Notes on the Unsteady Rectilinear Motion of a perfect gas

VII Some applications of the Stanyukovich Transformation

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SUMMARY

In this Report some straightforward applications of the Stanyukovich Transformation developed in the companion Report LR-329 (Feb. 1982, Ref [1]) are presented. The homentropic flows which form the point of departure are

1. A uniform steady homentropic flow (Ch. 3)
2. A normal shock-wave travelling with constant speed into a uniform gas (Ch. 4).
3. A homentropic flow, with pressure and velocity given by (Ch. 5).
   \[ p = p_0 + \beta_0 h, \quad u = u_0 - \beta_0 t \]
4. A homentropic centered simple wave (Ch. 6).

The cases 1, 2 and 4 had appeared already, for the greater part, in other papers, but here they are obtained from a systematic application of the Stanyukovich Transformation.
For case 1 also the infinitesimal transformations of the transformation group are considered.
The applications are preceded by a chapter (Ch. 2) where some additions to the general theory in Report LR-329 (Ref. [1]) are presented.
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1. INTRODUCTION

This Report is a companion to Report LR-329 (February, 1982). In Report LR-329 a transformation was developed, called the Transformation of Stanyukovich, which by simple manipulations generates the flow of a non-homentropic LMS-gas, when the flow of a homentropic gas is given.

By the term 'flow' is meant a (usually exact) solution of the equations of motion for the unsteady rectilinear motion of a perfect gas (without viscosity and heat conductivity) expressed in the independent variables t, the time, and h, the Lagrangian mass coordinate.

The discussion in Ref. [1] was mainly restricted to mathematical aspects. The transformation of different flow parameters was considered and the transformation was generalized into a 3-parameter continuous transformation group. The infinitesimal operators of the group were constructed together with several extended operators. It was found that the group induces linear transformations for some flow parameters and finally the invariance of several partial differential equations under the transformation group was tested and verified.

In order to find out whether interesting flow problems will become accessible by means of this transformation, applications have to be worked out.

In the present Report some simple and straightforward applications of the theory developed in Ref. [1] are presented. Several of them were already obtained earlier as distinct examples of flows of the LMS-gas. By considering these flows as the result of the Stanyukovich Transformation when applied to homentropic flows, the results appear in a more unified fashion and our insight in the structure of the theory is improved.

Also the infinitesimal transformations of the group will be applied. They yield flows which differ an infinitesimal amount from the original flow, which forms the starting point when applying the infinitesimal transformation.

It was found that the discussion of the infinitesimal transformations required as much space as the entire preceding discussion of the problem. Therefore the infinitesimal transformations are applied only to the problem in Chapter 3, while the application to the other problems is postponed.

The infinitesimal transformation may be obtained by adding infinitesimal increments to those values of the group-parameters, which generate the identity transformation, and by linearizing the resulting transformation in terms of these increments. They can be considered as linearized perturbations to the original flow, but perturbations within the group manifold.

Before starting with the discussion of applications it is desirable to add some points of a general character. These points supplement the theory of Ref [1] and have been found useful when applications were worked out. These matters form the content of Chapter 2 and they are followed by 4 Chapters where the Stanyukovich Transformation is applied to different homentropic flows. These flows are:

- Ch. 3. Application to a uniform steady flow,
- Ch. 4. Application to a uniformly travelling normal shock-wave,
- Ch. 5. Application to a homentropic gas, moving as a rigid body, with a constant acceleration,
- Ch. 6. Application to a homentropic centered simple wave.

The Transformation of Stanyukovich indeed unifies several of the results presented and clarifies the theory.

It is expected that further applications will appear in due time.

* This Report will be denoted as Ref. [1]. See List of References.
2. REMARKS ON THE GENERAL THEORY

2.1. The Transformation formulae valid for the entire $h_1$-plane

In Ref [1] the Stanyukovich Transformation is initially defined by the formulae

$$h_1 = \frac{1}{h}, \quad p_1 = \frac{p}{h}, \quad B_1(h_1) = B(h) h^{3Y-1}.$$  \hspace{1cm} (2.1)

Since the function $\phi$ transforms in the same way as $p$, and since the physical parameters are derivatives of $\phi$, or compositions of these derivations, the transformations of the parameters $p, u, v, x, K, L, a, \frac{a}{V}, ah$ etc. are easily deduced.

The transformation (2.1) is involutoric, while $h$ and $h_1$ are restricted to positive values in order to prevent the occurrence of negative pressures, etc.

The restriction of $h$ and $h_1$ to the first quadrant of the $h h_1$-plane seems artificial and was removed in Section 5 of Ref. [1] by the introduction of 3 further sets of transformation formulae, one set for each remaining quadrant of the $h h_1$-plane. The new sets of formulae differ from the formulae (2.1) only through the appearance of some – signs, resulting in positive, physically acceptable, pressures, specific volumes, etc. in each quadrant.

In this chapter we show that the 4 sets of transformation formulae can be taken together by introducing two constants $\alpha$ and $\alpha$ which assume the values $+1$ or $-1$ depending on the quadrant of the $h h_1$-plane considered in accordance with Table 1.

**Table 1.**

<table>
<thead>
<tr>
<th>Quadrant</th>
<th>$A$</th>
<th>$\alpha$</th>
<th>$A\alpha, A\alpha^{-1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-1</td>
<td>+1</td>
<td>-1</td>
</tr>
<tr>
<td>2</td>
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<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>3</td>
<td>-1</td>
<td>-1</td>
<td>+1</td>
</tr>
<tr>
<td>4</td>
<td>+1</td>
<td>+1</td>
<td>+1</td>
</tr>
</tbody>
</table>

The transformation formulae, valid for the entire $h h_1$-plane then take the form

$$h_1 = -\frac{A}{h}, \quad \phi_1 = \frac{\alpha}{h} \phi, \quad B_1(h_1) = \alpha^{Y+1} h^{3Y-1} B(h),$$  \hspace{1cm} (2.2)

and considering also the derivatives of $\phi$ one finds
\[ K_1 = \frac{a}{h} K, \quad x_1 = \frac{a}{A} (hx - \Phi), \quad L_1 = -\alpha u, \]

\[ p_1 = \frac{a}{h} p, \quad u_1 = \frac{a}{A} (hu + K), \quad V_1 = \alpha h^3 V, \quad (2.3) \]

\[ a_1 = a h, \quad V_1 = h^{-2} \frac{a}{V}, \quad a_1 h_1 = -\alpha A a. \]

It may be verified that the formulae (2.1) and those of Sections 4 and 5 in Ref. [1] are included in this set. The inverse transformations are found to take the form

\[ h = -\frac{A}{h_1}, \quad \phi = -\frac{A}{a} h_1 \phi_1, \quad B(h) = \frac{1}{\alpha h_1} \frac{h_1}{A} \left( \frac{1}{h_1} \right)^{3Y-1} B_1(h_1), \]

\[ K = -\frac{A}{a} h_1 K_1, \quad x = -\frac{1}{a} (h_1 \phi_1 - \phi), \quad L = \frac{A}{a} u_1, \quad (2.4) \]

\[ p = -\frac{A}{a} h_1 p_1, \quad u = -\frac{1}{a} (h_1 u_1 + K_1), \quad V = -\frac{1}{a A} h_1^3 V_1, \]

\[ a = -\frac{1}{a A} h_1 a_1, \quad \frac{a}{V} = h_1^{-2} \frac{a_1}{V_1}, \quad \alpha h = \frac{1}{a} a_1. \]

For orthogonal hyperbolaes with branches in the first and third quadrant \( A = -1, \frac{dh_1}{dh} < 0 \), and so \( h_1 \) is a decreasing function of \( h \). A consequence is that the right-running \( r \)-characteristics in the \( h,t \)-plane, are mapped into left-running \( s_1 \)-characteristics in the \( h_1,t \)-plane, while \( s \)-characteristics in the \( h,t \)-plane map into \( r_1 \)-characteristics in the \( h_1,t \)-plane.

For orthogonal hyperbolaes with branches in the second and fourth quadrant \( A = +1, \frac{dh_1}{dh} > 0 \) and \( r \)-characteristics map into \( r_1 \)-characteristics, while \( s \)-characteristics are mapped into \( s_1 \)-characteristics.

If the initial flow, denoted without subscripts, is homentropic \( B(h) = B_\infty \) = const. and along the \( r \)-characteristics

\[ r = u + \frac{2}{Y-1} a = \text{const.}, \quad (2.5) \]

while along the \( s \)-characteristics

\[ s = u - \frac{2}{Y-1} a = \text{const.}. \quad (2.6) \]
Application of the transformation formulae (2.4) then yields

\[
\begin{align*}
\frac{u + \frac{2}{\gamma - 1}}{a} & = -\frac{1}{a} \left[ K_{1} + h_{1} \left( u_{1} + \frac{2}{\gamma - 1} \frac{1}{A} a_{1} \right) \right], \\
\frac{u - \frac{2}{\gamma - 1}}{a} & = -\frac{1}{a} \left[ K_{1} + h_{1} \left( u_{1} - \frac{2}{\gamma - 1} \frac{1}{A} a_{1} \right) \right].
\end{align*}
\]

(2.7)

The transformed expressions in braces are the generalized Riemann-invariants for the LMS-gas denoted usually as \( r_{1}^{*} \) and \( s_{1}^{*} \). One checks that for \( A = -1 \) the formulae (2.7) take the form

\[
\begin{align*}
r & = -\frac{1}{a} s_{1}^{*}, \\
s & = -\frac{1}{a} r_{1}^{*},
\end{align*}
\]

(2.8)
in agreement with the remarks made earlier upon the mapping of the characteristics, while for \( A = +1 \) one obtains

\[
\begin{align*}
r & = -\frac{1}{a} r_{1}^{*}, \\
s & = -\frac{1}{a} s_{1}^{*}.
\end{align*}
\]

(2.9)

Also one may check that the expressions for the generalized Riemann invariants, when considered in the homentropic flow are transformed into the classical Riemann-invariants for the LMS-gas, i.e.

\[
\begin{align*}
K + h \left( u + \frac{2}{\gamma - 1} \right) a & = \frac{A}{\alpha} \left( u_{1} + \frac{2}{\gamma - 1} \frac{1}{A} a_{1} \right), \\
K + h \left( u - \frac{2}{\gamma - 1} \right) a & = \frac{A}{\alpha} \left( u_{1} - \frac{2}{\gamma - 1} \frac{1}{A} a_{1} \right).
\end{align*}
\]

(2.10)

With respect to the formulae (2.10) one has to note that in a homentropic gas the generalized Riemann-invariants in the left hand sides do not perform any special role, while the same remark applies to the classical Riemann-invariants in the right-hand sides when a non-homentropic LMS-gas is considered. When applications are considered it will usually be assumed that we start with a homentropic gas with \( B(h) = B_{0} = \text{const.} \), but one notes that the transformation formulae are valid without this restriction.

2.2. The Transformation Group

Upon completing the extension of the Stanyukovich-Transformation to the entire \( h_{1} \)-plane, some other limitations may be noted and then removed.

The origin of the \( h_{1} \)-plane and the coordinate axes represent the centre and the asymptotes of the orthogonal hyperbolae, generating the mapping. It might be useful if the centre of the hyperbolae could be located in an arbitrary point of the \( h_{1} \)-plane, with the asymptotes, parallel with the coordinate axes, passing through it, while it would also be of interest to magnify the curve and change its scale.

Both these desirable properties are obtained when we turn to the formulae of the Transformation Group developed in Sections 6-10 of Ref. [1].

The transformation then has the form
\[ h_1 = \frac{a_{11} h + a_{12}}{a_{21} h + a_{22}}, \quad \phi = \frac{\alpha}{a_{21} h + a_{22}}, \quad B(h_1) = \frac{\alpha^{r+1}}{a_{21} h + a_{22}} (a_{21} h + a_{22})^{3r-1} B(h), \quad (2.11) \]

with \( A \) representing the determinant

\[
A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} a_{22} - a_{12} a_{21}, \quad (2.12)
\]

To show that the mapping \( h \to h_1 \) represents a generalization of the formulae in the preceding section the first relation in (2.11) can be rewritten in the form

\[ a_{21} h_1 - a_{11} = -\frac{A}{a_{21} h + a_{22}}, \quad (2.13) \]

which no longer contains \( a_{12} \) except in the form \( A \) given by (2.12). Formula (2.13) represents an orthogonal hyperbola with asymptotes \( a_{21} h - a_{11} = 0, \ a_{21} h + a_{22} = 0 \).

It is convenient to make two assumptions with respect to the formulae presented so far. They are

1. It will be assumed throughout that \( a_{21} \) is positive i.e. \( a_{21} \geq 0 \).
2. The transformation \( h \to h_1 \) will be taken to be unimodular so that \( A = \pm 1 \).

If a set of \( a_{ij} \)'s is given with \( a_{21} < 0 \), the transformation (2.11) remains unaffected if the signs of all the \( a_{ij} \)'s are changed and \( \alpha \) is replaced by \( -\alpha \).

To determine the 4 \( a_{ij} \)'s uniquely in the first formula of (2.11) one condition between the 4 \( a_{ij} \)'s may be imposed (Cf. Ref. [1], p.18). This will be the condition (2). For \( A = -1 \) (and \( a_{21} \geq 0 \)) the formula (2.13) represents an orthogonal hyperbola with branches in the first and third quadrant determined by the asymptotes \( a_{21} h - a_{11} = 0, \ a_{21} h + a_{22} = 0 \).

Switching to \( A = +1 \) in formula (2.13), representing an orthogonal hyperbola with branches in the second and fourth quadrant, requires the replacement of \( a_{12} \) in the first formula of (2.11) and in (2.12) by \( \bar{a}_{12} \) determined from

\[ \bar{a}_{12} + a_{12} = \frac{2a_{11} a_{22}}{a_{21}}, \quad (2.14) \]

which may be checked easily.

With the condition \( A = \pm 1 \) it is clear that all even powers of \( A \) yield +1. As a consequence the \( A^{2r} \) in the third formula of (2.11) can be omitted.

To make the sign of \( \phi \) and of \( \phi_1 \), (and also of \( p \) and \( p_1 \)) the same we put \( \alpha = +1 \) for \( a_{21} h + a_{22} > 0 \) and \( \alpha = -1 \) for \( a_{21} h + a_{22} < 0 \). Inspection then shows that the formulae presented here are indeed the generalizations of the formulae presented in Section 2.1. In particular the \( A \) and \( \alpha \) here present a
generalization of A and $\alpha$ in Section 2.1 and their values assumed here can be taken from Table 1.

One checks, that the magnitude of the hyperbola is varied by modifying $a_{21}^{-2}$, when (2.13) is rewritten in the form

$$h_1 - \frac{a_{11}}{a_{21}} = - \frac{1}{a_{21}} \frac{A}{h + \frac{a_{22}}{a_{21}}}.$$  \hspace{1cm} (2.15)

From the formulae (2.11) the transformation formulae for the other parameters were obtained in Section 8 of Ref. [1]. With the conditions (1) and (2) of this Section they take the form

$$K_1 = \frac{\alpha}{a_{21} h + a_{22}} K,$$

$$x_1 = \frac{\alpha}{A} \left\{ (a_{21} h + a_{22})x - a_{21}\phi \right\} = \frac{\alpha}{A} (a_{21} \psi + a_{22} x),$$

$$p_1 = \frac{\alpha}{a_{21} h + a_{22}} p \hspace{1cm} V_1 = \alpha (a_{21} h + a_{22})^3 V,$$

$$u_1 = \frac{\alpha}{A} \left\{ (a_{21} h + a_{22})u + a_{21} K \right\} = \frac{\alpha}{A} (a_{21} L + a_{22} u),$$ \hspace{1cm} (2.16)

$$a_1 = \alpha (a_{21} h + a_{22}) a = \alpha (a_{21} h + a_{22} a),$$

$$\frac{a_1}{V_1} = (a_{21} h + a_{22})^{-2} \frac{a}{V},$$

$$a_1 h_1 = \alpha (a_{11} h + a_{12}) a = \alpha (a_{11} h + a_{12} a),$$

$$L_1 = K_1 + h_1 u_1 = \frac{\alpha}{A} (a_{11} L + a_{12} u).$$

The transformations of $x_1$ and $\psi_1 = h_1 x_1 - \phi_1$, those of $u_1$ and $L_1$, and those of $h_1 a_1$ and $a_1$ can be taken together in matrix form. The last two yield

$$\begin{bmatrix} L_1 \\ u_1 \end{bmatrix} = \frac{\alpha}{A} \begin{bmatrix} A \\ L \end{bmatrix} \begin{bmatrix} h \\ a \end{bmatrix}, \hspace{1cm} \begin{bmatrix} h_1 a_1 \\ a_1 \end{bmatrix} = \alpha \begin{bmatrix} A \\ a \end{bmatrix} \begin{bmatrix} h \\ a \end{bmatrix},$$ \hspace{1cm} (2.17)
with \([A]\) denoting the matrix

\[
[A] = \begin{bmatrix}
    a_{11} & a_{12} \\
    a_{21} & a_{22}
\end{bmatrix}.
\]  

(2.18)

It is of interest to note that the transformation (2.11) induces linear transformations for the parameters \(L\) and \(u\), and for the parameters \(h\alpha\) and \(a\), which are the elements of the classical Riemann-invariants \(r\) and \(s\) and of the generalized Riemann-invariants \(r^*\) and \(s^*\). The inverse transformations are easily checked to yield

\[
h = \frac{a_{22}h - a_{12}}{-a_{21}h + a_{11}}, \quad \phi = \frac{A}{\alpha} \frac{1}{-a_{21}h + a_{11}} \phi_1,
\]

\[
B(h) = \frac{1}{\alpha} \left( -\frac{a_{21}h + a_{11}}{A} \right)^{3Y-1} B_1(h_1),
\]

\[
K = \frac{A}{\alpha} \frac{1}{-a_{21}h + a_{11}} K_1,
\]

\[
x = \frac{1}{\alpha} \left\{ (-a_{21}h + a_{11})x_1 + a_{21}\phi_1 \right\} = \frac{1}{\alpha} \left( -a_{21}h_1 + a_{11}x_1 \right),
\]

\[
p = \frac{A}{\alpha} \frac{1}{-a_{21}h + a_{11}} p_1, \quad V = \frac{1}{\alpha A} \left( -a_{21}h + a_{11} \right)^3 V_1,
\]

\[
u = \frac{1}{\alpha} \left\{ (-a_{21}h + a_{11})u_1 - a_{21}K_1 \right\} = \frac{1}{\alpha} \left( -a_{21}h + a_{11}u_1 \right),
\]

\[
L = \frac{1}{\alpha} (a_{22}L_1 - a_{12}u_1),
\]

\[
a = \frac{1}{\alpha A} (-a_{21}h + a_{11})a_1 = \frac{1}{\alpha A} (-a_{21}h_1 a_1 + a_{11}a_1),
\]

\[
\frac{a}{V} = (-a_{21}h + a_{11})^{-2} \frac{a_1}{V_1},
\]

\[
ha = \frac{1}{\alpha A} (a_{22}h - a_{12})a_1 = \frac{1}{\alpha A} (a_{22}h_1 a_1 - a_{12}a_1).
\]
Denoting the matrix of the inverse transformation by \( A^{-1} \), with
\[
\begin{bmatrix}
A^{-1}
\end{bmatrix} = \begin{bmatrix}
a_{22} & -a_{12} \\
-a_{21} & a_{11}
\end{bmatrix}, \quad \begin{bmatrix}
A
\end{bmatrix} \begin{bmatrix}
A^{-1}
\end{bmatrix} = 1, \quad \begin{bmatrix}
\text{I}
\end{bmatrix} = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix},
\]
and \( \text{I} \) denoting the identity matrix, one deduces from (2.19) or from (2.17) and (2.20) that
\[
\begin{bmatrix}
L \\
u
\end{bmatrix} = \frac{1}{\alpha} \begin{bmatrix}
A^{-1}
\end{bmatrix} \begin{bmatrix}
L_1 \\
u_1
\end{bmatrix}, \quad \begin{bmatrix}
a
\end{bmatrix} = \frac{1}{\alpha A} \begin{bmatrix}
A^{-1}
\end{bmatrix} \begin{bmatrix}
h_1 a_1
\end{bmatrix}.
\]
(2.21)
which are again linear transformations as found before.

2.3. The Infinitesimal Transformations

In Sections 10 and 11 of Ref. [1] the infinitesimal transformations, their operators and extended operators were constructed and discussed. Some further points of a general kind will be added here.

The identity transformation in the group is
\[
h_1 = h, \quad \phi_1 = \phi, \quad B(h_1) = B(h),
\]
(2.22)
and requires \( a_{11} = a_{22} = 1, a_{12} = a_{21} = 0, \alpha = +1, A = +1 \). The infinitesimal transformation may be obtained by taking
\[
a_{11} = 1 + \delta a_{11}, \quad a_{12} = \delta a_{12}, \quad a_{21} = \delta a_{21}, \quad a_{22} = 1 - \delta a_{11},
\]
(2.23)
with \( \delta a_{ij} \) denoting an arbitrary infinitesimal increment of the coefficients of the group. The fourth relation in (2.23) follows from the condition \( A = +1 \) which requires
\[
\delta a_{11} + \delta a_{22} = 0,
\]
(2.24)
when only terms linear in \( \delta a_{ij} \) are retained.

When the transformation \( h + h_1 \) is worked out with the coefficients \( a_{ij} \) given by (2.23) one finds
\[
h_1 = h + \delta h = h + \delta a_{12} + 2h \delta a_{11} - h^2 \delta a_{21},
\]
(2.25)
with \( \delta h \) denoting the infinitesimal increment of \( h \).

The method employed here is slightly less general than the one used in Ref [1], as discussed for example on p.130 of Ref. [2]. The notation employed here, to
which we shall adhere, also differs from the one in Ref. [1], where formula (2.25) appears as (10.13) reading

$$\delta h_1 = \lambda \delta \tau + h_1 \mu \delta \tau + h_1^2 \nu \delta \tau,$$

(10.13)

The two notations are reconciled by putting

$$\delta a_{12} = \delta \tau, \quad \delta a_{11} = \frac{1}{2} \mu \delta \tau, \quad \delta a_{21} = - \nu \delta \tau.$$

(2.26)

By using formula (2.26) and the results found in Ref. [1], or by perturbing the identity transformation in the way used here for $h \rightarrow h$, one obtains the infinitesimal increments of the different parameters. They have been listed in Table 2.

**Table 2**

**Table of Infinitesimal Increments.**

<table>
<thead>
<tr>
<th>$\delta a_{12}$</th>
<th>$\delta a_{11}$</th>
<th>$\delta a_{21}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta h$</td>
<td>1</td>
<td>2h</td>
</tr>
<tr>
<td>$\delta b$</td>
<td>-</td>
<td>$-(3Y-1)B(h)$</td>
</tr>
<tr>
<td>$\delta \Phi$</td>
<td>-</td>
<td>$\Phi$</td>
</tr>
<tr>
<td>$\delta \Psi$</td>
<td>x</td>
<td>$\Psi$</td>
</tr>
<tr>
<td>$\delta \Theta$</td>
<td>-</td>
<td>$\Theta$</td>
</tr>
<tr>
<td>$\delta \chi$</td>
<td>x</td>
<td>$\chi$</td>
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<td>$\delta K$</td>
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<td>u</td>
<td>N</td>
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<td>-</td>
<td>$-x$</td>
</tr>
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<tr>
<td>$\delta G$</td>
<td>V</td>
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<td>V</td>
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<td>$-a$</td>
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<tr>
<td>$\delta (\frac{a}{V})$</td>
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<td>$\delta (ha)$</td>
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<td>ha</td>
</tr>
<tr>
<td>$\delta r$</td>
<td>-</td>
<td>$-r$</td>
</tr>
<tr>
<td>$\delta s$</td>
<td>-</td>
<td>$-s$</td>
</tr>
<tr>
<td>$\delta r^*$</td>
<td>r</td>
<td>$r^*$</td>
</tr>
<tr>
<td>$\delta s^*$</td>
<td>s</td>
<td>$s^*$</td>
</tr>
</tbody>
</table>

From the infinitesimal increments one easily writes down the infinitesimal operators of the group as demonstrated in Section 10 of Ref. [1].

This may be repeated here by taking successively one $\delta a_{ij}$ different from zero.

Considering a function $F(h,p,B)$ and denoting its infinitesimal increment, due to increments of the coefficients of the group by
\[ \delta F = \frac{\partial F}{\partial h} \delta h + \frac{\partial F}{\partial p} \delta p + \frac{\partial F}{\partial B} \delta B, \]  

(2.27)

one obtains for \( \delta a_{12} \neq 0 \)

\[ V_{1F} = \frac{\delta F}{\delta a_{12}} = \frac{\partial F}{\partial h}, \]  

(2.28)

for \( \delta a_{11} \neq 0 \)

\[ V_{2F} = \frac{\delta F}{\delta a_{11}} = 2h \frac{\partial F}{\partial h} + p \frac{\partial F}{\partial p} - (3Y-1)B \frac{\partial F}{\partial B}, \]  

(2.29)

and for \( \delta a_{21} \neq 0 \)

\[ V_{3F} = \frac{\delta F}{\delta a_{21}} = -h^2 \frac{\partial F}{\partial h} - hp \frac{\partial F}{\partial p} + (3Y-1)hB \frac{\partial F}{\partial B}, \]  

(2.30)

in agreement with (10.22) in Ref. [1], apart from a \(-\) sign and a factor \( \frac{1}{2} \).

Finally it should be observed that the infinitesimal increments of the different physical parameters are not independent. Since the equations of motion remain invariant under the group, the infinitesimal increments have to satisfy certain 'perturbation equations'.

Starting from the equations of motion in the form

\[ \frac{\partial V}{\partial t} - \frac{\partial u}{\partial h} = 0, \quad \frac{\partial u}{\partial t} + \frac{\partial p}{\partial h} = 0, \quad pV_1 \gamma = B_1(h_1), \]  

(2.31)

one obtains

\[ \frac{\partial}{\partial t} (V + \delta V) - \frac{\partial (u + \delta u)}{\partial (h + \delta h)} = 0, \]  

(2.32)

and upon expansion

\[ \left( \frac{\partial V}{\partial t} - \frac{\partial u}{\partial h} \right) + \left[ \frac{\partial}{\partial t} (\delta V) - \frac{\partial}{\partial h} (\delta u) + \frac{\partial u}{\partial h} \frac{\partial}{\partial h} (\delta h) \right] = 0, \]  

(2.33)
where the expression in braces represents the 'perturbation equation'.

Treating the other equations in (2.31) in the same way we obtain the three perturbation equations

\[
\frac{\partial}{\partial t} (\delta V) - \frac{\partial}{\partial h} (\delta u) + \frac{\partial u}{\partial h} \frac{\partial}{\partial h} (\delta h) = 0,
\]

\[
\frac{\partial}{\partial t} (\delta u) + \frac{\partial}{\partial h} (\delta p) - \frac{\partial p}{\partial h} \frac{\partial}{\partial h} (\delta h) = 0,
\]

\[
\gamma p V^{\gamma-1} \delta V + V^{\gamma} \delta p - \delta B = 0.
\]

These perturbation equations have not to be solved but are satisfied automatically when the increments of Table 2 are substituted. It is easily verified by taking in succession \(\delta a_{12} = 0\), \(\delta a_{11} = 0\) and \(\delta a_{21} = 0\).

Considering the characteristic equations in the same way one obtains the perturbation equations for the characteristics in the form

\[
d(\delta h) - \delta \left(\frac{a}{V}\right) d t = 0,
\]

\[
d(\delta p) + \delta \left(\frac{a}{V}\right) d u + \frac{a}{V} d(\delta u) = 0,
\]

when starting from the equations for the \(r\)-characteristics. From the equations for the \(s\)-characteristics one obtains

\[
d(\delta h) + \delta \left(\frac{a}{V}\right) d t = 0,
\]

\[
d(\delta p) - \delta \left(\frac{a}{V}\right) d u - \frac{a}{V} d(\delta u) = 0.
\]

3. THE TRANSFORMATIONS OF THE UNIFORM HOMENTROPIC FLOW

3.1. The homentropic flow

Consider a uniform gas at constant pressure \(p=p_o\) and with constant specific volume \(V=V_o\), moving with constant velocity \(u=u_o\). The gas is homentropic and so

\[
p V^{\gamma} = p_o V_o^{\gamma} = B_o = B_o^{\gamma},
\]

(3.1)
with $B_0$ and $b_0$ constant, while $\gamma$ is the constant ratio of the specific heats $c_p$ and $c_v$.

The mass of gas is considered to be of indefinite extent, but if desired one can imagine the flow to occur in a straight tube of constant cross section and of indefinite length. Also it may be convenient to consider the mass of gas to be semi-infinite and to be bounded on one side by a plane or piston moving with the speed of the gas $u = u_0$. This type of flow is the simplest flow situation imaginable.

One verifies that the flow is a proper solution of the equations of motion in the Lagrangian variables

$$\frac{\partial \nu}{\partial t} - \frac{\partial u}{\partial h} = 0 , \quad \frac{\partial u}{\partial t} + \frac{\partial p}{\partial h} = 0 . \quad (3.2)$$

In addition to the parameters mentioned most other flow parameters are constant. Examples are

the temperature \hspace{1cm} $T = T_0 = \frac{1}{R} \frac{p_0}{V_0}$,

the speed of sound \hspace{1cm} $a = a_0 = \sqrt{\gamma p_0 V_0}$, \hspace{1cm} (3.3)

the specific acoustic impedance \hspace{1cm} $\frac{a}{V} = \frac{a_0}{V_0} = \sqrt{\gamma \frac{p_0}{V_0}}$.

The Riemann invariants $r$ and $s$, in general constant along $r-$, respectively along $s$-characteristics and defined by

$$r = u + \frac{2}{\gamma-1} a , \quad s = u - \frac{2}{\gamma-1} a , \quad (3.4)$$

assume in this case values $r = r_0$ and $s = s_0$ also constant throughout the flow.

The characteristics of the flow are two sets of parallel straight lines, both in the $h,t-$ and in the $x,t-$plane. In the $h,t-$plane they are moreover symmetric with respect to lines $h = \text{const}$. This is also the case in the $x,t-$plane provided $u=0$, i.e. when the gas is at rest.

In order to apply the Stanyukovich Transformation the potentials of first and second order are required i.e. $E = x, K$ and $\phi$. These are defined in Refs. [3,1] and appeared already in Chapter 2. In the case here one finds

$$E = x = u_0 t + V_0 h + x_0 ,$$

$$K = p_0 t - u_0 h + K_0 , \quad (3.5)$$

$$\phi = -\frac{1}{2} p_0 t^2 + u_0 h t + \frac{1}{2} V_0 h^2 + x_0 h - K_0 t + \phi_0 ,$$

with $x_0, K_0$ and $\phi_0$ denoting integration constants, which determine the origin of $x$ etc. If the integration constants are all zero $E$ and $K$ are homogeneous of degree one in $h$ and $t$, while $\phi$ is homogeneous of degree two.
It is desirable to consider also the parameters $r^*$ and $s^*$, the generalized Riemann-invariants of the LMS-gas, defined by

$$r^* = K + h(u + \frac{2}{Y-1} a), \quad s^* = K + h(u - \frac{2}{Y-1} a). \quad (3.6)$$

In the homentropic flow considered here $r^*$ and $s^*$ assume the values

$$r^* = K_0 + p_0 t + \frac{2}{Y-1} a_0 h, \quad s^* = K_0 + p_0 t - \frac{2}{Y-1} a_0 h, \quad (3.7)$$

which are linear in $h$ and in $t$. Considering the values of $r^*$ and $s^*$ along a characteristic in the $h,t$-plane either $h$ or $t$ may be eliminated from (3.7) leaving expressions linear in $h$ or linear in $t$.

3.2. The Transformations of the homentropic flow

The original form of the transformation as presented by Stanyukovich in Ref. [4] applies to the parameters $h,p$ and $B(h) = B_0$ and reads

$$h_1 = \frac{1}{h}, \quad p_1 = \frac{p}{h}, \quad B_1(h_1) = B_0 h_1^{3Y-1}, \quad (3.8)$$

where the parameters of the transformed flow are denoted with subscript one. In Ref. [11] and in Chapter 2 the Stanyukovich Transformation applies to begin with to $h, \phi$ and $B(h)$ and appears in three forms of increasing complexity and generality. They may be summarized as Cases (i), (ii) and (iii).

Case (i) Only the first quadrant of the $h,h_1$-plane is used so that $h$ and $h_1$ are positive.

Case (ii) The entire $h,h_1$-plane may be used but the orthogonal hyperbolae have the coordinate axes as asymptotes and the magnitude of the curves is fixed.

Case (iii) The entire $hh_1$-plane may be used, the orthogonal hyperbolae can have their centre at an arbitrary point in the $hh_1$-plane and their magnitude can be varied.

The last form Case (iii) is the three parameter continuous transformation group.

In this Section we first apply the form of Case (i) upon the flow of Section 3.1., with $0 \leq h \leq + \infty$ and discuss the physical features of the transformed flow.

We then apply the transformation in the form of Case (ii). It will become clear that no new physical features appear and that four flows of the type obtained from Case (i) are generated.

Finally the most general form Case (iii) will be applied and it will be shown that the freedom this transformation offers has some advantages when applications are considered but the physical content remains as before.
Case (i) The mass $0 \leq h \leq + \infty$ of the homentropic flow in Section 3.1 is mapped into $0 \leq h_1 \leq + \infty$ and straightforward substitution of the parameters in Section 3.1 into the transformation formulae of Section 4 of Ref. [1] then yields

\[ \phi_1 = -\frac{1}{2} p_0 t^2 h_1 + u_0 t + \frac{1}{2} \frac{V_0}{h_1} - 1 + x_0 - K_0 h_1 t + \phi_0 h_1, \]

\[ K_1 = p_0 h_1 t - u_0 + K_0 h_1, \quad x_1 = -\frac{1}{2} p_0 t^2 - \frac{1}{2} \frac{V_0}{h_1} - 1 + K_0 t + \phi_0, \]

\[ p_1 = p_0 h_1, \quad u_1 = -p_0 t - K_0, \quad V_1 = V_0 h_1^{-3}, \]

\[ a_1 = a_0 h_1^{-1}, \quad \frac{a_1}{V_1} = \frac{a_0}{V_0} h_1^2, \quad T_1 = T_0 h_1^{-2}. \]

Equation (3.9)

The pressure $p_1$ in the transformed flow is a linear function of $h_1$ and so the pressure gradient

\[ \frac{\partial p_1}{\partial h_1} = p_0, \]  \hspace{1cm} (3.10)

is constant throughout the mass of gas and in time, resulting in a constant deceleration of the gas. As a consequence the velocity $u_1$ decreases linear with time as shown in (3.9). The flow can be described as a rigid-body motion with a constant deceleration and without time dependent compressions and expansions. Other physical properties of the transformed flow have been discussed in Ref. [5].

It may be verified that the relations in (3.9) represent a proper solution of the equations of motion for the LMS-gas

\[ \frac{\partial V_1}{\partial t} - \frac{\partial u_1}{\partial h_1} = 0, \quad \frac{\partial u_1}{\partial t} + \frac{\partial p_1}{\partial h_1} = 0, \quad p_1 V_1^{Y-1} = B_0 h_1^{-3(Y-1)}. \]  \hspace{1cm} (3.11)

The Riemann-invariants $r$ and $s$ in (3.4) transform into

\[ r = u + \frac{2}{Y-1} a = -K_1 - h_1 \left( u_1 - \frac{2}{Y-1} a_1 \right) = -s^*, \]

\[ s = u - \frac{2}{Y-1} a = -K_1 - h_1 \left( u_1 + \frac{2}{Y-1} a_1 \right) = -r^*, \]  \hspace{1cm} (3.12)

in agreement with (2.7) and (2.8). Since $r$ and $s$ are constant throughout the flow of Section 3.1 it follows that in the transformed flow $r^*_1$ and $s^*_1$ are constant throughout. The transformed flow obtained here was studied first in Ref. [5] by considering the flow of an LMS-gas where the generalized Riemann-
invariants \( r^* \) and \( s^* \) are both constant throughout. One also observes that the \( r \)-
characteristics of the homentropic flow transform into \( s_1 \)-characteristics in the
transformed flow, while \( s \)-characteristics transform into \( r_1 \)-characteristics.

Subjecting the expressions \( r^* \) and \( s^* \) in (3.6) and (3.7) to the transformation
one obtains

\[
\begin{align*}
    r^* &= -u_1 + \frac{2}{\gamma-1} \frac{a_1}{p_0 t + K_o} + \frac{2}{\gamma-1} h_1^{-1} a_o = -s_1, \\
    s^* &= -u_1 - \frac{2}{\gamma-1} \frac{a_1}{p_0 t + K_o} - \frac{2}{\gamma-1} h_1^{-1} a_o = -r_1,
\end{align*}
\]

(3.13)

which shows that \( s_1 \) and \( r_1 \) are linear in \( t \) and \( h_1^{-1} \).

The straight characteristics in the \( h,t \)-plane of the homentropic flow transform
into orthogonal hyperbolas in the \( h_1,t \)-plane of the transformed flow. It is
easily shown by direct integration of the characteristics equations

\[
\frac{dh_1}{h_1} + \frac{a_1}{V_1} dt = \frac{dh_1}{h_1} + \frac{a_o}{V_0} h_1^2 dt = 0.
\]

(3.14)

Employing the integrals of (3.14) one can eliminate either \( t \) or \( h_1^{-1} \) from (3.13)
to show that along a characteristic in the transformed flow \( s_1 \) and \( r_1 \) are linear
in \( t \) or in \( h_1^{-1} \), a result which is the counterpart of the property obtained in
connection with (3.7) in Section 3.1.

Case (ii) The mass of gas stretches now from \( -\infty \) to \( +\infty \) where at \( h=0 \) a plane
may be assumed to move with the gas velocity. Straightforward substitution of
the parameters for the flow in Section 3.1 into the formulae (2.3) and (2.4)
yields all the parameters for the transformed flow. Restricting us to the most
significant one's we find

\[
\begin{align*}
    h_1 &= -\frac{A}{h}, \quad p_1 = -\frac{a}{A} p_0 h_1^{-1}, \quad u_1 = \frac{a}{A} (p_0 t + K_o).
\end{align*}
\]

(3.15)

In each quadrant of the \( h_1 \)-plane this yields an MS-gas moving as a rigid body
with a constant acceleration and a pressure distribution linear in \( h_1 \) of the
same kind as found for Case (i). In Figure 1 the connections have been clarified
by plotting the original \( h \)-axis, the \( h,h_1 \)-plane and the \( p_1,h_1 \)-diagrams. The
letters and symbols in the different figures clarify the relationships.

Case (iii) Again the mass stretches from \( -\infty \) to \( +\infty \) but we use the transformation
formulae of the group in Section 2.2. Substitution of the flow parameters
of the homentropic flow in Section 3.1 then yields amongst others
\[ p_1 = \frac{\alpha}{a_{21} + a_{22}} p_0 = -\frac{\alpha}{A} (a_{21} h_1 - a_{11}) p_0 , \]

\[ u_1 = \frac{\alpha}{A} [a_{21} (h u_0 + K) + a_{22} u_0] = \]

\[ = \frac{\alpha}{A} [a_{21} p_0 t + (a_{21} K_0 + a_{22} u_0)] , \]

\[ \frac{\partial p_1}{\partial h_1} = -\frac{\alpha}{A} a_{21} p_0 , \quad \frac{\partial u_1}{\partial t} = \frac{\alpha}{A} a_{21} p_0 , \]

and the linear pressure distribution, together with the constant acceleration and deceleration, reappears.

However as pointed out before more freedom is obtained in the location of the four quadrants in the hh\_1-plane, the centres and asymptotes of the hyperbolae and their magnitudes.

It is clear from (3.16) with \( p_1 = 0 \) for \( h_1 = \frac{a_{11}}{a_{21}} \) etc. and in the appearance of \( a_{21} \) in the relations for the pressure gradients and accelerations.

To demonstrate this further we consider the flow studied in Ref. [5], where a finite mass \( H \) of an LMS-gas was considered, with a linear pressure distribution \( p_1 = \beta h_1 (0 \leq h_1 \leq H) \).

To map the domain \( 0 \leq h \leq +\infty \) in the hh\_1-plane on the finite segment of \( 0 \leq h_1 \leq H \), a hyperbola as shown in Fig. 2 can be used leading to

\[ h_1 = \frac{1}{a_{21}} \cdot \frac{1}{h + \frac{a_{22}}{a_{21}}} , \quad (3.17) \]

and to

\[ h \to +\infty , \quad h_1 = 0 , \quad (3.18) \]

\[ h = 0 , \quad h_1 = H = \frac{1}{a_{21} a_{22}} , \]

Since \( a_{11} = 0 \) in this case \( A = -a_{12} a_{21} = -1 \), we can express the non-zero \( a_{ij} \)'s as follows

\[ a_{12} = \frac{1}{a_{21}} , \quad a_{22} = \frac{1}{a_{21} H} , \quad (3.19) \]

where \( a_{21} \) can be chosen freely. Considering the transformation of the pressure and the pressure chosen
\[ p_1 = -\frac{\alpha}{\alpha} (a_{21} h_1 - a_{11}) = a_{21} h_1 p_o = \beta_o h_1, \]  
(3.20)

one finds
\[ a_{21} = \frac{\beta_o}{p_o}, \]  
(3.21)

and the transformation is now uniquely determined, when the operation is restricted to the first quadrant of the hh_*-plane, determined from the hyperbola (3.17). For the transformed velocity one now finds
\[ u_1 = -\left(\frac{\beta_o}{p_o} K_o + \frac{u_0 p_o}{\beta_o h_*}\right) = \beta_o t, \]  
(3.22)

in agreement with the earlier results.

3.3. The Infinitesimal Transformations

In this section the infinitesimal transformations will be considered. They will be applied first to the constant homentropic flow of Section 3.1 and then to the flows of the LMS-gas discussed in Section 3.2. In each case three different possibilities have to be distinguished given by

\[ \begin{align*}
(1) & \quad \delta a_{12} = 0 , \quad \delta a_{11} = 0 , \quad \delta a_{21} = 0 , \\
(2) & \quad \delta a_{12} = 0 , \quad \delta a_{11} \neq 0 , \quad \delta a_{21} = 0 , \\
(3) & \quad \delta a_{12} = 0 , \quad \delta a_{11} = 0 , \quad \delta a_{21} \neq 0,
\end{align*} \]  
(3.23)

and yielding together 6 sets of perturbations. Once these 6 sets of perturbations have been considered, also some linear combinations of (1) (2) and (3) in (3.23) will be discussed.

We begin with the three sets of perturbations for the uniform homentropic flow discussed in Section 3.1.

**Case 1.** \( \delta a_{12} = 0 \), \( \delta a_{11} = \delta a_{21} = 0 \).

From Table 2 it is observed that the infinitesimal increments of the flow parameters mostly vanish. A consequence is that the uniform homentropic flow, remains unmodified. The only significant change is a slight increment in \( h \). One has from Table 2
\[ \delta h = \delta a_{12} , \quad h_1 = h + \delta a_{12}, \]  
(3.24)

and for \( h = 0 \) one has \( h_1 = \delta a_{12} \), indicating a shift of the origin. As a consequence, the flow parameters in Table 2, which contain \( h \) explicitly are modified. One has
\[ \delta L = \delta (K + u h) = u \delta a_{12} = u_0 \delta a_{12}, \]
\[ \delta (ha) = a \delta a_{12} = a_0 \delta a_{12}, \]  \hspace{1cm} (3.25)
\[ \delta r^* = r \delta a_{12} = r_0 \delta a_{12}, \quad \delta s^* = s \delta a_{12} = s_0 \delta a_{12}. \]

Case 2. \[ \delta a_{12} = \delta a_{21} = 0, \quad \delta a_{11} = 0. \]

From the second column in Table 2 we have the increments
\[ \delta h = 2h \delta a_{11}, \quad \delta p = p \delta a_{11} = p_0 \delta a_{11}, \quad \delta B = -(3y-1)B_0 \delta a_{11}, \]
\[ \delta V = -3V \delta a_{11} = -3V_0 \delta a_{11}, \quad \delta u = -u_0 \delta a_{11}, \quad \delta a = -a_0 \delta a_{11}. \]  \hspace{1cm} (3.26)

From the increments in (3.26) it is noted that \( \delta h \) is proportional to \( h \), representing a uniform stretch of the \( h_1 \)-coordinate, with respect to the \( h \)-coordinate.

Since most physical parameters of the homentropic flow in Section 3.1. are constant, the infinitesimal increments are also constant throughout the flow. In particular \( \delta B \) is constant so that the flow after transformation is again homentropic. It follows that the physical character of the flow is essentially the same as before the transformation. One may verify that the perturbation equations (2.34) are satisfied. The slope of the characteristics in the \( h_1 \)-plane changes with the constant amount
\[ \delta \left( \frac{a}{V} \right) = 2 \frac{a}{V} \delta a_{11} = 2 \frac{a_0}{V_0} \delta a_{11}, \]  \hspace{1cm} (3.27)

but they remain straight and parallel.

Case 3. \[ \delta a_{12} = \delta a_{11} = 0, \quad \delta a_{21} = 0. \]

For the infinitesimal increments we now find
\[ \delta h = -h^2 \delta a_{21}, \quad \delta p = -hp_0 \delta a_{21}, \quad \delta B = (3h-1)hB_0 \delta a_{21}, \]
\[ \delta V = 3hV_0 \delta a_{21}, \quad \delta u = L \delta a_{21}, \quad \delta a = ha_0 \delta a_{21}, \quad \delta \left( \frac{a}{V} \right) = -2 \frac{a_0}{V_0} h \delta a_{21}, \]
\[ \delta L = 0, \quad \delta (ha) = 0, \quad \delta r^* = 0, \quad \delta s^* = 0, \]  \hspace{1cm} (3.28)
\[ \delta r = r^* \delta a_{21}, \quad \delta s = s^* \delta a_{21}. \]

One may verify that the perturbation equations (2.34) are satisfied. The flow is no longer homentropic since the perturbation \( \delta B \) is linear in \( h \). The increment \( \delta h \) is quadratic in \( h \), yielding parabolic curves for \( \delta h \) as function of \( h \).
An infinitesimal pressure decrement \( \delta p \) proportional to \( h \) is superposed upon \( p_o \), and since it may be checked that \( L = K_o + p_o t \), an infinitesimal acceleration \( p_o \delta a_{21} \), independent of \( h \) is established, in addition to an infinitesimal constant jump \( K_o \delta a_{21} \) in the velocity \( u = u_o \).

Considering the characteristic equations

\[
dh_1 + \frac{a_1}{V_1} \, dt = d(h - h^2 \delta a_{21}) + \frac{a_o}{V_o} (1 - 2h \delta a_{21}) \, dt = 0 ,
\]

one observes that the slope \( \frac{dh}{dt} = \frac{a_o}{V_o} \) is modified with an infinitesimal change, proportional to \( h \), yielding an infinitesimal change of \( h \), proportional to \( h^2 \).

The characteristics in the \( h,t \)-plane therefore no longer remain straight, since infinitesimal parabola's are superposed. They are sketched in Fig. 3.

For the Riemann invariants \( r_1 \) and \( s_1 \), and the generalized Riemann-invariants \( r_*^1 \) and \( s_*^1 \) one finds

\[
r_*^1 = r^* , \quad s_*^1 = s^* , \quad \delta r^* = 0 , \quad \delta s^* = 0 ,
\]

\[
r_1 = r_o + \delta r = r_o + r^* \delta a_{21} =
\]

\[
= (u_o + \frac{2}{\gamma - 1} a_o) + (K_o + p_o t + \frac{2}{\gamma - 1} a_o h) \delta a_{21} ,
\]

\[
s_1 = s_o + \delta s = s_o + s^* \delta a_{21} =
\]

\[
= (u_o - \frac{2}{\gamma - 1} a_o) + (K_o + p_o t - \frac{2}{\gamma - 1} a_o h) \delta a_{21} .
\]

It follows that \( r \) and \( s \) are no longer constant throughout the flow as in Section 3.1, since infinitesimal parts linear in \( h \) and \( t \) are superposed.

To clarify this we note that the perturbed flow is no longer homentropic. Therefore generally speaking no Riemann-invariants can be expected to exist. However by restricting the perturbations to the domain within the group, which contains the homentropic flow and the LMS-gas, the damage to the Riemann-invariants remains limited and only a perturbation linear in \( h \) and \( t \) has to be superposed upon the constant values \( r_o \) and \( s_o \) of the initial flow.

Considering \( r_1 \) along an \( r \)-characteristic, the characteristic can be taken in the unperturbed form since \( r_o \) is constant throughout the flow and since only terms linear in \( \delta a_{12} \) are retained. Eliminating either \( h \) or \( t \) then leaves increments along the characteristic which are either linear in \( t \) or linear in \( h \).

This completes the first part of the discussion. Next the infinitesimal transformations will be applied to the flow of the LMS-gas. We limit ourselves to Case (i) of the analysis presented there. Since this flow is considered here as the 'original flow', before the infinitesimal transformation is applied, we adjust the notation and rewrite the formulae (3.9) etc. without subscript '1' as follows.
\[
\phi = -\frac{1}{2} p_o t^2 h + u_0 t + \frac{1}{2} v_0 h^{-1} + x_0 - K_o h t + \phi_0 h,
\]

\[
K = p_o h t - u_0 + K_o h, \quad x = -\frac{1}{2} p_o t^2 - \frac{1}{2} v_0 h^{-2} - K_o t + \phi_o,
\]

\[
p = p_o h, \quad u = -p_o t - K_o, \quad V = v_0 h^{-3}, \tag{3.32}
\]

\[
a = a_0 h^{-1}, \quad \frac{a}{V} = \frac{a_0}{v_0} h^2, \quad T = T_0 h^{-2},
\]

\[
L = K = Hu - u_0, \quad ah = 0, \quad \psi = -v_0 h^{-1} - x_0 - u_0 t.
\]

Further we have \(0 \leq h \leq +\infty\) and

\[
B(h) = B_0 h^{-(3Y-1)}, \quad pV = B_0 h^{-(3Y-1)}, \tag{3.33}
\]

while (3.12) and (3.13) can be written without the subscripts '1' in the form

\[
s^* = K + h(u - \frac{2}{Y-1} a) = \text{const.},
\]

\[
r^* = K + h(u + \frac{2}{Y-1} a) = \text{const.},
\]

\[
s = -K_0 - p_o t - \frac{2}{Y-1} a_0 h^{-1} = u - \frac{2}{Y-1} a, \tag{3.34}
\]

\[
r = -K_0 - p_o t + \frac{2}{Y-1} a_0 h^{-1} = u + \frac{2}{Y-1} a.
\]

With these relations as starting point we discuss the three infinitesimal transformations listed in (3.23).

**Case 1.** \(\delta a_{12} = 0\), \(\delta a_{11} = \delta a_{21} = 0\).

It is again clear from the first column in Table 2 that most physical parameters remain unchanged, and therefore also the character of the flow remains the same. Only \(h\) changes into

\[
h + \delta h = h + \delta a_{12}, \quad \delta h = \delta a_{12}, \tag{3.35}
\]

and the parameters containing \(h\) explicitly are modified. We find for example
\[ \delta L = \delta(K + hu) = u \delta a_{12}, \quad \delta(ah) = a \delta a_{12}, \]

\[ \delta r^* = \delta(K + hr) = r \delta a_{12}, \quad \delta s^* = \delta(K + hs) = s \delta a_{12}. \quad (3.36) \]

It is of interest to note that although the physical parameters and properties of the flow remain unchanged the parameter \( r_1^* \) is no longer a constant, due to its dependence upon \( h \). We have from (3.34)

\[ r_1^* = r^* + \delta r^* = r^* + r \delta a_{12} = \]

\[ = (L + \frac{2}{Y-1} ah) + (u + \frac{2}{Y-1} a) \delta a_{12} = \quad (3.37) \]

\[ = (-u_0 + \frac{2}{Y-1} a_0) - (p_0 t + K_o - \frac{2}{Y-1} a_0 h^{-1}) \delta a_{12}, \]

a relation that shows some analogy with (3.31). Considering \( r_1^* \) along a characteristic \( t \) or \( h^{-1} \) can be eliminated, leaving an expression linear in \( h^{-1} \) or in \( t \), as far as the infinitesimal increment is concerned.

**Case 2.** \( \delta a_{11} = 0 \), \( \delta a_{12} = \delta a_{21} = 0 \).

The infinitesimal increments in this case are taken from the second column in Table 2. They are

\[ \delta h = 2h \delta a_{11}, \quad \delta p = p \delta a_{11}, \quad \delta B = -(3Y-1)B \delta a_{11}, \quad \delta V = -3V \delta a_{11}, \quad (3.38) \]

\[ \delta u = -u \delta a_{11}, \quad \delta a = -a \delta a_{11}, \quad \delta\left(\frac{a}{V}\right) = 2 \frac{a}{V} \delta a_{11}, \quad \delta L = L \delta a_{11}, \]

\[ \delta(ah) = ah \delta a_{11} = a_0 \delta a_{11}. \]

In the same way as for the earlier Case 2 the 'uniform stretch' of the \( h \)-coordinate may be noticed. The pressure \( p \), linear in \( h \) (Sec. (3.32)), obtains an infinitesimal increment proportional to \( p \). The pressure gradient \( \frac{\partial p}{\partial h} \), which is given by \( p_0 \) in the original flow, would seem to increase to \( (1 + \delta a_{11}) p_0 \). Considering also \( \delta u = -u \delta a_{11} \), and noticing that \( u = -K_o - p_0 t \) from (3.32), yielding \( \delta u = (p_0 t + K_o) \delta a_{11} \), one notices that \( \frac{\partial}{\partial t} (\delta u) = p_0 \delta a_{11} \) and so the retardation changes from the value \(-p_0\) to \(-p_0(1 - \delta a_{11}) \), which would seem to disagree with the increment of the pressure gradient. The feature we have overlooked so far is the 'stretching' of \( h \), which also affects the pressure gradient. Taking this into account, by considering the second equation in (2.34), it is easily checked that the change in the pressure gradient, represented by the second and third term in the equation cancel the term \( \frac{\partial}{\partial t} (\delta u) = p_0 \delta a_{11} \).
It is clear that the transformed flow is of the same kind as the original flow. In the original flow \( r^* \) and \( s^* \) are constant, while for the transformed flow

\[
\begin{align*}
    r_1^* &= r^* + \delta r^* = r^*(1 + \delta a_{11}) , \\
    s_1^* &= s^* + \delta s^* = s^*(1 + \delta a_{11}) .
\end{align*}
\] (3.39)

Finally also the other perturbation equations in (2.34) together with (2.35) and (2.36) may be verified.

**Case 3.** \( \delta a_{21} \neq 0 \), \( \delta a_{12} = \delta a_{11} = 0 \)

In this case the infinitesimal increments, taken from Table 2, are

\[
\begin{align*}
    \delta h &= - h^2 \delta a_{21} , \\
    \delta p &= - h p \delta a_{21} , \\
    \delta B &= (3Y-1) hB \delta a_{21} , \\
    \delta V &= 3hV \delta a_{21} , \\
    \delta u &= L \delta a_{21} , \\
    \delta a &= ha \delta a_{21} , \\
    \delta (\frac{a}{V}) &= - 2 \frac{a}{V} h \delta a_{21} , \\
    \delta L &= \delta (ha) = 0 \end{align*}
\] (3.40)

The coordinate \( h \) acquires an increment quadratic in \( h \). The pressure distribution linear in \( h \), changes with a part quadratic in \( h \)

\[
\begin{align*}
    p_1 &= p + \delta p = p - h p \delta a_{21} = p_0 h - p_0 h^2 \delta a_{21} .
\end{align*}
\] (3.41)

The pressure gradient would seem to change from its constant value when (3.41) is considered. However the increment of \( u \) is the constant \( L \delta a_{21} \), with no time dependent terms and no change of acceleration. Further consideration shows that the change of \( h \), due to the transformation, was again overlooked. When this is taken into account the second equation of (2.34) is satisfied with \( \frac{\partial}{\partial t} (\delta u) = 0 \).

Also in this case one has

\[
\begin{align*}
    \delta r^* &= 0 , \\
    \delta s^* &= 0 , \\
    \delta r &= r^* \delta a_{21} , \\
    \delta s &= s^* \delta a_{21} ,
\end{align*}
\] (3.42)

which completes the discussion.

Inspection of \( \delta h \) for the three cases, considered twice, once for the constant homogeneous flow and once for the LMS-gas, shows that for Cases 2 and 3 the increment \( \delta h \) vanishes only for \( h = 0 \). This indicates a preferential position for the location \( h = 0 \), and is unnatural. In order to remove this restriction linear combinations of Cases 1, 2 and 3 have to be considered, or expressed differently, we have to allow \( \delta h \) to take the general form

\[
\delta h = \delta a_{12} + 2h \delta a_{11} - h^2 \delta a_{21} .
\] (3.43)

Consider for example the constant stretch

\[
\delta h = 2h \delta a_{11} ,
\] (3.44)

of Case 2 with \( \delta h = 0 \) only for \( h = 0 \). If we desire \( \delta h = 0 \) for \( h = h_0 \neq 0 \) we have to take
\[ \delta h = \delta a_{12} + 2h \delta a_{11}, \]  
(3.45)

with
\[ 0 = \delta a_{12} + 2h_0 \delta a_{11}, \]  
(3.46)

and yielding
\[ \delta a_{12} = -2h_0 \delta a_{11}, \quad \delta h = 2(h-h_0) \delta a_{11}. \]  
(3.47)

From Table 2 the infinitesimal increments of the other parameters can now easily be obtained with \( \delta a_{12} \) given by (3.47).

In a similar fashion one can proceed with
\[ \delta h = -h^2 \delta a_{21}, \]  
(3.48)

which can be modified into
\[ \delta h = -(h-h_0)^2 \delta a_{21}, \]  
(3.49)

by taking
\[ \delta a_{12} = -h_0^2 \delta a_{21}, \quad \delta a_{11} = h_0 \delta a_{21}, \]  
(3.50)

or into
\[ \delta h = -(h-h_0)(h-h_1) \delta a_{21}, \]  
(3.51)

with
\[ \delta a_{12} = -h h_0 \delta a_{21}, \quad \delta a_{11} = \frac{1}{2}(h_0 + h_1) \delta a_{21}. \]  
(3.52)

It is of interest to note that with \( h \) ranging over the interval \( 0 \leq h \leq +\infty \), \( \delta h \) is only positive when (3.44) is used, but is both positive (for \( h > h_0 \)) and negative (for \( 0 \leq h \leq h_0 \)) when (3.47) is employed.

Again for (3.48) and (3.49) \( \delta h \) can only be negative but for (3.51) \( \delta h \) can be negative and positive (for \( h_0 \leq h \leq h_1 \)).

With the values of \( \delta a_{12} \) in (3.47) and the values of \( \delta a_{12} \) and \( \delta a_{11} \) in (3.50) and (3.52) the Tables of infinitesimal increments for the different parameters have been constructed from Table 2. They are shown in Table 3.
Table 3
The complementary Table of Inf. Increments

<table>
<thead>
<tr>
<th>$\delta a_{11}$</th>
<th>$\delta a_{21}$</th>
<th>$\delta a_{21}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta a_{12} = -2h_0 \delta a_{11}$</td>
<td>$\delta a_{11} = h_0 \delta a_{21}$</td>
<td>$\delta a_{12} = \frac{1}{2}(h_0 + h_1) \delta a_{21}$</td>
</tr>
<tr>
<td>$\delta a_{21} = 0$</td>
<td>$\delta a_{12} = -h_0^2 \delta a_{21}$</td>
<td>$\delta a_{12} = -h_0 h_1 \delta a_{21}$</td>
</tr>
<tr>
<td>$\delta h$</td>
<td>$2(h-h_0)$</td>
<td>$-(h-h_0)^2$</td>
</tr>
<tr>
<td>$\delta B$</td>
<td>$-(3Y-1)B(h)$</td>
<td>$(3Y-1)(h-h_0)B(h)$</td>
</tr>
<tr>
<td>$\delta \phi$</td>
<td>$\phi$</td>
<td>$-(h-h_0)\phi$</td>
</tr>
<tr>
<td>$\delta \psi$</td>
<td>$\psi - 2h_0x$</td>
<td>$h_0\psi - h_0^2x$</td>
</tr>
<tr>
<td>$\delta \theta$</td>
<td>$\theta$</td>
<td>$-(h-h_0)\theta$</td>
</tr>
<tr>
<td>$\delta x$</td>
<td>$x - 2h_0x$</td>
<td>$htK + h_0x - h_0^2x$</td>
</tr>
<tr>
<td>$\delta K$</td>
<td>$K$</td>
<td>$-(h-h_0)K$</td>
</tr>
<tr>
<td>$\delta L$</td>
<td>$L - 2h_0u$</td>
<td>$h_0L - h_0^2u$</td>
</tr>
<tr>
<td>$\delta M$</td>
<td>$M$</td>
<td>$-(h-h_0)M$</td>
</tr>
<tr>
<td>$\delta N$</td>
<td>$N - 2h_0u$</td>
<td>$htp + h_0N - h_0^2u$</td>
</tr>
<tr>
<td>$\delta E = \delta x$</td>
<td>$-x$</td>
<td>$\psi - h_0x$</td>
</tr>
<tr>
<td>$\delta F$</td>
<td>$-F$</td>
<td>$(h-h_0)F - \theta$</td>
</tr>
<tr>
<td>$\delta G$</td>
<td>$G - 2h_0V$</td>
<td>$(h^2-h_0^2)V + (h-h_0)G + \phi$</td>
</tr>
<tr>
<td>$\delta H$</td>
<td>$-H - 2h_0V$</td>
<td>$(h^2-h_0^2)V + (h-h_0)H + \theta$</td>
</tr>
</tbody>
</table>
Continue Table 3.

<table>
<thead>
<tr>
<th>$\delta a_{11}$</th>
<th>$\delta a_{21}$</th>
<th>$\delta a_{31}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta p$</td>
<td>$p$</td>
<td>$-(h-h_0)p$</td>
</tr>
<tr>
<td>$\delta u$</td>
<td>$-u$</td>
<td>$L - uh_0$</td>
</tr>
<tr>
<td>$\delta v$</td>
<td>$-3v$</td>
<td>$3(h-h_0)v$</td>
</tr>
<tr>
<td>$\delta a$</td>
<td>$-a$</td>
<td>$a(h-h_0)$</td>
</tr>
<tr>
<td>$\delta(\frac{a}{V})$</td>
<td>$2 \frac{a}{V}$</td>
<td>$-2 \frac{a}{V}(h-h_0)$</td>
</tr>
<tr>
<td>$\delta(ah)$</td>
<td>$a(h-2h_0)$</td>
<td>$ah_0(h-h_0)$</td>
</tr>
</tbody>
</table>

| $\delta r$ | $-r$ | $r^* - rh_0$ | $r^* - \frac{h_0^o + h_1}{2} r$ |
| $\delta s$ | $-s$ | $s^* - sh_0$ | $s^* - \frac{h_0^o + h_1}{2} s$ |
| $\delta r^*$ | $r^* - 2h_0 r$ | $h_0(r^*-h_0 r)$ | $r^* - \frac{h_0^o + h_1}{2} - rh_0 h_1$ |
| $\delta s^*$ | $s^* - 2h_0 s$ | $h_0(s^*-h_0 s)$ | $s^* - \frac{h_0^o + h_1}{2} - sh_0 h_1$ |

By substituting the different parameters of the constant homentropic flow, as given in Section 3.1, or of the non-homentropic LMS gas given in (3.32), again the infinitesimal increments or perturbations can be discussed. We shall refrain from a detailed discussion, which may be left to the reader, and restrict ourselves to one remark. For the perturbations of the LMS-gas, shown in (3.40), (3.41) and (3.42) it was found the $\frac{\partial}{\partial t}(\delta u) = 0$. Considering the non-homentropic LMS-gas case again but for the $\delta a_{21}$ in the third and second column of Table 3 one finds
\[ \delta u = \{- u_o + \frac{1}{2} (h_o + h_1)(p_o t + K_o)\} \delta a_{21}, \]

\[ \delta u = \{- u_o + h_o (p_o t + K_o)\} \delta a_{21}, \]

which now contain time dependent terms and yield infinitesimal accelerations given by

\[ \frac{\partial}{\partial t} (\delta u) = \frac{h_o + h_1}{2} p_o \delta a_{21}, \quad \frac{\partial}{\partial t} (\delta u) = h_o p_o \delta a_{21}. \]
4. THE TRANSFORMATION OF A NORMAL SHOCK-WAVE PROPAGATING INTO A UNIFORM HOMENTROPIC GAS

4.1. The homentropic flow

Consider a plane normal shock-wave propagating with constant speed into a uniform homentropic gas with pressure \( p_1 \), constant velocity \( u_1 \), and specific volume \( V_1 \). Denoting the parameters behind the shock-wave with subscript 2, and the shock trajectory in Lagrangian coordinates by

\[
\begin{align*}
  h &= H(t) = t H(t), \\
  \dot{H}(t) &= \frac{dH(t)}{dt} = \text{const.,}
\end{align*}
\]

the Lagrangian shock-wave relations may be written

\[
\begin{align*}
  p_2 - p_1 &= Z, \\
  u_2 - u_1 &= \frac{Z}{H(t)}, \\
  V_2 - V_1 &= -\frac{Z}{H(t)^2}, \\
  Z &= \frac{2V_2}{Y+1} \left[ \left( \frac{V_2}{V_1} \right)^2 - H(t)^2 \right] = \frac{2V_1}{Y+1} \left[ H(t)^2 - \left( \frac{a_1}{V_1} \right)^2 \right].
\end{align*}
\]

The mass of gas may stretch from \( h = -\infty \) to \( h = +\infty \), so that the shock-wave started a long time ago \( t = -\infty \) from a point far to the left \( h = -\infty \). Also one may take the mass to be semi-infinite \( 0 < h < +\infty \), bounded by a plane or piston at \( h = 0 \), which at \( t = 0 \), changed its velocity impulsively from the velocity \( u = u_1 \), to the larger value \( u = u_2 \). Also it is of interest to have the shock-wave travel from right to left \( H(t) < 0 \), with the higher pressure on the right and the lower pressure on the left of the shock-wave, the piston being located at \( h = +\infty \). The Lagrangian shock-wave relations (42) are equivalent to the more familiar Rankine-Hugoniot conditions and it is not difficult to convert the relations into each other.

Since the Lagrangian variables are used in the Stanyukovich transformation it is preferable to use the not so familiar Lagrangian shock-wave relations.

Comparing the problem here, with the one considered in Section 3.1, we have at each instant \( t \) two uniform domains of the type discussed there, separated by a normal shock-wave and resulting in a continuous growth of the domain behind the shock-wave at the cost of the domain in front of it. This holds true whether the shock-wave travels from left to right or from right to left.

It was found in Chapter 3 that the Stanyukovich Transformation transforms a homentropic uniform gas, moving at constant speed, into an LMS-gas with a linear pressure distribution, moving as a rigid body with a constant acceleration or deceleration.

It is clear for the problem here, that the uniform domains each transform into an LMS-gas with linear pressure distribution.
The question which arises is whether the shock-wave relations transform by the Stanyukovich Transformation in shock-wave relations for the LMS-gas. This question will be considered in the next section.

4.2. The Stanyukovich Transformation of the Lagrangian Shock-Wave relations

Since the two sides of the shock wave have been denoted by subscript 1 and 2, we denote the parameters after the Stanyukovich Transformation by a superscript *. The transformation applicable to the first quadrant of the \( h, h^* \)-plane then reads

\[
h^* = \frac{1}{h}, \quad p^* = \frac{p}{h}, \quad B^* = B_0 h^{3/2}, \tag{4.3}
\]

equivalent with (3.8).

The domains with constant pressures \( p_1 \) and \( p_2 \) before the transformation then pass into

\[
p_1^* = p_1 h^*, \quad p_2^* = p_2 h^*. \tag{4.4}
\]

The constant \( B_0 \) in (4.3) will be denoted as \( B_1 \), respectively \( B_2 \) for the two domains. Clearly we have two different cases to distinguish shown in Figs. 4 and 5. The first one corresponds with the shock-wave travelling to the right before the transformation, and to the left after the transformation while the second figure corresponds with the shock-wave travelling to the left before the transformation and to the right after the transformation.

If the shock-wave trajectory after the transformation is denoted by

\[
h^* = H^*(t), \tag{4.5}
\]

in analogy with (4.1), we have

\[
h^* = H^*(t) = \frac{1}{H(t)} = \frac{1}{t H(t)}. \tag{4.6}
\]

The speed of the transformed shock-wave is then

\[
\dot{H}^*(t) = \frac{dH^*(t)}{dt} = -\frac{H(t)}{H(t)^2} = -\frac{1}{t^2 H(t)}. \tag{4.7}
\]

since \( H(t) = \text{const} \). The shock trajectory in the \( h^*, t \)-plane is then an orthogonal hyperbola instead of the straight line before transformation in the \( h, t \)-plane.

To transform the shock-wave relations (4.2) we employ the transformation formulae

\[
h = \frac{1}{h^*}, \quad p = \frac{1}{h^*} p^*, \quad V = (h^*)^2 V^*, \tag{4.8}
\]

\[
u = -(h^* u^* + K^*), \quad K = \frac{1}{h^*} K^*.
\]

which may be taken from (2.3) or from Ref [1].
From relation (3.5) we have for the two uniform domains denoted with subscripts 1 and 2

\[ K_1 = p_1 t - u_1 h + K_{10}, \tag{4.9} \]
\[ K_2 = p_2 t - u_2 h + K_{20}, \]

with \( K_{10} \) and \( K_{20} \) denoting integration constants. At the shock-wave location at the instant \( t \), we have from (4.1)

\[ K_2 - K_1 = (p_2 - p_1) t - (u_2 - u_1) h H(t) + K_{20} - K_{10}, \tag{4.10} \]

and requiring \( K \), similar to \( x \), to be continuous at the shock-wave we find upon employing (4.3)

\[ K_{20} = K_{10}. \tag{4.11} \]

From the formula (4.8) applied at the shock-wave we now obtain

\[ p_2 - p_1 = \frac{1}{H^*(t)} (p_2^* - p_1^*), \]
\[ V_2 - V_1 = H^*(t)^3 (V_2^* - V_1^*), \tag{4.12} \]
\[ u_2 - u_1 = -H^*(t) (u_2^* - u_1^*) - (K_2^* - K_1^*). \]

From the last formula in (4.8) applied at the shock-wave it follows that the continuity of \( K \) at the shock-wave also leads to the continuity of \( K^* \) and so

\[ K_{2}^* - K_{1}^* = 0. \tag{4.13} \]

Finally from (4.6) we deduce

\[ \dot{H}(t) = -\frac{1}{H^*(t)} \frac{\dot{H^*}(t)}{H^*(t)^2} = \text{const}. \tag{4.14} \]

Substituting (4.12) and (4.14) into (4.2) then yields
\[ u_2^* - u_1^* = \frac{1}{H^*(t)} (p_2^* - p_1^*), \]
\[ v_2^* - v_1^* = -\frac{1}{H^*(t)^2} (p_2^* - p_1^*), \]
\[ p_2^* - p_1^* = \frac{2}{\gamma+1} v_2^* \left( \frac{p_2^*}{v_2^*} - H^*(t)^2 \right) = \frac{2}{\gamma+1} v_1^* \left( H^*(t)^2 - \frac{p_1^*}{V_1^*} \right), \]

which again are the Lagrangian shock-relations but for the parameters after transformation. The shock-relations therefore are invariant under the Stanyukovich Transformation, or expressed differently shock-waves transform into shock-waves. This is not altogether surprising since the shock-wave relations are obtained from the equations of motion, which are invariant under the transformation.

A restriction of the calculations presented in this Section is that the transformation formulae (4.8) are not the most general form of the Transformation formulae. It has been verified, and can be repeated by the reader, that the shock-wave relations also are invariant under the general Transformation formulae of Section 2.2, instead of the formulae (4.8) which correspond with (2.3).

### 4.3. The problem for a finite mass of gas

So far it was assumed that the mass of gas in the homentropic case and after the transformation for the LMS-case is semi-infinite and stretches over \( 0 \leq h \leq +\infty \) respectively \( 0 \leq h^* \leq +\infty \).

The simple inversion relation between \( h \) and \( h^* \) in (4.3) has the consequence that a piston located at \( h = 0 \) in the homentropic case is mapped into \( h^* = +\infty \) and is therefore 'removed from sight'. It will be more realistic, in particular for the LMS-gas, if everything remains visible i.e. in the range of finite \( h^* \).

In order to map the domain \( 0 \leq h \leq +\infty \) on a finite segment \( 0 \leq h^* \leq H^*_0 \), with \( H^*_0 \) denoting the total mass of LMS-gas, the general transformation formulae may be used, as already done in Section 3.2 for Case (iii). From the formulae (3.16) - (3.22) most of the answer required can be taken over with a slight change of notation.

Assuming that for \( t < 0 \), the homentropic gas has the constant velocity \( u = u_1^* \), and that at \( t = 0 \) the velocity of the piston at \( h = 0 \), impulsively jumps to the value \( u_2^* \), with \( u_2^* > u_1^* \), the shock-wave originates at \( t = 0 \) from \( h = 0 \). It is of interest to pursue in detail, what happens to the shock-wave and to the piston located at \( h^* = H^*_0 \) in the LMS-gas. We shall see that for a chosen total mass \( H^*_0 \) detailed relations for the speed and acceleration of shock-wave and piston appear.

It may be remarked that another way of considering a finite mass is to take the answers for the infinite mass \( 0 \leq h \leq +\infty \), \( 0 \leq h^* \leq +\infty \) and consider only the answers for the interval \( a < h < b \), with planes or pistons located at \( h = a \) and \( h = b \). Some shifts of origins of time etc. are then required to put these answers in the most convenient form. However this point of view will not be
pursued and we return to the former approach, mapping $0 \leq h \leq +\infty$ onto $0 \leq h^* \leq H_0$.

We begin with the formulation of the homentropic flow. A semi-infinite amount of homentropic uniform gas, stretching over $0 \leq h \leq +\infty$ has pressure $p = p_1$, speed $u = u_1$, and specific volume $V = V_1$, all assumed constant. At $h = 0$ the gas is bounded by a plane or piston, also moving with speed $u = u_1$. At time $t = 0$, the velocity of the piston jumps to the constant value $u = u_2$, with $u_2 > u_1$.

Instantaneously a shock-wave is formed, departing from the piston at $t = 0$ and travelling into the gas with a constant speed $H(t)$. The speed of the shock-wave the pressure $p = p_2$, behind the shock-wave, and the specific volume $V = V_2$, all constant for $t > 0$, are related by the equations (4.2), so that

$$u_2 - u_1 = \frac{2}{\gamma + 1} \frac{V_1}{H(t)} \left[ H(t)^2 - \left( \frac{a_1}{V_1} \right)^2 \right]. \quad (4.16)$$

From this relation one easily finds

$$H(t) = \frac{\gamma + 1}{4} \frac{u_2 - u_1}{V_1} + \sqrt{\left( \frac{\gamma + 1}{4} \frac{u_2 - u_1}{V_1} \right)^2 + \left( \frac{a_1}{V_1} \right)^2}. \quad (4.17)$$

Since $H(t) > 0$ the positive root has to be taken in (4.17). From the equation (4.2) now $p_2$ and $V_2$ easily follow.

Next the flow will be transformed into the flow of an LMS-gas. The total mass after transformation is chosen to be $h^* = H_0^*$, and to occupy $0 \leq h^* \leq H_0^*$.

We consider first the situation for $t < 0$, when the shock-wave has not yet been formed. From the discussion in Section 3.2 and the formulae (3.16) - (3.22) we have with

$$A = -1, \quad \alpha = +1, \quad a_{11} = 0, \quad (4.18)$$

that

$$a_{12} = \frac{1}{a_{21}}, \quad a_{22} = \frac{1}{a_{21}} H_0^*. \quad (4.19)$$

Selecting the total mass $H_0^*$ after transformation leaves only $a_{21}$ to be chosen.

In the discussion of Case (iii) in Section 3.2 the pressure gradient and the chosen acceleration determined $a_{21}$. Here we have from (3.16) - (3.22)
\[ a_{21} h^* = \frac{1}{a_{21} h + a_{22}}, \]

\[ p_1^* = a_{21} p_1 h^*, \]

\[ V_1^* = V_1 (a_{21} h^*)^{-3}, \]

\[ u_1^* = -a_{21} (p_1 t + K_{10}) - a_{22} u_1, \]

with \( a_{22} \) given by (4.19) and \( K_{10} \) an integration constant.

Next we turn to the situation for \( t > 0 \). For the homentropic case the shock-wave trajectory and speed are given in (4.1), valid for \( t > 0 \).

Denoting the shock-wave trajectory in the LMS-gas by (4.5)

\[ h^* = H^*(t), \]  

it follows from (4.20)

\[ a_{21} H^*(t) = \frac{1}{a_{21} H(t) + a_{22}}, \]  

with \( H(t) \) given by (4.1) and \( a_{22} \) by (4.19).

The relation (4.21) shows that at \( t = 0 \) the location of the shock-wave is \( h^* = H^*(0) = H_* \), as required, while for increasing \( t \), the values of \( H^*(t) \) decrease reaching \( h^* = 0 \) for \( t \to \infty \).

Differentiation of (4.21) to \( t \) yields the speed and the acceleration of the shock-wave expressed in the mass-coordinate i.e. mass-flux etc. One obtains

\[ a_{21} H^*(t) = -\frac{1}{(a_{21} H(t) + a_{22})^2} a_{21} \dot{H}(t), \]  

\[ a_{21} \ddot{H}^*(t) = -\frac{a_{21} \ddot{H}(t)}{(a_{21} H(t) + a_{22))^2} + \frac{2}{(a_{21} H(t) + a_{22})^3} [a_{21} \dot{H}(t)]^2 \]

Since the shock-wave is formed instantaneously at \( t = 0 \), due to the impulsive change in velocity of the piston, one can consider \( H(t) \) to have the character of a unit-step function, and its derivative \( \ddot{H}(t) \) to be a delta function. It follows that the acceleration of the shock-wave has to be considered infinite at \( t = 0 \), when it emerges. For \( t > 0 \) the acceleration \( \ddot{H}(t) \) is zero. From (4.22) and (4.1),
with \( H(t) = \text{const.} \), and \( \dot{H}(t) = 0 \) for \( t > 0 \) one obtains the values of \( H^*(t) \) and \( \dot{H}^*(t) \) in the LMS-gas, with \( H^*(t) \neq 0 \).

The flow between the piston located at \( h^* = H^*_0 \) and the shock-wave located at \( h^* = H^*(t) \), at time \( t \), is determined by the expressions

\[
\begin{align*}
p_2^* &= a_{21} p_2 h^*, \\
W_2^* &= V_2 \left(a_{21} h^*_0\right)^{-3}, \\
u_2^* &= -a_{21} (p_2 + K_0) - a_{22} u_2,
\end{align*}
\]

which again are obtained from (3.16) - (3.22).

To keep things simple we now select \( a_{21} \) by requiring

\[
\begin{align*}
p_1^* &= p_1 \quad \text{at} \quad h^* = H^*_0, \quad t < 0, \\
p_2^* &= p_2 \quad \text{at} \quad h^* = H^*_0, \quad t > 0,
\end{align*}
\]

Then

\[
a_{21} H^*_0 = 1, \quad a_{21} = \frac{1}{H^*_0}, \quad a_{12} = \frac{H^*_0}{H^*_0}, \quad a_{22} = 1,
\]

and the relations (4.20) - (4.23) can be rewritten with the \( a_{ij} \)'s from (4.25).

For the position velocity and acceleration of the shock-wave for \( t > 0 \) one obtains from (4.21) and (4.22)

\[
\begin{align*}
H^*(t) &= \frac{(H^*_0)^2}{H^*_0 + tH(t)}, \\
H^*(t) &= -\frac{\left(\frac{H^*(t)}{H^*_0}\right)^2}{H^*_0} H(t), \\
\dot{H}^*(t) &= 2 \left(\frac{H^*(t)}{H^*_0}\right)^3 \frac{H(t)^2}{H^*_0}\frac{H^*_0}{H^*}
\end{align*}
\]

with \( H(t) = \text{const.} \).

It is again observed from (4.26) that the shock-wave departs at \( t = 0 \) from \( h = H^*(0) = H^*_0 \) and arrives at \( h = 0 \) for \( t = +\infty \). Its velocity is negative, as it should, but the magnitude decreases when time proceeds, resulting in a positive acceleration.
For the velocities and accelerations of the gas in the two domains separated by the shock-wave we have from (4.20) and (4.23)

\[
\begin{align*}
  u_1^* &= - \frac{1}{H_0^*} (p_1 t + K_{10}) - u_1, \\
  u_2^* &= - \frac{1}{H_0^*} (p_2 t + K_{10}) - u_2,
\end{align*}
\] (4.27)

where we also used (4.11) and (4.25), and

\[
\frac{\partial u_1^*}{\partial t} = - \frac{p_1}{H_0^*}, \quad \frac{\partial u_2^*}{\partial t} = - \frac{p_2}{H_0^*},
\] (4.28)

in agreement with the pressure gradients in (4.20) and (4.23).

Considering the piston, located at \( h^* = 0 \), it is clear that at the instant \( t = 0 \), both its velocity, and its acceleration have to change impulsively, in agreement with (4.27) and (4.28) in order to realize the motion considered here. Finally from (4.20), (4.23), employing also (4.25) we observe that

\[
\begin{align*}
  p_1^* (V_1^*)^Y &= p_1 V_1 Y \left( \frac{h^*}{H_0^*} \right)^{- (3Y-1)} = B_1 \left( \frac{h^*}{H_0^*} \right)^{- (3Y-1)} = \exp \left( \frac{S_1}{c_v} \right), \\
  p_2^* (V_2^*)^Y &= p_2 V_2 Y \left( \frac{h^*}{H_0^*} \right)^{- (3Y-1)} = B_2 \left( \frac{h^*}{H_0^*} \right)^{- (3Y-1)} = \exp \left( \frac{S_2}{c_v} \right),
\end{align*}
\] (4.29)

demonstrating that the jump in entropy \( S_2 - S_1 \) is independent of \( h^* \), and constant throughout the entire motion of the shock-wave.

The shock-wave problems considered in this Chapter were already considered in Refs. [6,7], where some further points of discussion can be found.
5. THE TRANSFORMATION OF A HOMENTROPIC GAS WITH A CONSTANT ACCELERATION

5.1. The homentropic flow

It was found in Chapter 3, that the uniform homentropic gas, moving at constant speed, is transformed by the Stanyukovich Transformation into an LMS-gas, with a constant acceleration and a pressure distribution linear in the Lagrangian mass coordinate \( h \).

In this chapter we consider a homentropic gas moving as a rigid body with a constant acceleration and a pressure distribution linear in \( h \). We study its transformation when subjected to the Stanyukovich transformation.

Consider the flow of a homentropic gas with

\[
p = p_o + \beta_0 h, \quad u = u_o - \beta_0 t, \tag{5.1}\]

If the homentropic relation is written again as in (3.1)

\[
p V^Y = B_0 = b_o^Y, \tag{3.1}\]

with \( B_0 \) and \( b_o \) constants, one easily finds

\[
V = b_o p^{1 \over Y}, \quad T = {b_o \over \gamma} p^{Y-1 \over Y},
\]

\[
a = \sqrt{Y b_o p^{2Y \over Y-1}}, \quad {a \over V} = \sqrt{\gamma b_o p^{2Y \over Y+1}}. \tag{5.2}\]

One may check that the relations (5.1), (5.2) represent a proper solution of the equations of motion (3.2) and (3.1).

If the gas stretches from \( 0 \leq h \leq +\infty \) it is observed that for \( h = 0 \), \( p = p_o \) and at \( h = 0 \) a plane or piston moving with the speed \( u \) in (5.1) will be required to prevent the pressure from dropping to zero. Taking \( p_o = 0 \) the vacuum situation exists at \( h = 0 \) and the piston can be omitted.

In order to subject the solution (5.1), (5.2) to the Stanyukovich Transformation it is required to know also \( E(=x) \), \( K \) and \( \Phi \). They are defined in Refs. [3,1] and one easily finds

\[
E = x = x_o + u_o t - {\beta_0 \over 2} t^2 + {b_o \over \gamma - 1} p_o p^{Y-1 \over Y},
\]

\[
K = K_o - u_o h + p_o t + \beta_0 h t, \tag{5.3}\]

\[
\Phi = (\Phi_o + x_o h - K_o t) + u_o h t - {1 \over 2} \beta_0 h t^2 - {1 \over 2} p_o t^2 + {Y \over \gamma - 1} p_o p^{Y-1 \over 2Y-1} \beta_o^{2Y-1} p^{Y \over Y-1}.
\]
with $x_0$, $K_o$ and $\phi_o$ denoting integration constants. Two more parameters of interest are

$$L = K + hu = K_o + p_o t$$

$$h a = \sqrt{Yb_o h p} = \sqrt{Yb_o h (p_o + \beta_o h)^{2Y}}$$

(5.4)

For $\beta_o > 0$, which will be assumed throughout, from $h = 0$ onwards the pressure rises linear with $h$. The same observation applies to the temperature, though the temperature increase is not linear with $h$, and to the internal energy $U = c_v T$.

The situation differs essentially from the flow found in Section 3.2. While the relations for $p_1$ and $u_1$ in (3.9) are of the same form as $p$ and $u$ in [5.1], the relations in Section (3.2) apply to an LMS-gas.

As a consequence one finds from (3.9) at the vacuum location $h_1 = 0$ an infinite temperature, speed of sound and internal energy. Proceeding to increasing values of $h_1$ the temperature, speed of sound and internal energy steadily decrease to reach the values zero at $h_1 = +\infty$.

The difference in the distribution of the temperature and the internal energy for the homentropic- and LMS-gas, will give rise to a different behaviour of the gases, for example when they are allowed to expand.

It was pointed out to me some years ago by Professor F.A. Pillow of Brisbane and the late Professor J.M. Burgers of Maryland that the linear pressure distribution shown in (3.9) and in (5.1) is obtained naturally in a vertical column of gas at rest in a constant vertical gravitational field. To clarify this consider the equations of motion for a reference frame in which a constant gravitational acceleration $\beta_o$ acts downwards in the direction of increasing $h$. Denoting the velocity in this frame (positive in the direction of increasing $h$) by $\tilde{u}$, the equations of motion are

$$\frac{\partial v}{\partial t} - \frac{\partial \tilde{u}}{\partial h} = 0 \quad , \quad \frac{\partial \tilde{u}}{\partial t} + \frac{\partial p}{\partial h} = \beta_o$$

(5.5)

which can be rewritten in the form

$$\frac{\partial v}{\partial t} - \frac{\partial}{\partial h} (\tilde{u} - \beta_o t) = 0 \quad , \quad \frac{\partial}{\partial t} (\tilde{u} - \beta_o t) + \frac{\partial p}{\partial h} = 0$$

(5.6)

Upon setting

$$u = \tilde{u} - \beta_o t \quad , \quad \tilde{u} = u + \beta_o t$$

(5.7)

the equations of motion (3.2) are retrieved. Since $u$ and $\tilde{u}$ are counted positive in the direction of increasing $h$, which is also the direction of $\beta_o$, it is observed that the gas in the Section 3.2, formulae (3.9), will be at rest in the reference frame with gravity if

$$K_o = 0 \quad , \quad \beta_o = p_o$$

(5.8)
and for the gas in this section if \( u_0 = 0 \).

The reference frame for the homentropic gas, where (5.1) is \( u = -\beta_0 t \), is then 'falling-freely' with respect to the 'gravity-frame'.

This relationship appears again when the Cartesian coordinates \( x \) and \( \tilde{x} \) are introduced. From Ref. [1,3] one has

\[
\text{dx} = u \, dt + V \, dh ,
\]

and upon using the results (5.5) - (5.7) this can be rewritten

\[
\text{dx} = (\ddot{u} - \beta_0 t) \, dt + V \, ah = \ddot{u} \, dt + V \, dh - d\left(\frac{1}{2} \beta_0 t^2\right)
\]

\[
= \ddot{x} \, dt - d\left(\frac{1}{2} \beta_0 t^2\right) = d(\tilde{x} - \frac{1}{2} \beta_0 t^2) ,
\]

provided

\[
\ddot{x} = \ddot{u} \, dt + V \, dh , \quad x = \tilde{x} - \frac{1}{2} \beta_0 t^2 .
\]

The relation between \( \tilde{x} \) and \( x \) is again as in a frame with acceleration of gravity \( \beta_0 \) and in a frame, which is 'falling-freely'. This is easily verified provided the positive directions chosen are kept well in mind.

It is also of interest to consider the energy equation. From Ref. [3] one has

\[
\frac{\partial}{\partial t} \left[ c_v T + \frac{1}{2} u^2 \right] + \frac{3}{3h} (\text{up}) = 0 ,
\]

which contains the internal energy, the kinetic energy and the work term of the pressure force. Passing from the 'freely-falling' reference frame to the reference frame with gravity we obtain

\[
\frac{\partial}{\partial t} \left[ c_v T + \frac{1}{2} (\ddot{u} - \beta_0 t)^2 \right] + \frac{3}{3h} \left[ p(\ddot{u} - \beta_0 t) \right] = 0 .
\]

Expanding this equation, while using the expressions (5.5), (5.11), then yields

\[
\frac{\partial}{\partial t} \left[ c_v T + \frac{1}{2} \ddot{u}^2 - \beta_0 \ddot{x} \right] + \frac{3}{3h} (p \ddot{u}) = 0 .
\]

The third term in braces represents the potential energy in the gravity field, in agreement with the choice of the positive direction of \( x \) downwards.

Finally it should be pointed out that the homentropic flow given by (5.1) and (5.2) is not a constant flow or simple wave, but a 'general wave'. To demonstrate this consider the Riemann-invariants \( r \) and \( s \) and the characteristic equations.

The equations for the \( r \)-characteristics are

\[
\text{dh} - \frac{a}{V} \, dt = 0 , \quad \text{dp} + \frac{a}{V} \, du = 0 ,
\]

and for the \( s \)-characteristics
\[ dh + \frac{a}{V} \, dt = 0 \quad , \quad dp - \frac{a}{V} \, du = 0. \] (5.16)

The Riemann-invariants \( r \) and \( s \), constant along an \( r \)-characteristic, respectively an \( s \)-characteristic may be written

\[ r = u + \frac{2}{Y-1} \, a = u_0 - \beta_0 \, t + \frac{2}{Y-1} \sqrt{y b_o} \, p^{\frac{Y-1}{2Y}}, \] (5.17)

\[ s = u - \frac{2}{Y-1} \, a = u_0 - \beta_0 \, t - \frac{2}{Y-1} \sqrt{y b_o} \, p^{\frac{Y-1}{2Y}}, \]

with \( p \) given by (5.1). It is easily checked that integration of (5.15) and (5.16) using (5.1) yields essentially (5.17). From (5.17) one derives

\[ u = \frac{1}{2} \, (r + s) \quad , \quad t = \frac{u_0}{\beta_0} - \frac{1}{2 \beta_0} \, (r + s), \] (5.18)

\[ p = p_o + \beta_0 \, h = \left( \frac{y^{\frac{Y-1}{2Y}}}{4 \sqrt{y b_o}} \right) \, (r - s), \]

which are proper solutions in the two independent variables \( r \) and \( s \) of the Euler-Poisson-Darboux equations

\[ \frac{\partial^2 h}{\partial r \partial s} - \frac{1}{2} \, \frac{Y+1}{Y-1} \, \frac{1}{r-s} \left( \frac{\partial h}{\partial r} - \frac{\partial h}{\partial s} \right) = 0, \] (5.19)

\[ \frac{\partial^2 h}{\partial r \partial s} + \frac{1}{2} \, \frac{Y+1}{Y-1} \, \frac{1}{r-s} \left( \frac{\partial h}{\partial r} - \frac{\partial h}{\partial s} \right) = 0, \]

which is easily checked.

5.2. The Transformed flow

In this section the flow of the preceding section is subjected to the general Stanyukovich Transformation.

The transformation formulae for \( h \) are taken from Section 2.2 in the form

\[ h_{1} = \frac{a_{11} h + a_{12}}{a_{21} h + a_{22}}, \quad a_{21} h_{1} - a_{11} = - \frac{A}{a_{21} h + a_{22}}, \] (5.20)

together with its inverse

\[ h = \frac{a_{22} h_{1} + a_{12}}{a_{21} h_{1} - a_{11}}, \] (5.21)

Substitution in (5.1) yields
\[ p = p_0 - \beta_o \frac{a_{22} h_1 - a_{12}}{a_{21} h_1 - a_{11}}, \quad u = u_0 - \beta_o t, \quad (5.22) \]

and with the transformation formulae (2.16) one obtains upon using (5.2) and (5.3)
\[ p_1 = -\frac{a}{A} (a_{21} h_1 - a_{11}) p - \frac{a}{A} \left[(a_{21} p_0 - a_{22} \beta_o) h_1 - (a_{11} p_0 - a_{12} \beta_o)\right] \quad (5.23) \]

The transformations of \( L \) and \( u \) are the linear transformations
\[
\begin{bmatrix} L_1 \\ u_1 \end{bmatrix} = \frac{a}{A} \begin{bmatrix} A & 0 \\ A & 0 \end{bmatrix} \begin{bmatrix} L \\ u \end{bmatrix} = \frac{a}{A} \begin{bmatrix} K_0 + p_0 t \\ u_0 - \beta_o t \end{bmatrix},
\quad (5.24) \]

and this yields
\[ u_1 = \frac{a}{A} \left[(a_{21} K_0 + a_{22} u_0) + (a_{21} p_0 - a_{22} \beta_o) t\right]. \quad (5.25) \]

Clearly the transformed flow has again a pressure distribution linear in \( h_1 \), and a constant acceleration.

The relations now apply however to an LMS-gas with
\[ p_1 v_1^Y = B_1(h_1) = a^{Y+1} \left(-\frac{a_{21} h_1 - a_{11}}{A}\right)^{-3Y-1} B_0 \quad (5.26) \]

For \( \beta_o = 0 \) the flows in this Chapter reduce to the flows considered in Chapter 3.

If the transformation is selected such that
\[ a_{21} p_0 - a_{22} \beta_o = 0, \quad (5.27) \]
the transformed flow has a constant pressure \( p \), and the acceleration vanishes, yielding a constant \( u_1 \).

Flows with constant pressure and velocity, but non-uniform entropy, specific volume and temperature occur as contact regions in wave interaction experiments in shock-tubes. Also they can be produced with special efforts in a gas at rest \((u_1 = 0)\).

In a homentropic gas with \( p = \text{const}., u = \text{constant} \) the Riemann-invariants \( r \) and \( s \) are also constant. However for the LMS-gas with \( p_1 = \text{const.} \) and \( u_1 = \text{const.} \) this is not the case, and also the generalized Riemann invariants \( r_1^* \) and \( s_1^* \) are not constant. Since the non-homentropic flow of the LMS-gas with \( p_1 = \text{const.} \) and \( u_1 = \text{const.} \) results from the Stanyukovich transformation when applied to a homentropic 'general wave' this needs not come as a surprise.

To consider it we study the transformations of \( r \) and \( s \), and of \( r^* \) and \( s^* \).

From (2.17) we have
\[
\begin{bmatrix}
    h_1 & a_1 \\
    a_1
\end{bmatrix} = \alpha
\begin{bmatrix}
    A \\
    a
\end{bmatrix}
\begin{bmatrix}
    h_a \\
    a
\end{bmatrix},
\]
while we recall the definitions
\[
\begin{aligned}
    r &= u + \frac{2}{Y-1} a, &
    r^* &= K + hr = L - \frac{2}{Y-1} h_a, \\
    s &= u - \frac{2}{Y-1} a, &
    s^* &= K + hs = L - \frac{2}{Y-1} h_a.
\end{aligned}
\]

It is then not difficult to deduce the transformations
\[
\begin{aligned}
    \begin{bmatrix}
    r^*_1 \\
    r_1
\end{bmatrix} &= \alpha 
\begin{bmatrix}
    A \\
    a
\end{bmatrix}
\begin{bmatrix}
    L + \frac{2}{Y-1} A h_a \\
    u + \frac{2}{Y-1} A a
\end{bmatrix}, \\
    \begin{bmatrix}
    s^*_1 \\
    s_1
\end{bmatrix} &= \alpha 
\begin{bmatrix}
    A \\
    a
\end{bmatrix}
\begin{bmatrix}
    L - \frac{2}{Y-1} A h_a \\
    u - \frac{2}{Y-1} A a
\end{bmatrix},
\end{aligned}
\]

or written out in detail
\[
\begin{aligned}
    r^*_1 &= \frac{\alpha}{A} \left[ a_{11} (K_0 + p_0 t + \frac{2}{Y-1} A h_a) + a_{12} (u_0 - \beta_0 t + \frac{2}{Y-1} A a) \right] \\
    s^*_1 &= \frac{\alpha}{A} \left[ a_{11} (K_0 + p_0 t - \frac{2}{Y-1} A h_a) + a_{12} (u_0 - \beta_0 t - \frac{2}{Y-1} A a) \right] \\
    r_1 &= \frac{\alpha}{A} \left[ a_{21} (K_0 + p_0 t + \frac{2}{Y-1} A h_a) + a_{22} (u_0 - \beta_0 t + \frac{2}{Y-1} A a) \right] \\
    s_1 &= \frac{\alpha}{A} \left[ a_{21} (K_0 + p_0 t - \frac{2}{Y-1} A h_a) + a_{22} (u_0 - \beta_0 t - \frac{2}{Y-1} A a) \right].
\end{aligned}
\]

The condition (5.27) leaves \( r^*_1 \) and \( s^*_1 \) unchanged, but removes the time dependent terms in \( r_1 \) and \( s_1 \). Addition of \( r_1 \) and \( s_1 \) then shows that \( u_1 \) is indeed constant. However \( r_1 \) and \( s_1 \) still contain \( a \) and \( h_a \) which are varying through the flow, and the same applies for \( r^*_1 \) and \( s^*_1 \).

Before leaving this flow two further remarks should be made.
1. When studying homentropic general waves it is usual to select \( r \) and \( s \) as independent variables. One of the simplest problems is then to find the time \( t \) as function of \( r \) and \( s \). As an example one can consider the expressions (5.18).

The Stanyukovich Transformation leaves the time \( t \) unaffected, while the Riemann-invariants are subjected to linear transformations. It follows that the time \( t \) after transformation is obtained from \( t(r,s) \) by subjecting the independent variables \( r \) and \( s \) to the appropriate linear transformations.

2. The flow considered in this chapter, given by (5.1) is immediately seen to satisfy the momentum- and mass- conservation equations. A counterpart of this flow, which has not been considered, is the unsteady flow with

\[
V = V_o + \alpha_0 \cdot t \quad , \quad u = u_o + \alpha_0 \cdot h .
\] (5.32)

The mass conservation is then satisfied and since for a homentropic flow \( p \) then depends on the time only, also the momentum equation is satisfied with

\[
\frac{\partial u}{\partial t} = 0 \quad , \quad \frac{\partial p}{\partial h} = 0 .
\] (5.33)
6. THE TRANSFORMATION OF A HOMEOTROPIC CENTERED SIMPLE WAVE

6.1. The homentropic centered simple wave

In this Section the homentropic centered simple wave will be presented in the Lagrangian description.
Consider first a semi-infinite amount of homentropic uniform gas at rest in a straight tube of constant cross section and of infinite length. Denoting the Cartesian coordinate along the pipe by \( x \), the gas occupies the half \( x < 0 \), while for \( x > 0 \) there is vacuum. At \( x = 0 \) the gas is bounded by a plane or piston. At time \( t = 0 \) the piston is instantaneously removed and the gas expands into vacuum. The flow which develops is a complete centered simple wave. Denoting the Riemann-invariants by \( r \) and \( s \), with

\[
\begin{align*}
  r &= u + \frac{2}{\gamma - 1} a , \\
  s &= u - \frac{2}{\gamma - 1} a ,
\end{align*}
\]

(6.1)

both Riemann-invariants are constant throughout the gas before the removal of the piston with

\[
\begin{align*}
  r &= \frac{2}{\gamma - 1} a_0 , \\
  s &= -\frac{2}{\gamma - 1} a_0 ,
\end{align*}
\]

(6.2)

and \( u = 0 \). In the expansion wave which develops for \( t > 0 \) one has

\[
\begin{align*}
  r &= r_0 = u + \frac{2}{\gamma - 1} a = \frac{2}{\gamma - 1} a_0 ,
\end{align*}
\]

(6.3)

and only \( s \) varies, making the flow a simple wave. The wave is complete since the expansion proceeds to vacuum and centered since all the s-characteristics pass through one point \((h = 0, t = 0)\) due to the instantaneous removal of the piston.

Usually this flow is discussed in the Eulerian description, with the independent variables \( x, t \). Since we use the Lagrangian description with the variables \( h, t \), we follow this procedure also here. The Lagrangian variable \( h \) ranges over the domain \( -\infty < h < 0 \), and is proportional to \( x \) in the state of rest, when

\[
\begin{align*}
  p &= p_0 , \\
  u &= 0 , \\
  V &= V_0 , \\
  a &= a_0 ,
\end{align*}
\]

(6.4)

all assumed constant.
The expansion starts at \( t = 0 \), at \( h = 0 \), and the first sound wave, travelling into the gas from \( h = 0 \), which forms the boundary between the gas at rest and the expansion, is given by

\[
\begin{align*}
  h &= \frac{a_0}{V_0} t , \\
  \frac{V h}{\frac{a_0}{V_0}} &= 1.
\end{align*}
\]

(6.5)

The first sound wave is an s-characteristic and it is well-known, that for a homentropic simple wave, with \( r = r_0 = \text{const.} \), along the s-characteristics \( u, a, p, V \) and \( \frac{a}{V} \) are constant. Both in the \( x,t \)-plane and the \( h,t \)-plane the s-characteristics are then straight lines, which in this case all pass through \( h = 0, t = 0 \). From the equations for the s-characteristics

\[
\begin{align*}
  dh + \frac{a}{V} dt &= 0 , \\
  dp - \frac{a}{V} du &= 0 ,
\end{align*}
\]

(6.6)
one deduces along the $s$-characteristics

$$h + \frac{a}{V} t = 0, \quad p - \frac{a}{V} u = \text{const.} \quad (6.7)$$

From these relations, the relation (6.3) and the expressions

$$p \frac{V}{V_0} = p_0 \frac{V}{V_0}, \quad a^2 = \gamma p V, \quad (6.8),$$

one then deduces the parameters in the expansion wave

$$\frac{p}{p_0} = \left( - \frac{V o h}{a_o t} \right)^{\frac{2}{\gamma + 1}}, \quad \frac{V}{V_0} = \left( - \frac{V o h}{a_o t} \right)^{- \frac{2}{\gamma + 1}}, \quad \frac{a}{a_0} = \left( - \frac{V o h}{a_o t} \right)^{\frac{\gamma - 1}{\gamma + 1}}, \quad (6.9)$$

$$u = \frac{2a_o}{\gamma - 1} - \frac{2a_o}{\gamma - 1} \left( \frac{V o h}{a_o t} \right)^{\frac{\gamma - 1}{\gamma + 1}}.$$

all valid in the domain $0 < - \frac{V o h}{a_o t} < + 1$.

While the $s$-characteristics are straight lines it is easily checked that the $r$-characteristics are orthogonal hyperbolae. One has for the $r$-characteristics

$$dh - \frac{a}{V} dt = 0 \quad = dh + \frac{h}{t} dt. \quad (6.10)$$

and the statement follows.

A sketch of the expansion in the $h,t$-plane is given in Fig. 6.

The initial condition which specifies that at $t = 0$, the pressure at $h = 0$ is instantaneously reduced to zero, causes the fluid element there to pass instantaneously from a state of rest, to a vacuum state and a constant speed $u = \frac{2}{\gamma - 1} a_o$. The physical possibility of this operation will not be questioned.

If a fluid element with $h = h_o < 0$ is considered. The state of rest continues till the first sound wave arrives at

$$t = - \frac{V o h_o}{a_o} \quad (6.11)$$

The particle then begins to move and its speed as function of $t$ is given by the expression for $u$ in (6.9) with $h = h_o$. This allows the formulation of another problem as follows. Consider an amount of uniform gas stretching over $-\infty < h < h_o$ ($< 0$), bounded at $h = h_o$ by a piston, which is at rest for $t < - \frac{V o h_o}{a_o}$. At the instant $t = - \frac{V o h_o}{a_o}$, the piston starts to move according to the relation.
\[ u = \frac{2a_o}{Y-1} - \frac{2a_o}{Y-1} \left( - \frac{V_o h_o}{a_o t} \right)^{\frac{Y-1}{Y+1}}. \]  

(6.12)

The expansion flow which develops is now represented by the segment of the complete centered wave for \(-\infty < h \leq h_o\). It is possible to shift the origin of \(t\) to \(t = 0\) etc., to make the answer seem less dependent upon the complete centered simple wave.

In order to apply the Stanyukovich transformation also \(x\), \(K\) etc are required. One easily deduces from the definitions

\[ \begin{align*}
    dx &= u \, dt + V \, dh, \\
    dK &= p \, dt - u \, dh, \\
    d\phi &= x \, dh - K \, dt, \\
    L &= K + hu, 
\end{align*} \]

that

\[ \begin{align*}
    x &= \frac{2a_o t}{Y-1} \left\{ 1 - \frac{Y+1}{2} \left( - \frac{V_o h_o}{a_o t} \right)^{\frac{Y-1}{Y+1}} \right\}, \\
    K &= - \frac{2a_o h}{Y-1} \left\{ 1 - \frac{Y+1}{2Y} \left( - \frac{V_o h_o}{a_o t} \right)^{\frac{Y-1}{Y+1}} \right\}, \\
    \phi &= - \frac{2a_o h t}{Y-1} \left\{ \frac{(Y+1)^2}{4Y} \left( - \frac{V_o h_o}{a_o t} \right)^{\frac{Y-1}{Y+1}} - 1 \right\}, \\
    L &= - \frac{a_o h}{Y} \left( - \frac{V_o h_o}{a_o t} \right)^{\frac{Y-1}{Y+1}} = p_o t \left( - \frac{V_o h_o}{a_o t} \right)^{\frac{Y}{Y+1}}. 
\end{align*} \]

(6.14)

Along the first sound wave (6.5) applies and one finds from (6.9) and (6.14)

\[ \begin{align*}
    u &= 0, \quad p = p_o, \quad x = -a_o t, \quad K = p_o t, \quad L = p_o t. 
\end{align*} \]

(6.15)

6.2. The Stanyukovich Transformation applied to the complete centered wave

The simplest form of the Stanyukovich Transformation is given by the formulae (2.1), but since \(h\) has been chosen negative \((-\infty < h \leq 0)\), these formulae do not
apply and we have to go to the formulae (2.2) and (2.3) valid for the entire \(h_1\)-plane.

Selecting the branch of the orthogonal hyperbola in the second quadrant of the \(h_1\)-plane, with \(-\infty \leq h \leq 0\), \(0 \leq h_1 \leq +\infty\), we have to take \(A = +1\), \(a = -1\) and obtain for example from (2.2) and (2.3)

\[
h_1 = -\frac{1}{h} , \quad p_1 = -\frac{1}{h} p = h p , \quad u_1 = -L ,
\]

(6.16)

\[
B_1 = (-h)^{3Y-1} B_0 = B_0 h_1^{-(3Y-1)}.
\]

Since in the homentropic gas of Section 6.1 the pressure is constant and the velocity zero for \(t < -\frac{V_0}{a_0} h_1\), \(-\infty \leq h \leq 0\), it is clear that the transformation yields again the flow with the linear pressure distribution and constant acceleration considered in Chapter 3. The point \(h = 0\) corresponds with \(h_1 = +\infty\) and the infinite pressure there is at the instant \(t = 0\) reduced to zero, starting the expansion flow.

Substitution of the transformation formulae (6.16) in the relations (6.9) and (6.14) yields the parameters for the transformed expansion flow, valid for \(t \geq \frac{V_0}{a_0} h_1\), \(0 \leq h_1 \leq +\infty\). These relations show that the expansion begins at \(t = 0\), at the point \(h_1 = +\infty\), but only at \(t = \infty\), when \(h_1 = 0\). The straight s-characteristics transform into orthogonal hyperbolas, while the hyperbolic r-characteristics of the \(h,t\)-plane transform into straight lines in the \(h_1,t\)-plane. For the pressure and velocity in the transformed expansion flow one finds

\[
p_1 = p_0 h_1 \left(\frac{V_0}{a_0} \frac{2Y}{Y+1}\right),
\]

(6.17)

\[
u_1 = -L = -\frac{a_0}{Y h_1} \left(\frac{V_0}{a_0} \frac{Y-1}{Y+1}\right) = -p_0 t \left(\frac{V_0}{a_0} \frac{2Y}{Y+1}\right).
\]

To understand the last formula one remembers that before the arrival of the first sound wave, announcing the removal of the piston and the pressure drop at \(h_1 = \infty\), the LMS-gas executes the rigid body motion with pressure distribution \(p_1 = p_0 h_1\) and the constant deceleration \(-p_0\). For small values of \(h_1\) the gas has therefore acquired a very large negative velocity when the expansion there at last begins.

The relations (6.17) were already obtained earlier in Ref. [8] when solutions homogeneous in \(h\) and \(t\) were considered.

It has been remarked earlier (Section 4.3) that there is something unrealistic in a piston or plate at infinity, entirely 'out of sight'. It is therefore of interest to apply the general Stanyukovich Transformation to the expansion wave
and map the domain $-\infty < h < 0$ on a finite mass $H$ of LMS-gas. This will be considered next.

The general transformation formulae of Section 2.2 have to be used instead of the formulae (2.2) and (2.3). Considering only the pressure and velocity of the transformed flow one has from (2.16)

$$
p_1 = \frac{a}{a_{21} h + a_{22}} p = -\frac{a}{A} (a_{21} h_1 - a_{11}) p,
$$

$$
u_1 = \frac{a}{A} (a_{21} l + a_{22} u),
$$

with $p$, $L$ and $u$ representing the parameters in the homentropic flow. Since $p = p_0 =$ const. in the homentropic flow before the expansion begins, the transformed situation yields the linear pressure distribution, and the rigid body motion with constant acceleration discussed in Chapter 3. Substituting the general transformation formulae in the relations obtained in Section 6.1 one obtains

$$
-\frac{V_0 h}{a_0 t} = \frac{V_0}{a_0} \frac{a_{22} h_1 - a_{12}}{a_{21} h_1 - a_{11}} \left[ \frac{V_0}{a_0} \frac{a_{22} h_1 - a_{12}}{a_{21} h_1 - a_{11}} \right]^{\frac{2Y}{Y+1}},
$$

$$
p_1 = -\frac{a}{A} p_0 \left( a_{21} h_1 - a_{11} \right) \left[ \frac{V_0}{a_0} \frac{a_{22} h_1 - a_{12}}{a_{21} h_1 - a_{11}} \right]^{\frac{2Y}{Y+1}} \left[ \frac{V_0}{a_0} \frac{a_{22} h_1 - a_{12}}{a_{21} h_1 - a_{11}} \right]^{\frac{2Y}{Y+1}},
$$

$$
u_1 = -\frac{a}{A} \left( \frac{\gamma - 1}{\gamma - 1} a_{22} - \frac{a_0}{\gamma - 1} \right) \left[ \frac{V_0}{a_0} \frac{a_{22} h_1 - a_{12}}{a_{21} h_1 - a_{11}} \right]^{\frac{2Y}{Y+1}} \left[ \frac{V_0}{a_0} \frac{a_{22} h_1 - a_{12}}{a_{21} h_1 - a_{11}} \right]^{\frac{2Y}{Y+1}}.
$$

To map the domain $-\infty < h < 0$ into the finite segment $0 < h_1 < H$ one has to shift the hyperbolae as already discussed in Section 3.2. Here a branch of the hyperbola in the second quadrant of the $h h_1$-plane is required and $A = +1$, $\alpha = -1$. To achieve the mapping such that

$$
h = -\infty \rightarrow h_1 = 0,
$$

$$
h = 0 \rightarrow h_1 = H,
$$

one requires

$$
a_{11} = 0, \quad a_{12} = -\frac{1}{a_{21}}, \quad a_{22} = \frac{1}{a_{21} H},
$$

(6.21)
with $a_{21}$ still free to choose. Its value depends on the pressure gradient or acceleration before the expansion begins. Selecting the pressure in the LMS-gas at $h_1 = H$ to be $p_0$ we have from (6.18)

$$p_1 = p_0 = a_{21} H p_0, \quad a_{21} H = 1$$

(6.22)

and so with (6.21)

$$a_{21} = \frac{1}{H}, \quad a_{11} = 0, \quad a_{12} = -H, \quad a_{22} = -1.$$

(6.23)

Substitution of (6.23) into (6.19) then yields for the expansion flow of the LMS-gas

$$p_1 = p_0 \frac{h_1}{H} \left[ \frac{V_o H^2}{a_o t} \left( \frac{1}{h_1} - \frac{1}{H} \right) \right] \frac{2Y}{Y+1},$$

$$u_1 = -\frac{2}{Y-1} a_o + \frac{a_o}{Y(Y-1)} \left[ \left( Y+1 \right) + \left( Y-1 \right) \frac{H}{h_1} \right] x$$

$$x \left[ \frac{V_o H^2}{a_o t} \left( \frac{1}{h_1} - \frac{1}{H} \right) \right] \frac{Y-1}{Y+1}$$

(6.24)

valid in the domain

$$0 \leq \frac{V_o H^2}{a_o t} \left( \frac{1}{h_1} - \frac{1}{H} \right) \leq 1.$$

(6.25)

Apart from slightly different initial conditions and notation the relations (6.24) were already obtained in an entirely different way in Ref. [5]. (See formulae (6.1) and (6.2) of that paper). For further discussion we also refer to Ref [5].

Two more flows can be obtained from the general transformed formulae (6.19), by shifting the hyperbola in the $h_1$-plane to the left instead of to the right as done to obtain (6.24).

Consider therefore the $h_1$-plane in Figure 7 where the domain $-H \leq h \leq 0$ is mapped onto $0 \leq h_1 \leq +\infty$ in such a way that

$$h = -H \quad \Rightarrow \quad h_1 = +\infty,$$

$$h = 0 \quad \Rightarrow \quad h_1 = 0.$$

(6.26)

Since the branch of the hyperbola is in the first quadrant we have $A = -1, \quad \alpha = +1$ and from (6.26)
\[ a_{12} = 0, \quad a_{22} = a_{21} H, \quad a_{11} = -\frac{1}{a_{21} H}, \quad (6.27) \]

with \( a_{21} \) free to choose. The hyperbola has the equation

\[ a_{21} h_1 + \frac{1}{a_{21} H} = \frac{1}{a_{21} (h + H)}, \quad (6.28) \]

and its horizontal asymptote is at

\[ h_1 = -\frac{1}{a_{21} H}. \quad (6.29) \]

Before the expansion begins the pressure in the homentropic gas is \( p = p_o \), and so in the LMS-gas from (6.18)

\[ p_1 = (a_{21} h_1 + \frac{1}{a_{21} H}) p_o \quad (6.30) \]

The pressure has its lowest value at \( h_1 = 0 \) and then increases linear with \( h_1 \) before the expansion starts. Selecting the pressure \( p_1 \) in the LMS-gas to be \( p_o \) at \( h_1 = 0 \), for \( t \leq 0 \), we have again

\[ a_{21} H = 1, \quad a_{21} = \frac{1}{H} \quad (6.31) \]

and the relation (6.30) takes the form

\[ p_1 = p_o \left( 1 + \frac{h_1}{H} \right). \quad (6.32) \]

For the velocity one has from (6.18), with \( u = 0 \) in the homentropic gas

\[ u_1 = -a_{21} L = -\frac{1}{H} L = -\frac{p_o t}{H}. \quad (6.33) \]

The expansion flow in the LMS-gas now starts from \( h_1 = 0 \), at the location where the pressure is smallest, in contrast with the earlier case when the expansion was initiated at the point of highest pressure.

Combining (6.27) and (6.31) yields

\[ a_{12} = 0, \quad a_{11} = -1, \quad a_{22} = 1, \quad a_{21} = \frac{1}{H}, \quad (6.34) \]

and substituting (6.34) into (6.19) results in
\[ p_1 = p_0 \frac{H + h_1}{H} \left[ \frac{V_0^2}{a_0^2} \frac{h_1}{H + h_1} \right] \frac{2Y}{Y+1}, \]  

\[ u_1 = \frac{2}{Y-1} a_0 - \frac{a_o}{Y(Y-1)} \left[ (Y+1) + \frac{(Y-1)H}{H + h_1} \right] \left[ \frac{V_0^2}{a_0^2} \frac{h_1}{H + h_1} \right] \frac{Y-1}{Y+1}, \]

valid in

\[ 0 < \frac{V_0^2}{a_0^2} \frac{h_1}{H + h_1} < 1. \]

Another expansion flow is still obtained if the hyperbola in Fig. 8 is considered. The initial expansion at \( h = -H \) then differs from the sudden reduction of pressure to vacuum, but is to be taken from (6.12) and the discussion there. This will not be pursued further.
CONCLUSIONS

The examples presented in this Report show that the Stanyukovich Transformation clarifies and unifies relationships between well-known simple homentropic flows and corresponding flows of the non-homentropic LMS-gas. It gives a better insight in the structure of the theory, than can be obtained from a study of the individual non-homentropic flows.
It is expected that further applications, of interest in gasdynamics, can be worked out along the same lines.
LIST OF REFERENCES


\(-\infty < h < -H\)
\(0 < h_1 < +\infty\)

Fig. 8