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MASTER THESIS

Critical Percolation in Geometric Inhomogeneous Random Graphs

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Abstract

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by Zylan Benjert

Geometric inhomogeneous random graphs (GIRGs) are a class of random graphs where each vertex has an assigned weight coming from a power law distribution, and a spatial location according to a Poisson point process on the underlying geometric space. We consider GIRGs on the d-dimensional torus $\mathbb{T}^d := [-n^{1/d}/2, n^{1/d}/2)^d$ and perform percolation with edge-retention probability $\pi_n = \lambda n^{(3-\tau)/2}$ for some $\lambda > 0$. We prove that there exists a λ_c, such that for all $\lambda > \lambda_c$ a largest component unique in its order of magnitude exists within the highest $\lceil an^{(\tau-3)/2} \rceil$ weight vertices (the core) for some $a > 0$. Furthermore, we also prove novel results on the spatial distribution of the highest weight vertices. Starting from some fixed distance r_0, we sweep through the powers of 2 by defining $r_i := 2^i r_0$ and show that the amount of vertex pairs between r_i and r_{i+1} concentrates within its own expectation. We then consider the slightly supercritical regime, where we have already established the existence of a giant within the core. Extending the core to the entire graph, we establish that the largest component consists of mostly the emerging giant inside the core, together with the vertices reached outside the core, which we call its span. We then show a law of large numbers result on the size of the span of the giant component inside the core, where we show that for any $\varepsilon > 0$, we have that $\zeta^\lambda - \varepsilon \leq \frac{\text{Span}(\mathscr{C}^a_{(1)})|}{\sqrt{n}} \leq \zeta^\lambda + \varepsilon$.

Preface

This thesis is the conclusion to my master's degree in Applied Mathematics, and the culmination of almost six years of studying at TU Delft. With this time coming to an end, I would like to take this space to express my gratitude to a few special people that have been important to me during this time.

I would like to start by thanking the most important person in the development of my thesis, my supervisor Júlia Komjáthy. Working with her for almost 2.5 years now on various projects, she has opened an incredible amount of doors for me in the world of academia, resulting in papers being published, meeting many new people in the field, an internship in Zürich, and finally this thesis. She has helped me develop myself immensely as a mathematician and I would simply not be where I was today without her help, supervision, and advice. I cannot stress how thankful I am for everything she has done for me, and I do genuinely hope that this will not be the last time that our paths cross.

Next, as it is impossible to name everyone that deserves it, I would like to thank a specific few of my friends that have supported me and been there for me academically, socially, and in many other ways that I can't even begin to express over the past year. As to not turn any heads, in alphabetical order, a huge thank you to: Caoimhe, Charlotte, Elena, Julius, Mirte, and Quinten. My achievements are as much mine as they are yours, as I really could not have done all of this without you.

Lastly, I would like to thank my family for always supporting me in the ways they could and being there for me. You have all always had nothing but support for me in my pursuit of this degree and I will always be thankful for that.

Contents

1 Introduction

Networks are at the core of nature, sociology, technological advancements [42], and are a key structure of almost everything that exists. The way that networks appear in life are vast and has an incredibly broad area of applications and occurrences. For instance, consider mycelium, fungal networks that allow fungi to communicate with each other which have been around billions of years. Alternatively, we have the much more recent introduction of social media into our daily lives, where social networks have become increasingly larger and more important to the average person. Lest we forget about airport networks, or electrical grids [1], even the neuronal structure of the brain and how information passes through [45, 46]. Networks are everywhere, and when something is such an all-encompassing force throughout all aspects of life, leave it up to mathematicians to figure out a way to model and analyze the different properties that may occur.

1.1 Graph Theory

The introduction of graphs and graph theory is wildly regarded to have been in 1736, when Euler published a paper on the seven bridges of Königsberg [26]. The core idea of graph theory and why it is significant, is the analysis of objects (graphs) where there are directed or undirected relationships between nodes, or what we call vertices. These relationships can be for instance connections in a social network, but also a link between airports when there are flights happening between them.

A graph in its simplest form consists of only two things, its *vertices*, and its *edges*. We will commonly use the notation $G = (V, E)$ to indicate the vertex set V, and edge set E of graph G. Edges can be directed, meaning the link between vertices u and v only happens one way, i.e a hyperlink on a webpage that directs the user to a new website, or undirected, where the edge always exists in both directions, i.e in molecular structures, where atoms are connected to one another or are not. The vertex and edge sets together completely define a graph, and can be used to analyze different properties of different graphs.

A commonly studied property of graphs is the degree distribution of a network. Each vertex has a number of outgoing edges to other vertices, each node it connects to is called a neighbour, and the degree of vertex v, D_v, is its total number of neighbours. The degree distribution of a graph on n vertices is then defined as

$$p_k = \frac{n_k}{n}, \tag{1.1}$$

where n_k is the amount of vertices with degree k. Within real networks it is often observed that the degree distributions follow a power law distribution, i.e $p_k \propto k^{-\tau}$ for some τ which is called the power law parameter, where typically $\tau \in (2,3)$. When the degree distribution of a network follows a power law, we call the graph *scale-free* [48].

Scale free networks have been observed in multitude, ranging from paper citations, to webpage hyperlinks, to social networks [2, 6, 27, 34, 49]. The discovery even lead to a new model of random graphs called the preferential attachment model, introduced by Albert and Barabási [5], where vertices with high degree are more likely to connect to new vertices that join the graph.

Another property commonly studied is what we call the *small-world* property. First coined by Strogatz and Watts in 1998 [50], this is a property in graphs that more colloquially known as *six degrees of separation*, stemming from the famous experiment of Stanley Milgram in the 1960's [40], where he tried to connect any two randomly chosen people in the United States through letters to acquaintances. This showed the remarkable fact that on average it took around 5 or 6 connections to reach the intended target. Within graph theory, we consider this with respect to the *graph-distance* $d_G(u, v)$ between two vertices u and v, which is defined as the length of the shortest path. We then call a graph on n vertices *small-world*, if for two uniformly chosen vertices u and v, we have

$$d_G(u, v) \propto \log n. \tag{1.2}$$

1.1.1 Random graphs

Networks are naturally found, but the creation of one can be modelled in a stochastic manner. By connecting any two vertices u and v according to some connection probability, typically denoted by p_{uv}, we have created a random graph. The simplest example of this is the Erdős-Rényi model [24], where we independently flip a coin for each possible edge present in the graph with the same probability p.

However, the Erdős-Rényi random graph does not exhibit the scale-free property in its degree distribution. As every edge exists independently with probability p, the degree distribution ends up being being a binomial distribution, and does not follow a power law. Therefore, in the pursuit of finding models that mimic real networks, new advancements had to be made to incorporate the scale-free property which is so often observed.

There are many more random graph models that exist and are far more complex than the Erdős-Rényi model. However, there is a particular subset of them that we are especially interested in with regards to the connection to real networks.

Chung-Lu random graphs [17, 18, 19] are a precursor to the main model of this thesis, geometric inhomogeneous random graphs. On a set of n vertices, each vertex is prescribed a (random) weight such that the degree sequence follows a power law. Then we connect any two vertices u and v, with respective weights w_u and w_v with probability

$$p_{uv} = \left(1 \wedge \frac{w_u w_v}{W}\right), \tag{1.3}$$

where W is the total sum of all the weights $\sum_{i \in V(G)} w_i$.

There however, is no inclusion of any spatial dimension yet. As we will see, geometric inhomogeneous random graphs are a generalization of Chung-Lu graphs, where a spatial dimension is added.

1.2 Spatial Random Graphs

Random graphs as we have discussed them so far have simply been a collection of vertices where edges are formed between them randomly, yet what we often observe is that many networks have some underlying spatial structure. If we consider social networks, be it a social media network or simply considering the theoretical framework of people in your life, then there is a very obvious spatial component at play as we are much more likely to connect to people close to us geographically. This is where spatial random graphs come in, essentially all we add to the set of vertices is a location for each vertex on some space. The interesting part comes from how the connection probabilities change depending on the distance between two vertices.

The simplest model of a spatial random graph is the random geometric graph [44]. Here we first place n vertices randomly in some space according to some specified distribution, and then connect any two vertices if their distance is less than some R. This model however lacks long range edges and is purely formed by short (less than R) edges, taking R "too large" simply creates a complete graph and rids the network of any complexity.

Graphs formed from real networks often exhibit a few key characteristics, they are often scale-free, small-world, and include spatiality, where predominantly the networks are formed by short range edges but the long edges (sometimes called weak ties) do exist and create important connections between the local components [29].

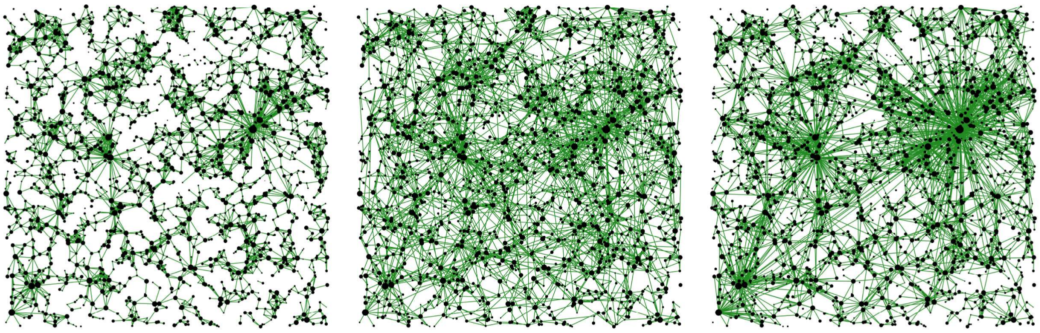
1.2.1 Geometric Inhomogeneous Random Graphs

Geometric Inhomogeneous Random Graph (GIRGs) [14] are a group of synthetic models that carry all of the aforementioned properties. GIRGs exhibit a scale-free degree distribution, the small-world property, high clustering (many triangles are present in the graph), and an existence of weak ties. Due to this they are an excellent candidate for modelling many real networks.

Vertices within a GIRG are ascribed a random weight, sampled from a power law distribution with parameter τ. Independently from their weight, all vertices are randomly placed in space, which we will always assume to be the d-dimensional volume-n torus $\mathbb{T}^d := [-n^{1/d}/2, n^{1/d}/2)^d$, according to a Poisson point process with intensity 1. Then, any two vertices are independently connected according to their weight, distance, and the *long-range parameter* $\alpha > 1$ in the following way:

$$p_{uv} := \min\left(1, \frac{w_u w_v}{|u-v|^d}\right)^{\alpha}. \tag{1.4}$$

This definition of the connection probability ensures that vertices very close together will always connect. However, it is also extremely likely that very high weight vertices also connect to each other. This creates a true hub structure within the graph, where hubs are formed from high weight vertices that connect to low weight vertices at small distances, but also connect to other high weight hubs far away.



(A) GIRG with parameters $\tau = 3.7$, $\alpha = 6$.

(B) GIRG with parameters $\tau = 3.7$, $\alpha = 1.7$.

(C) GIRG with parameters $\tau = 2.8$, $\alpha = 6$.

FIGURE 1.1: Illustration of the Geometric Inhomogeneous Random Graph model for three different settings of the parameters τ and α. The graphs were generated by assigning a position (chosen uniformly at random in the unit square $[0,1]^2$) and a random weight w (following a power-law with exponent τ) to 2000 vertices. Each pair of vertices $\{u, v\}$ draws an i.i.d. uniform variable $E_{uv} \in [0,1]$, and is connected by a link if $E_{uv} \leq c_1 \cdot (1 \wedge c_2 \cdot (w_u w_v / (\mathbb{E}[W] \|u - v\|^2))^\alpha)$. The constant c_2 was set to $c_2 = 0.7$ here, while the constant c_1 was varied so that the three graphs all have $\sim$ 5800 links. The position, weights, and link-existence random variables are coupled on the three graph instances. The size of a node is proportional to its weight.

Increasing the long-range parameter α penalizes edges further away more, and dictates how many weak links are present within the graph. In Figure 1.1, three different realizations of GIRGs are given with various parameters, to showcase the effect of changing both τ and α.

Many results about GIRGs are already known, such as the fact that the average distance is of order $O(\log \log n)$ [13] and thus the graph is ultra-small world. Furthermore, in [13] it is also shown that GIRGs exhibit a giant component and the diameter of the graph is poly-logarithmic, in the threshold model ($\alpha = \infty$) the diameter turns out to even be $\Theta(\log n)$ [8]. Another key algorithmic property that helps in applying GIRGs is the fact that they can be sampled in expected linear time [12], which was an original motivation of the introduction of the model as it was a significant improvement over the expected sampling time of hyperbolic random graphs.

1.3 Percolation

Percolation theory on random graphs is the study of a phase transition with respect to a giant component. Edges in a (random) graph on n vertices are deleted with probability p (possibly dependent on n), the phase transition occurs at some critical value p_c, where for every $p > p_c$ a giant component arises.

Originally introduced by Broadbent and Hammersly in 1957 [16], percolation processes were used to study the contrary of *diffusion processes* regarding fluid movement inside a medium. Percolation here focuses on the random properties of the medium, as opposed to diffusion processes that focus on the stochastic nature of the fluid.

Therefore, the problem now known as *bond percolation* was the first venture into percolation theory. Bond percolation is a problem posed as the question, can a fluid move its way through a porous material from top to bottom? Within this porous medium the gaps, or holes through which the fluid must pass are randomly created, making this a stochastic problem that very naturally lends itself to graph theory.

Earlier we mentioned the Erdős-Rényi (ER) graph, where for n vertices we add an edge with probability p independently for each possible edge. This is the same as simply performing percolation on the complete graph of n vertices, making the ER graph the same as a percolation problem. It has been shown in [25] that the critical percolation probability for Erdős-Rényi graphs is $1/n$, more specifically, if we let the percolation probability be of the form λ/n, then for $\lambda < 1$ we have that all components are the same order of size, namely $\Theta(\log n)$, where as when $\lambda > 1$ we have a giant component that arises which has a size that is linear n. This is what we call sub and super-criticality of the graph respectively.

Typically, the term super-criticality is used when there the giant that appears is of linear size. However, what we will see is that in GIRGs and in similar models the giant that arises is of lesser order, which is why we will use the term slightly supercritical to describe the graph.

In fact, the Erdős-Rényi random graph also exhibits slight supercritical behaviour in its phase transition, see for example [3]. Where for connection (percolation) probability with parameter $\lambda \to \infty$, $p = \frac{1+\varepsilon}{n}$, with $\varepsilon := \lambda n^{-1/3}$, we have that the largest connected component is of size $2\lambda n^{2/3}$. However, the second largest component is of order $\Theta(n^{2/3}\lambda^{-2}\ln\lambda)$. What we will show in GIRGs is that the slightly supercritical regime looks very different, in the sense that the largest component will have a magnitude of order $\Theta(\sqrt{n})$, and the second largest component $o(\sqrt{n})$.

1.3.1 Percolation on GIRGs

GIRGs are much more complex than the Erdős-Rényi model, with the added component of spatiality we have a lot more to consider when analyzing the critical percolation probability and the size of the components. If we consider the absence of a spatial dimension, and look at models that are also scale-free and have weights associated to its vertices, then we get a strong intuition for how we could proceed and on the conjectured orders of the components, and their corresponding critical percolation probabilities. This is exactly what is done in [9]. Bhamidi, Dhara, and van der Hofstad show that the critical percolation probability in a class of random graphs that are variants of the Chung-Lu model, meaning vertex weights follow a power law distribution, is of order $\Theta(\lambda n^{\frac{\tau-3}{2}})$, where there also exists a critical λ_c such that for all $\lambda > \lambda_c$ the graph is (slightly) supercritical and the giant that arises has size $\Theta(\sqrt{n})$.

1.3.2 First-passage Percolation and Applications

First-passage percolation (FPP) is a generalization of the classic percolation problem [32, 36]. Each edge is given a (random) time, and we are interested in the time it takes for a vertex to be reached from some source node. Here, we will focus mainly on its application to epidemiology, this is due to the fact that GIRGs are a natural candidate to model social networks. Social networks have been shown to exhibit for instance the small world property [40], contain weak ties [29], and high clustering [50]. These are all properties that GIRGs also exhibit.

The connection of FPP to epidemiology is also natural, consider patient zero being the source node of the network, then the analysis of how fast other vertices are reached and how the infection travels throughout the network is of utmost importance when trying to understand the spread of an epidemic.

To show this in practice we consider the Gowalla dataset [38], a social network with geographical locations for each user. We have modelled the Gowalla dataset as a GIRG in [7] and even a simple eye-test of Figure 1.2 shows the similar structure between the graphs.

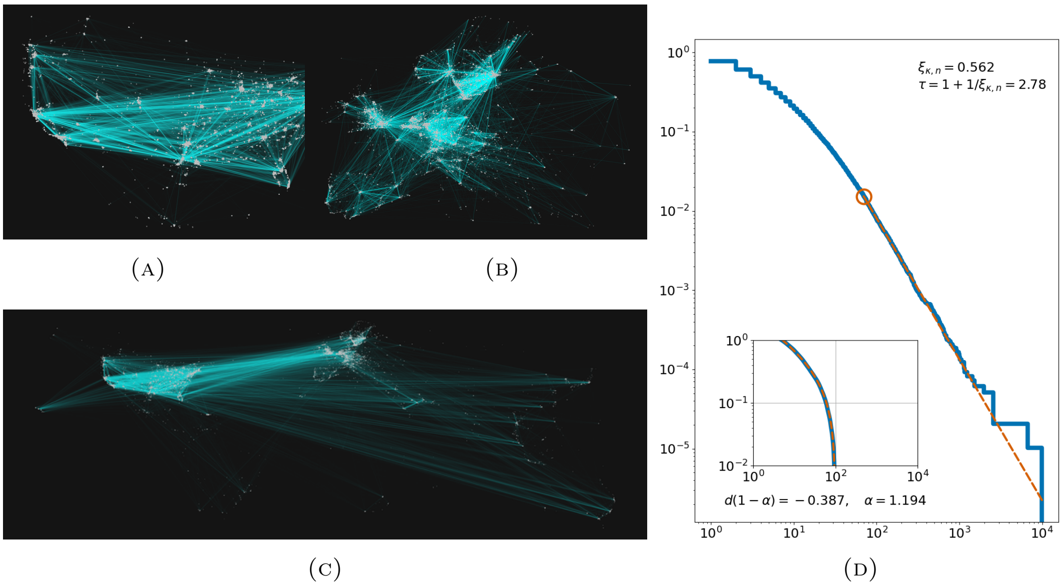


FIGURE 1.2: The Gowalla network. Visualizations of the network restricted to (a) the US and (b) Europe, as well as (c) the entire network. Figure (d) shows its cumulative degree distribution function, revealing a power-law tail. We estimated the power-law exponent using three different estimators (dashed), with consistent estimates around 1.78, yielding $\tau_{\text{Gow}} = 2.78$. The inset shows the link-length cumulative distribution, which follows a truncated power-law. We estimate the exponent using non-linear regression (that recovers the true parameter for synthetic GIRGs) on the well-behaved region of the plot from 10km to 100km, giving $\tau_{\text{Gow,link}} = 1.4 \pm 0.02$.

GIRGs have proven to model the Gowalla network relatively well, we can fit the parameters τ and α by examining the degree and edge-length distribution of a real network respectively. In Figure 1.2 (D) we can see how the degree distribution of the Gowalla network does indeed follow a power law distribution, and by methods used in [49] we estimate the τ of the network. In [7] we also perform a simple epidemic process on both synthetic GIRGs as well as the Gowalla network, and show that it yields similar results. This demonstrates the application of GIRGs to real networks, as we can go further than just the theoretical framework that we outline and indeed recreate the theory on real networks.

1.4 Thesis outline and Contributions

This thesis will focus on the slightly supercritical regime of GIRGs. We will start by defining the model as introduced by Bringmann, Keusch, and Lengler in [14] and describing some of its features and known results. We closely follow the work done in [9] on the multiscale genesis of the giant component, adding spatiality into the fold to apply it to the GIRG setting.

We aim to make progress on Conjecture 2.3.1 which describes when the graph is subcritical, at criticality, and slightly supercritical. Where this thesis focuses on the slightly supercritical part.

To deal with the added geometric component of GIRGs, there is a need to condition on the locations of high weight vertices. This is due to problems that can arise when many high weight vertices are clustered in space together, as this would remove the inter-hub connectivity that we so often see in typical GIRGs and also real networks. On this note in Chapter 3 we prove novel results on what we consider a "good configuration" of a GIRG, i.e one where the high weight vertices are dispersed through space evenly. We show that the probability that its complement (the bad event) occurs can be pushed below any δ that we desire.

The high weight vertices that we consider in the good configuration are the main component of the way we approach the problem. We consider the core of a GIRG, not to be confused with the typical k-core definition, we call the top $[an^{(3-\tau)/2}]$ vertices for some $a > 0$ the core of a graph. The idea throughout the whole thesis is that the induced subgraph formed from this core has a giant component which extends to the giant component of the full graph when considering all n vertices if we let $a \to \infty$. Which is why in Chapter 4 we start by expressing this core as an inhomogeneous random graph (IRG) using the theory from Bollobás, Janson, and Riordan from [11]. IRGs have a known phase transition and clear conditions on when a giant arises by making a connection to branching processes, which is formalized in Theorem 4.3.1. Here we show that an emerging giant exists in the core of a GIRG, and we can characterise the condition by finding an implicit expression for λ_c in terms of the associated integral operator.

Finally, we consider the aforementioned emerging giant of the core which has size $\Theta(\sqrt{n})$ and look towards at its embeddings in the full graph in Chapter 5. As we have left out the small weight vertices in every step until now we need to start carefully adding them back in. Effectively what we analyze is what we call the span of core, meaning the vertices reached outside of the high-weight subgraph. We prove a lower bound on the order of magnitude of the emerging giant, together with the vertices reached outside of the core withing a single edge in Section 5.3, which is of order $\Theta(\sqrt{n})$. In Section 5.4 we show an upper bound on the same object of the same order. There is however one caveat, within the high-weight subgraph there might be so-called return paths between two vertices. These are paths that connect two vertices within the core through vertices outside the core, and these need to be counted too for an upper bound. The idea is that these contributions are negligible, just as in [9], we manage to show that return paths of length 2 are negligible, but the problem of longer paths remains open. This means we are able to show that the largest component in the slightly supercritical regime is of order $\Theta(\sqrt{n})$, and that all other components are of order $o(\sqrt{n})$.

Commonly Used Notation in this Thesis

The following are commonly used notations used in this thesis and their meanings:

$G(V, E)$	G denotes the graph with vertex set V and edge set E, $G(V)$ and $G(E)$ are used respectively to indicate the vertex or edge set belonging to G.
w_u	The weight of vertex u is denoted by w_u.
p_{uv}	With p_{uv} we denote the probability of an edge existing between vertex u and v.
$u \leftrightarrow v$	When an edge exists between vertices u and v we denote it by $u \leftrightarrow v$.
$G(\pi_n)$	By $G(\pi_n)$ we denote the graph G percolated with probability π_n.
$\mathscr{C}_{(i)}$	The i-th largest component in a graph G is denoted by $\mathscr{C}_{(i)}$.
$f(n) = o(g(n))$	A function $f(n)$ is considered to be $f(n) = o(g(n))$ for some function $g(n)$ if $f(n)/g(n) \to 0$ as $n \to \infty$.
$f(n) = \Theta(g(n))$	A function $f(n)$ is considered to be $f(n) = \Theta(g(n))$ for some function $g(n)$ if there exists constants N, c_1, and c_2 such that for all $n \geq N$ we have $c_1 g(n) \leq f(n) \leq c_2 g(n)$.
$[n]$	$[n]$ denotes the integers starting from 1 to n, i.e $[n] = \{1, 2, \cdots, n\}$.
$a \wedge b$	We denote the minimum of a and b by $a \wedge b$.
$\mathbb{1}(x)$	The indicator function of x taking only values 0 or 1.
$\mathbb{T}^d$	The d-dimensional torus is denoted by $\mathbb{T}^d$.
With high probability (w.h.p)	We say a sequence of events E_n holds w.h.p if $\lim_{n\to\infty} \mathbb{P}(E_n) = 1$.

2 Geometric Inhomogeneous Random Graphs

We start by defining the model which will be studied in the rest of the paper. Then we discuss percolation in a general setting, and its application to the GIRG model by defining the restrictions under which we will work moving on. We also provide a heuristic argument for the intuition behind the critical percolation parameter and the size of the giant in the supercritical regime.

2.1 Model Definition

In the typical definition of a GIRG, the weights are sampled randomly and follow a power law (pareto) distribution. However, to study percolation, which is essentially the effect of deleting edges by flipping a coin for each one with a set percolation probability, it is very helpful to know the set of high weight vertices, and therefore have higher expected degrees. For this reason we will consider a GIRG where the weights still follow a power law distribution with exponent τ, but the randomness of the weight sequence is removed. This allows us to more carefully analyze the effect of high or low weight vertices on the percolation later on. This definition agrees with the typical stochastic definition of the weights and is also defined in a more general setting in [13].

Definition 2.1.1 (GIRG with Deterministic Weights).
For fixed parameters $\alpha > 1$, $\tau \in (2, 3)$, and dimension $d \in \mathbb{N}$, consider the vertex set $[n]$ for fixed n. For each $v \in [n]$ set its weight determined by its index:

$$w_v = c_F \left(\frac{n}{v}\right)^{\frac{1}{\tau-1}} \tag{2.1}$$

for some tunable constant $c_F > 0$. Furthermore, each $v \in [n]$ is uniformly placed at a location x_v on the volume-n torus $\mathbb{T}^d = [-n^{1/d}/2, n^{1/d})^d$. We connect any two vertices u and v independently with connection probability

$$p_{uv} = \min\left(1, \frac{w_u w_v}{|x_u - x_v|^d}\right)^{\alpha}, \tag{2.2}$$

with $|\cdot|$ denoting the $\infty-$norm on the torus between u and v. Furthermore, the following is an equivalent expression of the connection probability

$$p_{uv} = 1 - \exp\left(-\left(\frac{w_u w_v}{|u - v|^d}\right)^{\alpha}\right). \tag{2.3}$$

This results in a GIRG with deterministic weights for each vertex.

The different connection probabilities behave qualitatively similarly, however in certain cases it might be easier to analyze or bound one or the other, meaning we will use both formulations in the remainder of this thesis, and mention when we change definition. With this definition of a GIRG all of the known properties still follow through, i.e. expected degrees are proportional to weights, high clustering, and the small world property [14]. This is due to the fact that the geometric part of a traditional GIRG is still present in the same way, and the weights still follow a power-law distribution, in fact one that is even closer to a theoretical one than an i.i.d sample would. This implies all theorems from [14] and other relevant literature on GIRGs all follow through.

2.2 Percolation on GIRGs

We introduce the concept of percolation first by a few definitions, then show a short heuristic argument that provides intuition in the GIRG case. However, before we start we need to cover a little bit of notation. We use the Bachmann-Landau big O notation $O(\cdot), o(\cdot), \Theta(\cdot)$ to describe the asymptotics of functions. We also use $O_{\mathbb{P}}(\cdot), o_{\mathbb{P}}(\cdot), \Theta_{\mathbb{P}}(\cdot)$ for the probabilistic equivalent. Formally,

Definition 2.2.1 (Probabilistic Big O Notation)**.** *Let X_n and Y_n be sequences of random variables, we say that*

- $X_n = O_{\mathbb{P}}(Y_n)$ *if for all* $\varepsilon > 0$ *there exists an* $M > 0$ *and* $N > 0$ *such that* $\mathbb{P}(|X_n|/|Y_n| > M) < \varepsilon$ *for all* $n > N$,
- $X_n = o_{\mathbb{P}}(Y_n)$ *if* $X_n/Y_n \xrightarrow{\mathbb{P}} 0$,
- $X_n = \Theta_{\mathbb{P}}(Y_n)$ *if both* $X_n = O_{\mathbb{P}}(Y_n)$ *and* $Y_n = O_{\mathbb{P}}(X_n)$.

Definition 2.2.2. *Let G be a graph on n vertices. Let $\mathscr{C}_{(i)}$ denote the i-th largest connected component in G. We say G is* ***slightly supercritical*** *if*

$$|\mathscr{C}_{(2)}| = o_{\mathbb{P}}(|\mathscr{C}_{(1)}|), \tag{2.4}$$

$|\mathscr{C}_{(1)}|$ is unique and additionally $|\mathscr{C}_{(1)}| = o_{\mathbb{P}}(n)$. We call G ***slightly subcritical*** *if for any fixed $k \geq 2$ and some function f we have that*

$$\frac{(|\mathscr{C}_{(1)}|, |\mathscr{C}_{(2)}|, \cdots, |\mathscr{C}_{(k)}|)}{f(n)} \xrightarrow{\mathbb{P}} (c_1, c_2, \cdots, c_k), \tag{2.5}$$

for some constants $c_1, c_2, \cdots, c_k$. Lastly we say that G is ***at criticality*** *if there exists some function g, and* ***random variables*** *$X_1, X_2, \cdots, X_k$ such that*

$$\frac{(|\mathscr{C}_{(1)}|, |\mathscr{C}_{(2)}|, \cdots, |\mathscr{C}_{(k)}|)}{g(n)} \xrightarrow{\mathbb{P}} (X_1, X_2, \cdots, X_k) \tag{2.6}$$

The important distinction between the definitions of slightly subcritical and at criticality is that while in both cases the sizes of the components are of the same order and there is no unique giant, the sizes in the slightly subcritical case converge to a deterministic vector of constants. While at criticality the limit is a stochastic vector.

Typically in percolation, we consider a giant to be of linear size in the number of vertices, which is what we call *supercritical*, however as we will see in the GIRG case, the giant is not of linear order. Which is why the previous definition of slightly supercritical is more useful. Hence we only require the largest component $\mathscr{C}_1$ to be unique, and its size to be of higher order than that of $\mathscr{C}_2$. The problem that remains is under which conditions the graph G is slightly super, or subcritical.

Definition 2.2.3. *Given a graph G on n vertices, let $G(p)$ be the random graph obtained by keeping each edge of G independently with probability p. If there exists a p_c such that for all $p > p_c$, $G(p)$ is (slightly) supercritical, then we call $p_c(G)$ the* ***critical percolation probability*** *of G.*

GIRGs mimic lots of real world properties, including robustness under random failures [20], [37] when considering the infinite case. An infinite GIRG obtained by the vertices forming a Poisson point process has $p_c = 0$ [22]. Yet for any *finite* GIRG we can investigate the critical $p_c(G_n)$ such that the percolation makes the giant fall apart. Something noteworthy to mention is that this question can only be answered asymptotically on a sequence, and not on a single graph because of its requirement to be finite.

Analyses of giant components in percolated graphs have been done in many other models before, as for example Erdős-Rényi [25], Norros-Reittu [43], Chung-Lu [18], and generalized random graphs [15]. In particular, the Norros-Reittu and Erdős-Rényi random graphs and its critical percolation probabilities are of interest to us. The Norros-Reittu model is essentially a GIRG without a spatial component, meaning that it is natural to try and extend the work of [9], where they determine the critical percolation parameter, and analyze the conditions under where the graph has a giant component.

There is however a caveat in the case of GIRGs, the infinite graph G_∞ has an infinite largest component in the local weak limit sense of the graph when $\tau \in (2, 3)$, and is robust under percolation. However, for fixed n, G_n loses its robustness and the critical percolation probability $p_c > 0$. r percolation/ sparse just means expected degree finite This means that $p_c(G_\infty) = 0$ and $p_c(G_n) \downarrow 0$ as $n \to \infty$. But for a finite graph we can determine the critical percolation probability as a function of n.

The Erdős-Rényi graph with connection probability p and n vertices ($ER(n, p)$), provides a very natural intuition for the finite GIRG case. It is known that when $p = \lambda/n$, where n is the number of vertices, that when $\lambda > 1$, a unique giant component arises [25] with a size that is linear in the number of vertices. If we consider a GIRG and restrict to a subset of vertices that are of high enough weight, then we obtain a complete graph. Recall the connection probability between u and v from (2.2), the typical distance between u and v is of the form $n^{1/d}$ (see for example [44]), this makes it that if we restrict to vertices of at least weight $\sqrt{n}$, up to some constants, we obtain a complete graph. Ignoring constants, the order of the amount of vertices in this subset in the general GIRG case can be easily computed by

$$n \cdot \mathbb{P}\left(W \geq \sqrt{n}\right) = n \cdot \int_{\sqrt{n}}^{\infty} w^{-\tau} dw = n \cdot \left[w^{1-\tau}\right]_{\sqrt{n}}^{\infty} = n \cdot (\sqrt{n})^{1-\tau} = n^{\frac{3-\tau}{2}}. \tag{2.7}$$

In the formulation of Definition (2.1.1), we can compute the amount of vertices directly.

$$\begin{aligned}
&\text{We require } w_v \geq \sqrt{n} \text{ this gives:} \\
&c_F \left(\frac{n}{v}\right)^{\frac{1}{\tau-1}} \geq \sqrt{n} \\
&\frac{n}{v} \geq \frac{n^{\frac{1}{2}(\tau-1)}}{c_F^{\tau-1}} \\
&c_F^{\tau-1} n^{1-\frac{1}{2}(\tau-1)} \geq v \\
&c_F^{\tau-1} n^{\frac{3-\tau}{2}} \geq v
\end{aligned} \tag{2.8}$$

And the same result on the order of n appears.

As the $n^{(3-\tau)/2}$ vertices form a complete graph, if we decide to percolate the edges with probability π_n, it is now simply the $ER(n^{\frac{3-\tau}{2}}, \pi_n)$ problem. Which means a giant arises in the subgraph when

$$\pi_n > \frac{1}{n^{(3-\tau)/2}} = n^{-(3-\tau)/2}. \tag{2.9}$$

This is simply a heuristic argument, but it gives a strong intuition on the order of the critical percolation probability in GIRGs.

2.3 Main Results

We set out to make progress in proving the phase transition of graph in terms of the critical percolation parameter in GIRGs. This is summarized in the following conjecture;

Conjecture 2.3.1. *Let G_n be a GIRG as in definition 2.1.1, with $\tau \in (2,3)$ on n vertices. Let*

$$\pi_n = \lambda n^{\frac{\tau-3}{2}} \tag{2.10}$$

be the percolation probability for G_n. Then there exists a λ_c, such that for every $\lambda > \lambda_c$ we have that $G_n(\pi_n)$ is slightly supercritical. Furthermore, we have that as $n \to \infty$,

$$\frac{|\mathscr{C}_{(1)}(\pi_n)|}{\sqrt{n}} \xrightarrow{\mathbb{P}} \zeta^\lambda, \text{ and } \frac{|\mathscr{C}_{(2)}(\pi_n)|}{\sqrt{n}} \xrightarrow{\mathbb{P}} 0. \tag{2.11}$$

Where ζ^λ is the limit as $a \to \infty$ of

$$\zeta_a^\lambda = \lambda \int_0^a c_F u^{\frac{-1}{\tau-1}} \rho_a^\lambda(u) du, \tag{2.12}$$

and ρ_a^λ is the survival probability of a multi-type branching process which will be explicitly defined in Section 4.2.1.

Additionally, when $\pi_n = o(n^{-(3-\tau)/2})$, we have that G is slightly subcritical, and as $n \to \infty$ we have that for any fixed $i \in [n]$:

$$\frac{|\mathscr{C}_{(i)}(\pi_n)|}{\pi_n n^{1/(\tau-1)}} \xrightarrow{\mathbb{P}} \frac{c_F c_d}{i^{1/(\tau-1)}}. \tag{2.13}$$

Conjecture 2.3.1 tells us that when G is supercritical, our largest component is unique and of order $\sqrt{n}$. When we are in the subcritical case however, the sizes of the components are determined uniquely by the weights of the vertices. Meaning that the components are also disjoint from each other. In fact, it concentrates around $\pi_n w_i$, which is intuitively easily understood as in GIRGs the expected degrees of vertices are directly proportional to their weight [14]. Moreover, it is clear to see that there is no giant in this regime, and the largest percolated component will be formed by the vertex with the largest weight.

2.3.1 Our Contribution

We make progress in proving Conjecture 2.3.1, the novel results we do manage to show are summarized in the following theorem,

Theorem 2.3.2 (Summarization of Results). *Let G_n be a GIRG as in definition 2.1.1, with $\tau \in (2,3)$ on n vertices. Let*

$$\pi_n = \lambda n^{\frac{\tau-3}{2}} \tag{2.14}$$

be the percolation probability for G_n. Then there exists a λ_c, such that for every $\lambda > \lambda_c$ we have that

$$\frac{|\mathscr{C}_{(1)}(\pi_n)|}{\sqrt{n}} \geq \zeta^\lambda - \varepsilon. \tag{2.15}$$

The critical percolation probability λ_c can be implicitly expressed as the reciprocal of the integral operator norm

$$\lambda_c := ||T_\kappa||^{-1}, \text{ with}$$
$$(T_\kappa f)(u, \tilde{x}_u) = \int_{B_d} \int_0^a \Big(1 - \exp(-\Big(\frac{(uv)^{\frac{-1}{\tau-1}} c_F^2}{|\tilde{x}_u - \tilde{x}_v|^d}\Big)^\alpha\Big) f(v, \tilde{x}_v) \mathrm{d}v \mathrm{d}\tilde{x}_v. \tag{2.16}$$

Furthermore, the constant ζ^λ is the limit as $a \to \infty$ of the object

$$\zeta_a^\lambda = \lambda \int_0^a c_F u^{\frac{-1}{\tau-1}} \rho_a^\lambda(u) du, \tag{2.17}$$

and ρ_a^λ is the survival probability of a multi-type branching process which will be explicitly defined in Section 4.2.1.

Lastly, we show a partial upper bound on the largest component $\mathscr{C}_{(1)}(\pi_n)$ of the form

$$\frac{|\mathscr{C}_{(1)}(\pi_n)|}{\sqrt{n}} \leq \zeta^\lambda + \varepsilon + |\mathcal{R}^a_{(1)} \cup \mathrm{Span}_a(\mathcal{R}^a_{(1)})|, \tag{2.18}$$

where $\mathcal{R}^a_{(1)}$ is a set of vertices as defined in definition 5.2.3.

We give a brief outline on how we prove the results of Theorem 2.3.2. In the supercritical regime we will start by establishing the existence of a giant in a subset of high weight vertices. We will do this by first showing that the core (of high weight vertices) of a *percolated* GIRG can be expressed as an inhomogeneous random graph (IRG) as defined in [11]. Where we have an extra type for the spatial component. Using the theory of IRGs we can then show that under certain conditions of λ an emerging giant exists when restricting to the core of a GIRG, which we will call the tiny (or emerging) giant. We also connect the theory of branching processes to these graphs, by using the theory of [11] that states that the local weak limit of an IRG is, in our case, a multi-type branching process. We analyze the averaged survival probability of the BP ζ_a^λ and show that it converges as we extend the scope of the tiny giant by letting $a \to \infty$. However, the emerging giant clearly does not cover the full scope of the giant in the entire graph, which is why we investigate further outside of the core. We are interested in the component in the full graph that engulfs the tiny giant, we analyze this component by counting the paths that extend from the emerging giant into its complement, defining the set of vertices that are added as the emerging giants span. We prove a lower bound on the size of the tiny giant and its span by counting two step paths, a partial upper bound is found by showing that paths with length larger than two are of lesser order than the two step paths.

The spatial dimension of GIRGs is often an obstacle in these computations, this is why we also prove a number of results on the distribution of vertices in space. Namely, we show that what we call a "good configuration" of a GIRG, that is a configuration where the high weight vertices are well distributed in space, and not clustered together in a tiny pocket. In particular, in Chapter 3 we show that for the right choice of some starting distance, if we sweep through different distances by powers of 2, the bad event where vertices are clustered too much together can always be pushed below any arbitrary δ. We also compute the expected number of pairs of vertices in each distance range and its variance.

3 A Good Configuration of a GIRG

In this chapter we will prove novel results about what we consider to be a good configuration of a GIRG. This includes results on the expected amount of pairs within some distance r from each other, together with variance computations and concentration bounds.

3.1 The Geometric Picture

We will henceforth consider a GIRG on n vertices, as the results we will prove are solely based on the locations of the vertices, the result of Theorem 3.1.1, holds for the GIRG as defined in Definition (2.1.1) as for a GIRG with i.i.d random weights according to some power law distribution.

Our aim in this section is to establish a condition which we will call a "good configuration" of a GIRG. This entails that the vertices, specifically the high weight ones, are well spread out through space. The intuition behind this definition follows from the fact that if all high weight vertices are clustered in space, they can easily dominate the core and any extensions. This means that paths to lower weight vertices might cease to exist, causing us to obtain a graph that is unlike the typical GIRG we are used to seeing. We will show that the probability of such a distribution of vertices in space can be pushed below δ for any $\delta > 0$.

We will assume to be working on the volume-n torus $\mathbb{T}^d$. For a graph G, we fix some distance r_0, and let

$$r_i = 2^i r_0 \tag{3.1}$$

be distances sweeping through the powers of 2. Then we denote by

$$\mathcal{P}^{r_i}(V) := |\{(u,v) \in V : |u-v| \in (r_i, r_{i+1})\}|, \tag{3.2}$$

the amount of pairs within distance r_i and r_{i+1}. The main result we will be working towards proving is the following:

Theorem 3.1.1. *Let G be a GIRG where we order the vertices by weight such that $w_1 \geq w_2 \cdots \geq w_n$. Fix $a > 0$ and take the first $N_n(a) := \lceil an^{(3-\tau)/2} \rceil$ vertices, denote them by $V_{N_n(a)}$. For some $M > 0$, fix $r_0 = (Mn^{\tau-2})^{1/d}$, and then define the* ***good event***

$$\mathcal{A}_{good}(N_n(a)) := \left\{ \forall i \geq 0 \; \mathcal{P}^{r_i}(V_{N_n(a)}) \in \left[\frac{1}{2}\mathbb{E}[\mathcal{P}^{r_i}(V_{N_n(a)})], \frac{3}{2}\mathbb{E}[\mathcal{P}^{r_i}(V_{N_n(a)})] \right] \right\}. \tag{3.3}$$

Then for all $\delta > 0$ we have that

$$\mathbb{P}(\mathcal{A}^c_{good}(N_n(a)) < \delta. \tag{3.4}$$

3.2 Concentration of Distances Around their Mean

To prove Theorem 3.1.1 we will need a few intermediate results. To start we will compute the expectation and variance of $\mathcal{P}^{r_i}(V_{N_n(a)})$. We should mention that continuing on in this chapter we will always assume that the vertices in G are ordered by weight as stated before in Theorem 3.1.1.

Lemma 3.2.1. *Let G be a GIRG, and consider the first $\lceil an^{(3-\tau)/2}\rceil$ vertices. Then for all i with $2^i r_0 < n/2$*

$$\mathbb{E}\left[\mathcal{P}^{r_i}(V_{N_n(a)})\right] = \frac{a^2 n^{2-\tau} r_0^d}{2} 2^{id} A(1/2)(1 - o(1)), \tag{3.5}$$

$$Var(\mathcal{P}^{r_i}(V_{N_n(a)})) = \Theta(\mathbb{E}[\mathcal{P}^{r_i}(V_{N_n(a)})]), \tag{3.6}$$

where $A(1/2)$ denotes the volume of an d-dimensional annulus with inner and outer radii $1/2$ and 1 respectively.

Proof. We first consider a single pair of vertices (u, v). Due to the translation invariance of the torus, we can take, for example u, to be at the center of space. The probability of the pairs' distance being in the range (r_i, r_{i+1}) is then a simple geometric argument of v being in the annulus between r_{i+1} and r_i, this probability is given exactly by $\frac{r_0^d}{n} 2^{id} A(1/2)$. Because all points are independently placed in space, and as such the distances of each individual pair are also independent of each other, it suffices to consider all $\binom{an^{(3-\tau)/2}}{2}$ pairs. Expressing $\mathcal{P}^{r_i}(V_{N_n(a)})$ as a sum of indicators, and denoting $X_{uv} = \mathbb{1}(|u - v| \in (r_i, r_{i+1}))$, we find

$$\mathbb{E}\left[\mathcal{P}^{r_i}(V_{N_n(a)})\right] = \mathbb{E}\left[\sum_{(u,v)\in[an^{(3-\tau)/2}]} X_{uv}\right] = \binom{an^{(3-\tau)/2}}{2}\mathbb{P}(|u - v| \in (r_i, r_{i+1}))$$

$$= \frac{a^2 n^{3-\tau} r_0^d}{2n} 2^{id} A(1/2) = \frac{a^2 n^{2-\tau} r_0^d}{2} 2^{id} A(1/2). \tag{3.7}$$

Note that we use an approximation of the binomial coefficient in the third step which gives the expectation in (3.5). Next is the variance, which requires a little bit more insight into the independence of the pairs. Expressing $\mathcal{P}^{r_i}(V_{N_n(a)})$ as a sum of indicators again, we get that

$$Var(\mathcal{P}^{r_i}(V_{N_n(a)})) = \sum_{(u,v)\in[an^{(3-\tau)/2}]} Var(X_{uv}) + 2 \sum_{(u,v)\neq(k,l)} \text{Cov}(X_{uv}, X_{kl})$$

$$= \mathbb{E}[\mathcal{P}^{r_i}(V_{N_n(a)})] + 2 \sum_{(u,v)\neq(k,l)} \text{Cov}(X_{uv}, X_{kl}). \tag{3.8}$$

To now estimate the covariance term, we first observe that when u, v, k, l are all distinct vertices, their respective pairwise probabilities of being in the interval $(r_i, r_{i+1}$ are fully independent, and their covariance terms are 0. This means that we are left with the case where (u, v) and (k, l) share some vertex, i.e the case where $v = k$. Our claim is now that even in this case, because our vertices are placed on the torus, the covariance is still 0.

We analyze the term $\mathbb{P}(|u - v| \in (r_i, r_{i+1}),\ |v - l| \in (r_i, r_{i+1}))$. Because we work on the torus we can shift v to be the "center" vertex. Then this probability is simply the probability that both u and v lie in the annulus defined by r_i, which is completely independent of the location of v and of each other. We can do this for each triple of vertices by choosing the common vertex as the center, giving that the sum of covariances as a whole is 0. This concludes the expression in (3.6). □

Next we make use of Chebyshev's inequality to show concentration of $\mathcal{P}^{r_i}(V_{N_n(a)})$ around its mean.

Lemma 3.2.2. *Let G be a GIRG, and consider the first $\lceil an^{(3-\tau)/2}\rceil$ vertices. Then for r_0 as defiend in Theorem 3.1.1, and any i such that $2^i r_0 < n/2$*

$$\mathbb{P}\left(|\mathcal{P}^{r_i}(V_{N_n(a)}) - \mathbb{E}[\mathcal{P}^{r_i}(V_{N_n(a)})]| \geq \frac{1}{2}\mathbb{E}[\mathcal{P}^{r_i}(V_{N_n(a)})]\right) < \frac{4}{\mathbb{E}[\mathcal{P}^{r_i}(V_{N_n(a)})]}. \tag{3.9}$$

Proof. This follows immediately from applying Chebyshev's inequality and Lemma 3.2.1

$$\begin{aligned} &\mathbb{P}\left(|\mathcal{P}^{r_i}(V_{N_n(a)}) - \mathbb{E}[\mathcal{P}^{r_i}(V_{N_n(a)})]| \geq \frac{1}{2}\mathbb{E}[\mathcal{P}^{r_i}(V_{N_n(a)})]\right) \\ &\leq \frac{4Var(\mathcal{P}^{r_i}(V_{N_n(a)}))}{\mathbb{E}[\mathcal{P}^{r_i}(V_{N_n(a)})]^2} = \frac{4}{\mathbb{E}[\mathcal{P}^{r_i}(V_{N_n(a)})]} \end{aligned} \tag{3.10}$$

□

3.3 Few Pairs have Tiny Distance

Now all that is left for us to show is that the amount of pairs of vertices that are at distance less than $r_0 = (Mn^{\tau-2})^{1/d}$ is also small. For this we fix $\varepsilon > 0$ and first show that the amount of pairs at distances $[\varepsilon n^{(\tau-2)/d}, Mn^{(\tau-2)/d}]$ is small, we denote by $\mathcal{P}^{\varepsilon,M}(V_{N_n(a)})$ exactly this quantity. Then similarly we provide an argument for all distances less than $\varepsilon n^{(\tau-2)/d}$, denoted by $\mathcal{P}^{\varepsilon}(V_{N_n(a)})$

Theorem 3.3.1. *Let $\varepsilon > 0$. Then for any $\delta > 0$, we can find $M := M(\delta)$ such that*

$$\mathbb{P}(\mathcal{P}^{\varepsilon,M}(V_{N_n(a)}) \geq M^2 a^2 \mathrm{Vol}(1)) < \delta, \tag{3.11}$$

where Vol(1) denotes the volume of the d-dimensional ball with radius 1.

Proof. Let $\delta > 0$ and take $M > 1/\delta$. This time Markov's inequality suffices, $\mathbb{E}[\mathcal{P}^{\varepsilon,M}(V_{N_n(a)})]$ is computed in the same way as in Lemma (3.2.1), however we simply bound the expectation by the volume of the 1-ball instead of the annulus. Which gives us

$$\mathbb{E}[\mathcal{P}^{\varepsilon,M}(V_{N_n(a)})] < \frac{a^2 n^{2-\tau} r_0^d}{2} Vol(1) \tag{3.12}$$

Then by Markov's inequality

$$\mathbb{P}\left(\mathcal{P}^{\varepsilon,M}(V_{N_n(a)}) \geq M^2 a^2 \text{Vol}(1)\right) \leq \frac{\mathbb{E}[\mathcal{P}^{\varepsilon,M}(V_{N_n(a)})]}{a^2 M^2 \text{Vol}(1)} < \frac{1}{M} < \delta. \tag{3.13}$$

□

Lastly, we prove a statement for $\mathcal{P}^{\varepsilon}(V_{N_n(a)})$.

Theorem 3.3.2. *Let* $\varepsilon > 0$. *Then*

$$\mathbb{P}(\mathcal{P}^{\varepsilon}(V_{N_n(a)}) \geq 1) \leq \frac{\varepsilon^d a^2}{2} \text{Vol}(1). \tag{3.14}$$

Proof. As before, the expectation of $\mathcal{P}^{\varepsilon}(V_{N_n(a)})$ is given by $\frac{a^2\varepsilon^d}{2}\text{Vol}(1)$. Then Markov's inequality immediately provides the desired result. □

Now we can prove Theorem (3.1.1).

Proof of Theorem 3.1.1. Let $\delta > 0$. Then by Lemma 3.2.2 and our choice of $r_0^d = Mn^{\tau-2}$ we obtain

$$\mathbb{P}\left(|\mathcal{P}^{r_i} - \mathbb{E}[\mathcal{P}^{r_i}]| \geq \frac{1}{2}\mathbb{E}[\mathcal{P}^{r_i}]\right) \leq \frac{4}{\mathbb{E}[\mathcal{P}^{r_i}]} = \frac{8}{a^2 n^{2-\tau} r_0^d 2^{id} A(1/2)} = \frac{8}{a^2 M 2^{id} A(1/2)}$$

Now to upper bound the event $\mathcal{A}^c_{good}$ we use the previous inequality

$$\mathbb{P}(\mathcal{A}^c_{good}) \leq \sum_{i=0}^{r_{\max}} \mathbb{P}\left(\mathcal{P}^{r_i} \notin \left[\frac{1}{2}\mathbb{E}[\mathcal{P}^{r_i}], \frac{3}{2}\mathbb{E}[\mathcal{P}^{r_i}]\right]\right) < \frac{8}{a^2 M A(1/2)} \sum_{i=0}^{r_{\max}} 2^{-id} \tag{3.15}$$

$$\leq \frac{16}{a^2 M A(1/2)}. \tag{3.16}$$

Now, Theorem 3.3.2 shows the ε dependence, meaning that for tiny ε, there are also little pairs. The ε independency of Theorem 3.3.1 follows from the fact that we can bound it from above by the ball instead of the annulus, making it so that we simply have to set M to be such that $M > \max(\frac{1}{\delta}, \frac{16}{a^2\delta V})$, such that for any δ we can push the aforementioned probabilities to be tiny. □

4 The Core of a GIRG

In this chapter, we analyze the core of a GIRG by starting with discussing the theory of inhomogeneous random graphs (IRGs), we then show that a subset of the highest weight vertices of a percolated GIRG can be expressed as an IRG. The existing results of IRGs then tell us that a critical percolation parameter exists and a giant appears in the percolated core of a GIRG in the form of an operator norm. Furthermore, we connect the theory of branching processes to IRGs in the local weak limit sense, and finally show that the survival probability of the limiting branching process is convergent and finite.

4.1 Inhomogeneous Random Graphs

We start by treating the general theory of IRGs as created by Bollobás, Janson, and Riordan in [11], and defining generalized vertex spaces and kernels. By $\mathcal{S}$ we will denote seperable metric spaces which will be equipped with a probability measure μ. Continuing on in the thesis we will typically take it to be equal to the Lebesgue measure. We will consider sequences of n points in $\mathcal{S}$ that can be of random or deterministic origin, denoted by $\mathbf{u}_n = (u_1, u_2, \cdots, u_n)$.

Using the typical δ_u measure to ascribe mass 1 to u we can write

$$\nu_n := \sum_{i=1}^{n} \delta_{u_i}, \tag{4.1}$$

for the empirical distribution of $\mathbf{u}_n$. The probability measure μ that we equip $\mathcal{S}$ with will be assumed to be the limit of ν_n in the sense of probabilistic convergence. Formally we assume that

$$\nu_n(A) := \#\{i : x_i \in A\}/n \xrightarrow{\mathbb{P}} \mu(A). \tag{4.2}$$

Now we define kernels and vertex spaces formally

Definition 4.1.1. *Let $\mathcal{S}$ be a seperable metric space, μ denote a Borel probability measure, and for each $n \geq 1$ $(\mathbf{u}_n)_{n\geq 1} = (u_1, u_2, \ldots, u_n)$ is a (random) sequence of n points of $\mathcal{S}$, such that (4.2) holds. Then we call the tuple $\mathcal{V} = (\mathcal{S}, \mu, (\mathbf{u}_n)_{n\geq 1})$ a **vertex space**.*

Definition 4.1.2. *Given a vertex space $\mathcal{V} = (\mathcal{S}, \mu, (\mathbf{u}_n)_{n\geq 1})$, let $\kappa : (\mathcal{S}, \mathcal{S}) \to [0, \infty)$ be a function that is symmetric and Borel-measurable. Then we call κ a **kernel** on $(\mathcal{S}, \mu, (\mathbf{u}_n)_{n\geq 1})$.*

The space $\mathcal{S}$ is what we consider to be the type space. Each vertex in our graph has a type in $\mathcal{S}$, and it may be discrete or continuous. A famous example of a random graph that works with types specifically is the stochastic block model, see for example [21, 39, 41].

Given a vertex space $\mathcal{V}$ and kernel κ, we can now define a random graph as follows,

Definition 4.1.3 (Inhomogeneous Random Graph). *Let $\mathcal{V} = (\mathcal{S}, \mu, (\mathbf{u}_n)_{n\geq 1})$ be a vertex space and let κ be a kernel on $\mathcal{V}$. If we take $[n]$ to be the vertex set where we connect two vertices u_i and u_j with $i \neq j$ with probability*

$$p_{ij} = \min(1, \kappa(u_i, u_j)/n), \tag{4.3}$$

*independently, then with $G^{\mathcal{V}}(n, \kappa)$ we denote the random graph formed by κ on n vertices. We call $G^{\mathcal{V}}(n, \kappa)$ an **inhomogeneous random graph**.*

Definition 4.1.4. *Let $\mathcal{V} = (\mathcal{S}, \mu, (u_n)_{n\geq 1})$ be a vertex space, and. A kernel κ is called **graphical** if*

(i) κ is continuous a.e on $\mathcal{S} \times \mathcal{S}$,

(ii) $\kappa \in L^1(\mathcal{S} \times \mathcal{S}, \mu \times \mu)$,

(iii) $\frac{1}{n}\mathbb{E}\left[|E(G^{\mathcal{V}}(n, \kappa)|\right] \to \frac{1}{2}\int\int_{\mathcal{S}^2} \kappa(x, y)d\mu(x)d\mu(y)$.

Condition (iii) requires $G^{\mathcal{V}}(n, \kappa)$ to be sparse in the number of edges. This is an important distinction, and when moving forward we will show that the core of the percolated GIRG is in fact a sparse graph.

The definition of a graphical kernel naturally extends to a sequence of kernels in the following way:

Definition 4.1.5. *Let $\mathcal{V} = (\mathcal{S}, \mu, (\mathbf{u}_n)_{n\geq 1})$ be a vertex space, let κ be a kernel on $\mathcal{V}$. A sequence of kernels $(\kappa_n)_{n\geq 1}$ is graphical on $\mathcal{V}$ with limit κ, if for $(x, y) \in \mathcal{S}^2$ with sequences $(x_n) \to x$ and $(y_n) \to y$ we have that*

$$\kappa_n(x_n, y_n) \to \kappa(x, y) \tag{4.4}$$

and κ satisfies conditions (i) and (ii) of Definition (4.1.4). Furthermore we require

$$\frac{1}{n}\mathbb{E}\left[|E(G^{\mathcal{V}}(n, \kappa_n)|\right] \to \frac{1}{2}\int\int_{\mathcal{S}^2} \kappa(x, y)d\mu(x)d\mu(y) \tag{4.5}$$

4.2 The Percolated Core of a GIRG as an IRG

We now move on to define a kernel that satisfies the aforementioned conditions and gives us back our desired GIRG as defined in Definition 2.1.1. To do this we start by looking at a subset of only the highest weight vertices, specifically, for a fixed $a > 0$, we define

$$N_n(a) = \lfloor an^{(3-\tau)/2} \rfloor. \tag{4.6}$$

We will restrict our graph to the first $[N_n(a)]$ vertices, as our weights are deterministic these are the first $N_n(a)$ highest weight vertices. We then have that for any $u \in (0, a]$, a vertex has weight

$$w_{N_n(u)} = c_F \left(\frac{n}{N_n(u)}\right)^{\frac{1}{\tau-1}} = c_F u^{\frac{-1}{\tau-1}} \left(\frac{n}{n^{(3-\tau)/2}}\right)^{\frac{1}{\tau-1}} (1+o(1)) = c_F u^{\frac{-1}{\tau-1}} \sqrt{n}. \quad (4.7)$$

Where we get a small error as we ignore the integer part of $N_n(a)$ in the computation. However, as is clear to see this error is tiny and dependent on only n.

We then take $u = a$ to see that all vertices in $[N_n(a)]$ have at least weight $\sqrt{n}a^{\frac{-1}{\tau-1}}$.

The idea moving forward is the following, we will consider the percolated GIRG $G(\pi_n)$, with π_n as in Conjecture 2.3.1. As we are currently studying the supercritical phase, we will always assume that $\lambda > \lambda_c$. A GIRG itself is sparse in its number of edges, however the core, is not sparse. We will however show that the *percolated core* is again sparse. The induced subgraph on $[N_n(a)]$ will be denoted by $\mathcal{G}_{N_n(a)}$, we will define a kernel as in the theory discussed in Section 4.1 which will show that we can express $\mathcal{G}_{N_n(a)}$ as an IRG. We then move on to more results and theory of [11], which state that given an IRG we can find a unique giant component of linear size under some conditions.

Let us continue in the previous setting, we will consider the measure space $S_a = ((0,a], \mathscr{B}(0,a]), \Lambda_a)$, where $\mathscr{B}$ denotes the Borel sigma-algebra, and $\Lambda_a(dx) = \frac{dx}{a}$ is the normalized Lebesgue measure. With p_{uv} we denote the percolated connection probability in our GIRG. Note that the connection probability in Definition 2.1.1 is an upper bound to the alternative formulation that is

$$p_{uv} = \pi_n(1 - \exp(-\left(\frac{w_u w_v}{|u-v|^d}\right)^{\alpha}) \quad (4.8)$$

Theorem 4.2.1. *Let $S_a = ((0,a]\times[0,1)^d, \mathscr{B}(0,a]), \Lambda_a, (\mathbf{u}_n)_{n\geq 1})$ be a vertex space equipped with the Borel sigma-algebra and the normalized Lebesgue measure Λ_a. Let $(\mathbf{u}_n)_{n\geq 1}$ denote the vertices from the $N_n(a)$-discretized interval $(0,a]$ and their uniformly chosen positions in the d-dimensional unit torus $[0,1]^d$. Then for $(u,x_u),(v,x_v) \in S_a$, the kernel*

$$\kappa^{(a)}_{N_n}((u,\tilde{x}_u),(v,\tilde{x}_v)) = N_n(a) p_{(N_n(u),n^{1/d}\tilde{x}_u)(N_n(v),n^{1/d}\tilde{x}_v)}/\lambda, \quad (4.9)$$

characterizes an IRG on $N_n(a)$ vertices on the space S_a. Furthermore, the percolated core $\mathcal{G}_{N_n(a)}$ has the same distribution as this IRG, and $\kappa^{(a)}_{N_n}$ is a sequence of graphical kernels.

Essentially the idea here is that we can rewrite the connection probabilities of the GIRG within its core to fit the IRG definition. This is a purely formal rewrite to be able to use the machinery of IRGs from [11]. We then show that the number of edges in the percolated GIRG core is sparse and its expectation can be obtained by integrating the re-writing IRG kernel on a new state space. This proves at the same time that the sequence of kernels is graphical.

Proof. We set out to prove that $\kappa^{(a)}_{N_n}$ is a sequence of graphical kernels. We start by defining sequences for the vertices, we set $u_i = i/N_n(1)$, then note then that u_i is uniform on $(0,a]$ then for all $i \in [N_n(a)]$, we have that $p_{ij} = \lambda\kappa^{(a)}_{N_n}((u_i,x_i),(v_i,x_j))/N_n(a)$.

We can rescale the positions x_i and x_j to the d-dimensional unit box $[-1/2, 1/2]^d$, for any i, let $x_i = n^{1/d}\tilde{x}_i$, then $|x_i - x_j|^d = |n^{1/d}\tilde{x}_i - n^{1/d}\tilde{x}_j|^d = n|\tilde{x}_i - \tilde{x}_j|^d$.

Furthermore we know that $w_i = c_F u_i^{\frac{-1}{\tau-1}}\sqrt{n}$. Let us now look at the kernel as defined in (4.9), we express the connection probability in an alternative way first, using the equally valid representation of (4.8)

$$\begin{aligned} p_{(N_n(u),x_u)(N_n(v),x_v)} &= \lambda n^{\frac{-(3-\tau)}{2}}\Big(1 - \exp(-\Big(\frac{-u_i^{\frac{-1}{\tau-1}} v_j^{\frac{-1}{\tau-1}} c_F^2 \sqrt{n}\sqrt{n}}{n|\tilde{x}_u - \tilde{x}_v|^d}\Big)^\alpha) \\ &= \lambda n^{\frac{-(3-\tau)}{2}}\Big(1 - \exp(-\Big(\frac{(u_i v_j)^{\frac{-1}{\tau-1}} c_F^2}{|\tilde{x}_u - \tilde{x}_v|^d}\Big)^\alpha\Big). \end{aligned} \tag{4.10}$$

The prefactor term of $n^{\frac{-(3-\tau)}{2}}$ is almost the reciprocal of the number of vertices, lest we only need to multiply and divide by a to arrive at the correct order. This means that by multiplying with $N_n(a)/\lambda$ we find that

$$\kappa_{N_n}^{(a)}((u, x_u), (v, x_v)) = a\Big(1 - \exp(-\Big(\frac{(u_i v_j)^{\frac{-1}{\tau-1}} c_F^2}{|\tilde{x}_u - \tilde{x}_v|^d}\Big)^\alpha\Big). \tag{4.11}$$

Now, for any $(u_n)_{n\geq 1}, (v_n)_{n\geq 1} \subset (0, a]$ such that $u_n \to u \in (0, a]$ and $v_n \to v \in (0, a]$ we have that

$$\kappa_{N_n}^{(a)}((u, x_u), (v, x_v)) \to \kappa^{(a)}((u, x_u), (v, x_v)) = a\Big(1 - \exp(-\Big(\frac{(uv)^{\frac{-1}{\tau-1}} c_F^2}{|\tilde{x}_u - \tilde{x}_v|^d}\Big)^\alpha\Big). \tag{4.12}$$

Clearly $\kappa^{(a)}$ is bounded and continuous so all that is left to show is that $\kappa^{(a)}$ satisfies Equation (4.5). Let $B_d = [-1/2, 1/2]^d$, then

$$\begin{aligned} &\frac{1}{N_n(a)}\mathbb{E}[|E(G^{\mathcal{V}}(N_n(a), \kappa^{(a)})|] \\ &= \frac{1}{N_n(a)}\binom{N_n(a)}{2}\int_{B_d}\int_{B_d}\int_0^a\int_0^a \frac{\pi_n}{a}\kappa^{(a)}((u, x_u), (v, x_v))\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}\tilde{x}_u\,\mathrm{d}\tilde{x}_v \\ &= \frac{1}{N_n(a)}\binom{N_n(a)}{2}\int_{B_d}\int_{B_d}\int_0^a\int_0^a \pi_n\Big(1 - \exp(-\Big(\frac{(uv)^{\frac{-1}{\tau-1}} c_F^2}{|\tilde{x}_u - \tilde{x}_v|^d}\Big)^\alpha\Big)\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}\tilde{x}_u\,\mathrm{d}\tilde{x}_v \end{aligned} \tag{4.13}$$

Now we use the estimate that $\binom{N_n(a)}{2} \approx \frac{N_n(a)^2}{2}$ to find that continuing from (4.13) we have

$$= \frac{1}{2} \int\limits_{B_d} \int\limits_{B_d} \int_0^a \int_0^a \lambda\Big(1 - \exp(-\Big(\frac{(uv)^{\frac{-1}{\tau-1}} c_F^2}{|\tilde{x}_u - \tilde{x}_v|^d}\Big)^{\alpha}\Big) \,\mathrm{d}u \,\mathrm{d}v \,\mathrm{d}\tilde{x}_u \,\mathrm{d}\tilde{x}_v$$

$$= \frac{1}{2a} \int\limits_{B_d} \int\limits_{B_d} \int_0^a \int_0^a \lambda\kappa^{(a)}((u, x_u), (v, x_v)) \,\mathrm{d}u \,\mathrm{d}v \,\mathrm{d}\tilde{x}_u \,\mathrm{d}\tilde{x}_v$$

$$= \frac{1}{2} \int\limits_{B_d} \int\limits_{B_d} \int_0^a \int_0^a \lambda\kappa^{(a)}((u, x_u), (v, x_v)) \,\Lambda(\mathrm{d}u) \,\Lambda(\mathrm{d}v) \,\mathrm{d}\tilde{x}_u \,\mathrm{d}\tilde{x}_v, \tag{4.14}$$

(4.14) then shows the sparsity of the percolated core. With this we have now verified that $\kappa^{(a)}_{N_n}((u, x_u), (v, x_v))$ is a sequence of graphical kernels. As the sequence of kernels in (4.9) is graphical, we can now see that the percolated $[N_n(a)]$ core of a GIRG can be seen as an IRG as in Definition 4.1.3. □

We will now treat some more theory from [11], specifically to understand the conditions under which a giant emerges in $\mathcal{G}_{N_n(a)}$ and its properties. To this end, we start by defining the irreducibility of a kernel κ.

Definition 4.2.2. *A kernel κ on a vertex space $\mathcal{V} = (\mathcal{S}, \mu, (u_n)_{n\geq 1})$ is* ***irreducible*** *if*

$$A \subseteq \mathcal{S} \text{ and } \kappa = 0 \text{ a.e on } A \times \mathcal{S} \setminus A \implies \mu(A) = 0 \text{ or } \mu(\mathcal{S} \setminus A) = 0. \tag{4.15}$$

Intuitively this definition means that if a kernel is irreducible, there does not exist a nonempty set $A \subseteq \mathcal{S}$ such that the probability of edges forming from A to everywhere else, is zero. Effectively meaning that there can always be edges formed between any types of vertices.

4.2.1 Branching Processes

Next we need to discuss some general theory on branching processes. We will be adapting some theory from multiple sources, as for example classical ones [4] [33], and more modern ones [31, 48].

A single type branching process, sometimes also called a Galton-Watson process, is a simple model for the evolution of a population in time. We start with a root node, a single individual, who has a random amount of offspring, according to an *offspring distribution* $p_i = \mathbb{P}(\#\text{offspring} = i)$. Each individual offspring then independently follows the same offspring distribution and a branching process is born. A branching process naturally also defines a possibly infinite tree-graph. Branching processes are often found to be the local weak limits of random graph models. For a formal definition we refer you to for example [4, 48].

Multi-type branching process [33] are a natural next step. The number of offspring are independent across individuals given their types, and the distributions of each node with the same type are identical. We allow multiple offspring distributions to happen according their types, where types are part of a *type space*. The type describes the reproduction, i.e., it determines the probability for a node of type i to have a number of offspring of type j, and usually has a deterministic root node of which we know the offspring distribution. In our case we will consider a multi-type branching process where the offspring is determined by an inhomogeneous Poisson process, the advantage of the offspring being Poisson is that the joint distribution is fully determined by the intensity measure, which may depend on the type of the parent node.

In our application of multi-type branching processes to random graphs, each vertex in our graph has it's own type, uniquely determined by its weight, as well its location. Forming "offspring" is akin to forming edges between vertices according to the connection probability of one's graph. Visually one could represent this as one large $n \times n$ matrix for a graph of n vertices. This can be represented by the kernel which we use to form the connections. In this case we have that the kernel κ defined on the ground space $(\mathcal{S}, \mu)$, defines the branching process by having a vertex of type $x \in \mathcal{S}$ have offspring according to a Poisson process on $\mathcal{S}$ with intensity $\kappa(x, y)d\mu(y)$.

In the following sections we will make use of the following branching process with associated survival probabilities,

Definition 4.2.3. *Let $\mathcal{S}_a = ([N_n(a), [0,1))^d)$ be a type space and let $\Lambda_a(dx) = \frac{dx}{a}$ denote the normalized Lebesgue measure. For any vertex $u \in \mathcal{S}_a$ we write*

$$\overline{u} := (u, \tilde{x}_u), \tag{4.16}$$

for its type. Then we define $\mathcal{X}_a^\lambda$ to be the multi-type branching process on $\mathcal{S}_a$ where each vertex $\overline{u}$ produces offspring according to a Poisson process on $\mathcal{S}_a$ with intensity $\lambda\kappa(\mathbf{u}, \mathbf{x})\Lambda_a(\mathrm{d}y)\mathrm{d}\tilde{x}$. We write $\mathcal{X}_a^\lambda(\overline{u})$ to be the branching processes started from root $\overline{u}$.

The survival probability of $\mathcal{X}_a^\lambda$ is expressed as $\rho_a^\lambda(\overline{u})$, and we let $\rho_{a,\geq k}^\lambda$ denote the probability that $\mathcal{X}_a^\lambda(\overline{u})$ has at least k individuals. We define accordingly:

$$\rho_a^\lambda = \int\limits_{B_d}\int_0^a \rho_a^\lambda(\overline{u})\Lambda_a(\mathrm{d}u)\mathrm{d}x = \frac{1}{a}\int\limits_{B_d}\int_0^a \rho_a^\lambda(\overline{u})\mathrm{d}\overline{u}, \tag{4.17}$$

and

$$\rho_{a,\geq k}^\lambda = \int\limits_{B_d}\int_0^a \rho_{a,\geq k}^\lambda(\overline{u})\Lambda_a(\mathrm{d}\overline{u}) = \frac{1}{a}\int\limits_{B_d}\int_0^a \rho_{a,\geq k}^\lambda(\overline{u})\mathrm{d}\overline{u} \tag{4.18}$$

Now the integral operator defined by the kernel as in (4.12) is given by

$$\begin{aligned}(T_{\kappa^{(a)}}f)(u, \tilde{x}_u) &= \int\limits_{B_d}\int_0^a \kappa^{(a)}(u, \tilde{x}_u, v, \tilde{x}_v) f(v, \tilde{x}_v)\Lambda_a((\mathrm{d}v, \mathrm{d}\tilde{x}_v)) \\ &= \int\limits_{B_d}\int_0^a \Big(1 - \exp(-\Big(\frac{(uv)^{\frac{-1}{\tau-1}} c_F^2}{|\tilde{x}_u - \tilde{x}_v|^d}\Big)^\alpha\Big) f(v, \tilde{x}_v)\mathrm{d}v\mathrm{d}\tilde{x}_v. \end{aligned} \tag{4.19}$$

Consequently, we then have an operator norm of T_κ, which we will denote by $||T_\kappa||$. This operator can be intuitively understood as the expected sum over $f(\cdot)$ over all its children. Taking for instance $f = 1$ gives the expected number of children of a type x individual.

4.3 The Emergence of a Giant

We now turn to the main result of [11], which will provide us a clear condition under which a giant arises in an inhomogeneous random graph.

Theorem 4.3.1 (Existence of a giant component in an IRG [11]). *Let $(\kappa_n)_{n\geq 1}$ be a sequence of graphical kernels on a vertex space $\mathcal{V} = (\mathcal{S}, \mu, (u_n)_{n\geq 1})$ with limit $\kappa > 0$. Let T_κ denote the associated integral operator of κ. Then:*

(i) If $||T_\kappa|| \leq 1$ we have that

$$\mathscr{C}_{(1)}(G^{\mathcal{V}}(n, \kappa_n)) = o_{\mathbb{P}}(n). \tag{4.20}$$

However, If $||T_\kappa|| > 1$, then

$$\mathscr{C}_{(1)}(G^{\mathcal{V}}(n, \kappa_n)) = \Theta(n) \tag{4.21}$$

(ii) Furthermore, with $\rho(\kappa)$ as in (4.17)

$$\frac{1}{n}\mathscr{C}_{(1)}(G^{\mathcal{V}}(n, \kappa)) \xrightarrow{\mathbb{P}} \rho(\kappa). \tag{4.22}$$

(iii) If also $\sup_{x,y,n} \kappa_n(x, y) < \infty$, then with high probability we have

$$|\mathscr{C}_{(2)}|(G^{\mathcal{V}}(n, \kappa)) = O(\log n) \tag{4.23}$$

An immediate consequence of Theorem (4.3.1) is the following,

Corollary 4.3.2. *Let κ be a graphical kernel on a vertex space $\mathcal{V} = (\mathcal{S}, \mu, (u_n)_{n\geq 1})$, and consider the random graph associated to κ, $G^{\mathcal{V}}(n, \lambda\kappa)$, for some constant $\lambda > 0$. Then the threshold for the existence of a giant component as in (i) of Theorem 4.3.1 is precisely at $\lambda = ||T_\kappa||^{-1}$.*

As we have before shown that the percolated core can be viewed as an IRG, we would like to now define the corresponding multi-type branching process and use Corollary 4.3.2 to have an expression for the existence of the giant component inside the core.

Let $||T_{\kappa^{(a)}}||$ denote its operator norm and let $T^a_{\geq k}$ denote the set of vertices that belong to some component of size at least k in $\mathcal{G}_{N_n(a)}$.

We can now apply Corollary (4.3.2) to our case.

Proposition 4.3.3 (Existence of a Giant in $\mathcal{G}_{N_n(a)}$). *Let $\kappa^{(a)}$ be the limiting kernel of a sequence of graphical kernels $\kappa^{(a)}_{N_n}$ as defined in Theorem (4.2.1).*
Then for any $a > 0$ we have that if $\lambda > ||T_{\kappa^{(a)}}||^{-1}$ then

$$|\mathscr{C}^a_{(1)}| = N_n(a)\rho^\lambda_a(1 + o_{\mathbb{P}}(1)) \text{ and } |\mathscr{C}^a_{(2)}| = O_{\mathbb{P}}(\log(N_n(a))). \tag{4.24}$$

Furthermore, we also have that $\mathcal{T}^a_{\geq k}$ is stable, meaning that for every $\varepsilon > 0$, there exists a $\delta > 0$, such that with high probability removing at most δn edges from $\mathcal{G}_{N_n(a)}$ changes $\mathcal{T}^a_{\geq k}$ by at most εn vertices.

Proof. This follows immediately from Theorem 4.3.1 and Corollary 4.3.2, and the observation that clearly $\sup_{n,x,y} \kappa^{(a)}_{N_n}(x, y) < \infty$. The stability of the giant is proved in [11]. □

4.4 The Multi-Type Branching Process and its Survival Probability

In the previous sections we have shown that the percolated core of our GIRG can be characterized using the theory of inhomogeneous random graphs. The local weak limit (LWL) of an IRG is a multi-type branching process as defined before with inhomogeneous Poisson offspring reproduction [11].

In this section we will analyze the object

$$\zeta^{\lambda}_a := \lambda \int_0^a c_F u^{\frac{-1}{\tau-1}} \rho^{\lambda}_a(u) \mathrm{d}u, \tag{4.25}$$

where ρ^{λ}_a is as in (4.17). This object can be viewed as the 1-neighbourhood of the emerging giant, as the $c_F u^{\frac{-1}{\tau-1}}$ term is the rescaled weight within the IRG definition of the core, and $\rho^{\lambda}_a(u)$ is the survival probability of a vertex of type u. If we then let $a \to \infty$ and extend the core further, the one-neighbourhood of the emerging giant approximates the giant in the full graph.

We aim to show that the limit as $a \to \infty$ exists and is finite. Through observing ζ^{λ}_a we see that $u^{-1/(\tau-1)}$ is not integrable when we let $a \to \infty$, yet we will show that the limit still exists and is finite by upper bounding ζ^{λ}_a.

To show this convergence we will define a collapsed branching process that closely resembles the branching process obtained through the local weak limit of the IRG with the kernel defined in (4.9), and show that is an upper bound. The collapsed BP will be shown to be integrable and thus gives us the desired finiteness of ζ^{λ}_a.

Recall the setting of the IRG with kernel (4.9) with limit (4.12). Here we have a $d+1$ dimensional type space $\mathcal{S}_{N_n(a)} := ([\frac{1}{N_n}, \frac{2}{N_n}, \cdots, \frac{\lceil aN_n \rceil}{N_n}] \times [0,1)^d)$, which corresponds to the weight of the vertex given by the N_n discretized interval of $(0, a]$, which we will denote by $I^{(a)}_{N_n}$, and the d-dimensional position in the unit torus. Our type measure is a discrete uniform (DU) measure on $I^{(a)}_{N_n}$, and a d-dimensional Lebesgue measure on $(0,1)^d$. This causes the LWL of the IRG to be a multi-type branching process with Poisson reproduction, where the root has random type according to the type measure $\mu^{(a)}_{N_n} := DU(I^{(a)}_{N_n}) \otimes Leb((0,1)^d)$.

Given that a vertex has type $(u, \tilde{x}_u)$, its offspring form an inhomogeneous Poisson point process on the type space $\mathcal{S}_{N_n(a)}$ with intensity measure $\kappa((u, \tilde{x}_u), (v, \tilde{x}_v))\mathrm{d}\tilde{x}_v \mathrm{d}\Lambda_a(v)$, with κ as in (4.9). We will refer to this resulting branching process as defined in 4.2.3 by $BP(\kappa^{(a)})$ using the theory of Janson, Riordan and Bollobás from [11].

We want to first investigate the expected offspring of a type $(u, \tilde{x}_u)$ individual in $BP(\kappa^{(a)})$. Let $D^{(a)}_{u,\tilde{x}_u}$ denote its degree in the branching process. Then it follows from the offspring being Poisson distributed that

$$\mathbb{E}[D^{(a)}_{u,\tilde{x}_u}] = \int_0^a \int_{[0,1]^d} \kappa^{(a)}((u,\tilde{x}_u),(v,\tilde{x}_v)\mathrm{d}\tilde{x}_v \mathrm{d}\Lambda_a(v). \tag{4.26}$$

Before we move on we first define an important constant.

$$M_d := \int_{\mathbb{R}^d} 1 - \exp\left(\frac{-c_F^{2\alpha}}{\|\tilde{x}_v\|^{d\alpha}}\right) \mathrm{d}\tilde{x}_v. \tag{4.27}$$

Then, we need the following lemma

Lemma 4.4.1 (Scale and Translation Invariance)**.** *For any* $(u,\tilde{x}_u)$ *and* $(v,\tilde{x}_v)$ *we have that*

$$\int_{\mathbb{R}^d} 1 - \exp\left(-\left(\frac{c_F^2 (uv)^{\frac{-1}{\tau-1}}}{\|\tilde{x}_u - \tilde{x}_v\|^d}\right)^\alpha\right) \mathrm{d}\tilde{x}_v = (uv)^{\frac{-1}{\tau-1}} M_d. \tag{4.28}$$

Proof. We do a d-dimensional change of variables, we first define

$$y := x_v - x_u, \tag{4.29}$$

applying this yields

$$\int_{\mathbb{R}^d} 1 - \exp\left(-\left(\frac{c_F^2 (uv)^{\frac{-1}{\tau-1}}}{|\tilde{x}_u - \tilde{x}_v|^d}\right)^\alpha\right) \mathrm{d}\tilde{x}_v = \int_{\mathbb{R}^d} 1 - \exp\left(-\left(\frac{c_F^2 (uv)^{\frac{-1}{\tau-1}}}{|y|^d}\right)^\alpha\right) \mathrm{d}y. \tag{4.30}$$

We then set

$$z := y(uv)^{1/(\tau-1)d} \Leftrightarrow y = z(uv)^{-1/(\tau-1)d} \tag{4.31}$$

and note that

$$dy = (uv)^{-1/(\tau-1)} dz. \tag{4.32}$$

Using this in (4.30) we get

$$\begin{aligned}\int_{\mathbb{R}^d} 1 - \exp\left(-\left(\frac{c_F^2 (uv)^{\frac{-1}{\tau-1}}}{|y|^d}\right)^\alpha\right) \mathrm{d}y &= \int_{\mathbb{R}^d} 1 - \exp\left(\frac{-c_F^{2\alpha}}{|z|^{d\alpha}}\right)(uv)^{-1/(\tau-1)} \mathrm{d}z \\ &= (uv)^{\frac{-1}{\tau-1}} M_d, \end{aligned} \tag{4.33}$$

as desired. □

Then continuing on from (4.26) and using the scale and translation invariance, we have that

$$\begin{aligned}\mathbb{E}[D^{(a)}_{(u,\tilde{x_u})}] &\leq \int_0^a \lambda a (uv)^{\frac{-1}{\tau-1}} M_d \Lambda_a(\mathrm{d}v) \\ &= \lambda M_d u^{\frac{-1}{\tau-1}} \int_0^a v^{\frac{-1}{\tau-1}} \mathrm{d}v \\ &= \lambda M_d u^{\frac{-1}{\tau-1}} a^{1-\frac{1}{\tau-1}} \frac{\tau-1}{\tau-2}. \end{aligned} \tag{4.34}$$

This then means that the offspring of a type $(u, \tilde{x_u})$ individual in $BP(\kappa^{(a)})$ is stochastically dominated by a Poisson random variable with parameter $\lambda M_d u^{\frac{-1}{\tau-1}} a^{1-\frac{1}{\tau-1}} \frac{\tau-1}{\tau-2}$. A quick observation tells us that the expected degree grows as $a \to \infty$, which is a very intuitive statement by the fact that the core grows as a grows, and the giant has size $N_n(a)\rho_a^\lambda$.

We now define a new collapsed branching process BP^{col} that has a type space only consisting of $I_{N_n}^{(a)}$. The root has a uniformly chosen type from $I_{N_n}^{(a)}$ and each node u has offspring distributed according to a Poisson point process with intensity measure

$$\nu_u^{\text{col}}(dv) = M_d \lambda a (uv)^{\frac{-1}{\tau-1}} \Lambda_a(dv). \tag{4.35}$$

It is clear to see that the survival probability of the collapsed branching process $\rho_{\text{col}}^a(u)$ also stochastically dominates the survival probability of $BP(\kappa^{(a)})$, meaning we have $\rho_a^\lambda(\tilde{x}_u, u) \leq \rho_a^{\text{col}}(u)$.

The following well known theorem about Poisson point processes will help us in the analysis of BP^{col}, for a proof see for example [30].

Theorem 4.4.2. *Let μ be the intensity measure of a PPP, and let A be any set. Denote by N_A the amount of points in A, then given N_A, $(x_1, x_2, \cdots, x_{N_A})$ has the same distribution as the i.i.d sample of size N_A from the normalized measure*

$$\frac{\mu(\cdot)}{\mu(A)}. \tag{4.36}$$

We can then apply this theorem to prove the following lemma about the children in BP^{col}.

Lemma 4.4.3. *In BP^{col}, given the total number of children D_u^{col} of node u, each child has an i.i.d type from the normalized measure*

$$\nu_a^\star(dv) = a^{\frac{1}{\tau-1}-1} \frac{\tau-2}{\tau-1} v^{\frac{-1}{\tau-1}} dv \text{ on } [0, a). \tag{4.37}$$

Proof. We know by (4.34) that in BP^{col} a vertex of type u has Poisson many children with parameter $\lambda M_d u^{\frac{-1}{\tau-1}} a^{1-\frac{1}{\tau-1}} \frac{\tau-1}{\tau-2}$, which is also the total measure of $[0, a)$. Then given D_u^{col} and using Theorem 4.4.2 each child has an i.i.d type according to the measure

$$\nu_a^\star(dv) = \frac{\nu_u^{\text{col}}(dv)}{\nu_u^{\text{col}}([0, a))} = \frac{\lambda M_d u^{\frac{-1}{\tau-1}} a v^{\frac{-1}{\tau-1}} \Lambda_a(dv)}{\lambda M_d u^{\frac{-1}{\tau-1}} a^{1-\frac{1}{\tau-1}} \frac{\tau-1}{\tau-2}} = a^{\frac{1}{\tau-1}-1} \frac{\tau-2}{\tau-1} v^{\frac{-1}{\tau-1}} dv. \tag{4.38}$$

□

Importantly, $\nu_a^\star$ is completely independent of the type u. This means we can show the following proposition

Proposition 4.4.4. *BP^{col} is a rank-1 branching process on $(0, a]$ where a type u node has a total Poisson(λ_u) number of children, with*

$$\lambda_u := M_d \lambda \frac{\tau - 2}{\tau - 1} a^{1-\frac{1}{\tau-1}} u^{\frac{-1}{\tau-1}}. \tag{4.39}$$

and each child has an i.i.d type from the distribution $\nu_a^\star(dv)$ as defined in (4.37) on $(0, a]$.

Proof. This follows directly from (4.34) which shows that the expected number of offspring is simply a constant times $u^{-1/(\tau-1)}$, and the independence of the parent's type from Lemma 4.4.3, causing the expected offspring matrix to factorize. □

We now study the survival probability of BP^{col} when the root node is uniformly chosen from $\nu_a^\star(dv)$. No matter what the type of the root is, for each of its children the subtree that spans from them are all i.i.d with the same survival probability as per Lemma 4.4.3. If the root has given type u, then it produces $Poisson(\lambda_u)$ offspring, where each child's subtree survives with probability $\rho_a^\star$. Then

$$\begin{aligned}
\rho_a^{\text{col}}(u) &= \mathbb{E}[1 - (1 - \rho_a^\star)^{D_u}] \\
&= 1 - e^{-\lambda_u \rho_a^\star} \\
&= 1 - \exp\left(M_d \lambda_u \frac{\tau - 2}{\tau - 1} a^{1-\frac{1}{\tau-1}} u^{\frac{-1}{\tau-1}} \rho_a^\star \right).
\end{aligned} \tag{4.40}$$

If the root u is chosen uniformly at random from $\nu_a^\star$, then by the law of total probability we can find the survival probability by integrating over all possible types and find

$$\begin{aligned}
\rho_a^\star &= \int_0^a \rho_a^{\text{col}}(u) \nu_a^\star(du) \\
&= \int_0^a \rho_a^{\text{col}}(u) a^{\frac{1}{\tau-1}-1} \frac{\tau - 2}{\tau - 1} u^{\frac{-1}{\tau-1}} du \\
&= \int_0^a \left(1 - \exp\left(M_d \lambda_u \frac{\tau - 1}{\tau - 2} a^{1-\frac{1}{\tau-1}} u^{\frac{-1}{\tau-1}} \rho_a^\star \right)\right) a^{\frac{1}{\tau-1}-1} \frac{\tau - 2}{\tau - 1} u^{\frac{-1}{\tau-1}} du.
\end{aligned} \tag{4.41}$$

We now define the indicator

$$A_i := \mathbb{1}(\text{the i'th subtree survives}), \tag{4.42}$$

to express $\rho_a^{\text{col}}(u)$ as a sum of indicators.

$$\begin{aligned}
\rho_a^{\text{col}}(u) &= \mathbb{P}\left(\sum_{i=1}^{D_u^{\text{col}}} A_i \geq 1 \right) \leq \mathbb{E}\left[\sum_{i=1}^{D_u^{\text{col}}} A_i \right] = \mathbb{E}[D_u^{\text{col}}] \rho_a^\star \\
&= \lambda M_d u^{\frac{-1}{\tau-1}} a^{1-\frac{1}{\tau-1}} \frac{\tau - 1}{\tau - 2} \rho_a^\star.
\end{aligned} \tag{4.43}$$

We now define the following variable $\tilde{M}$

$$M^\star := a^{1-\frac{1}{\tau-1}} \rho_a^\star, \tag{4.44}$$

and prove the following proposition about $M^\star$.

Proposition 4.4.5. $\limsup_a M^\star$ *exists and is finite.*

Proof. We start by considering (4.41) and rewriting it such that we have

$$M^\star = a^{1-\frac{1}{\tau-1}}\rho_a^\star = \int_0^a \left(1-\exp\left(M_d\lambda_u \frac{\tau-1}{\tau-2}a^{1-\frac{1}{\tau-1}}u^{\frac{-1}{\tau-1}}\rho_a^\star\right)\right)\frac{\tau-2}{\tau-1}u^{\frac{-1}{\tau-1}}du \quad (4.45)$$

$$\leq \int_0^a \min\left(M_d\lambda_u\frac{\tau-1}{\tau-2}u^{\frac{-1}{\tau-1}}M^\star, 1\right)\frac{\tau-2}{\tau-1}u^{\frac{-1}{\tau-1}}du \quad (4.46)$$

$$= \int_0^{(M_d\lambda_u\frac{\tau-1}{\tau-2}M^\star)^{\tau-1}} \frac{\tau-2}{\tau-1}u^{\frac{-1}{\tau-1}}du \; + \int_{(M_d\lambda_u\frac{\tau-1}{\tau-2}M^\star)^{\tau-1}}^a M_d\lambda_u u^{\frac{-2}{\tau-1}}M^\star du \quad (4.47)$$

$$\leq C(M^\star)^{(\tau-1)(1-\frac{1}{\tau-1})} = C(M^\star)^{\tau-2}. \quad (4.48)$$

Here we end up collecting all constants in C, and we see that eventually after rearranging we end up with $M^\star \leq C(M^\star)^{\tau-2}$, as $\tau \in (2,3)$, we now see that $\limsup_a M^\star$ exists and must be finite. □

Now we prove the convergence of ζ_a^λ as defined in (4.25).

Proposition 4.4.6. *For any* $\lambda > \lambda_c$, *as* $a \to \infty$,

$$\zeta_a^\lambda \leq \lambda \int_0^a c_F u^{\frac{-1}{\tau-1}}\rho_a^{\mathrm{col}}(u)du \to \zeta^\lambda \in (0,\infty). \quad (4.49)$$

Proof. Recall from the definition of BP^{col} that the first inequality holds. Then, through the elementary upper bound

$$\rho_a^{\mathrm{col}}(u) \leq \min\left(1, \lambda M_d u^{\frac{-1}{\tau-1}}\frac{\tau-1}{\tau-2}M^\star\right), \quad (4.50)$$

we find that

$$\zeta_a^\lambda \leq C\int_0^a \min(1, u^{\frac{-1}{\tau-1}})u^{\frac{-1}{\tau-1}}du. \quad (4.51)$$

Where all constants are collected in C, and we can treat $M^\star$ as a constant by Proposition 4.4.5. Now this integral is integrable at both 0 and a as $a \to \infty$, and thus converges to the limit ζ^λ. □

5 The Slightly Supercritical Regime

Here we treat the slightly supercritical regime, where the percolation probability is given by $\pi_n = \lambda n^{(\tau-3)/2}$ for $\lambda > \lambda_c$. We first give some general estimates for the moments of the weights, and then move on to the analysis of the tiny giant by defining and investigating spans and return paths of the largest component in the core.

5.1 Moment Estimates for Weights

In the following section we will need the moment estimates of random variables that follow a power law distribution, sometimes truncated. For this we prove a few general results that are summarized in the following two lemmas.

Lemma 5.1.1. *Given a random variable W which follows a power law distribution with density $w^{-\tau}$ on $[1,\infty)$ and constants $r, d, w_{i_1}, w_{i_2}, n, b$, we have the following moment estimates:*

$$\mathbb{E}\left[W^{1+\alpha}\mathbb{1}\left(W \leq \min(\frac{r^d}{w_{i_2}}, \frac{\sqrt{n}}{b^{\frac{1}{\tau-1}}}\right)\right] = \left(\frac{r^d}{w_{i_2}} \wedge \frac{\sqrt{n}}{b^{\frac{1}{\tau-1}}}\right)^{2+\alpha-\tau},$$

$$\mathbb{E}\left[W^{2\alpha}\mathbb{1}(W \leq \min\left(\frac{r^d}{w_{i_2}}, \frac{\sqrt{n}}{b^{\frac{1}{\tau-1}}}\right)\right] = \left(\frac{r^d}{w_{i_2}} \wedge \frac{\sqrt{n}}{b^{\frac{1}{\tau-1}}}\right)^{2\alpha+1-\tau},$$

$$\mathbb{E}\left[W\mathbb{1}\left(\frac{\sqrt{n}}{b^{\frac{1}{\tau-1}}} \geq W \geq \frac{r^d}{w_{i_2}}\right)\right] = \frac{r^{d(2-\tau)}}{w_{i_2}^{2-\tau}}\mathbb{1}\left(\frac{\sqrt{n}}{b^{\frac{1}{\tau-1}}} \geq \frac{r^d}{w_{i_2}}\right),$$

$$\mathbb{E}\left[W^{\alpha}\mathbb{1}\left(\frac{r^d}{w_{i_2}} \leq W \leq \min\left(\frac{r^d}{w_{i_1}}, \frac{\sqrt{n}}{b^{\frac{1}{\tau-1}}}\right)\right)\right] = \begin{cases} \left(\frac{r^d}{w_{i_1}} \wedge \frac{\sqrt{n}}{b^{\frac{1}{\tau-1}}}\right)^{1+\alpha-\tau} & \text{if } \alpha > \tau-1 \\ \frac{r^{d(1+\alpha-\tau)}}{w_{i_2}^{1+\alpha-\tau}}\mathbb{1}\left(\frac{\sqrt{n}}{b^{\frac{1}{\tau-1}}} \geq \frac{r^d}{w_{i_2}}\right) & \text{if } \alpha < \tau-1 \\ \log\left(\left(\frac{r^d}{w_{i_1}} \wedge \frac{\sqrt{n}}{b^{\frac{1}{\tau-1}}}\right)\right) & \text{if } \alpha = \tau-1 \end{cases},$$

$$\mathbb{E}\left[W\mathbb{1}\left(\frac{r^d}{w_{i_2}} \leq W \leq \min\left(\frac{r^d}{w_{i_1}}, \frac{\sqrt{n}}{b^{\frac{1}{\tau-1}}}\right)\right)\right] = \frac{r^{d(2-\tau)}}{w_{i_2}^{2-\tau}}\mathbb{1}\left(\frac{\sqrt{n}}{b^{\frac{1}{\tau-1}}} \geq \frac{r^d}{w_{i_2}}\right).$$

Lemma 5.1.2. *For a uniformly chosen vertex in $[N_n(\varepsilon), N_n(a)]$ with weight distribution again a power law with parameter τ, we have that*

if $x < \tau - 1$, *then* $\mathbb{E}[W^x] = n^{x/2} a^{-x/\tau - 1}$.
If $x = \tau - 1$, *then* $\mathbb{E}[W^{\tau-1}] = n^{(\tau-1)/2} \log n$.
If $x > \tau - 1$, *then* $\mathbb{E}[W^x] = n^{x/2} \varepsilon^{-x/\tau - 1}$.

For proofs of these moment estimates see for instance [10].

5.2 Spans and Return Paths

In this section, we continue towards the proof of Conjecture 2.3.1 for the slightly supercritical regime. We will consider the giant in $[N_n(a)]$, which we have shown exists in the supercritical regime in Proposition 4.3.3. We want to show that this giant of size $N_n(a)\rho_a^\lambda(1 + o_{\mathbb{P}}(1))$ evolves into the giant of the full graph when letting $a \to \infty$ and investigate its size.

To approach this problem let G_n be a GIRG as in Definition 2.1.1, let us denote the induced subgraph of the top $N_n(a)$ vertices by $\mathcal{G}_{N_n(a)}$ as before, and we now write $\mathscr{C}^a_{(1)}$ for the largest component in $\mathcal{G}_{N_n(a)}$. Let $\mathscr{C}^{a,\star}_{(1)}$ denote the component of G_n that contains $\mathscr{C}^a_{(1)}$, and more generally let $\mathscr{C}^{a,\star}_{(i)}$ be the component in G that contains the i'th component in $\mathcal{G}_{N_n(a)}$. Clearly we always have $\mathscr{C}^a_i \subset \mathscr{C}^{a,\star}_{(i)}$, however the question is how much the components change in size. First, we have to account for the vertices in $[N_n(a)]^c$ that have connections to the ones in $[N_n(a)]$. To this end, define

Definition 5.2.1 (Span of a Vertex). *For some u in $[N_n(a)]$, let $\mathrm{Span}_a(u)$ be the set of vertices outside of $[N_n(a)]^c$ such that there exists a path from u, entirely in $[N_n(a)]^c$. Formally:*

$$\begin{aligned}\mathrm{Span}_a(u) = \{v \in [N_n(a)]^c : \exists\ (i_0, \cdots, i_k) \in [N_n(a)]^c \\ \text{s.t } (u \leftrightarrow i_0 \leftrightarrow \cdots \leftrightarrow i_k \leftrightarrow v) \in G_n\}.\end{aligned} \tag{5.1}$$

It is important to note that this definition requires the path that connects u and v to be *entirely* outside of $\mathcal{G}_{N_n(a)}$. The case where a path potentially returns to the subgraph will be treated later. Consequently, for a subset $V \subseteq [N_n(a)]$ we analogously define

Definition 5.2.2 (Span of a Component). *For $V \subseteq [N_n(a)]$ define,*

$$\mathrm{Span}_a(V) = \bigcup_{u \in V} \mathrm{Span}_a(u). \tag{5.2}$$

In the later analysis of the giant we will pick $V = \mathscr{C}^a_{(1)}$ and look at the span of the tiny giant to fully extend it to its component in G_n. The span of $\mathscr{C}^{(a)}_1$ however accounts for only some of the vertices that get added when considering the full graph. It is possible that through vertices in $[N_n(a)]^c$ a connection is formed between $\mathscr{C}^a_{(1)}$ and any other smaller component $\mathscr{C}^a_{(i)}$ with $i > 1$. In this case the whole components gets added to $\mathscr{C}^a_{(1)}$. To also cover this possibility, we define

Definition 5.2.3 (Return Path). *Let $u, v \in [N_n(a)]$, we say there exists a return path between u and v, if there exists a path between them with at least one vertex in $[N_n(a)]^c$. Define $\mathcal{R}^a_{(i)}$ as the set of vertices $v \in [N_n(a)]$ such that v is connected to some $u \in \mathscr{C}^a_{(i)}$ **only** by a return path.*

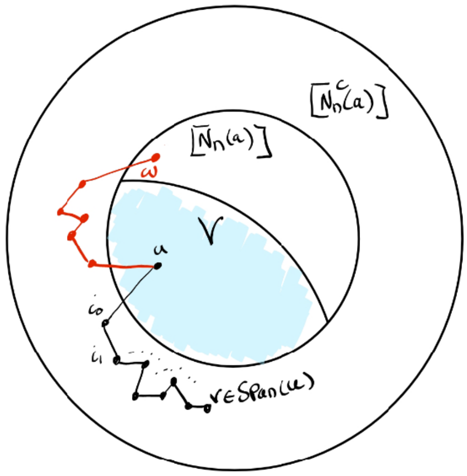


FIGURE 5.1: An example of a vertex $v \in \text{Span}(u)$ reached by the black path, and for instance a realization of a return path in red reaching the vertex w.

Using these definitions we can then express $\mathscr{C}_{(1)}^{a,\star}$, the component that contains $\mathscr{C}_{(1)}^{(a)}$ in G_n, as follows,

$$\mathscr{C}_{(1)}^{a,\star} = \mathscr{C}_{(1)}^{a} \cup \text{Span}_a(\mathscr{C}_{(1)}^{a}) \cup \mathcal{R}_{(1)}^{a} \cup \text{Span}_a(\mathcal{R}_{(1)}^{a}). \tag{5.3}$$

The idea is as follows, we will show that $\mathscr{C}_{(1)}^{a,\star}$ is mostly formed by $\mathscr{C}_{(1)}^{a}$ and its span. and that the contribution of return paths is negligible. In fact what we will see is that $\text{Span}_a(\mathscr{C}_{(1)}^{a})$ is asymptotically close to the one-neighborhood of $\mathscr{C}_{(1)}^{a}$ as $a \to \infty$. To show this we start by proving upper and lower bounds on the span of $\mathscr{C}_{(1)}^{a}$.

5.3 Lower Bound on the Span of $\mathscr{C}_{(1)}^{a}$

We start by fixing our $a > 0$ and work in the usual setting. For a subset $V \subseteq [N_n(a)]$ let $\mathcal{N}_l(V)$ denote the vertices in $\text{Span}_a(V)$ that are at graph-distance l from V, and let $\mathcal{N}_{\geq l}(V) = \cup_{l' \leq l} \mathcal{N}_{l'}(V)$. Then,

$$\text{Span}_a(V) = \mathcal{N}_1(V) \ \cup \ \mathcal{N}_{\geq 2}(V). \tag{5.4}$$

Clearly the lower bound $|\text{Span}_a(V)| \geq |\mathcal{N}_1(V)|$ holds. Estimating the size of $|\mathcal{N}_1(V)|$ is what will be our first step. In order to do this we will require the results of Chapter 3. By conditioning on the good configuration of a GIRG we reveal the vertices completely, their locations and their weights are known to fit the good event of Theorem 3.1.1 and we will use this to estimate the span of the largest component.

To do so we will apply a standard inclusion-exclusion argument, which is why we need an estimate on the expected number of two-step paths that leave V for one vertex outside of $[N_n(a)]$. We define the following for some $b \geq a$.

$$\mathcal{I}_2^{a,b}(V) = \{y : y > N_n(b), \exists i_1, i_2 \in V : i_1 \leftrightarrow y \leftrightarrow i_2\} \tag{5.5}$$

Where $i_1 \leftrightarrow y$ denotes an edge existing between both vertices. We then compute the expected size of $\mathcal{I}_2^{a,b}(V)$ through the following proposition.

Proposition 5.3.1. *Let $V \subseteq [N_n(a)]$. Then for every realisation of $\mathcal{G}_{N_n(a)}$ satisfying $\mathcal{A}_{\text{good}}$ event of Proposition 3.1.1 for fixed $a, b > 0$ and $b \geq a$, we have*

$$\mathbb{E}[|\mathcal{I}_2^{a,b}(V)| \mid \mathcal{G}_{N_n(a)}] = c(b)n^{(3-\tau)/2}, \tag{5.6}$$

where $c(b)$ is an explicitly computable constant that tends to 0 as $b \to \infty$.

We will prove this proposition in this general form, then we proceed to first apply it in the case where $b = a$ to prove a lower bound on the span of $\mathscr{C}_{(1)}^a$. Later on when considering the upper bound we will require the case where we no longer take $b = a$ and $b \to \infty$.

Proof. We fix $a, b > 0$ with $b \geq a$. Then $|\mathcal{I}_2^{a,b}|$ can simply be expressed as the sum of indicators

$$|\mathcal{I}_2^{a,b}| \leq \sum_{i \in V} \sum_{i_2 \in V} \sum_{y \notin [N_n(b)]} \mathbb{1}(i_1 \leftrightarrow y \leftrightarrow i_2) \tag{5.7}$$

Taking the conditional expectation over the good configuration event then gives us

$$\mathbb{E}[|\mathcal{I}_2^{a,b}(V)| \mid \mathcal{G}_{N_n(a)}] \leq \sum_{i \in V} \sum_{i_2 \in V} \sum_{y \notin [N_n(b)]} \pi_n^2 \overline{p}_{i_1 y} \overline{p}_{i_2 y}. \tag{5.8}$$

Where with $\overline{p}_{i_1 y}$ we denote the *conditional* connection probability in the case where the locations are revealed. We will define

$$\overline{p}_{I_1 y I_2} := \mathbb{P}(i_1 \leftrightarrow y \leftrightarrow i_2 | x_{i_1}, x_{i_2}) \tag{5.9}$$

To indicate the conditional probability over uniformly chosen $i_1, i_2 \in [N_n(a)]$. Note that in this conditioning it is now a random weight that we take expectation over regarding i_1 and i_2. By Lemma 3.2.1 we know that within $[N_n(a)]$ between any distance r and $2r$, there exist $\frac{a^2}{2} n^{2-\tau} r^d V$ many pairs. Because we condition on the entire graph being known we can upper bound by uniformly choosing i_1 and i_2 in $[N_n(a)]$ such that

$$\sum_{i \in V} \sum_{i_2 \in V} \sum_{y \notin [N_n(b)]} \pi_n^2 \overline{p}_{i_1 y} \overline{p}_{i_2 y} \leq \sum_{k=1}^{k_{max}} \frac{a^2}{2} n^{2-\tau} r_k^d V \sum_{y \notin [N_n(b)]} \overline{p}_{I_1 y I_2}. \tag{5.10}$$

We will WLOG assume that $w_{i_1} < w_{i_2}$. Furthermore we set $r := |i_1 - i_2|, r_1 := |y - i_1|, r_2 := |y - i_2|$. We now have to consider the distances of i_1 and i_2 to y. In the case that $r_1 \leq r_2$, we have that $r_2 \geq max(r/2, r_1)$. This means that we can replace r_2 in the connection probability for i_2 and be only dependent on r_1. Of course the opposite can also happen of $r_2 \leq r_1$. In this case the symmetric case holds, which we will both treat at the same time. Now by the percolated connection probabilities we upper bound $\overline{p}_{I_1yI_2}$

$$\overline{p}_{I_1yI_2} \leq \frac{\pi_n^2}{n}\mathbb{1}(r_1 \leq r_2)\int_{r_1=0}^{n^{1/d}} (\frac{w_yw_{i_1}}{r_1^d} \wedge 1)^\alpha (\frac{w_yw_{i_2}}{\max(r_k/2, r_1^d)} \wedge 1)^\alpha r_1^{d-1} dr_1$$
$$+\frac{\pi_n^2}{n}\mathbb{1}(r_2 \leq r_1)\int_{r_2=0}^{n^{1/d}} (\frac{w_yw_{i_1}}{\max(r_k/2, r_2^d)} \wedge 1)^\alpha (\frac{w_yw_{i_2}}{r_2^d} \wedge 1)^\alpha r_2^{d-1} dr_2. \quad (5.11)$$

We can immediately see that these cases are very symmetric, and we will usually only consider the integral over r_1, and adjust the symmetric results accordingly after.

Another thing to notice is the immediate obvious case distinction that happens in r_k, we have assumed that $w_{i_1} < w_{i_2}$, and with that as all weights are positive also that $w_y w_{i_1} < w_y w_{i_2}$. This makes it that there are three possible options for r_k, and we will split the sum over y into these three possible cases depending on the size of r_k, and on the restriction that $y \notin [N_n(b)]$. Note that all vertices in $[N_n(b)]$ have at least weight $c_F\sqrt{n}\, b^{\frac{-1}{\tau-1}}$, such that the restriction on the weight of y can be reformulated as $w_y \leq c_F\sqrt{n}\, b^{\frac{-1}{\tau-1}}$.

Continuing on we will consider each case separately, as the sum over y in equation (5.10) is simply a summation of the three case distinctions over r_k, it suffices to show that each case is of order $n^{(3-\tau)/2}$ and the constant $c(b)$ tends to 0 each time.

Before we move to the first case, let us recall the definition and discreteness of $r_k^d := 2^i r_0$, with r_0 the same as in Theorem 3.1.1, we will now first define the necessary constant $k^\star$ such that

$$2^{k^\star d} \leq \frac{n^{3-\tau}}{M(i_2b)^{1/(\tau-1)}}. \quad (5.12)$$

This $k^\star$ will prove to be useful in the next case.

Case 1: (Large r)

This is the case where $\min\left(\frac{r_k^d}{w_{i_2}}, \sqrt{n}\, b^{\frac{-1}{\tau-1}}\right) \geq w_y$. If we now consider the integral in (5.11) we can split it in the following way

$$\overline{p}_{I_1yI_2} \leq \int_{r_1=0}^{(w_yw_{i_1})^{1/d}} r_1^{d-1}\left(\frac{w_yw_{i_2}}{r_k^d}\right)^\alpha dr_1 + \int_{(w_yw_{i_1})^{1/d}}^{r_k/2} r_1^{d-1}\left(\frac{w_yw_{i_1}}{r_1^d}\right)^\alpha (\frac{w_yw_{i_2}}{r_k^d})^\alpha dr_1$$
$$+\int_{r_k/2}^{n^{1/d}} r_1^{d-1}(\frac{w_yw_{i_1}}{r_1^d})^\alpha(\frac{w_yw_{i_2}}{r_1^d})^\alpha dr_1, \quad (5.13)$$

multiplied by the indicator $\mathbb{1}(r_1 \leq r_2)$. The symmetrical case where $r_2 \leq r_1$ will show up in the final computation but as simply the maximum term switches we leave it out for now.

Evaluating this integral gives the following:

$$\overline{p}_{I_1yI_2} \leq \left(\frac{w_yw_{i_2}}{r_k^d}\right)^\alpha \left[r_1^d\right]_{r_1=0}^{r_1=(w_yw_{i_1})^{1/d})} + (w_yw_{i_1})^\alpha\left(\frac{w_yw_{i_2}}{r_k^d}\right)^\alpha\left[r_1^{d(1-\alpha)}\right]_{r_1=(w_yw_{i_1})^{1/d})}^{r_1=r/2}$$
$$+ (w_y^2w_{i_1}w_{i_2})^\alpha\left[r_1^{d(2-\alpha)}\right]_{r_1=r/2}^{r_1=n^{1/d}} \quad (5.14)$$

which gives us

$$\bar{p}_{I_1 y I_2} \leq \left(\frac{w_y w_{i_2}}{r_k^d}\right)^{\alpha} (w_y w_{i_1}) + (w_y w_{i_1})^{\alpha} (\frac{w_y w_{i_2}}{r^d})^{\alpha} (w_y w_{i_1})^{1-\alpha} + (w_y^2 w_{i_1} w_{i_2})^{\alpha} \frac{r_k^d}{r_k^{2d\alpha}} \tag{5.15}$$

After elementary rearrangements we arrive at

$$\bar{p}_{I_1 y I_2} \leq \frac{w_y^{1+\alpha}}{r_k^{d\alpha}} w_{i_2}^{\alpha} w_{i_1} + \frac{(w_y^2 w_{i_1} w_{i_2})^{\alpha} r_k^d}{r_k^{2d\alpha}}. \tag{5.16}$$

Now we consider also the case where $r_2 \leq r_1$, and it is easy to see that only the powers in the first term flip between w_{i_1} and w_{i_2} giving rise to the final solution of the case where $w_y \leq \frac{r k^d}{w_{i_2}}$:

$$\bar{p}_{I_1 y I_2} \leq \mathbb{1}(r_1 \leq r_2) \frac{w_y^{1+\alpha}}{r_k^{d\alpha}} w_{i_2}^{\alpha} w_{i_1} + \mathbb{1}(r_2 \leq r_1) \frac{w_y^{1+\alpha}}{r_k^{d\alpha}} w_{i_1}^{\alpha} w_{i_2} + \frac{(w_y^2 w_{i_1} w_{i_2})^{\alpha} r_k^d}{r_k^{2d\alpha}}. \tag{5.17}$$

We now take expectation over w_y and apply the moment estimates for the w_y from Lemma 5.1.1. This results in the following:

$$\bar{p}_{I_1 y I_2} \leq \frac{w_{i_2}^{\alpha}\ w_{i_1}}{r_k^{d\alpha}} \left(\frac{r_k^d}{w_{i_2}} \wedge \frac{\sqrt{n}}{b^{1/(\tau-1)}}\right)^{2+\alpha-\tau} + \frac{(w_{i_1} w_{i_2})^{\alpha} r_k^d}{r_k^{2d\alpha}} \left(\frac{r_k^d}{w_{i_2}} \wedge \frac{\sqrt{n}}{b^{1/(\tau-1)}}\right)^{2\alpha+1-\tau}. \tag{5.18}$$

Now we do a second case distinction with respect to the value of r_k using $k^\star$ as defined in (5.12). We first treat the term where $k \leq k^\star$ such that for all $k \leq k^\star$

$$\frac{r_k^d}{w_{i_2}} \leq \frac{\sqrt{n}}{b^{1/(\tau-1)}}, \tag{5.19}$$

this is due to the fact that by definition of $k^\star$ we have that

$$\frac{2^{k^\star d} n^{\tau-2} M}{w_{i_2}} \leq \frac{\sqrt{n}}{b^{1/(\tau-1)}} \implies 2^{k^\star d} \leq \frac{\sqrt{n} w_{i_2}}{n^{\tau-2} b^{1/(\tau-1)} M}. \tag{5.20}$$

Using that $w_{i_2} = \frac{\sqrt{n}}{i_2^{1/(\tau-1)}}$, then shows us that for all $k \leq k^\star$ the minimum is indeed obtained at $\frac{r_k^d}{w_{i_2}}$. We then find

$$\bar{p}_{I_1 y I_2} \mathbb{1}_{k \leq k^\star} \leq \frac{w_{i_2}^{\alpha}\ w_{i_1}}{r_k^{d\alpha}} \left(\frac{r_k^d}{w_{i_2}}\right)^{2+\alpha-\tau} + \frac{(w_{i_1} w_{i_2})^{\alpha} r_k^d}{r_k^{2d\alpha}} \left(\frac{r_k^d}{w_{i_2}}\right)^{2\alpha+1-\tau} \tag{5.21}$$

$$= w_{i_2}^{\tau-2} w_{i_1} r_k^{d(2-\tau)} + w_{i_1}^{\alpha} w_{i_2}^{\tau-\alpha-1} r_k^{d(2-\tau)}. \tag{5.22}$$

Notice here that in the second term we have the quotient $(\frac{w_{i_1}}{w_{i_2}})^{\alpha}$, due to our initial assumption that $w_{i_2} \geq w_{i_1}$, this ratio is clearly less than 1 and so it does not blow up for large α.

After using the moment estimates for w_{i_1} and w_{i_2} from Lemma 5.1.2 we then find that the order of this expression in terms of n is equal to

$$\log(n)n^{(\tau-1)/2}r_k^{d(2-\tau)}. \tag{5.23}$$

Recall the full summation over k from (5.10), we should not fail to mention the addition of the factor n that comes from taking the expectation over the w_y here that replaces the summation over said vertices. We then find, ignoring constants and simply looking at the order of n,

$$\sum_{i\in V}\sum_{i_2\in V}\sum_{y\notin[N_n(b)]}\pi_n^2\overline{p}_{i_1y}\overline{p}_{i_2y} \leq \sum_{k=1}^{k=k^\star} n^{2-\tau}r_k^{d(3-\tau)}n\frac{\pi_n^2}{n}n^{(\tau-1)/2}. \tag{5.24}$$

Now we can upper bound the sum of $\sum_{k=1}^{k=k^\star} r_k^{d(3-\tau)}$ by $2r_{k^\star}^{d(3-\tau)}$ which by definition of $k^\star$ is equal to

$$(r_k^\star)^{d(3-\tau)} = \left(\frac{n^{3-\tau}n^{\tau-2}}{b^{1/(\tau-1)}i_2^{1/(\tau-1)}}\right)^{3-\tau}. \tag{5.25}$$

Using this in (5.24) then gives the order of n to be

$$n^{2-\tau}\pi_n^2 n^{(\tau-1)/2}\sum_{k=1}^{k=k\star} r_k^{d(3-\tau)} \leq n^{2-\tau}n^{\tau-3}n^{(\tau-1)/2}n^{(3-\tau)(3-\tau)}n^{(\tau-2)(3-\tau)} = n^{(3-\tau)/2} \tag{5.26}$$

and the constant $c(b)$ clearly converges to 0 as $b\to\infty$.

Now in this first case we are left with the part of the sum that starts at $k^\star$ and the minimum is attained at $\frac{\sqrt{n}}{b^{1/(\tau-1)}}$. This gives us the remainder of the sum over k where

$$\begin{aligned}\sum_{i\in V}\sum_{i_2\in V}\sum_{y\notin[N_n(b)]}\pi_n^2\overline{p}_{i_1y}\overline{p}_{i_2y} \leq n^{2-\tau}\pi_n^2\Bigg(\sum_{k=k^\star}^{k=k_{\max}} r_k^{d(1-\alpha)}\left(\frac{\sqrt{n}}{b^{1/(\tau-1)}}\right)^{2+\alpha-\tau} w_{i_2}^\alpha w_{i_1}\\ + r_k^{d(2(1-\alpha))}\left(\frac{\sqrt{n}}{b^{1/(\tau-1)}}\right)^{2\alpha+1-\tau}(w_{i_1}w_{i_2})^\alpha\Bigg).\end{aligned} \tag{5.27}$$

Now we analyze the order of n similarly as before, taking the moment estimates for the weights w_{i_1} and w_{i_2} and as we see that the exponent of r_k in this case is negative, we can upper bound the sum by the lower boundary $k^\star$. This provides the following expression of powers of n

$$\begin{aligned}\sum_{i\in V}\sum_{i_2\in V}\sum_{y\notin[N_n(b)]}\pi_n^2\overline{p}_{i_1y}\overline{p}_{i_2y} \leq n^{2-\tau}\pi_n^2((n^{3-\tau}n^{\tau-2})^{1-\alpha}n^{\frac{2+\alpha-\tau}{2}}n^{\frac{\alpha+1}{2}}\\ + (n^{3-\tau}n^{\tau-2})^{2(1-\alpha)}n^{\frac{2\alpha+1-\tau}{2}}n^\alpha = n^{(3-\tau)/2},\end{aligned} \tag{5.28}$$

furthermore it is also again clear to see that $c(b)\to 0$ as $b\to\infty$. This concludes case 1. In the following cases we will use the same methodology to analyze the order of n and the limit of $c(b)$.

Case 2: (Small r)

Now we have entered the case where $\sqrt{n}\, b^{\frac{-1}{\tau-1}} \geq w_y \geq \frac{r^d}{w_{i_1}}$, this means that the product of the weights will always be larger than r. We split the integral in the following way, where again we first only treat the $r_1 \leq r_2$ case.

$$\overline{p}_{I_1 y I_2} \leq \int_{r_1=0}^{(w_y w_{i_1}))^{1/d}} r_1^{d-1} dr_1 + \int_{(w_y w_{i_1}))^{1/d}}^{(w_y w_{i_2})^{1/d}} r_1^{d-1} (\frac{w_y w_{i_1}}{r_1^d})^{\alpha} dr_1 + \int_{(w_y w_{i_2})^{1/d}}^{n^{1/d}} r_1^{d-1} (\frac{w_y w_{i_1}}{r_1^d})^{\alpha} (\frac{w_y w_{i_2}}{r_1^d})^{\alpha} dr_1 \tag{5.29}$$

Evaluating this integral gives

$$\overline{p}_{I_1 y I_2} \leq \left[r^d\right]_0^{(w_y w_{i_1})^{1/d}} + (w_y w_{i_1})^{\alpha} \left[r_1^{d(1-\alpha)}\right]_{(w_y w_{i_1})^{1/d}}^{(w_y w_{i_2})^{1/d}} + (w_y^2 w_{i_1} w_{i_2})^{\alpha} \left[r^{d(1-2\alpha)}\right]_{(w_y w_{i_2})^{1/d}}^{n^{1/d}} \tag{5.30}$$

which after some elementary analysis gives rise to

$$\begin{aligned} \overline{p}_{I_1 y I_2} \leq & w_y w_{i_1} + (w_y w_{i_1})^{\alpha} (w_y w_{i_1})^{1-\alpha} + (w_y^2 w_{i_1} w_{i_2})^{\alpha} (w_y w_{i_2})^{1-2\alpha} \\ = & \, w_y w_{i_1} + w_y w_{i_1}^{\alpha} w_{i_2}^{1-\alpha}. \end{aligned} \tag{5.31}$$

Note that in this case, it doesn't actually matter whether we have $r_1 < r_2$ or vice versa, the maximum of $r/2, r_1$ (or r_2) is never attained at $r/2$ except when the term in both connections probabilities is 1 anyways.

If we now want to apply the moment estimates for w_y there is no case distinction to make as there is no minimum that shows up. This means we can directly apply Lemma 5.1.1. Clearly if the indicator $\mathbb{1}\left(\frac{\sqrt{n}}{b^{\frac{1}{\tau-1}}} \geq \frac{r^d}{w_{i_2}}\right)$ is not met, then what we are trying to prove holds, so we will simply consider the case where it is met.

Then we find in the small r case that when ignoring constants we get the expression

$$\sum_{i \in V} \sum_{i_2 \in V} \sum_{y \notin [N_n(b)]} \pi_n^2 \overline{p}_{i_1 y} \overline{p}_{i_2 y} \leq \sum_{k=1}^{k=k_{\max}} n^{2-\tau} \pi_n^2 r_k^{d(3-\tau)} (w_{i_1} + w_{i_1}^{\alpha} w_{i_2}^{1-\alpha}),$$

which after applying Lemma 5.1.2 and taking the sum at its upper boundary, which is where k is such that $r_k = n^{1/d}$ gives us

$$n^{2-\tau} \pi_n^2 n^{3-\tau} n^{(\tau-1)/2} = n^{(3-\tau)/2}. \tag{5.32}$$

Case 3: (Medium r)

Recall that at the start WLOG we assumed that $w_{i_1} < w_{i_2}$, this means that now we are in the case that $\frac{r^d}{w_{i_1}} \leq w_y \leq \min(\sqrt{n}\, b^{\frac{-1}{\tau-1}}, \frac{r^d}{w_{i_2}})$. A quick observation tells us that in the case that $r_1 < r_2$, this integral is actually exactly the same as in case 2. As we can simply split it the exact same way and the maximums of the connection probabilities will be attained at the same places as in the small r case. However in the $r_2 < r_1$ case this integral does act slightly differently.

Here we get that

$$\overline{p}_{I_1 y I_2} \leq \int_{r_2=0}^{r/2} r_2^{d-1} (\frac{w_y w_{i_1}}{r^d})^\alpha dr_2 + \int_{r/2}^{(w_y w_{i_2})^{1/d}} r_2^{d-1} (\frac{w_y w_{i_1}}{r_2^d})^\alpha dr_2 + \int_{(w_y w_{i_2})^{1/d}}^{n^{1/d}} r_2^{d-1} (\frac{w_y w_{i_1}}{r_2^d})^\alpha (\frac{w_y w_{i_2}}{r_2^d})^\alpha dr_2 \tag{5.33}$$

Which comes out to be

$$\overline{p}_{I_1 y I_2} \leq (\frac{w_y w_{i_1}}{r^d})^\alpha r^d + (w_y w_{i_1})^\alpha \left[r_2^{d(1-\alpha)} \right]_{r/2}^{(w_y w_{i_2})^{1/d}} + (w_y^2 w_{i_1} w_{i_2})^\alpha \left[r_2^{d(1-2\alpha)} \right]_{(w_y w_{i_2})^1/d}^{n^{1/d}} \tag{5.34}$$

which gives us the following solution in the third case which is

$$\overline{p}_{I_1 y I_2} \leq \mathbb{1}(r_2 < r_1)((\frac{w_y w_{i_1}}{r^d})^\alpha r^d + (w_y w_{i_1})^\alpha r^{d(1-\alpha)} + w_y w_{i_2}^{1-\alpha} w_{i_1}^\alpha).$$

Note that in the second term there is no reason to do a case distinction on k, so we will now consider first the case again where $k \leq k^\star$ for the first term, and apply Lemma 5.1.1 where the minimum is attained at r^d/w_{i_1}. This gives us

$$\left(\frac{r_k^d}{w_{i_1}}\right)^{1+\alpha-\tau} w_{i_1}^\alpha r_k^{d(1-\alpha)} = r_k^{d(2-\tau)} w_{i_1}^{\tau-1}. \tag{5.35}$$

Then the summation over k becomes

$$\sum_{i \in V} \sum_{i_2 \in V} \sum_{y \notin [N_n(b)]} \pi_n^2 \overline{p}_{i_1 y} \overline{p}_{i_2 y} \leq \sum_{k=1}^{k=k^\star} n^{2-\tau} r_k^{d(3-\tau)} \pi_n^2 n^{(\tau-1)/2},$$

which after a quick observation tells us is the exact same equation as (5.26) and so the same holds. When $k \geq k^\star$, we then find in terms of powers of n

$$n^{2-\tau} \pi_n^2 (n^{(3-\tau)} n^{\tau-2})^{2-\alpha} n^{\frac{1+2\alpha-\tau}{2}} = n^{(3-\tau)/2}. \tag{5.36}$$

Although a small observation here might be made here by the reader that this only happens if $\alpha > 2$, however if $\alpha < 2$ and $\alpha > \tau - 1$, we can upper bound the sum through its upper boundary where $r_k^d = n^{1/d}$ and doing that we arrive at the same order of n either way. Another note is that when $\alpha < \tau - 1$, then by symmetry of the moment estimates for w_{i_1} and w_{i_2}, it is exactly the same order as in the $k \leq k^\star$ case.

Now all that is left is the second term of this case. Where we have $w_y w_{i_2}^{1-\alpha} w_{i_1}^\alpha$, here there is no minimum, so we simply apply Lemmas 5.1.1 and 5.1.2 to estimate the moments after taking expectation, and we find

$$\sum_{k=1}^{k=k_{\max}} n^{2-\tau} \pi_n^2 r_k^{d(3-\tau)} n^{(\tau-1)/2} = n^{(3-\tau)/2}. \tag{5.37}$$

Which is the same expression as in the small r case. This proves the proposition.

□

Lemma 5.3.2. *Let $V \subseteq [N_n(a)]$. Then for fixed $a > 0$, for every realisation of $\mathcal{G}_{N_n(a)}$ satisfying $\mathcal{A}_{\text{good}}$ event of Proposition 3.1.1, and $\varepsilon > 0$, as $n \to \infty$ we have that*

$$\mathbb{P}\left(\left| |\mathcal{N}_1(V)| - c_d \sum_{i \in V} \pi_n w_i \right| > \varepsilon \sqrt{n} \;\middle|\; G_{N_n(a)} \right) \xrightarrow{\mathbb{P}} 0, \tag{5.38}$$

where c_d is an explicitly computable constant.

Proof. By definition we have that

$$|\mathcal{N}_1(V)| = \sum_{j \notin [N_n(a)]} \mathbb{1}(i \leftrightarrow j \text{ for some } i \in V), \tag{5.39}$$

such that by union bound

$$\mathbb{E}\left[|\mathcal{N}_1(V)| \mid G_{N_n(a)} \right] \leq \sum_{i \in V} \sum_{j \notin [N_n(a)]} \pi_n \overline{p}_{ij}. \tag{5.40}$$

At the same time, by inclusion-exclusion we have

$$\mathbb{E}\left[|\mathcal{N}_1(V)| \mid \mathcal{G}_{N_n(a)} \right] \geq \sum_{i \in V} \sum_{j \notin [N_n(a)]} \pi_n p_{ij} - \sum_{i \in V} \sum_{i_2 \in V} \sum_{j \notin [N_n(a)]} \pi_n^2 p_{i_1 j} p_{i_2 j}. \tag{5.41}$$

We will now compute the value of $\sum_{i \in V} \sum_{j \notin [N_n(a)]} \pi_n \overline{p}_{ij}$ and furthermore show that

$$\sum_{i_1 \in V} \sum_{i_2 \in V} \sum_{j \notin [N_n(a)]} \pi_n^2 \overline{p}_{i_1 j} \overline{p}_{i_2 j} = o(\sqrt{n}). \tag{5.42}$$

Let us first compute $\sum_{i \in V} \sum_{j \notin [N_n(a)]} \pi_n \overline{p}_{ij}$. Note that in a GIRG, for a fixed vertex i we have that

$$\sum_{j \notin [N_n(a)]} \overline{p}_{ij} \leq \sum_{j \in [n]} \overline{p}_{ij} = \mathbb{E}\left[\deg(i) \right] = c_d w_i. \tag{5.43}$$

For an explicitly computable constant c_d, which we will compute later. Then

$$\sum_{i \in V} \sum_{j \notin [N_n(a)]} \pi_n \overline{p}_{ij} \leq c_d \sum_{i \in V} \pi_n w_i. \tag{5.44}$$

Now we apply Proposition 5.3.1 with $b = a$, to immediately see that

$$\sum_{i_1 \in V} \sum_{i_2 \in V} \sum_{j \notin [N_n(a)]} \pi_n^2 \overline{p}_{i_1 j} \overline{p}_{i_2 j} = \Theta(n^{(3-\tau)/2}) = o(\sqrt{n}). \tag{5.45}$$

Which proves the upper bound for the lemma, now all that is left to show is that we can find a lower bound close to this upper bound, however, a very naive upper bound already does the job,

$$\sum_{i_1 \in V} \sum_{j \in [N_n(a)]} \pi_n \overline{p}_{ij} \leq \pi_n (N_n(a))^2 = \Theta(n^{(3-\tau)/2}) = o(\sqrt{n}). \tag{5.46}$$

and this proves the lemma. □

We now need one more lemma before we can prove the main result that we need for the lower bound on the span of $\mathscr{C}^a_{(1)}$.

Lemma 5.3.3. *For any fixed $a > 0$, as $n \to \infty$, we have*

$$c_d \sum_{i \in \mathscr{C}^a_{(1)}} \frac{\pi_n w_i}{\sqrt{n}} \xrightarrow{\mathbb{P}} \zeta^\lambda_a, \tag{5.47}$$

with ζ^λ_a as defined in 4.25.

Proof. First note that for all vertices $i \leq \varepsilon N_n$, we have that

$$\frac{1}{N_n(a)} \sum_{i \leq \varepsilon N_n} \frac{w_i}{\sqrt{n}} = \frac{c_F n^{1/(\tau-1)}}{N_n(a)\sqrt{n}} \sum_{i \leq \varepsilon N_n} i^{-1/(\tau-1)} \leq \frac{C\varepsilon^{1-\frac{1}{\tau-1}}}{a}. \tag{5.48}$$

Then we can apply Theorem 9.10 from [11], such that we have

$$c_d \sum_{i \in \mathscr{C}^a_{(1)}} \frac{\pi_n w_i}{\sqrt{n}} = \frac{\lambda a c_d}{N_n(a)} \sum_{i \in \mathscr{C}^a_{(1)}} \frac{w_i}{\sqrt{n}} \xrightarrow{\mathbb{P}} c_d a \lambda \int_\varepsilon^a c_F u^{\frac{-1}{\tau-1}} \rho^\lambda_a(u) \Lambda_a(du) = \zeta^\lambda_a. \tag{5.49}$$

□

Now we prove the following lower bound on the span, recall the definition

Proposition 5.3.4. *For a fixed $\lambda > \lambda_c$, and any $\varepsilon > 0$, there exists an $a_0 := a_0(\varepsilon) > 0$, such that for all $a > a_0$,*

$$\lim_{n \to \infty} \mathbb{P}\left(\frac{|\mathrm{Span}_a(\mathscr{C}^a_{(1)})|}{\sqrt{n}} \geq \zeta^\lambda - \varepsilon \right) = 1. \tag{5.50}$$

Proof. As mentioned before we clearly have that $|\mathrm{Span}_a(V)| \geq |\mathcal{N}_1(V)|$. We now apply Lemma 5.3.2 and take $V = \mathscr{C}^a_{(1)}$, then together with Lemma 5.3.3 and Proposition 4.4.6 we can take a large enough such that $\zeta^\lambda_a \geq \zeta^\lambda - \varepsilon/2$, and this proves the proposition. □

We have now shown a lower bound on the span of $\mathscr{C}^a_{(1)}$, and all that is left to show is an upper bound of similar form.

5.4 Upper Bounding the Span of $\mathscr{C}^a_{(1)}$

We set out to prove the following law of large numbers result on the span of $\mathscr{C}^a_{(1)}$,

Theorem 5.4.1. *Fix $\lambda > \lambda_c$. For any $\varepsilon > 0$, there exists $a_1 := a_1(\varepsilon) > 0$, such that for all $a \geq a_1$, there exists a $k_0 = k_0(\varepsilon, a)$, such that for all $k \geq k_0$, with high probability we have*

$$\zeta^\lambda - \varepsilon \leq \frac{|\mathrm{Span}_a(\mathscr{C}^a_{(1)})|}{\sqrt{n}} \leq \zeta^\lambda + \varepsilon, \tag{5.51}$$

with ζ^λ as in Proposition 4.4.6.

The lower bound we have already proved in the previous section, for the upper bound we first need a few intermediate results to prove the following proposition, from which Theorem 5.4.1 immediately follows. Let $\mathcal{T}^a_{\geq k}$ denote the set of vertices that belong to some component of size at least k, and fix $\lambda > \lambda_c$ as always, then

Proposition 5.4.2. *For any $\varepsilon > 0$, there exists $a_1 := a_1(\varepsilon) > 0$, such that for all $a \geq a_1$, there exists a $k_0 = k_0(\varepsilon, a)$, such that for all $k \geq k_0$*

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{|\mathrm{Span}_a(\mathcal{T}^a_{\geq k})|}{\sqrt{n}} \leq \zeta^\lambda + \varepsilon\right) = 1. \tag{5.52}$$

To prove Proposition 5.4.2 we first show that the contribution to the span of any set $V \subseteq [N_n(a)]$ of vertices further than graph-distance 2 away, is negligible in comparison to the contribution of $\mathcal{N}_1(V)$. This is summarized in the following lemma:

Lemma 5.4.3. *Let $V \subseteq [N_n(a)]$ be such that $\sum_{i\in V} \pi_n w_i \leq C_0\sqrt{n}$ for some constant $C_0 > 0$. Then conditioned on the good event from Theorem 3.1.1, for any $\varepsilon > 0$, there exists $a_0 = a_0(\varepsilon) > 0$ such that for any $a > a_0$ as $n \to \infty$*

$$\mathbb{P}(|\mathcal{N}_{\geq 2}(V)| > \varepsilon\sqrt{n}|\ \mathcal{G}_{N_n(a)}) \xrightarrow{\mathbb{P}} 0. \tag{5.53}$$

Proof. As per usual we start by conditioning the graph on $\mathcal{A}_{\mathrm{good}}$ from Theorem 3.1.1. We will first show that

$$\mathbb{E}[|\mathcal{N}_{\geq 2}(V)||\mathcal{G}_{N_n(a)}] < \frac{\varepsilon}{2}\sqrt{n}. \tag{5.54}$$

Let $\mathcal{A}_l(i,j)$ be the event that there exists a path $(i, i_1, i_2, \cdots, i_{l-1}, j)$ between $i \in [N_n(a)]$ and $j \in [N_n(a)]^c$ of length l, such that the path lies entirely outside of $[N_n(a)]$, i.e $i_1 \cdots i_{l-1} \in [N_n(a)]^c$.

Now in the conditioning of the locations, we only reveal the locations within $[N_n(a)]$, such that when considering the probability $\mathbb{P}(\mathcal{A}_l(i,j)|\mathcal{G}_{N_n(a)})$ we have to integrate over the locations of the path. This gives us the following upper bound in the form of an l-fold integral:

$$\mathbb{P}(\mathcal{A}_l(i,j)|\mathcal{G}_{N_n(a)}) \leq \sum_{(i_1,i_2,\cdots,i_{l-1})\in[N_n(a)]^c} \int_{x_{i_1}} \cdots \int_{x_{i_{l-1}}} \int_{x_j} \pi_n^l \left(1 \wedge \frac{w_j w_{i_{l-1}}}{|x_j - x_{i_{l-1}}|^d}\right)^\alpha \cdots \left(1 \wedge \frac{w_i w_{i_1}}{|x_i - x_{i_1}|^d}\right)^\alpha \mathrm{d}x_{i_1} \cdots \mathrm{d}x_{i_{l-1}} \mathrm{d}x_j \tag{5.55}$$

Then evaluating each integral in the same way as we have done before in the proof of Proposition 5.3.1, we find that

$$\mathbb{P}(\mathcal{A}_l(i,j)|\mathcal{G}_{N_n(a)}) \leq \sum_{(i_1,i_2,\cdots,i_{l-1})\in[N_n(a)]^c} \frac{\pi_n^l c_d^l}{n^l} w_i w_{i_1}^2 w_{i_2}^2 \cdots w_{i_{l-1}}^2 w_j$$
$$= \frac{\pi_n c_d}{n} w_i w_j \left(\frac{\pi_n c_d}{n} \sum_{y\in[N_n(a)^c]} w_y^2 \right)^{l-1} = \frac{\pi_n c_d}{n} w_i w_j \overline{\nu}_n(a)^{l-1}, \tag{5.56}$$

where $\overline{\nu}_n(a) := \frac{\pi_n c_d}{n} \sum_{y\in[N_n(a)^c]} w_y^2$. Furthermore, we have that

$$\overline{\nu}_n(a) = \frac{\pi_n c_d}{n} \sum_{y>an^{(3-\tau)/2}} c_F^2 \left(\frac{n}{y}\right)^{2/(\tau-1)} = \frac{C\pi_n n^{2/(\tau-1)}}{n} \sum_{y>an^{(3-\tau)/2}} y^{-2/(\tau-1)}$$
$$\leq \frac{\tilde{C}\pi_n n^{2/(\tau-1)}}{n} (an^{(3-\tau)/2})^{1-\frac{2}{\tau-1}} = Ca^{\frac{\tau-3}{\tau-1}}, \tag{5.57}$$

which is clearly decaying in a. Then

$$\mathbb{E}[|\mathcal{N}_{\geq 2}(V)||\mathcal{G}_{N_n(a)}] \leq \sum_{l\geq 2}\sum_{i\in V}\sum_{j\in[N_n(a)]^c} \mathbb{P}(\mathcal{A}_l(i,j)|\mathcal{G}_{N_n(a)})$$
$$\leq \frac{c_d}{n}\overline{\nu}_n(a) \sum_{i\in V} \pi_n w_i \sum_{j\in[N_n(a)]^c} w_j \leq C\overline{\nu}_n(a)\sqrt{n}, \tag{5.58}$$

where in the last inequality we used the condition $\sum_{i\in V} \pi_n w_i \leq C_0\sqrt{n}$ which we assumed at the start. Now as $\overline{\nu}_n(a)$ is decaying in a, it is clear to see that we can find some $a_0(\varepsilon)$ such that for all $a > a_0$ we have that $\mathbb{E}[|\mathcal{N}_{\geq 2}(V)||\mathcal{G}_{N_n(a)}] \leq \frac{\varepsilon}{2}\sqrt{n}$.

As we wish to use a second moment method to show the convergence in probability we must next estimate the conditional variance of $|\mathcal{N}_{\geq 2}(V)|$.

We consider $|\mathcal{N}_{\geq 2}(V)|^2$, which can be formally expressed as

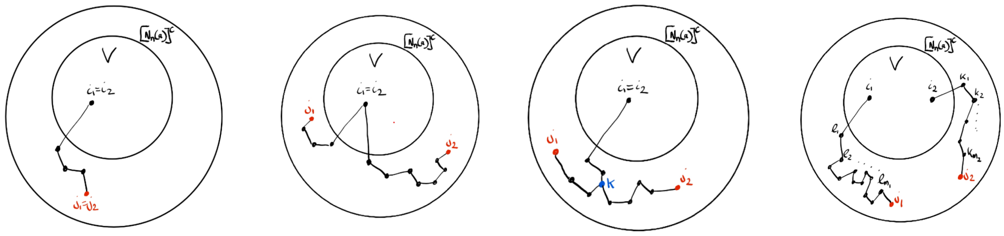
$$|\mathcal{N}_{\geq 2}(V)|^2 = |\{(j_1,j_2) : j_1, j_2 \in [N_n(a)]^c, j_1 \leftrightarrow [N_n(a)], j_2 \leftrightarrow [N_n(a)]\}. \tag{5.59}$$

The key idea here is that we are counting pairs (j_1, j_2) that both have a path to $[N_n(a)]$. At this point we are not counting paths yet, but really just the pairs themselves.

We can then decompose $|\mathcal{N}_{\geq 2}(V)|^2$ into the following three cases for choices of i_1, i_2, j_1, j_2:

- **Case 1:** $j_1 = j_2, j_1 \leftrightarrow [N_n(a)]$,
- **Case 2:** $j_1 \neq j_2 : \exists i_1 \in [N_n(a)], j_1 \leftrightarrow i_1, j_2 \leftrightarrow i_1$,
- **Case 3:** $j_1 \neq j_2 : \nexists i_1 \in [N_n(a)], j_1 \leftrightarrow i_1, j_2 \leftrightarrow i_1$.

In case 3, both j_1 and j_2 still connect to $[N_n(a)]$, which means that there are two vertex disjoint paths that connect them. Let I_{ij} be the indicator that $i \in V$ connects to $j \in [N_n(a)]^c$ through a path of length $l \geq 2$ completely in $[N_n(a)]^c$. Which is why in case 3 we consider $i_1, i_2 \in [N_n(a)]$. For a visual representation of the cases see Figure 5.2.



(A) (B) (C) (D)

FIGURE 5.2: These drawings depict the three possible cases for vertices counted in $|\mathcal{N}_{\geq 2}(V)|^2$. Sub-figure (A) is case 1: $i_1 = i_2$ and $j_1 = j_2$, sub-figures (B) and (C) are both case 2: $i_1 = i_2$ and $j_1 \neq j_2$, where in (B) the paths are completely vertex disjoint, and in (C) the paths are joint until vertex k, (D) is case 3: $i_1 \neq i_2$ and $j_1 \neq j_2$.

It will turn out that the case where $i_1 \neq i_2$ and $j_1 = j_2$ is already counted in case 2, which is why we do not treat it separately. It is clear to see case 1 is exactly the same as (5.58), so we start by treating case 2.

Let I_{ij} be the indicator that $i \in V$ connects to $j \in [N_n(a)]^c$ through a path of length $l \geq 2$ completely in $[N_n(a)]^c$.

Then starting with case 2 we can use the following union bound:

$$\mathbb{E}[|\mathcal{N}_{\geq 2}(V)|^2 \mathbb{1}(\text{Case 2 occurs})|\mathcal{G}_{N_n(a)}] \leq \sum_{i \in V} \sum_{j_1, j_2 \in [N_n(a)]^c} \mathbb{P}(I_{ij_1} = I_{ij_2} = 1 | \mathcal{G}_{N_n(a)}). \tag{5.60}$$

There are now two subcases that can happen, either i connects to j_1 and j_2 through completely disjoint vertex paths, or there is an intermediate vertex k such that we have the disjoint vertex paths $[i,k], [k,j_1], [k,j_2]$. We use the Van den Berg-Kesten (BK) inequality [47] to bound the probability in both cases, starting with case 1 we find

$$\begin{aligned}
&\sum_{i \in V} \sum_{j_1, j_2 \in [N_n(a)]^c} \mathbb{P}(I_{ij_1} = I_{ij_2} = 1 | \mathcal{G}_{N_n(a)}) \leq \sum_{i \in V} \sum_{j_1, j_2 \in [N_n(a)]^c} \mathbb{P}(I_{ij_1} = 1)\mathbb{P}(I_{ij_2} = 1) \\
&\leq \frac{\pi_n^2}{n^2} C^2 \sum_{i \in V} \sum_{j_1, j_2 \in [N_n(a)]^c} w_i^2 w_{j_1} w_{j_2} \leq \frac{\pi_n^2}{n^2} C^2 \sum_{i \in V} w_i^2 \left(\sum_{j \in [N_n(a)]^c} w_j \right)^2 \\
&\leq \pi_n^2 C^2 \sum_{i \in V} w_i^2 \leq \overline{C}(\pi_n n^{\frac{1}{\tau - 1}})^2 = o(n).
\end{aligned} \tag{5.61}$$

Where in the second inequality we use (5.56). Now in the second sub-case we can again use the BK-inequality to arrive at a similar bound.

$$\begin{aligned}
&\sum_{i\in V}\sum_{j_1,j_2\in[N_n(a)]^c}\mathbb{P}(I_{ik}=I_{kj_1}=I_{kj_2}=1|\mathcal{G}_{N_n(a)})\\
&\leq \sum_{i\in V}\sum_{j_1,j_2\in[N_n(a)]^c}\sum_{k\in[N_n(a)]^c}\mathbb{P}(I_{ik}=1)\mathbb{P}(I_{ki_1}=1)\mathbb{P}(I_{kj_2}=1)\\
&\leq \frac{\pi_n^3C^3}{n^3}\sum_{i\in V}\sum_{j_1,j_2\in[N_n(a)]^c}\sum_{k\in[N_n(a)]^c} w_iw_{j_1}w_{j_2}w_k^3 \leq \frac{C\pi_n^2}{n}\sqrt{n}\sum_{k\in[N_n(a)]^c} w_k^3\\
&\leq C\pi_n n a^{1-\frac{3}{\tau-1}}. && (5.62)
\end{aligned}$$

Where we once again used the assumption on $\sum_{i\in V}\pi_n w_i$ and estimate $\sum_{k\in[N_n(a)]^c} w_k^3$ in the same way as we did the second moment earlier in (5.57), this concludes case 2.

For the last case we assume that i_1, i_2, j_1, j_2 are all disjoint vertices. There is the possibility that their paths intersect, however these vertices have already been counted in case 2 and as such do not need to be considered anymore. Which is why we only treat the case where i_1 connects to j_1 and i_2 connects to j_2 through completely disjoint vertex paths.

We assume that the path from i_1 to j_1 is of length m_1 with vertices $l_1,\cdots,l_{m_1}$, and likewise the path from i_2 to j_2 is of length m_2 with vertices $k_1,\cdots,k_{m_2}$. Then as before we take conditional expectation:

$$\begin{aligned}
&\mathbb{E}[|\mathcal{N}_{\geq 2}(V)|^2\mathbb{1}(\text{Case 3 occurs})|\mathcal{G}_{N_n(a)}] \leq \sum_{i_1,i_2\in V}\sum_{j_1,j_2\in[N_n(a)]^c}\mathbb{P}(I_{i_1j_1}=I_{i_2j_2}=1)\\
&=\int_{x_{l_1}}\cdots\int_{x_{l_{m_1}}}\int_{x_{k_1}}\cdots\int_{x_{k_{m_2}}}\left(1\wedge\frac{w_{i_1}w_{l_1}}{|x_{i_1}-x_{l_1}|^d}\right)^\alpha\cdots\left(1\wedge\frac{w_{k_{m_2}}w_{j_2}}{|x_{k_{m_2}}-x_{j_2}|^d}\right)^\alpha && (5.63)
\end{aligned}$$

Now similarly as before we integrate over the locations and get out the weights such that

$$\begin{aligned}
&\int_{x_{l_1}}\cdots\int_{x_{l_{m_1}}}\int_{x_{k_1}}\cdots\int_{x_{k_{m_2}}}\left(1\wedge\frac{w_{i_1}w_{l_1}}{|x_{i_1}-x_{l_1}|^d}\right)^\alpha\cdots\left(1\wedge\frac{w_{k_{m_2}}w_{j_2}}{|x_{k_{m_2}}-x_{j_2}|^d}\right)^\alpha\\
&=\frac{w_{i_1}w_{j_1}\pi_nc_d}{n}\left(\sum_{l\in[N_n(a)]^c}\frac{w_l^2\pi_n}{n}\right)^{m_1}\cdot\frac{w_{i_2}w_{j_2}\pi_nc_d}{n}\left(\sum_{k\in[N_n(a)]^c}\frac{w_k^2\pi_n}{n}\right)^{m_2}\\
&=\frac{w_{i_1}w_{j_1}\pi_nc_d}{n}\overline{\nu}_n(a)^{m_1}\frac{w_{i_2}w_{j_2}\pi_nc_d}{n}\overline{\nu}_n(a)^{m_2}. && (5.64)
\end{aligned}$$

Summing over m_1 and m_2 now gives exactly $\mathbb{E}[|\mathcal{N}_{\geq 2}(V)||\mathcal{G}_{N_n(a)}]^2$. as in (5.58).

Now we are ready to use Chebyshev's inequality. Note that

$$\mathbb{E}[|\mathcal{N}_{\geq 2}(V)|^2 \mid \mathcal{G}_{N_n(a)}] = \mathbb{E}[|\mathcal{N}_{\geq 2}(V)|^2 \mid \mathcal{G}_{N_n(a)}] + o(n) + \mathbb{E}[|\mathcal{N}_{\geq 2}(V)| \mid \mathcal{G}_{N_n(a)}]^2 \quad (5.65)$$

such that

$$\begin{aligned}\mathrm{Var}(|\mathcal{N}_{\geq 2}(V)| \mid \mathcal{G}_{N_n(a)}) &= \mathbb{E}[|\mathcal{N}_{\geq 2}(V)|^2 \mid \mathcal{G}_{N_n(a)}] - \mathbb{E}[|\mathcal{N}_{\geq 2}(V)| \mid \mathcal{G}_{N_n(a)}]^2 \\ &= o(n) \end{aligned} \tag{5.66}$$

Then, on the event that $\mathbb{E}[|\mathcal{N}_{\geq 2}(V)| \mid \mathcal{G}_{N_n(a)}) < \frac{\varepsilon}{2}\sqrt{n}$, which occurs with high probability as shown in (5.58), we have that

$$\begin{aligned}&\mathbb{P}(|\mathcal{N}_{\geq 2}(V)| > \varepsilon\sqrt{n} \mid \mathcal{G}_{N_n(a)}) \leq \mathbb{P}(\big||\mathcal{N}_{\geq 2}(V)| - \mathbb{E}[|\mathcal{N}_{\geq 2}(V)| \mid \mathcal{G}_{N_n(a)}]\big| > \frac{\varepsilon}{2}\sqrt{n} \mid \mathcal{G}_{N_n(a)}) \\ &\leq \frac{4\mathrm{Var}(|\mathcal{N}_{\geq 2}(V)| \mid \mathcal{G}_{N_n(a)})}{\varepsilon^2 n} \xrightarrow{\mathbb{P}} 0. \end{aligned} \tag{5.67}$$

Which proves the claim. □

Furthermore, we also need the following result on the weight of small sets of vertices which follows verbatim from [9]:

Lemma 5.4.4. *For any $\delta > 0$ and $V \subseteq [N_n(a)]$ such that $|V| \leq \delta N_n$ we have that*

$$\frac{1}{\sqrt{n}} \sum_{k \in V} \pi_n w_k \leq \frac{c_F \delta^{1-\frac{1}{\tau-1}}}{1 - \frac{1}{\tau-1}}. \tag{5.68}$$

Proof. Recall the definition of the weights $w_k = c_F(\frac{n}{k})^{1/(\tau-1)}$, such that

$$\frac{1}{\sqrt{n}} \sum_{k \in V} \pi_n w_k \leq \frac{1}{\sqrt{n}} \sum_{k \leq \delta N_n} \pi_n w_k = c_F n^{\frac{\tau-3}{2} - \frac{1}{2} + \frac{1}{\tau-1}} \sum_{k \leq \delta N_n} k^{\frac{-1}{\tau-1}} \tag{5.69}$$

Approximating the sum by an integral we then find

$$c_F n^{\frac{\tau-3}{2} - \frac{1}{2} + \frac{1}{\tau-1}} \sum_{k \leq \delta N_n} k^{\frac{-1}{\tau-1}} \leq c_F n^{\frac{\tau-3}{2} - \frac{1}{2} + \frac{1}{\tau-1}} \frac{(\delta N_n)^{1-\frac{1}{\tau-1}}}{1 - \frac{1}{\tau-1}} = \frac{c_F \delta^{1-\frac{1}{\tau-1}}}{1 - \frac{1}{\tau-1}} \tag{5.70}$$

□

Now we are ready to prove Proposition 5.4.2.

Proof of Proposition 5.4.2. Let $\lambda > \lambda_c$ and fix any $\varepsilon > 0$. We will first show that for there exists an $a_1(\varepsilon)$, such that for all $a > a_1(\varepsilon)$ there exists a k_0, such that for all $k > k_0$ we have that

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{1}{\sqrt{n}} \sum_{i \in \mathcal{T}^a_{\geq k}} \pi_n w_i \leq \zeta^\lambda + \frac{\varepsilon}{2}\right) = 1. \tag{5.71}$$

To show this intermediate result we set $\delta = (\frac{\varepsilon}{4C_0})^{1/(1-\frac{1}{\tau-1})}$, with $C_0 = \frac{c_F}{1-\frac{1}{\tau-1}}$. Now by definition of $\rho^\lambda_{\geq k}$, it is decreasing in k, so that we can choose $k_0(\varepsilon, a)$ such that for all $k \geq k_0$, we have $\rho^\lambda_{a,\geq k} \leq \rho^a_{\geq k} + \delta/2a$. Now by Proposition 4.3.3 with high probability we have that

$$|\mathcal{T}^\lambda_{\geq k}| \leq |\mathscr{C}^a_{(1)}| + \delta N_n \implies |\mathcal{T}^\lambda_{\geq k} \setminus \mathscr{C}^a_{(1)}| \leq \delta N_n. \tag{5.72}$$

Now by Lemma 5.4.4 and the aforementioned δ, we have that

$$\frac{1}{\sqrt{n}} \sum_{i \in \mathcal{T}^a_{\geq k}} \pi_n w_i \leq \zeta^\lambda + \frac{\varepsilon}{2} \tag{5.73}$$

which shows (5.71).

Now taking $V = \mathcal{T}^a_{\geq k}$ in Lemma 5.3.2 and Lemma 5.4.3, gives that for all $a > \{a_0, a_1\}$, with a_0 as in Lemma 5.4.3, that

$$\frac{|\mathrm{Span}(\mathcal{T}^a_{\geq k})|}{\sqrt{n}} = (1 + o_{\mathbb{P}}(1)) \sum_{i \in \mathcal{T}^a_{\geq k}} \frac{c_d \pi_n w_i}{\sqrt{n}}, \tag{5.74}$$

which proves the proposition. □

6 Conclusion

As a conclusion to this thesis we briefly summarize the results and approaches we have taken, we then also discuss future steps that are yet to be taken in this particular field of research and provide a general outline on what approaches might work.

We have considered a GIRG with deterministic weights that follow a power law distribution with parameter $\tau \in (2,3)$. Our goal was to show that by percolating the graph with probability $\pi_n = \lambda n^{\frac{\tau-3}{2}}$, there exists a critical λ_c such that for all $\lambda < \lambda_c$ there is no unique giant component, and for all $\lambda > \lambda_c$ there does exists a unique giant component of size $\Theta(\sqrt{n})$. As a useful tool, we have defined a good configuration of a GIRG, one where the highest $N_n(a) := \lceil an^{(3-\tau)/2} \rceil$ weight vertices are well-distributed in space. We show that for this good event $\mathcal{A}_{good}$, the probability of its complement $\mathbb{P}(\mathcal{A}^c_{good}(N_n(a))$ can be pushed below any $\delta > 0$.

Continuing on, we show that the core of a GIRG can be expressed as an inhomogeneous random graph, using the established theory of IRGs we then show that within the core a giant component does exist, and with that λ_c exists, and we are able to express it in terms of the associated operator norm of the kernel of the IRG. After an excursion into the local weak limit of an IRG being a multi-type branching process, we show that ζ_a^λ has a converging limit as $a \to \infty$ which is finite.

Finally, we consider the giant in the full graph, where we condition the graph on the good event of Chapter 3. We establish that it consists of the giant inside the core, its span, the set of vertices connected by return paths, and its respective span as well. We then show an upper and lower bound on the size of the span of the largest component inside the core, $\mathscr{C}^a_{(1)}$. We prove that for any $\varepsilon > 0$

$$\zeta^\lambda - \varepsilon \leq \frac{|\mathrm{Span}(\mathscr{C}^a_{(1)})|}{\sqrt{n}} \leq \zeta^\lambda + \varepsilon. \tag{6.1}$$

This lower bound on the span already provides the lower bound on the magnitude of the emerging giant which is of the right order as claimed in Proposition 2.3.2.

To prove Conjecture 2.3.1 fully, it is only left to show the negligible contribution of the return paths and their span. This should be the first step in continuing the analysis of the supercritical case. Likely the ideas of [9] could be followed for the most part with some minor adjustments. Up until Lemma 5.11 in [9] could be adapted to the GIRG case easily as Proposition 5.3.1 is already in a generalized form. Given more time, this is the obvious next step as well as adapting the remainder of the proofs with regards to the added spatiality.

To show a true multiscale genesis akin to the one in [9] there would also need to be a consideration of the subcritical case, which is something that is not included in this thesis, also the possible existence of a critical window would need to be looked into. The proofs of Bhamidi, Dhara, and van der Hofstad in the subcritical case rely heavily on the "connectivity structure between hubs", where they show that when the graph is heavily percolated there asymptotically in n, exist no two-step paths between hubs in the subcritical regime. In GIRGs this is not immediately clear, as when hubs are close together they might still have a very good chance of connecting to each other, which is why the good configuration of Chapter 3 might prove to be useful once again. However, it is unclear exactly how to approach this problem.

Outside of the main subject of this thesis there is also another generalization of the problem that could be looked into that is much more complex. Consider for example GIRGs that have a power law parameter $\tau > 3$, this causes hubs to disappear almost immediately as the weights are now much less, causing the percolation to become much more difficult to analyze. It is still very much an open question how this affects the critical percolation probability and the size of the giant in the supercritical regime.

On a bigger scale, other similar models' phase transition could be studied as well, consider for instance the robust version of the spatial preferential attachment model [35], or the soft boolean model [28]. A multiscale genesis for the giant could be derived in these models perhaps similarly with the methods used in this thesis.

7 Bibliography

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