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The complexity of the vertex-minor problem

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A R T I C L E   I N F O

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A B S T R A C T

A graph H is a vertex-minor of a graph G if it can be reached from G by the successive application of local complementations and vertex deletions. Vertex-minors have been the subject of intense study in graph theory over the last decades and have found applications in other fields such as quantum information theory. Therefore it is natural to consider the computational complexity of deciding whether a given graph G has a vertex-minor isomorphic to another graph H. Here we prove that this decision problem is NP-complete, even when restricting H and G to be circle graphs, a class of graphs that has a natural relation to vertex-minors.

1. Introduction

A central problem in graph theory is the study of 'substructures' of graphs. These substructures are usually defined as the graphs that can be reached from a starting graph by a given set of graph operations. An well-studied example of such a substructure is the graph minor, where the central question is to decide whether a graph G can be transformed into a graph H through the successive application of vertex deletions, edge deletions, and edge contractions [1]. If this is the case we call H a minor of G. Many graph properties, such as planarity, can be tested by checking whether a graph has certain minors. In particular the Robertson-Seymour theorem [2] states that every set of graphs which is closed under taking minors is characterized by a finite set of forbidden minors.[1] To check if a graph is in the set, one can therefore check whether it contains one of the forbidden minors. For example, the set of planar graphs is closed under taking minors [3] and so is the set of graphs of tree-width at most k, since tree-width can not increase under taking minors [4]. The problem (MINOR) of deciding whether a graph H is a minor of G is NP-complete when both H and G are part of the input to the problem [5]. However, given a fixed H, we can define the problem (H-MINOR) of deciding whether H is a minor of G, where only G is part of the input. As shown in Robertson & Seymour's seminal series of papers [2], H-MINOR is solvable in cubic time for any graph H.

Since then a great variety of minor-relations has been defined and for many of those the complexity has been studied. Recently, minor-relations related to the graph operation of local complementation, i.e. vertex- and pivot-minors, have received particular attention. These two minor structures have been studied within the graph theory community [6–9] but have also found surprising applications outside of it, notably in the field of quantum information science [10–15]. Similarly to tree-width for minors, a complexity measure of graphs called *rank-width* can not decrease under taking vertex-minors. Furthermore, it has been shown that every set of graphs with bounded rank-width which is closed under taking vertex-minors is characterized by a finite set of forbidden vertex-minors. Examples of such sets include the set of distance-hereditary graphs, since these are exactly the graphs with rank-width one [6]. Another example of a set of graphs characterized by a finite set of forbidden vertex-minors are the circle

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[1] Also called obstructions.

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graphs [16], however, this set has unbounded rank-width ([17, Proposition 6.3] and [18]).

The complexity of the vertex- and pivot-minor decision problems was a notable open problem (see question 7 in [19]). Recently it was proven in [20] that the pivot-minor problem is NP-complete if both G and H are part of the input to the problem, but the complexity of the vertex-minor problem was left open. In [21] we proved, in the context of quantum information theory, the NP-completeness of the labeled version of the vertex-minor problem, i.e. the problem of deciding if H is a vertex-minor of the graph G, taking labeling into account. The labeled version of the vertex-minor problem is relevant in the context of quantum information theory since there the vertices of the graph correspond to physical qubits in, for example, a quantum network. However we did not discuss the complexity of the related problem of deciding whether G has a vertex-minor isomorphic to H (on any subset of the vertices). Here we close this gap, proving that the unlabeled version of the vertex-minor problem is also NP-complete. Moreover we prove that ISO-VERTEXMINOR remains NP-complete even when H is restricted to be a star graph and G a circle graph. To avoid confusion with the problems studied in [21] we will call the unlabeled version of the vertex-minor problem ISO-VERTEXMINOR.

In [21], we introduced the problem of deciding if a graph allows for a *semi-ordered Eulerian tour* (SOET) and used this to prove hardness of the labeled VERTEXMINOR-problem. Here we make use of a related reduction, however, using an unlabeled version of the SOET-problem (ISO-SOET). Although similar in nature, the hardness of the labeled version of the SOET-problem does not imply hardness of the unlabeled version and a substantially different proof is needed. The construction in [21] relies critically on the fact that the set of vertices V of the SOET are fixed in advance. Without this assumption the argument falls apart and additional ingredients are required. For example, in [21] we introduced a mapping, from 3-regular graphs to 4-regular multi-graphs, which we called the *triangular expansion*. The vertices from the original 3-regular graph have specific roles in the triangular expansion, which is not invariant under relabeling of the vertices. To take this into account, we introduce here a different mapping, also from 3-regular graphs to 4-regular multi-graphs, which we call the K_3-expansion.

Our work resolves the problem left open in [20]. However the related question posed in [19] where H is fixed and thus not part of the input to the problem, still remains open. Even though our work in this paper does not answer the second question directly, it excludes the possibility that the problem where H is part of the input is in P.

The paper is organized as follows: in section 2 we recall relevant graph theoretic notions such as vertex-minors and circle graphs. We also discuss the concept of semi-ordered Eulerian tours. In section 3 we prove the main result: the NP-completeness of the vertex-minor problem.

2. Preliminaries

In this section we review relevant graph theoretical notions. We begin by recalling the local complementation operation and the notion of vertex-minors before discussing a class of graphs called circle graphs. Here we also recall the notion of a semi-ordered Eulerian tour, which was introduced in [21], and connect it to the unlabeled version of the vertex-minor-problem on circle graphs.

We will denote graphs by capital letters: $G, H, F, R, \ldots$ Graphs are assumed to be simple unless otherwise indicated. The vertex-set of a graph G is denoted $V(G)$ and the edge-set is denoted $E(G)$. Give a vertex v in a graph G we denote the neighborhood of v (the set of vertices adjacent to v in G) by N_v. Given a graph G and a subset of its vertices V' we will denote the induced subgraph of G on those vertices by $G[V']$. We denote the fully connected graph on n vertices as K_n.

We denote words, i.e. ordered sequences of elements of a set (with repetition) by boldface letters, i.e. $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}, \ldots$. We denote the mirroring (reversing of the ordering of its letters) of a word $\boldsymbol{X}$ by $\widetilde{\boldsymbol{X}}$. Throughout this paper we use the following notation for sets of consecutive natural numbers

$$[n] \equiv \{i \in \mathbb{N} : 0 \leq i < n\} \tag{1}$$

2.1. Vertex-minors

We review the definition of local complementation:

Definition 1 *(Local complementation).* A local complementation τ_v is a graph operation specified by a vertex v, mapping a graph G to $\tau_v(G)$ by replacing the induced subgraph on the neighborhood of v, i.e. $G[N_v]$, by its complement. The neighborhood of any vertex u in the graph $\tau_v(G)$ is therefore given by

$$N_u^{(\tau_v(G))} = \begin{cases} N_u \Delta (N_v \setminus \{u\}) & \text{if } (u, v) \in E(G) \\ N_u & \text{else,} \end{cases} \tag{2}$$

where Δ denotes the symmetric difference between two sets.

Given a sequence of vertices $\boldsymbol{v} = v_1 \ldots v_k$, we denote the induced sequence of local complementations, acting on a graph G, as

$$\tau_{\boldsymbol{v}}(G) = \tau_{v_k} \circ \cdots \circ \tau_{v_1}(G). \tag{3}$$

A graph H that can be reached from another graph G using local complementations and vertex-deletions is called a *vertex-minor* [6] and is formally defined as:

Definition 2 *(Vertex-minor).* A graph H is called a vertex-minor of G if there exist a sequence of local complementations and vertex-deletions that maps G to H. If H is a vertex-minor of G we write this as

$$H < G. \tag{4}$$

Associated to the notion of vertex-minor is the natural decision problem:

Problem 1 *(ISO-VERTEXMINOR).* Given a graph G and a graph H decide whether there exists a graph $\tilde{H}$ such that (1) H and $\tilde{H}$ are isomorphic, and (2) $\tilde{H} < G$.

We can restrict this problem to a special case, where the graph H is a star-graph[2] on k vertices. We call this problem ISO-STARVERTEXMINOR. Note that we must only specify k as there exists only one star-graph on k vertices up to isomorphism. Formally we have

Problem 2 *(ISO-STARVERTEXMINOR).* Given a graph G and an integer k decide whether there exists a subset V' of $V(G)$ with $|V'| = k$ and a star graph on V' denoted $S_{V'}$ such that $S_{V'} < G$.

2.2. Circle graphs

Here we review circle graphs and representations of these under the action of local complementations. Circle graphs are also sometimes called alternance graphs since they can be described, as explained below, by a double-occurrence word such that the edges of the graph are the given by the alternances induced by this word. We will make use of this description here, which was introduced by Bouchet in [22] and also described in [16]. This description is also related to yet another way to represent circle graphs, as Eulerian tours of 4-regular multi-graphs, introduced by Kotzig in [23]. For an overview of the theory and history of circle graphs see for example the book by Golumbic [24].

2.2.1. Double-occurrence words

Let us first define double-occurrence words and equivalence classes of these. This will allow us to define circle graphs.

Definition 3 *(Double occurrence word).* A double-occurrence word $\boldsymbol{X}$ is a word with letters in some set V, such that each element in V occurs exactly twice in $\boldsymbol{X}$.

Given a double-occurrence word $\boldsymbol{X}$ we will write $V(\boldsymbol{X}) = V$ for its set of letters.

Definition 4 *(Equivalence class of double-occurrence words).* We say that a double-occurrence word $\boldsymbol{Y}$ is equivalent to another $\boldsymbol{X}$, i.e. $\boldsymbol{Y} \sim \boldsymbol{X}$, if $\boldsymbol{Y}$ is equal to $\boldsymbol{X}$, the mirror $\widetilde{\boldsymbol{X}}$ or any cyclic permutation of $\boldsymbol{X}$ or $\widetilde{\boldsymbol{X}}$. We denote by $\boldsymbol{d}_{\boldsymbol{X}} = \{\boldsymbol{Y} : \boldsymbol{Y} \sim \boldsymbol{X}\}$ the equivalence class of $\boldsymbol{X}$, i.e. the set of words equivalent to $\boldsymbol{X}$.

Next we define alternances of these equivalence classes, which will represent the edges of an alternance graph.

Definition 5 *(Alternance).* An *alternance* (u, v) of the equivalence class $\boldsymbol{d}_{\boldsymbol{X}}$ is a pair of distinct elements $u, v \in V$ such that a double-occurrence word of the form $\dots u \dots v \dots u \dots v \dots$ is in $\boldsymbol{d}_{\boldsymbol{X}}$.

[2] The problem remains the same if star graph is replaced with complete graph, since these are equivalent under local complementations.

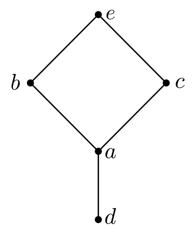


Fig. 1. An example of a circle graph induced by the double-occurrence word *adcbaebced*.

Note that if (u, v) is an alternance of $\boldsymbol{d}_{\boldsymbol{X}}$ then so is (v, u), since the mirror of any word in $\boldsymbol{d}_{\boldsymbol{X}}$ is also in $\boldsymbol{d}_{\boldsymbol{X}}$.

Definition 6 *(Alternance graph).* The alternance graph $\mathcal{A}(\boldsymbol{X})$ of a double-occurrence word $\boldsymbol{X}$ is a graph with vertices $V(\boldsymbol{X})$ and edges given exactly by the alternances of $\boldsymbol{d}_{\boldsymbol{X}}$, i.e.

$$E(\mathcal{A}(\boldsymbol{X})) = \{(u, v) \in V(\boldsymbol{X}) \times V(\boldsymbol{X}) : (u, v) \text{ is an alternance of } \boldsymbol{d}_{\boldsymbol{X}}\}. \tag{5}$$

Note that since $\mathcal{A}(\boldsymbol{X})$ only depends on the equivalence class of $\boldsymbol{X}$, the alternance graphs $\mathcal{A}(\boldsymbol{X})$ and $\mathcal{A}(\boldsymbol{Y})$ are equal if $\boldsymbol{X} \sim \boldsymbol{Y}$. Now we can formally define circle graphs.

Definition 7 *(Circle graph).* A graph G which is the alternance graph of some double-occurrence word $\boldsymbol{X}$ is called a circle graph.

As an example, consider the following double-occurrence word with letters in the set $V_0 = \{a, b, c, d, e\}$:

$$\boldsymbol{X}_0 = adcbaebced \tag{6}$$

The alternances of $d_{\boldsymbol{X}_0}$ are thus

$$(a, b),\ (a, c),\ (a, d),\ (b, e),\ (c, e) \tag{7}$$

and their mirrors. The alternance graph $\mathcal{A}(\boldsymbol{X}_0)$ is therefore the graph in Fig. 1.

2.2.2. Eulerian tours on 4-regular multi-graphs

There is yet another way to represent circle graphs, closely related to double-occurrence words, namely as Eulerian tours of 4-regular multi-graphs. A 4-regular multi-graph is a graph where each vertex has exactly four incident edges and can contain multiple edges between each pair of vertices or edges only incident to a single vertex.

Definition 8 *(Eulerian tour).* Let F be a connected multi-graph. An Eulerian tour U on F is a tour that visits each edge in F exactly once.

Any 4-regular multi-graph is Eulerian, i.e. has a Eulerian tour, since each vertex has even degree [25].

Furthermore, any Eulerian tour on a 4-regular multi-graph F traverses each vertex exactly twice, except for the vertex which is both the start and the end of the tour. Such a Eulerian tour induces therefore a double-occurrence word, the letters of which are the vertices of F, and consequently a circle graph as described in the following definition.

Definition 9 *(Induced double-occurrence word).* Let F be a connected 4-regular multi-graph on k vertices $V(F)$. Let U be a Eulerian tour on F of the form

$$U = x_0 e_0 x_1 \dots x_{2k-2} e_{2k-2} x_{2k-1} e_{2k-1} x_0, \tag{8}$$

with $x_i \in V(F)$ and $e_i \in E(F)$. Note that every element of V occurs exactly twice in U, except x_0. From a Eulerian tour U as in eq. (8) we define an induced double-occurrence word as

$$m(U) = x_0 x_1 \dots x_{2k-2} x_{2k-1}. \tag{9}$$

To denote the alternance graph given by the double-occurrence word induced by a Eulerian tour, we will write $\mathcal{A}(U) \equiv \mathcal{A}(m(U))$.

Similarly to double-occurrence words, we also introduce equivalence classes of Eulerian tours under cyclic permutation or reversal of the tour.

Definition 10 *(Equivalence class of Eulerian tours).* Let F be a connected 4-regular multi-graph and U be an Eulerian tour on F. We say that an Eulerian tour U' on F is equivalent to U, i.e. $U \sim U'$, if U' is equal to U, the reversal $\widetilde{U}$ or any cyclic permutation of U or $\widetilde{U}$. We denote by $\boldsymbol{t}_U$ the equivalence class of U, i.e. the set of Eulerian tours on F which are equivalent to U.

It is clear that if the Eulerian tours U and U' on a 4-regular multi-graph F are equivalent, then so are the double-occurrence words $m(U)$ and $m(U')$. Furthermore, as for double-occurrence words, two equivalent Eulerian tours on a connected 4-regular multi-graph induce the same alternance graph.

Consider for example the 4-regular multi-graph in Fig. 2(a). This graph has a tour U_0 with an induced double-occurrence word

$$m(U_0) = adcbaebced \tag{10}$$

Note, that this is equal to the word in eq. (6) which shows that $\mathcal{A}(U)$ is also the graph in Fig. 1.

2.2.3. Vertex-minors of circle graphs

When we are considering vertex-minors of circle graphs, it is useful to map the operations of local complementation and vertex deletion on an alternance graph of a double-occurrence word to operations on that double-occurrence word. Here we fix some notation and recap results also discussed in [21].

We start by considering local complementation. Let $\boldsymbol{X} = \boldsymbol{A} v \boldsymbol{B} v \boldsymbol{C}$ be a double-occurrence word with alternance graph $\mathcal{A}(\boldsymbol{X})$ and let v be an element in $V(\boldsymbol{X})$. Local complementation at the vertex v in the graph $\mathcal{A}(\boldsymbol{X})$ now corresponds to the mirroring of the sub-word $\boldsymbol{B}$ of $\boldsymbol{X}$ in between the two occurrences of v, i.e.

$$\tau_v\big(\mathcal{A}(\boldsymbol{X})\big) = \mathcal{A}(\boldsymbol{A} v \widetilde{\boldsymbol{B}} v \boldsymbol{C}) \tag{11}$$

Note that both the double-occurrence word $\boldsymbol{X} = \boldsymbol{A} v \boldsymbol{B} v \boldsymbol{C}$ and the double-occurrence word $\boldsymbol{A} v \widetilde{\boldsymbol{B}} v \boldsymbol{C}$ arise as words induced by Eulerian tours on the same 4-regular graph F. One can in fact show that two circle graphs are equivalent under the action of local complementation if and only if they arise as alternance graphs induced by Eulerian tours on the same 4-regular multi-graph [16].

Next we consider vertex deletion. We will denote by $\boldsymbol{X} \setminus v$ the deletion of the element v, i.e.

$$\boldsymbol{X} \setminus v \equiv (\boldsymbol{A} v \boldsymbol{B} v \boldsymbol{C}) \setminus v = \boldsymbol{ABC}. \tag{12}$$

The resulting word $\boldsymbol{ABC}$ is also a double-occurrence word and furthermore we have that

$$\mathcal{A}(\boldsymbol{X}) \setminus v = \mathcal{A}(\boldsymbol{X} \setminus v). \tag{13}$$

If $W = \{w_1, w_2 \dots, w_l\}$ is a subset of V, we will write $\boldsymbol{X} \setminus W$ as the deletion of all elements in W, i.e.

$$\boldsymbol{X} \setminus W = (\dots((\boldsymbol{X} \setminus w_1) \setminus w_2) \dots) \setminus w_l. \tag{14}$$

Connected to this we can also define an induced double-occurrence sub-word $\boldsymbol{X}[W] = \boldsymbol{X} \setminus (V \setminus W)$. The reason for calling this an induced double-occurrence sub-word stems from its relation to induced subgraphs of the alternance graph as

$$\mathcal{A}(\boldsymbol{X})[W] = \mathcal{A}(\boldsymbol{X}[W]). \tag{15}$$

We can decide if a circle graph has a certain vertex-minor by considering Eulerian tours of a 4-regular graph, which is captured in the following theorem, a proof of which can be found in [21]. This theorem states that vertex-minors of alternance graphs induced by a Eulerian tour on a 4-regular graph F are exactly the alternance graphs induced by sub-words formed by Eulerian tours on F.

Theorem 1. *Let F be a connected 4-regular multi-graph and let G be a circle graph such that $G = \mathcal{A}(U)$ for some Eulerian tour U on F. Then G' is a vertex-minor of G if and only if there exists a Eulerian tour U' on F such that*

$$G' = \mathcal{A}(m(U')[V(G')]). \tag{16}$$

2.2.4. Semi-ordered Eulerian tours

From the previous sections we have seen that circle graphs and their vertex-minors can be described by Eulerian tours on connected 4-regular multi-graphs. One can thus ask, given a graph H, what properties a 4-regular multi-graph F must possess such that any of its alternance graphs[3] has H as a vertex-minor. We answered this question in [10] for the case when H is a star graph by introducing the notion of a Semi-ordered Eulerian Tour (SOET), defined as

[3] Note that if $H < \mathcal{A}(U)$ for some Eulerian tour U on F then $H < \mathcal{A}(U')$ for all Eulerian tours U' on F.

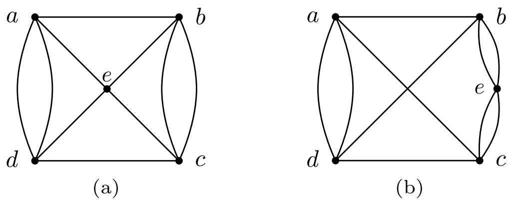


Fig. 2. Examples of two 4-regular multi-graphs. Fig. 2a is an example of a graph that allows for a SOET with respect to the set $V' = \{a, b, c, d\}$. A SOET for this graph is for example $m(U) = abcdaebced$. The graph in Fig. 2b on the other hand does not allow for any SOET with respect to the set $V' = \{a, b, c, d\}$.

Definition 11 *(SOET).* Let F be a 4-regular multi-graph and let $V' \subseteq V(F)$ be a subset of its vertices. Furthermore, let $\boldsymbol{s} = s_0 s_1 \dots s_{k-1}$ be a word with letters in V' such that each element of V' occurs exactly once in $\boldsymbol{s}$ and where $k = |V'|$. A semi-ordered Eulerian tour U with respect to V' is a Eulerian tour such that $m(U) = s_0 \boldsymbol{X}_0 s_1 \dots s_{k-1} \boldsymbol{X}_{k-1} s_0 \boldsymbol{Y}_0 s_1 \dots s_{k-1} \boldsymbol{Y}_{k-1}$ and where $\boldsymbol{X}_0, \boldsymbol{X}_1, \dots, \boldsymbol{X}_{k-1}, \boldsymbol{Y}_0, \dots, \boldsymbol{Y}_{k-1}$ are words (possibly empty) with letters in $V(F) \setminus V'$. This can also be stated as $m(U)[V'] = \boldsymbol{ss}$, for a word $\boldsymbol{s}$.

Note that the multi-graph F is not assumed to be simple, so multi-edges and self-loops are allowed. A SOET is a Eulerian tour on F that traverses the elements of V' in some order once and then again in the same order. The particular order in which the SOET traverses V' will not be important here, only that it traverses V' in the same order twice. As an example, the graph in Fig. 2(a) allows for a SOET with respect to the set $\{a, b, c, d\}$ but the graph in Fig. 2(b) does not.

The following theorem, a trivial corollary of Corollary 2.6.1 in [10], connects the problem of finding star graphs[4] as vertex-minors of circle graphs to the problem of finding SOETs on 4-regular multi-graphs.

Theorem 2. *Let F be a connected 4-regular multi-graph and let G be a circle graph given by the alternance graph of a Eulerian tour U on F, i.e. $G = \mathcal{A}(U)$. Furthermore let $S_{V'}$ be a star graph on the vertices V'. Then $S_{V'} < G$ if and only if F allows for a SOET (see Definition 11) with respect to V'.*

This gives rise to a natural decision problem which we denote ISO-SOET:

Problem 3 *(ISO-SOET).* Let F be a 4-regular multi-graph and let $k \leq |V(F)|$ be an integer. Decide whether there is a subset $V' \subset V(F)$ of size $|V'| = k$ such that there exists a SOET U on F with respect to the set V'.

In [21] we proved that a version of Problem 3 where V' is part of the input to the problem, is NP-complete. In the next section we prove that also the problem of deciding whether such a V' exists, i.e. Problem 3, is also NP-complete.

[4] The theorem also holds if one replaces star graphs with complete graphs since they are equivalent under local complementations.

One can see that a SOET on a 4-regular multi-graph F with respect to V', imparts an ordering on the subset of vertices V'. We will in particular be interested in vertices in V' that are 'consecutive' with respect to the SOET. Consecutiveness is defined as follows.

Definition 12 *(Consecutive vertices).* Let F be a 4-regular graph and U a SOET on F with respect to a subset $V' \subseteq V(F)$. Two vertices $u, v \in V'$ are called consecutive in U if there exist a sub-word $u\boldsymbol{X}v$ or $v\boldsymbol{X}u$ of $m(U)$ such that no letter of $\boldsymbol{X}$ is in V'.

We also define the notion of a "maximal sub-word" associated with two consecutive vertices.

Definition 13 *(Maximal sub-words).* Let F be a 4-regular multi-graph and U a SOET on F with respect to a subset $V' \subseteq V(F)$. The double-occurrence word induced by U is then of the form $m(U) = s_0 \boldsymbol{X}_0 s_1 \dots s_{k-1} \boldsymbol{X}_{k-1} s_0 \boldsymbol{Y}_0 s_1 \dots s_{k-1} \boldsymbol{Y}_{k-1}$, where $k = |V'|$, $\{s_0, \dots, s_{k-1}\} = V'$ and $\boldsymbol{X}_0, \dots, \boldsymbol{X}_{k-1}, \boldsymbol{Y}_0, \dots, \boldsymbol{Y}_{k-1}$ are words (possibly empty) with letters in $V(F) \setminus V'$.
For $i \in [k)$, we call $\boldsymbol{X}_i$ and $\boldsymbol{Y}_i$ the two maximal sub-words associated with the consecutive vertices s_i and $s_{(i+1 \pmod k)}$.

Given two consecutive vertices u and v, we will denote their two maximal sub-words as $\boldsymbol{X}$ and $\boldsymbol{X}'$, $\boldsymbol{Y}$ and $\boldsymbol{Y}'$ or similar.

3. NP-completeness of the vertex-minor problem

In this section we prove the NP-completeness of the vertex-minor problem. This we do in three steps. We will begin by (1) proving that ISO-SOET is NP-Hard. We do this by reducing the problem of deciding whether a 3-regular graph R is Hamiltonian to ISO-SOET. Next we (2) reduce ISO-SOET to ISO-STARVERTEXMINOR and ISO-STARVERTEXMINOR to ISO-VERTEXMINOR, thus proving the NP-hardness of ISO-VERTEXMINOR. Finally we (3) show that ISO-VERTEXMINOR is also in NP.

3.1. SOET is NP-hard

We first review the definition of a Hamiltonian graph and the associated CUBHAM decision problem.

Definition 14 *(Hamiltonian).* A graph is said to be Hamiltonian if it contains a Hamiltonian cycle. A Hamiltonian cycle is a cycle that visits each vertex in the graph exactly once.

Problem 4 *(CUBHAM).* Let R be a 3-regular graph. Decide whether R is Hamiltonian.

The reduction of CUBHAM to ISO-SOET is done by going through the following steps.

1. Introduce the notion of a (4-regular) K_3-expansion $\Lambda(R)$ of a 3-regular graph R. This is done in Definition 15.

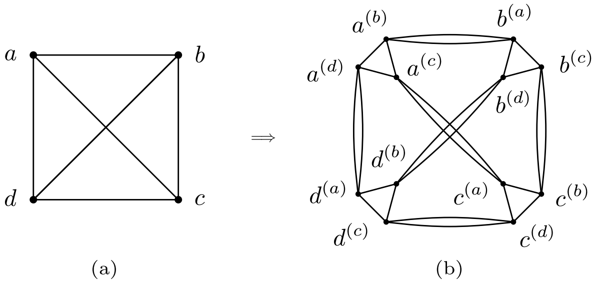


Fig. 3. Figure showing (a) the complete graph K_V on vertices $V = \{a, b, c, d\}$ and (b) its associated K_3-expansion $\Lambda(K_V)$.

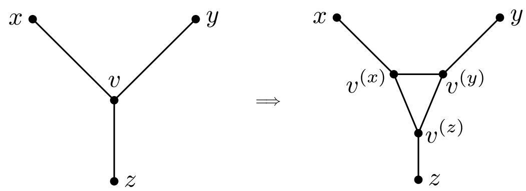
2. Prove that if a 3-regular graph R is Hamiltonian then the K_3-expansion $\Lambda(R)$ of R allows for a SOET of size $2|V(R)|$. This is done in Lemma 1.
3. Prove that if the K_3-expansion $\Lambda(R)$ of a 3-regular graph R allows for a SOET of size $2|V(R)|$ then R is Hamiltonian. This is done in Lemma 3.

Note that 2. and 3. above provide necessary and sufficient conditions for whether a 3-regular graph R is Hamiltonian in terms of whether $\Lambda(R)$ allows for a SOET of a certain size. This implies that CUBHAM reduces to ISO-SOET and hence that ISO-SOET is NP-hard.

We begin by introducing the K_3-expansion: a mapping from 3-regular graphs to 4-regular multi-graphs. Note that this expansion differs from the *triangular-expansion* used in [21].

Definition 15 *(K_3-expansion).* Let R be a 3-regular graph. A K_3-expansion $\Lambda(R)$ of a 3-regular graph R is constructed from R by performing the following two steps:

1. Replace each vertex v in R with a subgraph isomorphic to K_3 as below



(17)

where x, y and z are the neighbors of v. We will denote the K_3-subgraph associated to the vertex v with T_v, i.e. $T_v = G[\{v^{(x)}, v^{(y)}, v^{(z)}\}]$.

2. Double any edge that is incident on two subgraphs T_v, $T_{v'}$ for distinct v, v'.

The graph $\Lambda(R)$ will be called a K_3-*expansion* of R. A multi-graph F that is the K_3-expansion of some 3-regular graph R will also be referred to as a K_3-expanded graph. Furthermore, the number of vertices in $\Lambda(R)$ is $3 \cdot |V(R)|$ and the number of edges is $2 \cdot |E(R)| + 3 \cdot |V(R)|$. In Fig. 3 we show an example of a 3-regular graph and its K_3-expansion.

We now argue that if a 3-regular graph R is Hamiltonian then its K_3-expansion allows for a SOET on $2|V(R)|$ vertices and thus providing a necessary condition for a 3-regular graph being Hamiltonian.

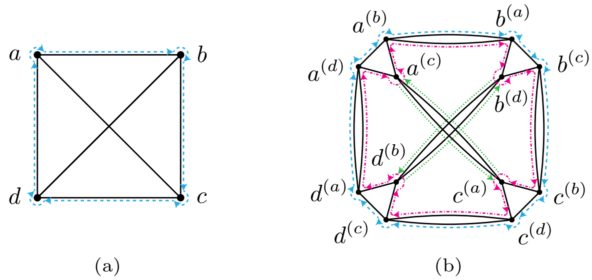


Fig. 4. Figure showing (a) a Hamiltonian path (blue dashed arrows) on the complete graph on vertices $V = \{a, b, c, d\}$ and (b) the corresponding disjoint trails described by the words $\mathbf{V}$ (blue dashed arrows) and $\mathbf{W}$ (pink dashed-dotted arrows) from eqs. (18) and (19) on the associated K_3-expansion $\Lambda(K_V)$. The edges used to extend the tour to a Eulerian tour as captured by Algorithm 1 are show as green dotted arrows.

Algorithm 1 Algorithm for lifting the tour U_{VW} to a Eulerian tour on $\Lambda(R)$.

for $i \in [k]$ **do**
  if $x_i^{(v_i)} v_i^{(x_i)} x_i^{(v_i)} \not\subset \mathbf{W}$ **then**
    Insert $v_i^{(x_i)} x_i^{(v_i)}$ into $\mathbf{W}$ right after $x_i^{(v_i)}$
  end if
end for

Lemma 1. *Let R be a 3-regular graph with k vertices and let $\Lambda(R)$ be its K_3-expansion. If R is Hamiltonian then $\Lambda(R)$ allows for a SOET of size $2k$.*

Proof. Let M be a Hamiltonian tour on R. Choose $x_0 \in V(R)$ and let $\mathbf{L} = x_0 x_1 \cdots x_{k-1}$ be the word formed by walking along M when starting on x_0. Note that $x_i, x_{(i+1 \pmod k)}$ are adjacent in R for all $i \in [k]$. Now consider the K_3-expansion $\Lambda(R)$ of R. We will argue that $\Lambda(R)$ allows for a SOET with respect to the set $V' = \{x_0^{(x_{k-1})}, x_0^{(x_1)}, x_1^{(x_0)}, x_1^{(x_2)}, \ldots, x_{k-1}^{(x_{k-2})}, x_{k-1}^{(x_0)}\}$. For all $i \in [k]$ let v_i be the unique vertex adjacent to x_i in $\Lambda(R)$ that is not $x_{(i-1 \pmod k)}$ or $x_{(i+1 \pmod k)}$. Now consider the following words on $V(\Lambda(R))$.

$$\mathbf{V} := x_0^{x_{k-1}} x_0^{(x_1)} x_1^{(x_0)} x_1^{(x_2)} x_2^{(x_1)} x_2^{(x_3)} \ldots x_{k-1}^{(x_{k-2})} x_{k-1}^{(x_0)} \tag{18}$$

$$\mathbf{W} := x_0^{x_{k-1}} x_0^{(v_0)} x_0^{(x_1)} x_1^{(x_0)} x_1^{(v_1)} x_1^{(x_2)} x_2^{(x_1)} x_2^{(v_2)} x_2^{(x_3)} \ldots x_{k-1}^{(x_{k-2})} x_{k-1}^{(v_{k-1})} x_{k-1}^{(x_0)} \tag{19}$$

These words describe disjoint trails on $\Lambda(R)$ as illustrated for an example graph in Fig. 4.

Now consider the word $\mathbf{VW}$. This word describes a trail U_{VW} on $\Lambda(R)$ that visits every vertex in V' exactly twice in the same order. This means U_{VW} is a semi-ordered tour. It is however not Eulerian. To make it Eulerian we have to extend the tour U_{VW} to include all edges in $\Lambda(R)$. Note that these edges are precisely the edges connecting the vertices $x_i^{(v_i)}, v_i^{(x_i)}$ for all $i \in [k]$. We can lift U_{VW} to a Eulerian tour by adding vertices to $\mathbf{W}$ by the following algorithm. It is easy to see that the tour described by $\mathbf{VW}$ after running Algorithm 1 is also Eulerian and is hence a SOET with respect to the set V'. This completes the lemma. □

Next we prove a necessary condition (Lemma 3) for the existence of a SOET on a subset V' of the vertices of a 4-regular graph F.

Lemma 2. *Let F be a 4-regular graph and $V' \geq 4$ be a subset of its vertices. If there exists three distinct vertices in V' which are all mutually adjacent, then F does not allow for a SOET with respect to V'.*

Proof. Assume that F has three vertices $\{u, v, w\} \subset V'$ which are all mutually adjacent. Let U be a Eulerian tour on F (note that U always exists). Assume by contradiction that U is a SOET with respect to V'. It is easy to see that since u and v are adjacent in F they must also be consecutive in U. However the same is true for u and w and also w and v. This means a tour starting at u and must traverse v, then w, and then immediately u again (up to interchanging u and w). Since $\{u, v, w\}$ is a strict subset of V' (since by assumption $|V'| \geq 4$) this means that, when starting at u, the tour U does not traverse all vertices in V' before returning to u. This gives a contradiction with the definition of SOET from which the lemma follows. □

Now we will leverage Lemma 2 to prove that if the K_3-expansion $\Lambda(R)$ of a 3-regular graph R allows for a SOET with respect to a vertex-set V' with $|V'| = 2|V(R)|$ then the graph R must be Hamiltonian.

Lemma 3. *Let R be a 3-regular graph and $\Lambda(R)$ its K_3-expansion. If there exists a set $V' \subset V(\Lambda(R))$ with $|V'| = 2|V(R)|$ such that $\Lambda(R)$ allows for a SOET with respect to V' then R must be Hamiltonian.*

Proof. Assume that there exists a subset V' of $V(\Lambda(R))$ with $|V'| = 2|V(R)|$ such that $\Lambda(R)$ allows for a SOET U with respect to V'.

Note first that since $|V(\Lambda)| = 3|V(R)|$ and $|V'| = 2|V(R)|$ we must have, by Lemma 2 that $|V[T_u] \cap V'| = 2$ for all $u \in V(R)$. This is easiest seen by contradiction. Assume that there exists a $u \in V(R)$ such that $|V[T_u] \cap V'| < 2$. From the fact that (1) $V(T_v) \cap V(T_{v'}) = \emptyset$ for all $v, v' \in V(R)$, (2) $|V(\Lambda)| = 3|V(R)|$ and (3) $|V'| = 2|V(R)|$ we then know that there must also a exist a $u' \in V(R)$ such that $|V(T_{u'}) \cap V'| = 3$. This means that $V(T_{u'}) \subset V'$. However the induced subgraph $\Lambda(R)[T_{u'}]$ is isomorphic to K_3 (this is easily seen from the definition of K_3-expansion). By Lemma 2 we must thus conclude that $\Lambda(R)$ does not allow for a SOET with respect to V' leading to a contradiction. Hence we must have that $|V(T_u) \cap V'| = 2$ for all $u \in V(R)$.

Now consider two vertices $x, x' \in V'$ such that x, x' are consecutive in the SOET U. Note that, by definition of $\Lambda(R)$, there must exist $w, w' \in V(R)$ such that $x \in T_w$ and $x' \in T_{w'}$. We will now argue that we must have either $w = w'$ or w, w' are adjacent in R. We argue this by contradiction. Assume thus that w, w' are neither equal nor adjacent in R. Now let $\mathbf{Y}$ be one of the two maximal sub-words of $m(U)$ associated to x, x'. Since w, w' are neither equal nor adjacent in R, the trail described by the word $\mathbf{Y}$ must pass through a triangle subgraph different from T_w and $T_{w'}$, i.e. there exists a vertex $w'' \in V(R)$ such that $|\mathbf{Y} \cap V(T_{w''})| \geq 2$. However since by construction $|V(T_{w''}) \cap V'| = 2$ (as shown above) and $|V(T_{w''})| = 3$ we must have that $|V' \cap \mathbf{Y}| \geq 1$. This, however, contradicts the assumption that $\mathbf{Y}$ is a maximal sub-word. Hence we must have that $w = w'$ or that $(w, w') \in E(R)$. Now consider the word $m(U)$ associated to the SOET U and the induced sub-word $m(U)[V']$. By the above, and the fact that if two vertices in V' are adjacent in $\Lambda(R)$, they must also be consecutive in U (this is a consequence of U being Eulerian and thus having to traverse the edge connecting these vertices), we have that $m(U)[V']$ must be of the form

$$m(U)[V'] = x_0x_0'x_1x_1'x_2x_2' \dots x_{k-1}x_{k-1}'x_0x_0' \dots x_{k-1}x_{k-1}' \tag{20}$$

where $x_i, x_i' \in T_{w_i}$ and $\{w_1, \dots w_k\} = V(R)$ and moreover that $(w_i, w_{i+1}) \in E(R)$ for all $i \in [k)$ and also $(w_k, w_0) \in E(R)$. This immediately implies that the word $\mathbf{M} = w_1w_2 \dots w_k$ describes a Hamiltonian tour on R, and hence that R is Hamiltonian. □

Since Lemma 3 and 1 provide necessary and sufficient conditions for a 3-regular graph being Hamiltonian in terms of whether a K_3-expanded graph allows for a SOET, we can now easily prove the hardness of Problem 3.

Theorem 3. *ISO-SOET is NP-Hard.*

Proof. Let R be an instance of CUBHAM, that is, a 3-regular graph on k vertices. From R we can construct the 4-regular K_3-expansion $\Lambda(R)$. Note that this can be done in poly-time in k. Now note that $(\Lambda(R), 2k)$ is an instance of ISO-SOET. If R is a YES instance of CUBHAM, that is, R is Hamiltonian, then by Lemma 1 we have that $(\Lambda(R), 2k)$ is a YES instance of ISO-SOET. On the other hand, if $(\Lambda(R), 2k)$ is a YES instance of ISO-SOET, then R is a YES instance of CUBHAM by Lemma 3. By contraposition this means that if R is a NO instance of CUBHAM, then $(\Lambda(R), 2k)$ is a NO instance of ISO-SOET. This means CUBHAM is Karp-reducible to SOET. Since CUBHAM is NP-complete [26], this implies that SOET is NP-hard. □

3.2. ISO-VERTEXMINOR is NP-hard

Note first that ISO-STARVERTEXMINOR trivially reduces to ISO-VERTEXMINOR, as it is a strict sub-problem. This means that if ISO-STARVERTEXMINOR is NP-hard then so is ISO-VERTEXMINOR. In this section we show that the ISO-SOET reduces to ISO-VERTEXMINOR, which follows from Theorem 2.

Theorem 2 states that a 4-regular multi-graph F allows for a SOET with respect to a subset of its vertices $V' \subseteq V(F)$ if and only if an alternance graph $\mathcal{A}(U)$ (which is a circle graph), induced by some Eulerian tour on F, has a star graph $S_{V'}$ on V' as a vertex-minor. We therefore have the following theorem.

Theorem 4. *The decision problem ISO-SOET reduces to ISO-STARVERTEXMINOR.*

Proof. Let (F, k) be an instance of ISO-SOET, where F is a 4-regular multi-graph and $k \le |V(F)|$ some integer. Also let G be a circle graph induced by some Eulerian tour U on F. From Corollary 2 we see that G has $S_{V'}$ as a vertex-minor for some subset of vertices V' of G if and only if F allows for a SOET with respect to this vertex set V'. Since an Eulerian tour U can be found in polynomial time [27] and since G can be efficiently constructed given U [21], considering the case of $|V'| = k$ concludes the reduction. □

3.3. ISO-VERTEXMINOR is in NP

Next we argue that the problem ISO-VERTEXMINOR is in NP. This just follows from the fact that the non-isomorphic vertex-minor problem is in NP.

Theorem 5. *The decision problem ISO-VERTEXMINOR is in NP.*

Proof. From [21] we know that there exists a polynomial-length witness for the problem of deciding if a labeled graph G has a vertex-minor equal to another graph H on some fixed subset of its vertices. Since GRAPHISOMORPHISM is in NP we can also construct a polynomial-length witness for ISO-VERTEXMINOR, i.e. to decide if G has a vertex-minor isomorphic to H. We thus conclude that ISO-VERTEXMINOR is in NP. □

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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