Technische Universiteit Delft
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De stelling van Reichert en netwerkmatrices

(Engelse titel: The theorem of Reichert and network matrices)

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"De stelling van Reichert en netwerkmatrices"

(Engelse titel: "The theorem of Reichert and network matrices")

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Chapter 1

Summary

In 1969, the theorem of Reichert was formulated [13]. Since then, two proofs have been found, both by Z. Jiang and Malcolm C. Smith, one of which in collaboration with Sara Y. Zhang [9][13]. These proofs are based on the transfer functions of the electrical networks.
The aim of this paper is to work toward a proof based on graph theory, linear algebra and differential equations. Whereas a proof has not been found, this paper provides insight on what the theorem of Reichert means, on ways to describe graphs and electrical networks with matrices and on the connection between those matrices. Lastly, we will take a look at Kron reduction: a way to remove resistors from resistive networks by manipulating their descriptive matrices.

Chapter 2

Preface

This report is the result of my BEP (Bachelor Eind Project) for the Bachelor Applied Mathematics at Delft University of Technology. I have read many papers and books about the theorem of Reichert, on impedance matrices, on RLC-networks and on reducing resistive networks. I have combined the knowledge all those articles and books brought me in this report. Although I have not reached the goal of proving the theorem through graph theory, linear algebra and differential equations, I hope that this report will be a helpful source of information should anyone else want to try to prove the theorem this way.

I would like to thank my supervisor Dr. J.W. van der Woude for his support and feedback thoughout the process.

Delft, 3 July 2017
Bodine van Leeuwen

Chapter 3

Definitions

There are a few types of (network) matrices that will be used throughout this paper. Their definitions can be found in this section.

3.1 Matrices general

In this subsection, some abstract definitions will be introduced, concerning only some theoretical properties of matrices (not their connections to graph theory or electrical networks).

Definition 1. *A* paramount matrix *is a symmetric matrix whose principal minors are not less in magnitude than any other minor built from the same rows [3].*

An example of a paramount matrix as provided by [4] is

$$\begin{pmatrix} 20 & -10 & 0 \\ -10 & 35 & -30 \\ 0 & -30 & 36 \end{pmatrix}.$$

Definition 2. *A* unimodular matrix A *is an* $n \times m$ *matrix such that* $a_{ij} \in \{0, -1, 1\}, \forall i = 1, ..., n,\ j = 1, .., m,$ *and all subdeterminants of* A *also equal either* $0, 1$ *or* -1 *[2].*

An example of a unimodular matrix is

$$\begin{pmatrix} -1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \end{pmatrix}.$$

3.2 Graphs and Matrices

It is useful to distinguish between different kind of graphs and different kinds of matrices to describe them. The kinds that will be used in this paper, will be described in this section.

Definition 3. *A* connected graph *is a graph with a path between any two vertices.*

The graph in Figure 3.1 is a connected graph.

Definition 4. *A* tree *is a connected graph that has no cycles.*

The graph in Figure 3.2 is a tree.

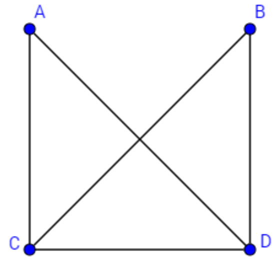

Figure 3.1: A connected graph

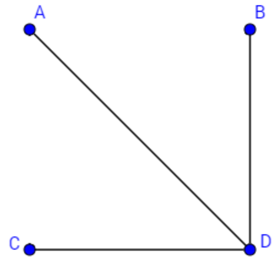

Figure 3.2: A tree

Definition 5. *A* weighted *graph is a graph all of whose edges are assigned a certain weight.*

The graph in Figure 3.3 is a weighted graph. The graph in Figure 3.1 is an unweighted graph.

Definition 6. *A* directed graph *is a graph whose edges can only be crossed in one direction.*

The graph in Figure 3.4 is a directed graph. The graph in Figure 3.1 is an undirected graph.

Definition 7. *An* incidence matrix *of an undirected graph is an $n \times m$ matrix A, representing a graph with n vertices and m edges. All elements of A equal either 0 or 1. If $a_{ij} = 1$ $(i = 1, ..., n; j = 1, ..., m)$, edge j has vertex i as endpoint. If $a_{ij} = 0$, edge j does not have vertex i as endpoint [1].*

For example, the undirected graph as shown in Figure 3.1 is represented by the following incidence matrix:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix}.$$

Definition 8. *An* incidence matrix *of a directed graph is an $n \times m$ matrix A, representing a graph with n vertices and m edges. All elements of A equal either $0, -1$ or 1. If $a_{ij} \in \{-1, 1\}$ $(i = 1, ..., n; j = 1, ..., m)$, edge j has vertex i as endpoint. The edge is directed from the element equal to 1 towards the vertex represented by the element equal to -1.*
If $a_{ij} = 0$, edge j does not have vertex i as endpoint.

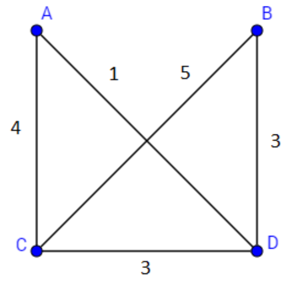

Figure 3.3: Weighted graph

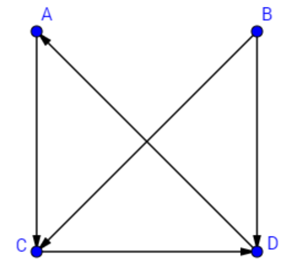

Figure 3.4: Directed graph

For example, the directed graph shown in Figure 3.4 is represented by the following incidence matrix:

$$\begin{pmatrix} -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 & -1 \\ 1 & -1 & 0 & -1 & 0 \end{pmatrix}.$$

Definition 9. *An unweighted* adjacency matrix *is an $n \times n$ matrix A all of whose elements equal* 0 *or* 1*. It represents an undirected, unweighted graph with n vertices: if an element $a_{ij} = 1$ $(i, j = 1, ..., n)$, vertices j and i are endpoints of the same edge; if $a_{i,j} = 0$ $(i, j = 1, ..., n)$, vertices j and i are not endpoints of the same edge.*

For example, the undirected, unweighted graph shown in Figure 3.1 is represented by the following adjacency matrix:

$$\begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

Definition 10. *A weighted* adjacency matrix *is an $n \times n$ matrix A. It represents an undirected, weighted graph with n vertices: if an element $a_{ij} > 0$ $(i, j = 1, ..., n)$, vertices j and i are endpoints of the same edge and the value of a_{ij} then equals the weight of the corresponding edge; if $a_{i,j} = 0$ $(i, j = 1, ..., n)$, vertices j and i are not endpoints of the same edge.*

For example, the undirected, weighted graph shown in Figure 3.3 is represented by the following adjacency matrix:

$$\begin{pmatrix} 0 & 0 & 4 & 1 \\ 0 & 0 & 5 & 3 \\ 4 & 5 & 0 & 3 \\ 1 & 3 & 3 & 0 \end{pmatrix}.$$

3.3 Matrices describing electrical networks

There are also matrices that describe electrical networks without providing straightforward information about the synthesis of the network they describe. The most important of those is the impedance matrix. Impedance is the classical term for the ratio between voltage and current [7].

Definition 11. *An* impedance matrix *is an $N \times N$ matrix Z that represents an N-port: it satisfies the equation $V = ZI$, where V is an $N \times 1$ vector where V_i $(i = 1, ..., N)$ represents the voltage at port i and I an $N \times 1$ vector where I_i $(i = 1, ..., N)$ represents the current at port i.*

An example of an impedance matrix is

$$Z = \frac{1}{24}\begin{pmatrix} 12 & 6 & 0 & 0 \\ 6 & 13 & 8 & -4 \\ 0 & 8 & 16 & -8 \\ 0 & -4 & -8 & 16 \end{pmatrix}.$$

The synthesis of an electrical network from an impedance matrix is not straightforward. It will be explained in section 5.2.2. The electrical network belonging to this particular matrix will be found in section 5.2.4.

Definition 12. *An* admittance matrix *is an $N \times N$ matrix Y that represents an N-port: it satisfies the equation $I = YV$, where V is an $N \times 1$ vector where V_i $(i = 1, ..., N)$ represents the voltage at port i and I an $N \times 1$ vector where I_i $(i = 1, ..., N)$ represents the current at port i.*

The synthesis of an electrical network from an admittance matrix is not straightforward. However, an admittance matrix is the inverse of the impedance matrix [6]. Therefore, once we know how to realise an impedance matrix, we will know how to realise an admittance matrix. For example, the inverse of the impedance matrix mentioned above is

$$Y = \begin{pmatrix} 3 & -2 & 1 & 0 \\ -2 & 4 & -2 & 0 \\ 1 & -2 & 3 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}.$$

Consequently, this admittance matrix will be realised in 5.2.4.

Chapter 4

The theorem of Reichert

Theorem 1 (Reichert)**.** *Any impedance of a one-port electrical network which can be realised with two reactive elements and an arbitrary number of resistors can be realised with two reactive elements and three resistors.*

Here, *impedance* refers to a 1×1 impedance matrix. Indeed, following definition in section 3.2 we find that for a one-port the impedance is a 1×1 matrix Z satisfying $V = ZI$, where V represents the port's voltage and I its current.

An N-port is a network with N connections to the outside world. We can view an N-port as a black box in which any number of resistors, capacitors and inductors may be tucked away; all we know is that out of that box, there are N ports connecting it to the outside world. These ports are made up of pairs of terminals [7].

For example, we can picture a 2-port as shown in Figure 4.1.

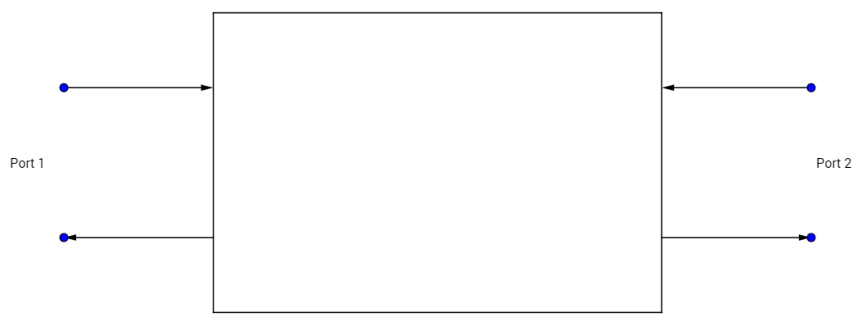

Figure 4.1: A 2-port

What happens inside the box is not directly clear from the network's impedance matrix. Moreover, there is still uncertainty as to what exactly are the sufficient and necessary conditions for impedance matrices to be realisable at all [12].

That leaves one term for explanation: reactive elements. There are two types of these: the capacitor and the inductor. The capacitor's symbol is a C, its law is $C\frac{dV}{dt} = I$ and its schematic representation is as shown in Figure 4.2 [11].

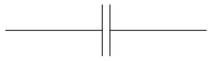

Figure 4.2: A capacitor

The inductor's symbol is L, its law is $V = L\frac{dI}{dt}$ and its schematic representation is as shown in Figure 4.3 [11].

Figure 4.3: An inductor

Lastly, a restistor's symbol is R, its law is $V = RI$ and its schematic representation is as shown in Figure 4.4. [11].

Figure 4.4: A resistor

So what the theorem of Reichert states, is that any impedance that can be realised with either

- one capacitor, one inductor, an arbitrary number of resistors;
- two capacitors, an arbitrary number of resistors;
- two inductors, an arbitrary number of resistors;

can be realised with, respectively,

- one capacitor, one inductor, three resistors;
- two capacitors, three resistors;
- two inductors, three resistors.

Chapter 5

Electrical networks and impedance matrices

5.1 From electrical network to impedance matrix

An impedance matrix is an $N \times N$ matrix Z that represents an N-port: it satisfies the equation $V = ZI$, where V is an $N \times 1$ vector where V_i $(i = 1, ..., N)$ represents the voltage at port i and I an $N \times 1$ vector where I_i $(i = 1, ..., N)$ represents the current at port i.
This provides us with the following equations:

$$\begin{cases} V_1 = Z_{11}I_1 + ... + Z_{N1}I_N \\ \vdots \qquad \vdots \qquad\qquad\quad \vdots \\ V_N = Z_{1N}I_1 + ... + Z_{NN}I_N. \end{cases}$$

It is clear that, in order to find element Z_{pq} of Z, all we have to do is open circuit (meaning $I = 0$) every port but p [10]. Then we can find Z_{pq} by using Kirckhoff's and Ohm's laws.

5.1.1 Example: 2-port

We will take a look at the network in Figure 5.1. This network is called a T-network because its resistors are ordered in a T-shape.

We let V_A denote the voltage across Z_A, V_B the voltage across Z_B and V_C the voltage across Z_C; I_A the current through Z_A, I_B the current through Z_B, I_C the current through Z_C; V_1 the voltage across port 1, V_2 the voltage across port 2 and I_1 the current through port 1, I_2 the current through port 2.

Since this network is a 2-port, its impedance matrix must be 2×2:

$$Z = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix}.$$

Now, to find Z_{11}, we open circuit port 2, meaning $I_2 = 0$. Using Kirckhoff's laws, we find $V_1 = (Z_A + Z_C)I_1$, so $Z_{11} = Z_A + Z_C$. To find Z_{21} we also open circuit port 2; first, we use Ohm's law to find $V_B = I_2 Z_B$ and $V_C = (I_1 + I_2)Z_C$. Then, we use Kirchhoff's law to find $V_2 = V_B + V_C = (I_1 + I_2)Z_C + I_2 Z_B = I_1 Z_C$ (because $I_2 = 0$). So $Z_{21} = Z_C$. We can find Z_{12} and Z_{22} in similar ways, which provides us with

$$Z = \begin{pmatrix} Z_A + Z_C & Z_C \\ Z_C & Z_B + Z_C \end{pmatrix}.$$

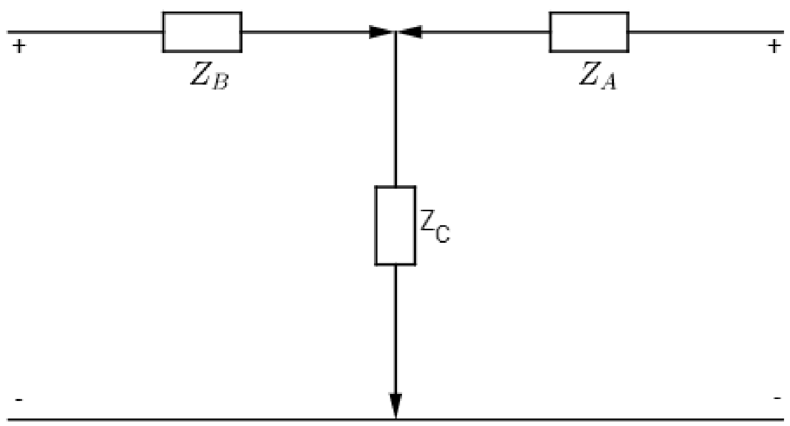

Figure 5.1: A T-network

5.2 From impedance matrix to electrical network

An impedance matrix provides information about the currents and the voltages of the ports of an electrical network. Such a network can be represented by a graph: edges represent elements such as resistors, capacitors or inductors. An incidence matrix provides information about what vertice is the endpoint of what edge. Therefore, both matrices describe a network. Yet it is easy to find a representing network out of an incidence matrix, whereas an impedance matrix does not directly tell us anything about its realisation. Many impedance matrices cannot be realised in the first place and research is still being carried out on the necessary and sufficient conditions that make an impedance matrix realisable [12].

5.2.1 Example: 2-port

We will take another look at the network in Figure 5.1. Since an impedance matrix has to be paramount [3] and all impedance matrices for reciprocal networks are symmetrical [10], we see that all 2×2 impedance matrices are realisable by a T-network: paramountcy means that the diagonal elements of a matrix will always be larger than (or equal to) its off-diagonal elements. So if we let $Z_C = Z_{21} = Z_{12}$, $Z_A = Z_{11} - Z_{21}$ and $Z_B = Z_{22} - Z_{12}$, we can realise the impedance matrix with the T-network shown in 5.1.

5.2.2 Step by step guide

We will now go through the process of retrieving the incidence matrix of a network from its impedance matrix step by step as described in [2] and [3].
Let Z be an $n \times n$ impedance matrix.

1. Find $Y = Z^{-1}$.

2. Find the element Y_{ij} of Y such that Y_{ij} is the off-diagonal non-zero element with the smallest absolute value.

3. Construct a vector v_1 of length n with $v_{1i}, v_{1j} \in \{-1, 1\}$ and all other elements in $\{0, -1, 1\}$. This construction has to follow a set of rules, which will be explained later on.

4. Substract the matrix $A_1 = |Y_{ij}| v_1 v_1^T$ from Y.

5. Work through step 1 to 3 again for $Y - A_1$, then $Y - A_1 - A_2$ etc, until you are left with a diagonal matrix $Y - A_1 - ... - A_k$.

6. For the diagonal elements $A_{k11}, ..., A_{knn}$ of $Y - A_1 - ... - A_k$ let $u_1, ..., u_n$ be the zero-vector if $A_{kii} = 0$ and e_i (the unit vector) if $A_{kii} \neq 0$.

7. Construct $\tilde{A} = [v_1\ v_2\ ...\ v_k\ u_1\ ...\ u_n]$.

8. Remove al zero-rows from $\tilde{A}$. Call this matrix A.

9. Find K, Q such that $A = -K^{-1}Q$, where K is the incidence matrix of a tree and Q the incidence matrix of a connected network. It is easiest to start with trying out a K and then solving the equation $A = -K^{-1}Q$ to find Q until you find a K that results in a unimodular Q with at most one 1 and one -1 in every column.

10. If Q and K are not yet incidence matrices, add a row with every element a $0, -1$ or 1 such that the result is an incidence matrix.

Q is the incidence matrix of the network. Using the definition in section 3.2, the electrical network can now easily be drawn.

We will now go into the rules described by [3] applying to picking $v_1, ..., v_k$. This means we assume that $Y - A_1, ..., Y - A_1 - ... - A_k$ is the diagonal matrix we were left with in step 5. We can then formulate the rules as follows:

1. The element Y_{ij} found in step 2 must be eliminated in step 4.

2. You may not choose four vectors such that A would have a submatrix (possibly after row permutations) of the following form: $\begin{pmatrix} * & * & 0 & * \\ * & 0 & * & * \\ 0 & * & * & * \end{pmatrix}$, where $*$ represents non-zero elements. This is called Tutte's characterisation [8].

3. The matrices $Y - A_1, ..., Y - A_1 - ... - A_k$ must all be paramount (note that Y is always paramount, since the inverse of an impedance matrix is an admittance matrix [6] and admittance matrices are always paramount [3]).

4. If $(Y - A_1 - ... - A_i)_{pq}$ $(p, q \in \{1, ..., n\})$ is negative, then $(Y - A_1 - ... - A_i - A_{i+1})_{pq}$ must either be negative or zero $\forall i \in \{0, ..., k-1\}$, where $A_0 = 0$. In other words, a negative elements may not undergo a sign switch in step 4.

5. If $(Y - A_1 - ... - A_i)_{pq}$ $(p, q \in \{1, ..., n\})$ is positive, then $(Y - A_1 - ... - A_i - A_{i+1})_{pq}$ must either be positive or zero $\forall i \in \{0, ..., k-1\}$, where $A_0 = 0$. In other words, positive elements may not undergo a sign switch in step 4.

6. If $(Y - A_1 - ... - A_i)_{pq}$ $(p, q \in \{1, ..., n\})$ equals zero, then $(Y - A_1 - ... - A_i - A_{i+1})_{pq}$ must also equal zero $\forall i \in \{0, ..., k-1\}$, where $A_0 = 0$. In other words, an element with value 0 must remain 0 in step 4.

7. For all $p, q \in \{1, ..., n\}$ the inequality $|(Y - A_1 - ... - A_i)_{pq}| \leq |(Y - A_1 - ... - A_i - A_{i+1})_{pq}|$ must hold $\forall i \in \{0, ..., k-1\}$, where $A_0 = 0$. In other words, an element may not increase in absolute value.

5.2.3 Example: 2-port

We will take a look at the impedance matrix we found in 5.2.1. Its inverse is

$$Z^{-1} = Y = \frac{1}{(Z_A+Z_C)(A_B+Z_C)+Z_C^2}\begin{pmatrix} A_B+Z_C & -Z_C \\ -Z_C & Z_A+Z_C \end{pmatrix}.$$

The off-diagonal element with the smallest absolute value is $-\dfrac{Z_C}{Z_AZ_B+Z_AZ_C+Z_BZ_C}$. There are four vectors we could choose from:

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The first two would not eliminate the choosen element in step 4, so we can choose either one of the other vectors (the effect will be the same). We choose $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Next, we compute

$$\begin{aligned} &\frac{1}{Z_AZ_B+Z_AZ_C+Z_BZ_C}\begin{pmatrix} A_B+Z_C & -Z_C \\ -Z_C & Z_A+Z_C \end{pmatrix} - \frac{Z_C}{Z_AZ_B+Z_AZ_C+Z_BZ_C}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \\ = &\frac{1}{Z_AZ_B+Z_AZ_C+Z_BZ_C}\begin{pmatrix} A_B & 0 \\ 0 & Z_A \end{pmatrix}. \end{aligned}$$

This means that

$$\tilde{A} = A = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}.$$

For the decomposition $A = -K^{-1}Q$, we choose

$$K = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, Q = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}.$$

In order to make incidence matrices out of these we have to add a row to both matrices:

$$K = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{pmatrix}, Q = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & -1 \end{pmatrix}.$$

Before we draw the network this results in, we want to know the values of the resistors on the three edges. We do this by decomposing Y as follows:

$$Y = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{Z_C}{Z_AZ_B+Z_AZ_C+Z_BZ_C} & 0 & 0 \\ 0 & \frac{Z_B}{Z_AZ_B+Z_AZ_C+Z_BZ_C} & 0 \\ 0 & 0 & \frac{Z_A}{Z_AZ_B+Z_AZ_C+Z_BZ_C} \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} [4].$$

Here, the values in the diagonal matrix represent the values of the resistors on the corresponding edges. It results in the network shown in Figure 5.2. Indeed, this network is equivalent with the one in Figure 5.1 we started with [1].

5.2.4 Example: 4-port

To illustrate the guide with a more complicated example, we will follow the example of the impedance matrix

$$Z = \frac{1}{24}\begin{pmatrix} 12 & 6 & 0 & 0 \\ 6 & 13 & 8 & -4 \\ 0 & 8 & 16 & -8 \\ 0 & -4 & -8 & 16 \end{pmatrix}$$

as shown briefly in [3].

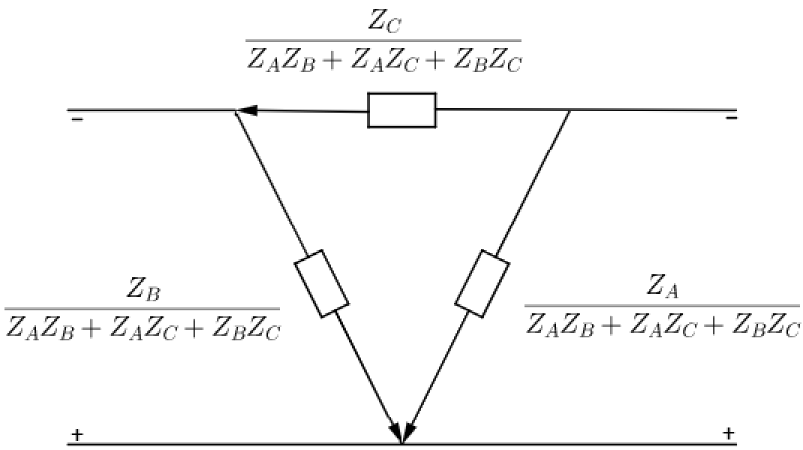

Figure 5.2: The resulting network

1. Inverting the matrix gives

$$Z^{-1} = Y = \begin{pmatrix} 3 & -2 & 1 & 0 \\ -2 & 4 & -2 & 0 \\ 1 & -2 & 3 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}.$$

2. Choose $Y_{13} = 1$.

3. There is only one essentially unique option for v_1, and that is

$$v_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}.$$

For the process of eliminating options, see appendix A.1.

4.

$$\begin{aligned} Y - |Y_{ij}|v_1v_1^T &= \begin{pmatrix} 3 & -2 & 1 & 0 \\ -2 & 4 & -2 & 0 \\ 1 & -2 & -3 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix} - \begin{pmatrix} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 3 & -1 & 0 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}. \end{aligned}$$

5. Going through the process again until we are left with a diagonal matrix, we find $v_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$, $v_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$ and $v_4 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$.

6. We are left with the diagonal matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

   It follows that $u_1 = e_1, u_2 = e_2, u_3 = 0, u_4 = e_4$.

7.
$$\tilde{A} = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

8.
$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ -1 & -1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

9. We start the last step by looking at options for K^{-1} and find that there is only one possibility. For this process, see appendix A.2.
   Once we have K, all we have to do is invert it and solve Q. We find:

$$K = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}, \ Q = \begin{pmatrix} -1 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

10. We notice that both K and Q contain rows with only one element. This means that, to realise the network, we have to add an extra vertex. We simply add an extra row to both K and Q and fill it such that every column contains exactly one -1 and one 1:

$$K = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \ Q = \begin{pmatrix} -1 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

This results in the network shown in Figure 5.3.

5.3 Impedance matrices of RLC-networks

So far, we have only looked at impedance matrices of networks with scalar inputs. However, as we recall the laws stated in chapter 4, when inductors and capacitors are involved, impedance matrices will have to contain functions. We call a network with resistors, capacitors and inductors an RLC-network.

Let us look, for example, at the simple one-port network in Figure 5.4. We cannot include the equations $C\frac{dV}{dt} = I, V = L\frac{dI}{dt}$ and $V = RI$ in our matrix immediately; we first have to

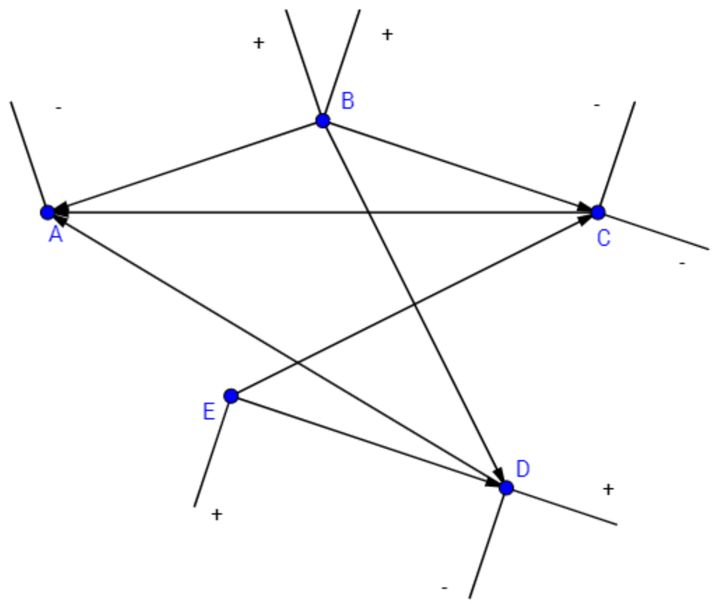

Figure 5.3: The resulting network

apply a Laplace transformation, which gives us $V = IR, V = ILs$ and $V = \frac{I}{Cs}$ [7]. So Ohm's law states, in this case, that $V = (R+Ls+\frac{1}{Cs})I$. Thus we find the impedance $Z = R+Ls+\frac{1}{Cs}$.

We know that an impedance Z can be represented as $Z = ADA^{-1}$ with A as found in the method described in section 5.2.2 and D a diagonal matrix [2]. If Z is the impedance matrix of a resistive matrix, element d_{ii} represents the weight of the edge represented by Q's ith column. If Z is the impedance matrix of an RLC-matrix, D can be written as $D = D_1 + \frac{1}{s}D_2 + sD_3$, where D_1 represents the values of the resistive edges, D_2 of the capacative edges and D_3 of the inductive branches [2]. This means that $Z = AD_1A^{-1} + A\frac{1}{s}D_2A^{-1} + AsD_3A^{-1}$.

In the next section we will go into Kron reduction on resistive networks, which means that theoretically we could reduce the network represented by AD_1A^{-1}. However, reducing it would imply changes in A; therefore, we cannot change the resistive edges without risking changing the capacitive and inductive edges.

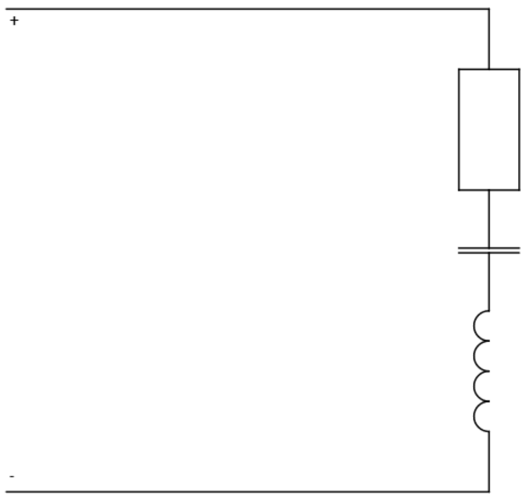

Figure 5.4: A simple RLC-network

Chapter 6

Kron reduction on resistive networks

Suppose we have a resistive network with n vertices. Kron reduction will help us reduce this to a network to a chosen number of vertices between 2 and $n-1$. The idea behind this, is that we rearrange a network's adjacency matrix such that we capture all the information about the vertices we want to keep in one submatrix (Q_{aa}) and all the information about the other vertices in three other submatrices, shaping Q as follows:

$$\begin{pmatrix} Q_{aa} & Q_{ab} \\ Q_{ba} & Q_{bb} \end{pmatrix}.$$

Then, we manipulate the four submatrices such that all information is captured a new matrix with the same size as Q_{aa}, meaning that we have found a new network with the same properties for the vertices we choose to keep and without all the other vertices.

In this chapter, we will give an explanation of the Kron reduction method as mentioned in [5] and illustrate it with an example.

6.1 Step by step guide

Given a network's adjancency matrix A $(n \times n)$, we reduce the network to a network with m vertices, with $m \in \{2, ..., n-1\}$ in the following way:

1. Choose the $m \geq 2$ vertices you want to keep and call those $v_1, ..., v_m$. Call the other vertices $v_{m+1}, ..., v_n$.
2. Reorder A such that its first m rows and columns represent $v_1, ..., v_m$ and rows $m+1, ..., n$ represent $v_{m+1}, ..., v_n$. Call this reordered matrix $\tilde{A}$.
3. Calculate $Q(\tilde{A})$ as follows: $q_{ij} = \begin{cases} -a_{ij} & \text{for } i \neq j \\ a_{ij} + \sum_{j=1}^{n} a_{ij} & \text{for } i = j \end{cases}$.
4. Let Q_{ab} denote the submatrix of $Q(\tilde{A})$ with the elements $q_{ij} : i \in a, j \in b$, where $a = \{1, ..., m\}, b = \{m+1, ..., n\}$.
5. Compute $\tilde{Q} = Q_{aa} - Q_{ab} Q_{bb}^{-1} Q_{ba}$.
6. The weighted adjacency matrix of the Kron reduced network is given by

$$-\tilde{Q} + \operatorname{diag}\left(\left\{\sum_{j=1, j\neq i}^{n} Q_{ij}\right\}_{i=1}^{n}\right).$$

6.2 Example

We will take a look at the network in Figure 6.1 from [5].

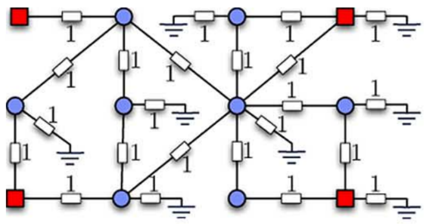

Figure 6.1: A resistive network [5]

First, we need to find its adjacency matrix:

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0
\end{pmatrix}.$$

Here, we have have labeled the vertices from left to right, then top to bottom. The red vertices are the ones we want to keep, the blue ones are the ones we want to keep. The white resistors are not vertices, they are edges.
Because we labeled the vertices before finding the adjacency matrix, we do not have to re-order it.
Now, we find that

$$\begin{aligned}
Q(A) &= \operatorname{diag}(1,3,2,3,4,3,3,3,7,4,4,3) + \operatorname{diag}(0,1,0,1,0,1,1,1,1,1,1,0) - A \\
&= \begin{pmatrix}
1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 3 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 \\
-1 & 0 & 0 & 0 & 4 & 0 & -1 & -1 & -1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 3 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & -1 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 3 & 0 & 0 & -1 & 0 \\
0 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & 7 & -1 & -1 & -1 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 4 & 0 & -1 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 4 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 3
\end{pmatrix}
\end{aligned}$$

This means that

$$
\begin{aligned}
Q_{aa} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}, \\
Q_{ab} &= \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 \end{pmatrix}, \\
Q_{bb} &= \begin{pmatrix} 4 & 0 & -1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 3 & 0 & 0 & -1 & 0 \\ -1 & -1 & 0 & 0 & 7 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 4 & 0 & -1 \\ 0 & 0 & 0 & -1 & -1 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & 3 \end{pmatrix}, \\
Q_{ba} &= \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.
\end{aligned}
$$

Next, we have to invert Q_{bb}:

$$
Q_{bb}^{-1} = \frac{1}{18760} \begin{pmatrix} 6072 & 396 & 2024 & 2316 & 1188 & 432 & 876 & 540 \\ 396 & 6653 & 132 & 253 & 1199 & 436 & 363 & 545 \\ 2024 & 132 & 6928 & 772 & 396 & 144 & 292 & 180 \\ 2316 & 253 & 772 & 7733 & 756 & 276 & 2123 & 345 \\ 1188 & 1199 & 396 & 759 & 3597 & 1308 & 1089 & 1635 \\ 432 & 436 & 144 & 276 & 1308 & 5592 & 396 & 2300 \\ 876 & 363 & 292 & 2123 & 1089 & 396 & 5493 & 495 \\ 540 & 545 & 180 & 345 & 1635 & 2300 & 495 & 7565 \end{pmatrix}.
$$

We find

$$
\tilde{Q} = \frac{1}{18760} \begin{pmatrix} 12688 & -1584 & -2900 & -972 \\ -1584 & 43632 & -1980 & -3924 \\ -2900 & -1980 & 24515 & -1215 \\ -972 & -3924 & -1215 & 38523 \end{pmatrix}.
$$

So we find the new weighted adjacency matrix

$$
\begin{aligned}
& -\tilde{Q} - \operatorname{diag}(5456, 7488, 6095, 6111) \\
= {} & \frac{1}{18760} \begin{pmatrix} 7232 & 1584 & 2900 & 972 \\ 1584 & 36144 & 1980 & 3924 \\ 2900 & 1980 & 18420 & 1215 \\ 972 & 3924 & 1215 & 32412 \end{pmatrix}.
\end{aligned}
$$

This corresponds to the network shown in Figure 6.2.

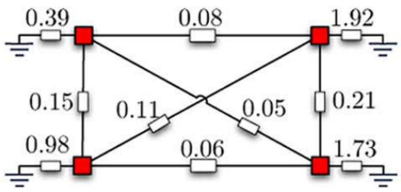

Figure 6.2: A reduced resistive network [5]

6.3 Kron reduction and impedance matrices

We have seen that an $N \times N$ impedance matrix represents a black-box with N connections to the outside world; it only discribes the ratio between voltage and current of the ports connecting the black box to the outside world. In section 5.2 we have seen how we can find a network to a realisable resistive impedance matrix. There is no telling how many resistors this process will leave our network with. Now, if there more than we want there to be, we can use Kron reduction to reduce the number of resistors without changing the currents and voltages at the ports and, therefore, without affecting the impedance matrix.

Chapter 7

Conclusions

The theorem of Reichert is a theorem that states that any one-port impedance that can be realised with two reactive elements and an arbitrary number of resistors can be realised with two reactive elements and three resistors.

Impedance matrices cannot be read as a straightforward representation of an electrical network. However, they can be decomposed as a product of incidence matrices and diagonal matrices that do. If the impedance matrix is resistive, after this process Kron reduction can be used to reduce the network to a network of two or more resistors. If the impedance matrix is also partially inductive and/or capacitive, Kron reduction cannot be applied on the resistive part of the network without risking affecting the inductive and/or capacitive part of the network.

Further understanding of RLC-networks is required if the theorem is to be solved through graph theory, linear algebra and differential equations.

Bibliography

[1] Bollobás, B. (1998). *Modern Graph Theory.* New York: Springer-Verlag.

[2] Cederbaum, I. (1958). Conditions for the Impedance and Admittance Matrices of n-Ports without Ideal Transformers. *Proceedings of the IEE - Part C: Monographs, Volume 105* (issue 7), pages 245-251.

[3] Cederbaum, I. (1959). Applications of Matrix Algebra to Network Theory. *IRE Transactions on Circuit Theory, Volume 6* (issue 5), pages 127-137.

[4] Cederbaum, I. (1961). Paramount Matrices and Synthesis of Resistive N-Ports. *IRE Transactions on Circuit Theory, Volume 8* (issue 1), pages 28-31.

[5] Dörfler, F., Bullo, F. (2013). Kron Reduction of Graphs With Applications to Electrical Networks. *IEEE Transactions on Circuits and Systems I: Regular Papers, Volume 60* (issue 1), pages 150-163.

[6] Easwaran, V., Gupta V.H., Munjal, M.L. (1993). Relationship between the Impedance Matrix and the Transfer Matrix with Specific Reference to Symmetrical, Reciporcal and Conservative Systems. *Journal of Sound and Vibration, Volume 161* (issue 3), pages 515-525.

[7] Guillemin, E.A. (1953). *Introductory Circuit Theory.* New York: Jorn Wiley & Sons, Inc.

[8] Schrijver, A. (1998). *Theory of Linear and Integer Programming.* Chichester: John Wiley.

[9] Jiang, J.Z., Smith, M.C. (2012). On the Theorem of Reichert. *Systems & Control Letters, Volume 61* (issue 12), pages 1124-1131.

[10] Whites, K.W., professor, lecture notes for Department of Electrical and Computer Engineering, South Dakota School of Mines and Technology (2016).

[11] Olsder, G.J., Woude, J.W. van der, Maks, J.G., Jeltsema, D. (2011). *Mathematical Systems Theory* 4th edition. Delft: VSSD.

[12] Wang, K., Chen, M.Z.Q. (2015). Minimal Realizations of Three-Port Resistive Networks. *IEEE Transactions on Circuits and Systems I: Regular Papers, Volume 62* (issue 4), pages 986-994.

[13] Zhang, S.Y., Jiang, J.Z., Smith, M.C. (2016). A New Proof of Reichert's Theorem. *2016 IEEE 55th Conference on Decision and Control (CDC)*, pages 2615-2619

Appendices

Appendix A

Justifying the choices in section 5.2.4

A.1 The choice of v_1

Now we find a suitable vector v_1. Since we picked element Y_{13}, we have to choose elements v_{11} and v_{13} from $\{-1, 1\}$; this leaves us with 36 options. However, all 24 options that have a non-zero element v_{14} cannot be chosen, for they would violate rule 6. We also find that $\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}$ are not options, for they would violate rule 1. We will illustrate this using $\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$:

$$\begin{aligned} Y - |y_{13}|v_1v_1^T &= \begin{pmatrix} 3 & -2 & 1 & 0 \\ -2 & 4 & -2 & 0 \\ 1 & -2 & -3 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 2 & -1 & 2 & 0 \\ -2 & 4 & -2 & 0 \\ 2 & -2 & -4 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}. \end{aligned}$$

Clearly, Y_{13} is not eleminated.

The vectors $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ -1 \\ -1 \\ 0 \end{pmatrix}$ violate rule 3. We will illustrate this using $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$:

$$\begin{aligned} Y - |Y_{13}|v_1v_1^T &= \begin{pmatrix} 3 & -2 & 1 & 0 \\ -2 & 4 & -2 & 0 \\ 1 & -2 & -3 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 2 & -3 & 0 & 0 \\ -3 & 3 & -3 & 0 \\ 0 & -3 & -4 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}. \end{aligned}$$

In the new matrix, $2 = |Y_{11}| < |Y_{12}| = 3$, meaning the matrix is no longer paramount.

Lastly, the vectors $\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$ will violate rule 7, which we will illustrate using $\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$:

$$\begin{aligned} Y - |Y_{13}|v_1 v_1^T &= \begin{pmatrix} 3 & -2 & 1 & 0 \\ -2 & 4 & -2 & 0 \\ 1 & -2 & -3 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 2 & -2 & 0 & 0 \\ -2 & 4 & -2 & 0 \\ 0 & -2 & -4 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}. \end{aligned}$$

We see that Y_{33} has increased in value.

Therefore, the only two vectors we can choose are $\begin{pmatrix} -1 \\ 1 \\ -1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}$. They will both result in the same matrix $|Y_{ij}|v_1 v_1^T$, which agrees with the statement that the matrix Q we will find will be essentially unique [3]: essential uniqueness allows differences in the order or sign (or both) of columns of Q [3].

A.2 The choice of K

In section 5.2.4 we mentioned that there is only one possibility for K and Q. We will illustrate this here.

There are 27 different possibilities for the network represented by K, as shown in Figure A.1. We will go through the process of checking whether a network can produce an incidence matrix Q for the top left network. Its incidence matrix is

$$K = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

Which means that

$$-K^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

Next, we solve the equation $A = -K^{-1}Q$. In this case, this results in

$$Q = \begin{pmatrix} -1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & -1 & -1 & 0 & -2 & -2 & 0 \\ 0 & 2 & 1 & 0 & 3 & 1 & 1 \\ 0 & -1 & 0 & -1 & -1 & 0 & 0 \end{pmatrix}.$$

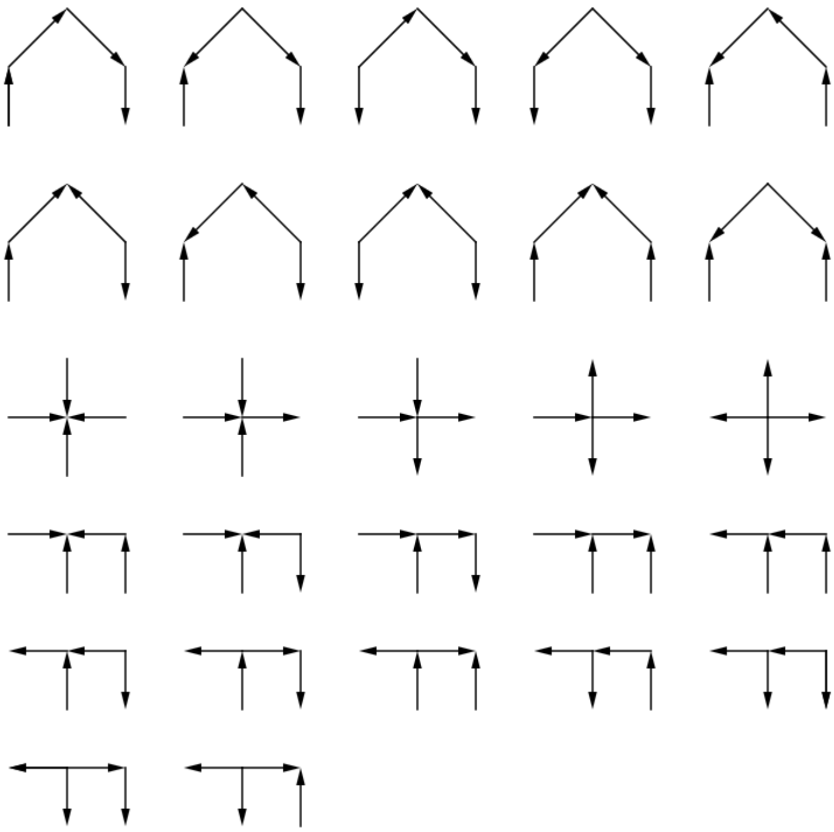

Figure A.1: Possible choices

We see that this matrix does not fit into the definition of an incidence matrix and therefore the top left network is not an option for the solution of this problem. In the top two rows of options, all but one have the same problem: they all have at least one element in Q with absolute value larger than one. The only one that provides us with an incidence matrix for Q is the network in A.2, with $K, -K^{-1}$ and Q as follows:

$$
\begin{aligned}
K &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}, \\
-K^{-1} &= \begin{pmatrix} -1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & -1 & -1 & 0 \\ -1 & -1 & -1 & -1 \end{pmatrix}, \\
Q &= \begin{pmatrix} -1 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & -1 \end{pmatrix}.
\end{aligned}
$$

All networks in the last four rows are unable to produce an incidence matrix Q, due to the

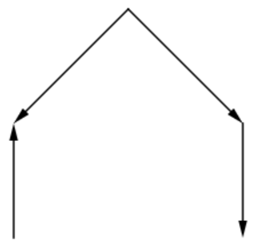

Figure A.2: Only possible network

column $\begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}$ in A: it will either cause Q to have a column with more than three elements, or it will cause Q to have a column with two elements of the same value. Both are not possible in an incidence matrix. Therefore, the only possibility for K is the matrix representing $A.2$.