Depth of Interaction in Columnar Scintillation Material

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Internship Report
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Abstract

Geometrical optics analysis is carried out in order to establish a relationship between the intensity distribution at the base plane of a column of scintillation material and the depth of interaction. Slabs of scintillation material are treated as a two dimensional model case, and the results are extended to approximations for circular and rectangular columns. The application of the detectable optical signal at the base plane of the columns for the reconstruction of the depth of interaction is also discussed.

1 Introduction

SPECT and PET are among the fastest growing diagnostic imaging techniques. In the recent years there have been significant improvement of resolution of both the techniques. In order to precisely determine the source of gamma ray, collimators are used for human scanners. On the other hand, small animal SPECT scanners depend on pinhole gamma cameras. One of the major problems with pinhole gamma cameras is the vertical and horizontal resolution of the point of interaction. It is necessary to determine both the vertical and horizontal position of the point of interaction with accuracy, in order to reconstruct the angle of the incident gamma ray, which is in turn used to reconstruct the position of the gamma source.

Columnar scintillation materials have been used to improve the horizontal resolution of the point of interaction. However, this leads to loss of resolution in the vertical depth of interaction. In a continuous scintillation crystal, it is possible to reconstruct the depth of interaction (DOI) from the spreading of light in the crystal. However, in a columnar scintillation crystal, the columns work as waveguides, and inhibits spreading. Thus, it becomes more difficult to reconstruct the depth of interaction.

In this report, we propose a method for the reconstruction of depth of interaction from the light radiated out of the waveguide columns of scintillation material. It depends on the ratio of the intensity of the light detected on the pixels adjacent to the columns and the pixels within the columns. First, we calculate the intensity distribution for different depths of interaction in case of two dimensional slabs of scintillation material spaced by air. Then we use this result to approximate the intensity distribution for different depths of interaction in case of circular and rectangular columns of scintillation material.
It is found that, only a limited range of vertical depth of interaction may be resolved by measuring the intensity of light adjacent to the columns of scintillation material, and the range of vertical depth depends on the width of the column. A combination of measurement of the intensity of light within and adjacent to the columns would facilitate better resolution of depth of interaction in wider columns.

2 Intensity distribution in slabs of scintillation material

First we consider slabs of scintillation material. In order to calculate the intensity distribution due to a single scintillation event, we make the following assumptions:

1. The time duration of a scintillation event is very short.
2. Light from the scintillation event is non-polarized, and radiates isotropically in all directions.
3. The scintillation material is homogeneous and absorption of light is negligible.
4. Intensity due to multiple reflection between two interfaces of scintillation material is negligible.
5. The dimensions of the columns are much larger compared to the wavelength of the light from scintillation events.

We consider slabs of scintillation material (Fig.1) of width $2r$ separated by distance $d$. The refractive index of the scintillation material is $n_c$, while the refractive index of the material in gaps between the slabs is $n_g$. We focus our attention on a scintillation event at the point $S(x, z)$ within the slab $AB$.

We trace the ray at an angle $\psi$ with the positive $x$ axis to determine the $x$ coordinate of the point where it intersects the $z = 0$ base line of the slabs. If the ray crosses the right boundary of the slab $AB$ at the point $P'$, then

$$
\begin{pmatrix}
  x_{P'} \\
  z_{P'}
\end{pmatrix} = \begin{pmatrix}
  x_s \\
  z_s
\end{pmatrix} + t \begin{pmatrix}
  \cos \psi \\
  -\sin \psi
\end{pmatrix} = \begin{pmatrix}
  r \\
  z_{P'}
\end{pmatrix}
$$

(2.1)
Figure 1: Slabs of scintillation material separated by gaps. O is the origin of the coordinate system. The vertical dashed line $OZ$ is the $z$ axis. The horizontal dashed lines indicate normals to the slab. Light radiated at an angle larger than $\theta_c$ with the $x$ axis is internally reflected and guided within the slab. The ray $SP'P$ is traced.
Thus,

$$ r = t \cos \psi + x_s $$
$$ t = \frac{r - x_s}{\cos \psi} $$  \hspace{1cm} (2.2)$$

From equations 2.1 and 2.2 we can express $z_P$ as,

$$ z_P = z_s - t \sin \psi = z_s - (r - x_s) \tan \psi $$  \hspace{1cm} (2.3)$$

Similarly, we can express the coordinates of the point $P$ as,

$$ \begin{pmatrix} x_P \\ z_P \end{pmatrix} = \begin{pmatrix} x_P' \\ z_P' \end{pmatrix} + \tau \begin{pmatrix} \cos \beta \\ -\sin \beta \end{pmatrix} = \begin{pmatrix} x_P \\ 0 \end{pmatrix} $$  \hspace{1cm} (2.4)$$

Thus,

$$ z_P' - \tau \sin \beta = 0 $$
$$ \tau = \frac{z_P'}{\sin \beta} $$  \hspace{1cm} (2.5)$$

From equations 2.3, 2.4 and 2.5 we find the expression for $x_P$,

$$ x_P = x_P' + \tau \cos \beta $$
$$ = r + z_P' \cot \beta $$
$$ = r + [z_s - (r - x_s) \tan \psi] \cot \beta $$  \hspace{1cm} (2.6)$$

However, from geometrical consideration the angle $\beta$ is related to the angle $\psi$ in the following way,

$$ n_g \sin \beta = n_c \sin \beta' $$
$$ n_g \sin \beta = n_c \sin \psi $$
$$ \beta = \arcsin \left( \frac{n_c}{n_g} \sin \psi \right) $$  \hspace{1cm} (2.7)$$

Therefore, from equations 2.6 and 2.7, we find the relationship between $x_P$ and $\psi$,

$$ x_P = r + [z_s - (r - x_s) \tan \psi] \cot(\beta) $$
$$ x_P = r + [z_s - (r - x_s) \tan \psi] \cot \left[ \arcsin \left( \frac{n_c}{n_g} \sin \psi \right) \right] $$
$$ \xi = [z_s - (r - x_s) \tan \psi] \times \tan \left[ \arccos \left( \frac{n_c}{n_g} \sin \psi \right) \right] $$  \hspace{1cm} (2.8)$$
where,

\[ \xi = x_P - r \] = distance of the point P from the right end of the slab.

Similarly, for a point Q at distance \( \zeta \) from the left end of the slab,

\[ \zeta = [z_s - (r + x_s) \tan \psi] \times \tan \left[ \arccos \left( \frac{n_c}{n_g} \sin \psi \right) \right] \]  

(2.9)

We can solve equation 2.8 in order to determine the angle \( \psi \) as a function of \( x_s, z_s \) and \( \xi \).

\[ \psi = \psi(x_s, z_s, \xi) = \text{solution of equation 2.8 for } \psi \]  

(2.10)

From geometrical consideration, the angle of incidence on the interface between the slab and the gap is equal to the angle of the radiated ray with the x axis. Thus, the fraction of incident energy transmitted to the gap is determined by the angle \( \psi \). For non-polarized light, the energy transmission coefficient \( T \) for an angle of incidence \( \psi \) is given by,

\[ T(\psi) = \frac{1}{2} \left[ T_\parallel(\psi) + T_\perp(\psi) \right] \]

\[ = \frac{1}{2} \left[ t_\parallel^2(\psi) + t_\perp^2(\psi) \right] \times \frac{n_g \cos \psi t}{n_c \cos \psi} \]  

(2.11)

where,

\( T_\parallel \) = energy transmission coefficient for P polarized light

\( T_\perp \) = energy transmission coefficient for S polarized light

\( t_\parallel \) = amplitude transmission coefficient for P polarized light

\( t_\perp \) = amplitude transmission coefficient for S polarized light

\( \psi_t \) = angle of refraction for the angle of incidence \( \psi \)

The amplitude transmission coefficients can be expressed as,

\[ t_\parallel = \frac{2 \sin \psi_t \cos \psi}{\sin(\psi_t + \psi) \cos(\psi - \psi_t)} \]  

(2.12)

\[ t_\perp = \frac{2 \sin \psi_t \cos \psi}{\sin(\psi_t + \psi)} \]  

(2.13)

Since, the scintillation event leads to isotropic radiation of light in all direction, the intensity of light at a point P at distance \( \xi \) from the right end of the
slab will be proportional to the energy transmission coefficient for the angle of incident given by $\psi(x_s, z_s, \xi)$.

$$I(\xi,0) = \text{constant} \times T(\psi(x_s, z_s, \xi)) \quad (2.14)$$

where, the functions $T(\psi)$ and $\psi(x_s, z_s, \xi)$ are defined by equations 2.11 and 2.10 respectively.

In figure 2, the relative intensity distribution is shown as a function of $\xi$ for CsI slabs of 160 $\mu$m width separated by 100 $\mu$m air gaps. It is clear from the plots that with increasing $z_s$, the intensity distribution becomes less dependent on $x_s$. In this particular case, the variation of intensity distribution due to different $x_s$ within the slab becomes negligible for $z_s = 500 \mu$m or larger. Thus, for thin slabs, we can neglect the horizontal position of the scintillation event in order to reconstruct the depth of interaction.

3 DOI in slabs of scintillation material

The reconstruction of the depth of interaction from intensity distribution would depend on the width and position of the pixels. If a pixel is on the right side of the slab, and extends from $x_1$ to $x_2$, then the total optical energy incident on the pixel due to a scintillation event within the slab is given by,

$$E(x_1, x_2) = \text{constant} \times \int_{\psi_1}^{\psi_2} T(\psi) d\psi \quad (3.1)$$

where,

$\psi_1 = \psi(x_s, z_s, (x_1 - r))$

$\psi_2 = \psi(x_s, z_s, (x_2 - r))$

Suppose, the total number of photons from a scintillation event is $N_p$. Since, the radiation is isotropic, the number of photons radiated in each radian angle is given by $\frac{N_p}{2\pi}$. Thus, the number of photons detected by a pixel on the right side of the slab, and extending from $x_1$ to $x_2$ is given by,

$$n_p \text{ ext}(x_1, x_2) = \frac{N_p}{2\pi} \times \int_{\psi_1}^{\psi_2} T(\psi) d\psi \quad (3.2)$$
Figure 2: Relative illumination as function of distance from the edge of the slab for different $x_s$ and $z_s$. 

(a) $z_s = 100\mu m$ 

(b) $z_s = 500\mu m$ 

(c) $z_s = 1\text{mm}$ 

(d) $z_s = 5\text{mm}$
On the other hand, the total number of photons guided by the slab of scintillation material is given by,

\[ n_p^{\text{(guided)}} = \frac{N_p}{2\pi} \times (2\pi - 4\theta_c) \quad (3.3) \]

Due to the isotropic nature of the radiation from the scintillation event, half of the photons will travel toward the top part of the slab and the other half will travel to the bottom part of the slab. If the top of the slab is perfectly reflective, the number of photons detected on a pixel extending from \( x_1 \) to \( x_2 \) is given by,

\[
n_p^{\text{int}}(x_1, x_2) = \begin{cases} 
\frac{N_p}{2\pi} \times (2\pi - 4\theta_c) \times \left( \frac{x_2 - x_1}{2r} \right) 
& \text{for } z_s > 2r \sin \theta_c \\
\frac{N_p}{2\pi} \times (\pi - 2\theta_c) \times 
\left\{ \left( \frac{x_2 - x_1}{2r} \right) + \left[ \sin^{-1} \left( \frac{x_2 - z_s}{z_s} \right) - \sin^{-1} \left( \frac{x_1 - z_s}{z_s} \right) \right] \right\} 
& \text{for } z_s < 2r \sin \theta_c 
\end{cases}
\] (3.4)

We can use equations 3.2 and 3.4 to calculate the ratio of the numbers of detected photons for different \( x_s \) and \( z_s \) and make a calibration curve to determine the depth of interaction from the measured ratio.

The relative fraction of total photons detected on a 1\( \mu m \) pixel located at different \( \xi \) is shown on figure 3 as a function of \( \xi \) for different horizontal position and depths of interaction in case of a 160\( \mu m \) CsI slab.

The relative fraction of total photons detected on pixels located at different positions as a function of the depth of interaction is shown on figure 4 for 160\( \mu m \) CsI slabs separated by 100 \( \mu m \) air gaps. Note the very fast decay of detected signal with increasing depth of interaction. Thus, the upper bound of the depth of interaction detectable by optical means in slabs of scintillation material is limited by the noise of the detector pixels.

### 4 DOI in circular columns of scintillation material

For a rotationally symmetric column of scintillation material, we can approximate the intensity distribution from the results in section 2.
(a) $x_s = -50\mu m$

(b) $x_s = 0$

(c) $x_s = 50\mu m$

Figure 3: Relative fraction of total photons detected on a $1\mu m$ pixel as a function of $\xi$. 
(a) Detector pixel extends from $\xi_1 = 20 \, \mu m$ to $\xi_2 = 70 \, \mu m$.

(b) Detector pixel extends from $\xi_1 = 50 \, \mu m$ to $\xi_2 = 100 \, \mu m$.

(c) Detector pixel extends from $\xi_1 = 70 \, \mu m$ to $\xi_2 = 120 \, \mu m$.

(d) Detector pixel extends from $\xi_1 = 100 \, \mu m$ to $\xi_2 = 150 \, \mu m$.

Figure 4: The relative fraction of total photons detected on pixels located at different positions as a function of $z_s$ for different $x_s$. 
Figure 5: For a very long cylindrical column we ignore the $x$ and $y$ coordinates of the scintillation event and consider it to be on the $z$ axis.
In addition to the assumptions in section 2, we assume that the radius $r$ of the circular columns is much smaller than their length $L$ (Fig. 5). Thus, we can neglect the horizontal distance of scintillation event from the $z$ axis in equation 2.8.

$$\rho = r + z_s \tan \left[ \arccos \left( \frac{n_c}{n_g} \sin \psi \right) \right]$$  \hspace{1cm} (4.1)

Therefore, we can solve equation 4.1 for the angle $\psi$ as a function of $\rho$ for a given $r$ and $z_s$.

$$\psi(z_s, \rho) = \text{solution of equation 4.1 for } \psi$$  \hspace{1cm} (4.2)

Then we can use equations 2.11 to compute the transmission coefficient for that angle. Thus, applying our assumption of isotropic radiation, the intensity of light at a point $P(x, y, 0)$ due the scintillation event can be expressed as,

$$I(x, y, 0) = \text{constant} \times T \left( \psi \left( z_s, \sqrt{x^2 + y^2} \right) \right)$$  \hspace{1cm} (4.3)

where, the functions $T(\psi)$ and $\psi(z_s, \rho)$ are defined by equations 2.11 and 4.2 respectively.

The number of photons detected on a pixel outside the column is given by,

$$n_{p \text{ ext}}(\text{pixel}) = \frac{\text{constant}}{2\pi} \times \iint_{\text{pixel}} T(\psi) d\psi d\phi$$  \hspace{1cm} (4.4)

However, we can express the angle $\psi$ as a function of the radial distance $\rho$. Thus, a change of variable of integration in equation 4.4 yields,

$$n_{p \text{ ext}}(\text{pixel}) = \frac{\text{constant}}{2\pi} \times \iint_{\text{pixel}} T(\psi(z_s, \rho)) \left| \frac{\partial \psi(z_s, \rho)}{\partial \rho} \right| d\rho d\phi$$  \hspace{1cm} (4.5)

Another change of variable of integration converts equation 4.5 into Cartesian coordinate,

$$n_{p \text{ ext}}(\text{pixel}) = \frac{\text{constant}}{2\pi} \times \iint_{\text{pixel}} \frac{1}{\rho(x, y)} T(\psi(z_s, \rho(x, y))) \left| \frac{\partial \psi(z_s, \rho(x, y))}{\partial \rho} \right| dx dy$$  \hspace{1cm} (4.6)

If the total number of photons from a scintillation event is $N_p$, the number of photons radiated in each radian of the angle $\psi$ is given by $\frac{N_p}{2\pi}$ because of the isotropic nature of the scintillation radiation.
Thus, we arrive at the following expression for the number of photons detected on an external pixel,

\[ n_{p\text{ ext}}(\text{pixel}) = \frac{N_p}{4\pi^2} \times \iint_{\text{pixel}} \frac{1}{\rho(x, y)} T(\psi(z_s, \rho(x, y))) \left| \frac{\partial \psi(z_s, \rho(x, y))}{\partial \rho} \right| \, dx \, dy \]  

(4.7)

On the other hand, the total number of photons guided by the column of scintillation material is given by,

\[ n_{p\text{ (guided)}} = \frac{N_p}{2\pi} \times (2\pi - 4\theta_c) \]  

(4.8)

Due to the isotropic nature of the radiation from the scintillation event, half of the photons will travel toward the top part of the column and the other half will travel to the bottom part of the column. If the top of the column is perfectly reflective, the number of photons detected on an internal pixel of area \( A_{\text{pixel}} \) is given by,

\[ n_{p\text{ int}}(\text{pixel}) = \left\{ \frac{N_p}{2\pi} \times (2\pi - 4\theta_c) \times \left( \frac{A_{\text{pixel}}}{\pi r^2} \right) \right\} \text{ for } z_s > 2r \sin \theta_c \]  

(4.9)

We can use equations 4.7 and 4.9 to calculate the ratio of the numbers of detected photons for different \( z_s \) and make a calibration curve to determine the depth of interaction from the measured ratio.

5 DOI in rectangular columns of scintillation material

Let us consider rectangular columns of scintillation material of width 2\( w \) and thickness 2\( l \). The vertical height of the columns is much larger compared to the other two dimensions (Fig. 6).

In addition to the assumptions in section 2, we make the following assumptions:

1. The light from a scintillation event radiated within a cone define by the angle \( \theta_c \) with the normal of a side of the rectangular column is partially transmitted to the gap region. All light radiated out of these cones is guided within the column.
Figure 6: For very tall rectangular columns we ignore the $x$ and $y$ coordinates of the scintillation event and consider it to be on the $z$ axis.
2. The light from the transmission cone is radiated in a hemisphere centered on the center of the circle formed the intersection of the transmission cone with the side of the column.

3. The intensity of light along each direction within the hemisphere is proportional to the transmission coefficient of the angle of incidence that gives rise to the angle of refraction corresponding to that direction.

4. The angle between a ray propagating to a given point and the normal to the side of the rectangular column is approximately equal to the angle between a line drawn from that point towards the center of the circle and the normal to that side. For example, in Fig. 6, $\angle PS'M \approx \angle PP'N$.

Suppose, a scintillation event takes place at a vertical height $z_s$ within a rectangular column. We approximate the position of the scintillation event at $S(0, 0, z_s)$. Then the angle of refraction $\psi_t$ for the ray towards the point $P(x, y, 0)$ is approximated by the angle $\angle PS'M$. The angle $\angle PS'M$ is given by,

$$\sin (\angle PS'M) = \frac{|PS' \times \hat{x}|}{|PS'| |\hat{x}|}$$

$$\angle PS'M = \sin^{-1} \left[ \frac{|((w - x)\hat{x} + (0 - y)\hat{y} + (z - 0)\hat{z}) \times \hat{x}|}{|((w - x)\hat{x} + (0 - y)\hat{y} + (z - 0)\hat{z})| |\hat{x}|} \right]$$

$$\psi_t(z_s, x, y) = \sin^{-1} \left[ \frac{n_g}{n_c} \sin (\psi_t(z_s, x, y)) \right]$$

Therefore, the angle of incidence for the ray towards the point $P(x, y, 0)$ is given by $\psi(z_s, x, y)$, where,

$$n_c \sin (\psi(z_s, x, y)) = n_g \sin (\psi_t(z_s, x, y))$$

$$\psi(z_s, x, y) = \sin^{-1} \left[ \frac{n_g}{n_c} \sin (\psi_t(z_s, x, y)) \right]$$

Substituting the expression for $\psi_t(z_s, x, y)$ from equation 5.1, we obtain,

$$\psi(z_s, x, y) = \sin^{-1} \left[ \frac{n_g}{n_c} \frac{|((w - x)\hat{x} + (0 - y)\hat{y} + (z - 0)\hat{z}) \times \hat{x}|}{\sqrt{(w - x)^2 + y^2 + z^2}} \right]$$

Thus, the intensity of light at the point $P(x, y, 0)$ due to the scintillation event will be,

$$I(x, y, 0) = \text{constant} \times T(\psi(z_s, x, y))$$
Figure 7 shows plots of intensity distribution adjacent to a 200\(\mu\)m\(\times\)200\(\mu\)m CsI column centered on the origin, due to scintillation events in the column at different depths of interaction.

The number of photons detected on an external pixel adjacent to the column is given by

\[
n_{p \text{ ext}}(\text{pixel}) = \int_{\text{pixel}} n_p(\psi(z_s, x, y)) T(\psi(z_s, x, y)) \, dx \, dy \tag{5.5}
\]

where, \(n_p(\psi(z_s, x, y))\) = number of photons per unit area incident at the angle \(\psi(z_s, x, y)\).

If the total number of photons from a scintillation event is \(N_p\), the number of photons radiated in each steradian solid angle is given by \(\frac{N_p}{4\pi}\). The number of photons radiated within a cone with apex angle \(2\theta_c\) is given by \(\frac{N_p}{2} (1 - \cos \theta_c)\). This number of photons will be distributed over a hemisphere of solid angle \(2\pi\). The number of incident photons per unit area will also follow the inverse square law.

\[
n_p(\psi(z_s, x, y)) = \frac{N_p}{2} (1 - \cos \theta_c) \times \frac{1}{2\pi r^2} \tag{5.6}
\]

where, \(r = \sqrt{(w - x)^2 + y^2 + z^2}\) is the distance of the point \(P(x, y, 0)\) from the center of the circle formed the intersection of the transmission cone with the side of the column.

Thus, equations 5.5 and 5.6 lead to the following expression for the number of photons detected on an external pixel adjacent to the column,

\[
n_{p \text{ ext}}(\text{pixel}) = \frac{N_p}{4\pi} (1 - \cos \theta_c) \int_{\text{pixel}} \frac{T(\psi(z_s, x, y))}{(w - x)^2 + y^2 + z^2} \, dx \, dy \tag{5.7}
\]

On the other hand, the total number of photons guided by the column of scintillation material is proportional to the solid angle left after removal of four cones with apex angle \(2\theta_c\). Thus, the total number of photons guided by the column of scintillation material,

\[
\begin{align*}
n_p(\text{guided}) &= \frac{N_p}{4\pi} \times \{4\pi - 4[2\pi(1 - \cos \theta_c)]\} \\
n_p(\text{guided}) &= N_p (2 \cos \theta_c - 1) \tag{5.8}
\end{align*}
\]
Figure 7: Intensity distribution adjacent to a rectangular column of scintillation material for different depths of interaction. The $x$ and $y$ axes units are in $10^{-5}$ m.

(a) $z_s = 500\,\mu$m

(b) $z_s = 1.5\,\text{mm}$

(c) $z_s = 3\,\text{mm}$
Due to the isotropic nature of the radiation from the scintillation event, half of the photons will travel toward the top part of the column and the other half will travel to the bottom part of the column. If the top of the column is perfectly reflective, the number of photons detected on an internal pixel of area $A_{\text{pixel}}$ is given by,

$$n_{p \text{ int}(\text{pixel})} = N_p (2 \cos \theta_c - 1) \times \left( \frac{A_{\text{pixel}}}{A_{\text{column}}} \right) \quad \text{for } z_s > 2r \sin \theta_c$$

$$n_{p \text{ int}(\text{pixel})} = N_p (2 \cos \theta_c - 1) \times \left( \frac{A_{\text{pixel}}}{4wl} \right) \quad \text{for } z_s > 2r \sin \theta_c \quad (5.9)$$

We can use equations 5.7 and 5.9 to calculate the ratio of the numbers of detected photons for different $z_s$ and make a calibration curve to determine the depth of interaction from the measured ratio.

The relative fraction of total photons detected on a $1\mu m \times 1\mu m$ pixel located at different $(\xi, \eta)$ is shown on figure 8 as a function of $(\xi, \eta)$ for different depths of interaction in case of a $200\mu m \times 200\mu m$ CsI column centered on the origin.

If the external pixel is rectangular extending from $x = \xi_1 + w$ to $x = \xi_2 + w$ and from $y = \eta_1$ to $y = \eta_2$, equation 5.7 reduces to

$$n_{p \text{ ext}(\text{pixel})} = \frac{N_p}{4\pi} (1 - \cos \theta_c) \int_{x=\xi_1+w}^{x=\xi_2+w} \int_{y=\eta_1}^{y=\eta_2} \frac{T(\psi(z_s, x, y))}{(w - x)^2 + y^2 + z^2} dxdy$$

$$(5.10)$$

Figure 9 plots the relative fraction of total photons detected on pixels located at different positions as a function of $z_s$ for a $200\mu m \times 200\mu m$ CsI column centered on the origin. Similar to the slab case, the detected signal decays very fast with increasing depth of interaction. Thus, the upper bound of depth of interaction detectable by optical means in an array rectangular columns of scintillation material is limited by the noise of the detector pixels.
(a) Colors indicate Relative fraction of total photons detected on a $1\mu m \times 1\mu m$ pixel for $z_s = 500\mu m$

(b) Colors indicate Relative fraction of total photons detected on a $1\mu m \times 1\mu m$ pixel for $z_s = 750\mu m$

(c) Colors indicate Relative fraction of total photons detected on a $1\mu m \times 1\mu m$ pixel for $z_s = 1\text{mm}$

(d) Colors indicate Relative fraction of total photons detected on a $1\mu m \times 1\mu m$ pixel for $z_s = 1.5\text{mm}$

(e) Colors indicate Relative fraction of total photons detected on a $1\mu m \times 1\mu m$ pixel for $z_s = 2.5\text{mm}$

(f) Colors indicate Relative fraction of total photons detected on a $1\mu m \times 1\mu m$ pixel for $z_s = 5\text{mm}$

Figure 8: Relative fraction of total photons detected on a $1\mu m \times 1\mu m$ pixel as a function of $(\xi, \eta)$. The $\xi$ and $\eta$ axes units are in $10^{-5}$m.
(a) 100 \, \mu m \times 100 \, \mu m detector pixel starting from $\xi_1 = 20 \, \mu m$ and $\eta_1 = -50, -30$ and $0 \, \mu m$.

$\xi_1 = 50 \, \mu m$ and $\eta_1 = -50, -30$ and $0 \, \mu m$.

(c) 100 \, \mu m \times 100 \, \mu m detector pixel starting from $\xi_1 = 70 \, \mu m$ and $\eta_1 = -50, -30$ and $0 \, \mu m$.

$\xi_1 = 100 \, \mu m$ and $\eta_1 = -50, -30$ and $0 \, \mu m$.

Figure 9: The relative fraction of total photons detected on pixels located at different positions as a function of $z_s$. 
6 Discussion

The theoretical analysis presented in this report indicates the feasibility of reconstruction of depth of interaction by comparing the intensity of light inside and adjacent to a column of scintillation material. However, optical reconstruction of depth of interaction in columnar scintillation material has a relatively small range. The detected signal in pixels adjacent to a scintillating column decays very fast with increasing depth of interaction. Thus, a large dynamic range of signal detection is necessary in order to differentiate larger values of depth of interaction. Additionally, a low noise level should be maintained to detect the faint signals from far away scintillation events.

Binning of intercolumn and intracolumn pixels might enhance the overall signal level. However, it would require either precise placement of the scintillation columns on the CCD chip. As an alternative, intensity distribution and curves representing detected fraction of photons at different DOI may be calculated for each of the columns, and used as calibration references. Since, the equations for calculating the intensity distribution and detected fraction of photons are relatively simple, it might become an easier method for the optical reconstruction of depth of interaction in columnar scintillation material.