson loops in terms of a weighted sum of certain surfaces. These surfaces are defined starting from the collection of loops using the four operations of merging, splitting, deformation and twisting. The proof proceeds by a complicated and lengthy calculation which hinges on Stein's method for random matrices. Analogous results have been obtained for $\mathrm{SU}(N)$ in [Jaf16] using similar methods. This development sparked renewed interest, leading to further progress for large-N gauge theories [CJ16, BG18, Cha21].

Another approach to a rigorous definition of a quantum Yang–Mills theory is the construction of the Yang–Mills measure and thus of the path integral using Brownian motion on the structure group. This direction has been pioneered by [Dri89, GKS89] in two dimensions. We refer the reader to [Sen08] for a relative recent review of two-dimensional Yang–Mills theory. In the physics literature, the Yang–Mills measure is taken to be the Lebesgue measure on the space of connections, weighted by a Boltzmann density involving the Yang–Mills action. To make sense of this formal description, one usually uses the holonomy mapping to define the Yang–Mills measure in terms of group-valued random variables indexed by embedded loops whose distribution is given by the heat density (at a "time" proportional to the area enclosed by the loop). For this reason, the calculation of expectation values of polynomials on Lie groups with respect to a Brownian motion attracted a lot of attention, especially in the large-N limit. In particular, combinatorial integration formulas for the expectation values of polynomials under the heat kernel measure have been obtained by [Xu97, Lé08] and recently generalized by [Dah17] to also allow polynomials in inverses of group elements.

The study in the aforementioned papers rely on heavy machinery from representation theory in the form of Schur–Weyl duality or Jucys–Murphy elements, or on a detailed probabilistic analysis using for example Stein's method. Due to this complexity, the results have usually been obtained first for the unitary group, and then generalized

5

in subsequent papers to other groups such as the orthogonal or symplectic group. Moreover, the methods have been tailored to the specific probability measure under study which made it hard to transfer progress from one scheme to another. In contrast, we here deduce and extend the main results of these papers from an elementary integration by parts formula. This allows us to generalize these results to arbitrary compact Lie groups and to analyze the Haar, Wilson and heat kernel cases simultaneously and on equal footing.

Our first main result is Theorem 5.3.6 which describes the expectation value of a product of Wilson loop observables in terms of other Wilson loops that are obtained from the initial family through two operations that we call twisting and merging[1]. This is a generalization of the results of [Cha19, Jaf16] to arbitrary compact Lie groups, arbitrary probability measures and arbitrary group representations. Our construction shows that the operations of twisting and merging are determined by an operator in the universal enveloping algebra of the Lie algebra that can be seen as an operator-theoretic counterpart to the so-called completeness relations. Moreover, these two operations can be represented in a diagrammatic way that resembles the Feynman path integrals rules. This diagrammatic calculus is similar but different to the one developed by [BB96] for the unitary group, cf. also [Cvi76]. For the Haar measure and for the Brownian motion, the resulting equations for the expectation of a product of Wilson loops lead to a recursive formula that can be solved using a straightforward algorithm. In the case of the Haar integral over the unitary group, we recover the recursion relations given in [Sam80, Section III]. For the Yang–Mills Wilson action, the equation takes a relative simple form which does not involve merging of Wilson loops with plaquette operators as in [Cha19] and which has the additional benefit to reduce the operations on the family of loops needed from four to two. Moreover, in the case of the unitary group, the structure of the equation is particularly well-suited to a large-N limit. As applications of our general framework, the result for the groups $\mathrm{SO}(N)$, $\mathrm{Sp}(N)$, $\mathrm{U}(N)$, and G_2 are discussed in more detail in Section 5.5.

In the second part of the chapter, we investigate the moment integrals (5.1.2) for an arbitrary compact Lie group. Theorem 5.4.2 shows that the moments satisfy an eigenvalue equation whose particular form depends on the probability measure. For the Haar measure, the moments yield a projection onto the subspace of invariants. Specialized to the unitary group, this result is a restatement of the well-known fact that the moments yield a conditional expectation onto the group algebra of the symmetric group [CS06, Proposition 2.2]. Moreover, Theorem 5.4.6 yields an explicit expansion of the moments as a sum over a spanning set of invariants. In particular, we define a Weingarten map for every compact Lie group (depending on the group representation and on the spanning set of invariants) and show that it gives the coefficients in this

[1] For the unitary group, these operations correspond to the Fission and Fusion processes of [Sam80, Section III], respectively.

expansion of the moments. This is similar in sprit to the definition of the Weingarten map as a pseudoinverse in [ZJ10] and equivalent to the results of [Col03, CS06] for the unitary, orthogonal and symplectic group. As a novel application, we determine in Section 5.5.5 the Weingarten map, and thus the moments, for the exceptional group G_2 in its natural 7-dimensional irreducible representation. In the case of Brownian motion, the moments can be calculated using the eigenvalues of the Casimir operator and converge for large times to the moments with respect to the Haar measure, see Corollary 5.4.4. This is a refinement and extension of the results of [Lé08, Dah17], where only the groups $\mathrm{U}(N)$, $\mathrm{O}(N)$, and $\mathrm{Sp}(N)$ were considered.

As we have mentioned above, at the heart of our approach lies a simple integration by parts formula. This is perhaps most similar to the derivation of the Schwinger–Dyson equation for the Gaussian unitary ensemble, see, e.g., [AGZ10, Equation (5.4.15)]. To illustrate how integration by parts can be used to calculate moments, consider the simple example of $T_{ijkl} = \int_G g_{ij} g^{-1}_{lk} \,\mathrm{d}g$. After inserting the Laplacian in the first factor, integration by parts yields

$$\int_G (\Delta g_{ij}) g^{-1}_{lk} \,\mathrm{d}g = -\int_G \langle \mathrm{d}g_{ij}, \mathrm{d}g^{-1}_{lk} \rangle \,\mathrm{d}g. \tag{5.1.3}$$

The Schur–Weyl lemma implies that g_{ij} is an eigenvector of the Laplacian with, say, eigenvalue λ. Thus, the left-hand side equals λT_{ijkl}. On the other hand, the right-hand side can be calculated by using an orthonormal basis ξ^a for the Lie algebra $\mathfrak{g}$ and introducing the operator $K_{ijkl} = \xi^a_{ij} \xi^a_{kl}$ (implicitly summing over a). In summary, we obtain

$$\lambda T_{ijkl} = K_{rjls} T_{irks}. \tag{5.1.4}$$

For the fundamental representation of $G = \mathrm{U}(N)$, the completeness relation is of the form $K_{rjls} = -\delta_{rs}\delta_{jl}$ and thus

$$\lambda T_{ijkl} = -\delta_{jl} T_{irkr} = -\delta_{jl}\delta_{ik}, \tag{5.1.5}$$

where the second equality follows from the definition of T_{ijkl}. The calculation of matrix coefficients of higher degree involves more delicate combinatorics, but the strategy remains the same: integration by parts yields an eigenvalue equation for the matrix coefficients, which is then solved by using a group-dependent completeness relation.

5.2. SETTING

5.2.1. DIFFERENTIAL GEOMETRY ON LIE GROUPS

In all that follows, we are using the Einstein summation convention where repeated indices are automatically summed over. Occasionally, this convention will be overridden by explicit summation symbols, when we want to more closely specify the range of summation.

For the rest of the chapter, consider the following setting: We are given a compact Lie group G and a finite-dimensional, complex, irreducible representation $\rho : G \to \mathrm{GL}(V)$. For all $g \in G$, write

$$\mathrm{tr}_\rho(g) := \mathrm{tr}(\rho(g)). \tag{5.2.1}$$

Equip the Lie algebra $\mathfrak{g} = T_e G$ of G with an Ad_G-invariant positive-definite symmetric bilinear form κ. In the examples we consider, G is usually semisimple and κ a negative multiple of the Killing form. By translation, the inner product κ induces a canonical bi-invariant Riemannian metric $\langle\cdot,\cdot\rangle$ on G. We normalize the volume form so that G as unit volume, and denote the corresponding probability measure by $\mathrm{d}g$.

We fix an orthonormal basis $\{\xi^a \in \mathfrak{g}\}$ of $\mathfrak{g}$ with respect to κ, and use it to define a global frame of TG by left translation. Analogously, the dual basis $\{\epsilon^a \in \mathfrak{g}^*\}$ associated to $\{\xi^a\}$ defines a global frame of the cotangent bundle T^*G.

We collect some abuses of notation: We will denote the global frames of TG and T^*G with the same letters ξ^a, ϵ^a as the pointwise objects. Similarly, we identify elements of $\mathfrak{g}$ with left-invariant vector fields on G and with derivations on $C^\infty(G)$. Further, the Lie group representation ρ induces a Lie algebra representation $\mathfrak{g} \to \mathrm{End}(V)$ which we will denote by the same letter ρ.

The Riemannian metric naturally induces the *musical isomorphisms*

$$\flat : TG \to T^*G, \quad v_p \mapsto \langle v_p, \cdot\rangle, \tag{5.2.2}$$

$$\sharp := \flat^{-1} : T^*G \to TG. \tag{5.2.3}$$

We extend the inner product on the fibers of $\mathrm{T}G$ to T^*G by declaring

$$\langle \alpha, \beta\rangle = \langle \alpha^\sharp, \beta^\sharp\rangle \tag{5.2.4}$$

for $\alpha, \beta \in \mathrm{T}^*G$ in the same fiber.

The *Laplace–Beltrami operator* is defined by

$$\Delta : C^\infty(G) \to C^\infty(G),\ f \mapsto \Delta f := \nabla\cdot\nabla f = \xi^a(\xi^a f), \tag{5.2.5}$$

where the sections ξ^a are viewed as a vector fields on G, hence derivations on $C^\infty(G)$. In our sign convention, Δ has negative eigenvalues.

The Laplace–Beltrami operator is tightly connected to the Casimir invariant $C := \xi^a\xi^a \in \mathrm{U}(\mathfrak{g})$. Under the identification of the universal enveloping algebra $\mathrm{U}(\mathfrak{g})$ with the left-invariant differential operators on G, the Casimir invariant maps to the Laplace–Beltrami operator.

Furthermore, consider the tensor product representation

$$\rho\otimes\rho : \mathfrak{g} \to \mathrm{End}(V\otimes V), \quad \xi \mapsto \rho(\xi)\otimes\mathrm{id} + \mathrm{id}\otimes\rho(\xi), \tag{5.2.6}$$

and the image of the Casimir invariants under ρ and $\rho\otimes\rho$

$$\begin{aligned} \rho(C) &= \rho(\xi^a)\cdot\rho(\xi^a), \\ (\rho\otimes\rho)(C) &= \big(\rho(\xi^a)\otimes\mathrm{id} + \mathrm{id}\otimes\rho(\xi^a)\big)\cdot\big(\rho(\xi^a)\otimes\mathrm{id} + \mathrm{id}\otimes\rho(\xi^a)\big). \end{aligned} \tag{5.2.7}$$

We immediately find

$$(\rho\otimes\rho)(C) = \rho(C)\otimes \mathrm{id} + \mathrm{id}\otimes \rho(C) + 2\rho(\xi^a)\otimes\rho(\xi^a). \tag{5.2.8}$$

The Casimir invariant C is well known to be independent of the choice of orthonormal basis, hence, by the above equation, so is the following important operator:

$$K := \rho(\xi^a)\otimes\rho(\xi^a) = \frac{1}{2}\big((\rho\otimes\rho)(C) - \rho(C)\otimes\mathrm{id} - \mathrm{id}\otimes\rho(C)\big) \in \mathrm{End}(V\otimes V). \tag{5.2.9}$$

Relative to a basis in V, it assumes the shape

$$K_{ijkl} = \xi^a_{ij}\xi^a_{kl}. \tag{5.2.10}$$

The operator K defined in Eq. (5.2.9) is independent of the choice of the basis ξ^a (but depends on κ) and it is sometimes called the split Casimir operator. Indeed, the nondegenerate bilinear form κ on $\mathfrak{g}$ yields the isomorphism $\mathrm{Hom}(\mathfrak{g},\mathfrak{g}) = \mathfrak{g}\otimes\mathfrak{g}^* \simeq \mathfrak{g}\otimes\mathfrak{g}$. Composing this with the representation $\rho : \mathfrak{g}\to\mathrm{End}(V)$ gives a map $\mathrm{Hom}(\mathfrak{g},\mathfrak{g})\to\mathfrak{g}\otimes\mathfrak{g}\to \mathrm{End}(V)\otimes\mathrm{End}(V)$. The image of the identity under this map is K. The operator K is related to the image of the Casimir invariant C through contraction:

$$\rho(C)_{ij} = K_{ikkj}. \tag{5.2.11}$$

If the representation ρ is unitary, then for each $\xi\in\mathfrak{g}$ the operator $\rho(\xi)$ is skew-Hermitian and the operator K has the following symmetry properties:

$$K_{ijkl} = \overline{K_{jilk}}, \qquad K_{ikkj} = \overline{K_{jkki}}. \tag{5.2.12}$$

In particular, $\rho(C)$ is a Hermitian operator.

Lastly, recall that, by Schur's Lemma, the images of the Casimir invariants under ρ and $\rho\otimes\rho$ are proportional to the identity on irreducible components of V and $V\otimes V$, respectively. By the second representation of K in Eq. (5.2.9), the same holds for K.

5.2.2. BROWNIAN MOTION ON LIE GROUPS

In this section we recall the definition of the Brownian motion on a compact Lie group. The systematic study of this subject goes back to the pioneering work of [Hun56, Yos52, Itô50], and we refer the reader to [Lia04] [RW94, Section V.35] for textbook treatments.

As before, G is a compact Lie group whose Lie algebra $\mathfrak{g}$ is endowed with an Ad_G-invariant scalar product κ, and $\{\xi^a\}$ denotes an orthonormal basis of $\mathfrak{g}$. Let $(W_t)_{t\geq 0}$ be the unique centered Gaussian process on $\mathfrak{g}$ with covariance matrix

$$\mathbb{E}\big(W^a_t W^b_s\big) = \min(t,s)\,\delta^{ab}, \qquad t,s\geq 0, \tag{5.2.13}$$

where $W^a_t = \kappa(\xi^a, W_t)$. The *(Riemannian) Brownian motion* on G starting at $g\in G$ is the unique G-valued stochastic process $(g_t)_{t\geq 0}$ which solves the Stratonovich stochastic differential equation

$$\mathrm{d}g_t = \xi^a(g_t)\circ\mathrm{d}W^a_t, \qquad g_0 = g. \tag{5.2.14}$$

That is, for every $f \in C^\infty(G)$,

$$f(g_t) = f(g) + \int_0^t (\xi^a f)(g_s) \circ \mathrm{d}W_s^a. \tag{5.2.15}$$

Converting into the Itô calculus yields

$$f(g_t) = f(g) + \int_0^t (\xi^a f)(g_s)\,\mathrm{d}W_s^a + \frac{1}{2}\int_0^t (\Delta f)(g_s)\,\mathrm{d}s. \tag{5.2.16}$$

Moreover, g_t is a Feller diffusion process on G whose infinitesimal generator, restricted to smooth functions, is one half of the Laplace operator Δ.

For $f \in C(G)$ and $g \in G$, we denote by $\mathbb{E}_g\big(f(g_t)\big) = \mathbb{E}\big(f(g_t)|g_0 = g\big)$ the conditional expectation of f given that g_t starts at g. The resulting semigroup is a contraction on $C(G)$ and satisfies

$$\mathbb{E}_g\big(f(g_t)\big) = \int_G f(a)\,p_t(g^{-1}a)\,\mathrm{d}a, \tag{5.2.17}$$

where, for $t > 0$, $p_t : G \to \mathbb{R}$ is the smooth probability density satisfying the heat equation

$$\frac{1}{2}\Delta p_t = \frac{\partial}{\partial t} p_t, \qquad \lim_{t\to 0} p_t = \delta_e. \tag{5.2.18}$$

Usually, we are interested only in processes starting at the identity and then abbreviate $\mathbb{E} \equiv \mathbb{E}_e$.

5.2.3. Wilson Loops

In this section, we recall basic elements of the lattice gauge theory. The reader is referred to the textbooks [RS17, MM97] for a detailed treatment.

Consider a directed graph (Λ^0, Λ^1), which one may think of as being embedded either in space or spacetime. Here, Λ^0 is the set of all vertices which we assume to be finite, and Λ^1_+ is the set of all directed edges (i.e., ordered pairs of vertices). For an edge e, let $s(e), t(e) \in \Lambda^0$ be its source and target vertices, respectively, and let e^{-1} be the edge going in the opposite direction. We denote by $\Lambda^1_- = \{e^{-1} : e \in \Lambda^1_+\}$ the set of all edges with their orientation reversed, and set $\Lambda^1_\pm = \Lambda_+ \cup \Lambda_-$. A (field) configuration is a map $g : \Lambda^1_+ \to G$ assigning to each edge e a group element $g(e)$ that should be thought of as the approximation of the parallel transport along that edge. We extend g to a map $g : \Lambda^1_\pm \to G$ by setting $g(e^{-1}) = g(e)^{-1}$.

An (oriented) path $l = (e_1, \dots, e_r)$ is an ordered tuple of edges $e_i \in \Lambda^1_\pm$ such that $t(e_i) = s(e_{i+1})$ for all $1 \le i < r$. A path is called a loop if the edges form a cycle, i.e. $t(e_r) = s(e_1)$. If, additionally, each edge occurs only once then the loop is called a plaquette (or face). Given a path l, the product of a configuration g along l is defined by $g(l) = g(e_1)\cdots g(e_r)$. Given a choice of a set Λ^2 of plaquettes, the probability density on the space of configurations is given by the Wilson action

$$g \mapsto \frac{1}{Z}\exp\Bigg(\beta \sum_{p\in\Lambda^2} \mathrm{tr}_\rho\big(g(p)\big)\Bigg), \tag{5.2.19}$$

where Z is a normalization factor (the partition function) and $\beta \in \mathbb{R}$ is the so-called inverse temperature. Here, the trace is taken with respect to a representation ρ of G, which usually is assumed to be irreducible or even to be the fundamental representation. One is mainly interested in expectation values of Wilson loop observables. These are functions W_l on the space of configurations indexed by loops $l = (e_1, \dots, e_r)$ and are given by

$$W_l(g) = \operatorname{tr}_\rho\big(g(l)\big) = \operatorname{tr}_\rho\big(g(e_1)^{\pm 1} \cdots g(e_r)^{\pm 1}\big), \tag{5.2.20}$$

where the sign in the factor $g(e_i)^{\pm 1}$ is determined based on whether e_i is an element of Λ^1_+ or Λ^1_-. In other words, one is lead to calculate integrals of the form

$$\int W_l(g) \exp\left(\beta \sum_{p \in \Lambda^2} \operatorname{tr}_\rho\big(g(p)\big)\right) \mathrm{d}g. \tag{5.2.21}$$

Since $\mathrm{d}g = \prod_{e \in \Lambda^1_+} \mathrm{d}g_e$ is the product of Haar measures, one can evaluate such an integral by successively integrating over copies of G.

In the following, we are mainly concerned with the resulting integral over a single edge. For this, it is convenient to change the notation and language slightly and consider restrictions to a single edge. For the lattice gauge theory calculations that one may want to perform in the end, it is good to remember that Wilson loops do, in fact, depend on many copies of G.

Definition 5.2.1 (Wilson Loops)**.** Let $\rho : G \to \operatorname{End}(V)$ be a finite-dimensional representation of G, and for some natural number $r \in \mathbb{N}$, fix an element

$$l = ((c_1, \sigma_1), \dots (c_r, \sigma_r)) \in (G \times \{\pm 1\})^r. \tag{5.2.22}$$

The *(single-argument) Wilson loop* $W_{\rho,l}$ associated with this data is given by

$$W_{\rho,l} : G \to \mathbb{C}, \qquad W_{\rho,l}(g) = \operatorname{tr}_\rho(c_1 g^{\sigma_1} c_2 g^{\pm 1} \cdots c_r g^{\sigma_r}). \tag{5.2.23}$$

Often the representation is clear from the context and we simply write W_l in this case. Moreover, we say that the $g^{\pm 1}$ between c_s and c_{s+1} is *in the s-th position*.

In this setting, we consider the probability measure to be a single-argument version of the *Wilson action*:

$$\nu_W(g) = \frac{1}{Z} \exp\left(\beta \sum_p W_p(g)\right), \quad g \in G, \tag{5.2.24}$$

where Z is a suitable normalization factor, $\beta \in \mathbb{R}$ a fixed number, the sum is over a finite set that is not further specified, and W_p is a single-argument Wilson loop of the form

$$W_p(g) = \operatorname{tr}_\rho(C_p g^{\pm 1}), \quad C_p \in G. \tag{5.2.25}$$

Below, we also need a slight generalization of a single-argument Wilson loop for which the coefficients are not necessarily elements of the same group.

Definition 5.2.2 (Generalized Wilson Loops)**.** Let $\rho : G \to \mathrm{End}(V)$ be a finite-dimensional representation of G, and for some natural number $r \in \mathbb{N}$, fix an element

$$l = ((c_1, \sigma_1), \dots (c_r, \sigma_r)) \in (\mathrm{End}(V) \times \{\pm 1\})^r. \tag{5.2.26}$$

The *generalized Wilson loop* $W_{\rho,l}$ associated with this data is given by

$$W_{\rho,l} : G \to \mathbb{C}, \qquad W_l(g) = \mathrm{tr}_V\big(c_1 \rho(g^{\sigma_1}) c_2 \rho(g^{\sigma_2}) \cdots c_r \rho(g^{\sigma_r})\big). \tag{5.2.27}$$

A (generalized) Wilson loop is called *linear* if there is only one factor of g in the above representation, i.e. $r = 1$; otherwise it is called *polynomial*.

The notion of a generalized Wilson loop is inspired by the concept of spin networks. In fact, a generalized Wilson loop can be visualized as a loop in a graph consisting of two vertices and $n+1$ directed edges, where one edge is decorated by $\rho(g)$ and the other edges by the endomorphisms c_i.

5

A linear generalized Wilson loop (with, say, positive exponent on the g factor) is completely determined by its coefficient $c \in \mathrm{End}(V)$. We thus obtain a map

$$\mathrm{End}(V) \to C^0(G, \mathbb{C}), \qquad c \mapsto W_{\rho,(c)} = \mathrm{tr}_V\big(c\, \rho(\cdot)\big). \tag{5.2.28}$$

Under the isomorphism $\mathrm{End}(V) \cong V^* \otimes V$ this is nothing but the usual embedding of matrix coefficients. In other words, linear generalized Wilson loops are just linear combinations of matrix coefficients, and every matrix coefficient is a linear generalized Wilson loop.

Proposition 5.2.3. *Every (generalized) Wilson loop $W_{\rho,l}$ can be written as a finite linear combination of linear generalized Wilson loops associated with irreducible representations. That is, there exists a finite set of irreducible representations (τ, V_τ) of G and a collection of endomorphisms $c_\tau \in \mathrm{End}(V_\tau)$ such that*

$$W_{\rho,l} = \sum_\tau W_{\tau,(c_\tau)}. \tag{5.2.29}$$

Proof. Note that a (generalized) Wilson loop transforms as $W_{\rho,l}(ag) = W_{\rho,a\cdot l}(g)$, where the action $a \cdot l$ of $a \in G$ on the coefficients c_i is either by left or right translation or conjugation depending on the signatures. This shows that the linear span of all generalized Wilson loops (relative to a given representation ρ) is a left G-translation invariant subspace of $C^0(G,\mathbb{C})$. By choosing a basis in $\mathrm{End}(V)$, we obtain a finite spanning set so that $\mathrm{span}\, W_{\rho,\cdot}$ is finite-dimensional. Hence, every (generalized) Wilson loop is a so-called *representative function*, see [BtD95, Definition III.1.1]. By [BtD95, Proposition III.1.5], every representative function is a finite linear combination of matrix coefficients with respect to irreducible representations. As we have remarked above, the latter are linear generalized Wilson loops. □

5.3. EXPECTATION VALUES OF WILSON LOOPS

The following identity is of fundamental importance for us, and it is derived by a simple application of integration by parts.

Lemma 5.3.1. *Let G be a compact Lie group and let ν be a probability density with respect to the normalized Haar measure on G. For smooth functions $F_1,\ldots,F_q$ on G,*

$$\begin{aligned}\sum_{r=1}^{q}\int_G (\Delta F_r)\, F_1\cdots\widehat{F_r}\cdots F_q\,\nu\,\mathrm{d}g &= \int_G F_1\cdots F_q\,\Delta\nu\,\mathrm{d}g\\ &\quad -2\sum_{\substack{r,s=1\\ r<s}}^{q}\int_G \langle \mathrm{d}F_r,\mathrm{d}F_s\rangle\, F_1\cdots\widehat{F_r}\cdots\widehat{F_s}\cdots F_q\,\nu\,\mathrm{d}g\end{aligned} \tag{5.3.1}$$

where the hat signifies omission of the corresponding term.

Proof. Using integration by parts twice, we obtain

$$\begin{aligned}&-\sum_r \int_G (\Delta F_r)\, F_1\cdots\widehat{F_r}\cdots F_q\,\nu\,\mathrm{d}g\\ &\qquad=\sum_r\int_G \langle \mathrm{d}F_r,\mathrm{d}\nu\rangle\, F_1\cdots\widehat{F_r}\cdots F_q\,\mathrm{d}g\\ &\qquad\quad+\sum_{r\neq s}\int_G \langle \mathrm{d}F_r,\mathrm{d}F_s\rangle\, F_1\cdots\widehat{F_r}\cdots\widehat{F_s}\cdots F_q\,\nu\,\mathrm{d}g\\ &\qquad=-\sum_r\int_G \Delta\nu\, F_1\cdots F_q\,\mathrm{d}g\\ &\qquad\quad+2\sum_{r<s}\int_G \langle \mathrm{d}F_r,\mathrm{d}F_s\rangle\, F_1\cdots\widehat{F_r}\cdots\widehat{F_s}\cdots F_q\,\nu\,\mathrm{d}g\end{aligned} \tag{5.3.2}$$

and the claimed equality follows immediately. □

In this section, we will make use of this basic lemma by applying it to a collection of single-argument Wilson loops $W_{l_1},\ldots,W_{l_q}$. For simplicity, we consider only the case where all Wilson loops are defined with respect to the same representation ρ and where the coefficients are elements of the group (i.e., normal Wilson loops instead of generalized ones). However, with minor modifications, everything we say generalizes to generalized Wilson loops with respect to possibly different representations, see Remark 5.3.10 below for more details. The significance of Lemma 5.3.1 lies in the fact that, for Wilson loops, both sides of the relation can be evaluated and this yields a non-trivial identity. Moreover, both sides have interpretations as systematic operations on Wilson loops: The Laplacian ΔW_l of a Wilson loop gives rise to what we call the *twisting* of W_l, and the inner product $\langle \mathrm{d}W_l,\mathrm{d}W_{l'}\rangle$ of two Wilson loops yields their *merging*.

5.3.1. MERGING: CALCULATION OF THE RIGHT-HAND SIDE

Within this subsection, we will focus on the calculation of the term involving the inner product of two Wilson loops. Given a Wilson loop W_l, let $E_+(l)$ be the positions j

where W_l has the identity g and $E_-(l)$ the positions where W_l has the inverse g^{-1}. Let $E(l) := E_+(l) \cup E_-(l)$. Similarly for $E_+(l'),E_-(l')$ and $E(l')$. Consider the matrix component functions $g_{ij} := \rho(g)_{ij} : G \to \mathbb{C}$ and $\xi_{ij} := \rho(\xi)_{ij} : \mathfrak{g} \to \mathbb{C}$. We find:

$$dg_{ij} = g_{il} \cdot \xi^a_{lj} \cdot \epsilon^a, \quad d(g^{-1})_{ij} = -\xi^a_{il} \cdot (g^{-1})_{lj} \cdot \epsilon^a. \tag{5.3.3}$$

Using these expressions for the differentials dg_{ij} and dg^{-1}_{ij}, and the expression (5.2.23) for a general single-argument Wilson loop, we find

$$\begin{aligned} dW_l = &\\ &\sum_{j\in E_+(l)} (c_1 g^{\pm1} \cdots c_{j-1} g^{\pm1} c_j g)_{k_1k_2} \xi^a_{k_2k_3} (c_{j+1} g^{\pm1} \cdots c_n g^{\pm1})_{k_3k_1}\, d\xi^a \\ &- \sum_{j\in E_-(l)} (c_1 g^{\pm1} \cdots c_{j-1} g^{\pm1} c_j)_{k_1k_2} \xi^a_{k_2k_3} (g^{-1} c_{j+1} g^{\pm1} \cdots c_n g^{\pm1})_{k_3k_1}\, d\xi^a. \end{aligned} \tag{5.3.4}$$

To simplify notation, we introduce the following definition.

Definition 5.3.2 (Merging loops in general representations)**.**
For two single-argument Wilson loops of the form $W_l(g) = \mathrm{tr}_\rho(Cg^{\sigma_1}), W_{l'}(g) = \mathrm{tr}_\rho(Dg^{\sigma_2})$ with $C,D \in G$ and exponents $\sigma_1,\sigma_2 \in \{\pm1\}$, we define their *merging* $\mathcal{M}(W_l,W_{l'}) : G \to \mathbb{C}$, depending on the value of the tuple of exponents (σ_1,σ_2), as follows:

$$\mathcal{M}(W_l,W_{l'})(g) = \begin{cases} +\mathrm{tr}_\rho(Cg\xi^a) \cdot \mathrm{tr}_\rho(Dg\xi^a) & \text{if } (\sigma_1,\sigma_2) = (+,+), \\ -\mathrm{tr}_\rho(Cg\xi^a) \cdot \mathrm{tr}_\rho(D\xi^a g^{-1}) & \text{if } (\sigma_1,\sigma_2) = (+,-), \\ -\mathrm{tr}_\rho(C\xi^a g^{-1}) \cdot \mathrm{tr}_\rho(Dg\xi^a) & \text{if } (\sigma_1,\sigma_2) = (-,+), \\ +\mathrm{tr}_\rho(C\xi^a g^{-1}) \cdot \mathrm{tr}_\rho(D\xi^a g^{-1}) & \text{if } (\sigma_1,\sigma_2) = (-,-). \end{cases} \tag{5.3.5}$$

Note the implicit sum over the Lie algebra index a in all of the above. The merging of two *generalized* Wilson loops is defined analogously. Since the above case distinctions depending on the tuple (σ_1,σ_2) will occur more often later, we will adopt the following equivalent notation for brevity:

$$\mathcal{M}(W_l,W_{l'})(g) = \begin{cases} (+,+): & +\mathrm{tr}_\rho(Cg\xi^a) \cdot \mathrm{tr}_\rho(Dg\xi^a), \\ (+,-): & -\mathrm{tr}_\rho(Cg\xi^a) \cdot \mathrm{tr}_\rho(D\xi^a g^{-1}) \\ (-,+): & -\mathrm{tr}_\rho(C\xi^a g^{-1}) \cdot \mathrm{tr}_\rho(Dg\xi^a), \\ (-,-): & +\mathrm{tr}_\rho(C\xi^a g^{-1}) \cdot \mathrm{tr}_\rho(D\xi^a g^{-1}). \end{cases} \tag{5.3.6}$$

For two arbitrary single-argument Wilson loops W_l and $W_{l'}$ with distinguished factors $g^{\pm1}$ in the, respectively, j-th and j'-th position, their *merging* $\mathcal{M}_{jj'}(W_l,W_{l'})$ *at the* j*-th and* j'*-th positions* is defined by the same formulas after the Wilson loops have been expressed in the above form[2] with C,D possibly depending on g. The *total merger* of

[2]That is, using the cyclicity of the trace: $W_l(g) = \mathrm{tr}_\rho(c_1 g^{\pm1} c_2 g^{\pm1} \cdots c_n g^{\pm1}) = \mathrm{tr}_\rho(c_{j+1} g^{\pm1} c_{j+2} g^{\pm1} \cdots c_n g^{\pm1} c_1 g^{\pm1} c_2 g^{\pm1} \cdots c_j g^{\pm1})$.

two loops $W_l, W_{l'}$ is defined as

$$\mathscr{M}(W_l, W_{l'}) := \sum_{\substack{j\in E(l),\\ j'\in E(l')}} \mathscr{M}_{jj'}(W_l, W_{l'}). \tag{5.3.7}$$

Remark 5.3.3. Note that this is *not* equal to what in [Cha19] is called the merging of loops when $G = SO(N)$. However, there is a relation between the notions, which is outlined in Section 5.5.1.

The particular form of the merge operation depends on the Lie algebra under study, and it is completely controlled by the operator K Eq. (5.2.10). In fact, we have

$$\mathscr{M}(W_l, W_{l'})(g) = K_{ijkl}\cdot\begin{cases}(+,+): & +C_{js}g_{si}\; D_{lt}g_{tk},\\ (+,-): & -C_{js}g_{si}\; D_{tk}g^{-1}_{lt},\\ (-,+): & -C_{si}g^{-1}_{js}\, D_{lt}g_{tk},\\ (-,-): & +C_{si}g^{-1}_{js}\, D_{tk}g^{-1}_{lt}.\end{cases} \tag{5.3.8}$$

Identities expressing K in terms of elementary matrices are called *completeness relations*. These relations usually allow one to rewrite $\mathscr{M}_{jj'}(W_l, W_{l'})$ as a linear combination of certain Wilson loops. Below in Section 5.5 we discuss this exemplarily for $G = \mathrm{O}(N)$, $\mathrm{Sp}(N)$, $\mathrm{U}(N)$, $\mathrm{SU}(N)$ in more detail. Note, however, that in general the merging of two Wilson loops is not a linear combination of Wilson loops again as the example of G_2 shows. On the other hand, the class of *generalized* Wilson loops is closed under the merging operation.

Proposition 5.3.4. *The merging of two generalized Wilson loops*

$$W_{C_1,\theta_1}(g) = \mathrm{tr}_V\big(C_1\rho(g^{\theta_1})\big), \quad W_{C_2,\theta_2}(g) = \mathrm{tr}_V\big(C_2\rho(g^{\theta_2})\big) \tag{5.3.9}$$

equals the generalized Wilson loop

$$\mathscr{M}(W_{C_1,\theta_1}, W_{C_2,\theta_2})(g) = \mathrm{tr}_{V^{\theta_1,\theta_2}}\big(K^{\theta_1,\theta_2}\,\psi^{\theta_1,\theta_2}(C_1,C_2)\,\rho^{\theta_1,\theta_2}(g)\big), \tag{5.3.10}$$

where, depending on the signatures (θ_1,θ_2)*, the representation* ρ^{θ_1,θ_2} *is defined by*

$$V^{\theta_1,\theta_2} = \begin{cases}(+,+): & V\otimes V,\\ (+,-): & V\otimes V^*,\\ (-,+): & V^*\otimes V,\\ (-,-): & V^*\otimes V^*,\end{cases} \qquad \rho^{\theta_1,\theta_2} = \begin{cases}(+,+): & \rho\otimes\rho,\\ (+,-): & \rho\otimes\rho^*,\\ (-,+): & \rho^*\otimes\rho,\\ (-,-): & \rho^*\otimes\rho^*,\end{cases} \tag{5.3.11}$$

and the map $\psi^{\theta_1,\theta_2} : \mathrm{End}(V)^2 \to \mathrm{End}(V^{\theta_1,\theta_2})$ *is defined by*

$$\psi^{\theta_1,\theta_2}(C,D) = \begin{cases}(+,+): & C\otimes D,\\ (+,-): & C\otimes D^*,\\ (-,+): & C^*\otimes D,\\ (-,-): & C^*\otimes D^*,\end{cases} \tag{5.3.12}$$

5

and $K^{\theta_1,\theta_2} = \rho^{\theta_1}(\xi^a) \otimes \rho^{\theta_2}(\xi^a)$ *with* $\rho^+ = \rho$ *and* $\rho^- = \rho^*$.

Proof. For simplicity, we only give the proof for the case $(\theta_1, \theta_2) = (+,-)$; the other cases are analogous. Since the trace is invariant under transposition, we have

$$\operatorname{tr}_V\big(C\rho(\xi)\rho(g^{-1})\big) = \operatorname{tr}_{V^*}\big(\rho(g^{-1})^*\rho(\xi)^*C^*\big) = -\operatorname{tr}_{V^*}\big(\rho^*(g)\rho^*(\xi)C^*\big) \tag{5.3.13}$$

for $C \in \operatorname{End}(V)$, $\xi \in \mathfrak{g}$ and $g \in G$. Thus, by (5.3.2), we find

$$\begin{aligned}\mathcal{M}(W_{C_1,\theta_1}, W_{C_2,\theta_2})(g) &= -\operatorname{tr}_V\big(C_1\rho(g)\rho(\xi^a)\big)\cdot\operatorname{tr}_V\big(C_2\rho(\xi^a)\rho(g^{-1})\big)\\ &= \operatorname{tr}_V\big(\rho(\xi^a)C_1\rho(g)\big)\cdot\operatorname{tr}_{V^*}\big(\rho^*(\xi^a)C_2^*\rho^*(g)\big)\\ &= \operatorname{tr}_{V\otimes V^*}\big(\rho(\xi^a)\otimes\rho^*(\xi^a)\circ C_1\otimes C_2^*\circ\rho(g)\otimes\rho^*(g)\big),\end{aligned} \tag{5.3.14}$$

which finishes the proof. □

5

With the above notation and Equation (5.3.4), we arrive at the following expression:

$$\langle dW_l, dW_{l'}\rangle = \sum_{\substack{j\in E(l),\\ j'\in E(l')}} \mathcal{M}_{jj'}(W_l, W_{l'}) = \mathcal{M}(W_l, W_{l'}) \tag{5.3.15}$$

5.3.2. TWISTING: CALCULATION OF THE LEFT-HAND SIDE

Let us now examine what the action of the Laplacian on a Wilson loop. We start off by using the higher-order product rule

$$\Delta(f\cdot h) = f\cdot\Delta h + \Delta(f)\cdot h + 2\langle \mathrm{d}f, \mathrm{d}h\rangle \tag{5.3.16}$$

for $f, h \in C^\infty(G)$. In the representation (5.2.23) for a single-argument Wilson loop W_l, this yields:

$$\begin{aligned}\Delta W_l = &\sum_{j=1}^{n}(c_1g^{\pm1}\cdots c_j)_{i_1i_2}\Delta(g^{\pm1}_{i_2i_3})(c_{j+1}\cdots c_ng^{\pm1})_{i_3i_1}\\ &+\sum_{j\neq k}\langle dg^{\pm1}_{i_2i_3}, dg^{\pm1}_{i_4i_5}\rangle(c_1g^{\pm1}\cdots c_j)_{i_1i_2}(c_{j+1}\cdots c_k)_{i_3i_4}(c_{k+1}\cdots c_ng^{\pm1})_{i_5i_1}.\end{aligned} \tag{5.3.17}$$

Recall that by the Peter–Weyl theorem, matrix elements of irreducible representations are eigenfunctions of the Laplacian, and matrix elements to the same irreducible representation lie in the same eigenspace. Hence, the first sum in the above is a scalar multiple of W_l. The mixed term takes a form that is very similar to the mergers of Definition 5.3.2, except that the loop is "merged with *itself*", in two different locations. Let us make this precise with the following definition.

Definition 5.3.5 (Twisting loops in general representations)**.** Given a single-argument loop $W_l(g) = \operatorname{tr}_\rho(Cg^{\sigma_1}Dg^{\sigma_2})$ with $C, D \in G$ and exponents $\sigma_1, \sigma_2 \in \{\pm1\}$. We define its *twisting* $\mathscr{T}(W_l): G \to \mathbb{C}$, depending on the value of the tuple of exponents (σ_1, σ_2), as follows:

$$\mathscr{T}(W_l)(g) = \begin{cases} (+,+): & +\operatorname{tr}_\rho(Cg\xi^a Dg\xi^a), \\ (+,-): & -\operatorname{tr}_\rho(Cg\xi^a D\xi^a g^{-1}), \\ (-,+): & -\operatorname{tr}_\rho(C\xi^a g^{-1} Dg\xi^a), \\ (-,-): & +\operatorname{tr}_\rho(C\xi^a g^{-1} D\xi^a g^{-1}). \end{cases} \tag{5.3.18}$$

For an arbitrary single-argument Wilson loops W_l with distinguished factors $g^{\pm 1}$ in the, respectively, j-th and j'-th position, its *twisting* $\mathscr{T}_{jj'}(W_l)$ *at the* j*-th and* j'*-th positions* is defined by the same formulas after the Wilson loops have been expressed in the above form with C, D possibly depending on g (cf. Definition 5.3.2). The *total twisting* of a loop W_l is defined as

$$\mathscr{T}(W_l) := \sum_{\substack{j,j' \in E(l) \\ j \neq j'}} \mathscr{T}_{jj'}(W_l). \tag{5.3.19}$$

Note that the twisting of a Wilson loop, too, is completely determined by the operator K:

$$\mathscr{T}(W_l)(g) = K_{ijkl} \cdot \begin{cases} (+,+): & +C_{ls}g_{si}\, D_{jt}g_{tk}, \\ (+,-): & -C_{ts}g_{si}\, D_{jk}g^{-1}_{lt}, \\ (-,+): & -C_{li}g^{-1}_{js}\, D_{st}g_{tk}, \\ (-,-): & +C_{ti}g^{-1}_{js}\, D_{sk}g^{-1}_{lt}. \end{cases} \tag{5.3.20}$$

This formula should be compared with the expression (5.3.8) for the merging, which has the same structure in K and $\rho(g^{\pm 1}) \otimes \rho(g^{\pm 1})$ but the contraction with the tensor $\rho(C) \otimes \rho(D)$ is different. As a consequence of Schur's lemma, the matrix elements are eigenfunctions of the Laplace operator Δ. Using its definition (5.2.5) and Eq. (5.2.11), we find

$$\lambda g_{ij} \stackrel{!}{=} \Delta g_{ij} = \xi^a(\xi^a g_{ij}) = g_{il}\xi^a_{lk}\xi^a_{kj} = g_{il}K_{lkkj} = g_{il}\rho(C)_{lj}. \tag{5.3.21}$$

Hence the eigenvalue λ of the Laplace operator equals the eigenvalue of the Casimir invariant $C \in \mathrm{U}(\mathfrak{g})$ in the representation ρ:

$$\rho(C)_{ij} = K_{ikkj} = \lambda \delta_{ij}. \tag{5.3.22}$$

Thus we can rewrite the Laplacian of a Wilson loop W_l in terms of the twisting $\mathscr{T}(W_l)$, the eigenvalue λ, and the number n counting the amount of $g^{\pm 1}$-factors contained in W_l:

$$\Delta W_l = \lambda \cdot n \cdot W_l + \sum_{j \neq j' \in E(l)} \mathscr{T}_{jj'}(W_l) = \lambda \cdot n \cdot W_l + \mathscr{T}(W_l). \tag{5.3.23}$$

5.3.3. SYNTHESIS

Combining the calculated terms with Lemma 5.3.1 we get the following theorem.

Theorem 5.3.6. *Let G be a compact Lie group equipped with a probability density ν, $\rho : G \to V$ an irreducible, finite-dimensional representation and $W_{l_1}, \dots, W_{l_q} : G \to \mathbb{C}$ a collection of single-argument Wilson loops. Let $\lambda \in \mathbb{C}$ be the eigenvalue of the Casimir $\rho(C)$, and denote the number of factors of g or g^{-1} in the canonical representation of the Wilson loop W_{l_r} by n_r. Then we have*

$$\begin{aligned}\lambda \sum_{r=1}^{q} n_r \cdot \int_G W_{l_1} \cdots W_{l_q}\, \nu \,\mathrm{dg} = \\ & -2 \sum_{\substack{r,s=1\\ r<s}}^{q} \int_G \mathscr{M}(W_{l_r}, W_{l_s}) \cdot W_{l_1} \cdots \widehat{W}_{l_r} \cdots \widehat{W}_{l_s} \cdots W_{l_q}\, \nu \,\mathrm{dg} \\ & - \sum_{r=1}^{q} \int_G \mathscr{T}(W_{l_r}) \cdot W_{l_1} \cdots \widehat{W}_{l_r} \cdots W_{l_q}\, \nu \,\mathrm{dg} \\ & + \int_G W_{l_1} \cdots W_{l_q}\, \Delta\nu \,\mathrm{dg}.\end{aligned} \tag{5.3.24}$$

As applications, let us state Theorem 5.3.6 for the three different choices of probability densities ν introduced above.

Corollary 5.3.7 (Haar measure)**.** *In the setting of Theorem 5.3.6, we have*

$$\begin{aligned}\lambda \sum_{r=1}^{q} n_r \cdot \int_G W_{l_1} \cdots W_{l_q} \,\mathrm{dg} = \\ & -2 \sum_{\substack{r,s=1\\ r<s}}^{q} \int_G \mathscr{M}(W_{l_r}, W_{l_s}) \cdot W_{l_1} \cdots \widehat{W}_{l_r} \cdots \widehat{W}_{l_s} \cdots W_{l_q} \,\mathrm{dg} \\ & - \sum_{r=1}^{q} \int_G \mathscr{T}(W_{l_r}) \cdot W_{l_1} \cdots \widehat{W}_{l_r} \cdots W_{l_q} \,\mathrm{dg}.\end{aligned} \tag{5.3.25}$$

Corollary 5.3.8 (Brownian motion)**.** *In the setting of Theorem 5.3.6, we have*

$$\begin{aligned}\int_G W_{l_1} \cdots W_{l_q}\, p_t \,\mathrm{dg} &= W_{l_1}(e) \cdots W_{l_q}(e) \\ & + \frac{1}{2} e^{\frac{t}{2}\lambda \sum_{r=1}^{q} n_r} \int_0^t \mathscr{M}\mathscr{T}(s)\, e^{-\frac{s}{2}\lambda \sum_{r=1}^{q} n_r} \,\mathrm{d}s,\end{aligned} \tag{5.3.26}$$

where

$$\begin{aligned}\mathscr{M}\mathscr{T}(t) = 2 \sum_{\substack{r,s=1\\ r<s}}^{q} \int_G \mathscr{M}(W_{l_r}, W_{l_s}) \cdot W_{l_1} \cdots \widehat{W}_{l_r} \cdots \widehat{W}_{l_s} \cdots W_{l_q}\, p_t \,\mathrm{dg} \\ + \sum_{r=1}^{q} \int_G \mathscr{T}(W_{l_r}) \cdot W_{l_1} \cdots \widehat{W}_{l_r} \cdots W_{l_q}\, p_t \,\mathrm{dg}.\end{aligned} \tag{5.3.27}$$

5

Proof. Using the heat equation, (5.3.24) reduces, for the Brownian motion, to a first-order linear differential equation of the form

$$c f(t) - 2f'(t) = h(t), \tag{5.3.28}$$

where $f(t)$ is the expectation of the product of Wilson loops, and $h(t)$ includes the merging or twisting terms. This equation has the general solution

$$f(t) = f(0) - \frac{1}{2} e^{\frac{c}{2}t} \int_0^t h(s) e^{-\frac{c}{2}s} \, \mathrm{d}s. \tag{5.3.29}$$

Since the heat kernel approaches the delta distribution at the identity as $t \to 0$, the initial value is $f(0) = W_{l_1}(e) \cdots W_{l_q}(e)$. This completes the proof. □

Corollary 5.3.9 (Wilson action)**.** *In the setting of Theorem 5.3.6, we have*

$$\begin{aligned}
\lambda \sum_{r=1}^{q} n_r \cdot \int_G W_{l_1} \cdots W_{l_q} \nu_W \, \mathrm{d}g = \\
&- 2 \sum_{\substack{r,s=1 \\ r<s}}^{q} \int_G \mathcal{M}(W_{l_r}, W_{l_s}) \cdot W_{l_1} \cdots \widehat{W_{l_r}} \cdots \widehat{W_{l_s}} \cdots W_{l_q} \nu_W \, \mathrm{d}g \\
&- \sum_{r=1}^{q} \int_G \mathcal{T}(W_{l_r}) \cdot W_{l_1} \cdots \widehat{W_{l_r}} \cdots W_{l_q} \nu_W \, \mathrm{d}g \\
&+ \beta\lambda \sum_p \int_G W_p \cdot W_{l_1} \cdots W_{l_q} \nu_W \, \mathrm{d}g \\
&+ \beta^2 \sum_{p,p'} \int_G \mathcal{M}(W_p, W_{p'}) \cdot W_{l_1} \cdots W_{l_q} \nu_W \, \mathrm{d}g.
\end{aligned} \tag{5.3.30}$$

Proof. The general identity $\Delta \exp(f) = \big(\Delta f + \langle \mathrm{d}f, \mathrm{d}f \rangle\big) \exp(f)$ implies for the Wilson action defined in (5.2.24) that

$$\begin{aligned}
\Delta \nu_W &= \Big(\beta \sum_p \Delta W_p + \beta^2 \sum_{p,p'} \langle \mathrm{d}W_p, \mathrm{d}W_{p'} \rangle\Big) \nu_W \\
&= \Big(\beta\lambda \sum_p W_p + \beta^2 \sum_{p,p'} \mathcal{M}(W_p, W_{p'})\Big) \nu_W,
\end{aligned} \tag{5.3.31}$$

where, in the second line, we used that Wilson loops with a single argument of g are eigenfunctions of the Laplacian and that the scalar product of two such loops equals their merging. Inserting this equality in (5.3.24) yields (5.3.30). □

Corollary 5.3.9 is essentially a generalization of [Cha19, Theorem 8.1], which studies the case $G = \mathrm{SO}(N)$ in the fundamental representation. The main difference between our and their presentation is that we have used integration by parts as the basic tool rather than Stein's method. One can obtain Chatterjee's result on the nose by carrying out integration by parts *once*. However, in our derivation of Theorem 5.3.6 we have used it *twice*. This has the added benefit that now the Wilson loop observables decouple from the plaquette variables and one no longer has mergers (or *deformations* in the terminology of [Cha19]) between Wilson loops and plaquettes.

Remark 5.3.10. In Theorem 5.3.6 and its corollaries, we have assumed that $W_{l_1}, \cdots, W_{l_q}$ are single-argument Wilson loops with respect to the same representation. A careful inspection of the calculation reveals that, with minor modifications, those results generalize to *generalized* Wilson loops in possibly different representations. For example, the merger of two Wilson loops with different representations ρ and ρ' is defined by essentially the same formula as in Definition 5.3.2 with the only difference that one trace is taken with respect to ρ and the other one with respect to ρ'. Similarly, the eigenvalue λ in Theorem 5.3.6 may now depend on the Wilson loop so that the factor $\lambda \sum_{r=1}^{q} n_r$ needs to be replaced by $\sum_{r=1}^{q} n_r \lambda_r$.

By Proposition 5.2.3, we have seen that every polynomial Wilson loop can be written as a linear combination of linear generalized loops with respect to different representations. Note that for linear loops Theorem 5.3.6 simplifies as there is no longer a twisting term. In particular, for the Haar measure and the Brownian motion, the relations in Corollaries 5.3.7 and 5.3.8 simplify to recursion relations involving less and less products of loops. This observation can be used to calculate the expectation values of the product of arbitrary Wilson loops $W_{l_1}, \dots, W_{l_q}$ according to the following algorithm.

1. Expand each Wilson loop W_{l_i} in terms of linear generalized loops (with respect to irreducible representations) as in Proposition 5.2.3.

2. For linear loops, solve the recursion relation in Corollaries 5.3.7 and 5.3.8 by induction over the number of loops involved.

3. For a single linear loop $W_{\rho,(c)}$ with respect to a non-trivial irreducible representation (ρ, V), the expectation value can be calculated using the results of the next section. In particular, the expectation

   $$\int_G W_{\rho,(c)} \,\mathrm{d}g = \mathrm{tr}_V\Big(c \int_G \rho(g)\,\mathrm{d}g\Big) \tag{5.3.32}$$

   vanishes since $\int_G \rho(g)\,\mathrm{d}g$ is the projection onto $V^G = \{0\}$. This serves as the induction start in the case of the Haar measure.

   For the Brownian motion, Corollary 5.4.4 below implies that

   $$\int_G W_{\rho,(c)}\, p_t \,\mathrm{d}g = \mathrm{tr}_V\Big(c \int_G \rho(g)\, p_t \,\mathrm{d}g\Big) = \exp\Big(\frac{1}{2} c_\rho t\Big)\,\mathrm{tr}_V(c), \tag{5.3.33}$$

   where c_ρ is the Casimir eigenvalue.

The following example illustrates this algorithm for the simplest case, namely $G = \mathrm{U}(1)$.

5

Example 5.3.11 (Circle group)**.** The irreducible representations of $\mathrm{U}(1)$ are one dimensional and given by $\rho_n(z) = z^n$ for some $n \in \mathbb{Z}$. Thus, irreducible Wilson loops are of the form $W_{n,c} = cz^n$ with $n \in \mathbb{Z}$ and $c \in \mathbb{C}$. Note that the Casimir invariant of ρ_n is $-n^2$.

As an example, let us calculate the expectation value of the product of two arbitrary Wilson loops W_1 and W_2 according to the above algorithm.

1. The expansion of W_1 according to Proposition 5.2.3, in this case, just amounts to writing it as a Fourier series:

$$W_1 = \sum_{n=-\infty}^{\infty} W_{n,c_{1,n}}, \tag{5.3.34}$$

where only finitely many constants $c_{1,n} \in \mathbb{C}$ are non-zero. Similarly, for W_2 with constants $c_{2,n}$.

2. By Corollary 5.3.7, we have

$$-(n^2+m^2) \int_{\mathrm{U}(1)} W_{n,c}(z) W_{m,d}(z)\, \mathrm{d}z = -2 \int_{\mathrm{U}(1)} \mathscr{M}(W_{n,c}, W_{m,d})(z)\, \mathrm{d}z. \tag{5.3.35}$$

The merger is given by $\mathscr{M}(W_{n,c}, W_{m,d})(z) = c(\mathrm{i}n)z^n \cdot d(\mathrm{i}m)z^m$, cf. Definition 5.3.2 and Remark 5.3.10. In line with Proposition 5.3.4, this is again a generalized Wilson loop, namely $\mathscr{M}(W_{n,c}, W_{m,d}) = W_{n+m,-cdnm}$.

3. Clearly, $\int_{\mathrm{U}(1)} W_{n,c}\, \mathrm{d}z$ vanishes except if $n = 0$.

Thus, in summary,

$$\begin{aligned} \int_{\mathrm{U}(1)} W_1(z) W_2(z)\, \mathrm{d}z &= \sum_{n,m=-\infty}^{\infty} \int_{\mathrm{U}(1)} W_{n,c_{1,n}} W_{m,c_{2,m}}(z)\, \mathrm{d}z \\ &= \sum_{n,m=-\infty}^{\infty} \frac{2}{n^2+m^2} \int_{\mathrm{U}(1)} W_{n+m,-c_{1,n}c_{2,m}nm}(z)\, \mathrm{d}z \\ &= \sum_{n=-\infty}^{\infty} c_{1,n} c_{2,-n}, \end{aligned} \tag{5.3.36}$$

which, of course, coincides with the result one gets by a direct calculation.

The same strategy can be used to calculate the mixed moments of the random variable $g \mapsto \mathrm{tr}_\rho(g^k)$. In this case, the expansion of $\mathrm{tr}_\rho(g^k)$ as a linear combination of linear Wilson loops can be achieved by decomposing the k-th tensor power into irreducible components. For the latter, Schur–Weyl duality can be used and then the second step in the above algorithm essentially boils down to an orthogonality relation of the characters of the dual group. For Haar distributed variables, this recovers [DS94, Theorem 2] [DE01, Theorem 2.1] for the unitary group, and [HR03, Theorem 3] for the orthogonal and symplectic group. We leave the details to the reader.

Remark 5.3.12. The same algorithm does not work when the Wilson probability measure is added, because the right-hand side of (5.3.30) contains terms with more Wilson loops than the original integral. In fact, it is a notoriously hard problem to calculate Wilson loop expectation values with respect to the Wilson action, and one has to resort to certain limits to obtain a reasonable result. The effect of the two most common limits, namely the strong-coupling expansion and the large N-limit for $\mathrm{U}(N)$ or $\mathrm{SU}(N)$, is readily apparent from (5.3.30). In the strong-coupling limit $\beta \to 0$, the additional terms with more Wilson loops are suppressed. Similarly, the merging and twisting terms as well as λ scale with N or simplify in the large N-limit. This limit has been extensively studied in [Cha19, Jaf16].

5.4. POLYNOMIALS OF MATRIX COEFFICIENTS

In this section, we discuss how the basic integration by parts formula of Lemma 5.3.1 can be used to determine polynomials of matrix coefficients.

5

As before, let G be a compact connected Lie group and ν be a probability density with respect to the normalized Haar measure on G. For a (not necessarily irreducible) real or complex representation $\rho : G \to \mathrm{GL}(V)$ of G on a finite-dimensional vector space, define $T(\nu) : V \to V$ by

$$T(\nu) = \int_G \rho(g)\, \nu(g)\, \mathrm{d}g\,. \tag{5.4.1}$$

Example 5.4.1. Let ϱ be a representation of G on the vector space W. Usually, this is taken to be the fundamental representation of G. Consider the tensor representation $\rho = \varrho^{\otimes n, \otimes n'} = \varrho \otimes \cdots \otimes \varrho \otimes \varrho^* \otimes \cdots \otimes \varrho^*$ on $V = W^{\otimes n} \otimes (W^*)^{\otimes n'}$ with n factors of ϱ and n' factors of the dual representation $\varrho^*(g) = \rho(g^{-1})^*$. Using bold-face multi-indices $\mathbf{i} = (i; i') = (i_1, \ldots, i_n; i'_1, \ldots, i'_{n'})$ to denote the components of elements of $W^{\otimes n} \otimes (W^*)^{\otimes n'}$, we find

$$T(\nu)_{\mathbf{i}\mathbf{j}} = \int_G g_{i_1 j_1} \cdots g_{i_n j_n}\, g^{-1}_{j'_1 i'_1} \cdots g^{-1}_{j'_{n'} i'_{n'}}\, \nu\, \mathrm{d}g\,, \tag{5.4.2}$$

where $g_{kl} = \varrho(g)_{kl}$ are the matrix coefficients of g in the representation ϱ. Thus, in this case, $T(\nu)$ completely encodes the ν-expectation value of polynomials in matrix coefficients and their inverses. The formulation in terms of the tensor product linearizes the problem of determining the polynomial coefficients on G to a study of the linear operator $T(\nu)$.

Somewhat surprisingly the simple integration by parts formula of Lemma 5.3.1 combined with basic representation theory of compact Lie groups allows us to determine $T(\nu)$. Before we discuss this in detail, let us recall the isotypic decomposition. Consider a representation ρ of G on a vector space V. Since G is compact, V decomposes into a direct sum of irreducible representations V_τ, see, e.g., [Kna02, Corollary IV.4.7][3]. For a

[3] [Kna02] only discusses the case of a complex representation. The proof in the real case is almost identical except that one uses a G-invariant real inner product.

given irreducible representation τ, define the isotypic component $V_{[\tau]}$ to be the sum of all $V_{\tau'}$ for which τ' is equivalent to τ; with the convention that $V_{[\tau]} = \{0\}$ if there is no such subrepresentation. The resulting direct sum decomposition

$$V = \bigoplus_{[\tau]\in\hat{G}} V_{[\tau]} \tag{5.4.3}$$

is called the *isotypic decomposition*. Here the sum is over the set $\hat{G}$ of equivalence classes of irreducible representations of G. Note that the isotypic component corresponding to the trivial representation is the set V^G of invariant elements.

Theorem 5.4.2. *Let G be a compact connected Lie group and ν be a probability density with respect to the normalized Haar measure on G. For a real or complex representation $\rho : G \to \mathrm{GL}(V)$ of G, the operator $T(\nu) : V \to V$ defined in (5.4.1) respects the isotypic decomposition (5.4.3) and satisfies*

$$c_{[\tau]}\, T(\nu)_{|V_{[\tau]}} = T(\Delta\nu)_{|V_{[\tau]}} \tag{5.4.4}$$

for each irreducible subrepresentation τ, where $c_{[\tau]} \in \mathbb{R}$ are non-positive constants depending only on the isomorphism type of the representation τ. Moreover, $c_{[\tau]} = 0$ if and only if τ is the trivial representation.

Proof. Lemma 5.3.1 applied to the matrix coefficients $g_{ij} = \rho(g)_{ij}$ yields

$$\int_G g_{ij}\,\Delta\nu(g)\,\mathrm{d}g = \int_G \Delta g_{ij}\,\nu(g)\,\mathrm{d}g = K_{lkkj}\int_G g_{il}\,\nu(g)\,\mathrm{d}g, \tag{5.4.5}$$

where the second equality follows from (5.3.21). Rewriting this equality in terms of operators gives $T(\Delta\nu) = T(\nu)\rho(C)$ with $\rho(C)$ being the Casimir invariant.

By going back to their definition, $T(\nu)$ and $\rho(C)$ respect the decomposition of V into irreducible representations and so also the isotypic decomposition. We have to show that the Casimir invariant $\rho(C)$ acts as a scalar multiple of the identity on each irreducible component V_τ and that the corresponding eigenvalue c_τ is real and non-positive. If τ is a complex representation, this is exactly [Bou05, Proposition IX.7.6.4]. Moreover, $c_\tau = 0$ if and only if τ is the trivial representation. By the same proposition, c_τ can be expressed in terms of the highest weight associated with τ and so it only depends on the isomorphism type of the representation τ.

Now, for an irreducible real representation $\tau : G \to \mathrm{End}(V_\tau)$, we can pass to its complexification $\tau^{\mathbb{C}} : G \to \mathrm{End}(V_\tau^{\mathbb{C}})$. Clearly, the complex-linear extension of the Casimir $\tau(C)$ is the Casimir $\tau^{\mathbb{C}}(C)$ of the complexified representation. By [BtD95, Theorem 6.3 and Proposition 6.6], the representation $V^{\mathbb{C}}$ is either irreducible or a direct sum of the form $U \oplus \bar{U}$ or $U \oplus U$ for an irreducible complex representation U. In either case, the above argument shows that the Casimir $\tau^{\mathbb{C}}(C)$ acts as a scalar multiplication, because the Casimir eigenvalue of the complex conjugate representation $\bar{U}$ is the same as the one of the representation U. By restricting to the real part, we conclude that $\tau(C)$ is a scalar multiple of the identity. This finishes the proof. □

Corollary 5.4.3 (Haar measure)**.** *Let G be a compact connected Lie group. For a real or complex representation $\rho : G \to \mathrm{GL}(V)$ of G, the operator $T(v = 1) : V \to V$ defined in (5.4.1) is the projection onto V^G along the isotypic decomposition (5.4.3).*

Proof. For $v = 1$, we have $\Delta v = 0$ and so $c_{[\tau]} T(1)_{|V_{[\tau]}} = 0$ for every irreducible subrepresentation τ. Because $c_{[\tau]}$ is strictly negative for non-trivial representations τ, the restriction $T(1)_{|V_{[\tau]}}$ has to vanish for such representations. Finally, the restriction of $T(1)$ to V^G is clearly the identity operator. □

Corollary 5.4.4 (Brownian motion)**.** *Let G be a compact connected Lie group and let $\rho : G \to V$ be a real or complex representation of G. The expectation value of the* $\mathrm{GL}(V)$*-valued random variable ρ relative to the Riemannian Brownian motion $(g_t)_{t\geq 0}$ respects the isotypic decomposition (5.4.3) and satisfies*

$$\mathbb{E}\big(\rho(g_t)\big)_{|V_{[\tau]}} = \exp\Big(\frac{1}{2} c_{[\tau]}\, t\Big)\mathrm{id}_{|V_{[\tau]}}. \tag{5.4.6}$$

for each irreducible subrepresentation τ, where $c_{[\tau]} \in \mathbb{R}$ are the same non-positive constants as in Theorem 5.4.2. Equivalently, $\mathbb{E}\big(\rho(g_t)\big) = \exp\Big(\frac{t}{2}\rho(C)\Big)$. Moreover,

$$\lim_{t\to\infty} \mathbb{E}\big(\rho(g_t)\big) = T(1). \tag{5.4.7}$$

Proof. By definition, $\mathbb{E}\big(\rho(g_t)\big) = T(p_t)$ with p_t being the heat density. Using the heat equation (5.2.18), Theorem 5.4.2 implies that the expectation value satisfies

$$c_{[\tau]}\, \mathbb{E}\big(\rho(g_t)\big)_{|V_{[\tau]}} = 2\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\big(\rho(g_t)\big)_{|V_{[\tau]}}. \tag{5.4.8}$$

For the initial condition, note that $\lim_{t\to 0}\mathbb{E}\big(\rho(g_t)\big) = \rho(e) = \mathrm{id}_V$. This shows that the expectation value is given by (5.4.6). Since $c_{[\tau]}$ are the eigenvalues of the Casimir operator, we get $\mathbb{E}\big(\rho(g_t)\big) = \exp\Big(\frac{t}{2}\rho(C)\Big)$.

For a non-trivial subrepresentation τ, the constant $c_{[\tau]}$ is strictly negative and thus $\mathbb{E}\big(\rho(g_t)\big)_{|V_{[\tau]}}$ converges to 0 as $t \to \infty$. On the other hand, $\mathbb{E}\big(\rho(g_t)\big)_{|V^G} = \mathrm{id}_{V^G}$. Thus, in summary, $\mathbb{E}\big(\rho(g_t)\big)$ converges to the projection onto V^G. □

In the case of the classical groups $G = \mathrm{U}(N), \mathrm{O}(N), \mathrm{Sp}(N)$, the expectation value formula $\mathbb{E}\big(\rho(g_t)\big) = \exp\Big(\frac{t}{2}\rho(C)\Big)$ has been obtained in [Lé08, Proposition 2.4], [Dah17, Lemma 4.1 and 4.2] using the explicit expression of the corresponding Casimir operator. Moreover, the long time asymptotic behavior has been established in this case using a rather complicated calculation, cf. [Dah17, Theorem 4.3 and Lemma 5.1]. In contrast, our proof shows that this is a direct and straight-forward consequence of the non-positivity of the spectrum of the Casimir.

For the tensor representation, the following result shows that the isotypic decomposition and the constants $c_{[\tau]}$ can be obtained from an eigenvalue problem for an operator determined by the operator K defined in Eq. (5.2.9). In particular, the isotypic decomposition of the tensor representation on $V^{\otimes n} \otimes (V^*)^{\otimes n'}$ for arbitrary integers n and n' is completely given in terms of data associated with the tensor representation on $V \otimes V^*$. This is particularly important for determining the decomposition in concrete examples using computer algebra systems.

Proposition 5.4.5. *Let G be a compact connected Lie group, and let $\varrho : G \to V$ be an irreducible representation of G. Let $\lambda \in \mathbb{R}$ be the eigenvalue of the Casimir invariant $\varrho(C)$. For non-negative integers n and n', the isotypic decomposition of the tensor representation $\varrho^{\otimes n, \otimes n'} = \varrho(g)^{\otimes n} \otimes \big(\varrho(g^{-1})^*\big)^{\otimes n'}$ on $V^{\otimes n} \otimes (V^*)^{\otimes n'}$ coincides with the eigenspace decomposition of the operator*

$$\begin{aligned}
C_{\mathbf{ij}} = {} & (n+n')\lambda\,\delta_{\mathbf{ij}} \\
& -2 \sum_{\substack{r,s=1\\ r<s}}^{n} K_{i_r j_r i_s j_s} \delta_{i_1 j_1} \cdots \hat{r} \cdots \hat{s} \cdots \delta_{i_n j_n} \delta_{i' j'} \\
& -2 \sum_{\substack{r,s=1\\ r<s}}^{n'} K_{j'_r i'_r j'_s i'_s} \delta_{i'_1 j'_1} \cdots \hat{r} \cdots \hat{s} \cdots \delta_{i'_{n'} j'_{n'}} \delta_{i j} \\
& +2 \sum_{r=1}^{n} \sum_{s=1}^{n'} K_{i_r j_r j'_s i'_s} \delta_{i_1 j_1} \cdots \hat{r} \cdots \delta_{i_n j_n} \delta_{i'_1 j'_1} \cdots \hat{s} \cdots \delta_{i'_{n'} j'_{n'}}.
\end{aligned} \tag{5.4.9}$$

Moreover, the constants $c_{[\tau]}$ of Theorem 5.4.2 are equal to the corresponding eigenvalues of C.

Proof. As discussed above, the isotypic decomposition coincides with the eigenspace decomposition of the Casimir element $\varrho^{\otimes n, \otimes n'}(C)$. To calculate the components $C_{\mathbf{ij}}$ of this operator, note that

$$\xi_{\mathbf{ij}} = \sum_{r=1}^{n} \xi_{i_r j_r} \delta_{i_1 j_1} \cdots \hat{r} \cdots \delta_{i_n j_n} \delta_{i' j'} - \sum_{r=1}^{n'} \xi_{j'_r i'_r} \delta_{i j} \delta_{i'_1 j'_1} \cdots \hat{r} \cdots \delta_{i'_{n'} j'_{n'}} \tag{5.4.10}$$

for $\xi \in \mathfrak{g}$. Consequently, the operator K (see Eq. (5.2.9)) for the tensor representation $\varrho^{\otimes n, \otimes n'}$ takes the form

$$\begin{aligned}
K_{\mathbf{ijkl}} = {} & \sum_{r,s=1}^{n} K_{i_r j_r k_s l_s} \delta_{i_1 j_1} \cdots \hat{r} \cdots \delta_{i_n j_n} \delta_{k_1 l_1} \cdots \hat{s} \cdots \delta_{k_n l_n} \delta_{i' j'} \delta_{k' l'} \\
& + \sum_{r,s=1}^{n'} K_{j'_r i'_r l'_s k'_s} \delta_{i'_1 j'_1} \cdots \hat{r} \cdots \delta_{i'_{n'} j'_{n'}} \delta_{k'_1 l'_1} \cdots \hat{s} \cdots \delta_{k'_{n'} l'_{n'}} \delta_{i j} \delta_{k l} \\
& - \sum_{r=1}^{n} \sum_{s=1}^{n'} K_{i_r j_r l'_s k'_s} \delta_{i_1 j_1} \cdots \hat{r} \cdots \delta_{i_n j_n} \delta_{k'_1 l'_1} \cdots \hat{s} \cdots \delta_{k'_{n'} l'_{n'}} \delta_{i' j'} \delta_{k l} \\
& - \sum_{r=1}^{n'} \sum_{s=1}^{n} K_{j'_r i'_r k_s l_s} \delta_{i'_1 j'_1} \cdots \hat{r} \cdots \delta_{i'_{n'} j'_{n'}} \delta_{k_1 l_1} \cdots \hat{s} \cdots \delta_{k_n l_n} \delta_{i j} \delta_{k' l'}
\end{aligned} \tag{5.4.11}$$

Our objective is to calculate the Casimir element $C_{\mathbf{ij}} = K_{\mathbf{ikkj}}$, with implicit summation over $\mathbf{k}$ understood. For this purpose, notice that, for each $r \neq s$ and with summation over k, we have

$$K_{i_r k_r k_s j_s} \delta_{i_1 k_1} \cdots \hat{r} \cdots \delta_{i_n k_n} \delta_{k_1 j_1} \cdots \hat{s} \cdots \delta_{k_n j_n} = K_{i_r j_r i_s j_s} \delta_{i_1 j_1} \cdots \hat{r}, \hat{s} \cdots \delta_{i_n j_n}. \tag{5.4.12}$$

On the other hand, for $r = s$, we obtain

$$K_{i_r k_r k_r j_r} \delta_{i_1 k_1} \cdots \hat{r} \cdots \delta_{i_n k_n} \delta_{k_1 j_1} \cdots \hat{r} \cdots \delta_{k_n j_n} = \lambda \delta_{ij}. \tag{5.4.13}$$

Using these and similar identities in each of the four summands yields

$$\begin{aligned}
C_{\mathbf{ij}} &= K_{\mathbf{ikkj}} \\
&= \sum_{\substack{r,s=1 \\ r\neq s}}^{n} K_{i_r j_r i_s j_s} \delta_{i_1 j_1} \cdots \hat{r}, \hat{s} \cdots \delta_{i_n j_n} \delta_{i' j'} + n\lambda \delta_{ij} \delta_{i' j'} \\
&\quad + \sum_{\substack{r,s=1 \\ r\neq s}}^{n'} K_{j'_r i'_r j'_s i'_s} \delta_{i'_1 j'_1} \cdots \hat{r}, \hat{s} \cdots \delta_{i'_{n'} j'_{n'}} \delta_{ij} + n'\lambda \delta_{ij} \delta_{i' j'} \\
&\quad - \sum_{r=1}^{n} \sum_{s=1}^{n'} (K_{i_r j_r j'_s i'_s} + K_{j'_s i'_s i_r j_r}) \delta_{i_1 j_1} \cdots \hat{r} \cdots \delta_{i_n j_n} \delta_{i'_1 j'_1} \cdots \hat{s} \cdots \delta_{i'_{n'} j'_{n'}} \\
&= (n + n')\lambda \delta_{\mathbf{ij}} \\
&\quad + 2 \sum_{\substack{r,s=1 \\ r<s}}^{n} K_{i_r j_r i_s j_s} \delta_{i_1 j_1} \cdots \hat{r} \cdots \hat{s} \cdots \delta_{i_n j_n} \delta_{i' j'} \\
&\quad + 2 \sum_{\substack{r,s=1 \\ r<s}}^{n'} K_{j'_r i'_r j'_s i'_s} \delta_{i'_1 j'_1} \cdots \hat{r} \cdots \hat{s} \cdots \delta_{i'_{n'} j'_{n'}} \delta_{ij} \\
&\quad - 2 \sum_{r=1}^{n} \sum_{s=1}^{n'} K_{i_r j_r j'_s i'_s} \delta_{i_1 j_1} \cdots \hat{r} \cdots \delta_{i_n j_n} \delta_{i'_1 j'_1} \cdots \hat{s} \cdots \delta_{i'_{n'} j'_{n'}}.
\end{aligned} \tag{5.4.14}$$

This finishes the proof. □

In applications, one can often use invariant theory to obtain a spanning set for the space of invariants $\left(V^{\otimes n} \otimes (V^*)^{\otimes n'}\right)^G$. This is well-studied for the classical groups, see [GW09], and also for some exceptional groups, see for example [Sch88] for the group $G = G_2$ and its 7-dimensional irreducible representation. The following theorem shows that such a spanning set is already enough to calculate the operator T for the Haar measure ($\nu = 1$). This generalizes the main results of [Col03, CS06] for the classical groups $G = \mathrm{U}(N), \mathrm{O}(N), \mathrm{Sp}(N)$ to arbitrary compact Lie groups.

Theorem 5.4.6. *Let G be a compact Lie group and let $\rho : G \to V$ be a finite-dimensional real or complex representation of G leaving the inner product $\langle \cdot, \cdot \rangle$ on V invariant. Let $\mathscr{A}$ be a finite-dimensional inner product space over the same field as V and let $\tau : \mathscr{A} \to$*

$V^{\otimes n} \otimes (V^*)^{\otimes n'}$ be a linear map. Denote by $\tau^* : V^{\otimes n} \otimes (V^*)^{\otimes n'} \to \mathscr{A}$ the adjoint of τ with respect to the following inner product[4] on $V^{\otimes n} \otimes (V^*)^{\otimes n'}$:

$$\langle v \otimes \alpha, w \otimes \beta \rangle = \langle v_1, w_1 \rangle \cdots \langle v_n, w_n \rangle \langle \alpha_1, \beta_1 \rangle \cdots \langle \alpha_{n'}, \beta_{n'} \rangle. \tag{5.4.16}$$

There exists a unique map $\mathrm{Wg} : \mathscr{A} \to \mathscr{A}$ satisfying the following properties:

1. $\tau^* \circ \tau \circ \mathrm{Wg} \circ \tau^* \circ \tau = \tau^* \circ \tau$,
2. $\mathrm{Wg} \circ \tau^* \circ \tau \circ \mathrm{Wg} = \mathrm{Wg}$,
3. $\mathrm{Wg}^* \circ \tau^* \circ \tau = \tau^* \circ \tau \circ \mathrm{Wg}$,
4. $\tau^* \circ \tau \circ \mathrm{Wg}^* = \mathrm{Wg} \circ \tau^* \circ \tau$.

If the image of τ is $\left(V^{\otimes n} \otimes (V^*)^{\otimes n'}\right)^G$, then

$$T(1) = \tau \circ \mathrm{Wg} \circ \tau^*, \tag{5.4.17}$$

with T defined as in (5.4.1) relative to the tensor representation $\rho^{\otimes n, \otimes n'}$. In particular, the coefficients of $T(1)$ with respect to an orthonormal basis of V are given by

$$T(1)_{\mathbf{ij}} = \sum_{k,l} \tau(a_k)_{\mathbf{i}} \overline{\tau(a_l)_{\mathbf{j}}} \langle \mathrm{Wg}(a_k), a_l \rangle, \tag{5.4.18}$$

where $\{a_k\}$ is an orthonormal basis of $\mathscr{A}$.

For the fundamental representation ρ of $G = \mathrm{U}(N)$, as we will discuss in detail in Section 5.5.3, a generating set of G-invariant elements of $V^{\otimes n} \otimes (V^*)^{\otimes n}$ is given in terms of permutations. That is, an orthonormal basis of $\mathscr{A}$ is indexed by permutations $\sigma \in S_n$ and the expression $\langle \mathrm{Wg}(\sigma), \varsigma \rangle$ in (5.4.18) recovers the so-called Weingarten function on S_n. For this reason, we will refer to Wg as the Weingarten map for the group G (relative to τ).

Proof. Recall that the pseudoinverse (or Moore–Penrose inverse) of an operator $A : H_1 \to H_2$ between finite-dimensional inner product spaces is an operator $A^+ : H_2 \to H_1$ satisfying

1. $AA^+A = A$,
2. $A^+AA^+ = A^+$,
3. AA^+ and A^+A are self-adjoint.

[4] Under the identification $V^{\otimes n} \otimes (V^*)^{\otimes n'} \simeq \mathrm{Hom}\left(V^{\otimes n'}, V^{\otimes n}\right)$, this inner product corresponds to the inner product

$$\langle S_1, S_2 \rangle = \mathrm{tr}\left(S_2^* S_1\right), \qquad S_1, S_2 \in \mathrm{Hom}\left(V^{\otimes n'}, V^{\otimes n}\right). \tag{5.4.15}$$

It is well known that every operator has a unique pseudoinverse (in the finite-dimensional setting). Moreover, the pseudoinverse satisfies $A^+ = (A^* A)^+ A^*$ and the operator $AA^+ : H_2 \to H_2$ is the orthogonal projector onto the image of A.

Now the properties (1) to (4) entail that Wg is the pseudoinverse of $\tau^* \circ \tau$. In particular, such an operator Wg exists and is uniquely defined by these properties. Moreover, $\tau^+ = (\tau^* \circ \tau)^+ \circ \tau^* = \mathrm{Wg} \circ \tau^*$. Hence,

$$\tau \circ \tau^+ = \tau \circ \mathrm{Wg} \circ \tau^* \tag{5.4.19}$$

is the orthogonal projector onto the image of τ, which is $\left(V^{\otimes n} \otimes (V^*)^{\otimes n'}\right)^G$ by assumption.

On the other hand, Corollary 5.4.3 shows that $T(1)$ equals the orthogonal projection onto $\left(V^{\otimes n} \otimes (V^*)^{\otimes n'}\right)^G$ along the isotypic decomposition. Since ρ leaves the inner product $\langle \cdot, \cdot \rangle$ invariant, $T(1)$ is easily seen to be self-adjoint. Thus, $T(1)$ is an orthogonal projector onto $\left(V^{\otimes n} \otimes (V^*)^{\otimes n'}\right)^G$ and thus coincides with $\tau \circ \mathrm{Wg} \circ \tau^*$. □

5

Remark 5.4.7. The proof shows that the Weingarten map Wg is the pseudoinverse of $\tau^* \circ \tau$. This observation can be used to calculate Wg using one of the well-known constructions of a pseudoinverse. For example, one could exploit the fact that $\tau^* \circ \tau$ is self-adjoint as follows. By the spectral theorem, we can write $\tau^* \circ \tau = UDU^*$ for a unitary operator U and a diagonal matrix D. Reordering the entries of D, we may assume that $D = \mathrm{diag}(\lambda_1, \dots, \lambda_k, 0, \dots, 0)$ where $\lambda_k \in \mathbb{R}$ is non-zero. Then

$$\mathrm{Wg} = U \,\mathrm{diag}(\lambda_1^{-1}, \dots, \lambda_k^{-1}, 0, \dots, 0)\, U^* \tag{5.4.20}$$

is the pseudoinverse of $\tau^* \circ \tau$. For the fundamental representation of the classical groups $G = \mathrm{U}(N), \mathrm{O}(N), \mathrm{Sp}(N)$, the decomposition $\tau^* \circ \tau = UDU^*$ can be calculated using character theory of a certain associated finite group (the Schur–Weyl dual group). In this way, we recover the description [CS06, Proposition 2.3 and 3.10] of the Weingarten map in these cases.

Of course, expectation values of products of Wilson loops can be calculated, at least in principle, once all polynomials in matrix coefficients are known. Thus, one could use Theorem 5.4.2 for the tensor representation to establish the factorization Theorem 5.3.6. On the other hand, Theorem 5.3.6 applied to well-chosen Wilson loops yields Theorem 5.4.2, showing that these two theorems are hence equivalent. The following remark explains this in more detail.

Remark 5.4.8. For every $1 \le i, j \le \dim V$, let $D : V \to V$ be defined by $D_{pq} = \delta_{pj}\delta_{qi}$ relative to a chosen basis of V. That is, D is the matrix whose only non-zero entry is in the j-th column in the i-th row. Then the generalized Wilson loop $W_l(g) = \mathrm{tr}(D\rho(g^{\pm 1}))$ evaluates to the matrix element $W_l(g) = g^{\pm 1}_{ij}$. This construction shows that the prescriptions $W_{l_k}(g) = g_{i_k j_k}$ and $W_{l'_k}(g) = g^{-1}_{j'_k i'_k}$ define generalized Wilson loops. Using (5.3.8), the merging of two such loops is given by

$$\mathcal{M}(l_r, l_s) = +K_{pj_rqj_s}\, g_{i_rp}\, g_{i_sq}, \tag{5.4.21a}$$

$$\mathcal{M}(l_r, l'_s) = -K_{pj_rj'_sq}\, g_{i_rp}\, g^{-1}_{qi'_s}, \tag{5.4.21b}$$

$$\mathcal{M}(l'_r, l'_s) = +K_{j'_rpj'_sq}\, g^{-1}_{pi'_r}\, g^{-1}_{qi'_s}. \tag{5.4.21c}$$

Thus Theorem 5.3.6 (cf., also Remark 5.3.10) implies

$$\begin{aligned}
\lambda(n+n')\, T_{ii'\,jj'}(\nu) - T_{ii'\,jj'}(\Delta\nu) = \\
&-2\sum_{\substack{r,s=1\\ r<s}}^{n} K_{pj_rqj_s} T_{ii'\,(j_1\ldots p\ldots q\ldots j_n)\,j'}(\nu)\\
&+2\sum_{r=1}^{n}\sum_{s=1}^{n'} K_{pj_rj'_sq} T_{ii'\,(j_1\ldots p\ldots j_n)(j'_1\ldots q\ldots j'_n)}(\nu)\\
&-2\sum_{\substack{r,s=1\\ r<s}}^{n'} K_{j'_rpj'_sq} T_{ii'\,j(j'_1\ldots p\ldots q\ldots j'_n)}(\nu),
\end{aligned} \tag{5.4.22}$$

where p and q always occur at the r-th and s-th position in the multi-indices, respectively. Comparing this equation with Proposition 5.4.5 establishes Theorem 5.4.2.

5.5. EXAMPLES

In this section, we explore how Theorem 5.3.6 reproduces important Wilson loop formulas from [Cha19, Jaf16] for the groups $G = \mathrm{SO}(N)$ and $\mathrm{SU}(N)$ in a straightforward way, using basic representation-theoretic information rather than the lengthy, explicit calculations employed in the cited papers. We also study equivalent Wilson loop formulas for other examples that, to the authors' knowledge, do not yet appear in the literature: the classical groups $\mathrm{Sp}(N)$ and $\mathrm{U}(N)$, and the exceptional group G_2.

5.5.1. DEFINING REPRESENTATION OF $\mathrm{SO}(N)$

Let us sketch how Theorem 5.3.6 reproduces [Cha19, Theorem 8.1] for $G = \mathrm{SO}(N)$. Consider the defining representation $\rho : \mathrm{SO}(N) \to \mathbb{R}^{N\times N}$. A basis $\{\xi^a\}$ of the associated Lie algebra $\mathfrak{so}(N) \subset \mathbb{R}^{N\times N}$ is given in terms of elementary antisymmetric matrices $\frac{1}{\sqrt{2}}\big(E_{ij} - E_{ji}\big)$ for $i \neq j$ (where $E_{ij} \in \mathbb{R}^{N\times N}$ with $(E_{ij})_{rs} = \delta_{ir}\delta_{js}$). The matrices ξ^a constitute an orthonormal basis of $\mathfrak{so}(N)$ relative to the following inner product

$$\kappa(X,Y) := -\operatorname{tr}(X\cdot Y) \quad \forall X, Y \in \mathfrak{so}(N), \tag{5.5.1}$$

which is $-\frac{1}{N-2}$ times the Killing form. A straightforward calculation gives the following completeness relation:

$$\xi^a_{ij}\xi^a_{kl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}. \tag{5.5.2}$$

5

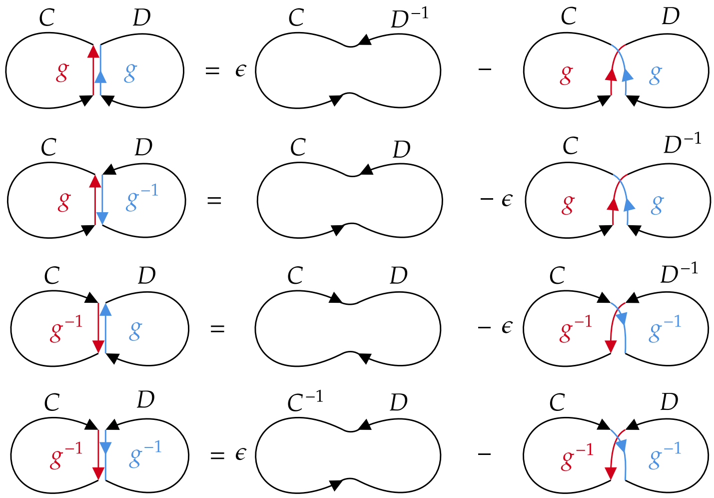

Figure 5.1: A visualization of the merging rules for the defining representations of Sp(N), SO(N), and U(N), see (5.5.3) and (5.5.10). The sets of rules only differ by the value of the scalar ϵ. One sets $\epsilon = 1$ for Sp(N) and SO(N), and $\epsilon = 0$ for U(N).

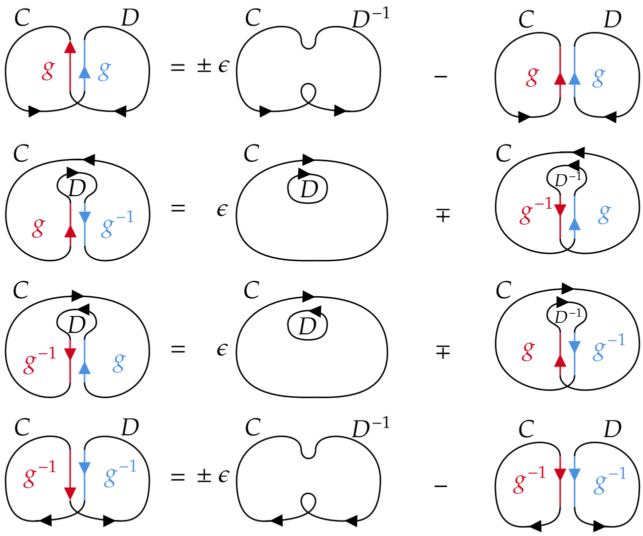

Figure 5.2: A visualization of the twisting rules for the defining representations of SO(N), Sp(N), and U(N), see (5.5.5) and (5.5.11). The sets of rules only differ by the choice of signs for the $\pm$ and $\mp$, and the value of the scalar ϵ. One chooses the upper signs for SO(N) and the lower signs for Sp(N) and U(N); further $\epsilon = 1$ for SO(N) and Sp(N), and $\epsilon = 0$ for U(N).

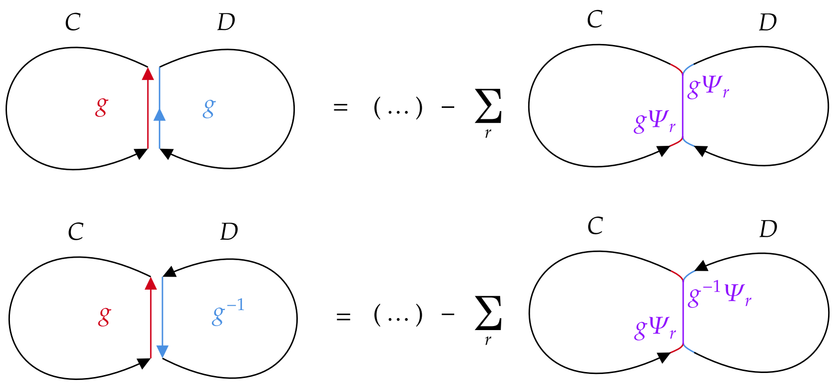

Figure 5.3: A visualization of some merging rules for the 7-dimensional irreducible representation of G_2, see (5.5.41). The ellipses represent similar merging rules as the ones for SO(N), but we also find terms which no longer are expressible as a simple linear combination of Wilson loops.

Given two Wilson loops $W_l(g) = \operatorname{tr}(Cg^{\pm 1}), W_{l'}(g) = \operatorname{tr}(Dg^{\pm 1})$ with $C, D \in G$. Then, depending on the given exponents, Equation (5.5.2) implies:

$$\mathscr{M}(W_l, W_{l'})(g) = \begin{cases} (+,+): & \operatorname{tr}(CD^{-1}) - \operatorname{tr}(CgDg), \\ (+,-): & \operatorname{tr}(CD) - \operatorname{tr}(CgD^{-1}g), \\ (-,+): & \operatorname{tr}(CD) - \operatorname{tr}(Cg^{-1}D^{-1}g^{-1}), \\ (-,-): & \operatorname{tr}(C^{-1}D) - \operatorname{tr}(Cg^{-1}Dg^{-1}), \end{cases} \tag{5.5.3}$$

This corresponds to a linear combination of what in [Cha19] is called the negative and positive mergers $W_{l\ominus_{i,j}l'}, W_{l\oplus_{i,j}l'}$ of the loops $W_l, W_{l'}$. In an analogous way, all other sums can be handled, and in their notation we find:

$$\langle dW_l, dW_{l'} \rangle = \sum_{\substack{j\in E(l),\\ j'\in E(l')}} \left(W_{l\ominus_{j,j'}l'} - W_{l\oplus_{j,j'}l'} \right). \tag{5.5.4}$$

Further, let $W_l(g) = \operatorname{tr}(Cg^{\pm 1}Dg^{\pm 1})$ be a single-argument Wilson loop with $C, D \in G$. Then we have, depending on the exponents:

$$\mathscr{T}(W_l)(g) = \begin{cases} (+,+): & \operatorname{tr}(CD^{-1}) - \operatorname{tr}(Cg)\operatorname{tr}(Dg), \\ (+,-): & -\operatorname{tr}(g^{-1}CgD^{-1}) + \operatorname{tr}(C)\operatorname{tr}(D), \\ (-,+): & -\operatorname{tr}(Cg^{-1}D^{-1}g) + \operatorname{tr}(C)\operatorname{tr}(D), \\ (-,-): & \operatorname{tr}(CD^{-1}) - \operatorname{tr}(g^{-1}C)\operatorname{tr}(g^{-1}D), \end{cases} \tag{5.5.5}$$

Again, this corresponds to linear combinations of what in [Cha19] is called *twistings* $W_{\propto_{j,j'}l}$ and *splittings* $W_{\times^1_{j,j'}}, W_{\times^2_{j,j'}}$ of W_l. Recall that the Laplacian can be written as $\Delta = \xi^a\xi^a$. Together with the completeness relation (5.5.2) this implies:

$$\Delta g_{il} = \sum_a \rho(\xi^a)_{ij}\rho(\xi^a)_{jk} g_{kl} = \left(\delta_{ij}\delta_{jk} - \delta_{ik}\delta_{jj}\right) g_{kl} = (1-N)g_{il}. \tag{5.5.6}$$

Note that by orthogonality, $(g^{-1})_{il} = g_{li}$ is itself just a generic matrix element, so the above equation also holds under the replacement $g \rightsquigarrow g^{-1}$. Thus, the Casimir eigenvalue λ equals $(1-N)$. In the defining representation of $\mathrm{SO}(N)$, Theorem 5.3.6 implies [Cha19, Theorem 8.1] as a corollary, the only differences coming from the fact that we used integration by parts *twice* rather than once in our derivation of Theorem 5.3.6.

5.5.2. DEFINING REPRESENTATION OF $\mathrm{Sp}(N)$

The compact symplectic group $\mathrm{Sp}(N) = \mathrm{Sp}(2N,\mathbb{C}) \cap \mathrm{U}(2N)$ consists of unitary $2N \times 2N$ matrices M satisfying $M^\intercal J M = J$, where $J = \left(\begin{smallmatrix} 0 & I_N \\ -I_N & 0 \end{smallmatrix}\right)$. Elements of its Lie algebra $\mathfrak{sp}(N)$ are block matrices of the form $\iota(a,b) = \left(\begin{smallmatrix} a & b \\ -\bar{b} & \bar{a} \end{smallmatrix}\right)$ with $a^* = -a$ and $b^\intercal = b$. Accordingly, a basis of $\mathfrak{sp}(N)$ is given by the following elements

$$\frac{1}{2}\iota\big(E_{ab} - E_{ba}, 0\big), \quad \frac{1}{2}\iota\big(\mathrm{i}E_{ab} + \mathrm{i}E_{ba}, 0\big), \quad \frac{1}{\sqrt{2}}\iota\big(\mathrm{i}E_{cc}, 0\big), \tag{5.5.7a}$$

$$\frac{1}{2}\iota\big(0, E_{ab} + E_{ba}\big), \quad \frac{1}{2}\iota\big(0, \mathrm{i}E_{ab} + \mathrm{i}E_{ba}\big), \quad \frac{1}{\sqrt{2}}\iota\big(0, E_{cc}\big), \quad \frac{1}{\sqrt{2}}\iota\big(0, \mathrm{i}E_{cc}\big) \tag{5.5.7b}$$

where $1 \le a < b \le N$ and $1 \le c \le N$ and where the matrix E_{ab} is defined by $(E_{ab})_{kl} = \delta_{ak}\delta_{bl}$ as before. These elements form an orthonormal basis with respect to the inner product

$$\kappa(X,Y) = -\operatorname{tr}(XY), \qquad X, Y \in \mathfrak{sp}(N), \tag{5.5.8}$$

which is $-\frac{1}{2N+2}$ times the Killing form. A direct calculation shows that the completeness relation for $\mathfrak{sp}(N)$ reads

$$K_{ijkl} = J_{ik}J_{jl} - \delta_{il}\delta_{jk}, \tag{5.5.9}$$

see also [Dah17, Appendix A].

Thus, the merging of the loops $W_l(g) = \operatorname{tr}(Cg^\sigma)$ and $W_{l'}(g) = \operatorname{tr}(Dg^\varsigma)$ with $C, D \in \mathrm{Sp}(N)$ is given by

$$\mathcal{M}(W_l, W_{l'})(g) = \begin{cases} (+,+): & \operatorname{tr}(CD^{-1}) - \operatorname{tr}(CgDg), \\ (+,-): & \operatorname{tr}(CD) - \operatorname{tr}(CgD^{-1}g), \\ (-,+): & \operatorname{tr}(CD) - \operatorname{tr}(Cg^{-1}D^{-1}g^{-1}), \\ (-,-): & \operatorname{tr}(C^{-1}D) - \operatorname{tr}(Cg^{-1}Dg^{-1}), \end{cases} \tag{5.5.10}$$

depending on the signatures (σ,ς). Similarly the twisting of the loop $W_l(g) = \mathrm{tr}(Cg^{\sigma}Dg^{\varsigma})$ with $C,D \in \mathrm{Sp}(N)$ takes the form

$$\mathcal{T}(W_l)(g) = \begin{cases} (+,+): & -\mathrm{tr}(CD^{-1}) - \mathrm{tr}(Cg)\,\mathrm{tr}(Dg), \\ (+,-): & \mathrm{tr}(CgD^{-1}g^{-1}) + \mathrm{tr}(C)\,\mathrm{tr}(D), \\ (-,+): & \mathrm{tr}(Cg^{-1}D^{-1}g) + \mathrm{tr}(C)\,\mathrm{tr}(D), \\ (-,-): & -\mathrm{tr}(CD^{-1}) - \mathrm{tr}(Cg^{-1})\,\mathrm{tr}(Dg^{-1}). \end{cases} \tag{5.5.11}$$

Comparing (5.5.3) and (5.5.10), we find that the rules for merging of SO(N) and Sp(N) are completely identical, whereas the twisting rules (5.5.5) and (5.5.11) are only *almost* identical, differing by a single sign per equation.

Finally, we find the Casimir eigenvalue by the following calculation:

$$\Delta g_{il} = (J_{ij}J_{jk} - \delta_{ik}\delta_{jj})g_{kl} = -(1+2N)g_{il}. \tag{5.5.12}$$

5

5.5.3. DEFINING REPRESENTATION OF U(N)

The Lie algebra of U(N) is the set of all skew-hermitian $N\times N$-matrices and a basis is given by the following elements:

$$\left\{\tfrac{i}{\sqrt{2}}\left(E^{ab}+E^{ba}\right)\right\}_{a<b} \cup \left\{\tfrac{1}{\sqrt{2}}\left(E^{ab}-E^{ba}\right)\right\}_{a<b} \cup \left\{iE^{aa}\right\}_{1\le a\le N} \tag{5.5.13}$$

where $E_{ab} \in \mathbb{R}^{N\times N}$ with $(E_{ab})_{rs} = \delta_{ar}\delta_{bs}$. The symmetric Ad-invariant bilinear form

$$\kappa(X,Y) := -\mathrm{tr}(XY), \quad \forall X,Y \in \mathfrak{u}(N), \tag{5.5.14}$$

is positive-definite because $\kappa(X,X) = \sum_{ij}|X_{ij}|^2$. Note that κ is not a scalar multiple of the Killing form of U(N) as the latter is degenerate. The basis introduced above is orthonormal with respect to κ.

A direct calculation shows that the completeness relation for $\mathfrak{u}(N)$ reads

$$K_{ijkl} = -\delta_{il}\delta_{jk}. \tag{5.5.15}$$

According to (5.3.8), the merging of two Wilson loops $W_l(g) = \mathrm{tr}(Cg^{\pm1})$, $W_{l'}(g) = \mathrm{tr}(Dg^{\pm1})$ with $C,D \in \mathrm{U}(N)$ is thus given by

$$\mathcal{M}(W_l,W_{l'})(g) = \begin{cases} (+,+): & -\mathrm{tr}(CgDg), \\ (+,-): & +\mathrm{tr}(CD), \\ (-,+): & +\mathrm{tr}(CD), \\ (-,-): & -\mathrm{tr}(Cg^{-1}Dg^{-1}). \end{cases} \tag{5.5.16}$$

Furthermore, by (5.3.20), the twisting of a Wilson loop $W_l(g) = \mathrm{tr}(Cg^{\pm1}Dg^{\pm1})$ with

$C, D \in \mathrm{U}(N)$ is:

$$\mathscr{T}(W_l)(g) = \begin{cases} (+,+): & -\operatorname{tr}(Cg)\operatorname{tr}(Dg), \\ (+,-): & +\operatorname{tr}(C)\operatorname{tr}(D), \\ (-,+): & +\operatorname{tr}(C)\operatorname{tr}(D), \\ (-,-): & -\operatorname{tr}(Cg^{-1})\operatorname{tr}(Dg^{-1}). \end{cases} \tag{5.5.17}$$

By (5.3.22), for the eigenvalue λ of the Laplace operator, we obtain

$$\lambda\delta_{ij} = K_{ikkj} = -N\delta_{ij}. \tag{5.5.18}$$

We now discuss the definition of the unitary Weingarten map based on our general result Theorem 5.4.6. For this we first recall a few well-known facts concerning the representation theory of the symmetric group S_n. By, e.g., [GW09, Section 9.1] the isomorphism classes of irreducible representations G_λ of S_n are bijectively indexed by partitions λ of n. Moreover, the group algebra of S_n decomposes as a direct sum

5

$$\mathbb{C}S_n \simeq \bigoplus_{\lambda\vdash n} \operatorname{End}(G_\lambda) \tag{5.5.19}$$

of simple algebras, see [GW09, Section 9.3.2]. Let $p_\lambda \in \mathbb{C}S_n$ be the minimal central idempotent that under this isomorphism acts by the identity on G_λ and by zero on the other components.

Let $V = \mathbb{C}^N$ be the fundamental representation of $G = \mathrm{U}(N)$. Every permutation $\sigma \in S_n$ acts on $V^{\otimes n}$ by mapping $v_1 \otimes \cdots \otimes v_n$ to $v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}$. Let $\tau : \mathbb{C}S_n \to \operatorname{End}(V^{\otimes n})$ be the linear extension of this representation. A moment's reflection convinces us that $\tau(\sigma)^* = \tau(\sigma^{-1})$. The Schur–Weyl theorem shows that the image of τ coincides with the set of invariants $\big(V^{\otimes n} \otimes (V^*)^{\otimes n}\big)^G \simeq \operatorname{End}_G(V^{\otimes n})$. Thus we are in the position to apply Theorem 5.4.6. For this purpose, endow $\mathbb{C}S_n$ with an inner product by declaring the basis $\sigma \in S_n$ to be orthonormal. Then a simple calculation shows that the adjoint of τ with respect to the inner pairing $\langle S_1, S_2\rangle = \operatorname{tr}(S_2^* S_1)$ on $\operatorname{End}(V^{\otimes n})$ is

$$\tau^*(S) = \sum_{\varsigma\in S_n} \operatorname{tr}\big(\tau(\varsigma^{-1})S\big)\,\varsigma. \tag{5.5.20}$$

According to Remark 5.4.7 the Weingarten map can be obtained by diagonalizing the operator

$$\tau^* \circ \tau(\sigma) = \sum_{\varsigma\in S_n} \operatorname{tr}\big(\tau(\varsigma^{-1})\tau(\sigma)\big)\,\varsigma = \sum_{\varsigma\in S_n} \operatorname{tr}\big(\tau(\varsigma^{-1}\sigma)\big)\,\varsigma = \sigma \sum_{\varsigma\in S_n} \operatorname{tr}\big(\tau(\varsigma)\big)\,\varsigma. \tag{5.5.21}$$

In fact, it turns out that $\sum_{\varsigma\in S_n} \operatorname{tr}\big(\tau(\varsigma)\big)\,\varsigma = \sum_{\lambda\vdash n} k_\lambda\, p_\lambda$ for some constants k_λ. In other words, the decomposition (5.5.19) is the eigenspace decomposition of $\tau^* \circ \tau$ with $\{k_\lambda\}$ as the associated eigenvalues. The above identity can be established in two different ways:

- First, we can use Schur–Weyl duality again to express $\varsigma \mapsto \operatorname{tr}\bigl(\tau(\varsigma)\bigr)$ in terms of the characters χ_λ of the irreducible representation G_λ and the dimension of the associated Weyl module F_λ^N with highest weight λ. Then the relation $p_\lambda = \frac{\dim G_\lambda}{n!}\chi_\lambda$, see [GW09, Theorem 9.3.10], yields

$$\sum_{\varsigma\in S_n} \operatorname{tr}\bigl(\tau(\varsigma)\bigr)\varsigma = n! \sum_{\substack{\lambda\vdash n\\ l(\lambda)\le N}} \frac{\dim F_\lambda^N}{\dim G_\lambda} p_\lambda\,, \tag{5.5.22}$$

where $l(\lambda)$ is the number of parts of the partition λ. This is the approach taken by [CS06, Proposition 2.3.2].

- Secondly, since $\tau(\varsigma)$ is a permutation matrix, $\operatorname{tr}\bigl(\tau(\varsigma)\bigr)$ coincides with the dimension of the set of fixed points. Thus,

$$\sum_{\varsigma\in S_n} \operatorname{tr}\bigl(\tau(\varsigma)\bigr)\varsigma = \sum_{\varsigma\in S_n} N^{\sharp\varsigma}\varsigma, \tag{5.5.23}$$

where $\sharp\varsigma$ is the number of cycles of ς. This equality can be further simplified by using the Jucys–Murphy elements X_k. In fact, $\sum_{\varsigma\in S_n} N^{\sharp\varsigma}\varsigma = \prod_{k=1}^n (N+X_k)$. Now using the fact the Gelfand–Tsetlin vectors indexed by standard Young tableaus are joint eigenvectors of the Jucys–Murphy elements one gets

$$\sum_{\varsigma\in S_n} \operatorname{tr}\bigl(\tau(\varsigma)\bigr)\varsigma = \sum_{\substack{\lambda\vdash n\\ l(\lambda)\le N}} \prod_{(i,j)\in\lambda} (N+j-i)\, p_\lambda\,. \tag{5.5.24}$$

The Hook length formula gives an expression for the dimensions of F_λ^N and G_λ, implying equality of the eigenvalues with the above description. Following this route leads to the relation of the Weingarten map with the Jucys–Murphy elements discovered in [Nov10, Theorem 1.1] and [ZJ10, Proposition 2].

Thus, in summary, Theorem 5.4.6 in combination with Remark 5.4.7 recovers [Col03, Theorem 2.1], [CS06, Corollary 2.4] in the following form.

Theorem 5.5.1. *Let $\rho : \mathrm{U}(N) \to \mathbb{C}^N$ be the fundamental representation of $\mathrm{U}(N)$. For non-negative integers n, n', define*

$$T^{n,n'}(S) = \int_G \rho^{\otimes n}(g)\circ S \circ \rho^{\otimes n'}(g)\,\mathrm{d}g \tag{5.5.25}$$

for $S \in \operatorname{Hom}\bigl((\mathbb{C}^N)^{\otimes n'}, (\mathbb{C}^N)^{\otimes n}\bigr)$. Then $T^{n,n'} = 0$ if $n \neq n'$, and otherwise

$$T^{n,n} = \tau\circ \mathrm{Wg}\circ\tau^*, \tag{5.5.26}$$

where τ and τ^* are defined above and $\mathrm{Wg} : \mathbb{C}S_n \to \mathbb{C}S_n$ is given by

$$\mathrm{Wg}(\sigma) = \sum_{\substack{\lambda\vdash n\\ l(\lambda)\le N}} \prod_{(i,j)\in\lambda} (N+j-i)^{-1}\, p_\lambda\,. \tag{5.5.27}$$

5.5.4. Defining representation of SU(N)

In [Jaf16], Wilson loop identities of the shape of Theorem 5.3.6 are derived for SU(N), using Stein's method and many technical, auxiliary lemmas. In comparison, we will see that our framework allows us to drastically reduce the amount of necessary calculation needed to arrive at the same conclusion.

Consider SU(N), with the Lie algebra $\mathfrak{su}(N)$ of skew-hermitian matrices with vanishing trace, and inner product

$$\kappa(X,Y) := -\operatorname{tr}(XY) \quad \forall X,Y \in \mathfrak{su}(N), \tag{5.5.28}$$

which is a renormalization of the Killing form by a factor $-\frac{1}{2N}$. By [BK08], an orthonormal basis $\{\xi^a\}$ is given by

$$\begin{aligned}\left\{\tfrac{i}{\sqrt{2}}\left(E^{jk}+E^{kj}\right)\right\}_{j<k} &\cup \left\{\tfrac{1}{\sqrt{2}}\left(E^{jk}-E^{kj}\right)\right\}_{j<k} \\ &\cup \left\{\tfrac{i}{\sqrt{2l(l+1)}}\left(\sum_{j=1}^{l}(E^{jj}-E^{l+1,l+1}\right)\right\}_{1\le l\le N-1}.\end{aligned} \tag{5.5.29}$$

The corresponding completeness relation turns out to be

$$\xi^a_{ij}\xi^a_{kl} = -\delta_{il}\delta_{jk} + \frac{1}{N}\delta_{ij}\delta_{kl}. \tag{5.5.30}$$

Again, the merging of two loops $W_l(g) = \operatorname{tr}(Cg^{\pm 1})$ and $W_{l'} = \operatorname{tr}(Dg^{\pm 1})$ turns out to be, depending on the exponents:

$$\mathscr{M}(W_l, W_{l'})(g) = \begin{cases} (+,+): & -\operatorname{tr}(CgDg) + \frac{1}{N}\operatorname{tr}(Cg)\operatorname{tr}(Dg), \\ (+,-): & +\operatorname{tr}(CD) - \frac{1}{N}\operatorname{tr}(Cg)\operatorname{tr}(Dg^{-1}), \\ (-,+): & +\operatorname{tr}(CD) - \frac{1}{N}\operatorname{tr}(Cg^{-1})\operatorname{tr}(Dg), \\ (-,-): & -\operatorname{tr}(g^{-1}Cg^{-1}D) + \frac{1}{N}\operatorname{tr}(Cg^{-1})\operatorname{tr}(Dg^{-1}). \end{cases} \tag{5.5.31}$$

These expressions are linear combinations of what, in [Jaf16], are called *positive mergers* $W_{l\oplus_{j,j'}l'}$, *negative mergers* $W_{l\ominus_{j,j'}l'}$ and the product of the unchanged loops $W_l \cdot W_{l'}$.

Similarly, the twisting of a loop $W_l(g) = \operatorname{tr}(Cg^{\pm 1}Dg^{\pm 1})$ yields:

$$\mathscr{T}(W_l)(g) = \begin{cases} (+,+): & -\operatorname{tr}(Cg)\operatorname{tr}(Dg) + \frac{1}{N}\operatorname{tr}(CgDg), \\ (+,-): & +\operatorname{tr}(C)\operatorname{tr}(D) - \frac{1}{N}\operatorname{tr}(CgDg^{-1}), \\ (-,+): & +\operatorname{tr}(C)\operatorname{tr}(D) - \frac{1}{N}\operatorname{tr}(Cg^{-1}Dg), \\ (-,-): & -\operatorname{tr}(g^{-1}C)\operatorname{tr}(g^{-1}D) + \frac{1}{N}\operatorname{tr}(g^{-1}Cg^{-1}D). \end{cases} \tag{5.5.32}$$

Similarly, these expressions are linear combinations of what Jafarov calls the *splitting* $W_{\times^1_{j,j'}} W_{\times^2_{j,j'}}$ and the original unchanged loop W_l.

Finally, the Casimir eigenvalue λ is gotten via

$$\lambda g_{ij} \stackrel{!}{=} \Delta g_{ij} = \left(-\delta_{ik}\delta_{ll} + \frac{1}{N}\delta_{ik}\right) g_{kj} = \left(-N + \frac{1}{N}\right) g_{ij}, \tag{5.5.33}$$

Inserting this information into Theorem 5.3.6 reproduces a version of [Jaf16, Theorem 8.1], the only differences coming from the fact that we used integration by parts *twice* rather than once in our derivation of Theorem 5.3.6.

5.5.5. IRREDUCIBLE 7-DIMENSIONAL REPRESENTATION OF G_2

In all previous examples, the merging and twisting of two Wilson loops were polynomials of Wilson loops again. This is not true in full generality, as we will demonstrate by considering the exceptional, real, compact Lie group G_2.

This Lie group is of dimension 14, and its smallest nontrivial irreducible representation (ρ, V) is of dimension 7. We cite from [Sch88] a construction of this representation based on the octonions $\mathbb{O}$. Recall the 8-dimensional, real, non-associative division algebra given by the octonions $\mathbb{O}$. With respect to the standard basis $\{e_i\}_{i=0}^{7}$ of $\mathbb{O}$, the multiplication is specified by

$$e_i e_j = \begin{cases} e_i & \text{if } j = 0, \\ e_j & \text{if } i = 0, \\ -\delta_{ij} e_0 + \psi_{ijk} e_k & \text{else,} \end{cases} \tag{5.5.34}$$

where $\{\psi_{ijk}\}_{i,j,k\in\{1,\ldots,7\}}$ denotes a totally antisymmetric symbol, assuming the value 1 on the ordered triples

$$(i,j,k) = (1,2,3),\ (1,4,7),\ (1,6,5),\ (2,4,6),\ (2,5,7),\ (3,5,4),\ (3,6,7), \tag{5.5.35}$$

and zero on all (i,j,k) which do not arise from the above triples by permutation. This algebra admits a linear involution by extension of

$$\overline{e_i} = \begin{cases} e_0 & \text{if } i = 0, \\ -e_i & \text{if } i > 0, \end{cases} \tag{5.5.36}$$

and a linear tracial map $\tau_{\mathbb{O}} : \mathbb{O} \to \mathbb{R}$ by extension of

$$\tau_{\mathbb{O}}(e_i) := \delta_{i0}. \tag{5.5.37}$$

Now G_2 is the Lie group of algebra automorphisms of $\mathbb{O}$. Set $V := \ker \tau_{\mathbb{O}}$, then G_2 acts irreducibly and unitarily on V with respect to the inner product

$$B(x,y) := \tau_{\mathbb{O}}(\overline{x} y) \quad \forall x, y \in V.$$

In [Mac01], an explicit basis $\{H^a\}_{a=1,\dots,14}$ of $\mathfrak{g}_2 := \mathrm{Lie}(G_2)$ as a subalgebra of $\mathfrak{su}(7)$ is constructed, fulfilling the following[5]:

$$\mathrm{tr}(H^a H^b) = \delta_{ab}, \quad (\overline{H^a})^T = -H^a, \quad \forall a, b \in \{1, \dots, 14\}. \tag{5.5.38}$$

These generators constitute an orthonormal basis with respect to the inner product

$$\kappa(X, Y) := \mathrm{tr}(\overline{X}^T Y). \tag{5.5.39}$$

By [Mac01], the completeness relation of $\mathfrak{g}_2$ is then equal to

$$(H^a)_{ij}(H^a)_{kl} = \frac{1}{2}\left(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}\right) - \frac{1}{6}\psi_{rij}\psi_{rkl}. \tag{5.5.40}$$

In the basis $\{e_i\}_{i\in\{1,\dots,7\}}$ of V, define the endomorphisms $\Psi_r := (\psi_{rij})_{i,j\in\{1,\dots,7\}} \in \mathrm{End}(V)$ for $r \in \{1, \dots, 7\}$. The merging of two Wilson loops $W_l(g) = \mathrm{tr}(Cg^{\pm 1})$, $W_{l'}(g) = \mathrm{tr}(Dg^{\pm 1})$ with $C, D \in G_2$ can be expressed as

$$\mathcal{M}(W_l, W_{l'})(g) = \begin{cases} (+,+): \frac{1}{2}\left(\mathrm{tr}(CD^{-1}) - \mathrm{tr}(CgDg)\right) - \frac{1}{6}\mathrm{tr}(Cg\Psi_r)\,\mathrm{tr}(Dg\Psi_r), \\ (+,-): \frac{1}{2}\left(\mathrm{tr}(CD) - \mathrm{tr}(CgD^{-1}g)\right) - \frac{1}{6}\mathrm{tr}(Cg\Psi_r)\,\mathrm{tr}(D\Psi_r g^{-1}), \\ (-,+): \frac{1}{2}\left(\mathrm{tr}(CD) - \mathrm{tr}(Cg^{-1}D^{-1}g^{-1})\right) - \frac{1}{6}\mathrm{tr}(C\Psi_r g^{-1})\,\mathrm{tr}(Dg\Psi_r), \\ (-,-): \frac{1}{2}\left(\mathrm{tr}(C^{-1}D) - \mathrm{tr}(Cg^{-1}Dg^{-1})\right) - \frac{1}{6}\mathrm{tr}(C\Psi_r g^{-1})\,\mathrm{tr}(D\Psi_r g^{-1}). \end{cases} \tag{5.5.41}$$

Here and in the following, summation over the common index r is understood.
The twisting of $W_l(g) = \mathrm{tr}(Cg^{\pm 1}Dg^{\pm 1})$ with $C, D \in G_2$ is given by

$$\mathcal{T}(W_l)(g) = \begin{cases} (+,+): \frac{1}{2}\left(\mathrm{tr}(CD^{-1}) - \mathrm{tr}(Cg)\,\mathrm{tr}(Dg)\right) - \frac{1}{6}\mathrm{tr}(Cg\Psi_r)\,\mathrm{tr}(Dg\Psi_r), \\ (+,-): \frac{1}{2}\left(-\mathrm{tr}(g^{-1}CgD^{-1}) + \mathrm{tr}(C)\,\mathrm{tr}(D)\right) - \frac{1}{6}\mathrm{tr}(Cg\Psi_r)\,\mathrm{tr}(D\Psi_r g^{-1}), \\ (-,+): \frac{1}{2}\left(-\mathrm{tr}(Cg^{-1}D^{-1}g) + \mathrm{tr}(C)\,\mathrm{tr}(D)\right) - \frac{1}{6}\mathrm{tr}(C\Psi_r g^{-1})\,\mathrm{tr}(Dg\Psi_r), \\ (-,-): \frac{1}{2}\left(\mathrm{tr}(CD^{-1}) - \mathrm{tr}(g^{-1}C)\,\mathrm{tr}(g^{-1}D)\right) - \frac{1}{6}\mathrm{tr}(C\Psi_r g^{-1})\,\mathrm{tr}(D\Psi_r g^{-1}), \end{cases} \tag{5.5.42}$$

It seems unlikely that the expressions involving the matrices Ψ_r can be simplified any further, and as such, we do not have a polynomial of Wilson loops, but only of generalized Wilson loops in the sense of Definition 5.2.2.
We can also apply Theorem 5.4.6 and Remark 5.4.7 to G_2 to calculate certain Weingarten functions: Denote by $G_2^{\mathbb{C}}$ the complex Lie group given by the automorphisms of the complexified octonion algebra. It also assumes an irreducible, unitary representation

[5] Our choice of generators $\{H^a\}$ differs from the ones in [Mac01] by a factor of $i/\sqrt{2}$, in order to achieve orthonormality and since we need to view $\mathfrak{su}(7)$ as a Lie algebra of *skew-hermitian* rather than hermitian matrices to achieve the right behavior under exponentiation.

on the complexification $V_{\mathbb{C}} = (\ker \tau_{\mathbb{O}})_{\mathbb{C}}$ and in [Sch88], generators for invariant spaces $(V_{\mathbb{C}}^{\otimes n} \otimes_{\mathbb{C}} (V_{\mathbb{C}}^*)^{\otimes n'})^{G_2^{\mathbb{C}}}$ have been calculated. Due to connectedness of $G_2^{\mathbb{C}}$, we have for every $G_2^{\mathbb{C}}$-module M

$$M^{G_2^{\mathbb{C}}} = M^{\mathrm{Lie}(G_2^{\mathbb{C}})} = M^{(\mathfrak{g}_2)_{\mathbb{C}}},$$

denoting by $(\mathfrak{g}_2)_{\mathbb{C}}$ the complexification of the Lie algebra $\mathfrak{g}_2$. As such, we can deduce from this the G_2 invariants $(V^{\otimes n} \otimes (V^*)^{\otimes n'})^{G_2} = (V^{\otimes n} \otimes (V^*)^{\otimes n'})^{\mathfrak{g}_2}$.
The representation space V admits an invariant, non-degenerate bilinear form

$$\alpha : V \otimes V \to \mathbb{R}, \quad e_i \otimes e_j \mapsto -\tau_{\mathbb{O}}(e_i e_j),$$

hence $V \cong V^*$ as G_2-modules and one may restrict to the invariants $(V^{\otimes n})^{G_2}$ without loss of generality.
For simplicity, let us study the case $n = 2$. An analysis of $n > 2$ is possible, but finding a basis of $(V^{\otimes n})^{G_2}$ becomes more difficult due to the presence of nontrivial relations between the generators determined in [Sch88]. Now, $(V \otimes V)^{G_2}$ is one-dimensional and is generated by $u := \sum_{i=1}^7 e_i \otimes e_i$. This element is dual to $\alpha \in V^* \otimes V^*$ due to the relation $\tau_{\mathbb{O}}(e_i e_j) = -\delta_{ij}$ for $1 \leq i, j \leq 7$. We set $\mathscr{A} = \mathbb{R}u \subset V \otimes V$, equip this subspace with the inner product $\langle u, u \rangle = 1$, and define $\tau : \mathscr{A} \to V \otimes V$ to be the embedding of this subspace. The adjoint τ^* with respect to the scalar product on $V \otimes V$ defined in Theorem 5.4.6 is given by

$$\tau^*(e_i \otimes e_j) = \delta_{ij} u, \quad \forall 1 \leq i, j \leq 7,$$

so $(\tau^* \tau)(u) = 7u$. Hence, by Remark 5.4.7, we find

$$\mathrm{Wg} = \frac{1}{7} \mathrm{id}_{\mathscr{A}}. \tag{5.5.43}$$

Now, Theorem 5.4.6 allows us to deduce that for all $1 \leq i, j \leq 7$ we have

$$\int_{G_2} \rho^{\otimes 2}(g)(e_i \otimes e_j) \, \mathrm{d}g = (\tau \circ \mathrm{Wg} \circ \tau^*)(e_i \otimes e_j) = \frac{1}{7} \delta_{ij} \sum_{k=1}^{7} e_k \otimes e_k$$

or, as a scalar integral,

$$\int_{G_2} \rho(g)_{i_1 j_1} \rho(g)_{i_2 j_2} \, \mathrm{d}g = \frac{1}{7} \delta_{i_1 i_2} \delta_{j_1 j_2}.$$

REFERENCES

[AGZ10] Greg W. Anderson, Alice Guionnet, and Ofer Zeitouni. *An introduction to random matrices*, volume 118 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2010.

5

[BB96] P. W. Brouwer and C. W. J. Beenakker. Diagrammatic method of integration over the unitary group, with applications to quantum transport in mesoscopic systems. *J. Math. Phys.*, 37(10):4904–4934, 1996.

[BG18] Riddhipratim Basu and Shirshendu Ganguly. SO(N) lattice gauge theory, planar and beyond. *Comm. Pure Appl. Math.*, 71(10):2016–2064, 2018.

[BK08] Reinhold A. Bertlmann and Philipp Krammer. Bloch vectors for qudits. *J. Phys. A*, 41(23):235303, 21, 2008.

[Bou05] Nicolas Bourbaki. *Lie groups and Lie algebras. Chapters 7–9*. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 2005. Translated from the 1975 and 1982 French originals by Andrew Pressley.

[BtD95] Theodor Bröcker and Tammo tom Dieck. *Representations of compact Lie groups*, volume 98 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. Translated from the German manuscript, Corrected reprint of the 1985 translation.

[Cha19] Sourav Chatterjee. Rigorous solution of strongly coupled $SO(N)$ lattice gauge theory in the large N limit. *Comm. Math. Phys.*, 366(1):203–268, 2019.

[Cha21] Sourav Chatterjee. A probabilistic mechanism for quark confinement. *Comm. Math. Phys.*, 385(2):1007–1039, 2021.

[CHJLY17] Jordan Cotler, Nicholas Hunter-Jones, Junyu Liu, and Beni Yoshida. Chaos, complexity, and random matrices. *J. High Energy Phys.*, (11):048, front matter+58, 2017.

[CJ16] Sourav Chatterjee and Jafar Jafarov. The $1/N$ expansion for SO(N) lattice gauge theory at strong coupling. *arXiv preprint arXiv:1604.04777*, 2016.

[CN16] Benoît Collins and Ion Nechita. Random matrix techniques in quantum information theory. *J. Math. Phys.*, 57(1):015215, 34, 2016.

[Col03] Benoît Collins. Moments and cumulants of polynomial random variables on unitary groups, the Itzykson-Zuber integral, and free probability. *Int. Math. Res. Not.*, (17):953–982, 2003.

[CS06] Benoît Collins and Piotr Śniady. Integration with respect to the Haar measure on unitary, orthogonal and symplectic group. *Comm. Math. Phys.*, 264(3):773–795, 2006.

[Cvi76] Predrag Cvitanović. Group theory for Feynman diagrams in non-Abelian gauge theories. *Phys. Rev. D (3)*, 14(6):1536–1553, 1976.

[Dah17] Antoine Dahlqvist. Integration formulas for Brownian motion on classical compact Lie groups. *Ann. Inst. Henri Poincaré Probab. Stat.*, 53(4):1971–1990, 2017.

[DE01] Persi Diaconis and Steven N. Evans. Linear functionals of eigenvalues of random matrices. *Trans. Amer. Math. Soc.*, 353(7):2615–2633, 2001.

[DFGZJ95] P. Di Francesco, P. Ginsparg, and J. Zinn-Justin. 2D gravity and random matrices. *Phys. Rep.*, 254(1-2):133, 1995.

[DM22] Tobias Diez and Lukas Miaskiwskyi. Expectation values of polynomials and moments on general compact lie groups. *arXiv preprint arXiv:2203.11607*, 2022.

[Dri89] Bruce K. Driver. YM_2: continuum expectations, lattice convergence, and lassos. *Comm. Math. Phys.*, 123(4):575–616, 1989.

[DS94] Persi Diaconis and Mehrdad Shahshahani. On the eigenvalues of random matrices. *J. Appl. Probab.*, 31A:49–62, 1994. Studies in applied probability.

[GKS89] Leonard Gross, Christopher King, and Ambar Sengupta. Two-dimensional Yang-Mills theory via stochastic differential equations. *Ann. Physics*, 194(1):65–112, 1989.

[GMGW98] Thomas Guhr, Axel Müller-Groeling, and Hans A. Weidenmüller. Random-matrix theories in quantum physics: common concepts. *Phys. Rep.*, 299(4-6):189–425, 1998.

[GW09] Roe Goodman and Nolan R. Wallach. *Symmetry, representations, and invariants*, volume 255 of *Graduate Texts in Mathematics*. Springer, Dordrecht, 2009.

[HR03] C. P. Hughes and Z. Rudnick. Mock-Gaussian behaviour for linear statistics of classical compact groups. *J. Phys. A*, 36(12):2919–2932, 2003. Random matrix theory.

[Hun56] G. A. Hunt. Semi-groups of measures on Lie groups. *Trans. Amer. Math. Soc.*, 81:264–293, 1956.

[Itô50] Kiyosi Itô. Brownian motions in a Lie group. *Proc. Japan Acad.*, 26(8):4–10, 1950.

[Jaf16] Jafar Jafarov. Wilson loop expectations in SU(N) lattice gauge theory. *arXiv preprint arXiv:1610.03821*, 2016.

5

[Kna02] Anthony W. Knapp. *Lie groups beyond an introduction*, volume 140 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, second edition, 2002.

[Lia04] Ming Liao. *Lévy processes in Lie groups*, volume 162 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2004.

[Lé08] Thierry Lévy. Schur-Weyl duality and the heat kernel measure on the unitary group. *Adv. Math.*, 218(2):537–575, 2008.

[Mac01] A. J. Macfarlane. Lie algebra and invariant tensor technology for g_2. *Internat. J. Modern Phys. A*, 16(18):3067–3097, 2001.

[MM97] István Montvay and Gernot Münster. *Quantum fields on a lattice*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge, 1997. Reprint of the 1994 original.

[MN13] Sho Matsumoto and Jonathan Novak. Jucys-Murphy elements and unitary matrix integrals. *Int. Math. Res. Not. IMRN*, (2):362–397, 2013.

[Nov10] Jonathan I. Novak. Jucys-Murphy elements and the unitary Weingarten function. In *Noncommutative harmonic analysis with applications to probability II*, volume 89 of *Banach Center Publ.*, pages 231–235. Polish Acad. Sci. Inst. Math., Warsaw, 2010.

[PV04] L. Pastur and V. Vasilchuk. On the moments of traces of matrices of classical groups. *Comm. Math. Phys.*, 252(1-3):149–166, 2004.

[RS17] Gerd Rudolph and Matthias Schmidt. *Differential geometry and mathematical physics. Part II*. Theoretical and Mathematical Physics. Springer, Dordrecht, 2017. Fibre bundles, topology and gauge fields.

[RW94] L. C. G. Rogers and David Williams. *Diffusions, Markov processes, and martingales. Vol. 1*. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. John Wiley & Sons, Ltd., Chichester, second edition, 1994. Foundations.

[Sam80] Stuart Samuel. $\mathrm{U}(N)$ integrals, $1/N$, and the De Wit-'t Hooft anomalies. *J. Math. Phys.*, 21(12):2695–2703, 1980.

[Sch88] Gerald W. Schwarz. Invariant theory of G_2 and Spin_7. *Comment. Math. Helv.*, 63(4):624–663, 1988.

[Sen08] Ambar N. Sengupta. Traces in two-dimensional QCD: the large-N limit. In *Traces in number theory, geometry and quantum fields*, Aspects Math., E38, pages 193–212. Friedr. Vieweg, Wiesbaden, 2008.

[tH74] G. 't Hooft. A planar diagram theory for strong interactions. *Nuclear Physics B*, 72(3):461–473, April 1974.

[Wei78] Don Weingarten. Asymptotic behavior of group integrals in the limit of infinite rank. *J. Mathematical Phys.*, 19(5):999–1001, 1978.

[Xu97] Feng Xu. A random matrix model from two-dimensional Yang-Mills theory. *Comm. Math. Phys.*, 190(2):287–307, 1997.

[Yos52] Kôsaku Yosida. On Brownian motion in a homogeneous Riemannian space. *Pacific J. Math.*, 2:263–270, 1952.

[ZJ10] Paul Zinn-Justin. Jucys-Murphy elements and Weingarten matrices. *Lett. Math. Phys.*, 91(2):119–127, 2010.

Appendices

A

Cosheaves and Čech homology

In this appendix, we will recall the definition of (pre-)cosheaves, Čech homology of precosheaves and simple properties thereof from [Bre97]. For the remainder of this section, fix a topological space X.

Definition A.1. [Bre97, Chapter V.1]

i) A *precosheaf* (of $\mathbb{R}$-vector spaces) $\mathscr{P}$ on X is a covariant functor from the category of open sets of X, morphisms given by inclusions, into the category of $\mathbb{R}$-vector spaces. Given an inclusion $U \subset V$ of open sets, we denote the associated mapping $\mathscr{P}(U) \to \mathscr{P}(V)$ by ι_U^V, called the *extension map* from U to V of the precosheaf $\mathscr{P}$.

ii) A *cosheaf* is a precosheaf $\mathscr{P}$ with the property that for every open cover $\mathscr{U} = \{U_i\}_{i \in I}$ of an open set $U \subset X$, the sequence

$$\bigoplus_{i,j} \mathscr{P}(U_i \cap U_j) \to \bigoplus_i \mathscr{P}(U_i) \to \mathscr{P}(U) \to 0 \tag{A.1}$$

is exact, where the maps are given by

$$(a_{ij})_{i,j} \mapsto \left(\sum_j \iota^{U_i}_{U_i \cap U_j}(a_{ij} - a_{ji}) \right)_i, \quad (b_i)_i \mapsto \sum_i \iota^U_{U_i} b_i. \tag{A.2}$$

iii) A *morphism of (pre-)cosheaves* is a natural transformation between the functors defining the (pre-)cosheaves.

Unless mentioned otherwise, all our (pre-)cosheaves take values in the category of $\mathbb{R}$-vector spaces.

A

Definition A.2. Let S be a precosheaf over a topological space X, and $\mathscr{U} = \{U_\alpha\}$ an open cover of X. Write $\alpha := (\alpha_1, \dots, \alpha_{p+1})$ for the p-simplex defined by a collection of indices α_i, and write

$$U_\alpha := U_{\alpha_1} \cap \dots \cap U_{\alpha_{p+1}}. \tag{A.3}$$

Define for a p-simplex α and a number $i \in \{1, \dots, p+1\}$ the $(p-1)$-simplex $\alpha^{(i)}$ as the simplex which arises from removing the i-th index from α. For all $p \geq 0$, we define the space of *Čech p-chains for S associated to the cover $\mathscr{U}$* as

$$\check{C}_p(\mathscr{U}_X; S) := \bigoplus_{\alpha=(\alpha_1,\dots,\alpha_{p+1})} S(U_\alpha). \tag{A.4}$$

We may then express elements $c \in \check{C}_p(\mathscr{U}_X; S)$ as formal linear combinations

$$c = \sum_\alpha c_\alpha, \quad c_\alpha \in S(U_{\alpha_1} \cap \dots \cap U_{\alpha_{p+1}}), \tag{A.5}$$

so that only finitely many c_α are nonzero.
The *Čech differential* $\check{\partial} : \check{C}_p(\mathscr{U}_X; S) \to \check{C}_{p-1}(\mathscr{U}_X; S)$ via

$$\check{\partial}(c_\alpha \alpha) := \sum_{i=1}^{p+1} (-1)^{i-1} \left(\iota_{U_\alpha}^{U_{\alpha^{(i)}}} c_\alpha \right) \alpha^{(i)}. \tag{A.6}$$

The *Čech homology associated to the cover $\mathscr{U}$ and the precosheaf S*, denoted by $\check{H}_\bullet(\mathscr{U}_X; S)$, is defined as the homology of the chain complex $\check{C}_\bullet(\mathscr{U}_X; P) := \bigoplus_{p \geq 0} \check{C}_p(\mathscr{U}_X; S)$.
The *Čech homology* of S is defined as

$$\check{H}_\bullet(X; S) := \varprojlim \check{H}_\bullet(\mathscr{U}_X; S) \tag{A.7}$$

where the inverse limit is taken with respect to refinement of covers.

Remark A.3. The symmetric group Σ_p acts on multiindices α of length p by permutation of the entries, and we denote this permutation by $\sigma \cdot \alpha$. Recall now the notation from (A.5). If $c = \sum_\alpha c_\alpha \cdot \alpha$ and $\alpha = (\alpha_1, \dots, \alpha_p)$ is one of the multiindices, we call c *skew-symmetric* if

$$c_{\sigma \cdot \alpha} = \operatorname{sign}(\sigma) c_\alpha \quad \forall \alpha. \tag{A.8}$$

The *skew-symmetrized* Čech complex is defined as the subcomplex

$$\check{C}^a_\bullet(\mathscr{U}_X; S) \subset \check{C}_\bullet(\mathscr{U}_X; S) \tag{A.9}$$

of skew-symmetric cochains. Dualizing the corresponding results for Čech cohomology of a sheaf [God58, Section 3.8], one finds that the inclusion $\check{C}^a_\bullet(\mathscr{U}_X; S) \hookrightarrow \check{C}_\bullet(\mathscr{U}_X; S)$ is a quasi-isomorphism.

We want to remark on a special class of cosheaves on which Čech homology is trivial.

Definition A.4. A cosheaf is called *flabby* if all its extension maps are injective.

Proposition A.5 ([Bre97, Chapter V, Proposition 1.6])**.** *Let P be a soft sheaf over a topological space X. Then compactly supported sections of P admit the structure of a flabby cosheaf over X, where the extension maps extend sections by zero.*

Proposition A.6 ([Bre97, Chapter VI, Corollary 4.5])**.** *Let S be a flabby cosheaf over X and $\mathcal{U}$ an open cover of X. Then*

$$\check{H}_p(\mathcal{U}_X; S) = \begin{cases} S(X) & \text{if } p = 0, \\ 0 & \text{else.} \end{cases} \tag{A.10}$$

Just like Čech cohomology of a sheaf can be calculated in terms of resolutions, we shall calculate Čech homology in terms of *coresolutions:*

Definition A.7. [Bre97, Chapter VI.7]

i) A precosheaf $\mathcal{P}$ on M is called *locally zero* if for every $x \in M$ and every open neighborhood U of x there is an open neighborhood $V \subset U$ so that $\iota_U^V = 0$.

ii) A sequence of precosheaves

$$\mathcal{P}_1 \xrightarrow{f} \mathcal{P}_2 \xrightarrow{g} \mathcal{P}_3 \tag{A.11}$$

is called *locally exact* is the precosheaf

$$U \mapsto \frac{\operatorname{Im} f(\mathcal{P}_1(U))}{\ker g(\mathcal{P}_2(U))} \tag{A.12}$$

is locally zero.

iii) A *coresolution* of a cosheaf $\mathcal{P}$ is a locally exact sequence of cosheaves

$$\cdots \to \mathcal{P}_2 \to \mathcal{P}_1 \to \mathcal{P}_0 \to \mathcal{P} \to 0. \tag{A.13}$$

The coresolution is called *flabby* if the $\mathcal{P}_0, \mathcal{P}_1, \dots$ (but not necessarily $\mathcal{P}$) are flabby.

To calculate Čech homology of cosheaves, the following result will be helpful:

Proposition A.8 ([Bre97, Theorems VI.7.2, VI.13.1])**.** *Let $\mathcal{P}$ be a cosheaf on M with flabby coresolution*

$$\cdots \to \mathcal{P}_2 \to \mathcal{P}_1 \to \mathcal{P}_0 \to \mathcal{P} \to 0. \tag{A.14}$$

A

i) Čech homology $\check{H}_\bullet(M;\mathscr{P})$ is equal to the homology of the complex

$$\cdots \to \mathscr{P}_2(M) \to \mathscr{P}_1(M) \to \mathscr{P}_0(M) \to 0. \tag{A.15}$$

ii) If $\mathscr{U}$ is an open cover of M with the property that

$$\cdots \to \mathscr{P}_2(U) \to \mathscr{P}_1(U) \to \mathscr{P}_0(U) \to \mathscr{P}(U) \to 0 \tag{A.16}$$

is exact whenever U is a finite intersection of elements of $\mathscr{U}$, then

$$\check{H}_\bullet(\mathscr{U};\mathscr{P}) = \check{H}_\bullet(M;\mathscr{P}). \tag{A.17}$$

One concept which one might hope for in the theory of cosheaves is a dual version of the well-known concept of sheafification, in other words, a way to universally assign to every precosheaf an appropriate cosheaf. For sheaves, one speaks of a left-adjoint functor to the inclusion of presheaves into sheaves, and sheafification exists for presheaves in most standard categories, e.g. the category of sets or abelian groups. Since sheafification respects stalks, locally, the original presheaf and its associated sheafification carry the same information.

Surprisingly, the dual concept of "cosheafification" is a lot more involved, and even existence of this concept in most standard categories is a difficult question, let alone constructing it explicitly, see for example [Cur14].

Instead, we will consider the concept of a *cosheaf on a base*. While the dual notion of sheaves on a base is well-studied, we are not aware of any mention in the literature of the cosheaf-theoretic version thereof.

Definition A.9. Let $\mathscr{B}$ be a topological base of M. In the following, view $\mathscr{B}$ as a subcategory of the category of open sets of M.

i) A *precosheaf S on $\mathscr{B}$* is a (covariant) functor from $\mathscr{B}$ to the category of abelian groups.

ii) For every $U \in \mathscr{B}$, choose an open cover $\{U_i\}_{i\in I}$ of U by elements in $\mathscr{B}$. Choose further for all indices $i, j \in I$ an open cover $\{V_{ij,k}\}_{k\in K}$ of $U_i \cap U_j$ by elements in $\mathscr{B}$. A precosheaf S on $\mathscr{B}$ is called a *cosheaf on $\mathscr{B}$* if, for all such choices, the Čech sequence

$$0 \leftarrow P(U) \leftarrow \bigoplus_i P(U_i) \leftarrow \bigoplus_{ijk} P(V_{ij,k}) \tag{A.18}$$

is exact.

iii) A morphism of (pre-)cosheaves on $\mathscr{B}$ is a natural transformation of the functors defining the (pre-)cosheaves.

The sequence is the analog of the cosheaf condition, but rather than working with all open sets, we just work with elements of a topological base $\mathscr{B}$. If $\mathscr{B}$ is chosen as the topology of M, then this definition is equivalent to the definition of a cosheaf on M.

This is precisely the dual of the well-studied concept of sheaves on a base, by viewing Ab-valued cosheaves as $\mathrm{Ab}^{\mathrm{op}}$-valued sheaves.

Theorem A.10. *Given a topological space M and a topological base $\mathscr{B}$ of M. An* Ab*-valued cosheaf on $\mathscr{B}$ extends, up to cosheaf isomorphism, uniquely to a cosheaf on M. A morphism between two cosheaves on $\mathscr{B}$ of M extends uniquely to a morphism between the induced cosheaves on M.*

Proof. The following proof is due to [jh]. The analog statement for $\mathscr{C}$-valued sheaves is true whenever $\mathscr{C}$ is a complete category (see [Vak17] or [Liu02, Lemma 2.2.7]). However, since Ab is a cocomplete category, $\mathrm{Ab}^{\mathrm{op}}$ is a complete category. This proves the statement. □

It is known that the setwise cokernels of cosheaf morphisms are again cokernels [Bre97, Proposition VI.1.2], the proof being a simple diagram chase. This straightforwardly extends to cosheaves on a base:

Proposition A.11. *Let $\mathscr{B}$ be a topological base of M.*

Let further $\phi : \mathscr{P} \to \mathscr{S}$ be a morphism of cosheaves on $\mathscr{B}$, and define a precosheaf $\operatorname{coker}\phi$ *by assigning to $B \in \mathscr{B}$*

$$\operatorname{coker}\phi(B) := \mathscr{S}(B)/\phi(\mathscr{P}(B)), \tag{A.19}$$

with extension maps induced by the cosheaf maps of $\mathscr{S}$. Then $\operatorname{coker}\phi$ *defines a cosheaf on $\mathscr{B}$.*

REFERENCES

[Bre97] Glen E. Bredon. *Sheaf theory*, volume 170 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1997.

[Cur14] Justin Michael Curry. *Sheaves, cosheaves and applications*. ProQuest LLC, Ann Arbor, MI, 2014. Thesis (Ph.D.)–University of Pennsylvania.

[God58] Roger Godement. Topologie algébrique et théorie des faisceaux. *Publications de*, 1, 1958.

[jh] jgon (https://math.stackexchange.com/users/90543/jgon). Cosheaf on a base. Mathematics Stack Exchange. URL:https://math.stackexchange.com/q/3642492 (version: 2020-04-24).

A

[Liu02] Qing Liu. *Algebraic geometry and arithmetic curves*, volume 6 of *Oxford Graduate Texts in Mathematics*. Oxford University Press, Oxford, 2002. Translated from the French by Reinie Erné, Oxford Science Publications.

[Vak17] R. Vakil. The rising sea: foundations of algebraic geometry. `http://math.stanford.edu/~vakil/216blog/index.html`, 2017.

B

The Hochschild–Serre Spectral Sequence for locally convex Lie algebras

In the finite-dimensional setting, the Hochschild–Serre spectral sequence is standard and a proof is laid out in [Fuk86, Chapter 1.5.1] and [HS53]. For general locally convex Lie algebras and continuous cohomology, one generally needs a number of topological assumptions. For example, restriction maps of continuous cochains like $C^q(\mathfrak{g}) \to C^r(\mathfrak{h}, \Lambda^{q-r}(\mathfrak{g}/\mathfrak{h})^*)$ are not necessarily surjective if the subspace $\mathfrak{h}$ is not complemented. We formulate some assumptions which suffice for the setting in this paper:

Theorem B.1. *Let $\mathfrak{g}$ be a complete, barreled, locally convex, nuclear Lie algebra whose strong dual space $\mathfrak{g}^*$ is complete, $\mathfrak{h} \subset \mathfrak{g}$ a finite-dimensional Lie subalgebra, and A a complete, locally convex space on which $\mathfrak{g}$ acts continuously. There is a cohomological spectral sequence $\{E_r^{p,q}, d_r\}$ converging to continuous cohomology $H^\bullet(\mathfrak{g})$ with*

$$E_1^{p,q} = H^q\left(\mathfrak{h}, C^p(\mathfrak{g}/\mathfrak{h}, A)\right), \tag{B.1}$$

where $C^p(X,Y)$ denotes skew-symmetric, jointly continuous, multilinear maps

$$\underbrace{X \times \cdots \times X}_{p\ times} \to Y, \tag{B.2}$$

and cohomology is taken with respect to continuous cochains. This spectral sequence is contravariantly functorial, in the sense that a diagram of continuous Lie algebra morphisms

B

$$\begin{tikzcd} \mathfrak{h} \arrow[r, hook] \arrow[d] & \mathfrak{g} \arrow[d] \\ \tilde{\mathfrak{h}} \arrow[r, hook] & \tilde{\mathfrak{g}} \end{tikzcd}$$

induces linear maps

$$E_r^{p,q}(\tilde{\mathfrak{g}},\tilde{\mathfrak{h}}) \to E_r^{p,q}(\mathfrak{g},\mathfrak{h}) \tag{B.3}$$

compatible with the differentials for all $p,q,r \geq 0$.

Proof. We define on the continuous cochains $C^\bullet(\mathfrak{g})$ the filtration

$$F^pC^{p+q}(\mathfrak{g};A) := \{c \in C^{p+q}(\mathfrak{g},A) : c(X_1,\dots,X_{p+q}) = 0 \text{ when } X_1,\dots,X_{q+1} \in \mathfrak{h}\}. \tag{B.4}$$

This is an ascending filtration with

$$C^r(\mathfrak{g},A) = F^0C^r(\mathfrak{g},A) \supset \cdots \supset F^rC^r(\mathfrak{g},A) \supset F^{r+1}C^r(\mathfrak{g},A) = 0, \tag{B.5}$$

and

$$dF^pC^{p+q}(\mathfrak{g};A) \subset F^pC^{p+q+1}(\mathfrak{g};A). \tag{B.6}$$

Denote by $\hat{\Lambda}^q$ the functor assigning to a locally convex vector space X the closure of the skew-symmetric tensors in its iterated projective tensor product $X^{\hat{\otimes}q}$, see for example [Sch71, Chapter III.7, IV.9]. We have a well-defined map

$$\begin{gathered} F^pC^{p+q}(\mathfrak{g},A) \to L(\hat{\Lambda}^q\mathfrak{h} \hat{\otimes} \hat{\Lambda}^p\mathfrak{g}/\mathfrak{h},A), \quad c \mapsto \tilde{c}, \\ \tilde{c}((h_1 \wedge \cdots \wedge h_q) \otimes [g_1] \wedge \cdots \wedge [g_p])) := c(h_1,\dots,h_q,g_1,\dots,g_p). \end{gathered} \tag{B.7}$$

This map is independent of the choices of representatives g_i by definition of the filtration and it is surjective because finite-dimensional subspaces are always complemented, so $\mathfrak{g} \cong \mathfrak{h} \oplus \mathfrak{g}/\mathfrak{h}$ as a direct sum of locally convex vector spaces. The kernel of this map equals $F^{p+1}C^{p+q}(\mathfrak{g},A)$. The image of this map is also indeed contained in the continuous linear maps by continuity of elements in the domain. Since $\mathfrak{h}$ is finite-dimensional, we trivially have

$$(\mathfrak{h} \hat{\otimes} \mathfrak{g}/\mathfrak{h})^* \cong \mathfrak{h}^* \hat{\otimes} (\mathfrak{g}/\mathfrak{h})^*. \tag{B.8}$$

By the assumptions on $\mathfrak{g}$ and A, we may apply [Trè67, Proposition 50.5] twice to find

$$L(\hat{\Lambda}^q\mathfrak{h} \hat{\otimes} \hat{\Lambda}^p\mathfrak{g}/\mathfrak{h},A) \cong L(\hat{\Lambda}^q\mathfrak{h}, L(\hat{\Lambda}^p\mathfrak{g}/\mathfrak{h},A)) \cong C^q(\mathfrak{h}, L(\hat{\Lambda}^p\mathfrak{g}/\mathfrak{h},A)). \tag{B.9}$$

Hence we get an isomorphism of vector spaces

$$F^pC^{p+q}(\mathfrak{g},A)/F^{p+1}C^{p+q}(\mathfrak{g},A) \cong C^q\left(\mathfrak{h}, L(\hat{\Lambda}^p\mathfrak{g}/\mathfrak{h},A)\right). \tag{B.10}$$

The differential of $C^\bullet(\mathfrak{g}, A)$ descends to the differential of this complex like in the purely algebraic case, so the spectral sequence associated to this filtration indeed has first page:

$$E_1^{p,q} = H^q\big(\mathfrak{h}, L(\hat{\Lambda}^p\mathfrak{g}/\mathfrak{h}, A)\big). \tag{B.11}$$

The functoriality with respect to Lie algebra pairs $(\mathfrak{g}, \mathfrak{h})$ is analogous to the purely algebraic setting. □

Remark B.2. This spectral sequence in the algebraic setting is generally also phrased with information about the second page if $\mathfrak{h}$ is an ideal. Adapting this to the continuous setting would require stronger assumptions, since this in particular requires commuting the projective tensor product with the cohomology.

REFERENCES

[Fuk86] D. B. Fuks. *Cohomology of infinite-dimensional Lie algebras.* Contemporary Soviet Mathematics. Consultants Bureau, New York, 1986. Translated from the Russian by A. B. Sosinskiĭ.

[HS53] G. Hochschild and J.-P. Serre. Cohomology of Lie algebras. *Ann. of Math. (2)*, 57:591–603, 1953.

[Sch71] Helmut H. Schaefer. Topological vector spaces. Graduate Texts in Mathematics, Vol. 3, pages xi+294. Springer-Verlag, New York-Berlin, 1971. Third printing corrected.

[Trè67] François Trèves. *Topological vector spaces, distributions and kernels.* Academic Press, New York-London, 1967.

C

The algebraic Loday–Quillen–Tsygan theorem

We recall from the presentation in [Lod92, Chapter 10] the outline of the proof of the algebraic Loday–Quillen–Tsygan theorem, originally developed in [LQ84] and even earlier in [Tsy83]. Fix a unital algebra A in this section.

Theorem C.1 (Loday, Quillen, Tsygan). *Let A be a unital algebra and*

$$\mathfrak{gl}(A) := \varinjlim \mathfrak{gl}_n(A) := \varinjlim \mathfrak{gl}_n(\mathbb{K}) \otimes A. \tag{C.1}$$

Then we have the following relation of the Lie algebra homology $H_\bullet(\mathfrak{gl}(A))$ and the cyclic homology $H^\lambda_\bullet(A)$:

$$H_\bullet(\mathfrak{gl}(A)) \cong \Lambda^\bullet H^\lambda_{\bullet-1}(A). \tag{C.2}$$

All above tensor products and homologies are taken algebraically.

Due to the unitality of A, for all finite $n \in \mathbb{N}$ the Lie algebra $\mathfrak{gl}_n(A)$ contains the reductive subalgebra $\mathfrak{gl}_n(\mathbb{K})$, and thus the reduction $C_\bullet(\mathfrak{gl}_n(A)) \to C_\bullet(\mathfrak{gl}(A))_{\mathfrak{gl}_n(\mathbb{K})}$ is a quasi-isomorphism, see [Lod92, Proposition 10.1.8].

Proposition C.2. *Denote by Σ_k the permutation group on k elements.*
If $n \geq k$, then there is an isomorphism

$$\phi_n : \left(\mathfrak{gl}_n(\mathbb{K})^{\otimes^k}\right)_{\mathfrak{gl}_n(\mathbb{K})} \to \mathbb{K}[\Sigma_k], \tag{C.3}$$

of Σ_k-modules, where Σ_k acts on the left-hand side by permutation of tensor factors and on the right-hand side by the adjoint action.

We can make this isomorphism map explicit (see [Lod92, Chapter 9.2]): Define

$$g := g_1 \otimes \cdots \otimes g_k \mapsto \sum_{\sigma \in \Sigma_k} T(\sigma)(g)\sigma \quad \forall g_i \in \mathfrak{gl}_n(\mathbb{K}), \tag{C.4}$$

where, if $\sigma \in \Sigma_k$ assumes a cycle decomposition

$$\sigma = (i_1 \ \dots \ i_k)(j_1 \ \dots \ j_r)\dots(t_1 \ \dots \ t_s), \tag{C.5}$$

we set

$$T(\sigma)(g) := \mathrm{tr}(g_{i_1}\dots g_{i_k})\,\mathrm{tr}(g_{j_1}\dots g_{j_r})\dots\mathrm{tr}(g_{t_1}\dots g_{t_s}). \tag{C.6}$$

By the invariance of the trace under cyclic permutations, it is straightforward to show that this map factors through to $C_\bullet(\mathfrak{gl}_n(A))_{\mathfrak{gl}_n(\mathbb{K})}$. A careful analysis then shows that this yields a bijection of the spaces. The family of isomorphisms $\{\phi_n\}_{n\geq k}$ is compatible with the inclusions induced by $\mathfrak{gl}_n(\mathbb{K}) \to \mathfrak{gl}_m(\mathbb{K})$ for $m \geq n \geq k$, meaning the following diagram commutes: Hence, since homology commutes with direct limits, we get

$$\begin{tikzcd} \left(\mathfrak{gl}_n(\mathbb{K})^{\otimes k}\right)_{\mathfrak{gl}_n(\mathbb{K})} \arrow[rr, hook] \arrow[dr, "\phi_n"] & & \left(\mathfrak{gl}_m(\mathbb{K})^{\otimes k}\right)_{\mathfrak{gl}_m(\mathbb{K})} \arrow[dl, "\phi_m"] \\ & \mathbb{K}[\Sigma_k] & \end{tikzcd}$$

$$H_\bullet(\mathfrak{gl}(A)) \cong \varinjlim H_\bullet(\mathfrak{gl}_n(A)) \cong \varinjlim H_\bullet\left(C_\bullet(\mathfrak{gl}_n(A))_{\mathfrak{gl}_n(\mathbb{K})}\right), \tag{C.7}$$

and as in the direct limit $n \to \infty$, every graded component of $C_\bullet(\mathfrak{gl}_n(A))_{\mathfrak{gl}_n(\mathbb{K})}$ becomes constant at some point, we can identify

$$\varinjlim C_\bullet(\mathfrak{gl}_n(A))_{\mathfrak{gl}_n(\mathbb{K})} \cong C_\bullet(\mathfrak{gl}(A))_{\mathfrak{gl}(\mathbb{K})} \cong \bigoplus_{k\geq 0}\left(\mathbb{K}[\Sigma_k]\otimes A^{\otimes k}\right)_{\Sigma_k}. \tag{C.8}$$

The Σ_k-action on the last term is given by the tensor product of the signed permutation action on $A^{\otimes k}$ and the adjoint action on $\mathbb{K}[\Sigma_k]$ The last ingredient is to relate the last cochain complex space and the differential it inherits to the cyclic complex of A and the cyclic differential. Recall that $(1 \ \cdots \ k) \in \Sigma_k$ denotes the cyclic permutation of k elements.

Proposition C.3. *Consider the map*

$$\begin{aligned} C^\lambda_{\bullet-1}(A) &\to \bigoplus_{k\geq 0}\left(\mathbb{K}[\Sigma_k]\otimes A^{\otimes k}\right)_{\Sigma_k}, \\ [a_1\otimes\cdots\otimes a_k] &\mapsto [(1 \ \cdots \ k)\otimes(a_1\otimes\cdots\otimes a_k)]. \end{aligned} \tag{C.9}$$

This map is well-defined, intertwines the differentials, and extends to an isomorphism of chain complexes

$$\theta : \Lambda^{\bullet} C^{\lambda}_{\bullet-1}(A) \to \bigoplus_{k\geq 0} \left(\mathbb{K}[\Sigma_k] \otimes A^{\otimes^k}\right)_{\Sigma_k} \tag{C.10}$$

by setting, for $[u_1], \dots, [u_l]$ *with* $[u_i] \in C^{\lambda}_{k_i-1}(A)$ *and* $N := \sum_i k_i$*:*

$$\begin{aligned}\theta([u_1] \wedge \cdots \wedge [u_l]) := \Big[\Big((1 \ \cdots \ k_1) \circ (k_1+1 \ \cdots \ k_2) \circ \cdots \circ (k_{l-1}+1 \ \cdots \ k_l)\Big) \\ \otimes (u_1 \otimes \cdots \otimes u_l)\Big] \in \left(\mathbb{K}\left[\Sigma_N\right] \otimes A^{\otimes^N}\right)_{\Sigma_N}.\end{aligned} \tag{C.11}$$

C

Remark C.4. The reader is invited to check that the final product map is well-defined on all levels: It is independent of the ordering of $[u_1] \wedge \cdots \wedge [u_l]$, as a different ordering is equivalent to a permutation by Σ_N, and we map into the Σ_N-coinvariants. It is also independent of the choice of representatives $u_i \in [u_i] \in C^{\lambda}_{k_i-1}(A)$, since this is equivalent to a cyclic permutation acting on u_i, and the cycle $(k_{i-1}+1 \ \cdots \ k_i)$ is fixed under conjugation by itself.

The differential on the domain of θ is simply the differential of tensor product complexes. Hence, the Künneth theorem finally implies

$$\Lambda^{\bullet} H^{\lambda}_{\bullet-1}(A) \cong \varinjlim H_{\bullet}\left(C_{\bullet}(\mathfrak{gl}_n(A))_{\mathfrak{gl}_n(\mathbb{K})}\right) \cong H_{\bullet}(\mathfrak{gl}(A)). \tag{C.12}$$

REFERENCES

[Lod92] Jean-Louis Loday. *Cyclic homology*, volume 301 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1992. Appendix E by María O. Ronco.

[LQ84] Jean-Louis Loday and Daniel Quillen. Cyclic homology and the Lie algebra homology of matrices. *Comment. Math. Helv.*, 59(4):569–591, 1984.

[Tsy83] B. L. Tsygan. Homology of matrix Lie algebras over rings and the Hochschild homology. *Uspekhi Mat. Nauk*, 38(2(230)):217–218, 1983.

D

THE FLAT DE RHAM COMPLEX

Let M be an smooth, n-dimensional manifold (possibly with boundary), $N \subset M$ a closed submanifold, both of

$$\dim H^k(N), \dim H^k(M) < \infty \quad \forall k \in \mathbb{N}_0. \tag{D.1}$$

In this section, we want to make a short excursion in understanding the flat de Rham complex

$$\Omega^\bullet_{\text{flat}}(M,N) := \{\omega \in \Omega^\bullet(M) : (j^\infty \omega)|_N = 0\}, \tag{D.2}$$

i.e. the subcomplex of $\Omega^\bullet(M)$ given by forms which are flat on N.

Lemma D.1. *Consider the complex*

$$0 \to \Omega^0_{\text{flat}}(M,N) \to \cdots \to \Omega^n_{\text{flat}}(M,N) \to 0, \tag{D.3}$$

with the differential given by the restriction of the de Rham differential. Its homology equals relative singular cohomology of the pair (M,N), and the differential has closed range.

The proof utilizes some basic knowledge of sheaves and sheaf cohomology, for which we direct the reader to [Bre97].

Proof. The proof idea uses ideas from [hn]. Denote by $\underline{\mathbb{R}}$ the constant sheaf on M and by Ω^k the soft sheaf of k-forms on M. Then there is the well-known resolution

$$0 \to \underline{\mathbb{R}} \to \Omega^0 \to \Omega^1 \to \dots \tag{D.4}$$

Assigning to a sheaf S on M the sheaf $S_{M\setminus N}$ on M with stalks

$$(S_{M\setminus N})_x := \begin{cases} S_x & \text{if } x \notin N, \\ 0 & \text{if } x \in N \end{cases} \tag{D.5}$$

is an exact functor, since exactness of sequences of sheaves may be checked on stalks, see [Bre97, Section I.2]. Further, if S was soft, then so is $S_{M\setminus N}$ [Bre97, Prop II.9.13]. Then the sheaf $\Omega^\bullet_{M\setminus N}$ assigns to an open $U \subset M$ the set $\Omega^\bullet_{\mathrm{flat}}(U, U\cap N)$, and, if $U \subset M$ is diffeomorphic to $\mathbb{R}^n$ then for $\underline{\mathbb{R}}_{M\setminus N}$ we have

$$\underline{\mathbb{R}}_{M\setminus N}(U) := \begin{cases} \mathbb{R} & \text{if } U\cap N = \emptyset, \\ 0 & \text{if } U \cap N \neq \emptyset. \end{cases} \tag{D.6}$$

Thus

$$0 \to \underline{\mathbb{R}}_{M\setminus N} \to \Omega^0_{M\setminus N} \to \Omega^1_{M\setminus N} \to \cdots \to \Omega^n_{M\setminus N} \to 0 \tag{D.7}$$

is a soft resolution of the sheaf $\underline{\mathbb{R}}_{M\setminus N}$, and the complex (D.3) calculates its sheaf cohomology. But by standard sheaf-theoretical arguments using resolutions by singular cochain spaces, the sheaf cohomology of $\underline{\mathbb{R}}_{M\setminus N}$ equals relative singular cohomology of the pair (M, n), see for example [Bre97, Chapter III.1]. Now, since we assumed M and N to have finite-dimensional cohomology, the image of the de Rham differential is, in every degree, cofinite-dimensional within its kernel $\ker d_{\mathrm{dR}}$. Together with Theorem 3.3.9 and closedness of $\ker d_{k+1}$, this shows the statement. □

Remark D.2. It is likely possible to drop the assumption of finite-dimensionality of the cohomology groups of M and N and proceed in a similar manner as [Pal72, Proposition 5.4], but we do not need this here.

REFERENCES

[Bre97] Glen E. Bredon. *Sheaf theory*, volume 170 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1997.

[hn] Liviu Nicolaescu (https://mathoverflow.net/users/20302/liviu nicolaescu). De Rham model for relative cohomology. MathOverflow. URL:https://mathoverflow.net/q/122900 (version: 2016-03-31).

[Pal72] V. P. Palamodov. On a Stein manifold the Dolbeault complex splits in positive dimensions. *Mat. Sb. (N.S.)*, 88(130):287–315, 1972.

Curriculum Vitæ

Lukas Miaskiwskyi

01-04-1994	Born in Neumarkt i.d.OPf., Germany.

Education

2004–2012	Secondary education Ostendorfer Gymnasium, Neumarkt i.d.OPf.
2012-2015	Bachelor of Science in Mathematical Physics Julius Maximilian University of Würzburg *Thesis:* Applications of SU(2) generalized coherent states *Advisor:* Prof. dr. T. Ohl
2015-2018	Master of Science in Mathematical Physics Julius Maximilian University of Würzburg *Thesis:* Invariant Hochschild Cohomology *Advisor:* Prof. dr. S. Waldmann
2018-2022	PhD in Mathematics University of Technology Delft *Thesis:* Cohomological local-to-global principles and integration in finite- and infinite-dimensional Lie theory *Promotor:* Prof. dr. J. M. A. M. van Neerven *Copromotor:* Dr. B. Janssens

List of Publications

4. **L. Miaskiwskyi**, *Continuous cohomology of gauge algebras and bornological Loday–Quillen–Tsygan theorems.* arXiv preprint arXiv:2206.08879 (2022).

3. **L. Miaskiwskyi**, *A local-to-global analysis of Gelfand–Fuks cohomology.* arXiv preprint arXiv:2202.09666 (2022).

2. **M. Caspers, B. Janssens, A. Krishnaswamy-Usha, & L. Miaskiwskyi**, *Local and multilinear noncommutative de Leeuw theorems.* arXiv preprint arXiv:2201.10400 (2022).

1. **L. Miaskiwskyi**, *Invariant Hochschild cohomology of smooth functions*, Journal of Lie Theory **31**, 2 (2021).