Clustering and Uniqueness in Mathematical Models of Percolation Phenomena

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PROEFSCHRIFT

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CONTENTS

INTRODUCTION

Suppose a connected graph $\mathcal{G}$ with countably many vertices is given and consider a nonempty subgraph. Some important global characteristics of the subgraph are the following: it could contain all the vertices of the original graph (1); or it could have at least one connected component containing infinitely many vertices (2); or it could fall apart into infinitely many finite components (3).

Suppose next that the subgraph is given in a random way. A precise definition of what we mean by random will be stated in the next section and it involves countably additive probability measures on a suitable σ-algebra of subsets of all possible subgraphs of the given graph.

Depending on the original graph and on the type of random mechanism generating the subgraph we will be able to indicate in some cases the probabilities with which (1), (2) or (3) occur.

A useful tool is the concept of percolation, which is just the existence of an infinite connected component (property (2) above): we say that a given vertex percolates if it belongs to such an infinite component.

The theory of percolation has been developed as a model for the spread of fluid or gas through random media. A system of channels, say, is modelled by the edges of a random subgraph and the fluid spreads from a given vertex through the edges in a deterministic way. Therefore if this vertex percolates then the fluid leaves any finite region, and literally percolates through the medium.

Another source of interest in the modern theory of percolation is that the probability measures describing the random subgraphs often depend on one or more parameters. These parameters specify quantities which are local, for example the probability that a certain edge belongs to the subgraph; nevertheless for some values of the parameters percolation does not occur and a small change in the parameters (which determine *local* quantities) causes the onset of percolation (which is a *global* phenomenon). This is the kind of thing that researchers in statistical mechanics are hunting for, because such phenomena are related to phase transitions. By analogy with statistical mechanics we say that a random graph is in the subcritical phase if no vertices percolate, and in the supercritical phase if at least one vertex percolates. Deeper connections with statistical mechanics will be discussed later.

In this thesis we discuss mainly the supercritical phase. Our main result is that, for a large class of graphs and probability measures, two or more disjoint infinite maximal components cannot coexist with positive probability. This has consequences for other quantities which are of interest in describing the supercritical phase, such as the probability that two distant vertices are in the same component of the random subgraph. The four articles reproduced below are mainly concerned with these problems and they, together with many other very important contributions quoted in the references, prove the above-mentioned non-coexistence in considerable generality. For some of the examples below we also describe the shape of the infinite component in terms of the ratio between boundary and volume.

In the introduction we review the models which are of major interest in the theory of percolation and in statistical mechanics. Our results are then placed in the wider context of present knowledge of these models.

After introducing some notation we begin with relatively simple models, and progress to more complicated systems towards the end of the introduction.

The relation between percolation and the random graphs theory will be discussed in section 5. When a graph is given and a random subgraph of it is described by a probability measure, we say that we are considering a random graph. We say that the random graph is totally connected if the subgraph satisfies condition (1), i.e. contains all vertices of the original graph, with probability one. In the introduction we consider some examples and describe conditions sufficient for a graph to be totally connected. The contribution of the present work to random graphs theory consists of simplified proofs that certain conditions are sufficient to ensure that the graph is totally connected.

We stress throughout this thesis the supercritical phase, and only a few of the recent results concerning the subcritical phase will be mentioned.

0. Definitions and notation

We now fix the notation which will be used in the introduction and indicate the relation with the notation used in the articles reproduced below.

The *graphs* G we consider consist of a countable set V, whose elements are called *vertices,* and a set $E \subseteq V_2 = \{\{v_1, v_2\} : v_1 \neq v_2, v_i \in V \text{ for } i = 1, 2\}$, whose elements are called *edges*. We say that the edge $\{v_1, v_2\}$ is the edge between the vertices v_1 and v_2.

Let us further specify the graphs before we proceed. Given the set of vertices V, interesting phenomena occur when E is the set of all edges between vertices in V or, if V is embedded in a metric space, when E consists of the set of edges between vertices at a fixed distance from each other. We limit ourselves to these two cases and denote them as *long range models* and *nearest neighbour models*, respectively. Specific choices for V will be d-dimensional spaces $\mathbb{Z}^d, d \geq 1$, or half spaces $\mathbb{Z}^d \times \mathbb{Z}_+, \mathbb{Z}_+ = \{0, 1, 2, \cdots\}$ and $d \geq 1$, or subsets of $\mathbb{Z}^d$ such as orthants, slabs, quarter slices or wedges, which are described in 1.3 below. We consider the special case $V = \mathbb{Z}_+$ separately. More general subexponential graphs, in which the boundary of the appropriate sphere of radius n is of lower order than its volume as n tends to infinity, are considered in Gandolfi, Keane and Newman [1989] below.

When $V = \mathbb{Z}^d \times \mathbb{Z}_+^k$, for $d + k \geq 1$, we denote the *origin* by $\mathbf{0} = (0, \ldots, 0)$ (with $d+k$ zero's here) and *boxes* by $B_n = \{(x_1, \ldots, x_{d+k}) \in V : |x_i| \leq n; i = 1, \ldots, d+k\}, n \geq 1$. Consider a sequence of boxes $\{B_{n_k}\}_{k \in \mathbb{N}}$ and a function $f_{n_k} = f(B_{n_k})$, with values in a topological space, which depends on these boxes; we say that f_{n_k} converges to an element f in the topological space when B_{n_k} *diverges* if $\lim_{k \to \infty} f_{n_k} = f$. Sometimes we omit to indicate the index n_k and we simply say that a box B diverges to indicate what we just described. Under the present assumptions on V we define the *boundary* ∂S of a set $S \subseteq V$ as the set of vertices $v \in V \backslash S$ such that there exists at least one vertex $w \in S$ whose Euclidean distance from v is one. In this context nearest neighbour models are such that all edges have Euclidean length one.

We now consider a fixed subgraph (V', E') of $\mathcal{G}$ and introduce the notions of paths, connections between vertices, and clusters.

A *path* γ' in (V', E') between two vertices v and v' in V is a finite sequence $(v_1, l_1, v_2, l_2, \ldots, l_{n-1}, v_n)$ of vertices $v_i \in V'$ and edges $l_i \in E'$ such that $v_1 = v, v_n = v'$ and $l_i = \{v_i, v_{i+1}\}$ for all $i = 1, \ldots, n-1$. A *self-avoiding path* γ between two vertices is a path between them such that any two vertices v_i and v_j in γ are distinct if $i \neq j$. Given a path γ' between two vertices it is easy to see that one can always construct a self-avoiding path γ between the two vertices which uses only vertices and edges of γ'. From now on all the paths will be self-avoiding. If $V = \mathbb{Z}^2$ we define a circuit as a path which is self-avoiding with the exception of the first and the last vertex which coincide. A circuit divides the plane into two subsets, the interior and the exterior of the circuits; if a path contains infinitely many vertices of which one belongs to the interior of the circuits then the path has a nonempty intersection with the circuit itself.

Two vertices v and v' in V are *connected* in (V', E') if there is a path in (V', E') between them. In general two subsets A and B of V are said to be *connected* in (V', E') if there are two vertices $v \in A$ and $v' \in B$ which are connected in (V', E'); the set of subgraphs of a given graph in which A and B are connected is denoted by $(A \to B)$.

Clusters or *components* of the fixed subgraph (V', E') are subsets C of V' satisfying the following two properties: C is connected, i.e. all pairs of vertices of C are connected by at least one path in (V', E'), and it is maximal with respect to this property, i.e. it is not properly contained in any other connected subset of V.

We indicate the particular cluster containing a given vertex $v \in V'$ by C_v and sometimes, if it contains infinitely many vertices, by I_v, as in Gandolfi, Keane and Newman [1989] below. Given a cluster C of (V', E'), the *boundary* of C is the set $\partial C \subset V$ of vertices $v \notin C$ such that there exist $e \in E$ and $w \in C$ with $e = \{v, w\}$. Note that in defining the boundary of a cluster C we have considered vertices in V and edges in E.

In the introduction we consider two types of models, which correspond to two ways of choosing V' and E'. In the *site percolation model* a variable V' is considered and E' is taken to be the set of edges between vertices in V'. In the *bond percolation model* $V' = V$ is fixed and a variable subset $E' \subseteq E$ is taken. The variable subsets are chosen in a random way as further discussed below. Note that the names of the models are not entirely appropriate, but are those used in the literature.

A bond percolation model can be seen as a site percolation model on a different graph, called the covering graph (see Kesten [1982], §2.5); in general it is not possible to represent a site model by a bond model. Nevertheless we treat bond percolation models separately because they are more suitable for visualizing long range models and they are more directly related to the theory of random graphs.

We now introduce the mathematical tools by which we describe the random subgraphs of a fixed graph $G = (V, E)$. We describe a subgraph in a bond percolation model by means of a *configuration* $\eta \in H = \{0,1\}^E$; in fact V' will be equal to V and E' will contain the edges $e \in E$ such that $\eta(e) = \eta_e = 1$. Similarly a subgraph in a site percolation model is described by a *configuration* $\omega \in \Omega = \{-1, 1\}^V$, assuming that V' consists of the vertices $v \in V$ such that $\omega(v) = \omega_v = 1$ and E' consists of the edges between vertices in V'. Therefore we will identify H or Ω with the sets of possible subgraphs in bond or site percolation models, respectively.

In H a *topology* is generated by the set of *cylinders*, which are subsets of H of the form $E_{e_1,\ldots,e_n}^{\alpha_1,\ldots,\alpha_n} = \{\eta \in H : \eta_{e_1} = \alpha_1, \ldots, \eta_{e_n} = \alpha_n\}$ for some $n \in \mathbb{N}$, where the set $\{e_1, \ldots, e_n\} \subset E$ forms the *base* of the cylinder and $\alpha_i \in \{0,1\}$ for all $i = 1, \ldots, n$. A topology in Ω is also generated by the set of similarly defined cylinders.

The sets of cylinders generate the Borel σ-algebras of H and Ω and random subgraphs are described by *probability measures* defined on these σ-algebras. When referring to site or bond percolation models we will refer to the probability measure and the σ-algebra as well as to the graph from now on. Subsets of H or Ω are called events when they belong to these σ-algebras, even if no explicit proof of this fact is given. If A is an event then I_A indicates its *characteristic function*. We need some more notation in connection with probability measures. The Borel σ-algebras on H or Ω contain the sub-σ-algebras generated by the cylinders with base contained in a fixed subset of H or Ω. We say that a probability measure is defined on a given set to indicate that it is defined on the Borel σ-algebra generated by the cylinders whose bases are contained in that set. Let $\mathcal{A}$ be a σ-algebra and P be a probability measure defined on $\mathcal{A}$. Given a sub-σ-algebra $\mathcal{B} \subseteq \mathcal{A}$, we indicate the conditional probability of an event $A \in \mathcal{A}$ given $\mathcal{B}$ by $P(A|\mathcal{B})$. When the sub-σ-algebra is generated by the cylinders with base in a set S the function $P(A|\mathcal{B})$ can be viewed as a function of the configurations in $\{0,1\}^S$ or $\{-1,1\}^S$, according to which model we are considering. Fix a subset S of V (or of E). Given a probability measure P defined on the set S and a function $f : S \to \mathbb{R}$, measurable with respect to $\mathcal{A}$, we denote the integral of f with respect to P by $\int f dP$ or $P(f)$, if it exists.

Suppose a site or a bond percolation model is given and indicate by P its probability measure. We are first interested in the probability that the random subgraph contains all vertices of V. If this happens with probability one we say that the random graph is *totally connected.*

Secondly we are interested in the event that a fixed vertex $v \in V$ belongs to a cluster containing infinitely many vertices, an infinite cluster. We refer to this event by saying that v is in an infinite cluster, or that v *percolates* and we denote this event by $A_{v,\infty}$ or by $\{|C_v| = \infty\}$ or by $\{|I_v| = \infty\}$, where, for a given set C, $|C|$ indicates the cardinality of the set C. Given a probability measure P we denote the probability that v percolates by $\vartheta(P,v) = P(A_{v,\infty})$; this quantity is called *probability of percolation* from v.

We consider also other relevant quantities. The *expected size of the cluster* of v is defined by $\chi(P,v) = \sum_{n\geq 1} nP(|C_v| = n) = \sum_{w\in V} P(w \in C_v)$; the *mean perimeter of the cluster* of v is defined by $\Xi(P,v) = \sum_{w\in V} P(w \in \partial C_v)$ and the *mean number of clusters per vertex* is given by:

$$\kappa(P,v) = \sum_{n\geq 1} \frac{1}{n} P(|C_v| = n)$$

(this last name is appropriate when the probability measure is invariant under translations, see Kesten [1982], §9.1, or Grimmett [1989], §4.1). When $V = \mathbb{Z}^d \times \mathbb{Z}_+^k$, with $d + k \geq 1$, we are also interested in the *connectivity functions* defined by

$$\Upsilon_P(n) = \Upsilon_P(\mathbf{0}, (0,\dots,0,n)) = P(\mathbf{0} \to (0,\dots,0,n))$$

(with $d + k - 1$ zero's here). Other functions, which coincide with the connectivity functions if there are no infinite clusters, are the *truncated connectivity functions*, defined by

$$\begin{aligned}\Upsilon_P^f(n) &= \Upsilon_P^f(\mathbf{0}, (0,\dots,0,n)) \\ &= P(\mathbf{0} \to (0,\dots,0,n) \text{ and } |C_{\mathbf{0}}| < \infty),\end{aligned}$$

where the suffix f indicates that the origin $\mathbf{0}$ and the vertex $(0,\dots,0,n)$ (with $n+k-1$ zero's) belong to the same finite component of the random subgraph.

Of fundamental importance is the invariance of the models under some transformations of the graphs and under their induced actions on events, measures etc.

The most relevant are the *translations*, which are maps, whose form is to be specified in the various cases, $T_v : V \longrightarrow V$, for $v \in V$. We now describe the induced actions which can be obtained once a translation T_v is given. We will denote these induced maps with the same symbol, even though they are maps acting on different sets: it will be always clear which of the maps we are using. Given T_v there is an induced action on E defined by $T_v(e) = T_v(\{v_1, v_2\}) = \{T_v v_1, T_v v_2\}$, $e \in E$. We say that the set of edges E is T_v-invariant whenever the following holds: $\{v_1, v_2\} \in E$ if and only if $\{T_v v_1, T_v v_2\} \in E$.

Given a T_v-invariant set of edges there are induced maps on H defined by $(T_v\eta)_e = \eta_{(T_v(e))}$, on Ω given by $(T_v\omega)_w = \omega_{(T_v(w))}$, on the subsets A of H given by $T_v(A) = \{T_v(\eta) : \eta \in A\}$ and on the subsets B of Ω given by $T_v(B) = \{T_v(\omega) : \omega \in B\}$. Given a σ-algebra $\mathcal{A}$ of subsets of H (or of Ω) we say that T_v is $\mathcal{A}$ measurable if $T_v^{-1}(A) \in \mathcal{A}$ for all $A \in \mathcal{A}$ and we say that a probability measure P defined on $\mathcal{A}$ is T_v-invariant if $P(T_v^{-1}(A)) = P(A)$ for all $A \in \mathcal{A}$.

If the probability measure we consider is invariant under some *invertible* translation T_v, then many results from ergodic theory can be applied. We now introduce the relevant concepts and results.

If a probability measure P is invariant under an invertible transformation, or, more generally, under a group of transformations, then it is possible to decompose the measure itself into *ergodic measures*. These are measures which are invariant under the same group, but which cannot be expressed as a convex combination of other measures invariant under the same group. Denote by $\mathcal{P}$ the set of probability measures invariant under the group and by Σ the ergodic ones, then there exists a probability measure ρ_P, defined on the Borel σ-algebra in $\mathcal{P}$ with the topology of weak convergence, such that

$$P = \int_{\mathcal{P}} \mu \rho_P(d\mu) = \int_{\Sigma} \mu \rho_P(d\mu),$$

in the sense that for all events $A \in \mathcal{A}$

$$P(A) = \int_{\Sigma} \mu(A) \rho_P(d\mu).$$

The measures μ are the *ergodic components* of P.

An ergodic measure can be recognized by the fact that either $P(A) = 0$ or $P(A) = 1$ for any event $A \in \mathcal{A}$ which is invariant under the whole group. If P is ergodic under an invertible transformation T and f is a function integrable with respect to P, then

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} f \circ T^i = \int f dP \qquad P - \text{almost everywhere.}$$

For an introduction to ergodic theory see, for instance, Walters [1976].

We now discuss some inequalities for probability measures defined on a σ-algebra of subsets of a partially ordered set Δ. In particular our prototype of a partially ordered set is $\{0,1\}^X$, where X is a finite or a countably infinite set. For instance the following discussion applies to $\Delta = H$ or $\Delta = \Omega$. Let $\delta, \delta' \in \Delta = \{0,1\}^X$. We obtain a partial order in Δ by defining that $\delta \succ \delta'$ whenever $\delta_x \geq \delta'_x$ for all $x \in X$. For any two configurations δ and δ' in Δ let $\delta \vee \delta'$ be a configuration such that $(\delta \vee \delta')_x = \max(\delta_x, \delta'_x)$ and let $\delta \wedge \delta'$ be such that $(\delta \wedge \delta')_x = \min(\delta_x, \delta'_x)$. We say that a real-valued function f defined on X is *increasing* if $f(\delta) \geq f(\delta')$ for all $\delta \succ \delta'$; an event is said to be *positive* if its characteristic function is increasing.

We now define the *Holley's stochastic inequality* between two probability measures μ and ν defined on Δ. We say that μ *stochastically dominates* ν in the Holley's sense if we have $\mu(f) \geq \nu(f)$ for all increasing functions f defined on Δ for which $\mu(f)$ and $\nu(f)$ exist. Furthermore we say that a probability measure μ defined on Δ satisfies the *FKG correlation inequalities* if $\mu(fg) \geq \mu(f)\mu(g)$ for all pairs of increasing functions defined on Δ for which $\mu(f^2) < \infty$.

We now discuss some sufficient conditions for the stochastic inequalities to hold.

Suppose a total order in S is given and identify S with $\mathbb{N}$. Given a configuration $\delta \in \Delta$ we denote by $[\delta]_n$ the cylinder of configurations which coincide with δ in the set $\{1, \ldots, n\}$. We say that two probability measures defined on Δ satisfy the *Russo's condition* if there exists an order in S such that the conditional probabilities of μ and ν satisfy the following inequality:

$$\mu(\delta_n = 1 | [\delta]_{n-1}) \geq \nu(\delta_n = 1 | [\delta']_{n-1})$$

for all $\delta \succ \delta'$ and for all $n \in \mathbb{N}$. If two measures satisfy the Russo's conditions then they satisfy the Holley's stochastic inequality (Russo [1982]).

A stronger condition, which is very useful in practice, is the following. Let $v \in X$ and denote by $\mathcal{B}_v$ the σ-algebra generated by the cylinders with base in $X \backslash \{v\}$. Let E_v^1 be the cylinder $\{\delta \in \Delta : \delta_v = 1\}$. Suppose that two probability measures μ and ν are such that $\inf_\delta \mu(E_v^1 | \mathcal{B}_v)(\delta) \geq \sup_\delta \nu(E_v^1 | \mathcal{B}_v)(\delta)$. It is easy to see that in this case the Russo's condition holds and therefore μ stochastically dominates ν.

Another sufficient condition is as follows. Suppose that for all finite subsets $F \subseteq X$ two probability measures μ and ν satisfy

$$\mu(\delta \vee \delta')\nu(\delta \wedge \delta') \geq \mu(\delta)\nu(\delta') \tag{1}$$

for all $\delta, \delta' \in \{0,1\}^F$. Then it is easy to see that Russo's condition and Holley's inequality hold. In particular if X is a finite set we say that two measures satisfy the *Holley's condition* if (1) holds for all $\delta, \delta' \in \Delta$. This implies that (1) holds for all subsets of X and therefore that the Holley's stochastic inequality holds (Holley [1974], Preston [1974]).

We now describe a sufficient condition for the FKG correlation inequality to hold. Suppose X is as before and let ν be a probability measure defined on Δ. Suppose that for all finite subsets $F \subseteq X$ the probability measure ν satisfies

$$\nu(\delta \vee \delta')\nu(\delta \wedge \delta') \geq \nu(\delta)\nu(\delta')$$

for all $\delta, \delta' \in \{0,1\}^F$. Then ν satisfies the FKG correlation inequality (see Grimmett [1989], §2.2).

1. Independent percolation

We consider here a simple way of realizing a random subgraph of $\mathcal{G} = (V, E)$. For site percolation models we make for every vertex, randomly and independently of the other vertices, the choice of whether or not the vertex belongs to V'; a similar choice is made for the edges in bond percolation models: if we think of this procedure being complete we have realized a random subgraph of $\mathcal{G}$. We call these models independent percolation models.

As described above the formal definition is by means of certain probability measures $P_{\mathcal{P}}^{\text{site}}$ and $P_{\mathcal{P}}^{\text{bond}}$ defined on $\{-1,1\}^V$ and $\{0,1\}^E$ respectively. These measures are called Bernoulli probability measures. Here $P_{\mathcal{P}}^{\text{site}}$ and $P_{\mathcal{P}}^{\text{bond}}$ are the product measures $\prod_{v\in V} P_{p_v}^v$ and $\prod_{e\in E} P_{p_e}^e$, where $P_{p_v}^v$ and $P_{p_e}^e$ are the elementary probability measures defined on the cylinders with base in $\{e\}$ or $\{v\}$ by $P_{p_v}^v(\omega_v = 1) = p_v$ and $P_{p_e}^e(\eta_e = 1) = p_e$, respectively. The symbol $\mathcal{P}$ stands for the set of parameters $\{p_v\}_{v\in V}$ or $\{p_e\}_{e\in E}$.

We first *consider nearest neighbour models* with $V = \mathbb{Z}^d, d \geq 2$, in sections 1.1 and 1.2; then nearest neighbour models with V being a subset of $\mathbb{Z}^d$ in section 1.3 and finally we treat long range models in the case $V = \mathbb{Z}$ in section 1.4.

We now restrict our attention to nearest neighbour models and introduce some specific notation.

We first limit the number of parameters to one by assuming $p_v \equiv p$ for all $v \in V$ or $p_e \equiv p$ for all $e \in E$, with $0 \leq p \leq 1$. The probability measures related to these models will be denoted by P_p^{site} and P_p^{edge}.

It is now natural to study the various quantities we introduced, such as the probability of percolation from the origin or the mean cluster size, as a function of the parameter p. Let $V = \mathbb{Z}^d$ and consider a nearest neighbour site or bond percolation model. Then if p is small enough the probability $\vartheta(p)$ of percolation from the origin is zero, while if p is large enough $\vartheta(p) > 0$ (see Broadbent and Hammersley [1957] or Kesten [1982] or Grimmett [1989]). Furthermore it is easy to see that $\vartheta(p)$ is a non-decreasing function of p. We define the *critical point* for percolation as

$$p_c^\varepsilon = \sup\{p : \vartheta(P_p^\varepsilon) = 0\},$$

where ε stands for site or bond (note that we have suppressed the dependence on the underlying graph).

It is interesting to mention the relation between the probabilities of percolation in independent bond and site percolation models defined on the same graph and with the same value of the parameter p. A simple inequality shows that in this case the probability of percolation from a given vertex (the origin, for instance) is larger in the bond model than in the site model. Indeed $\vartheta^{\text{ site}}(p) \leq p(\vartheta^{\text{ bond}}(p))$. Therefore $p_c^{\text{site}} \geq p_c^{\text{bond}}$. These inequalities are valid for more general graphs than those considered here (see Kesten [1982], §10.1).

Before proceeding to the discussion of the various examples, let us remark that from now on we omit the superscript "site" or "bond", in the notation of the probability measures and of the other quantities, when statements are valid for both site and bond percolation models. Furthermore, if $V = \mathbb{Z}^d$, the Bernoulli probability measures are invariant under all maps T_v, defined by $T_v(w) = v + w$. This allows us to apply the results developed in ergodic theory, as already mentioned in the introduction. Moreover we can always refer to the origin when considering quantities like the probability of percolation or the mean cluster size. Finally these measures satisfy the FKG correlation inequalities (as already proved by Harris [1960]).

1.1 Independent percolation in $\mathbb{Z}^2$.

The study of random subgraphs of $\mathbb{Z}^2$ with edges of Euclidean length one is simplified by the fact that the graph is planar. We begin with a description of independent percolation in this case.

First of all the independent nearest neighbour bond percolation model with parameter p is the dual of the same model with parameter $1-p$ (for definitions of duality and related properties see Kesten [1982] or Grimmett [1989]). This suggested that the critical point p_c^{bond} should be equal to 1/2 and this was proved by Kesten ([1980]); furthermore there is no percolation at $p = p_c^{\text{bond}}(2) = 1/2$ (see Kesten ([1980], Harris [1960] and Russo [1978]).

The critical point for site percolation is strictly higher; rigorous bounds are $p_c^{\text{site}}(2) > 0,503478\ldots$ (Tóth [1985]) and $p_c^{\text{site}}(2) \leq 3/4$ (Broadbent and Hammersley [1957]); these bounds are still far from the value of $p_c^{\text{site}}(2)$, which simulations indicate to be around 0,59. When p equals $p_c(2)$ the probability of percolation in the site model is also zero (Russo [1978], Seymour and Welsh [1978]). The fact that the graph is planar is very important in the proof that the probability of percolation vanishes at $p = p_c$; whether the same also holds for higher dimensions is one of the most interesting unsolved questions (see the discussion in section 1.2 below).

We now describe the subcritical phase, i.e. when $p < p_c(2)$, and the supercritical phase, i.e. when $p > p_c(2)$. Statements in the rest of this section will be valid for both bond and site percolation models.

In the *subcritical phase* the mean cluster size $\chi(p)$ is finite (Kesten [1981]). Other properties of $\chi(p), \vartheta(p)$ and $\kappa(p)$ are known and they will be mentioned in the section devoted to higher dimensions (1.2 below). The connectivity functions $\Upsilon_p(n)$ satisfy the following property: there exists $\xi(p)$ such that

$$\lim_{n\to\infty} -\frac{1}{n}\log(\Upsilon_p(n)) = \xi(p)^{-1}$$

and $\xi(p) \in (0,\infty)$ if $p \in (0, p_c)$ (see Grimmett [1989], §6.6).

In the *supercritical phase* there is exactly one infinite cluster and all other clusters are finite. Furthermore each vertex is surrounded by infinitely many circuits (see Harris

[1960] for bond percolation and Fisher [1961] for site percolation models). Many extensions of these results are available and a quite general one is in Gandolfi, Keane and Russo [1988] below.

The connectivity functions $\Upsilon_p(n)$ do not decay because uniqueness of the infinite cluster and the FKG correlation inequalities imply that $\Upsilon_p(n) \geq \vartheta^2(p)$ for $n \geq 1$. For the bond percolation model the truncated connectivity functions $\Upsilon_p^f(n)$ decay exponentially and the correlation length $\xi(p)$, defined by

$$\xi(p)^{-1} = \lim_{n\to\infty} -\frac{1}{n} \log \Upsilon_p^f(n)$$

satisfies $\xi(p) = \frac{1}{2}\xi(1-p)$, being therefore finite for all $p \neq p_c^{\text{bond}}\ (2) = 1/2$ (see Grimmett [1989], §9.4).

It is also possible to describe the shape of finite clusters in the supercritical phase. We mention that, among other results, Alexander, Chayes and Chayes [1989] prove that the limit

$$\lim_{n\to+\infty} -n^{1/2} \log P_p(|C_0| = n) = \delta(p)$$

exists and $\delta(p) \in (0,\infty)$ for $p \in (p_c, 1)$.

1.2 Higher dimensions.

The values of the *critical points* for three-dimensional short range independent percolation have not been exactly computed. We know that $p_c^{\text{bond}}\ (3) < 1/2$ (Kesten [1982]) and $p_c^{\text{site}}\ (3) < \frac{1}{2}$ (Campanino and Russo [1985]). It is easy to see that the critical points for d-dimensional site and bond percolation satisfy $p_c^{\text{site}}\ (d) \geq p_c^{\text{bond}}\ (d) \geq \frac{1}{2d-1}$. The asymptotic behaviour is the following: there exists $k > 0$ such that

$$\frac{1}{2d-1} \leq p_c^{\text{bond}}\ (d) \leq p_c^{\text{site}}\ (d) \leq \frac{1}{2d} + k\frac{(\log\log d)^2}{d \log d}$$

for all $d \geq 2$ (Kesten [1988a]).

It is not known whether in general $\vartheta(p_c)$ is positive or not. It has been proved that it is zero if the dimension is sufficiently large (Hara and Slade [1989]).

In the *subcritical phase* the expected cluster size $\chi(p)$ is finite. In fact this is implied by the exponential decay of the cluster radius: if $p < p_c$ then there exists $\Psi(p) > 0$ such that

$$P_p(0 \to \partial B_n) \leq e^{-n\Psi(p)}$$

for all boxes $B_n, n \geq 1$ (see Menshikov [1986], Menshikov, Molchanov and Sidorenko [1986] and Aizenman and Barsky [1987]). This also implies that the connectivity functions $\Upsilon_p(n)$ decay exponentially: $\lim_{n\to\infty} -\frac{1}{n} \log \Upsilon_p(n) = \varphi(p)$, with $\varphi(p) > 0$ if $p \in (0, p_c)$ (Grimmett [1989], §5.2).

More work shows that also the cluster size decays exponentially; in fact the following inequality holds:

$$P_p(|C_0| \geq n) \leq 2\exp(-\frac{1}{2}n\chi(p)^{-2}) \quad \text{for all } n > \chi(p)^2;$$

if $p < p_c$ then $\chi(p) < \infty$ and therefore the left hand side of the last display decays exponentially (Kesten [1981], Aizenman and Newman [1984]).

The relation between cluster size and boundary can be expressed by means of the mean perimeter of the cluster as follows:

$$\chi(p)p^{-1} - \Xi(p)(1-p)^{-1} = 1$$

for all p such that $\vartheta(p) = 0$ and $\chi(p)$ and $\Xi(p)$ are finite (Coniglio and Russo [1979]).

We discuss now the *supercritical phase.* Both in bond percolation and site percolation models two infinite clusters of the same type cannot coexist (Aizenman, Kesten and Newman [1987], with simplified versions in Gandolfi, Grimmett and Russo [1988] below, Burton and Keane [1989] and also Gandolfi, Keane and Newman [1989] below). Ergodicity under any of the maps T_v, which leave the number of infinite clusters invariant, implies that in case of percolation there is one infinite cluster with probability one.

Uniqueness of the infinite cluster implies that the connectivity functions do not decay, because the FKG correlation inequalities yield $\Upsilon_p(n) \geq \vartheta^2(p)$ for all p, which bounds $\Upsilon_p(n)$ away from zero if percolation occurs.

Uniqueness of the infinite cluster can also be used to show that $\vartheta(p)$ is continuous for $p \in (p_c, 1]$ (van den Berg and Keane [1984]). Recent results (Grimmett and Marstrand [1989]) show further that $\kappa(p), \vartheta(p)$ and $\chi^f(p)$ (the mean size of the finite cluster of the origin) are infinitely differentiable functions of p in $(p_c, 1]$, in any dimension; proofs of these statements can be obtained for $p \in (p_c^+, 1]$ (for a suitable p_c^+), from Russo [1978] combined with Zhang [1989] as done in Grimmett [1989], §6.8; Grimmett and Marstrand [1989] were afterwards able to show that $p_c^+ = p_c$.

The radius of a finite open cluster decays exponentially, i.e. the limit

$$\sigma(p) = \lim_{n\to\infty} \frac{-1}{n} \log P_p(0 \to \partial B_n \text{ and } |C_0| < \infty)$$

exists and $\sigma(p) \in (0, \infty)$ for $p \in (p_c, 1)$ (as before see Grimmett [1989], §6.5, and Grimmett and Marstrand [1989]). The truncated connectivity functions converge to the same limit:

$$\lim_{n\to\infty} -\frac{1}{n} \log \Upsilon_p^f(n) = \sigma(p) \quad \text{for } p \in (p_c, 1).$$

The size distribution of finite clusters does not decay at the same rate in the supercritical phase as it does in the subcritical phase; let

$$\delta^+(p) = \limsup_{n\to+\infty} -n^{\frac{1-d}{d}} \log P_p(|C_0| = n)$$

and

$$\delta^-(p) = \liminf_{n\to+\infty} -n^{\frac{1-d}{d}} \log P_p(|C_0| = n),$$

then $\delta^+(p), \delta^-(p) \in (0,\infty)$ for $p \in (p_c, 1)$. (See Grimmett [1989], §6.7). It is not known whether that $\delta^+ = \delta^-$ when the dimension d is larger than two.

There are various results describing the ***shape of the infinite clusters***, all showing that, roughly speaking, the perimeter of the cluster is approximately $(1-p)/p$ times the size of the cluster itself.

We refer to site percolation models, but analogous results hold for the bond models. Suppose p is such that $\vartheta(p) > 0$ and let C be the infinite cluster. Let B_n be a fixed box and define the random variables $m_n = |C \cap B_n|$ and $\ell_n = |\partial(C \cap B_n) \cap B_n|$. We express the relation between boundary and volume of the infinite cluster by $M_n = \left(\frac{m_n}{p(m_n+\ell_n)} - \frac{\ell_n}{(1-p)(m_n+\ell_n)}\right)$ if $m_n + \ell_n \neq 0$ (otherwise let $M_n = 0$). We express the same relation also by the random variable defined by $R_n = \frac{\ell_n}{m_n}$ if $m_n \neq 0$ (let R_n be equal to 0 if $m_n = 0$).

We have that $\lim_{n\to\infty} R_n = \frac{1-p}{p}$ P_p-almost everywhere (Grimmett [1989], §6.9) and therefore $\lim_{n\to\infty} \int R_n dP_p = \frac{1-p}{p}$, because $R_n \leq 2d$ for all $n \in \mathbb{N}$. Similarly it is possible to show that $\lim_{n\to\infty} M_n = 0$ P_p-almost everywhere and that $\lim_{n\to\infty} \int M_n dP_p = 0$, because $|M_n| \leq \max(p^{-1}, (1-p)^{-1})$ for all $n \in \mathbb{N}$.

It is also possible to evaluate the rate of convergence of M_n to 0 and of R_n to $\frac{1-p}{p}$. This is achieved by combining the following two properties of the infinite cluster. First it is known that if the value of $m_n + \ell_n$ is fixed, then the probability that $|M_n| \geq \varepsilon$ is less than $e^{-c\varepsilon^2(m_n+\ell_n)}$, for a suitable $c > 0$. This large deviation property was first shown in the proof of non-coexistence of two infinite clusters in Aizenman, Kesten and Newman [1987] (an equivalent version can be found in Gandolfi, Grimmett and Russo [1988], below). Further below we restate this property in an abstract formulation. The second property we use to estimate the rate of convergence of M_n to 0 refers to the probability that m_n is not too small. In fact, combining the work of Grimmett and Marstrand [1989] with the construction in Durrett and Schonmann [1987], it is possible to show that, roughly speaking, the probability that m_n is comparable to $\vartheta(p)|B_n|$ is larger than $1 - e^{-c_1 n^{d-1}}$, for a suitable $c_1 > 0$.

In the next Theorem we rigorously state these properties and indicate how to prove them.

THEOREM. *Let P_p be the Bernoulli probability measure of a nearest neighbour site percolation model in $\mathbb{Z}^d$, $d \geq 1$. For all $p > p_c(d)$ and for all $\varepsilon > 0$ there exist $k_1(\varepsilon) > k_2(\varepsilon) > 0$, which depend on p, such that*

$$e^{-k_1(\varepsilon)n^{d-1}} \leq P_p(|M_n| \geq \varepsilon) \leq e^{-k_2(\varepsilon)n^{d-1}}$$

and

$$e^{-k_1(\varepsilon)n^{d-1}} \leq P_p(|R_n - \frac{1-p}{p}| \geq \varepsilon) \leq e^{-k_2(\varepsilon)n^{d-1}}$$

for all $n \geq 1$.

PROOF: First we indicate how to prove the lower bounds. With a suitable adjustment of the configurations in the boundary of B_n it is possible to obtain, for instance, that $M_n > 1/2$ or $R_n \leq (1-p)/2p$. Since the cardinality of the boundary of B_n is approximately n^{d-1}, the probability of these configurations is at least $\pi^{k_1 n^{(d-1)}} = e^{-k_1 n^{(d-1)}}$, where π equals $\inf(p,(1-p))$ and k_1 is a suitable positive real value. Following these indications it is easy to show that the lower bounds hold.

We discuss next the upper bounds. In order to show that they hold we prove that with probability close to one, exponentially in n^{d-1}, the cardinality m_n of the intersection $C \cap B_n$ between the infinite cluster C and the box B_n is of the order of $|B_n|$. More precisely we show that given $\rho > 0$ there exists $k_3 > 0$ such that

$$P_p(m_n \geq (1-\rho)\vartheta(p)|B_n|) \geq 1 - e^{-k_3 n^{d-1}} \quad \text{for all } n \geq 1. \tag{2}$$

Once this is proved we use the above-mentioned large deviations property of M_n. We use the version presented in the next Theorem. To apply the large deviation property we observe that given a configuration $\overline{\omega} \in \Omega_{\backslash B_n} = \{-1,1\}^{\mathbb{Z}^d \backslash B_n}$ we can partition the configurations of Ω_{B_n} according to the form of $C \cap B_n$. If we consider clusters C such that $m_n + \ell_n = h$, then we can partition Ω_{B_n} as required by the next Theorem. This yields that there exists $k_4 = k_4(p) > 0$, independent of $\overline{\omega}$, n and h, such that

$$P_p\left(|M_n| \geq \varepsilon, m_n + \ell_n = h | \mathcal{A}_{\backslash B_n}\right)(\overline{\omega}) \leq e^{-k_4 \varepsilon^2 h}$$

for all $\varepsilon > 0$, where $\mathcal{A}_{\backslash B_n}$ indicates the σ-algebra generated by the cylinders with base in $V \backslash B_n$.

Combining this last inequality with (2) we have that for suitable $k_2(\varepsilon), k_5 > 0$ the following holds:

$$
\begin{aligned}
&P_p(|M_n| \geq \varepsilon)\\
&\quad \leq P_p(m_n \leq (1-\rho)\vartheta(p)|B_n|)+\\
&\quad\ P_p\left(|M_n| \geq \varepsilon, m_n > (1-\rho)\vartheta(p)|B_n|\right)\\
&\quad \leq e^{-k_3 n^{d-1}} + \sum_{h>(1-\rho)\vartheta(p)|B_n|} P_p\left(|M_n| \geq \varepsilon, m_n + \ell_n = h\right)\\
&\quad = e^{-k_3 n^{d-1}} + \sum_{h>(1-\rho)\vartheta(p)|B_n|} \int_{\Omega_{\setminus B_n}} P_p\left(|M_n| \geq \varepsilon, m_n + \ell_n = h | \mathcal{A}_{\setminus B_n}\right)(\overline{\omega}) P_p(d\overline{\omega})\\
&\quad \leq e^{-k_3 n^{d-1}} + \sum_{h>(1-\rho)\vartheta(p)|B_n|} e^{-k_4 \varepsilon^2 h}\\
&\quad \leq e^{-k_3 n^{d-1}} + e^{-k_5 \varepsilon^2 n^d}\\
&\quad \leq e^{-k_2(\varepsilon) n^{d-1}}
\end{aligned}
$$

for all $n \geq 1$.

On the other hand if $m_n > (1-\rho)\vartheta(p)|B_n|$ then, for sufficiently large n, $|R_n - \frac{(1-p)}{p}| \geq \varepsilon$ implies $|M_n| \geq \frac{\varepsilon}{(1-p)} m_n \geq \frac{\varepsilon}{(1-p)}(1-\rho)\vartheta(p)|B_n| > \varepsilon$. Therefore $k_2(\varepsilon)$ can be taken such that also

$$P_p(|R_n - \frac{(1-p)}{p}| \geq \varepsilon) \leq e^{-k_2(\varepsilon) n^{d-1}}$$

holds for all $n \geq 1$.

To show that the upper bounds hold it remains only to prove (2). This will be achieved in the following way. First we fix an hyperplane $Y = \{(x_1, \ldots, x_{d-1}, 0) \in \mathbb{Z}^d\}$. Using invariance under translations it is now easy to see that to prove (2) it is sufficient to prove the following: there exists $k_6 > 0$ such that

$$(2') \qquad P_p\left(|Y \cap B_n|^{-1} \sum_{v \in Y \cap B_n} I_{A_{v,\infty}} \leq (1-\rho)\vartheta(p)\right) \leq e^{-k_6 |Y \cap B_n|} \leq e^{-k_6 n^{d-1}},$$

for all $n \geq 1$, where, as defined above, $A_{v,\infty}$ is the event that v percolates and $I_{A_{v,\infty}}$ is its characteristic function.

The proof of this last inequality involves some technicalities. We will only indicate the relevant steps. Let r_1 and r_2 be integers, whose values will be specified later on, such that $r_1 < r_2$, and consider the subset of Y defined by $\mathcal{L}_{r_2} = r_2 Y = \{(r_2 x_1, \ldots, r_2 x_{d-1}, 0) : (x_1, \ldots, x_{d-1}, 0) \in Y\}$. Let B_{r_1} be a box and consider the event $\{B_{r_1} \to \infty\}$ that there is a vertex in B_{r_1} which percolates. For $n \in \mathbb{N}$ define

$$U_n = |\mathcal{L}_{r_2} \cap B_n|^{-1} \sum_{v \in \mathcal{L}_{r_2} \cap B_n} (1 - I_{(T_v\{B_{r_1} \to \infty\})});$$

this is the frequency of vertices v in $\mathcal{L}_{r_2} \cap B_n$ such that from the box B_{r_1} centered in v no vertex percolates.

Next define the following events which dependent only on the values of the configurations in B_{r_2}:

$$A'_{0,\partial B_{(r_2-r_1)}} = (\ 0 \ \to \partial B_{(r_2-r_1)} \text{ and there is a unique cluster of } B_{(r_2-r_1)} \backslash B_{2r_1} \\ \text{connecting } \partial B_{2r_1} \text{ to } \partial B_{(r_2-r_1)}).$$

We now choose the values of r_1 and r_2. Grimmett and Marstrand [1989] construct a dynamic renormalization of the percolation process, by means of which it is proved that if $p \geq p_c(d)$ then percolation occurs also in the half space $\mathbb{Z}^{d-1} \times \mathbb{Z}_+$ and in the orthant $\mathbb{Z}_+^d$ (we discuss this in more details in section 1.3.1 below). Once we know that percolation occurs in the orthant $\mathbb{Z}_+^d$ it is possible to perform an other dynamic renormalization as in Barsky, Grimmett and Newman [1989a]. Then we make use of a suitable multidimensional (dynamic) Peierls' argument, similar to the one used in a two-dimensional situation by Durrett [1984], §10. Combining the dynamic renormalization and the Peierls' argument it is not difficult to see that we can choose $\overline{r_1}$ and $\overline{r_2}$ such that if $r_1 > \overline{r_1}$ and $r_2 > \overline{r_2}$ then the following holds: there exists $k_7, k_8 > 0$ such that

$$(*) \qquad P_p(U_n \geq \vartheta(p)\rho/2) \leq e^{-k_7|B_n \cap \mathcal{L}_{r_2}|} \leq e^{-k_8 n^{d-1}},$$

for all $n \geq 1$. Uniqueness of the infinite cluster implies that given r_1 we can choose r_2 such that

$$P_p(A'_{0,\partial B_{(r_2-r_1)}}) > (1-\rho/2)\vartheta(p).$$

In fact we choose r_1 and r_2 such that this last inequality and (*) are simultaneously satisfied. Note that (*) already provides an estimate, similar to the one in (2) but with $(1-\rho)$ replaced by a smaller value, which is sufficient for our purposes. We nevertheless consider $(1-\rho)$ because in this way we also obtain a large deviation property for m_n.

From now on we follow Durrett and Schonmann [1987].

Let $L_{r_1} = B_{r_1} \cap Y$ and define

$$S_n = \cup_{v \in \mathcal{L}_{r_2} \cap B_n} T_v L_{r_1},$$
$$V_w = |T_w L_{r_1}|^{-1} \sum_{v \in T_w L_{r_1}} I_{(T_v A'_{0,\partial B_{r_2-r_1}})}$$

and

$$W_n = |\mathcal{L}_{r_2} \cap B_n|^{-1} \sum_{w \in \mathcal{L}_{r_2} \cap B_n} V_w = |S_n|^{-1} \sum_{v \in S_n} I_{(T_v A'_{0,\partial B_{r_2-r_1}})}.$$

Using the invariance of the model under translations it is not difficult to see that to prove that (2′) holds for a suitable k_6, and therefore that (2) holds for a suitable k_3, it is sufficient to show the following: there exists $k_9 > 0$ such that

$$(**) \qquad P_p\left(|S_n|^{-1}\sum_{v\in S_n} I_{A_{v,\infty}} \le \theta(p)(1-\rho)\right) \le e^{-k_9 n^{d-1}}$$

for all $n \ge 1$. To prove this exponential bound we use the following inequality; its proof follows directly from the definitions:

$$P_p\left(|S_n|^{-1}\sum_{v\in S_n} I_{A_{v,\infty}} \le \vartheta(p)(1-\rho)\right) \le P_p(W_n \le (1-\rho/2)\vartheta(p)) + P_p(U_n \ge \rho/2\vartheta(p)).$$

We only need a bound for the first term in the right hand side of the last display since (*) gives a bound for the second term. But W_n is the average of the i.i.d. random variables V_j and $\int V_j dP_p > (1-\rho/2)\vartheta(p)$. Therefore the large deviation theorem for bounded i.i.d. random variables yields that there exists $k_{10} > 0$ such that

$$P_p(W_n \le (1-\rho/2)\vartheta(p)) \le e^{-k_{10}n^{d-1}} \qquad \text{for all } n \ge 1.$$

Combining the last inequality with (*) we have that (**) holds and therefore (2′) holds for a suitable $k_9 > 0$. This proves the Theorem. ∎

The large deviation property required in the Theorem above and proved in Aizenman, Kesten and Newman [1987] or in Gandolfi, Grimmett and Russo [1988] below, can be rephrased in an abstract form as follows. We refer to the mentioned articles for a proof of the following Theorem.

THEOREM. *Let F be a finite set and let P_p be the Bernoulli probability measure with parameter p defined on (all subsets of) $\{0,1\}^F$. Denote by C_k a family of disjoint cylinders whose basis are all of size k. For $D \in C_k$ let D^1 be the subset of F where all the configurations of D assume value 1 and D^{-1} the rest of the basis of D; note that $|D^1|+|D^{-1}| = k$ and put $h(D) = \frac{|D^1|}{p} - \frac{|D^{-1}|}{(1-p)}$. Then there exists a constant $a = a(p) > 0$ such that*

$$P_p(D \in C_k : h(D) \ge \varepsilon k) = \sum_{\substack{D\in C:\\ h(D)\ge\varepsilon k}} P_p(D) \le e^{-k\varepsilon^2(4a)^{-1}}$$

for all $\varepsilon > 0$ and for all $k \ge 1$. ∎

1.3 Percolation in subsets of $\mathbb{Z}^d$.

The relevant subsets we discuss are half spaces, orthants, slabs of the form $S_k = \mathbb{Z}^{d-1} \times \{0, \ldots, k\}$, quarter slices $Q_k = \mathbb{Z}_+^2 \times \{1, \ldots, k\}^{d-2}$ and wedges defined in 1.3.2 below.

The first four types are interesting because of the relation between their critical points and the critical point p_c of the whole space; the last is of interest because of possible discontinuity at its critical point.

The following results hold for both site and bond percolation. Fix a dimension $d \geq 3$. Let p_c be the critical point for $\mathbb{Z}^d$, $p_c(\mathbb{H})$ the one for $\mathbb{Z}^{d-1} \times \mathbb{Z}_+$, $p_c(\mathbb{O}^l)$ the one for $\mathbb{Z}^{d-l} \times \mathbb{Z}_+^l, l = 2, \cdots, d$, and finally $p_c(S_k)$ and $p_c(Q_k)$ the critical points for slabs S_k and quarter slices Q_k. Then:

THEOREM. *For all l such that $2 \leq l \leq d$ holds:*
$\lim_{k \to +\infty} \quad p_c(Q_k) = \lim_{k \to +\infty} \quad p_c(S_k) = p_c(\mathbb{O}^l) = p_c(\mathbb{H}) = p_c$ ∎

From left to right these equalities have been proved in various steps (Aizenman, Chayes, Chayes, Frölich and Russo [1983], Barsky, Grimmett and Newman [1989a, 1989b], Kesten [1988c]) and finally the last equality has been proved in Grimmett and Marstrand [1989].

1.3.1 Half spaces and slices.

Percolation at a value p in $\mathbb{Z}^d$ implies percolation in the corresponding half space $\mathbb{Z}^{d-1} \times \mathbb{Z}_+$ at any value $p' > p$; furthermore for any $p' > p$ there is $k \in \mathbb{N}$ such that percolation occurs in S_k and in Q_k (Grimmett and Marstrand [1989]). This is how the equality of critical points is proved, but does not answer the question of continuity of $\vartheta(p)$ at p_c.

If we suppose percolation occurs in the half space $\mathbb{H}$ at a given value p then there exists $k \in \mathbb{N}$ such that percolation occurs in S_k and Q_k for *the same p* (Barsky, Grimmett and Newman [1989a. 1989b]). From the construction in Barsky, Grimmett and Newman [1989b] we know that percolation occurs also at $p - \varepsilon$, if $\varepsilon > 0$ is sufficiently small. This therefore implies that the percolation function for the half space $\vartheta_{\mathbb{H}}(p)$ vanishes at $p_c(\mathbb{H})$; the same properties for percolation functions of orthants $\mathbb{O}^l$ can be derived using the techniques of inducing percolation on subsets. A similar argument shows that the relative percolation functions are continuous at the critical points of quarter slices. We do not know how to prove absence of percolation at the critical point of the space or of slabs.

The infinite cluster of half spaces, slabs, orthants or quarter slices is unique. For the half space a proof of uniqueness first appeared in Kesten [1988c] and is given in a different and more general fashion in Gandolfi, Keane and Newman [1989], below. For slabs it is possible to adapt any of the proofs given for the full space $\mathbb{Z}^d$. Uniqueness for orthants and quarter slices is proved in Barsky, Grimmett and Newman [1989a].

Adapting van den Berg and Keane [1984] we obtain that continuity of the percolation functions follows from the uniqueness of the infinite cluster also for subsets of $\mathbb{Z}^d$.

1.3.2 Wedges.

For special subsets of $\mathbb{Z}^d$ interesting phenomena can occur. For instance there can be a discontinuity of the percolation function at the critical point or there can be intermediate phases. This is the case for *wedges* of $\mathbb{Z}^d$. Their definition requires a constant G, the growth rate, a function $c : \mathbb{N} \to \mathbb{N}$, the correction to the exponential growth, and a divergent sequence $A = \{a_i\}_{i \in \mathbb{N}}$ of positive integers $a_1 < a_2 < \dots a_i < \dots$. Then the wedge $\mathbb{W}_{G,c,A}$ consists of the set of vertices

$$V_{G,c,A} = \{(x_1, \dots, x_d) \in \mathbb{Z}^d : \\ \exists j \text{ s.t.} \sum_{i=1}^{j-1} c(a_i) e^{G a_i} \leq x_1 < \sum_{i=1}^{j} c(a_i) e^{G a_i}, x_l \leq a_j \, , \, l = 2, \dots, d\}$$

and the set $E_{G,c,A}$ of edges of length one between the vertices of $V_{G,c,A}$. We discuss bond percolation.

First we have that for any wedge $\mathbb{W} = \mathbb{W}_{G,c,A}$, i.e. for any $A, c > 0$ and $G > 0$, $\lim_{n\to\infty} \frac{(-1)}{n} \log \Upsilon_{p,\mathbb{W}}(n) > 0$ if and only if $\lim_{n\to\infty} \frac{(-1)}{n} \log \Upsilon_p(n) > 0$, where $\Upsilon_{p,\mathbb{W}}(n)$ are the connectivity functions for the wedge $\mathbb{W}$ defined in analogy to $\Upsilon_p(n)$. In other words the values of p for which there is exponential decay of the connectivity functions are the same for $\mathbb{Z}^d$ and for any d-dimensional divergent wedge $\mathbb{W} \subset \mathbb{Z}^d$ (Hammersley and Whittington [1984], Chayes and Chayes [1986]).

To have a transition to percolation we need a value of p generally bounded away from the percolation point p_c of $\mathbb{Z}^d$. If $G > 0$ then the critical point $p_c(\mathbb{W})$ for the wedge $\mathbb{W} = \mathbb{W}_{G,c,A}$ satisfies $p_c \leq p_c(\mathbb{W}) < 1$ (Hammersley and Whittington [1984]) and there are wedges for which the first inequality is strict (Chayes and Chayes [1986]). Therefore for the values of $p \in (p_c, p_c(\mathbb{W}))$ there is no percolation in $\mathbb{W}$, but the connectivity functions decay slower than exponentially.

If p is larger than the critical point p_c of $\mathbb{Z}^d$, then there is $G > 0$ such that wedges with growth rate larger than G exhibit percolation at the value p of the parameter (Grimmett [1981a], Grimmett [1983], Chayes and Chayes [1986], with final aid from Grimmett and Marstrand [1989]).

The value of the critical point for wedges depends on the growth rate G and behaviour of percolation at $p_c(\mathbb{W})$ depends on the correction $c(n)$: Chayes and Chayes [1986] show that if the dimension is two and if $\sum_{j=1}^{\infty} c(a_j) < \infty$ then the percolation function as function of p is discontinuous at $p_c(\mathbb{W})$, i.e. there is a positive probability of percolation at criticality.

It is not known whether the infinite cluster is unique in wedges.

1.4 Long range one-dimensional independent percolation.

Of the many results available on these models we briefly review only those which are closer to the arguments of this thesis.

First of all we limit the number of possible parameters to three only and our model, necessarily a bond percolation model, will be as follows.

The set of vertices is $V = \mathbb{Z}$ and the set of edges $E = \{\{v_1, v_2\}, v_i \in V, i = 1, 2\}$ is the set of all possible edges. Edges belong to the subgraph with independent stationary probabilities given by $P(\{m, n\}$ belongs to $E') = p_{|n-m|}$. We assume $p_1 = p$ to be one of the parameters and we take $p_n = \beta/n^s$ for $n \geq 2$, where $\beta > 0$ and $s \geq 0$ are the other two real parameters. Let us assume also that p, β and s are such that $p_n < 1$ for all n, otherwise percolation would always occur.

We begin by describing for what values of p, β and s percolation occurs.

If $s > 2$ percolation does not occur for any value of the other parameters (Schulman [1983]).

If $1 < s \leq 2$ the most interesting situations occur. For any such an s there is $\beta_c(s)$ such that if $\beta > \beta_c(s)$ then there exists $p_c(\beta)$ such that if $p > p_c(\beta)$ percolation occurs. In other words for β sufficiently large there is a phase transition for percolation (Newman and Schulman [1986]).

If $s \leq 1$ than any vertex belongs to an infinite cluster; in fact given a vertex v the Borel-Cantelli Lemma implies that the subgraph contains edges between v and infinitely many other vertices with probability one. A new situation occurs: with probability one the cluster of the origin contains all the other vertices and the graph is therefore totally connected (Grimmett, Keane and Marstrand [1984]); in section 5 we discuss this phenomenon in greater details.

We now review the behaviour in the various phases of the functions which are typical for percolation.

The *subcritical phase* refers to the following values of the parameters: $1 < s \leq 2$ and either $\beta < \beta_c(s)$ or $\beta > \beta_c(s)$ and $p < p_c(\beta)$; in particular it refers to all values of p if $s = 2$ and $\beta < 1$. In this case the connectivity functions $\Upsilon(n)$ decay as a power of n: there exists $c > 0$ such that $\Upsilon(n) \leq c/n^s$ for all n, furthermore $\lim_{n\to\infty} \frac{-\log \Upsilon(n)}{\log n} = s$; this implies that the expected cluster size χ is finite (Aizenman and Newman [1986]).

We discuss the behaviour of the random subgraph at *criticality* only for $s = 2$, which is the boundary value for the existence of a phase transition and the most interesting case. We have that for $s = 2, \beta > 1$ and $p = p_c(\beta)$ the probability of percolation $\vartheta(p)$ satisfies:

$$\vartheta(p_c(\beta)) \geq \beta^{-\frac{1}{2}} > 0.$$

This means that there is a discontinuity at criticality in the percolation function as function of the parameter p (proofs of these statements are corollaries of a more general inequality in Aizenman and Newman [1986]).

The *supercritical phase* corresponds to $1 < s \leq 2$, $\beta > \beta_c$ and $p \geq p_c(\beta)$. Uniqueness of the infinite cluster (Aizenman, Kesten and Newman [1987] or Gandolfi, Keane and Newman [1989] below) together with the FKG inequality imply that the connectivity functions do not decay: $\Upsilon(n) \geq \vartheta^2(p) > 0$. The truncated connectivity functions $\Upsilon^f(n) = P(0$ is connected to n by a finite cluster) are conjectured to exhibit a peculiar behaviour for $s = 2$. We have seen that in the subcritical phase they decay as n^{-2}: the same is expected, but not proved, for large β (at fixed p) (Imbrie [1982]), but for any $\varepsilon > 0$ there are values of p, β and $c > 0$ for which $\Upsilon^f(n) \geq c/n^{\varepsilon}$ (Imbrie and Newman [1988], Corollary 1.5).

2. Gibbs states and Ising ferromagnets

We discuss here random subgraphs realized in a more elaborated way. In this case it is natural to consider nearest neighbour site percolation models with $V = \mathbb{Z}^d$. Therefore we need a mechanism to produce the random subset $V' \subseteq V$. To this purpose we introduce a probability measure on $\{-1, 1\}^V$. A natural way of defining a non-independent probability measure leads to Markov processes, which satisfy the Markov property (see, for instance, Feller [1950]). We discuss here a broader class of dependent probability measures satisfying a Markov-like property. These measures are well known in statistical mechanics and are called Gibbs states. Simple examples of Gibbs states can be considered as multidimensional versions of one-step Markov processes and are called Ising ferromagnets. We introduce the Gibbs states first and then we discuss the properties of percolation, in particular in the special case of Ising ferromagnets. A special attention is devoted to the Ising ferromagnets in $\mathbb{Z}^2$ since the properties of percolation in these models are interesting for various reasons; we mentioned here only the relation with the proof of absence of non-stationary states (see Russo [1979], Higuchi [1982] and Aizenman [1980]), and the relation with a model of three-dimensional wetting (see Abraham and Newman [1988]).

First we need some notation. Consider any subset S of the set of vertices V and define $\Omega_S = \{-1, 1\}^S$. Suppose $R \subseteq S \subseteq V$ and let $\alpha_{R,S} : \Omega_S \to \Omega_R$ be the map obtained by restricting configurations of Ω_S to R. We denote with the same symbol $\alpha_{R,S}$ the natural extension of $\alpha_{R,S}$ to the probability measures defined on Ω_S.

To define the Gibbs states we need an *interaction*

$$\Phi : \bigcup_{R \text{ finite},\ R \subset V} \Omega_R \to \mathbb{R},$$

which we take to be invariant under the translations T_v, i.e. $\Phi(T_v\omega) = \Phi(\omega)$ for all $\omega \in \Omega_R$ and for all finite $R \subset V$ (note that $T_v(\omega) \in \Omega_{T_v^{-1}(R)}$). Let B be a box and for $\omega \in \Omega_B$ define the *energy*

$$U_B(\omega) = \sum_{R \subseteq B} \Phi(\alpha_{R,B}\omega).$$

If $\overline{\omega} \in \Omega_S$, with $S \cap B = \emptyset$, then define the *interaction energy* by

$$W_{B,S}(\omega \sqcup \overline{\omega}) = \sum_{R \text{ finite},\ R \subseteq (B \cup S), R \cap B \neq \emptyset, R \cap S \neq \emptyset} \Phi(\alpha_{R,(B \cup S)}(\omega \sqcup \overline{\omega}))$$

for all $\omega \in \Omega_B$.

A *Gibbs state* for a given interaction Φ is a probability measure on Ω such that for all boxes B the following holds:

$$\alpha_{B,V}\,\mu(\omega) = \int_{\Omega_{V \setminus B}} m_{B,\overline{\omega}}(\omega)\,\alpha_{V \setminus B, V}\,\mu(d\overline{\omega}) \tag{3}$$

where, for all $\overline{\omega} \in \Omega_{V\setminus B}$, $m_{B,\overline{\omega}}$ is the finite volume Gibbs state on Ω_B defined by

$$m_{B,\overline{\omega}}(\omega) = (Z_{B,\overline{\omega}})^{-1} \exp(-U_B(\omega) - W_{B,V\setminus B}(\omega \sqcup \overline{\omega})),$$

where $Z_{B,\overline{\omega}}$ is a normalizing factor.

There is a second equivalent way of defining Gibbs states. The space of probability measures, taken as dual of the Banach space of continuous functions on Ω and endowed with the topology of weak convergence, is a topological vector space in which the set of probability measures is compact. Therefore once a configuration $\overline{\omega} \in \Omega$ is fixed, the *sequence* of probability measures $m_{B_n,\alpha_{B_n,V}(\overline{\omega})}$, considered as measures on Ω, has at least one subsequence which is weakly convergent. All these weak limits satisfy (3) and are therefore Gibbs states. On the other hand all Gibbs states are obtained as convex combinations of such weak limits for various $\overline{\omega}$. A rigorous account of the theory of Gibbs states, proofs of the previous statements and remarks on the physical significance, are in Ruelle [1978].

Gibbs states exhibit the phenomenon of phase transitions. In fact it is possible that there exist more than one probability measure satisfying (3) for a given interaction. We discuss this further below, after introducing the specific case of Ising ferromagnets.

Even if the interaction is translation invariant the Gibbs states ought not to be translation invariant (see Dobrushin [1972] or van Beijeren [1975]), because of the effect of the boundary conditions $\overline{\omega}$. On the other hand, given an interaction Φ the Gibbs states which are translation invariant can be decomposed into ergodic states, as described in section 0 above.

To give examples of the possible phenomena which can occur for Gibbs states we now introduce the Ising ferromagnets. Afterwards we briefly review what are the possible Ising ferromagnets for the different values of β and h in the various dimensions.

To obtain *Ising ferromagnets* we make a specific choice of the interaction. If $|R| \geq 3$ then $\Phi(\omega) = 0$ for all $\omega \in \Omega_R$. Next consider $h \in \mathbb{R}$ and $\beta \in \mathbb{R}_+$; if $R = \{v, w\}$ then $\Phi(\omega_v, \omega_w) = \beta\omega_v\omega_w$ and if $R = \{v\}$ then $\Phi(\omega_v) = h\omega_v$. The parameter h is called external field and the parameter β is called inverse temperature (for a review see Gallavotti [1972] or Ellis [1985]). Given β and h we indicate by $\mu_{\beta,h}$ any Ising ferromagnets satisfying (3) for the interaction Φ defined as above with the given β and h. Since we have taken $\beta \geq 0$ all Ising ferromagnets satisfy the FKG correlation inequality (see Fortuin, Kasteleyn and Ginibre [1971]). Furthermore $\mu_{\beta,h}$ stochastically dominates $\mu_{\beta,h'}$ if $h \geq h'$ (see Holley [1974]).

In any dimension, if $h \neq 0$ then there is a unique Ising ferromagnet, irrespective of the boundary conditions $\overline{\omega} \in \Omega$. This unique probability measure is also translation invariant.

If the dimension is two and if $h = 0$ then there exists a critical value $\beta_c(2)$ of β. If $\beta \leq \beta_c(2)$ then there is a unique Ising ferromagnet, which is also translation invariant, and if $\beta > \beta_c(2)$ then there are two different Ising ferromagnets $\mu^+_{\beta,0}$ and $\mu^-_{\beta,0}$ obtained with two different $\overline{\omega}$; $\mu^+_{\beta,0}$ is obtained if $\overline{\omega}_v = +1$ for all $v \in V$, and $\mu^-_{\beta,0}$ if $\overline{\omega}_v = -1$ for all $v \in V$. All the other Ising ferromagnets are convex combinations of $\mu^+_{\beta,0}$ and $\mu^-_{\beta,0}$ (Higuchi [1982], Aizenman [1980]) and they are all translation invariant. By analogy we use the symbols $\mu^+_{\beta,0}$ and $\mu^-_{\beta,0}$ also for $h \neq 0$, even if they indicate the same probability measure.

In higher dimensions if $h = 0$ then there is a unique Ising ferromagnet if $\beta \leq \beta_c(d)$, for a suitable $\beta_c(d)$, and more than one if $\beta > \beta_c(d)$. In particular if β is large enough there are states which are not translation invariant (Dobrushin [1972], van Beijeren [1975]).

Note that for $\beta = 0$ the Ising ferromagnet $\mu_{0,h}$ coincides with the Bernoulli probability measure with parameter $p = e^h(e^h + e^{-h})^{-1}$.

In the next two sections we discuss the properties of the site percolation model obtained by choosing the random subgraph of (V, E) according to a configuration $\omega \in \Omega$ with a distribution given by an Ising ferromagnet.

2.1 Percolation in the two-dimensional Ising ferromagnets.

As usual let us first analyze for which values of the parameters β and h percolation occurs in the random subgraph described by an Ising ferromagnet $\mu_{\beta,h}$.

For $\beta = 0$ the model corresponds to independent site percolation; as discussed above, there is a critical value, which we denote here by h_c, such that only for $h > h_c$ percolation occurs. Also for $0 < \beta < \beta_c$ there exists a critical $h_c(\beta)$, but estimates on $h_c(\beta)$ are not accurate and whether percolation occurs or not at $h_c(\beta)$ is not known. Simple arguments, called of Peierls' type (Peierls [1936]), show that h_c is neither $-\infty$ nor $+\infty$, but they provide rough estimates of the value of $h_c(\beta)$.

If $h = 0$ the measure $\mu_{\beta,h}$ is invariant under spin reversal, i.e. $\mu_{\beta,h}$ is invariant under the action induced on the probability measure by the transformation which interchanges 1 and -1. This implies that for $\beta \leq \beta_c$ (and $h = 0$) there is no percolation. Let us indicate how to prove this result. To this purpose we introduce a new way of realizing a random subgraph of the given graph; this different model is used in this section only. Consider the random subgraph consisting of the vertices $v \in V$ such that $\omega_v = -1$ and of all edges between these vertices; if there is a connected component containing infinitely many vertices we say that *percolation of minus signs* occurs. When discussing both models of percolation we indicate the usual one by *percolation of plus signs*. If percolation of plus signs occurs then there is a unique cluster which forms infinitely many circuits around each vertex (Coniglio, Nappi, Peruggi and Russo [1976]), so that percolation of minus signs cannot occur. Invariance under spin reversal implies therefore that neither of the two types of percolation can occur. The result on the behaviour of the infinite cluster if percolation occurs has been generalized in Gandolfi, Keane and Russo [1988], as we discuss below. If $\beta > \beta_c$ percolation does occur for the measure $\mu^+_{\beta,0}$ (see Coniglio, Nappi, Peruggi and Russo [1976] or our discussion on FK representation in the next section) and it does not for $\mu^-_{\beta,0}$ (Russo [1979], this follows also from Gandolfi, Keane and Russo [1988] below). The absence of percolation for $\mu^-_{\beta,0}$ can be shown by the following argument: percolation of both signs cannot coexist for $\mu^-_{\beta,0}$; furthermore the FKG correlation inequality implies that under $\mu^-_{\beta,0}$ percolation of minus signs has a higher probability than percolation of plus signs and it is therefore the only one which can occur.

Applying the FKG correlation inequality we can deduce that if $\beta > \beta_c$ then percolation under $\mu^+_{\beta,h}$ occurs for $h \geq 0$ and does not for $h < 0$.

Let us remark that also in this model there is , in some sense, a discontinuity of the percolation function at criticality. Take a fixed $\beta > \beta_c$ and consider $\mu^+_{\beta,h}$; note that the boundary conditions are in fact relevant only for $h = 0$. If the probability of percolation $\mu^+_{\beta,h}(A_{0,\infty})$ is considered as a function of h and h varies, then there is a discontinuity at $h_c = 0$. Such a discontinuity is not expected for $\beta \leq \beta_c$; especially for $\beta = \beta_c$ it is believed that $h_c = 0$, with absence of percolation at the critical point.

We discuss now the *subcritical phase*. Consider the measure $\mu^-_{\beta,h}$; if $\beta > \beta_c$ and $h \leq 0$ then the connectivity functions decay exponentially: there exists a constant $c = c(h) > 0$ such that $\Upsilon(n) \leq e^{-cn}$ (see Chayes, Chayes and Schonmann [1987] for $h = 0$; use the Holley's stochastic inequality for $h < 0$); therefore the expected cluster size is finite. For $\beta \leq \beta_c$ Higuchi [1986] shows that for β and h satisfying $h < -4(\beta_c - \beta) \leq 0$ there is a sub-exponential decay of the connectivity functions: for all $n \in \mathbb{N}$ it is the case that $\Upsilon(n) \leq e^{-cn(\log n)^{-1}}$ for some $c > 0$. This also implies that the expected cluster size is finite; these facts, or even an exponential bound of the connectivity functions, are believed to hold in the whole subcritical phase, up to $h_c(\beta)$; the critical value $h_c(\beta)$ is believed to be positive at least for $\beta < \beta_c$ (see Higuchi [1988] for further comments).

We consider the *supercritical phase*. If percolation occurs then the infinite cluster is unique and with probability one there are infinitely many circuits around each vertex (see Coniglio, Nappi, Peruggi and Russo [1976] for $h = 0$ and Gandolfi, Keane and Russo [1988], below, for the other cases).

Uniqueness of the infinite cluster together with the FKG correlation inequality implies that the connectivity functions are bounded below by the square of the probability of percolation: $\Upsilon(n) \geq (\mu_{\beta,h}(A_{0,\infty}))^2 > 0$ for all $n \geq 1$ and all possible Ising states $\mu_{\beta,h}$ for which percolation occurs.

2.2 Percolation in higher dimensional Ising ferromagnets.

We already mentioned that Ising ferromagnets with $\beta = 0$ are equivalent to Bernoulli probability measure s. Therefore, if we consider the random graph describe by these probability measure s, the critical point, in dimension three and higher, is smaller than 1/2 (Campanino and Russo [1985]). This fact suggests that percolation occurs for all $\beta \geq 0$ if $h = 0$, but only partial results are available.

Suppose $h = 0$. If β is sufficiently small then percolation occurs (Campanino and Russo [1985]). On the other hand a general inequality discussed in section 3.1 below shows that percolation occurs for $\beta > \beta_c$ (Coniglio, Nappi, Peruggi and Russo [1976] or Fortuin and Kasteleyn [1972a, 1972b]). It is also believed that there exists $\overline{\beta} > \beta_c$ such that $h_c(\beta) < 0$ if $\beta < \overline{\beta}$. This is the case if the dimension d is large enough. The existence of $\overline{\beta}$ seems to be related to the so-called roughening transitions and is in contrast with the situation of two dimensional case, where coexistence of percolation of opposite signs cannot occur.

In the *subcritical phase* it is possible to show that there is sub-exponential decay of the connectivity functions, i.e. $\Upsilon(n) \leq e^{-cn(\log n)^{-1}}$, for a suitable $c > 0$, for the following values of the parameters: $\beta \geq \beta_c(d)$ and $h < 0$, or $\beta < \beta_c(d)$ and $h < -2d(\beta_c(d) - \beta)$ (Higuci [1986]).

To describe the shape of finite clusters consider the mean cluster size χ and the mean cluster perimeter Ξ. Let $S = \sum_{v \in V}(\mu_{\beta,h}(\omega_0\omega_v) - \mu_{\beta,h}(\omega_0)\mu_{\beta,h}(\omega_v))$ be the so called susceptibility, which is one of the thermodynamical quantities relevant in the description of the Ising ferromagnets. Furthermore let $m = \mu_{\beta,h}(\omega_0)$ be the magnetization. Then for all β and h for which there is no percolation and χ and Ξ are finite the following holds:

$$(1-m)\chi - m\Xi \geq S$$

with equality if $\beta = 0$:

$$(1-m)\chi - m\Xi = m(1-m)$$

(Coniglio and Russo [1979]). If $\beta = 0$ this is the result we discussed in 1.2 above. These results show that, also in the Ising ferromagnets, the mean perimeter of the finite clusters is proportional to the mean cluster size, but the difference (suitably rescaled) may be unbounded; this is likely to be the case, for instance, in the two-dimensional Ising model when $h = 0$ and β approaches β_c from below; in this case S diverges (Ellis [1985]) and χ and Ξ are believed to be finite.

We discuss now the *supercritical phase*. If the Ising ferromagnet is translation invariant and percolation occurs, then the infinite cluster is unique. A proof can be found in Gandolfi [1987] below, or in Burton and Keane [1988] or in Gandolfi, Keane and Newman [1989] below. The connectivity functions do not decay, in fact $\Upsilon(n) \geq (\mu_{\beta,h}(A_{0,\infty}))^2 > 0$ for all $n \geq 1$.

If the Ising ferromagnet is not translation invariant there is no evidence of whether the infinite cluster is unique. We can only show that the mean density of the vertices $v \in V$ which belong to the boundary of more than one infinite cluster is 0. More precisely fix $v \in V$ and let $\{N_v \geq 2\}$ be the event that v belongs to the boundary of at least two distinct infinite clusters. Consider a box B_n and let μ indicate a generic Ising ferromagnet. Then

$$\lim_{n\to\infty} \mu(|B_n|^{-1} \sum_{v \in B_n} I_{\{N_v \geq 2\}}) = \lim_{n\to\infty} |B_n|^{-1} \sum_{v \in B_n} \mu(\{N_v \geq 2\}) = 0$$

A proof can easily be obtained by Gandolfi [1987], below. In particular consider the subset $L \subset V$ of all vertices the sum of whose coordinates is an even number and consider a suitable box $A(B_n) \subset B_n$ such that $lim_{n\to\infty} |A(B_n)|/|B_n| = 1$. Then it is easy to see that the required result is equivalent to $\limsup_{n\to\infty} |A(B_n) \cap L|^{-1} \sum_{v\in A(B_n)\cap L} \mu(\{N_v \geq 2\}) = 0$. Then follow the second part of Lemma 1 and the proof of Theorem 1 in Gandolfi [1987], below. In this way it is possible to show this last result for more general Gibbs states, including the Ising ferromagnets.

The shape of the infinite cluster has been described in section 1.2 above by means of the random variables $R_n = \frac{|\partial(C\cap B_n)\cap B_n|}{|C\cup B_n|} = \ell_n/m_n$, where C and ∂C are the sets of vertices belonging to the infinite cluster and to its boundary, respectively, and B_n is a box. In the Ising ferromagnets we can no longer expect that $lim_{n\to\infty} R_n = (1-q)/q$, if $q = q_{\beta,h} = \mu_{\beta,h}(\omega_0 = 1)$, as in the independent case (see Coniglio and Russo [1979]). On the other hand it is the case that if $\mu_{\beta,h}$ is translation invariant then the limit $lim_{n\to\infty} R_n = R_{\beta,h}$ exists $\mu_{\beta,h}$ – almost everywhere . The random variable $R_{\beta,h}$ is non-negative and satisfies $R_{\beta,h} \leq (1-q_{\beta,h})/q_{\beta,h}$, $\mu_{\beta,h}$ – almost everywhere , with equality if $\beta = 0$. Furthermore $R_{\beta,h}$ is constant $\mu_{\beta,h}$ – almost everywhere if $\mu_{\beta,h}$ is ergodic. In fact these are easy consequences of the ergodic theorem and of the FKG correlation inequality (see section 4.1 below).

2.3 Other Gibbs states.

It is also possible to consider a random subgraph described by other Gibbs states than Ising ferromagnets. There are only few results concerning percolation in this case.

No particular results of interest concerning the subcritical phase are available. On the other hand it is possible to give conditions which ensure that the infinite cluster is unique if percolation occurs. Let Φ be a given interaction. Let $S \subset V$ be a finite set and suppose $\overline{\omega} \in \Omega_{V\setminus S}$. Consider the finite volume Gibbs state $m_{S,\overline{\omega}}$ for the given Φ and suppose it satisfies the following condition for all finite S: if $m_{S,\overline{\omega}}(\omega) > 0$ and $\omega' \succ \omega$ then $m_{S,\overline{\omega}}(\omega') > 0$. Then take a Gibbs state μ for Φ which is invariant under all translations $T_v, v \in V$. If percolation occurs for μ then the infinite cluster is unique. A proof can be found in Gandolfi, Keane and Newman [1989], below. In fact the above-mentioned condition implies that μ satisfies the positive finite energy condition, as defined in section 4.2 below; this last condition is used in the mentioned article to prove uniqueness of the infinite cluster.

We mention another general result concerning the shape of the infinite cluster. Suppose the interaction Φ has *finite range*, i.e. there exists $k > 0$ such that $\Phi(\omega) = 0$ if $\omega \in \Omega_S$ with $|S| \geq k$. Next let F be the set of vertices, not equal to $\mathbf{0}$, at distance less than k from $\mathbf{0}$. Let $\Omega(0) = \{-1,1\}^F$ and $\sigma \in \Omega(0)$. Given a vertex $v \in V$ and a configuration $\omega \in \Omega$ we say that the local environment of v is σ if ω coincides with σ in $T_v F$. For $\omega \in \Omega, \sigma \in \Omega(0)$ and C the infinite cluster define
$m_\sigma^n(\omega)$ = number of vertices $v \in B_n \cap C$ such that the local environment around v is σ and
$\ell_\sigma^n(\omega)$ = number of vertices $v \in \partial(B_n \cap C) \cap B_n$ such that the local environment around v is σ.

Let furthermore $m_n = \sum_{\sigma \in \Omega(0)} m_{\sigma,n}$ and let $\ell_n = \sum_{\sigma \in \Omega(0)} \ell_{\sigma,n}$.
Define
$T_\sigma = \mu(E_0^1 | \sigma$ is the local environment)
and
$U_\sigma = \mu(E_0^{-1} | \sigma$ is the local environment).
Next define

$$M_n(\omega) = (m_n + \ell_n)^{-1} \sum_{\sigma \in \Omega(0)} \left(T_\sigma^{-1} m_\sigma^n(\omega) - U_\sigma^{-1} \ell_\sigma^n(\omega) \right).$$

Suppose μ is invariant under all translations T_v, $v \in V$, and it is ergodic under the full group of translations. Then it is not difficult to see that the ergodic theorem implies that

$$\lim_{n \leftarrow +\infty} M_n(\omega) = 0$$

for μ-almost all $\omega \in \Omega$, if percolation occurs. If μ is not ergodic this result holds for each ergodic component separately and therefore for μ itself. In 1.2 we discussed this result for $\beta = 0$.

In Gandolfi [1987], below, there are partial results about the rate of convergence of M_n^C to 0. Suppose Φ is a finite range interaction, i.e. there exists $k_1 > 0$ such that $\Phi(\omega) = 0$ if $\omega \in \Omega_S$ with $|S| \geq k_1$. Next let μ be a Gibbs state for Φ, invariant under all translations T_v, $v \in V$. Then it is not difficult to deduce, from the proof of Lemma 3 in the mentioned article, that there exists $h > 0$ such that for all $s \geq 1$ the following holds: $\mu(|M_n| \geq \epsilon, \quad m_n + \ell_n = s) \leq \exp(-h\epsilon^2 s)$ for all $n \geq 1$. These results can not be used to give a satisfactory estimate of the large deviations of $|M_n|$, because it is not known how to give a good estimate of the probability that $m_n + \ell_n$ is of the order of $|B_n|$. Such an estimate has been used in the proof of the last Theorem in section 1.2 above: in that case it was in fact possible to estimate the large deviations of $|M_n|$.

We now introduce some models which are related to the Ising ferromagnets of the previous section, to their extensions to long range interactions and to models of spin glasses.

The original formulation is due to Fortuin and Kasteleyn (Fortuin and Kasteleyn [1972a, 1972b], Kasteleyn and Fortuin [1969]) and gives a different way of defining the Ising ferromagnets with external field $h = 0$. Many generalizations have been developed since then (Swendsen and Wang [1987], Edwards and Sokal [1988], Kasai and Okiji [1988], Newman [1988a]) and we discuss here a quite general version of them.

After introducing the model we will turn to single applications: there we will complete the description of the properties of the model and discuss bond percolation.

We first introduce a probability measure on the configurations of edges in a finite box. This probability measure is called *finite volume random interaction random cluster model*. Let V be a countable set and let E be a subset of the set V_2 of all edges between vertices in V. Let $J = \{a, f\}^E$; in this way a configuration $j \in J$ is a prescription of edges being ferromagnetic (f) or antiferromagnetic (a). A choice of $j \in J$ is made in a way to be specified in the single examples.

Next let $D \subset V$ be a finite set, whose precise form will be specified in the examples, and let $E_D \subset E$ be the set of edges of E which are between vertices in D. Once $j \in J$ is fixed we construct the probability measure corresponding to a finite volume random interaction random cluster model on the configurations of edges described by $H_D = \{0,1\}^{E_D}$. Let $\eta_{\setminus D} \in H_{\setminus D} = \{0,1\}^{E \setminus E_D}$ be a fixed choice of the configuration of the edges outside E_D. Once $\eta_{\setminus D}$ and $\eta_D \in H_D$ are given let $\eta = \eta_{\setminus D} \sqcup \eta_D$ be the joint configuration which coincides with $\eta_{\setminus D}$ in $E \setminus E_D$ and with $\eta_D \in E_D$. Note that the subgraph formed by V and $E' = \{e \in E : \eta_e = 1\}$ can be partitioned into maximal connected components. Consider how many of these components intersect D and denote this number by $cl(\eta_D) cl(\eta_D \sqcup \eta_{\setminus D}) = cl(\eta) =< \infty$. The next step is the "colouring" with q colours of the vertices of D once a colouring of the vertices in $V \setminus D$ is given. The rules for both colourings will be that each vertex is assigned a colour r_i, $i = 1, \ldots, q$, with the following requirements: if $e = \{v_1, v_2\}$ belongs to the subgraph, then the colours in v_1 and v_2 have to be equal if $j_e = f$ and to be different if $j_e = a$; if e does not belong to the subgraph any of the q^2 possible combinations for the colours of v_1 and v_2 is admissible. This can lead to contradictions for some choices of j and η. Once $j \in J$ is given we can choose $\eta_{\setminus D}$ such that at least one colouring is possible and we fix one such possible colouring $\omega_{\setminus D} \in \{r_1, \ldots, r_q\}^{V \setminus D} = \Omega_{\setminus D}$. Next we denote by $U(\eta_D) = U_{j, \eta_{\setminus D}}(\eta_D)$ a function defined on H_D as follows: U assumes value 1 if η_D is *unfrustrated*, i.e. $\eta = \eta_D \sqcup \eta_{\setminus D}$ can be coloured once j is given, and value 0 if η_D is *frustrated*, i.e. η cannot be coloured. The finite volume finite interaction random cluster model is defined by the following measure on H_D:

$$
\begin{aligned}
&P_{\mathcal{P},D}^{j,\eta_{\setminus D},\omega_{\setminus D}}(\eta_D) = \\
(5)\quad &= \left(\prod_{e\in E_D:(\eta_D)_e=1} p_e\right)\left(\prod_{e\in E_D:(\eta_D)_e=0} (1-p_e)\right) q^{cl(\eta_D)} U_{j,\eta_{\setminus D}}(\eta_D) Z_{\mathcal{P}}^{-1}(j,\eta_{\setminus D},\omega_{\setminus D})
\end{aligned}
$$

where $\mathcal{P} = \{p_e\}_{e\in E}$, with $0 \le p_e \le 1$; $j \in J$, $\eta_{\setminus D} \in H_{\setminus D}$, $\omega_{\setminus D} \in \Omega_{\setminus D}$ and $Z_{\mathcal{P}}^{-1}(j,\eta_{\setminus D},\omega_{\setminus D})$ is a normalizing factor.

The probability measure introduced above is defined on H_D and related to a colouring with q colours. Specifying the graph, the set of parameters $\mathcal{P}$, the choice of j, of $\omega_{\setminus D}$, of $\eta_{\setminus D}$ and of q we find independent bond percolation or dependent bond percolation models. The dependent models will give, with a suitable transformation of the measure to construct a site model, the Ising ferromagnets for nearest neighbour and long range interaction, Potts models and spin glasses. We shall discuss this further below.

3.1 The FK model for the Ising ferromagnets.

We now specify the model introduced in the previous section. Let $V = \mathbb{Z}^d$ and let E be the set of edges of Euclidean length one. Let $p_e = p \in (0,1)$, for all $e \in E$. The most relevant choice we make in this section is $j_e = f$ for all $e \in E$. In this way the model becomes entirely ferromagnetic: in fact in this case colours tend either to be equal or to ignore each other, which results in a ferromagnetic attraction between them. The function $U(\eta_D)$ is then identically one because no frustration can arise. We further take D to be a box. With these assumptions the measure defined in (5) becomes a finite volume random cluster measure, which we denote by $P_{p,D}^{\eta_{\setminus D},\omega_{\setminus D}}$.

Before taking weak limits of $P_{p,D}^{\eta_{\setminus D},\omega_{\setminus D}}$ as D diverges, we relate $P_{p,D}^{\eta_{\setminus D},\omega_{\setminus D}}$ to probability measures on $\{r_1,\ldots,r_q\}^D$. For $q = 2$, with $r_1 = 1$ and $r_2 = -1$, we obtain the Ising ferromagnets, and for $q \ge 3$ we obtain the so called Potts models. Let $B = B_n$ be a box and $\Omega_B = \{r_1,\ldots,r_q\}^B$. Fix $\eta_{\setminus B}$. We say that $\omega_{\setminus B}$ is compatible with $\eta_{\setminus B}$, for the given $j \in J$, if it is a possible colouring of the vertices of $V\setminus B$ following the rules described above. Given also $\eta_B \in H_B$, we say that $\omega_B \in \Omega_B$ is compatible with η_B if it is a possible colouring of the vertices of B, once $j, \eta_{\setminus B}, \eta_B$ and $\omega_{\setminus B}$ are given. We denote this last relation by $\eta_B \sim \omega_B$ Next fix $\omega_{\setminus B}$ compatible with $\eta_{\setminus B}$. Define the following measure on Ω_B:

$$
\begin{aligned}
\mu_{B,p}^{\eta_{\setminus B},\omega_{\setminus B}}(\omega_B) &= \sum_{\eta_B:\eta_B\sim\omega_B} P_{p,B}^{\eta_{\setminus B},\omega_{\setminus B}}(\eta_B) q^{-cl(\eta)} = \\
(6)\quad &= Z_p^{-1}(\eta_{\setminus B},\omega_{\setminus B}) \sum_{\eta_B:\eta_B\sim\omega_B} p^{\#\{e\in B:(\eta_B)_e=1\}}(1-p)^{\#\{e\in B:(\eta_B)_e=0\}}
\end{aligned}
$$

where the term $Z_p^{-1}(\eta_{\setminus D},\omega_{\setminus D})$ is a suitable normalizing factor.

If $q = 2$ and $p = 1 - e^{-2\beta}$, then the probability measure in (6) is the finite volume Ising state $\mu^{\overline{\omega}}_{\beta,h,B}$, with external field $h = 0$ and boundary conditions $\overline{\omega}$ restricted to $V \backslash B$ given by $\omega_{\backslash B}$.

If $q = 3, 4, \ldots$ then weak limits of $\mu^{\eta_{\backslash B}, \omega_{\backslash B}}_{B,p}$, for different $\omega_{\backslash B}$ and $\eta_{\backslash B}$, are the probability measures corresponding to the Potts model (see Fortuin and Kasteleyn [1972a, 1972b]).

We now examine some weak limits of finite volume random cluster measures. Let $q = 2$. For $B = B_n$ suppose that $(\eta_{\backslash B})_e = 1$ for all $e \in E \backslash E_B$ and $(\omega_{\backslash B})_v = 1$ for all $v \in V_{\backslash B}$; we define then the *wired* probability measure P^w_p as the weak limit of $P^{\eta_{\backslash B}, \omega_{\backslash B}}_{p,B}$ when $B = B_n$ diverges. If, on the other hand, $(\eta_{\backslash B})_e = 0$ for all $e \in E \backslash E_B$, then we define the *free boundary conditions* probability measure P_p^{free} as the weak limit of $P^{\eta_{\backslash B}, \omega_{\backslash B}}_{p,B}$ when $B = B_n$ diverges; note that in the definition of the free boundary condition probability measure the values assumed by $\omega_{\backslash B}$ are not relevant.

Let a random subgraph be described by one of the random cluster measures P_p^{free} or P^w_p and consider the quantities which are relevant in the theory of percolation. There is a relation between these quantities and those which are involved in the description of the thermodynamical properties of the Ising ferromagnets $\mu_{\beta,0}$ and $\mu^+_{\beta,0}$. We now discuss this relation.

Spontaneous magnetization of $\mu^+_{\beta,0}$, which is defined as the expected value of ω_0, corresponds to the probability of percolation under P^w_p:

$$\mu^+_{\beta,0}(\omega_0) = P^w_p(A_{0,\infty}),$$

if $p = 1 - e^{-2\beta}$.

Correlation functions under $\mu_{\beta,0}$, defined as $\mu_{\beta,0}(\omega_0 \omega_v)$, $v \in V$, are related to connectivity functions of P_p^{free} :

$$\mu_{\beta,0}(\omega_0 \omega_v) = P_p^{\text{free}}(0 \to v) = \Upsilon(v),$$

if $p = 1 - e^{-2\beta}$ (see Fortuin and Kasteleyn [1972a, 1972b], or Aizenman, Chayes, Chayes and Newman [1988]).

These relations show a deep interconnection between percolation and thermodynamical properties. We discuss now the properties of percolation when the random subgraph is described by a random cluster measure with $q = 2$. The case $q = 1$ has been already discussed as independent bond percolation model; on the other hand we do not enter the details of the Potts model, $q \geq 3$.

In particular we consider the wired probability measure P^w_p. Let $V = \mathbb{Z}^d$. Define the critical point $p_c(d) = \sup\{p : P^w_p(A_{0,\infty}) = 0\}$. This value corresponds to the critical temperature $\beta_c(d)$ of the Ising ferromagnets via $p_c(d) = 1 - e^{-2\beta_c(d)}$.

In dimension two $p_c(2)$ is exactly known from the solution of the model proposed by Onsager [1949] (see also Lebowitz [1972]): $p_c(2) = 2 - \sqrt{2}$. In general there are inequalities between the present model and the independent bond model ($q = 1$), stating that $p_c^{bond}(d) \leq p_c(d) \leq \frac{2p_c^{bond}(d)}{2-p_c^{bond}(d)}$ (Fortuin and Kasteleyn [1972a, 1972b]).

An asymptotic expression has been given by Kesten [1988b]: there exists $k > 0$ such that

$$\frac{1}{d} - \frac{k}{d^2} \leq p_c(d) \leq \frac{1}{d} + \frac{k(\log\log d)^2}{d \log d}$$

for all $d \geq 2$.

The *subcritical phase* is characterized by exponential decay of connectivity functions $\Upsilon(n)$: there exists $c > 0$ such that

$$\Upsilon(n) \leq e^{-cn} \qquad \text{for all } n \geq 0,$$

(see Aizenman [1985]). This and many other results are known because of the relation between connectivity functions and correlations of the Ising ferromagnets.

In the *supercritical phase* the infinite cluster is unique. This *per se* does not seem to correspond to any thermodynamical property, but has consequences for the connectivity functions. A proof of the uniqueness of the infinite cluster can be obtained from Burton and Keane [1988], or from Gandolfi, Keane and Newman [1989] below. Uniqueness implies that the connectivity functions do not decay:

$$\Upsilon_p(n) \geq (P_p^w(A_{0,\infty}))^2 \qquad \text{for all } n \geq 0,$$

because the measure P_p^w satisfies the FKG correlation inequality.

Truncated connectivity functions $\Upsilon_p^f(n)$ are known to decay exponentially if the dimension is two: there exists $c = c(p) > 0$ such that

$$lim_{n\to\infty} \frac{-1}{n} \log \Upsilon_p^f(n) = c(p)$$

(Chayes, Chayes and Schonmann [1987]).

3.2 FK models for long range Ising ferromagnets.

Although it has not been done above, it is clear that, given an interaction Φ it is possible to define Ising ferromagnets when the graph consists of $V = \mathbb{Z}$ and E consists of the set of all edges V_2. It is also possible to derive such a model of Ising ferromagnets from a certain random cluster model; we do not perform the derivation here, but we study instead the random cluster model.

Let a finite volume random cluster measure $P_{\mathcal{P},n}^{\eta\setminus D,\omega\setminus D}$ be defined as in (5) with $V = \mathbb{Z}$, $E = V_2$, D consisting of a box and $j_e = f$ for all $e \in E$. The probabilities $p_e \in \mathcal{P}$ depend only on the length of $e = \{v_1, v_2\}$: $p_e = p_{|v_1 - v_2|}$. Suppose furthermore that they satisfy $p_n = \beta/n^s$, for $n \geq 2$, and for all β and $s \in \mathbb{R}^+$; let us denote p_1 by p. If $q = 1$ then this is the independent long range percolation model discussed in section 1.4 above. This fact, together with inequalities relating the models for different values of q, allowed the solution of many problems concerned with the long range Ising ferromagnets, which is obtained with $q = 2$. In spite of the historical order we discuss only the properties of percolation when the random subgraph is described by a random cluster probability measure with $q = 2$, specifically the wired probability measure $P^w_{s,\beta,p}$. The probability measure $P^w_{s,\beta,p}$ is obtained as a weak limit, as D diverges, of the finite volume random cluster measures $P_{\mathcal{P},n}^{\eta\setminus D,\omega\setminus D}$, with the assumptions described above and the following boundary conditions: $(\eta_{\setminus D})_e = 1$ for all $e \in E\setminus E_D$ and $(\omega_{\setminus D})_v = 1$ for all $v \in V\setminus D$ (see Aizenman, Chayes, Chayes and Newman [1988]).

The behaviour of the functions describing the percolation phenomenon does not differ very much from the corresponding behaviour in the independent case.

If $s \leq 2$ then there exists $\beta_c = \beta_c(s) > 0$ such that for $\beta > \beta_c$ there exists $p_c(\beta) > 0$ such that percolation occurs for $p > p_c(\beta)$ and does not for $p < p_c(\beta)$ (see Dyson [1969] for $s < 2$; Aizenman, Chayes, Chayes and Newman [1988], Newman and Schulman [1986] and Imbrie and Newman [1988] for $s = 2$; see also Berbee [1989]); for $s = 2$, which is the more interesting case, the critical value is $\beta_c(2) = 1$.

If $s > 2$ then there is no percolation for any value of β and p (Dobrushin [1968], Ruelle [1968]).

The decay of connectivity functions in the *subcritical phase* has been described for $q = 2$ and $s = 2$ by Aizenmam, Chayes, Chayes and Newman [1988]:

$$\lim_{n\to\infty} -\log(\Upsilon(n)/\log n) = 2.$$

Therefore the expected cluster size is finite.

If $s = 2$ and $\beta = 1$ there is a discontinuity in the percolation probability as a function of p at $p = p_c(1)$, in the sense that the probability of percolation is strictly positive at $p_c(1)$ (Aizenman, Chayes, Chayes and Newman [1988]).

The *supercritical phase* is characterized for all values of the parameters for which percolation occurs by a unique infinite cluster: this is proved in Gandolfi, Keane and Newman [1989] below. Since the FKG correlation inequality holds, the connectivity functions are bounded below: $\Upsilon(n) \geq \vartheta^2(s,\beta,p) > 0$, where $\vartheta(s,\beta,p)$ indicates the probability of percolation. For $s = 2$ the peculiar behaviour of the truncated connectivity functions conjectured in the independent case has been proved: for all $\varepsilon > 0$ there are values of β, p and $c > 0$ such that $\Upsilon^f(n) \geq c/n^\varepsilon$ (Imbrie and Newman [1988]), but for other values of β and p there is $c > 0$ such that $\Upsilon^f(n) \leq c/n^2$ for all $n \geq 1$ (Imbrie [1982]).

3.3 FK models for spin glasses.

In this section the (finite volume) random interaction random cluster measure is considered in its full generality. Let $V = \mathbb{Z}^d, d \geq 2$, or $V = \mathbb{Z}^d \times \{0, 1, \ldots, m\}^k$, with k and $m \geq 1$.

We consider either nearest neighbour models, whose edges are those of Euclidean length one, or long range models, with $E = V_2$.

The choice of $j \in J$ is random. For instance we can assume that each $j_e = f$ with probability 1/2, independently of $j_{e'}$ for the other $e' \in E$. This corresponds to a Bernoulli measure $m_{1/2}$ on $\{a, f\}^E$. For each given $j \in J$ a finite volume random interaction random cluster measure is constructed, according to (5), for a given compatible choice of boundary conditions. Then weak limits are taken when D diverges. These weak limits correspond to random cluster measures for the given $j \in J$ (see §4 of Gandolfi, Keane and Newman [1989], below, for a more detailed account).

The theory of spin glasses deals with a probability measure on $\{-1, 1\}^V$. This probability measure is such that there is a randomly chosen interaction between neighbour vertices; this interaction favours either equal values or opposite values of the configuration in two neighbour vertices. We consider two vertices neighbouring each other if they is an edge between them. We next take $q = 2$. The same procedure which has been used to derive the Ising ferromagnets from the random cluster model (with $j_e \equiv f$) can be applied now to derive models of spin glasses from the random cluster measure described at the beginning of this section. For each $j \in J$ we obtain a measure on $\{-1, 1\}^V$; the randomness in the interaction comes from the random choice of j. We do not give further details since here the interest is in the percolation properties of the random cluster measure. These are related, even if less directly than in the case of the Ising ferromagnets, to the thermodynamical properties of the induced measure on $\{-1, 1\}^V$ (see, for instance, Newman [1988a]).

Given $j \in J$, boxes $B = B_n$, $\eta_{\setminus B} \in H_{\setminus B}$ and the probabilities $\mathcal{P} = \{p_e\}_{e \in E}$, let $P^j_{\mathcal{P}}$ be a weak limit, as B_n diverges, of the finite volume measures defined in (5). In general these weak limits are not invariant under the translations of $\mathbb{Z}^d$.

Before discussing the properties of percolation in the bond percolation model described by $P^j_{\mathcal{P}}$ let us remark that the possibility that η is frustrated, once j is given, makes the transition to percolation more difficult. In fact frustration occurs more easily if we try to consider configurations corresponding to subgraphs containing "many" edges; on the other hand subgraphs in which percolation occurs are, in some sense, likely to contain "many" edges.

In Gandolfi, Keane and Newman [1989] below we give an inequality which implies the occurrence of percolation for certain random cluster models related to spin glasses. Consider a nearest neighbour model with $V = \mathbb{Z}^d$, $d \geq 3$, or with $V = \mathbb{Z}^2 \times \{0, \ldots, m\}$, $m \geq 1$ and let p be the parameter describing the model (i.e. $p_e = p$ for all $e \in E$). If p is taken to be large enough then percolation occurs.

The same inequality can be easily extended to the long range case, where p_e depend on $e \in E$. This extension, together with Theorem 1 in the same article, implies that in the supercritical phase of $P_{\mathcal{P}}^{j}$ the infinite cluster is unique for $m_{1/2}$-almost all $j \in J$, provided that the integrated measure $P_{\mathcal{P}}^{j} m_{1/2}(dj)$ is translation invariant under T_v, for all $v \in V = \mathbb{Z}^d$. Note that the measures $P_{\mathcal{P}}^{j}$ are not assumed to be translation invariant, neither is translation invariance assumed for $j \in J$. Examples and details are given in Gandolfi, Keane and Newman [1989], below.

An easy extension of the above-mentioned Theorem 1 covers the case of $V = \mathbb{Z}^2 \times \{0, 1, \ldots, n\}$; for instance, take all the vertices of the set R in the last part of the proof of Theorem *1 to* be in $\mathbb{Z}^2 \times \{0\}$, then suitably increase the constant k to compensate for m: in this way it is still possible to prove the Theorem by contradiction) and shows that the infinite cluster is unique in this case too.

There are no results available about the occurrence of percolation in dimension two.

4. General dependent percolation models: uniqueness of the infinite cluster

Motivated by the examples of the previous sections we now discuss random subgraphs described by probability measures which satisfy some general property. Some of these properties will be sufficient to ensure that there is at most one infinite component in the random subgraph. For some other of these properties we show examples of measures satisfying them in which two or more infinite clusters coexist with positive probability. Roughly speaking, we will conclude that non coexistence of infinite clusters is ensured by good properties of the conditional probabilities and not at all by good mixing properties. In section 4.1 we also discuss the boundary/volume ratio of the infinite cluster in the models satisfying the FKG correlation inequality.

4.1. The FKG correlation inequality.

We first consider uniqueness of the infinite cluster in two-dimensional models. Two general results are available if $V = \mathbb{Z}^2$. The first one refers to a property which ensures uniqueness of the infinite cluster for both nearest neighbour and long range percolation models: it will be discussed in the next section. The second result refers to the FKG property, is based on geometrical arguments and applies to two-dimensional nearest neighbour models only: we treat it here below.

Let $V = \mathbb{Z}^2$ and let E be the set of edges of Euclidean length one. The result we now mention applies to both site and bond percolation models and we generically denote the probability measure in one of these models by P.

Suppose P satisfies the FKG correlation inequality and is invariant both under the translations T_v, for all $v \in V$, and under the maps induced on the measure by the reflections of $\mathbb{Z}^2$ with respect to the axis. Suppose further that P is ergodic for both translations $T_{(1,0)}$ and $T_{(0,1)}$ along the x-axis and the y-axis, *separately*. If there is an infinite cluster in the random subgraph described by P then it is unique. Furthermore each vertex $v \in V$ is surrounded by infinitely many circuits with probability one and the connectivity functions are bounded below by the square of the probability that the origin percolates:

$$\tau_P(n) \geq P^2(A_{0,\infty}) > 0 \qquad \text{for all } n \geq 1.$$

These results are the contents of Gandolfi, Keane and Russo [1988] below.

We next consider the boundary/volume ratio of the infinite cluster. For simplicity we discuss site percolation models on the graph $\mathcal{G} = (V, E)$, with $V = \mathbb{Z}^d$ and E consisting of the edges of Euclidean length one. Suppose the random subgraph is described by a probability measure P invariant under T_v for all $v \in V$ and satisfying the FKG correlation inequality. Consider the random variables R_n. In section 1.2 we have seen that if P is the Bernoulli probability measure with parameter p then $\lim_{n\to\infty} R_n = (1-p)/p$. In the present case we can prove a one-sided inequality.

THEOREM. *Let P be a probability measure on $\Omega = \{-1,1\}^{\mathbf{Z}^d}$. Suppose P is invariant under T_v for all $v \in V$ and satisfies the FKG correlation inequality. Then the following limit*

$$lim_{n\to\infty} R_n = R$$

exists P – almost everywhere. The random variable R is non-negative, constant P – almost everywhere if P is ergodic and satisfies

$$R \leq \frac{(1-q)}{q} \quad P - \text{almost everywhere},$$

where q indicates the probability $P(\omega_0 = 1) = P(E_0^1)$.

PROOF:

If P is not ergodic it is enough to prove the result for all ergodic probability measures in the ergodic decomposition of P; the properties of the ergodic decomposition ensures then that the result holds for P as well. Therefore we suppose P to be ergodic.

Denote the infinite cluster by C and its boundary by ∂C. Then apply the ergodic theorem to the indicator functions I_C and $I_{\partial C}$ of the events $A_\infty = \{0 \in C\}$ and $B_\infty = \{0 \in \partial C\}$. This implies that

$$lim_{n\to\infty}\frac{|C\cap B_n|}{|B_n|} = lim_{n\to\infty}\frac{1}{|B_n|}\sum_{v\in B_n} I_c o T_v^{-1} = P(A_\infty)$$

and

$$lim_{n\to\infty}\frac{|\partial(C\cap B_n)\cap B_n|}{|B_n|} = lim_{n\to\infty}\frac{1}{|B_n|}\sum_{v\in B_n} I_{\partial C} o T_v^{-1} = P(B_\infty)$$

hold P-almost everywhere. Therefore R_n converges P-almost everywhere to the constant $P(B_\infty)/P(A_\infty)$. We estimate this limit. Let A'_∞ be the event depending on $\mathbf{Z}^d\backslash\{0\}$ that the infinite cluster C is at distance one from $\mathbf{0}$, in other words A'_∞ is the event that one of the nearest neighbour vertices of $\mathbf{0}$ percolates.

The FKG correlation inequality implies that

$$P(E_0^{-1}\cap A'_\infty) \leq P(E_0^{-1})P(A'_\infty)$$

and

$$P(E_0^1\cap A'_\infty) \leq P(E_0^1)P(A'_\infty).$$

Therefore

$$\begin{aligned}\lim_{n\to\infty} R_n &= R = P(B_\infty)/P(A_\infty)\\ &= P(E_0^{-1}\cap A'_\infty)/P(E_0^1\cap A'_\infty)\\ &\leq P(E_0^{-1})/P(E_0^1) = (1-q)/q.\end{aligned}$$

4.2. Positive finite energy.

We now state a condition which allows us to prove *uniqueness* of the infinite cluster for random graphs generated by a wide class of probability measures. Results hold for site as well as for bond percolation models, both for nearest neighbour and long range models. Let $V = \mathbb{Z}^d$ and E be a subset of V_2 which is invariant under T_v for all $v \in V$. To simplify the discussion we refer to bond percolation models and therefore we consider a probability measure defined on $\{0,1\}^E$.

We say that P satisfies the ***positive finite energy condition*** for $e \in E$ if the conditional probabilities of $\eta_e = 1$, given the σ-algebra $\mathcal{A}_{\setminus e}$ generated by the cylinders with base in $E\setminus\{e\}$, are positive for almost all $\eta_{\setminus e} \in H_{\setminus e} = \{0,1\}^{E\setminus\{e\}}$:

$$P(\eta_e = 1|\mathcal{A}_{\setminus e})(\eta_{\setminus e}) > 0 \qquad \text{for } P\text{-almost all } \eta_{\setminus e} \in H_{\setminus e}.$$

If a probability measure P satisfies the positive finite energy condition for all $e \in E$ and is invariant under T_v for all $v \in V$ then there is at most one infinite cluster in the random graph described by P (Gandolfi, Keane and Newman [1989] below). Furthermore the connectivity functions are bounded away from zero: there exists $c > 0$ such that $\Upsilon(n) \geq c$ for all $n \geq 1$.

The proof of the above mentioned result has been achieved in various steps. Newman and Schulman [1981] consider a probability measure which is ergodic under the full group of translations of $\mathbb{Z}^d$ and satisfies the ***finite energy condition***, i.e. besides the conditional probabilities that $\eta_e = 1$ also the conditional probabilities that $\eta_e = -1$ are positive. They show that under these conditions if $n \geq 2$ there cannot coexist n distinct infinite clusters with positive probability, but they could not exclude that with positive probability infinitely many infinite clusters occur.

Burton and Keane [1988] proved that in a nearest neighbour model there is at most one infinite cluster in the random subgraph described by a translation invariant probability measure which satisfies the finite energy condition. Their argument shows that if two (and hence many, as proved in Newman and Schulman [1981]) distinct infinite clusters occur, then the surface of an cube can not accommodate all the disjoint open paths which are nevertheless forced by the regularities of the measure to intersect that surface.

In Gandolfi, Keane and Newman [1989] below, this argument is extended to show that the volume of the cube itself in not sufficient to accommodate the disjoint open paths still forced by the geometrical properties of the random subgraph to intersect it. This shows, then, that there can be at most one infinite cluster in random subgraphs described by a stationary probability measure satisfying the positive finite energy condition, both for nearest neighbour and for long range percolation models.

The proof that connectivity functions do not decay is a simple consequence of the non-coexistence of more than one infinite cluster and can be rigorously proved as in Gandolfi [1987], Proposition 1, below.

The previous results are further extended in Gandolfi, Keane and Newman [1989] below, in the sense that it is shown how to prove uniqueness of the infinite cluster in the random subgraph described by a probability measure for which percolation occurs and which satisfies weaker conditions. In fact Theorem 1 in the mentioned paper only assumes that $V = \mathbb{Z}^d \times \mathbb{Z}_+^k$, for $d \geq 1$ and $k \in \{0,1\}$, and that the set of edges $e \in E$ for which the probability measure satisfies the positive finite energy condition is large enough to connect every pair of vertices in V.

A different direction is explored in Theorem 1′ in Gandolfi, Keane and Newman [1989] below. If a graph "grows too rapidly" then coexistence of two or more infinite clusters can occur with positive probability: examples which illustrates this statement are given in the next section, while the above-mentioned Theorem 1′ states conditions of "sufficiently slow growth" of the graph which are sufficient, together with finite energy and invariance under suitable translations, to yield non-coexistence of two or more infinite clusters.

4.3 Ergodicity, mixing, one-dependence and trees: contrexamples to uniqueness.

We have seen that translation invariance and suitable properties of the conditional probabilities of the probability measure describing the random subgraph are sufficient to ensure non-coexistence of infinite clusters. On the other hand there are many examples in which more than one infinite cluster occur with positive probability.

Let $V = \mathbb{Z}^2$ and consider nearest neighbour site percolation models. There are examples satisfying all conditions given in Gandolfi, Keane and Russo [1988] below, except separate ergodicity or except the FKG correlation inequality, for which more than one infinite cluster occur with probability one. Such an example of a probability measure violating the FKG correlation inequality is given in Burton and Keane [1989]. Here we describe a simple example of a probability measure which violates separate ergodicity. In particular this probability measure satisfies the FKG correlation inequalities but not the (positive) finite energy condition. We describe a nearest neighbour site percolation model in $\mathbb{Z}^2$. Choose independently of each other the values of ω_v for $v = (n,0)$ and when this is done for all $n \in \mathbb{Z}$ just copy the first line on all the other lines, in such a way that $\omega_{(n,m)} = \omega_{(n,0)}$ for all $m \in \mathbb{Z}$ (see Gandolfi, Keane and Russo [1988] below) In this model there are infinitely many infinite clusters with probability one. The probability measure is ergodic under the full group of translations, even though not with respect to the translation $T_{(0,1)}$.

Other examples of models depending on one parameter whose probability measure is separately ergodic and such that for some values of the parameter infinitely many infinite clusters occur with positive probability are the so called ergodic percolation models (Meester [1988]).

Not only ergodicity, but also strong mixing properties are not sufficient to ensure uniqueness of the infinite cluster. The strongest possible mixing property (which is not independence) is satisfied by probability measures which are *one-dependent* (for a review see de Valk [1988]). Let us define this concept for a probability measure defined on $\{0,1\}^E$, where E is the set of edges of Euclidean length one in $\mathbb{Z}^d$. Two subsets of

E are considered to be at distance greater than one apart if no edges in the first set can share a vertex with edges in the second set. We say that a probability measure is one-dependent if for any two sets S_1 and S_2 contained in E which are at distance greater than one apart, two events A_1 and A_2 are independent if they belong to the σ-algebras generated by the cylinders with base in S_1 and S_2 respectively. Examples of random graphs described by a one-dependent probability measure such that there are $n \geq 2$ infinite clusters with probability one are mentioned in the introduction of Gandolfi, Keane and Newman [1989],pg. 3, below. These probability measures are invariant under T_v for all $v \in \mathbb{Z}^d$, but they do not satisfy positive finite energy.

These examples and the results discussed in the previous sections show that in general uniqueness of the infinite cluster in a random graph is ensured by suitable properties of the conditional probabilities, which are in some sense Markov properties, and not at all by good mixing properties.

In an other direction it is possible to define a random subgraph on an infinite labelled tree T with degree k. Both site and bond percolation models on the graph described by the tree are possible. For instance let us take a site percolation model in which vertices belong to the random subgraph with probability p, independently of each other. Then the existence of an infinite cluster corresponds to the survival of a branching process and the values of many functions can be computed rigorously (see, for instance, Harris [1961]).

We are here interested in the number of infinite clusters. One can easily see that when percolation occurs in the random subgraph of a tree than there are infinitely many clusters with probability one.

In other models, such as $\mathbb{Z} \times T$, the cartesian product of $\mathbb{Z}$ with a tree T, the number of infinite clusters depend on the value of the parameters involved in the description of the model. In the case of $\mathbb{Z} \times T$ with independent choice of each edge, this number is either zero, one or infinity (see Grimmett and Newman [1988]).

A reason for the appearance of infinitely many infinite clusters is the following. Given a connected graph $\mathcal{G} = (V, E)$ we define distance on the graph between two vertices $v_1, v_2 \in V$ the number of edges in the shortest path connecting them. Let v be a fixed vertex in V; let B_n be the set of vertices at distance smaller than n from v and ∂B_n the set of vertices at distance n from v. Trees and similar sets are such that $lim_{n \to \infty} \frac{|\partial B_n|}{|B_n|} > 0$; in other words the surface ∂B_n of B_n is of the same order of B_n, as n tends to infinity. On the other hand the above mentioned limit is zero for $V = \mathbb{Z}^d$ or similar sets. It is exactly this slow growth rate of the graph that prevent the coexistence of more than one infinite cluster (see also Theorem 1′ in Gandolfi, Keane and Newman [1989] below).

5. Random graph theory

In this section we discuss conditions which ensure that a random graph is totally connected.

We consider bond percolation models on a graph $\mathcal{G} = (V, E)$.

IN the first case we examine the graph consists of the set of sites $V = \mathbb{Z}^d$ and of the set $E = V_2$ of all pairs of edges between vertices in V. Suppose that each edge $e = \{v_1, v_2\} \in E$ belongs to the random subgraph E' independently of the other edges and with probability $p_e = p_{\{v_1,v_2\}} = p_{|v_2-v_1|}$ which depends only on the Euclidean length $|e| = |v_2 - v_1|$ of e.

If $p_{\{0,v\}} < 1$ for all $v \in V$, and

$$\sum_{v \in V} p_{\{0,v\}} < \infty \tag{6}$$

then each vertex is connected by edges incident to it to finitely many vertices only. We say that a vertex is directly connected to another if the edge between them belongs to the subgraph. If the sum in (6) is finite then there is a positive probability that the origin 0 is not directly connected to any other vertex, and therefore is isolated. The probability measure is translation invariant and ergodic under T_v, for all $v \in V$, and therefore there are infinitely many isolated vertices with probability one.

If the sum in (6) is infinite then each vertex is directly connected to infinitely many other vertices and the graph is totally connected with probability one. This was first proved in Grimmett, Keane and Marstrand [1984]. It is also an easy consequence of Gandolfi, Keane and Newman [1989] below. In fact this is a long range percolation model, the probability measure describing the graph is invariant under T_v for all $v \in V$, and connectedness of the graph is equivalent to uniqueness of the infinite cluster.

We now discuss generalized versions of this result. As before let E be the set of all possible edges between vertices in V and let e be in E' with probability $p_e < 1$ depending only on the Euclidean length of $e = \{v_1, v_2\}$. The graph is totally connected if and only if the sum in (6) is infinite for the following choices of V: V is an half space, i.e. $V = \mathbb{Z}^d \times \mathbb{Z}_+, d \geq 1$ (this is again an easy consequence of Theorem 1 in Gandolfi, Keane and Newman [1989] below) or V is an orthant, i.e. $V = \mathbb{Z}^d \times \mathbb{Z}_+^m, d \geq 1$ and $m \geq 0$ (Kesten [1989]). A proof of this last statement is based on induction, starting from $V = \mathbb{Z}_+$, for which the equivalence of divergence in (6) and connectivity of the graph was first proved by Kalikow and Weiss [1988]. A different proof of Kalikow and Weiss' result is a consequence of Theorem 2 in Gandolfi, Keane and Newman [1989] below, in which it is more generally proved that the infinite cluster is unique.

In the random subgraphs considered above each edge e belongs to E' independently of the others and with a probability which is a function of the Euclidean length of e. There are interesting models with $V = \mathbb{Z}^d$ for which independence still holds, but with probabilities depending also on the position of the edge, for instance on the vertex with smallest distance from the origin; some of these models have been completely solved recently (see Kalikow and Weiss [1988], Shepp [1988] and Durrett and Kesten [1989]).

We consider the opposite direction and discuss dependent models in the case $V = \mathbb{Z}_+$. In this case we can not use the Borel-Cantelli Lemma and condition (6) does not necessarily implies that the origin is directly connected to infinitely many other vertices with probability one. Nevertheless it might happen that for some reasons we have this information. In this case uniqueness of the infinite cluster implies connectedness or the random graph. To make this section self-contained we recall here a set of conditions sufficient for the graph to be totally connected. Let the random subgraph be described by a probability measure P which is invariant under the maps T_v, for all $v \in V = \mathbb{N}$. Suppose he measure P is such that the events $\{\eta_e = 1\}$ and $\{\eta_{e'} = 1\}$, corresponding to $e \in E'$ and $e' \in E'$, are not necessarily independent. Suppose furthermore P satisfies positive finite energy for a set $\overline{E} \subseteq E$ of edges such that any two vertices are connected by a chain whose edges are in $\overline{E}$. If the origin is connected to infinitely many vertices with probability one, then the graph is totally connected with probability one.

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ON THE UNIQUENESS OF THE INFINITE OCCUPIED CLUSTER IN DEPENDENT TWO-DIMENSIONAL SITE PERCOLATION

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We consider dependent site percolation on the two-dimensional square lattice, the underlying probability measure being invariant and ergodic under each of the translations and invariant under axis reflections. If this measure satisfies the FKG condition and if percolation occurs, then we show that the infinite occupied cluster is unique with probability 1, and that all vacant star-clusters are finite.

1. Introduction. Consider a probabilistic situation in which each of the sites of the two-dimensional square lattice (i.e., each point in Z^2) is either occupied or vacant, the stochastic nature being specified by a probability measure μ on the set of all such configurations. Regarding the nearest-neighbour bonds (i.e., the line segments of length 1 joining two points of Z^2) as connections, the set of occupied sites of a given configuration falls apart into maximal connected subsets called occupied clusters. The theory of site percolation on the square lattice deals with the description of these clusters.

In this article we shall be concerned with the number N of occupied clusters that contain an infinite number of sites. Clearly, N is a random variable invariant under the group of transformations of configuration space induced by the group of translations of Z^2. If we restrict our attention to those probability measures which are ergodic under this group, this is sufficient to ensure that N is constant with probability 1. Then if N is not 0, we say that percolation occurs.

The first investigations in percolation theory were of Bernoulli percolation, which arises when each site is occupied with probability p and vacant with probability $1 - p$, independent of the other sites. Then there is a critical value p_c for the parameter p, strictly between 0 and 1, below which N is 0 and above which N is nonzero. More recently, it has been shown that N is 0 for $p = p_c$ [14]. The exact value of p_c is unknown; the best lower bound is 0.503478 ([15]) and heuristic calculations ([4]) indicate that p_c is approximately 0.59. We remark that, for the same model in higher dimensions, the value of N at p_c is not known.

Early in the development of percolation theory, Harris [9] showed for a model similar to the preceding one, the independent bond percolation model on Z^2, that $N = 1$ above p_c. Fisher [6] then noted that Harris' techniques work equally well for independent site percolation on Z^2. Later, an article by Coniglio, Nappi, Peruggi and Russo [3], based on ideas in Miyamoto [12], led to $N = 1$ when μ

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describes a percolating Ising model with no external field. General references containing many recent results on independent percolation are Kesten [10] and [11].

Our goal in this article is to extend these uniqueness results to a wide class of probability measures μ. More precisely, suppose that

(1) μ is invariant under horizontal translation, vertical translation, horizontal axis reflection and vertical axis reflection;
(2) μ is ergodic with respect to horizontal translation and vertical translation (separately);
(3) increasing events are positively correlated under μ; and
(4) N is nonzero.

(The second condition implies that N is constant with probability 1; the third is the so-called ferromagnetic or FKG condition [7]).

Then we shall show that $N = 1$ with probability 1. In fact, we obtain, as Harris did, a bit more, namely, that any finite set of sites is surrounded by an occupied circuit with probability 1. Now, if we consider two points star-connected when their distance is less than or equal to $\sqrt{2}$, we can define the connected components of the set of vacant sites as vacant star-clusters. Our result implies that all vacant star-clusters are finite with probability 1.

Examples of measures satisfying our conditions are given by extremal Gibbs states in Z^2, in particular, Ising states with nonzero external fields, and restrictions of higher-dimensional Ising states to a suitable plane. However, our results are quite general and the question arises as to whether the set of given conditions is minimal, i.e., it is not possible to omit any single condition and obtain the same result. In this direction, it is not hard to see that the separate ergodicity is a necessary part of the set of conditions, as the ergodicity under the whole group is not sufficient. We do not know whether the FKG condition can be omitted.

We mention three related results. In [13], Newman and Schulman have shown, under conditions more general than ours and for arbitrary dimensions, that the only possible values for N are 0, 1 and ∞. For Bernoulli percolation in all dimensions, a relation between uniqueness and qualitative properties of thermodynamic functions was first shown in [16]; more recently the combined work of Aizenman, Kesten and Newman [1] has strengthened this relation and has yielded a proof of uniqueness. A corresponding result under our conditions may well also hold, but other methods will certainly be necessary for this.

2. Preliminaries. We begin by fixing our notation for points and subsets of the discrete plane. Let $P = Z^2$ and set

$$0 = (0,0),$$

$$x_m = (m,0), \qquad m \in Z,$$

$$y_n = (0,n), \qquad n \in Z.$$

We shall need the following subsets of Z^2, described in terms of points $z = (z_1, z_2)$

belonging to them:

(1) The upper and lower half-planes

$$H^+ = [z_2 \geq 0]$$

and

$$H^- = [z_2 \leq 0].$$

(2) The horizontal strips

$$Q_m = [-m \leq z_2 \leq m], \qquad m \geq 0.$$

(3) The horizontal line and half-lines

$$L_m = [z_2 = m],$$
$$L_m^+ = [z_2 = m,\ z_1 \geq 0],$$
$$L_m^- = [z_2 = m,\ z_1 \leq 0], \qquad m \in Z.$$

(4) *Boxes* B of the form $[-m \leq z_1, z_2 \leq m]$, $m \geq 0$. The set of all such boxes is denoted by $\mathbb{B}$.
(5) *Chains*, defined to be finite or infinite sequences of elements of P whose successive terms are at distance 1 from each other. A chain is *self-avoiding* if any two of its elements are distinct, and a finite chain is a *circuit* if its first and last element are at distance 1 from each other. A chain is said to *join* two (points or) subsets if it contains elements belonging to each, and it joins a subset to ∞ if it contains an infinite number of distinct terms, one of which belongs to the subset.
(6) *Star-chains*, obtained by replacing "distance 1" in (5) by "distance 1 or $\sqrt{2}$."

In the following we shall make no distinction between z and $\{z\}$ for notational convenience, and occasionally we shall confuse a chain with the curve in R^2 obtained by connecting successive elements of the chain to each other with straight line segments. We shall also make use of the elementary properties of the *index* or *winding number* of a finite chain C around a point $z \in P$ not belonging to C. The index, denoted by

$$i(C, z),$$

is intuitively defined as $1/2\pi$ times the total change in the angle of the vector $z' - z$ as z' proceeds from the initial point of the chain to the final point along the curve corresponding to the chain. For the precise definition and the elementary properties, we refer to Beardon [2].

Next we introduce the probability space

$$\Omega = \{0,1\}^P,$$

provided with the σ-algebra $\mathbf{A}$ generated by all finite cylinders and a fixed probability measure μ. An element $\omega = (\omega_z)_{z \in P} \in \Omega$ is a *configuration*; $z \in P$ is said to be *vacant* or *occupied* in ω according to whether $\omega_z = 0$ or $\omega_z = 1$. An event $A \in \mathbf{A}$ is *increasing* if $\omega \in A$ and $\omega \leq \omega'$ imply $\omega \in A$, where "$\leq$" denotes the coordinatewise partial ordering on Ω, and *decreasing* if the complement of A is increasing.

Our notation for connection by occupied site chains will be as follows. Let U, V and W be subsets of P, with possibly also $V = \infty$. Then

$$[U, V; W]$$

denotes the set of all configurations ω for which U and V are joined by a chain each of whose elements is occupied in ω and belongs to W. If $W = P$, then we simply write

$$[U, V].$$

It is easily seen that the occupied chain joining U and V may be assumed to start in U, to end in V, and to be self-avoiding.

If $U \subseteq P$ is finite, then U denotes the set of all configurations ω for which there exists a self-avoiding circuit C (disjoint from U) each of whose elements is occupied in ω and such that for each $z \in U$, $i(C, z) = +1$. It is well known that this is equivalent to requiring that U is not star-joined to ∞ by a chain of vacant elements (see, e.g., [10]).

If $\omega \in \Omega$, an *occupied cluster* in ω is a subset of P such that any two points of the subset can be joined by a chain all of whose elements are occupied, and such that it is maximal with respect to this property. The set of occupied clusters forms a partition of the set of occupied sites of ω, and we denote by $N(\omega)$ the number of occupied clusters in ω that contain an infinite number of sites.

Points and subsets of P, configurations and events in Ω, will be moved around using the horizontal and vertical translations

$$S(z) = z + (1,0),$$

$$T(z) = z + (0,1).$$

Note that S translates points and subsets of P to the right, but configurations and events to the left:

$$(S\omega)_z = \omega_{S(z)}, \qquad z \in P.$$

T acts in a similar fashion.

We can now state our assumptions A concerning the probability measure μ on $(\Omega, \mathbf{A})$.

(A.1) μ is invariant under horizontal and vertical translations and axis reflection.
(A.2) μ is ergodic (separately) under horizontal and vertical translation.
(A.3) For any increasing events E and F,

$$\mu(E \cap F) \geq \mu(E)\mu(F).$$

(A.4) $0 < \mu([0, \infty]) < 1$.

A few remarks are useful. Assumption (A.2) is more than enough to imply that N is constant with probability 1, and assumption (A.4) rules out $N = 0$, as well as the trivial measure for which all sites are occupied with probability 1. Assumption (A.3), the ferromagnetic or FKG condition, implies the same inequality if E and F are both decreasing, and the reverse inequality if one is decreasing and the other increasing, as follows from the definition of decreasing events.

THEOREM. *If assumptions* (A.1)–(A.4) *hold, then*

$$\mu(N = 1) = 1.$$

Moreover, any finite set of sites is surrounded by an occupied circuit with probability 1 *and, equivalently, all vacant star-clusters are finite with probability* 1.

The main part of the proof will be given in the next section. The remainder of this section will be devoted to preparations for the proof. Our first lemma allows us to prove a bit less. In the sequel we always assume that (A.1)–(A.4) hold.

BOX LEMMA. *Suppose that there exists a positive number δ such that for each box $B \in \mathbb{B}$,*

$$\mu(\mathbf{B}) \geq \delta.$$

Then any finite set of sites is surrounded by an occupied circuit with probability 1 *and*

$$\mu(N = 1) = 1.$$

PROOF. Any finite set is contained in a box $B \in \mathbb{B}$ and the event $\mathbf{B}$ decreases as the size of B increases. Therefore,

$$\mu\Bigl(\bigcap_{B \in \mathbb{B}} \mathbf{B}\Bigr) \geq \delta,$$

and since this event is translation invariant, it has by ergodicity measure 1, and only $\mu(N = 1) = 1$ remains to be shown. Let $z, z' \in P$. Then the set $\{z, z'\}$ is surrounded by an occupied circuit with probability 1, so the probability that z and z' belong to different infinite occupied clusters is 0. Since there are only countably many pairs $z, z' \in P$, we conclude that $\mu(N = 1) = 1$. □

Next we show that percolation cannot occur in a strip.

STRIP LEMMA. *For each $z \in Q_m$ we have*

$$\mu([z, \infty; Q_m]) = 0.$$

PROOF. By (A.4), vacant sites occur with positive probability, and then by (A.3) line segments of vacant sites also occur with positive probability, since occurrence of a vacant site is a decreasing event. Hence by ergodicity of S the strip Q_m will be closed off infinitely often in each direction by a line segment of vacant sites with probability 1, and the lemma follows. □

The following lemma is ergodic–theoretic in nature and is a mild adaptation of an interesting theorem of Furstenberg [8].

MULTIPLE ERGODIC LEMMA. *If* A_0, A_1 *and* A_2 *are monotonic* (*i.e., increasing or decreasing*) *events, then*

$$D\text{-}\lim_{N\to\infty} \mu(A_0 \cap S^{-N}A_1 \cap S^{-2N}A_2) = \mu(A_0)\mu(A_1)\mu(A_2),$$

where $D\text{-}\lim_{N\to\infty}\alpha_N = \alpha$ *if* α_N *tends to* α *along a sequence of density* 1.

PROOF. First suppose that $A_2 = \Omega$. Then by ergodicity of S,

$$\lim_{N\to\infty} N^{-1} \sum_{n=0}^{N-1} \mu(A_0 \cap S^{-n}A_1) = \mu(A_0)\mu(A_1),$$

and by (A.3) the sign of

$$\mu(A_0 \cap S^{-n}A_1) - \mu(A_0)\mu(A_1)$$

is constant. Thus the lemma follows for $A_2 = \Omega$ by standard arguments. Now imitate the proof on page 85 of [8], noting that the events obtained by translation of monotonic events and the intersection of monotonic events of the same type are still monotonic. □

Clearly, the same lemma holds for T. The result leads to a lower bound independent of the box size for percolation probabilities outside large boxes which are far away. We shall use it in the following form.

COROLLARY. *Let* $z \in P$, U *and* V *be finite subsets of* P, *and* W *an infinite subset of* P. *Then there exists a positive integer* N *such that*

$$\mu([z, \infty; W \setminus (S^{-N}U \cup S^N V)]) \geq \tfrac{1}{2}\mu([z, \infty; W]).$$

PROOF. Define the events

$$\begin{aligned} A_0 &= [\text{all sites in } U \text{ are vacant}], \\ A_1 &= [z, \infty; W], \\ A_2 &= [\text{all sites in } V \text{ are vacant}], \\ \tilde{A}_N &= [z, \infty; W \setminus (S^{-N}U \cup S^N V)]. \end{aligned}$$

Then A_0 and A_2 are decreasing, A_1 and $\tilde{A}_N$ are increasing, and clearly

$$S^N A_0 \cap A_1 \cap S^{-N}A_2 = S^N A_0 \cap \tilde{A}_N \cap S^{-N}A_2.$$

Then (A.3) yields

$$\begin{aligned} \mu(\tilde{A}_N)\mu(S^N A_0 \cap S^{-N}A_2) &\geq \mu(S^N A_0 \cap \tilde{A}_N \cap S^{-N}A_2) \\ &= \mu(S^N A_0 \cap A_1 \cap S^{-N}A_2). \end{aligned}$$

By the multiple ergodic lemma, we have

$$\underset{N\to\infty}{D\text{-lim}}\, \mu(S^N A_0 \cap A_1 \cap S^{-N} A_2) = \mu(A_0)\mu(A_1)\mu(A_2)$$

and

$$\underset{N\to\infty}{D\text{-lim}}\, \mu(S^N A_0 \cap S^{-N} A_2) = \mu(A_0)\mu(A_2).$$

Hence

$$\underset{N\to\infty}{D\text{-lim sup}}\, \mu(\tilde{A}_N) \geq \mu(A_1)$$

and the corollary follows. □

Our last lemma will provide a tool for constructing circuits around boxes.

TOPOLOGICAL LEMMA. *Let $\bar{z} \in B \in \mathbb{B}$, and let z_1 and z_2 be two different points outside B. Suppose that C_1 and C_2 are finite chains both starting at z_1, ending at z_2 and disjoint from B. If $i(C_1, \bar{z})$ is different from $i(C_2, \bar{z})$, then there exists a self-avoiding circuit C (disjoint from B) such that*

$$i(C, \bar{z}) = 1$$

and such that any site of C is a site of either C_1 or C_2.

PROOF. Construct the circuit $\overline{C}$ by going from z_1 to z_2 via C_1 and then from z_2 back to z_1 via C_2. By additivity of the index,

$$i(\overline{C}, \bar{z}) = i(C_1, \bar{z}) - i(C_2, \bar{z})$$

and this last term is different from 0.

It then follows that the component of $R^2 \setminus \overline{C}$ containing $\bar{z}$ is bounded and contains B. Moreover, it is simply connected and its boundary, traversed in the correct direction, yields the desired self-avoiding circuit C. □

3. Proof of the theorem. We divide the proof into two parts, according to whether percolation occurs in the upper half-plane or not. A proof of the first part can be obtained by adapting Harris [9], as was done in Ferrari [5], but the proof we give here is more closely related to assumptions (A.1)–(A.4) and indicates the direction to be followed in the second part of the proof.

First part of the proof. Assume that percolation occurs in the upper half-plane, that is,

$$\mu([0, \infty; H^+]) = p > 0.$$

We intend to apply the box lemma of the preceding section. Choose any box $B \in \mathbb{B}$. Using the corollary with $z = 0$, $U = \varnothing$, $V = B$, $W = H^+$ and T in place of S, we obtain a positive integer N such that

$$\mu([0, \infty; H^+ \setminus T^N B]) \geq p/2,$$

and T-invariance of μ then yields

$$\mu([y_{-N}, \infty; T^{-N} H^+ \setminus B]) \geq p/2.$$

The strip lemma implies that

$$\mu([y_{-N}, \infty; Q_N]) = 0,$$

so that we have

$$\mu([y_{-N}, L_N; Q_N \setminus B]) \geq p/2.$$

This event is the union of the events

$$[y_{-N}, L_N^+; (L_{-N} \cup Q_{N-1} \cup L_N^+) \setminus B]$$

and

$$[y_{-N}, L_N^-; (L_{-N} \cup Q_{N-1} \cup L_N^-) \setminus B],$$

which have the same measure by (A.1) since they are reflections of each other with respect to the vertical axis. Hence

$$\mu([y_{-N}, L_N^+; (L_{-N} \cup Q_{N-1} \cup L_N^+) \setminus B]) \geq p/4,$$

and by horizontal reflection then also

$$\mu([y_N, L_{-N}^+; (L_N \cup Q_{N-1} \cup L_{-N}^+) \setminus B]) \geq p/4.$$

The events in the last two inequalities are increasing, so that, by (A.3) their intersection J satisfies

$$\mu(J) \geq p^2/16.$$

We now claim that

$$J \subseteq [y_{-N}, y_N; P \setminus B],$$

since if $\omega \in J$, then there are occupied self-avoiding chains from y_{-N} to L_N^+ and from y_N to L_{-N}^+, both lying in Q_N and avoiding B. Let $z_+ \in L_N^+$ and $z_- \in L_{-N}^+$ be their end-points. If y_N or z_- belongs to the chain from y_{-N} to L_N^+, then there is the connection we required. Otherwise, this last chain can be enlarged to a self-avoiding circuit C with $i(C, y_N) = 0$ and $i(C, z_-) = \pm 1$, which the chain from y_N to z_- must then intersect in a point of the other chain, and we have verified the claim.

It follows that

$$\mu([y_{-N}, y_N; P \setminus B]) \geq p^2/16.$$

Now choose any $x \in B$ and note that the index of a chain from y_{-N} to y_N avoiding B around x is an odd multiple of $\frac{1}{2}$, and changes sign under reflection with respect to the vertical axis. Hence if we define J^+ (J^-) to be the event that there is a chain of occupied sites from y_{-N} to y_N outside B with positive (negative) index around x, then

$$\mu(J^+) = \mu(J^-) \geq p^2/32,$$

and since both events are increasing,

$$\mu(J^+ \cap J^-) \geq p^4/1024.$$

But now the topological lemma yields

$$J^+ \cap J^- \subseteq \mathbf{B},$$

so

$$\mu(\mathbf{B}) \geq p^4/1024$$

independently of the size of B, and the box lemma with $\delta = p^4/1024$ is then applied to complete the proof.

Second part of the proof. Assume now that

$$\mu([0, \infty]) = p > 0$$

and

$$\mu([0, \infty; H^+]) = 0.$$

Together with translation invariance this implies in particular that with probability 1, any infinite self-avoiding chain of occupied sites intersects each horizontal line in an infinite number of distinct sites. Again we intend to apply the box lemma. Choose any box $B \in \mathbb{B}$. Using the corollary with $z = 0$, $U = V = B$ and $W = P$, we obtain an integer $R > 0$ such that

$$\mu\big(\big[0, \infty; P \setminus (S^R B \cup S^{-R} B)\big]\big) \geq p/2.$$

REMARK. The full strength of the multiple ergodicity seems to be necessary here. The point is that the intersection of the events

$$\big[0, \infty; P \setminus S^R B\big]$$

and

$$\big[0, \infty; P \setminus S^{-R} B\big]$$

is not, contrary to what one might suppose at first sight, contained in the event

$$\big[0, \infty; P \setminus (S^R B \cup S^{-R} B)\big].$$

Our penultimate aim now is to bound $\mu([0, x_{2R}; P \setminus S^R B])$ from below. By horizontal invariance,

$$\mu\big(\big[x_{2R}, \infty; P \setminus (S^R B \cup S^{3R} B)\big]\big) \geq p/2.$$

Next, let

$$I = \{x_m \colon 0 \leq m \leq 4R\}$$

and apply the corollary again with $z = x_{2R}$, $U = V = I$ and $W = P \setminus (S^R B \cup S^{3R} B)$. This yields a positive integer M such that

$$\mu([x_{2R}, \infty; K_{2R}]) \geq p/4,$$

where for notational convenience

$$K_{2R} = P \setminus (S^R B \cup S^{3R} B \cup S^M I \cup S^{-M} I).$$

Put

$$K = P \setminus (S^R B \cup S^{-R} B).$$

Now if $\omega \in [0, \infty; K]$, then there is an infinite self-avoiding chain of occupied sites from the origin in ω, which by assumption must (with full probability) intersect the union of half-lines $S^M L_0^+ \cup S^{-M} L_0^-$ infinitely often, and hence at least once. The first intersection can lie either in $S^M L_0^+$ or $S^{-M} L_0^-$ and vertical axis reflection then yields

$$\mu\big([0, S^M L_0^+; K \setminus S^{-M} L_0^-]\big) \geq p/4.$$

Similarly,

$$\mu\big([x_{2R}, S^{-M} L_0^-; K_{2R} \setminus S^{M+4R} L_0^+]\big) \geq p/8,$$

by reflection around the vertical axis at $2R$. Note that here the symmetry argument depends essentially on the presence of the "spurious" boxes $S^{-R}B$ and $S^{3R}B$, and that the length of I has been chosen to preserve the necessary symmetries.

Next, let A^+ denote the event that there exists an occupied chain C from 0 to $S^M L_0^+$ contained in $K \setminus S^{-M} L_0^-$, such that it either contains x_{2R} or is such that

$$i(C, x_{2R}) > 0,$$

and denote by A^- the corresponding event with negative index. By horizontal axis reflection,

$$\mu(A^+) = \mu(A^-) \geq p/8,$$

since

$$A^+ \cup A^- = [0, S^M L_0^+; K \setminus S^{-M} L_0^-]$$

has measure at least $p/4$. Now (A.3) yields

$$\mu\big(A^+ \cap A^- \cap [x_{2R}, S^{-M} L_0^-; K_{2R} \setminus S^{M+4R} L_0^+]\big) \geq p^3/512,$$

as these three events are increasing. We now claim that

$$A^+ \cap A^- \cap [x_{2R}, S^{-M} L_0^-; K_{2R} \setminus S^{M+4R} L_0^+] \subseteq [0, x_{2R}; P \setminus S^R B].$$

Indeed, if ω is a configuration belonging to the intersection on the left, then there exist occupied chains C^+, C^- and $\tilde{C}$ such that

(1) C^+ and C^- begin at 0 and end in $S^M L_0^+$;
(2) either x_{2R} belongs to $C^+ \cup C^-$ or $i(C^+, x_{2R}) > 0$ and $i(C^-, x_{2R}) < 0$;
(3) $\tilde{C}$ begins at x_{2R} and ends in $S^{-M} L_0^-$;
(4) C^+ and C^- do not intersect $S^R B \cup S^{-M} L_0^-$;
(5) $\tilde{C}$ does not intersect $S^M I \cup S^{M+4R} L_0^+ \cup S^R B = S^R B \cup S^M L_0^+$.

Now form a circuit $\bar{C}$ by first traversing C^+, then the straight line segment J in $S^M L_0^+$ between the terminal elements of C^+ and C^-, and then returning to 0 via C^- in the reverse direction. Since the index is additive and since $i(J, x_{2R})$ is clearly 0, we have that either x_{2R} belongs to $C^+ \cup C^-$ or

$$i(\bar{C}, x_{2R}) = i(C^+, x_{2R}) - i(C^-, x_{2R}) \geq 1.$$

In the latter case x_{2R} belongs to a bounded component of $R^2 \setminus \bar{C}$ and $S^{-M} L_0^-$ lies in the unbounded component of $R^2 \setminus \bar{C}$, as C^+ and C^- do not intersect $S^{-M} L_0^-$. Hence $\tilde{C}$, which connects x_{2R} and $S^{-M} L_0^-$, must intersect $\bar{C}$, and since it does not intersect $S^M L_0^+$, it must intersect C^+ or C^-. Thus in both cases 0 and x_{2R} are connected by a chain lying in $\tilde{C} \cup C^+ \cup C^-$ and noting that these chains do not intersect $S^R B$ then yields $\omega \in [0, x_{2R}; P \setminus S^R B]$. It follows that

$$\mu\big([0, x_{2R}; P \setminus S^R B]\big) \geq p^3/512,$$

and an application of the topological lemma as in the first part of the proof gives

$$\mu(\mathbf{B}) = \mu(\mathbf{S}^{\mathbf{R}}\mathbf{B}) \geq p^6/2^{20}.$$

Taking $\delta = p^6/2^{20}$ in the box lemma finishes the proof. □

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On the Uniqueness of the Infinite Cluster in the Percolation Model

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Abstract. We simplify the recent proof by Aizenman, Kesten and Newman of the uniqueness of the infinite open cluster in the percolation model. Our new proof is more suitable for generalization in the direction of percolation-type processes with dependent site variables.

1. Introduction

It has long been conjectured that the infinite open cluster of the percolation model is unique (almost surely) whenever it exists. This conjecture was verified affirmatively in the recent paper of Aizenman, Kesten and Newman [1], which is couched in the context of (possibly long-range) percolation on any lattice $\mathscr{L}$. Their proof utilizes several distinct ideas and techniques, some of which have their origins in statistical mechanics. Furthermore, the proof has certain consequences for the "thermodynamic functions" of percolation theory, such as the number of clusters per site and the connectivity functions. On the other hand, there are certain miraculous aspects to the method of proof in [1], and it was in attempting to understand this proof that the ideas of this note evolved. In this note, we present a proof of the uniqueness of the infinite open cluster which uses essentially only one of the main ingredients of [1], namely a large-deviation estimate for a certain random variable defined on large but finite open clusters.

The principal motivation for this work was to understand how one may prove the uniqueness theorem for more general processes than "Bernoulli" percolation. Such a generalization to a class of Gibbs measures will appear in [3]. In addition, we hope that our argument may be useful in approaching the question of the uniqueness of the "incipient infinite cluster" of the percolation process; see [2] and [8]. In related work, Gandolfi, Keane and Russo [4] have shown that the infinite cluster is unique for a certain class of two-dimensional models; their techniques are similar to those of Harris [6] for Bernoulli percolation, and are quite different from the general arguments of [1] and the present paper.

We refer the reader to [1] and [4] for motivation and background.

2. The Result

We state and prove the result for site percolation on $\mathbb{Z}^d$, where $d \geq 2$, but the same proof is valid for all bond, site, and mixed bond/site models as well as long-range bond models.

Let μ_p be the Bernoulli product measure on the configuration space $\{-1, 1\}^{\mathbb{Z}^d}$ with density p, where $0 < p < 1$. We denote by E_p the expectation operator related to μ_p, and we call a site *open* if its state is 1 and *closed* otherwise. Writing I for the event that there exists an infinite open cluster, we shall prove the following result.

Theorem. *If p is such that $\mu_p(I) > 0$, then there exists almost surely a unique infinite open cluster.*

Proof. We fix a large positive integer n, and let Λ be the box $\{x \in \mathbb{Z}^d : |x_i| \leq n$ for all $i\}$. Later we shall take the limit as $n \to \infty$; we express this limit as the limit "as $\Lambda \to \infty$". For $x \in \Lambda$, we write $C(x)$ for the open cluster of Λ containing x, and $\mathscr{C}_\Lambda$ for the set of all open clusters of Λ which intersect the internal boundary $\{x \in \mathbb{Z}^d : |x_i| = n$ for some $i\}$ of Λ. We consider the following random subsets of Λ:

$$F_\Lambda(\omega) = \bigcup_{C \in \mathscr{C}_\Lambda} C; \quad G_\Lambda(\omega) = \bigcup_{C \in \mathscr{C}_\Lambda} \partial C;$$

$$H_\Lambda(\omega) = \bigcup_{\substack{C_1, C_2 \in \mathscr{C}_\Lambda \\ C_1 \neq C_2}} \{\partial C_1 \cap \partial C_2\}.$$

Here, ∂C represents the external boundary (in Λ) of C, being the set of sites of Λ which do not belong to C but which are adjacent to some site in C. Thus F_Λ is the set of sites which are joined by open paths to the internal boundary of Λ, G_Λ is the set of closed sites which are adjacent to sites in F_Λ, and H_Λ is the set of closed sites which are adjacent to sites in two or more of the clusters in $\mathscr{C}_\Lambda$. We write L_x for the event that the site x belongs to the external boundary of two or more infinite open clusters of the lattice. It is easy to see that

$$\lim_{\Lambda \to \infty} \frac{1}{|\Lambda|} E_p(|H_\Lambda|) = \mu_p(L_x), \tag{1}$$

where $|A|$ is the cardinality of the set A. On the other hand (see [1] and [9]), the number of infinite open clusters equals 0 or 1 almost surely if $\mu_p(L_x) = 0$, and so it suffices to prove that

$$\limsup_{\Lambda \to \infty} \frac{1}{|\Lambda|} E_p(|H_\Lambda|) \leq 0; \tag{2}$$

the remainder of the proof is devoted to showing this.

From the definition of G_Λ and H_Λ, we have that

$$|H_\Lambda(\omega)| \leq \left(\sum_{C \in \mathscr{C}_\Lambda} |\partial C| \right) - |G_\Lambda(\omega)|. \tag{3}$$

On the other hand,

$$\mu_p(x \in F_\Lambda | x \in F_\Lambda \cup G_\Lambda) = p$$

if x is not a boundary vertex of Λ, so that

$$E_p\left(\sum_{C\in\mathscr{C}_\Lambda}|C|\right)=E_p|F_\Lambda|=\frac{p}{1-p}E_p|G_\Lambda|+o(|\Lambda|), \tag{4}$$

giving from (3) that

$$\frac{1}{|\Lambda|}E_p(|H_\Lambda|)\leqq\frac{1}{|\Lambda|}E_p\left(\sum_{C\in\mathscr{C}_\Lambda}\left\{|\partial C|-\frac{1-p}{p}|C|\right\}\right)+o(1). \tag{5}$$

In order to estimate the right-hand side here, we require (as in [1]) the large-deviation estimate

$$\mu_p(h(C(x))\geqq\varepsilon k,|\bar{C}(x)|=k)\leqq\exp\left(-\frac{k\varepsilon^2}{4a}\right), \tag{6}$$

valid for all $x\in\Lambda, k\geqq 1$, and $\varepsilon>0$, where $h(C(x))=|\partial C(x)|-p^{-1}(1-p)|C(x)|$, $\bar{C}(x)=C(x)\cup\partial C(x)$, and $a=a(p)$ is a function of p which is strictly positive on $(0,1)$. For the convenience of the reader we include a proof of (6), although this proof differs only slightly from the corresponding step in [1]; these differences arise from the facts that we are considering clusters of Λ only, and we need only an estimate which is one-sided in $h(C(x))$. Let $a_{m\ell}(x)$ be the number of connected subgraphs of Λ with m sites in their interiors and ℓ sites in their external boundaries and which contain x. For any $r\geqq 0$, we have that

$$\begin{aligned}\mu_p(h(C(x))\geqq\varepsilon k,|\bar{C}(x)|=k)&\leqq e^{-\varepsilon kr}E_p(e^{rh(C(x))};|\bar{C}(x)|=k)\\&=e^{-\varepsilon kr}\sum_{\substack{m,\ell:\\m+\ell=k}}a_{m\ell}(x)(pe^{-rp^{-1}(1-p)})^m\{(1-p)e^r\}^\ell\\&\leqq e^{-\varepsilon kr}f(r,p)^k\mu_\pi(|\bar{C}(x)|=k),\end{aligned}$$

where

$$f(r,p)=pe^{-rp^{-1}(1-p)}+(1-p)e^r,$$

and $\pi=pe^{-rp^{-1}(1-p)}/f(r,p)$. Now $f(r,p)=1+O(r^2)$ as $r\downarrow 0$, so that

$$\mu_p(h(C(x))\geqq\varepsilon k,|\bar{C}(x)|=k)\leqq e^{-\varepsilon kr+akr^2} \tag{7}$$

for some function $a=a(p)$ which is strictly positive and finite on $(0,1)$. We choose r to minimize the right-hand side of (7) and obtain (6).

Finally we show that (5) and (6) imply (2). We fix $\varepsilon>0$ and δ such that $0<\delta<1/d$, and we write

$$\mathscr{C}'_\Lambda(\omega)=\{C\in\mathscr{C}_\Lambda(\omega):|\bar{C}|\geqq|\Lambda|^\delta\}$$

and B_Λ for the event that $h(C)=|\partial C|-p^{-1}(1-p)|C|\leqq\varepsilon|\bar{C}|$ for all $C\in\mathscr{C}'_\Lambda$. From (6),

$$1-\mu_p(B_\Lambda)\leqq|\Lambda|\sum_{k=|\Lambda|^\delta}^{\infty}\exp\left(-\frac{k\varepsilon^2}{4a}\right)\to 0\quad\text{as}\quad\Lambda\to\infty, \tag{8}$$

since, if B_Λ does not occur, then there exists a site x in Λ such that $h(C(x))>\varepsilon|\bar{C}(x)|$

and $|\bar{C}(x)| \geqq |\Lambda|^\delta$. Thus

$$\frac{1}{|\Lambda|}E_p\left(\sum_{C\in\mathscr{C}'_\Lambda}\left\{|\partial C| - \frac{1-p}{p}|C|\right\}\right) \leqq \left(\frac{1}{|\Lambda|}E_p\sum_{C\in\mathscr{C}'_\Lambda}\varepsilon|\bar{C}|\right) + 2d\{1-\mu_p(B_\Lambda)\} \to 0, \tag{9}$$

as $\Lambda \to \infty$ and $\varepsilon \downarrow 0$. On the other hand, any cluster in $\mathscr{C}_\Lambda(\omega)\backslash\mathscr{C}'_\Lambda(\omega)$ is contained in a d-dimensional "annulus" of Λ with thickness of order $|\Lambda|^\delta$, giving that

$$\lim_{\Lambda\to\infty}\frac{1}{|\Lambda|}E_p\left(\sum_{C\in\mathscr{C}_\Lambda\backslash\mathscr{C}'_\Lambda}\left\{|\partial C| - \frac{1-p}{p}|C|\right\}\right) = 0. \tag{10}$$

We combine this with (5) and (9) to deduce (2), and the proof is complete.

We note that similar calculations appear in [5], and a related approach to the large-deviation argument may be found in [7].

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UNIQUENESS OF THE INFINITE CLUSTER FOR STATIONARY GIBBS STATES

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(Abbreviated title: Uniqueness of the cluster for Gibbs states)

Abstract

We prove, in all dimensions, that for a stationary Gibbs state with finite range or rapidly decreasing interaction, there is at most one infinite percolation cluster. This implies that the connectivity function is bounded away from zero.

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1. Introduction

This paper deals with global features of percolation in Gibbs models. We shall consider the d-dimensional lattice $\mathbb{Z}^d$, in each site of which there is a spin variable which can be up or down. The stochastic distribution of spins will be described by a Gibbs state, a probability measure on the set of all possible configurations of spins, the conditional probabilities of which are given by the Gibbs formula ((2.2) in sect.2); for a general introduction see [1], [2] or [3]. The model is built up starting from a given interaction between the spins which determines the conditional probabilities. For some interactions there can be more than one state having the same conditional probabilities. This, then, represents the phenomenon of phase transition ([1], [4]) and hence the interest of the Gibbs models in equilibrium statistical mechanics. Some of these states may not be stationary ([5], [6]), but in the present paper we will consider only the ones which are.

We also assume that the interaction has finite range or decreases sufficiently rapidly (viz. it satisfies (2.1) in sect. 2).

Percolation is described as follows. In any configuration of spins we consider two nearest neighbour sites as connected if both the spins are up. The theory of percolation, then, deals with the probabilistic description of the maximal connected components of the set of sites in which the spins are up. These components are called clusters and percolation arises when there is at least one cluster containing infinitely many sites (an infinite cluster) with positive probability. (See [26] for a general reference). To study the global properties of percolation we consider the number N of infinite clusters. We prove that in fact the infinite cluster is unique when it exists (meaning that the possible values of N are only zero and one) for the Gibbs states we are considering (sect.3). This implies that when percolation arises the probability of any two points being connected by a chain of spins up is bounded away from zero (sect. 4).

The study of percolation in Gibbs models is interesting not only in itself but also for the techniques it has contributed. These have been used for example to prove the absence of a non-stationary two-dimensional Ising measure ([7], [8], [9]) and in the study of large deviations ([21]). We hope that the techniques developed in this paper will also prove of use in statistical mechanics.

Next we review the previous results in the study of the number of infinite clusters. For the Bernoulli model, where the probability of each spin being up equals p independent of the other spins, the solution to the uniqueness problem (i.e. the proof that N can assume only the values 0 or 1 with probability one) was given in [10] and [11] for d=2 and in [12] for any dimension d. This last work deals with many other quantities related to N as functions of p. But in spite of the remarkable findings of a relation between the analytical properties of some thermodynamical quantities and the uniqueness of the cluster (see also [13], [14] and especially [16] for a similar point of view) the proof is quite involved. A simplified proof which does not make use of the variation of the parameter p is given in [23]. We follow some ideas of these works and in particular we generalize a sort of large deviation property proved in [12] and adapted in [23] (see sect. 3, lemma 3, below).

Our paper generalizes also [17] where the uniqueness problem was solved for some Gibbs models in dimension 2. A generalization of [17] in a different direction (and for dimension 2) was given in [18] under the assumption that the measure is FKG ([24]), ergodic and has some geometrical properties.

A different approach was used in [19] to show that under general conditions N can assume only one of the values 0, 1 or ∞. No non trivial system satisfying the conditions in [19] has so far been shown to have $N=\infty$ with positive probability. The present paper shows that this case does not occur for the stationary Gibbs states with finite range or rapidly decreasing interaction.

The assumption that the Gibbs state is stationary seems essential for uniqueness. On the other hand we believe that the result holds (in the stationary case) also when certain local configurations of spins are excluded or for long range interactions not satisfying (2.1) (see the discussion in sect. 4). Nevertheless we cannot treat these cases.

2. Preliminaries

We consider the d-dimensional lattice $Z=\mathbb{Z}^d$.

A configuration is an element $\omega=(\omega_x)_{x\in Z}$ of the configuration space $\Omega = \{-1, 1\}^Z$. For every subset $S \subseteq Z$ define $\Omega_S = \{-1,1\}^S$; Ω_S can be endowed with the product topology and all the measures μ on Ω_S will be considered as defined on its Borel σ-algebra; E_μ will denote the expectation with respect to μ. For a subset $M \supset S$ we have maps $\alpha_{S,M}: \Omega_M \to \Omega_S$ defined by $\alpha_{S,M}\omega=(\omega_x)_{x\in S}$, transforming the measure μ on Ω_M into the measure $\alpha_{S,M}\mu = \mu \circ \alpha^{-1}_{S,M}$ on Ω_S; we denote $\alpha_S = \alpha_{S,Z}$. We also have the group G=Z of translations of Z which generates maps $\tau^a : \Omega_S \to \Omega_{S-a}$ defined by $(\tau^a\omega)_x = \omega_{x+a}$ for all $a \in G$, $S \subseteq Z$ and $x \in S-a: =\{x \in Z : x+a \in S\}$. Then consider boxes $B = B_m = \{(x_1,\dots,x_d) \in Z : -m < x_i < m, i=1,\dots,d\}$ of linear size $m \in \mathbb{N}$ and denote espressions like $\lim_{m\to\infty} a_{B_m}$ as $\lim_{B\uparrow\infty} a_B$. This will replace the usual limit in the sense of van Hove (see [1], Chapt. 3.9).

To introduce the measures which we shall be dealing with we need an interaction

$$\Phi : \bigcup_{S \text{ finite } \subset Z} \Omega_S \to \mathbb{R}$$

invariant under translations, i.e. satisfying $\Phi(\tau^a\omega) = \Phi(\omega)$ for all $a \in G$.

Let $B=B_R$ be a box, $R \in N$, and let

$$a_R = \sum_{S \text{ finite},\ \underline{0} \in S,\ S\cap(Z/B) \neq \emptyset} \sup_{\omega\in\Omega_S} |\Phi(\omega)|.$$

We will mainly be concerned with interactions for which

$$R^{2d-2} a_R \to 0 \text{ as } R \to \infty. \tag{2.1}$$

This represents our (rapidly decreasing) interaction; we say that an interaction has finite range if there exists an integer R such that $a_R=0$; the smallest of these integers will be called the length of interaction.

For any two disjoint sets B, $M \subseteq Z$, B finite, the energy U_B and the interaction energy $W_{B,M}$ for an interaction Φ are real functions on Ω_B and Ω defined by, respectively,

$$U_B(\omega) = \sum_{S \subseteq B} \Phi(\alpha_{S,B}\omega)$$

and

$$W_{B,M}(\omega) = \sum_{S \text{ finite } \subseteq (B \cup M)\ :\ S \cap M \neq \emptyset,\ S \cap B \neq \emptyset} \Phi(\alpha_S\omega).$$

Definition 1. Let Φ be a (stationary) interaction. A Gibbs state for Φ is a probability measure μ on Ω such that

$$\alpha_S\, \mu(\omega) = \int_{\Omega_{Z/S}} M_{S,\eta}(\omega)\, \alpha_{Z/S}\, \mu(d\eta) \tag{2.2}$$

for all finite $S \subset Z$, where for all $\eta \in \Omega_{Z/S}$, $M_{S,\eta}$ is the probability measure on Ω_S defined by

$$M_{S,\eta}(\omega) = (Z_{S,\eta})^{-1} \exp[-U_S(\omega) - W_{S,Z/S}(\omega\vee\eta)]$$

where $Z_{S,\eta}$ is a normalizing factor and $\omega v\eta \in \Omega$ is defined by

$\alpha_S(\omega v\eta) = \omega$ and $\alpha_{Z/S}(\omega v\eta) = \eta$.

We say that a Gibbs state is stationary if $\tau^a\mu = \mu$ for all $a \in G$.
Note that the stationarity of a Gibbs state is not implied by that of Φ (see [5], [6]).

If μ is stationary we say that it is a finite range Gibbs state if Φ has finite range and we call it a long range Gibbs state if Φ satisfies (2.1).
Observe that the measures $M_{S,\eta}$ are the conditional probabilities $\mu(\bullet \mid \eta)$ on Ω_S (see [1], Sec 1.7).

We will consider percolation of nearest-neighbour points, which are elements of Z at distance one. Let S be a subset of Z. A chain in S is a sequence of elements of S such that successive terms are nearest neighbours, and two points x, y $\in$ S are connected by a chain in S if the chain contains the two points. S is connected if every two of its points are connected by a chain in S.
Let $M \supset S$ be a subset of Z and let $\omega \in \Omega_M$ be any configuration. A cluster of ω in S is a maximal connected subset $C \subseteq S \cap \omega^{-1}(+1)$; if S = Z we say that C is a cluster of ω. Percolation occurs for a measure P on Ω when there is a positive probability to have a cluster containing infinitely many points, an infinite cluster. The number of distinct infinite clusters in ω is denoted by $N(\omega)$, where $N : \Omega \to \mathbb{N} \cup \{\infty\}$, and we study this quantity for the Gibbs states.

Theorem 1. Let μ be a long range Gibbs state on Ω. Then with probability one N assumes only the values 0 or 1.

The theorem will be proved in two steps. First we prove the result for finite range Gibbs states (sect. 3); in this way the main ideas will be more clearly exposed and the result for the long range states will follow by imitating the first step (sect. 4).

Note that no ergodicity has been assumed for μ and thus N is not forced to assume one value with probability one (see [19]). Of course this is only a matter of exposition, as our results carry over from the ergodic states to all the stationary states via the ergodic decomposition (see [1], Appendix 5, or [22]).

Let x be a point in Z. First we reduce the problem to the study of the number of distinct infinite clusters containing at least one nearest-neighbour of x, which we denote by $N_{x,\infty}(\omega)$, $\omega \in \Omega$. Note that $N_{x,\infty} \in \{0,1,\dots,2d\}$. For a measure μ the quantity we are interested in is $E(x,\infty) = E_\mu\big((N_{x,\infty} - 1)\, I_{\{N_{x,\infty}>0\}}\big)$, where I_Q denotes the characteristic function of the event Q. The next lemma applies obviously to finite and long range Gibbs states.

<u>Lemma 1</u> Let μ be a Gibbs state with interaction Φ whose interaction energy $W_{S,Z/S}(\omega)$ is finite for all finite $S \subset Z$, $\omega \in \Omega$. Then $\mu(N>1) = 0$ is equivalent to $E(\underline{0},\infty) = 0$.

Proof. Since $E(0,\infty)>0$ implies immediately that $\mu(N>1)>0$ we simply have to prove the reversed implication, for which it is enough to show that $\mu(N_{0,\infty}>1)>0$ if $\mu(N>1)>0$. A proof of this is based on the σ-additivity and the stationarity of μ and the fact that changing the configurations from an event with positive probability in a finite nonrandom collection of sites leaves the probability of the event positive, which is provided by the finiteness of $W_{S,Z/S}$.

A detailed proof is given in proposition 1.1 in [12] and in proposition 9 and theorem 1 in [19].

■

In view of lemma 1 the proof of theorem 1 is equivalent to $E(\underline{0},\infty) = 0$. To make the exposition more elegant we introduce a measure μ^x on Ω defined by $\mu^x = \mu + \mu \circ \beta_x$ where $\beta_x : \Omega_S \to \Omega_S$ is defined by $(\beta_x(\omega))_x = -\omega_x$ and $(\beta_x(\omega))_y = \omega_y$ for $y \neq x$, $x \in Z$.

Then we have $E(\underline{0},\infty) < E^{\underline{0}}(\underline{0},\infty) := E_{\mu^{\underline{0}}}\big((N_{\underline{0},\infty}-1)\, I_{\{N_{\underline{0},\infty}>0\}}\big)$.

For all Gibbs states whose interaction energy $W_{S,Z/S}$ is finite for any finite set S there exists a constant $K>0$ such that $E^{\underline{0}}(\underline{0},\infty) < K\, E(\underline{0},\infty)$ and thus theorem 1 holds if and only if $E^{\underline{0}}(\underline{0},\infty) = 0$.

Let $R \in \mathbb{N}$. We will also consider the sublattice $L = L_R \subset Z$ of points $(x_1,\dots,x_d)$ such that $x_i = KR$, for $i=1,2,\dots,d$ and $K \in \mathbb{Z}$. When no confusion arises we will omit the index R.

Let now $B=B_m$ be a given box and consider a second box $A(B)$

of linear size equal to the integer part of $m-\sqrt{m}$; note that $\lim_{B\uparrow\infty} \frac{|A(B)|}{|B|} = 1$, where $|B|$ denotes the cardinality (the volume) of B.

Next we introduce the family $\underline{C}_B$ of subsets of B which are connected and contain a nearest neighbour of a point of Z/B. For $C \in \underline{C}_B$ we denote by $\bar{C}$ the set of all the points of B contained in C or nearest neighbours of a point of C, and by $F_C \subset \Omega_B$ the set of all the configurations ω of which C is a cluster in B; a fortiori $\omega_x=-1$ for x in $\bar{C}/C$.

For x in a finite set $S \subset Z$ we shall also consider the number $N_{x,\partial S}(\omega)$ of distinct clusters in S of a configuration $\omega \in \Omega_S$ containing at least one nearest neighbour of x and one neighbour of some point in Z/S. Then let

$$E(x,\partial S) = E_{\alpha_B\mu^x}\left((N_{x,\partial S}-1)\, I_{\{N_{x,\partial S}>0\}}\right).$$

The next lemma again holds for measures that are more general than long range Gibbs state.

<u>Lemma 2</u>. Let μ be any stationary probability measure on Ω. Then

$$E(\underline{0},\infty) < E^{\underline{0}}(\underline{0},\infty) = \lim_{B\uparrow\infty} |A(B)\cap L|^{-1} \sum_{\omega\in\Omega_B} \sum_{C\in\underline{C}_B:\omega\in F_C} \sum_{x\in\bar{C}\cap L\cap A(B)} (-\omega_x)\alpha_B\mu^x(\omega).$$

<u>Proof</u>. For any $K \in \{0,1,\dots,2d\}$ the sequence of events $\{N_{\underline{0},\partial B}=K\}$ converges to $\{N_{\underline{0},\infty}=K\}$ as $B\uparrow\infty$ and this yields

$$E^{\underline{0}}(\underline{0},\infty) = \lim_{B\uparrow\infty} E^{\underline{0}}(\underline{0},\partial B).$$

Next we make use of the stationarity of μ. Let $x \in Z$ and let Λ^1, Λ^2 and B be boxes such that $\Lambda^1+x \subset B \subset \Lambda^2+x$. Then $N_{x,\partial(\Lambda^1+x)} > N_{x,\partial B} > N_{x,\partial(\Lambda^2+x)}$ yields

$$\lim_{B\uparrow\infty} E^{\underline{0}}(\underline{0},\infty) = \lim_{B\uparrow\infty} \sum_{K=2}^{2d} \alpha_{(B+x)}\mu^x(N_{x,\partial(B+x)} \geq K) \tag{2.3}$$

$$= \lim_{B\uparrow\infty} E^x(x,\partial B) = \lim_{B\uparrow\infty} |A(B)\cap L|^{-1} \sum_{x\in A(B)\cap L} E^x(x,\partial B)$$

where $E^x(x,\partial B)$ is defined to be zero if $x \notin B$. The last equality can be obtained by observing that dist$(\partial B, \partial A(B)) \to +\infty$ as $B\uparrow\infty$.

For a box B and $x \in B$ the cylinder $F_x^1 \subset \Omega_B$ is the set of all the configurations ω such that $\omega_x = i$, $i \in \{-1, +1\}$. Furthermore, recall that $N_{x,\partial B}(\omega) > 2$ implies $\omega_x = -1$. Then it is easy to see that

$$\sum_{x \in A(B)\cap L} E^x(x,\partial B) =$$

$$\sum_{x\in A(B)\cap L} \left[\sum_{\omega\in F_x^{-1}} I_{\{N_{x,\partial B}(\omega) > 2\}} N_{x,\partial B}(\omega)\ \alpha_B\ \mu^x(\omega) - \alpha_B\ \mu^x(F_x^{-1} \cap \{N_{x,\partial B} > 2\}) \right] =$$

$$\sum_{x\in A(B)\cap L} \left[\sum_{\omega\in F_x^{-1}} N_{x,\partial B}(\omega)\ \alpha_B \mu^x(\omega) - \alpha_B \mu^x(F_x^{-1} \cap \{N_{x,\partial B} > 1\}) \right].$$

Since $\alpha_B\ \mu^x\ (F_x^{-1} \cap \{N_{x,\partial B} > 1\}) = \alpha_B \mu^x (F_x^{+1} \cap \{N_{x,\partial B} > 1\})$,

the last expression equals

$$\sum_{\omega\in\Omega_B} \left[\sum_{x\in A(B)\cap L\cap\omega^{-1}(-1)} N_{x,\partial B}(\omega)\ \alpha_B\ \mu^x(\omega) - \sum_{x\in A(B)\cap L\cap\omega^{-1}(+1)} I_{\{N_{x,\partial B} > 1\}}(\omega)\ \alpha_B \mu^x(\omega) \right] =$$

$$\sum_{\omega\in\Omega_B}\ \sum_{C\in \underline{C}_B:\ \omega\in F_C} \left[\sum_{x\in\bar{C}\cap(A(B)\cap L)} (-\omega_x) \right] \alpha_B\ \mu^x(\omega).$$

In the last equality note that in the sum over $C\in\underline{C}_B$ the sites $x\in A(B)\cap L$ such that $\omega_x=+1$ are counted only once if they belong to a cluster, while if $\omega_x=-1$ the site x is counted exactly $N_{x,\partial B}(\omega)$ times.
This proves the lemma.

■

3. Uniqueness of the infinite cluster for finite range Gibbs states.

Throughout the section we consider only finite range Gibbs states, with R length of interaction for such a state. Fix a box $\bar{B}=B_R$ of linear size R: we shall always consider boxes B such that $\bar{B}+x\subset B$ for all $x\in A(B)$. We shall also consider the sublattice $L=L_R$.

Define the set $\Omega(x)$ of the <u>local environments</u> around a point $x \in L$ as $\Omega_{\{(\bar{B}+x)/\{x\}\}}$. Let $\sigma \in \Omega(\underline{0})$; define for a Gibbs state μ:

$$T_\sigma = \mu(\omega_{\underline{0}} = +1 \mid \alpha_{\bar{B}/\{\underline{0}\}}\omega = \sigma)$$

$$U_\sigma = \mu(\omega_{\underline{0}} = -1 \mid \alpha_{\bar{B}/\{\underline{0}\}}\omega = \sigma);$$

furthermore for $C \in \underline{C}_B$ and $\omega \in F_C$ let

$m_\sigma^C(\omega)$ = number of sites $x \in C \cap L$ such that the local environment around x is σ.

$\ell_\sigma^C(\omega)$ = number of sites $x \in (\bar{C}/C) \cap L$ such that the local environment around x is σ.

$$M^C(\omega) = \sum_{\sigma \in \Omega(\underline{0})} \left[U_\sigma^{-1}\, \ell_\sigma^C(\omega) - T_\sigma^{-1}\, m_\sigma^C(\omega) \right]$$

In the independent case we can choose $L = Z$, $\bar{B} = \{\underline{0}\}$ so that

$M^C(\omega) = \left[\dfrac{|\bar{C}-C|}{1-p} - \dfrac{|C|}{p} \right]$. In this case the property of M^C stated in the next lemma was already proved in [12] and adapted in [23]. Note that we will make no use of the stationarity of the measure.

<u>Lemma 3</u>. Let μ be a Gibbs state with finite range interaction (with length R) Then there exists $H = H(\{T_\sigma\}_{\sigma \in \Omega(\underline{0})}) > 0$ such that for all B, $n \in N$ and $\varepsilon > 0$ there exists a probability measure $\bar{\mu}_\varepsilon$ on Ω_B satisfying

$$Q_{B,n}(\varepsilon) = \sum_{C,n}^{*} \alpha_B \mu(\{M^C > \varepsilon n\} \cap F_C) < e^{-\varepsilon^2 nH} \sum_{C,n}^{*} \bar{\mu}_\varepsilon(F_C)$$

where $\sum_{C,n}^{*}$ means $\sum_{C \in C_B : |\bar{C}| = n}$.

<u>Proof</u>. First note that for all $\gamma > 0$

$$(3.1) \qquad Q_{B,n}(\varepsilon) < e^{-\varepsilon n \gamma} \sum_{C,n}^{*} E_{\alpha_B \mu}\left[e^{\gamma M^C} I_{(F_C)} \right].$$

Next rewrite the measure of a configuration $\omega \in \Omega_B$ using the choice of L and the Markov property induced by the finiteness of the range of $\mho$, to obtain

(3.2) $$E_{\alpha_B\mu}\left[e^{\gamma M^C} I_{F_C}\right] = \sum_{\omega\in F_C} e^{\gamma M^C(\omega)} \prod_{\sigma\in\Omega(0)} (T_\sigma)^{m_\sigma^C(\omega)} (U_\sigma)^{\ell_\sigma^C(\omega)} \alpha_{B/(L\cap\bar{C})}\mu(\alpha_{B/(L\cap\bar{C}),B}\omega).$$

Define for $C \in \underline{C}_B$ and for all $\omega \in \Omega_B$

$$\bar{\mu}_C(\omega) = \prod_{\sigma\in\Omega(\underline{0})} \mathcal{S}(\gamma)^{\ell_\sigma^C(\omega)} \mathcal{T}(\gamma)^{m_\sigma^C(\omega)} \alpha_{B/(L\cap\bar{C})}\mu(\alpha_{B/(L\cap\bar{C}),B}\,\omega)$$

where

$$\mathcal{S}(\gamma) = \left[\frac{U_\sigma e^{\gamma U_\sigma^{-1}}}{Z_\sigma}\right]$$

$$\mathcal{T}(\gamma) = \left[\frac{T_\sigma e^{-\gamma T_\sigma^{-1}}}{Z_\sigma}\right]$$

$$Z_\sigma = U_\sigma e^{\gamma U_\sigma^{-1}} + T_\sigma e^{-\gamma T_\sigma^{-1}}$$

Then

(3.3) $$E_{\alpha_B\mu}\left[e^{\gamma M^C} I_{F_C}\right] = \sum_{\omega\in F_C} \prod_{\sigma\in\Omega(\underline{0})} (Z_\sigma)^{m_\sigma^C(\omega)+\ell_\sigma^C(\omega)} \bar{\mu}_C(\omega).$$

It is easy to see that for any $a\in[0,1]$ there exists $H(a)>0$ such that

$$\left(a\, e^{\gamma a^{-1}}+(1-a)e^{-\gamma(1-a)^{-1}}\right) < e^{\gamma^2/H(a)} \quad \text{for all } \gamma \in \mathbb{R}.$$

For any $\sigma \in \Omega(\underline{0})$ observe that $T_\sigma + U_\sigma = 1$ and let $H' = \min_{\sigma\in\Omega(\underline{0})} H(T_\sigma)>0$ which yields $Z_\sigma < e^{\gamma^2/H'}$.

Let $C \in \underline{C}_B$ be such that $|\bar{C}| = n$ and let $\omega\in F_C$; then $\sum_{\sigma\in\Omega(\underline{0})} \left(m_\sigma^C(\omega) + \ell_\sigma^C(\omega)\right) < \frac{2n}{R}$, where R is the number which occurs in the definition of the sublattice L.

Collecting (3.1), (3.3) and the last remark we have

(3.4) $$Q_{B,n}(\varepsilon) < e^{-\varepsilon n\gamma} \sum_{C,n}^{*} \exp\left(\frac{\gamma^2\, 2\, n}{RH'}\right) \bar{\mu}_C(F_C)$$

Next put $\gamma = \gamma_\varepsilon = \dfrac{H'R\varepsilon}{4}$ and $H = \dfrac{H'R}{8}$.

Then define

$$\mu_\varepsilon(\omega) = \prod_{\sigma\in\Omega(\underline{0})} \mathcal{S}(\gamma_\varepsilon)^{\ell_\sigma(\omega)} \; \mathcal{T}(\gamma_\varepsilon)^{m_\sigma(\omega)} \; \alpha_{B/L}\mu(\alpha_{B/L,B}\omega)$$

where

$m_\sigma(\omega)$ = number of points x of $A(B)\cap L$ such that ω_x =+1 and the local environment is σ.

$\ell_\sigma(\omega)$ = number of points x of $A(B)\cap L$ such that ω_x =-1 and the local environment is σ.

Using the Markov property of μ it is easy to see that

$$\bar{\mu}_C(F_C) = \bar{\mu}_\varepsilon(F_C)$$

which yields

$$Q_{B,n}(\varepsilon) \leq e^{-\varepsilon^2 nH} \sum_{C,n}^{*} \bar{\mu}_\varepsilon(F_C)$$

and proves the lemma.

By collecting the previous lemmas the proof of theorem 1 for finite range Gibbs states is now straightforward.

<u>Proof</u> (of theorem 1 for finite range Gibbs states). In view of lemma 1 we have to prove $E(\underline{0},\infty)=0$. First we rewrite the estimation made in lemma 2 for the Gibbs measure μ.

Let B be a box, $\omega\in\Omega_B$, $x\in A(B)$ and $\sigma \in \Omega(\underline{0})$. If $\omega_x=+1$ (or $\omega_x=-1$) and the local environment around x is σ then $\alpha_B\mu^x(\omega)=T_\sigma^{-1}\alpha_B\mu(\omega)$ (resp. $U_\sigma^{-1}\alpha_B\mu(\omega)$). Furthermore perform the sum in $C\in \underline{C}_B$ according to the size of $\bar{C}$, noting that from the definition of A(B) it follows that the size of $\bar{C}$ is at least the integer part of $m-\sqrt{m}$, where $B=B_m$, which we will call n(B).

This yields

$$(3.5)\quad E(\underline{0},\infty) \leq \lim_{B\uparrow\infty} |A(B)\cap L|^{-1} \sum_{n=n(B)}^{|B|} \sum_{C,n}^{*} \sum_{\omega\in F_C} M^C(\omega)\, \alpha_B\mu(\omega).$$

Now we want to make use of the estimation given in lemma 3. Let $\varepsilon>0$ be a positive constant. For $C\in\underline{C}_B$ such that $|\bar{C}|= n$ denote by Q^+ the event $\{M^C > \varepsilon n\} \cap F_C$ and by Q^- the event $\{M^C < \varepsilon n\} \cap F_C$.

In (3.5) divide the sum over $\omega\in F_C$ in Q^+ and Q^-. The first term can be estimated from lemma 3 by noting that $M^C < \frac{2n}{T}$, where $T=\min_{\sigma\in\Omega(\underline{0})}(T_\sigma, U_\sigma)$ and

$n=|\bar{C}|$, as

$$\sum_{n=n(B)}^{|B|} \sum_{C,n}^{*} \sum_{\omega\in Q^+} M^C(\omega)\ \alpha_B\mu(\omega) < \sum_{n=n(B)}^{|B|} \frac{2n}{T}\ Q_{B,n}(\varepsilon) <$$

$$\sum_{n=n(B)}^{|B|} \frac{2n}{T}\ e^{-\varepsilon^2 nH} \sum_{C,n}^{*} \bar{\mu}_\varepsilon(F_C) < \frac{2}{T}\ e^{-\varepsilon^2 n(B)\ H}\ 2d|B|$$

where $\bar{\mu}_\varepsilon$ and H were defined in lemma 3; in the last inequality we have used that for any probability measure μ on Ω_B, $\sum_{n=1}^{|B|} n \sum_{C,n}^{*} \mu(F_C) < 2d\ |B|$.

Then $\lim_{B\uparrow\infty} |A(B)\cap L|^{-1} \frac{2}{T}\ e^{-\varepsilon^2 n(B)H} 2d|B| = 0$ because $n(B) \to \infty$ and $|B|\,|A(B)\cap L|^{-1} \to R^d$ when $B\uparrow\infty$, where R is the length of interaction.

The sum over ω in Q^- can be estimated similarly using that $M^C<\varepsilon n$ and that $Q^- \subset F_C$ to obtain

$$\lim_{B\uparrow\infty} |A(B)\cap L|^{-1} \sum_{n=n(B)}^{|B|} \sum_{C,n}^{*} \sum_{\omega\in Q^-} M^C(\omega)\ \alpha_B\mu(\omega) < \lim_{B\uparrow\infty} |A(B)\cap L|^{-1}\varepsilon 2d|B| = \varepsilon 2dR^d$$

Thus $E(\underline{0},\infty)$ is smaller than any positive value and this proves the theorem.

■

4. Uniqueness for long range Gibbs states and related results.

We show in this section that also for the long range Gibbs states the infinite cluster is unique if it exists and this ends the proof of theorem 1. The proof is essentially an imitation of the one above for finite range Gibbs states.

We will therefore sketch a proof which follows step by step the proof of sections 2 and 3. The main changes will be in the choice of the constants in lemma 3.

Proof (of theorem 1). Consider a long range Gibbs state μ, lemma 1 applies and thus we have to prove $E(\underline{0},\infty) = 0$. Consider again the lattice $L_R \cap Z$. Later we will have to let $R\to\infty$ as the interaction now has infinite range. Lemma 2 obviously applies and the next step is to rewrite the estimation it provides.

Consider $\sigma_\infty \in \Omega_{Z/\{\underline{0}\}}$ as environment (no-longer local) of a point and define the conditional probabilities T_{σ_∞} and U_{σ_∞}, the functions $m^C_{\sigma_\infty}$ and $\ell^C_{\sigma_\infty}$ for

$C \in \underline{C}_B$ as before, and define $M^C = \sum_{\sigma_\infty \in \Omega_{Z/\{\underline{0}\}}} \left[U^{-1}_{\sigma_\infty} \ell^C_{\sigma_\infty} - T^{-1}_{\sigma_\infty} m^C_{\sigma_\infty} \right]$

(the sum is finite as we only consider points of a box B). We start from the proof of the first part of theorem one. Formula (3.5) holds replacing

$\sum_{\omega \in F_C} M^C(\omega)\, \alpha_B \mu(\omega)$ with $\int_{F_C} M^C(\omega)\, \mu(d\omega)$ where F_C is considered as event of Ω.

Let $\varepsilon>0$ and define Q^+ and Q^- as above. To estimate the expression concerning the integral in Q^+ we imitate lemma 3. Equation (3.1) still holds. Let σ_R be an R-local environment (in $\Omega_R=\Omega_{B_R/\{\underline{0}\}}$ and let $\sigma_0=\sigma_{\{\underline{0}\}}=1$; define

$$T_{\sigma_R} = \frac{\exp\left(-U_{\{\underline{0}\}}(\sigma_0) - W_{\{\underline{0}\},\, B_R/\{\underline{0}\}}(\sigma_0 v \sigma_R)\right)}{\sum_{\omega_0=1,-1} \exp\left(-U_{\{\underline{0}\}}(\omega_0) - W_{\{\underline{0}\},\, B_R/\{\underline{0}\}}(\omega_0 v \sigma_R)\right)}.$$

T_{σ_R} and $U_{\sigma_R} = 1-T_{\sigma_R}$ are formal conditional probabilities given σ_R. To estimate

the espectation which appears in (3.1) we approximate T_{σ_∞} and U_{σ_∞} by T_{σ_R} and U_{σ_R}.

Recall the definition of a_R and observe that $T_{\sigma_R} e^{-2a_R} < T_{\sigma_\infty} < T_{\sigma_R} e^{2a_R}$

and that a similar inequality holds for U_{σ_R} and U_{σ_∞} when $\alpha_{B_R/\{\underline{0}\},\, Z/\{\underline{0}\}}\sigma_\infty=\sigma_R$.

Let $M^C_R = \sum_{\sigma_R \in \Omega_R} \left[U^{-1}_{\sigma_R} \ell^C_{\sigma_R} - T^{-1}_{\sigma_R} m^C_{\sigma_R} \right]$ and note that $M^C < M^C_R + \left(\frac{e^{2a_R}-1}{T}\right) \frac{2n}{R}$,

where $T = \inf_R \inf_{\sigma_R} (T_{\sigma_R}, U_{\sigma_R}) > 0$ and we have used that

$\frac{2n}{R}$ is larger than $|L_R \cap \bar{C}|$. Thus in (3.2) we can replace M^C by M^C_R and

T_{σ_∞}, U_{σ_∞} by T_{σ_R}, U_{σ_R} by adding an extra factor $\left(e^{2\gamma(e^{2a_R}-1)T^{-1}} e^{4a_R}\right)^{\frac{n}{R}}$.

The following steps are made by substituting σ_R to σ. Only note that H' can be taken as the infimum over σ_∞, so it no longer depends on R; again $H'>0$ as $T>0$. Equation (3.4) now reads

$$Q_{B,n}(\varepsilon) < \sum_{C,n}^{*} \exp[n(-\varepsilon\gamma + \frac{\gamma^2 2}{RH'} + \frac{2\gamma(e^{2a_R}-1)}{RT} + \frac{4a_R}{T})]\bar{\mu}_C(F_C).$$

Next choose $\gamma = \gamma_R = 2\sqrt{a_R H}$, independent of ε, and $H = \frac{H'}{2}$. Define $\bar{\mu}_R$ as $\bar{\mu}_\varepsilon$

where γ_R replaces γ_ε and σ_R replaces σ. Then $\bar{\mu}_C(F_C) = \bar{\mu}_R(F_C)$ still holds. The main difference is that now we fix ε depending on R, viz.

$\varepsilon = \frac{1}{R}\left(\frac{9\sqrt{a_R}}{2\sqrt{H}} + 2\frac{e^{2a_R}-1}{T}\right)$. Then $Q_{B,n}(\varepsilon) < e^{-a_R n} \sum_{C,n}^{*} \bar{\mu}_\varepsilon(F_C)$ so that the term concerning Q^+ converges to zero for every fixed R.

For the other term the choice of ε implies that the limit is

$\varepsilon\, 2dR^d = 2d\, R^{d-1}\left(\frac{9\sqrt{a_R}}{2\sqrt{H}} + 2\frac{e^{2a_R}-1}{T}\right)$. This term converges to zero as $R\to\infty$ as a result of the assumption on the interaction Φ, viz. (2.1). ∎

For measures having longer interaction Φ (a natural assumption is that $a_R \to 0$, see [1] pg. 13) we do not have much information. In the stationary case we expect the infinite cluster to be always unique, while for the non-stationary Gibbs states we do not have any evidence of whether uniqueness may hold or not. The only information in the latter case is provided by the following remark. The stationarity of the measure has been used only to derive (2.2), thus the second term in this formula is zero even for the non-stationary Gibbs states (provided they satisfy (2.1).

As in lemma 2 it is easy to see that therefore

$$0 = \lim_{B\uparrow\infty} |A(B)\cap L|^{-1} \sum_{x\in A(B)\cap L} E^x(x,\partial B) > \lim_{B\uparrow\infty} |A(B)\cap L|^{-1} \sum_{x\in A(B)\cap L} \mu(N_{x,\infty}>) > 0.$$

In words, the density of sites from the neighbours of which there is more than one distinct infinite cluster is zero (a.e. with respect to any long range Gibbs state).

The last part of this section is devoted to the study of the connectivity function $\tau(x,y)$ defined as the probability that there is a chain connecting the two points $x,y \in Z$. The uniqueness of the infinite cluster implies that if percolation arises then $\tau(x,y)$ is bounded away from zero uniformly in x and y. In the Bernoulli model this represents a sharp transition from the non-percolating phase, where $\tau(x,y)$ decreases exponentially with the distance x-y (see [25] and [15]; the behaviour for $p=p_c=\inf\{p$:percolation arises for the Bernoulli model with parameter $p\}$ is excluded in this scheme).

The uniqueness of the cluster holds also for non-attractive potentials, for which the FKG inequality ([24]) in general does not hold. For the Gibbs states satisfying the FKG inequality it is easy to see that $\tau(\underline{0},x) \geq [\mu(N_{\underline{0},\infty}=1)]^2 > 0$; in general we have a less explicit bound as stated in the next proposition.

Proposition 1. Let μ be a finite range Gibbs state on Ω and let $x,y \in \underline{Z}$ If percolation occurs there exists a constant $c>0$ such that

$$\tau(x,y) \geq c > 0.$$

Proof. Given a box B let B^∞ be the event {there is an infinite cluster intersecting B}. If percolation occurs then there exists a box B such that

$$\mu((B+x)^\infty \mid N > 0) > \frac{1}{2}$$

$$\mu((B+y)^\infty \mid N > 0) > \frac{1}{2}$$

and therefore

$$\mu((B+x)^\infty \cap (B+y)^\infty) > 0.$$

We simply have to connect the points x and y to the (unique) infinite cluster intersecting the two boxes B+x and B+y. Let $\phi:\Omega \to \Omega$ be defined by $\phi(\omega)_z=+1$ for all $z \in (B+x)\cup(B+y)$ and $\phi(\omega)_z=\omega_z$ elsewhere. As already remarked in the proof of lemma 1, the finiteness of the energies U_B and $W_{B,Z/B}$ provides $\mu(\phi(Q))>0$ if $\mu(Q)>0$ for all events $Q\subset\Omega$. (A detailed proof is in prop. 9 of [19]).

Hence there exists c such that

$$\tau(x,y) \geq \mu(\phi((B+x)^\infty \cap (B+y)^\infty)) \geq c > 0.$$

■

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Uniqueness Of The Infinite Component In A Random Graph With Applications To Percolation And Spin Glasses

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Abstract. We extend the theorem of Burton and Keane on uniqueness of the infinite component in dependent percolation to cover random graph on $\mathbb{Z}^d$ or $\mathbb{Z}^d \times \mathbb{N}$ with long range edges. We also study a short range percolation model related to nearest neighbor spin glasses on $\mathbb{Z}^d$ or on a slab $\mathbb{Z}^d \times \{0, \cdots, K\}$ and prove both that percolation occurs and that the infinite component is unique for $V = \mathbb{Z}^2 \times \{0,1\}$ or larger.

1. Introduction

Consider a countable set V and a subset E of the set of unordered pairs of elements in V: we call elements in V vertices and elements in E edges. Consider a subset $\tilde{E} \subset E$ and suppose vertices are connected through the elements of $\tilde{E}$; then V falls apart into connected components. To have a good definition we will only refer to maximal connected components, having the property that they are not properly contained in any other connected component. The global connectivity picture can now be given by one of the following possibilities:

(1) all components are finite;
(2) there are some infinite components, but also finite ones;
(3) all vertices of V are connected.

Different terminologies have been developed to describe the various cases, so we define case (1) as absence of percolation, case (2) as occurrence of percolation and case (3) as *connectedness of the graph* $(V, \tilde{E})$. A further study can be carried out in case (2) counting the number of distinct infinite components, called infinite clusters in percolation theory, which do occur: if there is only one infinite cluster we say *uniqueness of the infinite component* holds. Note that we adopt terminologies coming both from random graph theory and from percolation theory.

We now suppose that $\tilde{E}$ is described by a probability measure on $\{0,1\}^E$. Having an assignment η of the values 0 and 1 to the edges, we declare an edge $e \in E$ open, and thus belonging to $\tilde{E}$, if the value assumed by η in e is 1 and closed, and not belonging to $\tilde{E}$, if this value is 0.

In this paper we want to show that for a broad class of choices of $V$and E and of probability measures on $\{0,1\}^E$ two dichotomic laws hold:

(a) *connectedness of the random graph*: when with probability one the graph described by P is either totally connected or falls apart into infinitely many components.

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(b) *uniqueness of the random infinite component*: if with P probability one there is either no infinite component or a unique one.

Actually we will prove results concerning (b) and those concerning (a) will be an immediate consequence.

These kind of results, together with the search for conditions to ensure that one of the two possibilities described in (a) or (b) occurs, have already a long history, and we now review some of the more relevant steps. But let us first remark that we are considering infinite random graphs and the results are thus different from those obtained in the graphs studied originally by Erdös and Rényi ([9]) and surveyed for example by Bollobás ([3]), in which the number of vertices is finite and in which asymptotic properties are considered when the number of vertices approaches infinity. Nevertheless there are some features which can be considered common: in particular, also in the finite case, for a large class of probability distributions, either the graph has a probability tending to one to be totally connected or the number of components tends to infinity (see [3]).

Another difference, this time not so significant, is between bond percolation models, which are those considered here, and site ones, in which vertices are randomly distributed and all edges of a set E are open. Indeed many arguments can be easily translated from one model to the other. We prefer the bond (edge) one because the relation between graph theory and percolation theory is somewhat clearer, because of the relevance to statistical mechanical models of bond percolation (see section 4 below), and finally because long range models, which we treat in this paper, are intrinsic to bond percolation.

We recall that n.n.(nearest neighbor) models are those in which only edges between vertices at distance one are considered (with the distance always taken to be the Euclidean distance for $V \subseteq \mathbb{Z}^d$) and long-range models will be for us those in which all edges are considered. We use the term connectedness when (a) holds and uniqueness when (b) holds. The problems of connectedness and uniqueness were successively solved in the following settings.

First we have n.n. stationary models in $\mathbb{Z}^2$ for which uniqueness was proved for distributions of edges which are independent (Harris [18] and Fisher [10]), Markov (Coniglio *et al* [6]) or simply positively correlated models (the so called FKG condition) with some additional geometrical requirements (Gandolfi, Keane and Russo [15]). We mention that some of the results of these papers are not included in the present one.

As we move to $\mathbb{Z}^d$ we first have results for n.n. and long range distributions for the independent case yielding connectedness of the infinite graph (Grimmett, Keane and Marstrand [16], in which also necessary and sufficient conditions are given to decide which of the possibilities occurs) and uniqueness (Aizenmann, Kesten and Newman [1], with a simplified proof in Gandolfi, Grimmett and Russo [14]). Then uniqueness was obtained for Gibbs measures (with some additional requirements, Gandolfi [13]).

In the meantime n.n. percolation in $\mathbb{Z}^d$ was analyzed with the only requirements of finite energy (and stationarity) in a first attempt by Newman and Schulman ([27]) and successfully by Burton and Keane ([4]), whose argument gives a very simple proof of uniqueness for n.n. measures in $\mathbb{Z}^d$ with only the assumptions of stationarity and finite energy (finite energy essentially means that it is possible to change locally an event preserving its positive probability). Their argument shows that if two (and hence many) distinct infinite clusters occur, then the surface of a cube can not accommodate all the disjoint open paths which are nevertheless forced by the regularities of the measure to intersect that surface.

We want to push forward their argument first of all to long range models. This will be achieved by realizing that the volume of the cube itself is not sufficient to accommodate the disjoint open paths still forced by the geometrical properties of the measure to intersect it (see the proof of Theorem 1).

We are also able to reduce the requirements about the set of vertices; Theorem 1, for example, covers $\mathbb{Z}^d \times \mathbb{N}$ as well as $\mathbb{Z}^d$ and Theorem 1′ covers other vertex sets. Nonetheless we cannot treat in generality other classes of graphs: the first example to which our results do not apply is long range percolation in the quadrant $\mathbb{N} \times \mathbb{N}$. Previous results for the n.n. independent problem in $\mathbb{Z}^d \times \mathbb{N}$ are given by Kesten where he proves uniqueness for these models; Kesten's results were extended to $\mathbb{N}^k$ and $\mathbb{Z}^d \times \mathbb{N}^k$ by Barsky *et al* [2]. Kesten ([24]) also shows connectedness for long range independent percolation in $\mathbb{Z}^d \times \mathbb{N}^k, d \geq 0$, finding the conditions under which the graph is totally connected.

Theorem 1 and our other results also do not entirely require finite energy, but only a one-sided version of it. We will say that P has positive finite energy for $e \in E$ if the conditional probability that e is open, given the configuration of all other edges in E, is almost surely positive; we will say that P obeys the positive finite energy condition if the set of such e's is large enough to connect any pair of vertices in V. Theorem 1 is stated in the next section after a Lemma, which will be needed in its proof. Meanwhile we state a special case of Theorem 1:

THEOREM 0. *Let $V = \mathbb{Z}^d$ and E = the set of all pairs of vertices from V. The random graph determined by a probability measure P on $\{0,1\}^E$ satisfies uniqueness if P is (a) stationary and (b) obeys the positive finite energy condition.*

We remark that if one further restricts Theorem 0 to the case where P is assumed ergodic, then most of the technical issues which arise in the proof of Theorem 1 are eliminated. The reader is encouraged to consider this special case while looking at the proof of Theorem 1 given in the next section.

It may be appropriate here to mention some examples in which (a) or (b) is violated. Infinitely many infinite clusters can occur even with stationarity and the positive finite energy condition for percolation on a graph where the number of vertices at distance R from a fixed vertex grows exponentially, such as a homogeneous tree T or $T \times \mathbb{Z}^d$ (Grimmett and Newman [17]). Other examples can be found in certain exactly solved percolation models in $\mathbb{Z}^d$ called ergodic percolation (Meester [25]) where the positive finite energy condition does not hold. A nice class of examples in which one obtains a finite number of distinct infinite clusters in $\mathbb{Z}^d$ may be constructed by considering i.i.d. variables $\{X_v : v \in \mathbb{Z}^d\}$ taking values in $\{1, \cdots, q\}$ with probabilities $p_i \equiv P(X_v = i)$ for $i = 1, \cdots, q$. Take $E = \{$ n.n. edges $\}$ and then define $\eta_e = 1$ if and only if $e = \{v, v'\}$ has $X_v = X_{v'}$. There will be one infinite cluster for each i such that there is independent n.n. site percolation in $\mathbb{Z}^d$ at density p_v. For example, with $d = 3, q = 2$ and $p_1 = p_2 = \frac{1}{2}$, there will be exactly two distinct infinite clusters since the critical value for n.n. site percolation in $\mathbb{Z}^3$ is strictly below 1/2 (Campanino and Russo [5]).

There is a second line of development for the connection-uniqueness problem which is concerned with models in $\mathbb{N}$. Connectedness for long range distributions on $\mathbb{N}$ has been shown in great generality by Kalikow and Weiss ([19]). They also find for independent distributions the conditions under which the graph is completely connected or there are infinitely many finite components with probability one, work generalized by Kesten to $\mathbb{Z}^d \times \mathbb{N}^k$ as already mentioned ([24]). Conditions for this transition, including the explicit computation of the critical value of the parameter in one parameter families, has been found for non-homogeneous distributions by Shepp ([29]) and Durrett and Kesten([7]), thus "solving" a large class of these models. We limit ourselves to the homogeneous case by which we mean that the distribution is invariant under the (induced map given by the one sided) shift $n \to n+1$. In section 3 we give a proof of uniqueness for all these models, provided they have finite energy: results on connectedness for homogeneous models are an immediate consequence.

As an application of Theorem 1 (or the original Burton-Keane theorem) we show in section 4 that the cluster of edges is unique in a model related to spin glasses. In the process of verifying the conditions of Theorem 1, we show that indeed percolation occurs when $V = \mathbb{Z}^d$, $d \geq 3$ or even in a slab with $V = \mathbb{Z}^2 \times \{1, \ldots, K\}$ with $K \geq 2$ for n.n. models (so that it has a meaning to worry about the number of infinite components): this result is in accordance with the belief that phase transitions should occur for dimensions higher than 2 in these models. We have not determined whether or not percolation occurs when $V = \mathbb{Z}^2$.

In the next section we begin with some definitions and an introductory lemma, before stating and proving Theorem 1 and the closely related Theorem 1'.

2. The main result.

We now proceed by fixing the notation. Let V be a countable set; the elements of V will be called *vertices*. The set of *edges* between vertices in V will be a subset E of $V_2 = \{\{v_1, v_2\}, v_i \in V, i = 1,2\}$. Each $e \in E$ will be identified by the two vertices which define it and these will be called the *end-points* of e : $e = \{v_1, v_2\}$, for $v_1, v_2 \in V$. The edges are not directed. A (vertex self-avoiding) *path* γ in E between distinct vertices v and v' in V is a finite sequence of distinct vertices $(v_o = v, v_1, \cdots, v_n = v')$ such that the edge $\{v_{i-1}, v_i\} \in E$ for $i = 1, \cdots, n$. v and v' will be said to be connected by γ. We will identify γ with the set of these edges and write for example that $\gamma \subset E$.

To represent edges which are open or closed we consider $H = \{0,1\}^E$ in which a topology is given by the *cylinders* of the form $C = \{\eta \in H : \eta_{e_{(1)}} = \alpha_1, \ldots, \eta_{e_{(n)}} = \alpha_n\}$ with *base* $\{e_{(1)}, \ldots, e_{(n)}\} \subset E$ for $n \in \mathbb{N}, \alpha_i \in \{0,1\}$. For a subset $E' \subset E$ a topology is similarly obtained by the cylinders with base contained in E', to which we shortly refer as cylinders in E'. Given $\eta \in H$ and a vertex $v \in V$ we define the *component* I_v of v (in η) on (V, E) as the maximal subset of V which has the property that all its vertices are connected to v by a path γ whose edges are open, i.e. $\eta_e = +1$ when $e \in \gamma$. We will also consider *components on a subset* $V' \subset V$, by which we mean maximal connected components on (V', E') in η' where $E' \subset E$ is the set of edges whose end-points are both in V' and $\eta' \in (0,1)^{E'}$.

For $v \in V$ we consider maps (to be specified) $T_v : V \to V$. Note that given such a map T_v we have an induced action on E defined by $T_v(e) = T_v(\{v_1, v_2\}) = (T_v v_1, T_v v_2), e \in E$, where we use the same notation for the induced maps because it will be always clear at which level we are considering the map. We will say that E is T_v invariant if $\{v_1, v_2\} \in E$ if and only if $(T_v v_1, T_v v_2)$ is in E. There is then an induced map on H defined by $T_v(\eta)_e = \eta_{T_v(e)}, \eta \in H$ and on the subsets A of H by $T_v(A) = \{T_v(\eta) : \eta \in A\}$. If a σ-algebra $\mathcal{A}$ of subsets of H is given, then T_v is $\mathcal{A}$ measurable if $T_v^{-1}(A) \in \mathcal{A}$ for all $A \in \mathcal{A}$ and a probability measure P defined on $\mathcal{A}$ is T_v invariant if $P(T_v^{-1}(A)) = P(A)$ for all $A \in \mathcal{A}$.

We start now with a technical Lemma. It relates the conditional probabilities of a probability measure P given a sub σ-algebra with the same conditional probabilities of the ergodic components of P, when P is decomposed into probability measures which are ergodic with respect to a map T. This result essentially shows that if finite energy (or positive-finite energy) holds for P then it holds also for its ergodic components.

LEMMA 1. *Let E be a countable set and T an invertible map of E onto itself. Let $H = \{0,1\}^E, e \in E$ and let $\mathcal{A}$ and $\mathcal{A}_{E\setminus\{e\}}$ be the σ-algebras generated by the cylinders in E and $E\setminus\{e\}$ respectively. As usual let us denote by the same letter the extension of the map T to H and $\mathcal{A}$.*

Suppose that $T^n(e) \neq e$ for all $n \geq 1$. Then there exists a function φ on H, measurable with respect to $\mathcal{A}_{E\setminus\{e\}}$, such that for any T-invariant probability measure P defined on $\mathcal{A}$,

$$\varphi(\eta_{E\setminus\{e\}}) = P(\eta_e = 1|\mathcal{A}_{E\setminus\{e\}})(\eta_{E\setminus\{e\}}) \qquad P - \text{almost everywhere}$$

and therefore for almost all ergodic components $\tilde{P}$ in the ergodic decomposition of P related to T

$$\tilde{P}(\eta_e = 1|\mathcal{A}_{E\setminus\{e\}}) = P(\eta_e = 1|\mathcal{A}_{E\setminus\{e\}})$$

$\tilde{P}$-almost everywhere.

PROOF: the main idea in the proof is a proper use of the ergodic theorem. Indeed if $C \subset H$ is a cylinder set and I_c denotes its indicator function, then $lim_{n\to+\infty}\frac{1}{n}\sum I_C(T^n\eta) = \mu_\eta(C)$, exists for P almost all $\eta \in H$ whenever P is a T-invariant probability measure, and if P is ergodic this limit no longer depends on η. From now on we consider only $\eta \in H$ for which $\mu_\eta(C)$ exists for all cylinders C and since there are only countably many cylinders, the set of these η's has probability one for any T-invariant measure. Choose a sequence of finite subsets $B_m \subset E$ such that $e \in B_m \subset B_{m+1}$ and $\cup_m B_m = E$. We now define cylinders depending on a given η. For $m \in \mathbb{N}$ let $C_m(\eta)$ be the cylinder set of the configurations coinciding with η in $B_m\setminus\{e\}$; i.e., $C_m(\eta) = \{\eta' : \eta'_{e'} = \eta_{e'}$ for all $e' \in B_m\setminus\{e\}\}$. Furthermore, let $C_m^1(\eta)$ be the cylinder set with the same properties as $C_m(\eta)$ but with e forced to be open: $C_m^1(\eta) = \{\eta' \in C_m(\eta) : \eta'_e = 1\}$.

Let $\varphi_m(\eta) = \frac{\mu_\eta(C_m^1(\eta))}{\mu_\eta(C_m(\eta))}$ when this is defined (i.e. for the η's we are considering for which the denominator is non zero). Because of the assumption on $T^n(e)$, $\mu_\eta(C)$ does not depend on η_e for any C. If $\tilde{P}$ is ergodic, then with $\tilde{P}$ probability one, $\mu_n(C) = \tilde{P}(C)$ and hence $\varphi_m(\eta)$ depends only on the values of η in $B_m\setminus\{e\}$ and

$$\varphi_m(\eta) = \tilde{P}(\eta_e = 1|\mathcal{A}_{B_m\setminus\{e\}})(\eta)$$

for $\tilde{P}$ almost all $\eta \in H$. Now apply the martingale convergence theorem to the sequence $\varphi_m(\eta)$ of functions measurable with respect to $\mathcal{A}_{B_m \setminus \{e\}}$, where $\mathcal{A}_{B_m \setminus \{e\}}$ converges as a sequence of σ-algebras to $\mathcal{A}_{E \setminus \{e\}}$, to see that

$$\varphi(\eta) = lim_{m \to +\infty} \varphi_m(\eta) = \tilde{P}(\eta_e = 1 | \mathcal{A}_{E \setminus \{e\}})(\eta)$$

holds $\tilde{P}$ almost everywhere. To extend the equality to all invariant measures we can use uniqueness (a.e.) of the conditional probability and remark that using the ergodic decomposition and denoting by ρ_P the probability measure on the space of probability measures that realizes this decomposition, the following holds for any $A \in \mathcal{A}_{E \setminus \{e\}}$:

$$\int_A \varphi dP = \int \left(\int_A \varphi d\tilde{P} \right) \rho_P(d\tilde{P})$$

$$= \int \tilde{P}(A \cap \{\eta_e = 1\}) \rho_P(d\tilde{P}) = P(A \cap \{\eta_e = 1\}).$$

This shows that φ satisfies the equation claimed in the Lemma. Next consider P and note that for ρ_P almost all $\tilde{P}$ in the ergodic decomposition of P the set in which $P(\eta_e = 1 | \mathcal{A}_{E \setminus \{e\}}) = \varphi$ has probability one. If $\tilde{P}$ is one of these components then the second equality of the Lemma holds for $\tilde{P}$ almost all $\eta \in H$. ∎

We are now ready to state our main result for percolation on $\mathbb{Z}^d$ or $\mathbb{Z}^d \times \mathbb{N}$.

THEOREM 1. *Let V be $\mathbb{Z}^d$ or $\mathbb{Z}^d \times \mathbb{N}$ and let $T_v : V \to V$ be defined by $T_v(\omega) = v + w$. Let E be a subset of $V_2 = \{\{v_1, v_2\} : v_i \in V, i = 1, 2\}$ such that E is T_v-invariant for each $v \in V$ and let P be a probability measure on $H = \{0,1\}^E$ invariant under T_v for all $v \in V$. Assume P satisfies the positive finite energy condition; i.e., assume that the set E' of edges e such that*

$$P(\eta_e = 1 | \mathcal{A}_{E \setminus \{e\}}) > 0 \quad P - \textit{almost everywhere},$$

is large enough so that for every $v_1, v_2 \in V$, there is some path $\gamma \subset E'$ connecting v_1 and v_2. Then, in the random graph defined by P, there is at most one infinite component with P probability one.

PROOF: The proof is divided into several steps.

1. First we reduce the range of values the number of infinite components can assume, using Lemma 1, to 0, 1 or ∞.

Denote the origin $(0, ..., 0)$ by v_0. First note that for some $v \in V, v \neq v_0$ also $-v \in V$ and thus T_v becomes an invertible map of V onto itself, whose extension to E has also such a property. This was the requirement of Lemma 1, together with the fact, here obviously satisfied, that $T_v^n(e) \neq e$ for all $e \in E'$, for $n \geq 1$.

The second conclusion of Lemma 1 shows then that almost all ergodic components $\tilde{P}$ of P in the decomposition related to T_v satisfy positive finite energy on the same subset E'.

For an ergodic measure $\tilde{P}$ the number of infinite components is constant almost everywhere and positive finite energy on E' easily implies this number must be $0, 1$ or ∞ (as in [1] or [27], think of another number, find this many components intersecting a finite region and join them through E', which gives a prohibited positive probability to a smaller number).

Considering P again, it remains only to exclude the possibility of a positive probability for having infinitely many components. We turn immediately to P to stress that almost the whole proof can be done without using ergodic properties (in some cases they do not appear at all as in Theorem 2 below).

2. To achieve a contradiction suppose there are infinitely many infinite components with P positive probability. The strategy is now to find at least three of these distinct infinite components and join them to a fixed vertex, the origin, in such a way that they remain disjoint apart from vertices near the origin; this has to occur with positive probability.

Under the present assumptions there exist three vertices v_1, v_2 and $v_3 \in V$ with the property that $P(v_1, v_2, v_3$ are in three distinct infinite components $) > 0$, since there are only countably many triplets of vertices. We want to connect these vertices to v_0. Therefore consider a path $\gamma_{v_0} \subset E'$ connecting v_1, v_2, v_3 and v_0 and use repeatedly the positive finite energy condition for the edges of γ_{v_0} starting from the event just described. The conclusion is that the event τ'_{v_o}, as defined in the next equation has nonzero probability:

$$\begin{aligned}
&P(\tau^1_{v_0})\\
&\quad = P(\gamma_{v_0} \text{ is open, each of the three } v_i\text{'s is in the same infinite component,}\\
&\qquad \text{but there is an altered configuration of the } \eta_e\text{'s for } e \in \gamma_{v_0}\\
&\qquad \text{such that } v_1, v_2 \text{ and } v_3 \text{ would be in three disjoint infinite components }) = \sigma > 0.
\end{aligned}$$

Consider the set of edges $E \backslash \gamma_{v_0}$. When $\tau^1_{v_0}$ occurs there are at least three distinct infinite components $C^{(1)}_{v_0}, \ldots, C^{(i_{v_0}(\eta))}_{v_0}$ on $(V, E\backslash\gamma_{v_0})$ which are then connected through edges in γ_{v_0}. These components of $(V, E\backslash\gamma_{v_0})$ will be called *branches* of v_0.

3. The event just defined for the origin can occur also around other vertices and this will be our definition of τ^1_v in $\mathbb{Z}^d$; but since we want to preserve the probability of the events, in the case of $\mathbb{Z}^d \times \mathbb{N}$, a better definition is necessary.

To define similar events for $v \in V$ let $\tau^1_v = T_v^{-1}(\tau^1_{v_0})$. Note that τ^1_v only refers to edges having both end-points in $T_v(V)$ and nothing is assumed about edges having one end-point at least in $V \backslash T_v(V)$ (if this set is not empty). The branches $C^{(i)}_v$ of v are related to those of v_o by $C^{(i)}_v(\eta) = T_v(C^{(i)}_{v_o}(T_v(\eta)))$ for $\eta \in \tau^1_v$. Furthermore define $C_v = \cup_i C^{(i)}_v$ for $v \in V$.

4. We now want to show that if $\tau^1_v \cap \tau^1_w$ occurs for two sites v and w the branches $C^{(i)}_v$ and $C^{(j)}_w$ will satisfy the hypotheses of Lemma 2 (given below) when w and v are "far enough apart".

We first define a certain subset $V' \subseteq V$; pairs of sites from V' will be far enough apart. First choose a box B_1 containing $v_i, i = 0, \ldots, 3$ and all end-points of edges in γ_{v_0}, where box in the context of $\mathbb{Z}^d$ means a set of vertices of the form $[-k, k]^d \cap V$ and in the context of $\mathbb{Z}^d \times \mathbb{N}$ means $([-k, k]^d \times [0, k]) \cap V$. For $v, w \in V$, if $v \in T_w(V)$ and $w \in T_v(V)$, a relation which we denote by $v \sim w$, we simply ask that $T_v(B_1) \cap T_w(B_1) = \emptyset$ and if, for instance, $v \notin T_w(V)$, which implies $w \in T_v(V)$, we ask that $T_v(B_1) \cap T_w(V) = \emptyset$. Choose a subset $V' \subseteq V$ such that for any $v, w \in V'$ the previous requirements are fulfilled. V' can and will be chosen to have positive density. Now we want to see what the simultaneous occurrence of τ_v^1 and τ_w^1, for $v, w \in V'$, implies.

Let us first take the case where $v \sim w$ (this is the only case in $\mathbb{Z}^d$). Either $w \notin I_v$ (the component of v), and thus $v \notin I_w$, or there is an open path γ' connecting w and v. If $w \notin I_v$ or if every such γ' contains a vertex in $V \backslash T_v(V) = V \backslash T_w(V)$ then we have $(\{v\} \cup C_v) \cap (\{w\} \cup C_w) = \emptyset$, because branches are defined only using vertices in $T_v(V)$. Suppose therefore that there is such an open γ' whose end-points are all vertices in $T_v(V)$. First let us remark that w is in a branch of v. Indeed the open path γ' connects v and w and there are at least three distinct infinite components of $(T_w(V), T_wE\backslash\gamma_w)$ which are connected only through $\gamma_w(\gamma_w = T_w(\gamma_{v_o}))$; therefore at most one can contain vertices which are end-points of edges of γ_v since γ_v and γ_w are disjoint (since $T_v(B_1) \cap T_w(B_1) = \emptyset$), and γ_v would connect two branches of w outside γ_w. This implies that at least two infinite paths emanate from w without using edges of γ_v and thus end-points of edges in these paths have to be part of a branch, say $C_v^{(1)}$, of v, together with end-points of γ' and w itself. For the same reason v is in a branch, say $C_w^{(1)}$, of w. No other branch of v can have a common vertex with the remaining branches $C_w^{(i)}$, $i \neq 1$, of w, which are those branches not containing any end-point of edges in γ_v, since otherwise there would be a path not containing edges in γ_v and connecting two branches of v. Thus $C_v^{(1)} \supseteq \{w\} \cup C_w \backslash C_w^{(1)}$ and, for the same reason, $C_w^{(1)} \supseteq \{v\} \cup C_v \backslash C_v^{(1)}$.

Now we consider the case where $w \in T_v(V)$ but $v \notin T_w(V)$; if $w \notin I_v$ or if every open path connecting v and w contains edges whose end-points are vertices of $V \backslash T_v(V)$, then again $(C_v \cup \{v\}) \cap (C_w \cup \{w\}) = \emptyset$. Suppose instead that there is an occupied path γ' from v to w whose end-points are all contained in $T_v(V)$. This time we conclude immediately that w is in a branch of v, which we denote again by $C_v^{(1)}$, and that $C_v^{(1)} \supseteq \{w\} \cup C_w \backslash C_w^{(1)}$; indeed no vertex of C_w can be an end-point of an edge in γ_v since we assumed $T_w(V) \cap T_v(B_1) = \emptyset$, so they are all contained in $C_v^{(1)}$, together with w.

5. The next step in the argument will be to show, based on probabilistic reasons, that for a positive density of vertices $v \in V'$, τ_v^1 occurs. This would already be conclusive for n.n. models; to overcome the problem of having a possibly long range model we first provide a box around v in which each of the branches of v contains many vertices. Together with the density of occurrence of τ_v^1's and the disjointness of the various branches (from Lemma 2 below) this will require the existence of more vertices than there actually are.

Take $V' \subset V$ as before. Now V' was chosen to have a positive density, i.e. $\lim_{n \to +\infty} |B_n \cap V'|/|B_n \cap V| = \rho > 0$, where $\{B_n\}_{n \in \mathbb{N}}$ is a sequence of boxes such that $B_n \supset B_{n-1}$ and $\cup_{n \geq 0} B_n = V$. Let $K \in \mathbb{N}$ be such that $K(\sigma/2)\rho/2 > 1$. (Recall σ from the definition of $\tau_{v_o}^1$). Then we can choose a box $B_2 \supset B_1$ such that $\tau_{v_o}^2$, defined in the next equation, satisfies

$$
\begin{aligned}
&P(\tau_{v_0}^2) \\
&\quad = P(\tau_{v_0}^1 \text{ occurs and all the branches of } v_0 \text{ contain} \\
&\qquad \text{at least } K \text{ vertices in } B_2 \backslash B_1)) \geq \sigma/2 > 0.
\end{aligned}
$$

Now define $\tau_v^2 = T_v^{-1}(\tau_{v_0}^2)$, for $v \in V$. Next consider a box $B_3 \supset B_2$ such that $\overset{o}{B}_3 = \{v \in B_3 : T_v(B_2) \in B_3\}$ satisfies

$$\frac{|\overset{o}{B}_3 \cap V'|}{|B_3|} \geq \rho/2.$$

Such a box exists because of the positive density of V'. (*Remark.* We use here the fact that our graph (V, E) is sub-exponential: this is the reason why this proof cannot work for trees, see also Theorem 1′ below). The mean number of sites v in $\overset{o}{B}_3 \cap V'$ for which τ_v^2 occurs is $|\overset{o}{B}_3 \cap V'|\sigma/2$ and thus there must be at least this number with positive probability. Thus, τ^3, the event that this number exceeds $|B_3|(\sigma/2)\rho/2$, occurs with nonzero probability.

To apply Lemma 2 given below take $\eta \in \tau^3$ and let R be the set of $v \in \overset{o}{B}_3 \cap V'$ for which τ_v^2 occurs. To define S take $v \in R$ and let us remove v from C_v if it was contained in it; denote with the same symbol $C_v^{(i)}$ the branches of v intersected with B_3 and from one of which v has been removed (if it was contained in C_v). Now let $S = \cup_{v \in R}(C_v \cup \{v\})$. From the previous discussion, we have seen that condition (a) of Lemma 2 is satisfied when as subsets of S we take for each $v \in R$ the branches of v. When $v \sim w$ conditions (I) or (IV) of (b) are satisfied; otherwise, conditions (I), (III) or (III) of (b) are satisfied. Lemma 2 then implies that $|S| \geq K(|R| + 2)$, since the definition of τ_v^2 implies that every branch contains at least K vertices in B_3. Therefore we can conclude that with P positive probability

$$|B_3 \cap S| \geq K(|B_3|(\sigma/2)\rho/2 + 2) > |B_3|.$$

This, by contradiction, proves the theorem. ∎

The next Lemma which was used in the proof of Theorem 1 is a combinatorial one which extends Lemma 2 of Burton and Keane [4].

LEMMA 2. *Given a set S and a finite subset $R \subset S$, suppose that*

(a) for all $v \in R$ there exists a finite family
$(C_v^{(1)}, C_v^{(2)}, \ldots, C_v^{(n_v)})$ of $n_v \geq 3$ disjoint
non-empty subsets of S not containing v;

(b) for all $v, w \in R$ one of the following cases occurs (where we define $C_w = \cup_\ell C_w^{(\ell)}$ for any $w \in R$):

(I) $(\{v\} \cup C_v) \cap (\{w\} \cup C_w) = \emptyset$

(II) $\exists i$ *s.t.* $C_v^{(i)} \supset \{w\} \cup C_w$

(III) $\exists j$ *s.t.* $C_w^{(j)} \supset \{v\} \cup C_v$

(IV) $\exists i, j$ *s.t.*
$C_v^{(i)} \supset \{w\} \cup C_w \backslash C_w^{(j)}$ *and*
$C_w^{(j)} \supset \{v\} \cup C_v \backslash C_v^{(i)}$.

Then $|S| \geq (\min_{v \in R}(\min_i |C_v^{(i)}|))(|R| + 2)$.

PROOF: First we observe that the assumptions imply that there exist $v \in R$ and $i \in \mathbb{N}$, $i \leq n_v$, such that $C_v^{(i)} \cap R = \emptyset$. In fact choose any $w_1 \in R$ and $j_1 \in \mathbb{N}$. If $C_{w_1}^{(j_1)} \cap R = \emptyset$

take $v = w_1$ and $i = j_1$; otherwise consider $w_2 \in C_{w_1}^{(j_1)} \cap R$ and j_2 such that $C_{w_2}^{(j_2)} \subset C_{w_1}^{(j_1)}$. The existence of such a j_2 can be easily derived by (b); in fact for $w_2 \in C_{w_1}^{(j_1)}$ case (I) cannot occur and neither can case (II) with $w_2 = v$ and $w_1 = w$ because by the assumptions $w_2 \in C_{w_1}^{(j_1)} \subset C_{w_1}$ but $w_2 \notin C_{w_2}$ and thus $C_{w_1} \not\subseteq C_{w_2}^{(i)}$ for any i. Then clearly $|C_{w_2}^{(j_2)} \cap R| < |C_{w_1}^{(j_1)} \cap R|$ so that repeating this procedure we obtain v and i with the requested properties.

Next take $R\backslash\{v\}$ and $S\backslash C_v^{(i)}$ and note that properties (a) and (b) still hold. In fact $R\backslash\{v\} \subset S\backslash C_v^{(i)}$ since $C_v^{(i)} \cap R = \emptyset$. Property (a) still holds since for $w \in R\backslash\{v\}$, of the previous family $\{C_w^{(1)}, C_w^{(2)}, \dots\}$ only one set, say $C_w^{(1)}$, may have changed into $C_w^{(1)}\backslash C_v^{(i)}$, but in this case $\exists j \neq i$ such that $C_w^{(1)} \supset C_v^{(j)}$ and thus $C_w^{(1)}\backslash C_v^{(i)} \neq \emptyset$. This follows easily from (b). Property (b) still holds since of the previous families related to two points w_1 and w_2 not equal to v no element has become empty and inclusion relations are unchanged under the transformation $C_w^{(j)} \to C_w^{(j)}\backslash C_v^{(i)}$.

Furthermore the proof that property (a) still holds shows that $\min_{v\in R}(\min_i|C_v^{(i)}|)$ can only have been increased. This results in an application of induction, noting that if $|R| = 1$, then $|S| \geq 3\min_{v\in R}(\min_i C_v^{(i)})$. ∎

The next theorem is proved by essentially the same arguments used for Theorem 1; details are left to the reader. Neither Theorem 1 nor Theorem 1′ contains the other. We include Theorem 1′ because it covers some examples, such as percolation on the graphs of finitely generated groups, which may be of some interest.

THEOREM 1′. *Let V be a countable set and let $E = V_2$, the set of all edges between pairs of elements of V. Let P be a probability measure on $\{0,1\}^E$ and let G be a set (or without loss of generality, a group) of bijections, $T : V \to V$, whose natural extensions to $\{0,1\}^E$ leave P invariant. Suppose there are elements T_* of G and 0 of V such that the following properties hold:*

(i) (Finite Energy) The set E', of edges $e = \{v_1, v_2\}$ such that P has finite energy for e, i.e.,

$$0 < P(\eta_e = 1|A_{E\backslash\{e\}}) < 1 \quad P \text{ almost everywhere,}$$

and such that $\{T_*^n v_1, T_*^n v_2\} \neq \{v_1, v_2\}$ for all $n \geq 1$, is large enough so that all points in V are connected by paths in V'.

(ii) (Subexponential Growth of Volumes) There is an increasing sequence of finite subsets C_n of V containing D and converging to V such that for any K,

$$\lim_{L\to\infty} \frac{|\{x \in V : \exists T \in G \text{ s.t. } T(0) = x \text{ and } T(C_K) \subseteq C_L\}|}{|C_L|} = 1,$$

where $|A|$ denotes the cardinality of A.

Then in the random graph defined by P, there is at most one infinite component with P probability one.

REMARKS. 1. It is probably possible to write down a theorem sufficiently general to include both Theorems 1 and 1′ as special cases (and prove it). We shall spare the reader the agony of reading such a theorem by not writing one down.

2. We mention a simple consequence for connectedness of random graphs. Under the condition of Theorem 1 or Theorem 1′, if the random graph has the property that every vertex $v \in V$ is connected to infinitely many other vertices with probability one, than connectedness holds. This gives a different proof of the result in [16], as well as an extension of it to half spaces (see also [24]) and dependent measures.

3. Long range models on $\mathbb{N}$.

There has recently been some interest in long-range models on $\mathbb{N}$, where $V = \mathbb{N}$ and E is the set of all edges $\mathbb{N}_2$ (see [7], [19], [29]). In this specific case the proof of Theorem 1 and Theorem 1′ can not be directly applied because the space is not invariant under an invertible map. Nevertheless it is possible to modify the proof to show that also for long range models on $\mathbb{N}$ there is at most one infinite component. We only require that the probability measure has finite energy, which is a very natural assumption, and is invariant under translations. This second assumption rules out interesting cases ([7] and [29]) in which a transition occurs from connection to non-connection of the graph, but it includes, for instance, the case when all edges $e \in E$ have independent probabilities $p_e = p_{\{v_1,v_2\}} = p_{|v_1-v_2|}$ to be open and $\sum_{i\in\mathbb{N}} p_i = \infty$, which is studied in [19] with different techniques; the result proven there that the graph is totally connected with probability one is achieved here as an easy consequence of the next theorem, which more generally shows that the infinite component is unique.

THEOREM 2. *Let $E = \mathbb{N}_2$ be the set of all edges between vertices in $\mathbb{N}$ and let P be a probability measure on $\{0,1\}^E$. Suppose that:*

(a) P is invariant under T_n, in the sense that $P(T_n^{-1}(A)) = P(A)$ for all events A, where T_n is the map on the σ-algebra generated by the cylinder sets in E induced by the map $T_n : T_n(m) = m + n$, for $n, m \in \mathbb{N}$.

(b) P has finite energy, i.e. for all $e \in E$

$$1 > P(\eta_e = 1|\mathcal{A}_{E\backslash\{e\}})(\eta_{E\backslash\{e\}}) > 0 \quad P-\text{almost everywhere},$$

where $\eta \in \{0,1\}^E, \eta_{E\backslash\{e\}}$ is its restriction to $E\backslash\{e\}$ and $\mathcal{A}_{E\backslash\{e\}}$ is the σ-algebra generated by the cylinder sets in $E\backslash\{e\}$.

Then there is at most one infinite component with P probability one.

Proof: Since the result is achieved by adapting Theorem 1 we will only indicate the principal modifications. The main difference is that we now directly prove that there can not be *two* or more disjoint infinite components with positive probability. So we start by supposing that there exist two vertices $v_1, v_2 \in \mathbb{N}$ such that

$$P(v_1 \text{ and } v_2 \text{ are in two distinct infinite components}) > 0.$$

Finite energy allows us to assume now that with positive probability $\sigma > 0$ the origin v_0 is in two distinct branches $C_{v_0}^{\prime(1)}$ and $C_{v_0}^{\prime(2)}$; denote this event by τ_{v_0}.

For $n \in \mathbb{N}$, let $\tau_n = T_n^{-1}(\tau_{v_0})$ and suppose τ_n and τ_m occur for $n, m \in \mathbb{N}$, with $n < m$.

Note that by our definition of τ_n the two branches $C_n^{\prime(1)}$ and $C_n^{\prime(2)}$ (which are the τ_n analogues of $C_{v_0}^{\prime(1)}$ and $C_{v_0}^{\prime(2)}$) are infinite components of the graph restricted to $[n, \infty)$ and

we think of them as not containing the vertex $n \in \mathbb{N}$ and being therefore completely disjoint. Then two possible cases can occur:
(1) $m \notin C_n = C_n^{(1)} \cup C_n^{(2)}$ in which case $(C_m \cup \{m\}) \cap (C_n \cup \{n\}) = \emptyset$
(2) $m \in C_n$ in which case there exists a branch of n, say $C_n^{(1)}$ such that $C_n^{(1)} \supset \{m\} \cup C_m$.
Recall that $P(\tau_{v_0}) = \sigma > 0$ and let $K \in \mathbb{N}$ be such that $K(\sigma/2) > 1$. As in the proof of Theorem 1 we find sets B_2 and $B_3 \subset \mathbb{N}$ such that $\tau_{v_o}^2$, the event defined as

$$\tau_{v_o}^2 = \{\tau_{v_o} \text{ occurs and both branches } C_{v_0}^{(1)} \text{ and } C_{v_0}^{(2)} \text{ contain at least } K \text{ vertices in } B_2\},$$

has $P(\tau_{v_o}^2) > \sigma/2$ and $\overset{o}{B}_3 = \{n \in B_3 : T_n(B_2) \subset B_3\}$ satisfies $\frac{|\overset{o}{B}_3|}{|B_3|} \geq \frac{1}{2}$.

Let $\tau_n^2 = T_n^{-1}(\tau_{v_0}^2)$. The mean number of integers in $\overset{o}{B}_3$ for which τ_n^2 occurs is $|\overset{o}{B}_3|\sigma/2$ and thus there must be at least this number with positive probability. We want now to imitate Lemma 2 to show that in this case there are at least $|\overset{o}{B}_3|K\sigma/2$ vertices in B_3 and achieve a contradiction.

Consider the set $R \subset \overset{o}{B}_3$ of vertices for which τ_n^2 occurs and the set $S = \bigcup_{n \in R}(\{n\} \cup C_n \cap B_3)$. Take $m, n \in R$ with, say, $n < m$. Note that (1) or (2) must occur. Then it is not difficult to see that

$$|S| \geq (\min_{n \in R} \min_{i=1,2} |C_n^{(i)} \cap B_3|) \cdot (|R| + 1) \geq K \cdot (|R| + 1).$$

This is true if $|R| = 1$ and it can be shown by induction for all values of $|R|$. Indeed it is enough to take $m \in R$ such that there is a branch, say $C_m^{(1)}$, with the property $C_m^{(1)} \cap R = \emptyset$ and to note that for each element $n \in R\backslash\{m\}$ there are two branches in $S\backslash C_m^{(1)}$ satisfying (1) and (2). Application of induction follows again by observing that the procedure we just described does not decrease the minimum size of the branches left.

Since $|S| \geq K|R| > |B_3|K\sigma/2 > |B_3|$ and $S \subset B_3$ we have achieved the required contradiction and proved the theorem. ∎

4. Spin glass models - random cluster models

We consider in this section dependent percolation models which are related to spin glasses. The interest in spin glass models is in the behaviour of configurations of spins under a suitable Gibbs distribution, but as Fortuin and Kasteleyn have discovered ([11],[12],[21]) for Ising ferromagnets it is possible to obtain the Gibbs distribution for the spins (which are located at the vertices of a graph) by a construction which starts from a distribution of variables defined on the edges between spins. Following Kasai and Okiji ([20]) and Swendsen and Wang ([30]) (see also Edwards and Sokal ([8]) and Newman ([26])), we start from the definition of such a distribution and we study some of its percolation properties as they are related, even if less directly than in the Ising ferromagnet, to the magnetic properties of the spin distribution.

Consider again a set of vertices V, which we assume to be $\mathbb{Z}^d$, and the set of edges $E = V_2$. We start from a finite box $B \subset V$, which we take of the form $B = \{x \in \mathbb{Z}^d :\| x \|\leq$

$k\}, k \in \mathbb{N}$, $\| \,.\, \|$ being the sup-norm in $\mathbb{Z}^d$. Let E_B be the set of all edges between vertices of B. Let $J_B = \{a, f\}^{E_B}$, so that $j \in J_B$ is a prescription of the edges being ferromagnetic (f) or antiferromagnetic (a) and denote J_B by J when $B = \mathbb{Z}^d$; the choice of the values $a's$ or $f's$ is made independently for each edge with a and f equally likely. Once $j \in J$ is fixed we construct the random cluster measure on the edge variables $\eta \in H = \{0,1\}^E$. Let therefore $\eta = \eta_B \in H_B = \{0,1\}^{E_B}$ be a prescription of occupation variables for the edges of B. Once η is given the set of vertices of B can be split into maximal connected components (where the connection is once again through open edges) called *clusters* and we denote by $cl(\eta)$ the *number of* these *components*. The next step is the "colouring" of the vertices of B, i.e. to assign to each vertex a spin value ± 1, with the prescription that if an edge e is closed, i.e. such that $\eta_e = 0$, then the spin variables ω can assume in the two vertices v_1 and v_2 any of the two values ± 1, but if the edge is open then $\omega_{v_1} = \omega_{v_2}$ if $j_e = f$ and $\omega_{v_1} \neq \omega_{v_2}$ if $j_e = a$; this can lead to a contradiction for some choices of j and η, which we then call *frustrated*, and we denote by $U(\eta) = U_j(\eta)$ a function being 1 if η is *unfrustrated* for the given j, i.e. it is possible to assign the spin variables without contradiction, and 0 if η is frustrated. As usual we think here of two values of the spin variables, but the results are valid when a different number is considered.

For a given j, the probability distribution on the edge variables will be

$$P^j_{\mathcal{P},B}(\eta) = \left(\prod_{e:\eta_e=1} p_e \right) \left(\prod_{e:\eta_e=0} (1-p_e) \right) 2^{cl(\eta)} U_j(\eta) Z^{-1}_{\mathcal{P}}(j)$$

where $\mathcal{P} = \{p_e\}_{e\in E}, 0 \leq p_e \leq 1, j \in J_B, \eta \in H_B$ and $Z^{-1}_{\mathcal{P}}(j)$ is a normalizing factor.

If $j = j_F \equiv f$ (i.e., in the totally ferromagnetic case) we have the Fortuin-Kasteleyn representation for the Ising model (see [26]) which is then obtained by independently and symmetrically colouring each cluster of B and letting $B \uparrow \mathbb{Z}^d$.

There are many possible choices for $\mathcal{P}$ and two of interest are:

(1) nearest neighbor (n.n.) models: $p_e = \begin{cases} p \text{ if } \| e \| = 1 \\ 0 \text{ otherwise} \end{cases}$ where $0 \leq p \leq 1$.

(2) long range models: $p_e = p_{\|e\|}$, for example with $\lim_{x\to+\infty} x^2 p_x = \beta \in \mathbb{R}$

To simplify the exposition we will limit ourselves to n.n. models from now on, but it will be clear that similar computations, and in particular the uniqueness described in Corollary 2, can be obtained for long range models as well. In the notations we will therefore replace $\mathcal{P}$ by the *single parameter p.*

We have not mentioned so far the boundary conditions we are taking. This amounts to fixing edge variables and spin values off B and effects the definition of $cl(\eta)$ as well as the ensuing colouring process. It is possible that for fixed values of the parameter p different boundary conditions can produce different weak limits when $B = B_n \uparrow \mathbb{Z}^d$ (i.e. when $B_n \subsetneq B_{n+1}$ and $\mathbb{Z}^d = \bigcup_n B_n$). This is related to the existence of more than one (infinite volume) Gibbs distribution for the spin variables. We will generally take *periodic boundary conditions*; i.e. B will be a rectangle in $\mathbb{Z}^d$ treated as a torus (i.e. with "opposite" sites in B identified).

An important remark is that when we consider the joint distribution of j and η, any weak limit obtained with periodic boundary conditions will be invariant under all $\mathbb{Z}^d$-translations.

For a given j, a sufficient condition for the occurence of more than one Gibbs distribution for the spin variables is that in some Gibbs distribution, Cov $(\omega_o, \omega_x) \not\to 0$ as $\| x \| \to \infty$. We will consider this possibility for the Gibbs distributions that arise, from a translation invariant distribution of j and η obtained as described above, by conditioning on j and using the independent colouring process to construct w from η. We will use a subscript p to keep track of the parameter p in the model (in physical terms, $p = 1 -$ exp $(-K/T)$, where T is the temperature and K is a positive constant, so that $p \to 1$ as $T \to 0$.); thus conditioned on a fixed j, Cov_p will denote the covariance in the Gibbs distribution of w and P_p^j will denote the corresponding distribution for η.

One of the results of Fortuin and Kasteleyn ([11, 12, 21]) is that in the totally ferromagnetic case

$$\begin{aligned}\mathrm{Cov}_p(\omega_0, \omega_x) &= P_p^{j_F}(0 \text{ is connected to } x \text{ by a path of open edges }) \\ &=: P_p^{j_F}(A_{0,x})\end{aligned}$$

where $P_p^{j_F}$ is the weak limit of $P_{p,B}^{j_F}$ with periodic boundary conditions. In the general case we have only a one sided result (see e.g. [26]):

$$|\mathrm{Cov}_p(\omega_0, \omega_x)| \leq P_p^j(A_{0,x}).$$

Nevertheless, it is clear that percolation is necessary for $\mathrm{Cov}_p(\omega_o, \omega_x) \not\to 0$ as $\| x \| \to \infty$, and as a first partial result we study $P_p^j(A_{0,x}), j \in J$ and the behaviour of the infinite components of open edges.

Since we are interested in properties which are true for almost all configurations $j \in J$ we consider the joint distribution $P_{p,B}$ of j and η variables for edges in E_B, which we call the (finite volume) *random interaction random cluster model.* As explained above, consider periodic boundary conditions and let P_p be the weak limit of P_{p,B_n} along a suitable subsequence $B_n \uparrow \mathbb{Z}^d$. Let P_p^{edge} be the marginal of P_p on the edge variables. The next Theorem provides a lower bound to the conditional probability, under P_p^{edge}, that $\eta_e = +1$ given the η values for $e' \neq e$. As a consequence we obtain that for p large enough, in dimension $d \geq 3$ (and in fact for the slab $\mathbb{Z}^2 \times \{0,1\}$), there will be percolation of open edges under P_p^j for almost all $j \in J$ (Coroll. 1). The lower bound implies also that the positive finite energy condition of Theorem 1 holds for P_p; we will therefore be able to conclude that the infinite cluster of open edges is unique P_p^j almost always for almost all $j \in J$ (Coroll. 2).

THEOREM 3. *Let P_p^{edge} be the marginal on the edge variables of a weak limit P_p of finite volume random interaction random cluster models. Then for $e \in E$*

$$P_p^{edge}(\eta_e = +1 | \mathcal{A}_{E \setminus \{e\}})(\eta_{E\setminus\{e\}}) \geq p/2,$$

for P_p^{edge}-almost all $\eta_{E\setminus\{e\}} \in \{0,1\}^{E\setminus\{e\}}$, where $\mathcal{A}_{E\setminus\{e\}}$ is the σ-algebra generated by the variables in $E\setminus\{e\}$.

PROOF: Let us call e' the edges in $E\backslash\{e\}$ and let $P_{p,B}$ be the finite volume measure with edges marginal $P_{p,B}^{edge}$ converging to P_p^{edge}. To prove the Lemma it is enough to estimate, uniformly in B, the following:

$$(*) \qquad \min_{\overline{\eta}_e} \frac{P_{p,B}^{edge}(\eta_e = +1, \overline{\eta}_{e'})}{P_{p,B}^{edge}(\eta_e = 0, \overline{\eta}_{e'})},$$

where $\overline{\eta}_e = \overline{\eta}_{e'}^B$ means a choice of $\eta_{e'}$ for $e' \in E\backslash\{e\} \cap B = E_B\backslash\{e\}$

The explicit form of the measure $P_{p,B}^{edge}$ is

$$\begin{aligned} P_{p,B}^{edge}(\eta_e = +1, \overline{\eta}_{e'}) &= \sum_{j \in \{a,f\}^{E_B}} P_{p,B}(\eta_e = +1, \overline{\eta}_{e'}, j) \\ &= \sum_{j \in \{a,f\}^{E_B}} p^{N_1}(1-p)^{N_0} 2^{cl(\eta)} U_j(\eta) Z_p^{-1}(j) \end{aligned}$$

where η is the configuration assuming the values $+1$ and $\overline{\eta}_{e'}$ in e and e' respectively, N_0 and N_1 are the numbers of edges $e'' \in E_B$ such that $\eta_{e''} = 0$ or 1 respectively.

To study (*) we can thus estimate

$$(**) \qquad \min_{\overline{\eta}_{e'}, \overline{j}_{e'}} \frac{\sum_{j_e=a,f} P_{p,B}(\eta_e = +1, j_e, \overline{\eta}_{e'}, \overline{j}_{e'})}{\sum_{j_e=a,f} P_{p,B}(\eta_e = 0, j_e, \overline{\eta}_{e'}, \overline{j}_{e'})}$$

where $\overline{j}_{e'}$ is a configuration of the j variables for $e' \in E_B\backslash\{e\}$.

Let us consider separately the four cases $\eta_e = 1, 0$ and $j_e = a, f$. There are still three different situations which can occur once $\overline{\eta}_{e'}$ and $\overline{j}_{e'}$ are given. Either v_1 and v_2, the two end-points of e_1, are not connected (I) by any path of edges in $\overline{\eta}_{e'}$ or they are; in this second case either $U_{j_e=f,\overline{j}_{e'}}(\eta_e = +1, \overline{\eta}_{e'}) = 0$ or $U_{j_e=a,\overline{j}_{e'}}(\eta_e = +1, \overline{\eta}_{e'}) = 0$, in other words if v_1 and v_2 are connected in $\overline{\eta}_{e'}$ then the edge variable η_e being $+1$ forces either $j_e = a$ (II) or $j_e = f$ (III).

This explains the following table which gives the value of $P_{p,B}(\eta_e, j_e, \overline{\eta}_{e'}, J_{e'})$ for the indicated values of η_e and j_e and the three possible cases $(I), (II)$ and (III). In each entry the factor $C_i, i = I, II, III$ depends only on $\overline{j}_{e'}, \overline{\eta}_{e'}$ and is constant on each line and thus irrelevant in (**).

The normalizing factor $Z_p(j)$ depends on the entire j, but we denote it by $Z(a)$ or $Z(f)$ to mean that $j_e = a$ or f and $\overline{j}_{e'}$ is fixed.

	$j_e = f$		$j_e = a$	
	$\eta_e = 0$	$\eta_e = 1$	$\eta_e = 0$	$\eta_e = 1$
I	$\frac{(1-p)}{Z(f)} C_I$	$\frac{p}{2Z(f)} C_I$	$\frac{(1-p)}{Z(a)} C_I$	$\frac{p}{2Z(a)} C_I$
II	$\frac{(1-p)}{Z(f)} C_{II}$	0	$\frac{(1-p)}{Z(a)} C_{II}$	$\frac{p}{Z(a)} C_{II}$
III	$\frac{(1-p)}{Z(f)} C_{III}$	$\frac{p}{Z(f)} C_{III}$	$\frac{(1-p)}{Z(a)} C_{III}$	0

We can evaluate (**) as

$$(***)\qquad \begin{aligned} &\min\Big(\Big(\frac{p}{2Z(f)}+\frac{p}{2Z(a)}\Big)\frac{1}{(1-p)(\frac{1}{Z(f)}+\frac{1}{Z(a)})}, \\ &\frac{p}{(1-p)}\frac{\frac{1}{Z(a)}}{\frac{1}{Z(a)}+\frac{1}{Z(f)}}, \frac{p}{(1-p)}\frac{1}{\frac{Z(f)}{Z(a)}+1}\Big) = \\ &= \min\Big(\frac{p}{2(1-p)}, \frac{p}{(1-p)}\frac{1}{1+\frac{Z(a)}{Z(f)}}, \frac{p}{(1-p)}\frac{1}{1+\frac{Z(f)}{Z(a)}}\Big)\end{aligned}$$

Next it remains to evaluate the ratio $\frac{Z(a)}{Z(f)}$ and we follow the same scheme which led from (**) to table 1.

Indeed writing $Z(a) = Z(j_e = a, \bar{j}_{e'})$ explicitly, separating the cases $\eta_e = +1$ and $\eta_e = -1$ and thinking $\bar{\eta}_{e'}$ fixed we achieve the same computation as in table 1 apart from the normalizing factors: this is summarized in table 2.

	$Z(a)$	$Z(f)$
I	$\frac{2(1-p)+p}{2}D_I$	$\frac{2(1-p)+p}{2}D_I$
II	$(1-p)D_{II}$	D_{II}
III	D_{III}	$(1-p)D_{III}$

We conclude that

$$\min_{\bar{j}_{e'}} \frac{Z(a)}{Z(f)} = (1-p), \max_{\bar{j}_{e'}} \frac{Z(a)}{Z(f)} = \frac{1}{1-p}.$$

We evaluate (***) as $\min(\frac{p}{2(1-p)}, \frac{p}{2-p}) = \frac{p}{2-p}$ which is also the minimum in (*) and yields

$$P^{edge}_{p,B}(\eta_e = +1|\mathcal{A}_{E\setminus\{e\}})(\eta_{E\setminus\{e\}}) \geq \frac{1}{1+\frac{2-p}{p}} = \frac{p}{2}$$

uniformly in B. ∎

The first consequence concerns the occurrence of percolation. Consider a finite box B and its edges E_B. It is not difficult to see that for two probability measures μ_1 and μ_2 on $\{0,1\}^{E_B}$ the inequality

$$\begin{aligned} &\min_{\eta_{E\setminus\{e\}}} \mu_1(\eta_e = +1|\mathcal{A}_{E_{B\setminus\{e\}}})(\eta_{E\setminus\{e\}}) \geq \\ &\geq \max_{\eta_{E\setminus\{e\}}} \mu_2(\eta_e = +1|\mathcal{A}_{E_{B\setminus\{e\}}})(\eta_{E\setminus\{e\}})\end{aligned}$$

implies that μ_1 stochastically (or FKG) dominates μ_2 in the sense that for any increasing function f on $\{0,1\}^{E_B}$ we have $\int f d\mu_1 \geq \int f d\mu_2$ (see for instance Russo [28]).

Theorem 3 implies that $P_{p,B}^{edge}$ stochastically dominates uniformly in B the Bernoulli measure $\mu_{p/2}$, where $p/2$ is the density of edge variable assuming the value $+1$; since for $d \geq 3$ for bond percolation in $\mathbb{Z}^d$ (or for percolation in $\mathbb{Z}^2 \times \{0,1\}$) the density at which edge percolation occurs is strictly smaller than 1/2 (see [22]) we can now prove the following:

Corollary 1. *If $d \geq 3$ and p is large enough then P_p^j (the origin is in an infinite cluster of open edges) > 0 for almost all $j \in J$, where P_p^j is the weak limit of $P_{p,B}^j$ along a sequence of boxes converging to $\mathbb{Z}^d$ for which $P_{p,B}$ weakly converges.*

Proof: From Theorem 3 and the previous remarks we know that P_p^{edge}(there exists an infinite cluster of open edges)=1. The same holds for P_p and applying Fubini's theorem P_p^j(there exists an infinite cluster of open edges)=1 for almost all $j \in J$. Additivity of the measure and finite energy yield the result. ∎

A second consequence of Theorem 3 is about uniqueness of the infinite cluster of open edges.

Corollary 2. *Let P_p^j be as in Corollary 1 in any dimension. Then P_p^j (there exists an unique infinite cluster of open edges)=1 for almost all $j \in J$.*

Proof: We apply Theorem 1 to P_p^{edge} which is preserved by the full group of translations of $\mathbb{Z}^d$ and has (positive) finite energy on the full lattice E by Theorem 3. A subsequent application of Fubini's theorem yields the result. ∎

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CLUSTERVORMING EN UNICITEIT IN WISKUNDIGE MODELLEN

VAN PERCOLATIE-VERSCHIJNSELEN

SAMENVATTING

Dit proefschrift behandelt de volgende situatie. Van een gegeven samenhangende graaf $\mathcal{G}$, met aftelbaar veel knooppunten, beschouwt men een niet lege deelgraaf $\mathcal{G}'$. De volgende situaties kunnen optreden: (1) $\mathcal{G}'$ bevat alle knooppunten van de oorspronkelijke graaf $\mathcal{G}$; (2) $\mathcal{G}'$ bevat een samenhangende component met oneindig veel knooppunten; (3) $\mathcal{G}'$ is op te splitsen in oneindig veel eindige componenten.

Stel dat de deelgraaf $\mathcal{G}'$ gegenereerd wordt door een kansmechanisme. Voor bepaalde grafen en kansmechanismen beschrijft dit proefschrift de kansen op de gebeurtenissen (1), (2) en (3).

Als gebeurtenis (1) optreedt dan zegt men dat de graaf totaal samenhangend is; als gebeurtenis (2) positieve kans heeft dan zegt men dat percolatie optreedt, en als gebeurtenis (3) optreedt met kans één, dan zegt men dat er geen percolatie is.

Percolatie-processen komt men voor het eerst tegen in een artikel van Broadbent en Hammersley (1957), die een wiskundig model probeerden te maken voor de verspreiding van een gas of vloeistof door een poreus materiaal. Percolatie-processen zijn ook van belang bij de beschrijving van geleiders, communicatie netwerken en polymeren.

Een andere bron van belangstelling in percolatie vindt men in de statistische mechanica. Veel modellen van percolatie-processen zijn afhankelijk van één of meerdere parameters en percolatie vindt wel of niet plaats afhankelijk van de waarden van deze parameters. Dit verschijnsel wordt in de statistische mechanica beschouwd als een eenvoudig voorbeeld van fase overgang.

Dit proefschrift bestaat uit een algemene inleiding en vier artikelen. Behandeld wordt het gedrag van de random deelgraaf in het geval dat percolatie optreedt. Er wordt voor veel verschillende grafen en kansmechanismen bewezen dat nooit meer dan één oneindige maximale component kan optreden. Dit verschijnsel noemt men uniciteit van het oneindige cluster. Uniciteit heeft gevolgen voor de samenhangende eigenschappen van de deelgraaf.

In de inleiding van dit proefschrift bespreken we ook soortgelijke resultaten verkregen door andere onderzoekers. In het bijzonder worden "short" en "long range" modellen besproken in de d-dimensionale ruimte $\mathbb{Z}^d$, $d \geq 1$, of in bepaalde deelgrafen van $\mathbb{Z}^d$. De kansmechanismen, die behandelt worden, zijn: onafhankelijke kansverdelingen, Ising modellen en FK "random cluster" modellen. Er worden ook voorbeelden gegeven van modellen waarin meer dan één (oneindige maximale samenhangende) component met positieve kans kan optreden.

Vervolgens wordt het gedrag bestudeerd van de oneindige component. Hiervoor worden resultaten gegeven over de verhouding tussen rand en volume van de oneindige component in het geval van onafhankelijke kansmechanismen en Ising modellen.

Artikel A bewijst dat er in $\mathbb{Z}^2$ en voor algemene kansmaten die aan de FKG ongelijkheid voldoen, slechts één oneindige maximale component kan optreden en dat, als dit gebeurt, er oneindig veel circuits rond elke knooppunt bestaan.

Artikel B geeft een eenvoudig bewijs voor de uniciteit van het oneindige cluster bij onafhankelijke percolatie in $\mathbb{Z}^d$, welk resultaat eerst werd verkregen door Aizenman, Newman en Kesten [1987].

In artikel C word het bovenstaande resultaat uitgebreid tot alle Gibbs maten. Tevens word er een grote afwijikingen stelling voor het oneindige cluster bewezen.

Artikel D is een vervolg op een artikel van Burton en Keane [1988]. Hier worden "short" en "long range" modellen behandeld in $\mathbb{Z}^d$ en in $\mathbb{Z}^d \times \mathbb{N}$, $d \geq 1$. Er wordt voor deze grafen bewezen dat voor alle kansmechanismen die aan de "positive finite energy condition" voldoen, uniciteit geldt. Tevens bewijzen we deze stellingen met de graaf $\mathbb{Z}^d$ vervangen door de graaf $\mathbb{N}$. Toepassingen worden gegeven om condities te vinden die gebeurtenis (1) impliceren. Tenslotte wordt een "FK random cluster" model van spin glasses bestudeerd. We tonen aan dat in de gevallen $\mathbb{Z}^d (d \geq 3)$ of $\mathbb{Z}^2 \times \{1, \cdots, k\} (k \geq 2)$ percolatie optreedt. Bovendien bewijzen we dat al deze modellen aan de "positive finite energy condition" voldoen.

CURRICULUM VITAE

De schrijver van dit proefschrift werd geboren op 11 april 1961 te Modena, Italië. In 1979 behaalde hij het diploma Tecnico Commerciale aan het Instituut Meucci te Carpi (Modena), waarna hij wiskunde ging studeren aan de Universiteit van Modena. In 1985 studeerde hij af, met als hoofdrichting kansrekening, bij Prof. L. Russo. Zijn scriptie was getiteld "Sulla dipendenza dalla temperatura della probabilita' di percolazione nel modello di Ising".

Uitgezonderd de periode juni 1987-mei 1988, gedurende welke de auteur in Italië zijn militaire dienstplicht vervulde, is hij sinds september 1985 verbonden aan de Technische Universiteit Delft. Gedurende deze periode was hij medewerker bij het project "percolatie modellen" onder leiding van Prof. dr. M.S. Keane. Uit dit project is dit proefschrift voortgekomen.

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6 - Een *random Markov proces* is een stationair proces in $\mathbb{Z}$ zodanig dat de voorwaardelijke kans van de waarde in de oorsprong gegeven de waarden in het verleden alleen af hangt van de laatste m waarden, net als in een gewoon m-staps Markov proces, maar waarbij de waarde van m random is.

Een random Markov proces is niet altijd een random Markov proces met betrekking tot de toekomst, d.w.z. met omgedraaide tijd (zie [1]).

Hierdoor is het onduidelijk hoe dit begrip uitgebreid kan worden tot dimensie twee.

[1] S. Kalikow: Random Markov processes and uniform martingales. Preprint (1989).

7 - De Nobelprijs voor natuurkunde in 1985 is een duidelijke illustratie van de verhouding tussen de vernuftige Nederlandse doelmatigheid en de abstracte Italiaanse Weltanschauung.

8 - Een Nederlandse vertaling van het boek "De morbis artificum diatriba" van Bernardino Ramazzini zou van belang zijn niet alleen voor de geneeskundehistorici, maar ook voor de wetenschappers in het algemeen. De in dit boek aangegeven uiteindelijke fundering van de geneeskunde in de praktijk geldt namelijk voor de gehele wetenschap.

9 - In het kader van de Europese eenwording verdient het aanbeveling om het Nederlandse fietspadennet tot de gehele Europese Gemeenschap uit te breiden.

[1] A. Frigessi, M. Piccioni: Parameter estimation for two-dimensional Ising fields corrupted by noise. Preprint (1988).
[2] A. Frigessi, A. Gandolfi: in preparation.

4 - Zij $x = (x_0, x_1, \dots) = 2211212212211\dots$ de rij gedefinieerd door de volgende eisen ([1]):

(1) $x_0 = 2$, $x_n = 1$ of 2,

(2) de rij van lengten van opeenvolgende runs in x is gelijk aan x, waarbij runs als volgt zijn gedefinieerd. Een run ter lengte k in x is een verzameling R van opeenvolgende indices $\{i, i+1, \dots, i+k-1\}$ waarvoor geldt: (I) $x_i = x_{i+1} = \cdots = x_{i+k-1}$ en (II) de verzameling R is maximaal met betrekking tot eigenschap (I), d.w.z. R is niet strict bevat in een verzameling R' die aan eigenschap (I) voldoet. Een run R volgt een andere run R' als de indices van R volgen die van R'.

Voor alle $n \in \mathbb{N}$, zij $d_n(1)$ het aantal indices $i \in \{0, \dots, n-1\}$ waarvoor $x_i = 1$; zij $\overline{d}(1) = \limsup_{n\to\infty} d_n(1)/n$ en $\underline{d}(1) = \liminf_{n\to\infty} d_n(1)/n$. Elementaire simulaties met een computer doen vermoeden dat $\underline{d}(1) = \overline{d}(1) = 1/2$. Er kan worden bewezen dat $\overline{d}(1) \geq (3-\sqrt{5})/2$.

[1] W. Kolakoski: Self generating runs. Problem 5304. Amer. Math Monthly **72**, 674, (1965).
[2] F.M. Dekking: Stellingen bij het Proefschrift "Combinatorial and statistical properties of sequencies generated by substitutions". Delft (1980)

5 - In een spel kiest een speler uit twee identieke enveloppen er een uit. De enveloppen bevatten geld en de speler weet dat één enveloppe bevat twee maal het bedraag die de ander bevat; zijn doel is zoveel mogelijk geld te verdienen. Nadat hij het bedrag x van de gekozen enveloppe heeft gezien, wordt hem de mogelijkheid geboden om de andere enveloppe te nemen.

Voor diegenen die van mening zijn dat bij wisseling van de enveloppe de verwachte winst $5x/4$ is, kunnen we de volgende opmerkingen maken.

Stel dat een (groot) aantal geisoleerde spelers dit spel uitvoeren met twee enveloppen die allemaal y respectievelijk $2y$ bevatten. Als iedereen zijn enveloppe wisselt, is de gemiddelde winst gelijk aan $3y/2$, evenals in het geval dat niemand zijn enveloppe wisselt.

Een betere strategie is de volgende. Nadat in de eerste enveloppe x was gevonden trekken we een random getal z uit een standaard exponentiele verdeling. We wisselen de enveloppe als $z > x$ en wisselen niet als $z \leq x$. Dit geeft een gemiddelde winst $3y/2 + (e^{-y} - e^{-2y})y/2$.

Wie van mening is dat bij het vinden van x in de eerste enveloppe de verwachte winst na wisseling $5x/4$ is, zal aan deze strategie van wisseling de voorkeur geven boven de genoemde vergelijking met een random getal.

3 - Zij $\Omega = \{-1,1\}^{\mathbb{Z}^2}$. Definieer de volgende twee kansmaten op twee kopien $\Omega^{(1)}$ en $\Omega^{(2)}$ van Ω. Laat $\mu_{\beta,0}$, $\beta > 0$, een kansmaat op $\Omega^{(1)}$ zijn die bij een Ising ferromagneet met uitwendig veld nul behoort (zie blz. 24 van dit Proefschrift). Laat verder $P_{1-\varepsilon}$, $0 \leq \varepsilon < 1/2$, een Bernoulli kansmaat op $\Omega^{(2)}$ zijn waarvoor de kans op 1 gelijk is aan $1-\varepsilon$. Zij $B_k = \{x \in \mathbb{Z}^2 : |x| \leq k\}$, $k \in \mathbb{N}$. Laat het *echte beeld* $\omega^{(1)}_{B_k}$ de beperking van $\omega^{(1)} \in \Omega^{(1)}$ tot B_k zijn en laat de *storing* $\omega^{(2)}_{B_k}$ de beperking van $\omega^{(2)} \in \Omega^{(2)}$ tot B_k zijn. Zij verder *het ontvangen beeld* ω_{B_k} een combinatie van het echte beeld en de storing, gedefinieerd als de beperking tot B_k van het puntgewijze product $\omega^{(1)}\omega^{(2)}$.

Een probleem in de statistiek is een schatting van de parameters ε en β uit te voeren door het analyzeren van het ontvangen beeld.

Er bestaan twee schatters $\overline{\varepsilon}_k = \overline{\varepsilon}(\omega_{B_k})$ en $\overline{\beta}_k = \overline{\beta}(\omega_{B_k})$ zodat met kans één ten opzichte van de productkans $\mu_{\beta,0} \times P_{1-\varepsilon}$ geldt: $\lim_{k\to\infty} \overline{\varepsilon}_k = \varepsilon$ en $\lim_{k\to\infty} \overline{\beta}_k = \beta$ (zie [1]).

Neem aan dat voor de gegeven waarde van β er fase overgang plaats vindt (zie blz. 24 van dit Proefschrift). Dan is het mogelijk een schatter $\tilde{\varepsilon}_k$ van ε, en daarmee ook een schatter $\tilde{\beta}_k$ van β, te vinden die eenvoudiger zijn te berekenen dan de schatters in [1].

Laat $n_j^{(i)} = n_j^{(i)}(\omega_{B_k})$, $i \in \{-1,1\}$ en $j = 0,\ldots,4$, het aantal punten x in het inwendige van B_k zijn waarvoor ω_{B_k} waarde 1 in x heeft en waarvoor er precies j punten zijn op afstand 1 van x waarvoor ω_{B_k} waarde 1 heeft. Zij $\delta_j^{(i)}$ de verhouding tussen $n_j^{(i)}$ en het aantal punten van B_k.

Laat

$a_k = a_k(\omega_{B_k}) = -(c_k)^{-1}(3(\delta_3^{(1)} + \delta_1^{(1)} - \delta_3^{(-1)} - \delta_1^{(-1)}) + 4(\delta_2^{(1)} - \delta_2^{(-1)}))$,

$b_k = b_k(\omega_{B_k}) = (c_k)^{-1}(\delta_2^{(1)} - \delta_2^{(-1)})$,

en

$c_k = c_k(\omega_{B_k}) = 6(\delta_4^{(1)} - \delta_4^{(-1)} + \delta_3^{(1)} - \delta_3^{(-1)} + \delta_2^{(1)} - \delta_2^{(-1)} + \delta_1^{(1)} - \delta_1^{(-1)} + \delta_0^{(1)} - \delta_0^{(-1)})$.

Laat verder

$$\tilde{\varepsilon}_k = \begin{cases} \frac{1}{2} - \sqrt{\frac{1}{2}(\frac{1}{2} + a_k + \sqrt{a_k^2 - 4b_k})}, & \text{als } c_k \neq 0 \\ 0, & \text{als } c_k = 0 \end{cases}$$

en

$$\tilde{\beta}_k = -\frac{1}{8}\log\left(\frac{(\sum_{i=0}^{4}(1-\tilde{\varepsilon}_k)^i(\tilde{\varepsilon}_k)^{4-i}(-1)^i(\delta_i^{(1)} + \delta_i^{(-1)} + \delta_i^{(-1)}/\tilde{\varepsilon}_k))}{(\sum_{i=0}^{4}(1-\tilde{\varepsilon}_k)^i(\tilde{\varepsilon}_k)^{4-i}(-1)^i(\delta_i^{(1)} + \delta_i^{(-1)} - \delta_i^{(-1)}/\tilde{\varepsilon}_k))}\right).$$

Dan geldt $\lim_{k\to\infty} \tilde{\varepsilon}_k = \varepsilon$ en $\lim_{k\to\infty} \tilde{\beta}_k = \beta$ met $\mu_{\beta,0} \times P_{1-\varepsilon}$-kans één ([2]).

Als β zodanig is dat er geen fase overgang is dan convergeren a_k, b_k en c_k naar 0, als k naar oneindig nadert, met $\mu_{\beta,0} \times P_{1-\varepsilon}$-kans één.

STELLINGEN

behorende bij het proefschrift "Clustering and uniqueness in mathematical models of percolation phenomena" van Alberto Gandolfi.

1 - Laat $X = \{X_n\}_{n\in\mathbb{Z}}$ een stationaire oneindige rij zijn van identiek verdeelde stochastische variabelen met waarden in $\{0,1\}$. Stel dat X één-afhankelijk is, d.w.z. X_i en X_j zijn onafhankelijk zodra $|i-j| \geq 2$. Als de verwachting van X_i gelijk is aan α, dan geldt

$$(*) \qquad \mathrm{Cov}(X_i, X_{i+1}) \leq \begin{cases} (2\alpha-1)+(1-\alpha)^{3/2}-\alpha^2, & \text{als } 0 \leq \alpha \leq 1/2 \\ \alpha^{3/2}-\alpha^2, & \text{als } 1/2 \leq \alpha \leq 1 \end{cases}$$

voor alle $i \in \mathbb{Z}$ (zie [1]).

Voor ieder $\alpha \in [0,1]$ bestaat er tenminste één één-afhankelijke rij van stochastische variabelen waarvoor gelijkheid in (*) geldt (zie [1]).

Een soortgelijk minimum voor de covariantie is niet bekend.

[1] A. Gandolfi, M.S. Keane, V. de Valk: Extremal two-correlations of two-valued stationary one-dependent processes. Probab. Th. Rel. Fields **80**, 475-480 (1989).

2 - Laat $X = \{X_n\}_{n\in\mathbb{Z}}$ een stationaire oneindige rij zijn van identiek verdeelde stochastische variabelen met waarden in $\{0,1\}$. Zij $N = \min\{n : n \geq 0, X_n = 1\}$ de stochastische variabele die de positie geeft van de eerste 1 onder de stochastische variabelen X_i met $i \geq 0$.

Als de stochastische variabelen X_i onafhankelijk van elkaar zijn en α de verwachting van X_i is, dan is de verwachting van N gelijk aan $(1-\alpha)/\alpha$.

Laat $\alpha \in [0,1/2]$. Dan voldoet de verwachting $E(N)$ van N aan de volgende ongelijkheid:

$$E(N) \leq \alpha^{-1}((1-\alpha)+(1-\alpha)^{3/2})$$

voor alle rijen van stochastische variabelen X die één-afhankelijk zijn (zie Stelling 1) en met verwachting van X_i gelijk aan α; deze bovengrens wordt aangenomen door tenminste één dergelijke één-afhankelijke rij.

Een soortgelijk maximum voor de verwachting van N in het geval $\alpha \in (1/2, 1]$ is niet bekend.

In het algemeen kunnen er niet-triviale ondergrenzen voor de verwachting van N worden gegeven, maar geen bovengrenzen, d.w.z. voor ieder $\alpha \in [0,1)$ en $M > 0$ is er tenminste één rij van stochastische variabelen met verwachting van X_i gelijk aan α en verwachting van N groter dan M (zie [1]).

[1] P.W. Kasteleyn: Variations on a theme by Mark Kac. J.Stat.Phys. **46**, 811-827 (1987).