Stellingen
ENHANCED CONFORMAL MAPPING METHOD
FOR AIRFOIL DESIGN AND ANALYSIS

Stellingen

Djoko Sardjadi

11 december 1996

1. Het vermogen van een vleugelprofiel of een vleugel om circulatie te produceren kan worden vergroot door het toevoegen van een wig aan de onderzijde van de achterrand. Naarmate de lengte/hoogte verhouding kleiner is, is de wig meer effectief.

2. Het is effectiever om een wig toe te voegen aan de achterrand van de achterste klep van een multi-element profiel of vleugel dan de wig te plaatsen op een van de andere componenten.

3. Een ontwerpmethode die in staat is om dikke profielachterranden te behandelen, geeft meer vrijheid om de drukverdeling voor te schrijven, bijvoorbeeld om profielen te genereren met een grote circulatie.

4. Het toevoegen van een wig aan de achterkant van een profiel in transsonne stroming verbetert waarschijnlijk de prestaties, aangezien dezelfde draagkrachtscoëfficiënt kan worden verkregen bij een kleinere invalshoek, waardoor een zwakkere schokgolf ontstaat. Hierdoor kan ook een toeneming van het 'drag divergence March number' worden verwacht.

5. Bestaande computerprogramma's die het cirkelvlak gebruiken als het rekendomein kunnen 'eenvoudig worden gerenoveerd' om profielen met dikke achterranden te kunnen behandelen bij directe en inverse problemen.

6. Bij het gebruik van panelenmethoden voor het oplossen van inverse problemen is het mogelijk een achterrand dikte voor te schrijven door de coördinaten van de twee achterrandhoekpunten aan boven- en onderzijde apart op te geven.

7. Wil je correct en sterk zijn, dan moet je niet aarzelen om met iemand te praten die je niet mag, want alleen dan krijg je een duidelijker beeld van je zwakheden.

8. Wie dicht bij een groot man gaat staan, verliest een deel van zijn uitzicht op de horizon.

9. Wie jong is wil de wereld veranderen.
Wie macht heeft wil verandering dirigeren.
Bij het ouder worden willen mensen hun instelling veranderen om van veranderingen te kunnen profiteren.
De meeste mensen wachten tot lang, zij zijn dan niet meer in staat hun eigen instelling te veranderen.

10. Als natuur is de zee.
Als natuur is de lucht.
Als natuur is de woestijn.
Als natuur is het oerwoud.
Als natuur is het moeras of de heuvel.
Als natuur alles is behalve de beschaving.
Dan is de natuur niet democratisch maar orde.

11. Bij tien opdrachten die ik heb uitgevoerd zegt mijn Nederlandse vriend: 'hé, je maakt een fout'.
Mijn Indonesische vriend zal mij feliciteren: 'mooi, je hebt er negen perfect gedaan'.
Beiden hebben gelijk, niettemin is het met beiden moeilijk samenwerken.
ENHANCED CONFORMAL MAPPING METHOD
FOR AIRFOIL DESIGN AND ANALYSIS

Propositions

Djoko Sardjadi
December 11, 1996

1. The capability of an airfoil or a wing to produce circulation can be increased by adding a wedge on the trailing edge lower surface. The shorter the wedge, for a fixed height, the more effective in increasing the circulation. However, one should be aware of a possible increase in base drag.

2. It is more effective to add a wedge to the trailing edge of the last flap of a multi-element airfoil for a wing to increase the maximum lift coefficient than adding the wedge to the main or the middle components.

3. A design method which is able to deal with thick trailing edges provides more freedom in specifying pressure distributions, e.g. to produce airfoils with large circulation.

4. Adding a trailing edge wedge to an airfoil transonic flows is probably improving the performance, since the same lift coefficient can be obtained at a lower angle of attack, hence resulting in a weaker shock wave. Thus, an increase in drag divergence Mach number can also be expected.

5. Existing computer codes which use the circle plane as the computing domain can be 'easily renovated' to deal with thick trailing edge airfoils in the direct and inverse problems.

6. When using panel methods for solving inverse problems it is feasible to prescribe a trailing edge gap by fixing the coordinates of the trailing-edge upper and lower corners.

7. Should you want to be correct and strong, don't hesitate to talk to somebody who doesn't like you, you will hear your weaknesses more clearly.

8. Standing up close to a great man you will lose part of your view on the horizon.

9. Being young, a man like to change the world,
   Being in power, a man wants to direct the change.
   Getting old, people want to change their mind to take the benefits from the change.
   Most people wait too long, they cannot change their mind anymore.

10. If nature is the sea,
    If nature is the sky,
    If nature is the desert.
    If nature is the jungle.
    If nature is the marsh over the hill.
    If nature is everything except civilization.
    Nature is not democratic but order.

11. Ten jobs I have made,
    My Dutch friend tells me: 'hey, you did one wrong'.
    My Indonesian friend congratulates me: 'good, you made nine perfect'.
    Both are right, nevertheless both are difficult to cooperate with.
ENHANCED CONFORMAL MAPPING METHOD
FOR AIRFOIL DESIGN AND ANALYSIS

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dedicated to
my son, Johar (11)
and my daughter, Khailan (4)
with love

Holy books definition: truth does not belong to the majority
Acknowledgments

I would like to thank all the colleagues at the Low Speed Aerodynamics Laboratory for providing a nice atmosphere and support which enabled me to complete this thesis. You are really nice people! To Mr. Blom in particular, I'd like to convey my thanks for his endless optimism and support. I'm grateful to Professor Narayana (from IIT Madras) for his encouragement to complete the last lap of this journey, and for his assistance in editing the draft. I thank my wife, Hartini, for her never-turbulent-love in the severely corrupted time, and with her 486.50.
ABSTRACT

A new method of conformal mapping is presented for the solution of direct and inverse problems of incompressible flow about two-dimensional wing sections. By adding a logarithmic term to the classical transformation formulae an arbitrary trailing edge gap can be represented. This also allows to model an airfoil thickened by the boundary layer displacement thickness. For the inverse problem, a general compatibility condition is developed, of which the well known Lighthill closure constraint for zero trailing edge thickness becomes a particular case. In the present method an arbitrary trailing edge gap can be specified. The present method provides a way of design with more freedom in specifying the pressure distribution than customary procedures. In the practical engineering application the specified pressure distribution need not be adjusted to satisfy the trailing edge gap. Whether the resulting geometry is realistic or not, should be checked after the solution is obtained.

It is also possible to handle mixed problems where a part of the solution is given geometrically and another part is determined by certain flow conditions. As an example can be mentioned an airfoil with a large separation region.

Thwaites' method and Green's lag entrainment method are used to take into account the boundary layer development in the laminar and turbulent parts respectively. As a transition criterion the "New Michel" or "extended Granville" method is used. Laminar separation bubbles are analyzed using the Gordon-Schmidt method. Examples for the direct, inverse, and mixed problems are presented. Experimental results are also presented to assess the usefulness of the method and to illustrate the effect of thickening the trailing edge of airfoils.
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<tr>
<td>$A_{1,2,...}$</td>
<td>coefficient in the Laurent series, eq.(1.21)</td>
</tr>
<tr>
<td>$A$</td>
<td>surface angle of a given airfoil (Chapter 2),</td>
</tr>
<tr>
<td>$B_{1,2,...}$</td>
<td>coefficient in the Laurent series, eq.(1.1a)</td>
</tr>
<tr>
<td>$C_{1,2,...}$</td>
<td>coefficient in the Laurent series, eq.(1.1b)</td>
</tr>
<tr>
<td>$C_d$</td>
<td>airfoil drag coefficient</td>
</tr>
<tr>
<td>$C_E$</td>
<td>flow entrainment coefficient</td>
</tr>
<tr>
<td>$D$</td>
<td>airfoil drag,</td>
</tr>
<tr>
<td>$D$</td>
<td>domain of mapping,</td>
</tr>
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<td>$D$</td>
<td>trailing edge gap as complex vector, eq.(4.7)</td>
</tr>
<tr>
<td>$D_{2,3,...}$</td>
<td>coefficient in the Laurent series, eq.(4.7)</td>
</tr>
<tr>
<td>$E_{1,2}$</td>
<td>coefficient in the complex velocity, eq.(4.6)</td>
</tr>
<tr>
<td>$F$</td>
<td>force</td>
</tr>
<tr>
<td>$H_{1,2,...}$</td>
<td>coefficient in the Laurent series, eq.(1.17)</td>
</tr>
<tr>
<td>$H$</td>
<td>boundary layer shape factor, $H = \delta_1 / \delta_2$</td>
</tr>
<tr>
<td>$H_1$</td>
<td>boundary layer shape factor, $H_1 = (\delta-\delta_1) / \delta_2$</td>
</tr>
<tr>
<td>$H_{32}$</td>
<td>energy shape factor, $H_{32} = \delta_3 / \delta_2$</td>
</tr>
<tr>
<td>$H()$</td>
<td>function of variables inside the bracket, eq.(1.17)</td>
</tr>
<tr>
<td>$Im()$</td>
<td>imaginary part of a function inside the bracket</td>
</tr>
<tr>
<td>$J$</td>
<td>number of coordinates</td>
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<tr>
<td>$K$</td>
<td>second Pohlhausen parameter, eq.(7.34)</td>
</tr>
<tr>
<td>$\bar{K}$</td>
<td>average of $K$, eq.(7.33)</td>
</tr>
<tr>
<td>$L$</td>
<td>airfoil lift,</td>
</tr>
<tr>
<td>$L$</td>
<td>arbitrary reference length,</td>
</tr>
<tr>
<td>$M$</td>
<td>moment about a point</td>
</tr>
<tr>
<td>$N$</td>
<td>number of terms in series,</td>
</tr>
<tr>
<td>$P$</td>
<td>a fixed point in the circle plane, fig.2.3</td>
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<tr>
<td>$P_{1,2}$</td>
<td>a fixed point on the airfoil surface, fig.2.4</td>
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<tr>
<td>$P_t$</td>
<td>total pressure</td>
</tr>
<tr>
<td>$Q$</td>
<td>a fixed point image of $P$, fig.2.3</td>
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<tr>
<td>$Q$</td>
<td>flow entrainment into the boundary layer, eq.(7.41)</td>
</tr>
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</table>
R  
radius of the circle

Re  
Reynolds number defined as $\rho u x / \mu$

Re()  
real part of a function inside the bracket

SA  
perimeter of the given airfoil, (section 5.1.1)

SI  
total length of the contour of the intermediate airfoil, eq.(5.4)

SP  
stagnation point

TE  
trailing edge

Tu  
free stream turbulence, %

V  
local flow speed

Z  
complex vector

a  
coefficient given by eq.(2.25) to (2.28),

ac  
critical point in the Joukowsky mapping, (section 3.2)

ac  
aerodynamic center

arg()  
argument of a complex variable (or a complex function) inside the bracket,

b  
coefficient given by eq.(2.25) to (2.28)

b  
is the coefficient of the logarithmic term, eq.(3.2)

c  
airfoil chord,

c  
critical point of mapping function, eq.(3.4)

c_d  
drag coefficient (associated with momentum loss),

c_d  
dissipation coefficient, eq.(7.28)

c_F  
force coefficient

c_f  
surface friction coefficient

c_{10}  
c_f for flat plate at zero pressure gradient

c_l  
lift coefficient

c_m  
moment coefficient

c_p  
pressure coefficient

e  
base of natural logarithm

f  
function

f'  
derivative of function f

h  
function, $h = \log(f')$, eq.(2.2)

i  
imaginary number, $i = \sqrt{-1}$

k  
$k = 1 - \delta / \pi$, used in section 1.5

l  
p parameter defined by eq.(7.16)
\[ l_{am} \] length of the laminar part of a separation bubble
\[ l_{tur} \] length of the turbulent part of a separation bubble
\[ \log \] natural logarithm
\[ n \] index of the series term
\[ n \] factor defined by eq.(7.36)
\[ p \] coefficient in the Fourier series, eq.(2.57)
\[ p \] pressure on the airfoil surface
\[ q \] coefficient in the Fourier series, eq.(2.57)
\[ r \] radius of an arc in the physical (airfoil) plane,
\[ r \] modulus of \( z \),
\[ r \] modulus of \( \zeta \)
\[ s \] station along the airfoil contour
\[ t \] running coordinate in the circle plane
\[ u \] velocity component in the direction tangential to the surface, also \( x \)-component of velocity
\[ u' \] fluctuation of \( u \)
\[ v \] velocity in the direction normal to the surface, also \( y \)-component of velocity
\[ v' \] fluctuation of \( v \)
\[ w \] complex velocity, \( u+iv \)
\[ \overline{w} \] conjugate of \( w \), \( u-iv \)
\[ x \] real part of \( z \)
\[ y \] imaginary part of \( z \)
\[ z \] complex coordinate in the physical (airfoil) plane \( z = x + iy \)

\[ \alpha \] angle of attack of the free stream flow with respect to the \( x \)-axis,
\[ \alpha \] factor defined by eq.(2.46)
\[ \chi \] complex potential
\[ \Delta TE \] modulus of the trailing edge gap, eq.(3.16)
\[ \delta \] trailing edge included angle,
\[ \delta \] boundary layer thickness,
\[ \delta_1 \] boundary layer displacement thickness, eq.(7.2)
\[ \delta_2 \] boundary layer momentum thickness, eq.(7.13)
\[ \delta_3 \] boundary layer energy thickness, eq.(7.27)
\[ \varepsilon \] an arbitrarily small number
\( \phi \) zero lift direction of a given airfoil, fig.2.1
\( \phi \) potential function, fig.3.4

\( \Gamma \) circulation (closed contour)
\( \Gamma^* \) line integral of speed on the airfoil contour (over part of a contour)

\( \gamma \) direction of trailing edge gap, fig.3.6
\( \gamma \) angular position in the physical plane, fig.2.2

\( \eta \) imaginary part of \( \zeta \)
\( \psi \) angular position in the circle plane, fig.2.3
\( \psi \) stream function

\( \Lambda \) first Pohlhausen parameter, eq.(7.37)
\( \lambda \) parameter defined by eq.(7.18)
\( \mu \) coefficient of viscosity

\( \nu \) coefficient of kinematic viscosity, \( \nu = \mu / \rho \)
\( \theta \) angular position in the circle plane,
\( \rho \) air density,

\( \sigma \) real part of function \( h \), eq.(2.2)
\( \tau \) imaginary part of function \( h \), eq.(2.2)
\( \tau = \pi/2 - \theta + \omega \) defined in eq.(5.22)

\( \omega \) surface angle on the airfoil, fig.1.6.
\( \xi \) real part of \( \zeta \)
\( \zeta \) complex coordinate in the circle plane

\( \int \) indefinite integral
\( \oint \) integral around a closed contour

\( \int \) integral along a curve
\( \Sigma \) summation
Subscripts

R  reattachment point
S  separation point
SP stagnation point
T  transition point
TE trailing edge
a  airfoil
c  circle
c  critical point of mapping function, eq.(3.4)
c  critical point = point of instability within the boundary layer
e  at the boundary layer edge
i  inviscid
j  station number
l  lower
lam laminar part of the separation bubble
tur turbulent part of the separation bubble
u  upper
\infty at infinity,

Abbreviations

SP  stagnation point
TE  trailing edge
V-I-I Viscous Inviscid Interaction
t.e. trailing edge
CHAPTER 1
INTRODUCTION

1.1. General observations and objectives of the thesis

In the present days of competitive practical aerodynamic design and analysis, the contribution of numerical aerodynamics is important and very much rewarding. The main factors contributing to this importance are:
- the tough competition to sell good aircraft at the right time in the international market,
- the better understanding of the aerodynamics of wings,
- the availability of high performance computers [1].

In an earlier study on numerical aircraft aerodynamic design, particularly connected with wing sections, the present author concluded [2]:
- The weak Viscous Inviscid Interaction (V-I-I)\(^1\) method plays an important role in the aerodynamic design of transport aircraft, due to the capability of attaining a high standard of accuracy, coupled with relative simplicity of programming.
- Aerodynamic design by using the Navier-Stokes equations is still not practical and has to be developed further because of the difficult problem of turbulence modeling and the dependence on the availability of high performance computers.
- The weak V-I-I method will have a bright future and it is still worthwhile to develop refined versions.

Concerning the overall drag of aircraft, up to this day, sufficiently accurate predictions are still dependent on wind tunnel measurements. Since wind tunnel experiments are expensive in time and money, any improvement in computational accuracy and / or speed is welcomed by aerodynamic design engineers. This fact

---

\(^1\) V-I-I is a flow computation scheme where the flow region is subdivided into two regions, one close to the surface where the direct viscous effects are taken into account through boundary layer theory and the other, outer region where the flow is considered to be inviscid. The pressure distribution on the surface is governed by the inviscid region, while the effective body surface and the surface friction are determined by the viscous region. The computations may be made iteratively one after the other, starting with the inviscid calculation of the pressure distribution (as generally adopted). Solving the governing equations at the same time and coupling the pressure distribution to the distribution of the displacement thickness has also been tried [3], in that case the term strong V-I-I is used.
encourages engineers to exploit computational methods, in parallel to wind tunnel tests. In this respect attention is given to:

- physical model of air flow over the wing,
- governing equations, and
- numerical schemes [4].

This thesis is aimed at contributing to a better understanding of the aspects referred to above in airfoil design and analysis. The discussion will be limited to the relatively simple case of a two dimensional single element wing in an incompressible flow. A classical technique, a conformal mapping from a circle to an airfoil, provides a relatively easy way to perform the present analysis. In general, the application of conformal mapping has been limited to sharp or rounded trailing edge airfoils. In this thesis, a more general mapping function will be presented, capable of handling airfoils with a physically thick trailing edge (the definition of the trailing edge geometry can be seen in figure 1.1), airfoils thickened by boundary layer effects, airfoils with a large separated flow region and airfoils with a spoiler.

1.1.1. Arrangement of the thesis

The thesis is arranged as follows. Chapter 1 describes the basic ideas in generalizing the mapping function, to accommodate the more general airfoil geometry. Chapter 2 presents Timman's mapping method (which is comparable to Lighthill's method), as an example of existing methods. The essential feature of the general mapping function, is the addition of a logarithmic term. The effect of this "log term" will be presented in Chapter 3. Chapter 4 presents the fundamental equations and the compatibility relations between the airfoil-domain and the circle-domain. Some operational equations needed to develop a practical mapping technique are given in Chapter 5. Chapter 6 discusses the extension of the method to the case of airfoils with massive separation. In Chapter 7, this mapping method is coupled with integral boundary layer calculation methods to produce results for some practical cases. Finally, Chapter 8 describes some results of an experimental program for evaluation and validation of the theory.
1.2. An introductory note on conformal mapping

It is interesting to note the evolution of conformal mapping methods to describe flow fields around given airfoils (direct method), and its later use to reconstruct the airfoil when the speed distribution on the yet unknown surface is prescribed (inverse method). The following note is not meant to present this development in detailed historical order but just to highlight some aspects for later reference.

Before the thirties, airfoil theories available were the conformal transformations of Joukowsky (1910) [5], R.von Mises (1917) [6], von Kármán-Trefftz (1918) [7], J.Geckeler (1922) [8], and W.Müller (1924) [9]. These theories did not find much practical application, since they were neither able to evaluate a given arbitrary airfoil nor to reconstruct an airfoil from a specified pressure distribution. They involve only a few degrees of freedom and therefore deal only with a limited class of airfoils with corresponding limitations on the pressure distribution. The Joukowsky transformation, although only a two-degree of freedom mapping, was the founding idea of thick airfoil theory. The von Mises transformation offered the possibility to develop stable airfoils with s-shaped mean lines. The von Kármán - Trefftz transformation alleviated the restriction of the Joukowsky transformation to cusped trailing edges only. Finally the Geckeler-Müller transformation offered all possibilities provided by earlier transformations.

Beside this, the simpler thin airfoil theory was able to estimate some basic aerodynamic characteristics of a given airfoil, e.g. the zero lift angle and the pitching moment. Among the contributors to this development were M.M.Munk (1924) [10], Glauert (1928) [11], and Th.Theodorsen (1931) [12].

In the mean time, since the introduction of the lifting line theory by Prandtl in 1911, it was understood that the performance of a three dimensional wing would directly benefit from the performance of the wing sections. The need for aircraft with better performance, in the early thirties, was the driving force to develop better airfoils. These airfoils were developed by systematically varying the combination of the camber line shape, and basic thickness distribution (including nose radius). This resulted in some well known airfoil catalogues (e.g.[13]) with a wealth of experimentally determined aerodynamic characteristics. It is clear that the information contained in these catalogues has required a huge amount of work. It is also evident that the selection of airfoils would benefit enormously if, for a given airfoil shape, the pressure distribution and of course the boundary layer development could be determined theoretically.
Therefore much credit is due to Th. Theodorsen of NACA, who showed that an arbitrary airfoil shape may be analyzed theoretically to obtain the pressure distribution, and the lift and pitching moment coefficients [14]. Theodorsen applied the Joukowsky mapping inversely to an arbitrary airfoil, resulting in a nearly circular shape. Subsequently, this nearly circular shape was transformed iteratively into a circle. Through this transformation, an airfoil may be related to a circle for which the surface speed distribution is known analytically. Such a scheme of determining the pressure distribution over a given body, was later denoted as a "direct problem". However, at that time, the method did not really solve the practical problem of airfoil selection, since the use of such a sophisticated method was hindered by the limited power of computation. The effort needed for such a computation was still too much. Because of this situation the thin airfoil theories were developed further.

With regard to the Theodorsen method the following remark is in order. When the Joukowsky transformation is applied to an airfoil with a non zero trailing edge included angle, a near circle is obtained with a kink at the point corresponding to the trailing edge of the airfoil. Therefore it would have been better to apply the von Kármán-Trefftz transformation to avoid the kink.

In Theodorsen's time, it was already understood that the pressure distribution on the airfoil surface plays an important role in its performance. The location of minimum pressure may be used to indicate the portion of the (smooth) airfoil surface having a laminar boundary layer. An effort to increase the laminar portion on an airfoil was made to obtain airfoils with low drag. Knowing the relation between the speed and the geometry, and restricting himself to a small perturbation, Theodorsen was able to make a small shape modification through modifying the speed distribution. This scheme of finding the airfoil geometry from the speed distribution, is referred to as the "inverse problem". The development of knowledge during this period resulted in the well known NACA 6-series "laminar flow airfoils".

In the mid forties, the understanding of the inverse problem was enhanced, especially due to the famous paper by Lighthill [15]. The correspondence between the speed distribution on an airfoil (but specified on the polar angle in the circle plane) with that on a circle and the procedure of determining the airfoil geometry was described in detail. At about the same time as Lighthill, Timman independently developed a similar method [16]. Later Eppler [17] published also papers which discussed the direct and inverse problems of airfoil theory.

The method developed by Timman is essentially the same as that due to Lighthill, as stated by him self as "The method developed appeared afterwards to be closely related to Lighthill's method which again involves the same basic ideas
as Mangler’s”. Eppler used essentially the same basic ideas as Timman and Lighthill, but he limited himself to deal only with cusped trailing edge airfoils.

A contribution to the inverse method has also been provided by Strand [18] in 1973, where the speed distribution is prescribed along the surface of the airfoil to be designed, instead of along the circle surface. The “elemental compatibility condition” (see Chapter 4 of the present thesis) was employed to transfer the speed on the airfoil as a function of the arc length on the airfoil $V_a(s)$, into the same $V_a$, but now as a function of the angle in the circle plane $V_a(\theta)$.

It was known already in the thirties that the inviscid flow theory does not produce exactly the experimentally determined pressure distribution. This deviation is attributed to the boundary layer which reduces the mass flow rate close to the surface. The boundary layer effectively thickens the airfoil body and in an inviscid flow model this effect must be taken into account by thickening the airfoil up to the trailing edge, resulting in a thick trailing edge and a “tail” downstream.

The first attempt (known to the present author) to handle airfoils with thick trailing edges, using conformal mapping, was made in 1974 by Bauer et.al. [19] by introducing a $1/\zeta$ term into the derivative of the mapping function (see section 1.5). The coefficient of $1/\zeta$, upon integration around the circle, results in a gap in the physical plane. (The word gap is used here in the mathematical sense to note a relative position between two edges. In the present discussion, a gap is the vector from the upper trailing edge corner to the lower one, figure 1.1.) However, Bauer used this mapping function only in the solution of direct problems. Halsey [20] in 1980 introduced a pre-mapping using a logarithmic function, which reduces a thick trailing edge to a pointed one; then a corner-removing transformation is used to obtain a near circle and afterwards an exact circle is obtained iteratively. The procedure due to Halsey is applicable only for direct problems. These methods were proposed to analyze airfoils with viscous effects, where the airfoil would be thickened by the boundary layer displacement effect. Volpe (1981) [21], made use of the mapping function introduced by James [22], (compare section 1.5 of the present thesis), in an inverse problem which could result in an airfoil with a thick trailing edge. Nevertheless, the author admitted that he did not have full control on the trailing edge gap.

In response to the need of designing a physically thick trailing edge airfoil (see e.g. the new airfoil design concept due to Henne [23]), it is necessary to develop a method having full control over the trailing edge gap. This thesis presents a theory and procedure to solve airfoil problems with an arbitrary trailing edge gap.
Other remarkable developments of airfoil mappings were the methods of Ives [24], James [25], and Halsey [26]. The first, based on the Theodorsen method, can be used to analyze two-element airfoils, the second was developed for the design and analysis of two element airfoils, and the last one can be used to analyze airfoils with any number of elements.

A well known difficult problem in airfoil design is the question which freedom there is in specifying the speed distribution. Many investigators still believe that the speed distribution can not be specified "sufficiently arbitrarily" if a closed curve corresponding to it is to be obtained. Section 1.6. of the present thesis, describes that a closed curve corresponding to a "sufficiently arbitrary" speed distribution can be obtained.

1.3. Physical modeling of the real situation

The inviscid flow model helps in computing the pressure distribution on the body surface. In a real flow, the velocity field close to the body surface, namely the boundary layer, effectively displaces the streamlines near the surface outward. The distribution of this displacement depends on the development of the boundary layer. The boundary layer displacement thickness represents by definition the amount of the effective surface displacement. Hence, when an inviscid flow model is used to approximate a real flow, we see that all bodies in a real flow will have to be artificially thickened in an equivalent inviscid flow model. The viscous wake downstream of the body in the real flow should be represented by a tail-like displacement body in the inviscid model. The tail is cut off, in the present method, at the upper and lower trailing edge corners. The effective airfoil, the airfoil plus the boundary layer displacement thickness, is here referred to as an equivalent airfoil. Hence, an equivalent airfoil will always have a thick trailing edge. Consequently, there is no sharp trailing edge airfoil in the inviscid model of a real flow. With this in view, an airfoil in a real flow may be represented by an equivalent airfoil in an inviscid flow. Figure 1.1 shows some equivalent airfoils in an inviscid flow.

When a boundary layer is separated from the body surface before it reaches the trailing edge, the effective displacement of the surface becomes very large. Boundary layer methods fail in this situation to compute the effective displacement. A large separation problem may be handled with the present extended mapping function, since the airfoil effective surface in the separated region may be represented by a separated "inviscid stream line". The separated stream line is not known a priori. The separation point can be determined using boundary layer
theory. The separated stream line can be determined by the inverse procedure where the pressure distribution in the separated region is specified, e.g. constant and equal to the pressure at the separation point. This type of problem, where one part of the airfoil is represented by its geometry and the other part by its corresponding speed distribution, is referred to as a mixed-problem. This problem is elaborated in Chapter 6.

![Diagram of airfoils](image)

**Figure 1.1. Physical and equivalent airfoils.**

1.4. Mathematical modeling
1.4.1. Geometrical relations

This section will discuss the essential equations and definitions needed to develop a more general mapping function useful for airfoil problems. First the basic ideas of mapping are considered. Following the Riemann fundamental mapping theorem, and accepting that self-overlapping domains may occur [27], a circle in the \( \zeta \)-plane can be transformed to a curve in the \( z \)-plane, which may self-intersect (figure 1.2). Such a self-overlapping curve is not acceptable to represent an airfoil
in practical cases. However, a mathematical formulation that is capable to handle this situation may give some additional insight into the relationship of pressure distributions and the corresponding airfoils (see section 1.4.2.).

As discussed earlier (figure 1.1), an airfoil will have a thick trailing edge in inviscid modeling. Therefore special attention should be given to the definition of the boundary of the domain in the $\zeta$-plane. Normally it is considered that the $\zeta$-domain is bounded inside by a circle and unbounded outside. In the present consideration however, the boundary of the $\zeta$-domain includes the rear stagnation streamline in the circle-plane. One should not cross this streamline in going from the upper to the lower side, but should make a $2\pi$ radian turn around the circle. Hence the upper and lower sides of this streamline have a $2\pi$ radian polar distance as shown in figure 1.3. For later use, the rear stagnation stream line in the circle plane will be considered as two coincident streamlines.

![Diagram of $\zeta$ and $z$ planes](image)

**Figure 1.2. Mathematically acceptable solution.**

For a regular and single-valued mapping function $z=f(\zeta)$, a boundary in the $\zeta$-domain would be mapped to a corresponding boundary in the $z$-domain. Upon mapping the $\zeta$-domain onto the $z$-domain, the rear stagnation streamlines of the circle will be transformed correspondingly to the rear stagnation streamlines of the airfoil. Since these streamlines are at two different sides in the $\zeta$-plane, the mapping allows to devise a function which separates them in the $z$-plane. In the next section a logarithmic term will be added to a Laurent series to do the separation.
When an airfoil has a non zero trailing edge thickness, and we want the upper and lower trailing edge corners to correspond to the rear stagnation point of the circle, the two rear streamlines will trail from these two corners. It should be understood that the region in between the trailing streamlines (referred to as slit), as well as the interior of the airfoil, do not belong to the mapping domain in consideration (figure 1.3). Also, the slit behind the airfoil should not be interpreted as a model of the physical wake, but rather as a body-like extension which in Chapter 3 is referred to as a material wake. Moreover, the shape of this body-like extension will depend on the direction of the oncoming flow.

**Figure 1.3. Definition of the domain.**
In Chapter 6, it is shown that the present approach shows a good representation of real viscous effects along the body surface. Therefore no further discussions on a possible relation between the slit and the real wake will be made.

Special attention should be given to the Kutta condition determining the circulation. For airfoils with a closed (sharp) trailing-edge, the Kutta condition states that the flow leaves the sharp trailing-edge smoothly, which is along the bisector, (figure 1.4a). Then the circulation should be such that the rear stagnation point in the circle plane is at the point corresponding to the airfoil trailing-edge. In the present concept, the Kutta condition is also interpreted as "the circulation must be such that the flow leaves the airfoil at the trailing edge corners", (figure 1.4b).

![Diagram of Kutta conditions for airfoils](image)

**Figure 1.4. Kutta condition applied to a thick trailing edge.**

This Kutta condition will be satisfied automatically since the rear stagnation point of the circle corresponds to the upper and lower corners of the airfoil trailing edge. The angle between each trailing streamline and the surface upstream of the corner is found to be equal to $\pi - \delta/2$, just as for the sharp trailing-edge.

### 1.4.2. Mathematical expression for the mapping

Following Laurent's theorem [28], an analytic mapping function $z = f(\zeta)$ can be represented by a "Laurent series". When the $\zeta$-domain is unbounded outside the circle of radius $R$, for $|\zeta| > R$, the series reduces to
\[ z = f(\zeta) = B_0 + B_1 \zeta + B_2 \frac{\zeta^2}{2} + \ldots \]  \hspace{6cm} (1.1a)

To obtain compatibility with the notation of the series for the derivative of (1.1a) to be used later, we rewrite equation (1.1a) in the (at present less elegant) form

\[ z = f(\zeta) = C_0 + C_1 \log(\zeta) - \frac{C_2}{\zeta} - \frac{1}{2} \frac{C_3}{\zeta^2} - \frac{1}{3} \frac{C_4}{\zeta^3} - \ldots \]  \hspace{6cm} (1.1b)

This mapping function would be sufficient to map a circle (centered at the origin) in the \( \zeta \)-plane onto any airfoil having a closed trailing edge in the \( z \)-plane. However, when the airfoil trailing edge is physically thick, or the boundary layer displacement thickness is added to the surface, or the boundary layer separates from the surface, equation (1.1) is not applicable. Nevertheless, the same conformal mapping method can still be applied by adding a logarithmic term (a natural logarithm) to the mapping function which leads to

\[ z = f(\zeta) = C_0 + C_1 \log(\zeta) - \frac{C_2}{\zeta} - \frac{1}{2} \frac{C_3}{\zeta^2} - \frac{1}{3} \frac{C_4}{\zeta^3} - \ldots \]  \hspace{6cm} (1.2)

The logarithmic term provides a "uniform" split of \( 2\pi iC_1 \) to the rear stagnation streamlines as shown in figure 1.5. Each point of the rear stagnation streamline in the plane of the circle leads to two points in the physical plane, separated by the same vector \( 2\pi iC_1 \), between upper and lower streamline.

![Figure 1.5. The slit.](image)
The coefficients $C_H$ in general are complex numbers. The coefficient $C_{-1}$ will be set to 1 (this is not a restriction on the generality of the transformation because $C_{-1}$ means a scaling and rotation only), to normalize both planes at infinity to have an identical flow field $(dz/d\zeta = 1)$. The coefficient $C_0$ will be set to a certain value such that the airfoil (upper) trailing edge will be in the origin of the $z$-plane.

When the airfoil geometry is given and the velocity field is to be computed (a direct problem) using the conformal mapping method, the determination of the points on the circle corresponding to the points on the airfoil is the crux of the problem. Equation (1.2) provides point to point relations from the $\zeta$-domain to the $z$-domain. The coefficient $C_1$ follows from the trailing edge gap (figure 1.5). Equation (1.2) is readily solved (iteratively) for $C_H$ by using residue analysis (see Chapter 5).

1.4.3. Mapping of flow fields

The application of conformal mapping to fluid flow problems is not limited to incompressible flows only. It may also be used in grid generation to provide an easier computational domain for general flow equations. In 2-dimensional incompressible flows, particularly, conformal mapping is an attractive choice since the Laplace equation governs the flow. The mapping leaves the Laplace equation invariant which means that the problem reduces to solving the Laplace equation also in the circle plane independent of the details of the mapping. (It may be noted that in other applications of mapping, such as in compressible flows, the mapping parameters appear in the resulting flow equations to be solved. The flow solution in the physical plane then follows from the mapping.) The discussion in the remaining part of this thesis will be limited to incompressible flows.

For two-dimensional inviscid incompressible flows the continuity equation takes the form,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (1.3a)$$

and the condition of zero rotation,

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \quad (1.3b)$$

Introducing a potential function $\phi$ and a stream function $\psi$, where
\[ u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \quad (1.4a) \]
\[ v = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \quad (1.4b) \]

transforms equations (1.3) into the following Laplace equations
\[ \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \quad (1.5a) \]
\[ \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \quad (1.5b) \]

It is seen that equations (1.4) are identical to the Cauchy-Riemann equations. Hence, a complex potential function
\[ \chi(z) = \chi(x + iy) = \phi(x, y) + iv(x, y) \quad (1.6) \]

is analytic in the whole region of the z plane. Taking the derivative of equation (1.6)
leads to:
\[ \frac{d\chi}{dz} = \frac{\partial \phi}{\partial x} + i \frac{\partial \psi}{\partial x} = -i \frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial y} = \frac{\partial \phi}{\partial x} - i \frac{\partial \phi}{\partial y} = i \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y} \quad (1.7) \]

Introducing (1.4) into (1.7) yields
\[ \frac{d\chi}{dz} = u - iv = \overline{w} \quad (1.8) \]

where \( \overline{w} \) is the conjugate of the velocity vector \( w \)

The velocity at a point in the z-domain can be related to the velocity at the
 corresponding point in the \( \zeta \)-domain by defining a relation between the complex
 potentials in the two domains. We will follow the usual practice in defining the
 complex potential at a point in the z-domain to be equal to that at the corresponding
 point in the \( \zeta \)-domain,
\[ \chi(z) = \chi(\zeta) \quad (1.9) \]
where \( z \) and \( \zeta \) are related by equation (1.2). Having defined the complex potential, the conjugate complex velocities in the airfoil and circle planes follow from
\[
\overline{w}_c(\zeta) = \frac{d\chi}{d\zeta} \tag{1.10}
\]
and
\[
\overline{w}_a(z) = \frac{d\chi}{dz} \tag{1.11}
\]
where the subscripts \( c \) and \( a \) stand for circle and airfoil respectively. On the circle and airfoil contours we obtain
\[
\overline{w}_c(\theta) = V_c(\theta) \cdot e^{-i\alpha} \tag{1.12}
\]
and
\[
\overline{w}_a(s) = V_a(s) \cdot e^{-i\phi(s)} \tag{1.13}
\]
where \( V_c = |w_c| \) and \( V_a = |w_a| \) (figure 1.6). The velocity relation between both planes follows from equations (1.10) and (1.11) in the form
\[
\frac{dz}{d\zeta} = f' = \frac{\overline{w}_a}{\overline{w}_c} \tag{1.14}
\]

Figure 1.6. Surface velocities.
Hence the geometrical mapping relation also determines the ratio between the conjugate complex velocities in both planes. Substituting equation (1.2) into the left hand side of equation (1.14) and using $C_{1} = 1$ results in

$$f' = \frac{\tilde{w}_{a}}{\tilde{w}_{a}} = 1 + \frac{C_{1}}{\zeta} + \frac{C_{2}}{\zeta^{2}} + \frac{C_{3}}{\zeta^{3}} + ...$$  \hspace{1cm} (1.15)$$

which is valid on and outside the circle centered at the origin. It will be shown in Chapters 4 and 5 that in an inverse problem, the quantity $|\tilde{w}_{c}/\tilde{w}_{a}|$ on the contour can be determined. This equation may be readily solved (iteratively) for $C_{n}$ by using residue analysis. With the known series coefficients, equation (1.2) may be used to obtain the airfoil coordinates. It will be shown in Chapter 5 that in engineering practice there is some freedom such that even a trailing edge gap may be specified.

### 1.5. Trailing edge included angle and its relation to the trailing edge gap

At a sharp trailing edge, which corresponds to the critical point (to be discussed in Chapter 3) on the circle surface, the mapping function should transform an angle $\pi$ into a wedge of angle $2\pi - \delta$, where $\delta$ is the trailing edge included angle. In principle the Laurent series with an infinite number of terms allows to do this transformation. In practice, numerically, it may raise a problem since the series must be truncated to a certain number of terms, which will result in a rounded trailing edge. Whether it gives a practical problem or not, will be discussed in Chapter 5.

This section discusses a more careful mathematical treatment in constructing a wedge from the series. It has been discussed by James [22], Bauer et.al. [19], and Volpe [21], that such a series may take the form

$$f'(\zeta) = \frac{dz}{d\zeta} = \left(1 - \frac{\zeta_{TE}}{\zeta}\right)^{1-\delta/x} \cdot H(\zeta)$$  \hspace{1cm} (1.16)$$

for $|\zeta| \geq R$, where $H(\zeta)$ is an analytic function which then can be expressed in a Laurent series,

$$H(\zeta) = H_0 + \frac{H_1}{\zeta} + \frac{H_2}{\zeta^2} + \frac{H_3}{\zeta^3} + ...$$  \hspace{1cm} (1.17)$$
It has been shown also by Bauer and Volpe\(^2\), that equation (1.16) may be used also to transform airfoils with a thick trailing edge. By writing \(1-\delta/\pi=k\), the factor in equation (1.16) can be expanded into

\[
\left(1 - \frac{\zeta_{\text{TE}}}{\zeta}\right)^k = 1 - \frac{k\zeta}{1!\zeta} + \frac{k(k-1)\zeta^2}{2!\zeta^2} - \frac{k(k-1)(k-2)\zeta^3}{3!\zeta^3} + \ldots \tag{1.18}
\]

This series is convergent for \(|\zeta_{\text{TE}}/\zeta|\leq 1\) since \(k\geq 0\); in general \(0\leq k\leq 1\). Multiplying (1.17) with (1.18) we have,

\[
f'(\zeta) = H_0 + \left(\frac{H_1}{1!} - \frac{k}{1!} H_0 \zeta_{\text{TE}}\right) \frac{1}{\zeta} + \left(\frac{H_2}{2!} - \frac{k}{2!} H_1 \zeta_{\text{TE}} + \frac{k(k-1)}{2!} H_0 \zeta_{\text{TE}}^2\right) \frac{1}{\zeta^2} + \ldots \tag{1.19}
\]

This equation (1.19) can be equated to equation (1.15) and hence \(H_n\) can be expressed in the \(C_n\) and \(k\). In a direct problem, the coefficient \(C_1=(H_1+H_0 k \zeta_{\text{TE}})\) and \(k\) can be determined from the geometry. In an inverse problem, the coefficient \(C_1\) will be specified and \(k\) may be determined from the behavior of the prescribed speed distribution in the neighborhood of the trailing edge. However, to the experience of the author, the deviation of the trailing edge geometry from having a sharp angle, affects the speed distribution only locally as will be shown in Chapter 5. Further, the relation between \(k\) and the speed distribution in the neighborhood of the trailing edge would need more detailed discussions. The author has tried (not shown in this thesis) to utilize equation (1.16), but since it involves more numerical approximations, it appeared to deliver less accurate practical results than using equation (1.15). Hence, in the present study, we use equation (1.15) which enables us already to deal with the coefficient \(C_1\) which has more global effects than the parameter \(k\).

\(^2\) Volpe [21] has used equation (1.16) (and (1.17)) in inverse problems. However to the knowledge of the present author, Volpe indicated that he did not have a direct control on the trailing edge gap.
1.6. The geometry related to a given speed distribution: The inverse problem

Today, half a century after the first application of theoretical airfoil design procedures, there is still some discussion about the question whether "any" speed distribution would result in a closed geometry solution. It will be discussed in this section that a given speed distribution can result in various possible geometry solutions and that the trailing edge gap (see section 5.3.3) or the nose radius can be used to single out one particular geometry solution. In the present thesis we will not discuss the intricate and difficult existence and uniqueness problems. We will instead rely on the numerical "construction" of solutions. If the numerical solution has an infinite precision it may lead to an analytical solution. However, since the precision of the computer and the method are limited, the numerical solution may not be able to reconstruct the local behavior e.g. speed distribution in the neighborhood of the trailing edge.

The discussion on this matter will be limited to "practical" speed distributions, which e.g., do not show more than one forward stagnation point and are sufficiently smooth. This follows from the mathematical assumption used in developing the procedure, that is, a Laurent series is used to derive an airfoil surface from a circle contour. Consequently, the airfoil contour as well as the velocity distribution must be (piecewise) smooth and each is expandable into a Laurent series. Furthermore, the Laurent series has to be truncated up to a certain number of terms. Airfoils with spikes, although they could be in the class of piecewise smooth, and possibly could be used to result in a certain pressure distribution, are excluded from the discussion.

Lighthill [15] and Timman [16] derived a set of 3-integral equations, (given in section 2.5), to satisfy the following requirements (a) equal free stream flow in the airfoil and the circle planes, (b) zero trailing edge thickness. Later this set of 3-integral equations was apparently considered as a restriction which prevents a given speed distribution along a surface, \( V_a(s) \), (but specified as a function of the angular position on the circle) from having a closed geometry solution. It is possibly this reason why inverse methods remain less developed when compared to the direct problem. In Chapter 4 this set of 3-integral equations of Lighthill and Timman will be derived from equation (1.14).

For convenience, particles moving from the stagnation point towards the upper side, are assumed to have a positive sign for the speed, while those moving in the other direction are assigned a minus sign (figure 1.7).

Combining equations (1.10) to (1.14), leads to
\[ e^{-i\omega(s)} = \frac{\overline{W}_o(\theta)}{f'(\zeta)V_o(s)} \] (1.20)

where \( \omega(s) \), defined in figure 1.6, is a function related to the airfoil shape (geometry solution).

![Diagram](image)

Figure 1.7. Speed correspondence with sign convention.

The Riemann mapping theorem states:

*If \( D \) and \( D' \) are two simply-connected domains, there always exists an analytic function \( f(\zeta) \) which is regular in \( D \) and maps \( D \) conformally onto \( D' \) [27].*

Since there always exists a mapping function \( f(\zeta) \), as stated by the above theorem, the following statement for a known \( f(\zeta) \) is put forward:

An analytic function \( f'(\zeta) \), the derivative of \( f(\zeta) \), always exists which is relating the complex conjugate velocities \( \overline{W}_D \) (of the domain \( D \)) to \( \overline{W}_{D'} \) (that of domain \( D' \)).

Applying the above statements to equation (1.20), and the fact that \( V_B(s) \) is given, from which \( \overline{W}_o(\theta) \) is determined, it can be concluded that equation (1.20) can be solved for \( \omega(s) \) to construct the airfoil geometry. This mathematical solution does not guarantee, however, to exclude solutions which can be physically un-realistic. Indeed the determination of \( \omega(s) \) is the crux of an inverse problem.
Lighthill expanded the logarithm of the derivative of a mapping function, 
\[ \log(f') = \log(V_a) - i\omega \] (see Chapter 4), in Fourier series in terms of the polar angle \( \theta \)
on the circle. For a specified distribution of \( \log(V_a(\theta)) \), the vector direction \( \omega(\theta) \) maybe determined by finding the conjugate Fourier series of \( \log(V_a(\theta)) \). It would bebetter if \( V_a(\theta) \) were specified on \( s \), \( V_a(s) \), which then can be transferred to \( V_a(\theta) \),since the aerodynamic characteristics of the yet unknown airfoil shape can then becomputed. The transfer of \( V_a(s) \) to \( V_a(\theta) \) can be done using the compatibilitycondition discussed in Chapter 4 in this thesis.

Returning to equation (1.20), for \( \omega(s) \) to be the geometry solution of an airfoil
in the \( z \)-plane, corresponding to a circle in the \( \zeta \)-plane, the function \( 1/f'(\zeta) \) must beanalytic on the contour (except at the critical point i.e. where \( f'(\zeta) = 0 \), whichgenerally corresponds to the airfoil trailing edge) as well as in the flow field.Therefore the function \( 1/f'(\zeta) \) must be expandable in a Laurent series which can bewritten as

\[
\frac{1}{f'(\zeta)} = A_0 + A_1 \frac{1}{\zeta} + A_2 \frac{1}{\zeta^2} + A_3 \frac{1}{\zeta^3} + \ldots \tag{1.21}
\]

From equation (1.15) we obtain,

\[
\frac{1}{f'(\zeta)} = 1 - C_1 \frac{1}{\zeta} - C_2 \frac{1}{\zeta^2} - C_3 \frac{1}{\zeta^3} + \ldots \tag{1.22}
\]

Comparing equation (1.21) with (1.22) we observe that the coefficients \( C_n \) of theearlier terms in equation (1.22) (\( A_n \) in equation (1.21)), are present in thecoefficients of the next terms. It shows that the coefficient of a certain term can befixed while the coefficient of the other terms will follow. Here we choose to fix \( C_1 \)since it gives a practical meaning, i.e. specifying the trailing edge gap of the airfoil.Nevertheless, instead of specifying the trailing edge gap, some part of thegeometry shape can also be fixed (see Chapter 6).

Equation (1.22) indicates that an infinite number of solutions of \( \omega(s) \) can beobtained. It can be elaborated as follows. Combining equations (1.12) to (1.15) we have

\[
f' = \frac{V_c(\theta)}{V_a(\theta)} e^{(\pi/2 - \theta + \omega(\theta))} = 1 + C_1 \frac{1}{\zeta} + C_2 \frac{1}{\zeta^2} + C_3 \frac{1}{\zeta^3} + \ldots \tag{1.23}
\]
where $\omega$ has been written in $\theta$. The relation between the geometry solution, $\omega(\theta)$, and the series coefficients is given by

$$C_1 = \frac{iR}{2\pi} \int \frac{V_c(\theta)}{V_a(\theta)} e^{i\omega(\theta)} d\theta$$

$$C_2 = \frac{iR^2}{2\pi} \int \frac{V_c(\theta)}{V_a(\theta)} e^{i\omega(\theta)} d\theta$$

(1.24)

etc.

We should keep in mind that $V_c(\theta)/V_a(\theta)$ can be determined from $V_a(s)$, and what we are aiming to obtain is the solution for $\omega(\theta)$, and that $C_n$, $n=1,2,\ldots$ are not known. It should be realized that there is an infinite number of combinations of the coefficient $C_n$ and $\omega(\theta)$ satisfying equation (1.24). We can assume a (complex) number to fix one of the coefficient $C_n$ to select one combination of $C_n$ and $\omega(\theta)$. We choose here to fix $C_1$ since it has a more direct physical meaning, i.e. fixing the trailing edge gap. (Section 5.3.3. discusses further whether $C_1$ is to be fixed or not.) We will then assume, initially, a certain form of (not necessarily the correct) $\omega(\theta)$. Then, with the assumed $\omega(\theta)$, from the second equation of (1.24) we can obtain $C_2$, and so on, up to $C_n$. The fact that there is a specific relation between the known $V_c(\theta)/V_a(\theta)$ and the $\omega(\theta)$ is taken into account implicitly in this residue analysis. Having $C_1$ up to $C_n$, a more refined $\omega(\theta)$ then can be obtained from the series. Through an iteration process, finally we have a set of $C_n$ to form the Laurent series from which the airfoil coordinates can be computed.
CHAPTER 2
TIMMAN'S MAPPING METHOD

2.1. Basis

This chapter introduces Timman's conformal mapping method as used in the Low Speed Aerodynamic Laboratory, Faculty of Aerospace Engineering of Delft University of Technology. This method represents, more or less, the state of the art of the application of conformal mapping to airfoil theory. It is presented here to provide a comparison between the principles and procedures of the method proposed in this thesis and a method currently well accepted. The presentation in this chapter follows mainly from [29]. More extensive information is available in [16].

An element of the airfoil contour in the \( z \)-plane is related to the corresponding element of the circle contour in the \( \zeta \)-plane by the relation

\[
\frac{dz}{d\zeta} = f'(\zeta) = e^{h(\zeta)} \tag{2.1}
\]

where

\[
h(\zeta) = \sigma(r, \theta) + i\tau(r, \theta) \tag{2.2}
\]

This relation is defined in the whole domain. From the theory of complex functions it is known that if \( f'(\zeta) \) is an analytic function, \( h(\zeta) \) is also an analytic function. From equation (2.1) it follows that a line element in the \( \zeta \) plane is stretched by a factor of \( e^\sigma \)

\[
|dz| = e^\sigma |d\zeta| \tag{2.3}
\]

and rotated by an angle \( \tau \),

\[
\arg(dz) = \arg(d\zeta) + \tau \tag{2.4}
\]

From equation (1.14) of chapter 1 it follows that

\[
|\bar{w}_a| = e^{-\sigma} |\bar{w}_c| \tag{2.5}
\]

where \( \bar{w}_a \) and \( \bar{w}_c \) are the conjugate complex velocities in the \( z \)- and \( \zeta \)-planes respectively, as shown in figure 2.1. The free stream flows are taken to be identical in both planes. In this chapter, the trailing edge of the airfoil is chosen to correspond to the
point \((R,0)\) on the circle. Hence the zero-lift line coincides with the horizontal axis. The airfoil chord is defined to be the longest chord and \(\phi\), the so called zero lift angle, is defined as the angle between the longest chord and the zero lift line and hence with the horizontal axis. In the later chapters, however, we will allow the trailing edge of the airfoil to correspond to a point \((R,\theta)\) on the circle where \(\theta\) need not be zero.

![Diagram of airfoil transformation](image)

**Figure 2.1.** The transformation from the \(\zeta\) - to the \(z\)-plane.

### 2.2. The airfoil geometry

When \(e^{H(\zeta)}\) is known on the circle, the airfoil geometry may be computed from

\[
Z_1 - Z_0 = \int_{\zeta_0}^{\zeta} e^{H(\zeta')} d\zeta'
\]

where the lower integral limit is \(\zeta_0\), the rear stagnation point, which corresponds to the airfoil trailing edge \(z_0\). Without loss of generality, the circle may be centered at the origin in the \(\zeta\)-plane. With \(\zeta = \text{Re}^{i\theta}\), \(d\zeta = i \text{Re}^{i\theta} d\theta\), \(|dz| = ds\), and translating the airfoil such that the trailing edge is at the origin (figure 2.1), the coordinates of the point \(z\) are given by
\[ z = x + iy = \int_0^\theta e^{-i\theta}iRe^{i\theta}d\theta = iR\int_0^\theta e^{-i\theta}e^{i(\theta + 1)}d\theta \] (2.7)

or
\[ s/R = \int_0^\theta e^{i\sigma}d\theta \] (2.8)

\[ x/R = -\int_0^\theta e^{i\sigma}\sin(\theta + \tau)d\theta \] (2.9)

\[ y/R = \int_0^\theta e^{i\sigma}\cos(\theta + \tau)d\theta \] (2.10)

where \( s \) is measured from the origin along the airfoil contour.

Figure 2.2. The radius of curvature.

The radius of curvature of the airfoil contour may be found by comparing two corresponding arcs on the circle and on the airfoil, as shown in figure 2.2. Using equations (2.3) and (2.4), and observing that
- the length of a small arc in the \( z \)-plane is \( e^\sigma \) times as large as the corresponding arc length in the \( \zeta \)-plane,
- a line element in the \( z \)-plane is turned (anti-clockwise) over an angle \( \tau \) with respect to the corresponding element in the \( \zeta \)-plane,

it follows that
\[ r \Delta \gamma = e^\circ R \Delta \theta \]

or

\[ r / R = e^\circ \frac{\Delta \theta}{\Delta \gamma} = e^\circ \frac{\Delta \theta}{\Delta \theta + \Delta \tau} \quad (2.11) \]

In the limit as \( \Delta \theta \to 0 \), \( r \) becomes the local radius of curvature, given by

\[ r / R = \frac{e^\circ}{1 + \frac{d\tau}{d\theta}} \quad (2.12) \]

The sign-convention is such that \( r \) is taken positive when the outside surface of the airfoil is convex. From equation (2.12) it is also seen that the radius of curvature is zero where \( e^\circ = 0 \).

2.3. Pressure distribution, lift, moment, and aerodynamic center

The complex potential describing the flow around a circle in the \( \zeta \)-plane with circulation \( \Gamma \) in a uniform flow \( V_\infty \) at an angle \( \alpha \) with respect to the horizontal axis (figure 2.1), is

\[ \chi(\zeta) = V_\infty e^{-i\alpha} \zeta + V_\infty R^2 e^{i\alpha} \frac{1}{\zeta} + \frac{i\Gamma}{2\pi} \log(\zeta) \quad (2.13) \]

the circulation is

\[ \Gamma = 4\pi RV_\infty \sin(\alpha) \quad (2.14) \]

The velocity on the circle contour is then

\[ |\bar{w}_c| = 2V_\infty \left( \sin(\theta - \alpha) + \sin(\alpha) \right) \quad (2.15) \]

The velocity on the airfoil contour follows from equation (2.5).

\[ |\bar{w}_a| = 2e^{-\circ}V_\infty \left( \sin(\theta - \alpha) + \sin(\alpha) \right) \quad (2.16) \]

The non dimensional pressure coefficient for the airfoil is
\[ c_p = \frac{P - P_\infty}{1/2 \rho V_\infty^2} = 1 - \left| \frac{w_a}{V_\infty} \right|^2 = 1 - 4e^{-2\alpha} (\sin(\theta - \alpha) + \sin(\alpha))^2 \] (2.17)

The lift may be computed by integrating the pressure distribution, or alternatively using the Kutta-Joukowsky theorem,

\[ L = \rho V_\infty \Gamma \] (2.18)

Inserting \( \Gamma \) from equation (2.14) leads to

\[ c_i = 8\pi \frac{R}{c} \sin(\alpha) \] (2.19)

The lift curve slope is obtained as,

\[ \frac{dc_l}{d\alpha} = 8\pi \frac{R}{c} \cos(\alpha) \] (2.20)

For thin and small camber airfoils (where \( R/c \equiv 1/4 \)) and for small \( \alpha \) we have

\[ \frac{dc_l}{d\alpha} \approx 2\pi \] (2.21)

The moment coefficient around the aerodynamic center, as derived in [29], is given by

\[ c_{m,ac} = 4\pi (R/c)^2 b_2 \] (2.22)

where the position of the aerodynamic center is given by

\[ x_{ac} / R = a_2 + a_0 / R \] (2.23)

\[ y_{ac} / R = b_2 + b_0 / R \] (2.24)

where \( a_0, b_0, a_2, \) and \( b_2 \) are given by (as given in [29])

\[ a_0 / R = \frac{1}{2\pi} \int_0^\pi \frac{x}{R} d\theta \] (2.25)

\[ b_0 / R = \frac{1}{2\pi} \int_0^\pi \frac{y}{R} d\theta \] (2.26)

\[ a_2 = \frac{1}{\pi} \int_0^\pi \sigma(R, \theta) \cos(2\theta) d\theta = -\frac{1}{\pi} \int_0^\pi \tau(R, \theta) \sin(2\theta) d\theta \] (2.27)
2.4. Relation between \( \sigma \) and \( \tau \)

From the preceding sections it follows that all important characteristics of the airfoil are known when \( \sigma \) and \( \tau \) are known on the circle. It is even sufficient to know only one of these functions, since \( \sigma \) and \( \tau \) (the real and imaginary parts of an analytic function) are not independent. The relation between \( \sigma \) and \( \tau \) can be derived as follows. Applying Cauchy's integral theorem to the contour ABCDEFA (figure 2.3) where \( \zeta \) refers to a fixed point \( P = Re^{i\psi} \) "inside" of the contour, and \( t \) is the running point on the contour, leads to

\[
\sigma(t) = \frac{1}{2\pi i} \oint_{AB, FA} \frac{\sigma(t)}{t - \zeta} dt
\]

(2.29)

![Figure 2.3. The contour ABCDEF](image)

Along the contour of integration we have,

1. the integrals along CD and FA cancel each other
2. on ABC we have \( t = R e^{i\theta}, dt = iRe^{i\theta} d\theta, \)
3. on DEF we have \( t = R e^{i\psi}, dt = iR e^{i\psi} d\theta, \)
4. \( \oint_{ABC} = -\oint_{CBA} \)

then equation (2.29) reduces to the following relation,
\[ h(\zeta) = \frac{1}{2\pi i} \oint_{DEF} \frac{h(R_1 e^{i\theta}) R_1 e^{i\phi}}{R_1 e^{i\phi} - re^{i\psi}} d\theta - \frac{1}{2\pi} \oint_{CBA} \frac{h(Re^{i\theta}) Re^{i\phi}}{Re^{i\phi} - re^{i\psi}} d\theta \quad (2.30) \]

Another useful relation can be obtained by applying Cauchy's theorem for a fixed point \( Q = \zeta^* = (R^2/r) e^{i\psi} \) (the image point of P). Since \( Q \) is "outside" the contour ABCDEFA we obtain,

\[ 0 = \frac{1}{2\pi i} \oint_{AB, FA} \frac{h(t)}{t - \zeta^*} d\theta \quad (2.31) \]

Using the same procedure as for (2.29) the following relation is obtained

\[ 0 = \frac{1}{2\pi} \oint_{DEF} \frac{h(R_1 e^{i\phi}) R_1 e^{i\phi}}{R_1 e^{i\phi} - \frac{R^2}{r} e^{i\psi}} d\theta - \frac{1}{2\pi} \oint_{CBA} \frac{h(Re^{i\theta}) Re^{i\phi}}{Re^{i\phi} - \frac{R^2}{r} e^{i\psi}} d\theta \quad (2.32) \]

If \( R_1 \to \infty \) the first integrals in equations (2.30) and (2.32) reduce to zero since it is required that at infinity the free streams in both planes are the same, i.e. \( f'(\infty) = 1 \) so that \( h(\infty) = 0 \).

Subtracting equation (2.32) from (2.30), as \( R_1 \to \infty \), the following relations are obtained

\[ \sigma(r, \psi) = -\frac{1}{2\pi} \oint \frac{\sigma(R, \theta)(R^2 - r^2)}{R^2 - 2rR \cos(\psi - \theta) + r^2} d\theta \quad (2.33) \]

\[ \tau(r, \psi) = -\frac{1}{2\pi} \oint \frac{\tau(R, \theta)(R^2 - r^2)}{R^2 - 2rR \cos(\psi - \theta) + r^2} d\theta \quad (2.34) \]

Adding equation (2.32) to (2.30), as \( R_1 \to \infty \), the corresponding relations are

\[ \sigma(r, \psi) = \frac{Rr}{\pi} \oint \frac{\tau(R, \theta) \sin(\psi - \theta)}{R^2 - 2rR \cos(\psi - \theta) + r^2} d\theta \quad (2.35) \]

\[ \tau(r, \psi) = \frac{Rr}{2\pi} \oint \frac{\sigma(R, \theta) \sin(\psi - \theta)}{R^2 - 2rR \cos(\psi - \theta) + r^2} d\theta \quad (2.36) \]
In the limit of \( r \to R \) equations (2.35) and (2.36) may be written as the so called Poisson integrals

\[
\sigma(\psi) = \frac{1}{2\pi} \oint \tau(\theta) \cot \left( \frac{1}{2} (\psi - \theta) \right) d\theta
\]

(2.37)

\[
\tau(\psi) = \frac{1}{2\pi} \oint \sigma(\theta) \cot \left( \frac{1}{2} (\psi - \theta) \right) d\theta
\]

(2.38)

Note that the integral sign \( \oint \) has been changed to \( \int \) which denotes that we use the so called Cauchy principal value

\[
\operatorname{Lim}_{\varepsilon \to 0} \left[ \int_0^{\varepsilon} (\cdots) d\theta + \int_0^{2\pi - \varepsilon} (\cdots) d\theta \right]
\]

since at \( \theta \to \psi \) the cotangent tends to infinity. Equations (2.37) and (2.38) provide a way to determine \( \sigma \) (or \( \tau \)) from the known \( \tau \) (or \( \sigma \)) on the circle.

### 2.5. Timman's conditions

It was already mentioned that at infinity, the free stream in the \( z \)-plane is made identical to that in the \( \zeta \)-plane. This requirement leads to the condition that \( e^{h(\zeta)} = 1 \) as \( \zeta \to \infty \), hence \( h(\infty) = 0 \). If now it is also required that the airfoil in the \( z \)-plane should have a closed contour, it follows that

\[
\oint \frac{dz}{z} = \oint e^{h(\zeta)} d\zeta = 0
\]

(2.39)

where the integrations are taken on the surfaces of the airfoil and on the circle respectively. This implies that if \( e^{h(\zeta)} \) is expanded in a Laurent series, the result should not contain a term with \( 1/\zeta \), hence \( h(\zeta) \) should not contain a term with \( 1/\zeta \) too. These restrictions can be elaborated as follows. Multiplying equation (2.36) by \( i \) and adding to equation (2.33) leads to
\[(\sigma + i\tau)_{r,\psi} = -\frac{1}{2\pi} \int \frac{\sigma(R, \theta) \left( R^2 - r^2 + 2iRr \sin(\psi - \theta) \right)}{R^2 - 2rR \cos(\psi - \theta) + r^2} \, d\theta \]
\[= -\frac{1}{2\pi} \int \frac{\sigma(R, \theta) \left( R e^{i\theta} + r e^{i\psi} \right)}{R e^{i\theta} - r e^{i\psi}} \, d\theta \tag{2.40} \]

Writing \( t = R e^{i\theta} \) for the coordinate on the circle and \( \zeta = r e^{i\psi} \) for the fixed point outside the circle, equation (2.40) can be written as

\[(\sigma + i\tau)_{r,\psi} = -\frac{1}{2\pi} \int \sigma(R, \theta) \frac{t + \zeta}{t - \zeta} \, d\theta \tag{2.41} \]

Observing that

\[\frac{t + \zeta}{t - \zeta} = \frac{1 + \frac{t}{\zeta}}{1 - \frac{t}{\zeta}} \tag{2.42} \]

and for \( |t/\zeta| < 1 \)

\[\frac{1}{1 - \frac{t}{\zeta}} = 1 + \left( \frac{t}{\zeta} \right) + \left( \frac{t}{\zeta} \right)^2 + \left( \frac{t}{\zeta} \right)^3 + \ldots \tag{2.43} \]

we can write

\[\frac{t + \zeta}{t - \zeta} = -1 - 2 \sum_{n=1}^{\infty} \left( \frac{t}{\zeta} \right)^n \tag{2.44} \]

Inserting equation (2.44) into (2.41) leads to

\[(\sigma + i\tau)_{r,\psi} = \frac{1}{2\pi} \int \sigma(R, \theta) \, d\theta + \sum_{n=1}^{\infty} \alpha_n \left( \frac{r}{R} \right)^n e^{-in\psi} \tag{2.45} \]

where

\[\alpha_n = \frac{1}{\pi} \int \sigma(R, \theta) e^{in\theta} \, d\theta \tag{2.46} \]
Now, requiring that \((\sigma + i\tau)\omega = 0\), equation (2.45) yields

\[ \oint \sigma(R,\theta) d\theta = 0 \]  
(2.47)

With the second requirement, that the expansion of \(h(\zeta)\) should not contain a term with \(1/\zeta\), that \(a_1\) should be zero and hence equation (2.45) yields

\[ \oint \sigma(R,\theta) e^{i\theta} d\theta = 0 \]  
(2.48)

which may be split up into its real and imaginary parts as

\[ \oint \sigma(R,\theta) \cos(\theta) d\theta = 0 \]  
(2.49)

\[ \oint \sigma(R,\theta) \sin(\theta) d\theta = 0 \]  
(2.50)

Now multiplying equation (2.34) by \(i\) and adding to equation (2.35), using the same procedure, we find similar relations for \(\tau\). Summarizing Timman’s conditions which \(\sigma\) and \(\tau\) have to satisfy on the circle are:

\[ \oint \sigma(R,\theta) d\theta = 0 \quad \oint \tau(R,\theta) d\theta = 0 \]
\[ \oint \sigma(R,\theta) \cos(\theta) d\theta = 0 \quad \oint \tau(R,\theta) \cos(\theta) d\theta = 0 \]  
(2.51)

\[ \oint \sigma(R,\theta) \sin(\theta) d\theta = 0 \quad \oint \tau(R,\theta) \sin(\theta) d\theta = 0 \]

It should be observed that the first row requires the identity of the free stream flow in both planes, and the last two rows require the closure of the airfoil contour. Further note that \(\sigma(\theta)\) and \(\tau(\theta)\) are already related by the Poisson integrals (2.37) and (2.38). Hence using only one of the columns of (2.51) in addition to the Poisson integrals should be sufficient. However, for numerical work it may be convenient to use the complete set of (2.51).

2.6. The determination of \(\sigma\) and \(\tau\) for an arbitrary given airfoil

From the preceding discussions we have seen that it is convenient to use \(\sigma\) and \(\tau\) as characteristic functions describing the airfoil. The coordinates are easily found from equations (2.9) and (2.10). On the other hand the pressure distribution follows from the function \(\sigma\).
Since at a sharp trailing edge $dz/d\zeta=0$ and hence $e^\sigma=0$ (hence $\sigma \to -\infty$) and $\tau$ makes a jump, it is necessary to pay special attention to the trailing edge region. The present section describes how the functions $\sigma$ and $\tau$ can be obtained for an arbitrary given airfoil normalized on its longest chord. When the included angle of the trailing edge is $\delta$, the jump in $\tau$ is equal to $\pi-\delta$. It is clear therefore that the numerical evaluation of $\sigma$ and $\tau$ at the trailing edge needs special care. To get rid of the numerical complication due to the singular behavior at the trailing edge, a von Kármán-Trefftz or Muller or any airfoil with exactly the same trailing edge angle as the given airfoil, can be selected as an initial airfoil to approximate the given airfoil. Then corrections $\Delta\sigma(\theta)$ and $\Delta\tau(\theta)$ can be added to obtain the given airfoil. Since the singularity at the trailing edge is already taken care of by the initial airfoil, the corrections are free of singularities.

If (as required by Timman) the trailing edge of the airfoil is made to correspond to a fixed point on the circle, the point $(R,0)$ for example, the angular attitude of the airfoil in the $z$-plane is not free. Since the direction of the oncoming flow in both planes is the same, the airfoil will finally appear at zero lift angle. It should be noted that the zero lift angle and the ratio between the airfoil chord and the radius of the circle are not yet known.

For the initial airfoil, $e^\sigma$ and $\tau$ at $N$ equally spaced points on the circle can be found from the known transformation formulae. Let the given airfoil be normalized on its longest chord and compare it to the initial airfoil. In general the two airfoils will not be equal (figure 2.4). To change the initial airfoil with functions $\sigma$ and $\tau$ and zero lift angle $\phi_1$, into the given airfoil with functions $\sigma + \Delta\sigma$ and $\tau + \Delta\tau$ and zero lift angle $\phi_2$, we have to apply corrections $\Delta\sigma$ and $\Delta\tau$ to $\sigma$ and $\tau$. The point $P_1$ on the initial airfoil, corresponding to the angle $\theta$ on the circle, has to shift to some point $P_2$ on the given airfoil, corresponding to the same angle $\theta$ on the circle. Since the function $\tau$ is related to the surface angle of the airfoil contour, we obtain an equation for $\Delta\tau$. The surface angle with respect to the longest chord is (figure 2.1) $\theta + \pi/2 + \tau + \phi$. Then from figure 2.4 we obtain, for the upper surface

$$\theta + \tau + \Delta\tau + \frac{\pi}{2} + \phi - \pi = A(x_2)$$  \hspace{1cm} (2.52)

and for the lower surface

$$\theta + \tau + \Delta\tau + \frac{\pi}{2} + \phi - 2\pi = A(x_2)$$  \hspace{1cm} (2.53)

where $A$ is the local surface angle, with respect to the longest chord, of the given airfoil. Since $x_2$ and $\phi_2$ are not yet known, we rewrite equations (2.52) and (2.53) as
\[ \Delta \tau_{\text{upper}} = \left( -\theta - \tau + \frac{\pi}{2} + A(x_1) - \phi_1 \right) + \left( A(x_2) - A(x_1) - (\phi_2 - \phi_1) \right) \] (2.54)

\[ \Delta \tau_{\text{lower}} = \left( -\theta - \tau + \frac{3\pi}{2} + A(x_1) - \phi_1 \right) + \left( A(x_2) - A(x_1) - (\phi_2 - \phi_1) \right) \] (2.55)

Figure 2.4. Surface angle of the given and intermediate airfoils

To determine \( \Delta \tau \) we can use the terms inside the first ( ) as a first approximation. The correction for \( \Delta \sigma \) may then be obtained from the Poisson integral, equation (2.37), by writing \( \Delta \tau \) and \( \Delta \sigma \) in place of \( \tau \) and \( \sigma \) respectively. Finally new functions \( \sigma + \Delta \sigma \) and \( \tau + \Delta \tau \) may be obtained, thus leading to a better approximation to the given airfoil. This procedure can be repeated a few times until a sufficiently accurate approximation has been obtained. To ensure that each iteration will lead to a closed trailing edge airfoil, the approximate \( \Delta \tau \) should satisfy Timman's conditions,

\[ \int \Delta \tau(\theta)d\theta = 0 \quad \int \Delta \tau \sin(\theta)d\theta = 0 \quad \int \Delta \tau \cos(\theta)d\theta = 0 \] (2.56)

This can be done by developing the approximate \( \Delta \tau \) function in a Fourier series

\[ \Delta \tau = \frac{1}{2} p_0 + p_1 \cos(\theta) + p_2 \cos(2\theta) + \ldots + q_1 \sin(\theta) + q_2 \sin(2\theta) + \ldots \] (2.57)
Comparing equation (2.57) with equation (2.56) it follows that terms containing \( p_0 \), \( p_1 \), and \( q_1 \) should be subtracted from the approximate \( \Delta \tau \) function.

2.7. Changing the pressure distribution

Some times it is necessary to change the airfoil geometry to obtain an improved pressure distribution. It requires the possibility to specify a new pressure distribution and then to obtain the corresponding modification of the airfoil shape. The analysis can be started from an initial airfoil for which the pressure distribution is known for a certain angle of attack with respect to the zero lift angle. The velocity on the circle contour, follows from equation (2.15),

\[
\frac{\overline{w_c}}{V_c} = 2\sin(\theta - \alpha) + \sin(\alpha) \quad (2.58)
\]

as the airfoil will appear at the zero lift angle. For the speed distribution on the airfoil we have

\[
\frac{|\overline{w}_a|}{V_\infty} = e^{-\sigma} \frac{|\overline{w}_c|}{V_c} \quad (2.59)
\]

Comparing the existing and the proposed speed distributions,

\[
|\overline{w}_a(\theta)|_{existing} = e^{-\sigma} |\overline{w}_c(\theta)|
\]

\[
|\overline{w}_a(\theta)|_{proposed} = e^{-(\sigma + \Delta \sigma)} |\overline{w}_c(\theta)|
\]

with the same \( |\overline{w}_c(\theta)| \), the following relation is obtained

\[
\Delta \sigma_{proposed} = -\log\left(\frac{|\overline{w}_a(\theta)|_{existing}}{|\overline{w}_a(\theta)|_{proposed}}\right) \quad (2.60)
\]

The corresponding function \( \Delta \tau \) can be computed from the Poisson integral, equation (2.38), by writing \( \Delta \tau \) and \( \Delta \sigma \) in place of \( \tau \) and \( \sigma \) respectively. Finally new functions \( \sigma + \Delta \sigma_{proposed} \) and \( \tau + \Delta \tau_{proposed} \) can be obtained. To obtain a closed trailing edge airfoil, the function \( \Delta \sigma \) is made to satisfy Timman's conditions in the same way as in equations (2.56) and (2.57). The airfoil coordinates can then be computed using equations (2.9)
and (2.10). While in the direct problem an iterative procedure had to be used to obtain the converged result, this is not used here.

It should be observed that Timman [16] and following him van Ingen [29] have accepted restrictions on their freedom to choose improved pressure distributions. This is due to the following explicit or implicit choices in the method they developed.

a. The initial and the modified pressure distribution are taken at the same aerodynamic angle of attack, hence the resulting lift coefficient remains (nearly) the same.

b. The modified pressure distribution has been specified as a function of the angle θ on the circle and not (as we do in the present thesis) as a function of the coordinate s along the (unknown) airfoil contour.

c. In applications of the method van Ingen kept the corrections Δτ (and Δα) regular, which means implicitly that the trailing edge angle δ was not allowed to change.

In the new method proposed in this thesis we remove the above mentioned restrictions by specifying the Vx in s, which need not be obtained from an initial airfoil, and leaving the trailing edge angle to adjust itself. Furthermore, a logarithmic term is added to the mapping function which results in a flexibility to deal with thick trailing edge airfoils. This will allow us more freedom in choosing new pressure distributions.
CHAPTER 3
CONFORMAL MAPPING WITH Log TERM

3.1. Introduction

A recent study [23] shows that airfoils with a thick trailing edge are considered more attractive and practical for the following reasons. Trailing edge thicknesses of the order of 0.5% of the chord can be utilized with drag penalties only as low as one drag count (one drag count = \( \Delta c_d = 0.0001 \)). A thick trailing edge enables one to design an airfoil with a small adverse pressure gradient over the rear part of the airfoil upper surface and with a higher pressure over the rear part of the airfoil lower surface. The key towards the success is the ability to maintain the large pressure difference between the upper and lower surface on the rear part of the airfoil up to very close to the trailing edge. Specifying pressure distributions of this kind and using a design procedure capable only to deal with thin trailing edges may lead to an airfoil with a self-intersecting trailing edge. There is a need, therefore, to develop a design procedure capable of producing thick trailing edge airfoils.

A conformal mapping function incorporating a logarithmic term, as introduced in chapter 1, can handle this problem. Some essential aspects of this enhanced conformal mapping function are discussed in the present chapter.

3.2. Adding a logarithmic term to the Joukowsky transformation

This section is devoted to a discussion of the geometrical effect of adding a logarithmic term to a Joukowsky mapping function. Consider the following function,

\[
z = \zeta + \frac{a^2}{\zeta}
\]

in which, without loss of generality, the constant \( a \) can be set to unity, leading to

\[
z = \zeta + \frac{1}{\zeta}
\]  \hspace{1cm} (3.1)

and
\[
\frac{dz}{d\zeta} = 1 - \frac{1}{\zeta^2}
\]

Hence it is seen that critical points\(^1\) of this transformation are at \(\zeta = +1\) and \(\zeta = -1\). Applying this transformation to a unit circle, centered at the origin in the \(\zeta\)-plane, will result in a flat plate of chord length 4 as seen in figure 3.1. A slightly larger circle, through \(\zeta = (1,0)\) and centered on the negative \(x\)-axis is mapped into the symmetric airfoil as indicated. We may note that the trailing edge of the airfoil is sharp since it corresponds to the critical point of the mapping function, \(\zeta_{c1} = (1,0)\). The other critical point \(\zeta_{c2} = (-1,0)\) is confined within the circle and hence the resulting airfoil has a rounded leading edge.

![Figure 3.1. Joukowsky transformation](image)

Adding a (natural) logarithmic term to equation (3.1) with a (complex) coefficient \(-b/2\pi\), we obtain the following relation.

\(^1\) According to [28], some definition are given below.
1. A function \(f\) of the complex variable \(\zeta\) is analytic at a point \(\zeta_0\) if its derivative exists not only at \(\zeta_0\) but at each point in some neighborhood of \(\zeta_0\).
2. If a function fails to be analytic at a point \(\zeta_0\), but is analytic at some point in every neighborhood of \(\zeta_0\), then \(\zeta_0\) is called a singular point of the function.
3. At each point \(\zeta\) where a function \(f\) is analytic and \(f'(\zeta)\) not equal to zero the mapping \(z = f(\zeta)\) is conformal.
4. If \(f\) is an analytic function at a point \(\zeta_0\), and \(f'(\zeta_0) = 0\), \(\zeta_0\) is called critical point where the transformation is not conformal.
\[ z = \zeta + \frac{1}{\zeta} \frac{b}{2\pi} \log(\zeta) \]
\[ = \zeta + \frac{1}{\zeta} \frac{ib\theta}{2\pi} \frac{b}{2\pi} \log(r) \]  
(3.2)

The derivative of equation (3.2) can be written as
\[ \frac{dz}{d\zeta} = 1 - \frac{b}{\zeta} \left( \frac{2\pi}{b} \right) - \frac{1}{\zeta^2} \]  
(3.3)

This equation shows that the logarithmic term does not disturb the flow field at large distances from the origin. Nevertheless, the logarithmic term displaces the critical points to other locations which should be all confined within or lie at most on the contour of the circle to be mapped. Equation (3.2) contains two critical points which are

\[ \zeta_{c1} = \frac{b}{4\pi} + \sqrt{1 + \left( \frac{b}{4\pi} \right)^2} \]  
(3.4)

\[ \zeta_{c2} = \frac{b}{4\pi} - \sqrt{1 + \left( \frac{b}{4\pi} \right)^2} \]

As an example, for this discussion, b will be taken as a real number. It may be observed that, with b=0.4, \( \zeta_{c1} = (1.0323, 0) \) is outside the unit circle in figure 3.1. This situation should be avoided, otherwise there will be a critical point in the flow field. A circle with radius slightly larger than unity, which passes through \( \zeta_{c1} \), can be chosen to be transformed onto the z-plane. Figure 3.2 shows the mapping of two circles through the critical point \( \zeta_{c1} \) with centers at the origin and on the negative real axis respectively. Mapping of circles where \( \zeta_{c1} \) falls inside the circles are shown in figure 3.3.

We will denote by “trailing edge gap”, the vector \( Z_{TE} \) drawn from the point \( z_{TE_U} \) to \( z_{TE_L} \) (figure 3.2). This “gap” is controlled by the coefficient of the logarithmic term. Since \( z_{TE_U} = z_{TE_L} \), the vector \( Z_{TE} \) may be found from equation (3.2) as

\[ Z_{TE} = z_{TE_L} - z_{TE_U} \]
\[ = -\frac{ib}{2\pi} (2\pi - 0) \]
\[ = -ib \]

or

\[ b = iZ_{TE} \]  
(3.5)
Note that $b$ may be complex and hence the "gap" may contain a real component.

Figure 3.2. Joukowsky transformation + log term with $b=0.4$ and $\zeta_{TE} = \zeta_{c1}$

Figure 3.3. Joukowsky transformation + log term with $b=0.4$ and $|\zeta_{c1}| < |\zeta_{TE}|$
3.3. Kutta condition

Figure 3.4 shows the correspondence between flow regions in the circle-plane and the airfoil-plane at zero angle of attack for an airfoil with a trailing edge gap. The Kutta condition states that the flow should leave the airfoil at the trailing edge smoothly. In the case of a thick trailing edge, the Kutta condition states, as discussed in chapter 1, that the flow should leave the airfoil at the upper and lower trailing edge corners smoothly. The extended definition of the domain in the circle-plane proposed in chapter 1 is helpful to cope with the Kutta condition statement above. Each point along the stagnation streamline leaving the circle is transformed onto two corresponding points in the airfoil-plane (z-plane) each pair is separated by the same vector $Z_{TE}$. Since the flow in the circle-plane is continuous and the mapping function is regular and the derivative is single valued, the two corresponding points mentioned would have the same physical flow properties. Hence, the two corresponding streamlines leaving the corners of the airfoil trailing edge would have the same pressure distribution along those lines.

![Diagram showing flow regions in the circle plane and in the airfoil plane](image)

*Figure 3.4. Flow regions in the circle plane and in the airfoil plane*
3.4. Aerodynamic force and moment

The force and moment on an airfoil may be obtained by integrating the pressure distribution as follows. Consider an airfoil, in the z-plane, with known pressure distribution \( p(s) \) as shown in figure 3.5.

\[
dF = ipdz
\]

(3.6)

where \( dF = dX + idY \) is directed into the airfoil, and the airfoil is to the left of the vector element \( dz \). The airfoil surface, in this present study, is defined as the contour around the physical airfoil from \( Z_{TEu} \) along \( s \) to \( Z_{TEl} \) and hence excluding the tail and the "base". The non-dimensional force and pressure are related as

\[
dc_F = ic_p \frac{dz}{c}
\]

(3.7)
where \( c \) is the airfoil chord. The force on the surface can be obtained by integrating equation (3.7) along the airfoil surface. As the physical airfoil and the "material" wake, in a steady state, behave as a continuous impervious body, the net force and moment on the airfoil can be obtained from the integration along the airfoil surface alone (figure 3.6). The interior pressure of the material wake, referred to as a slit discussed in chapter 1, cannot be determined by the mapping since it does not belong to the mapping domain. However, the material wake is not present in the real situation. Instead, at the trailing edge base we find a base pressure which in general is not equal to the free stream pressure. In the present thesis, the base pressure is determined empirically and taken into account in computing the forces. The base force component perpendicular to the free stream flow contributes to the lift and the tangential one contributes to the drag.

**Figure 3.6. Cutting the airfoil from the body extension**

What we propose to do is finding the force and moment due to the "inviscid" pressure distribution on the airfoil surface and adding to it, the force due to the base pressure at the trailing edge base. The moment reference point is taken at the
upper corner of the trailing edge. In this chapter we will exclude the base force
which will be discussed in chapter 7, hence the force here refers only to the
"inviscid" pressure distribution, with the free stream pressure (at infinity) as the
reference. In this respect the base pressure will be of concern only insofar it will be
different from the free stream static pressure. The force follows from,

$$c_F = \int_c c_P \, \frac{dz}{c}$$  \hspace{1cm} (3.8)

where the integration is taken along the airfoil surface s in the anti clockwise
direction as indicated in figure 3.5.

The moment M about a point z may be evaluated from (Appendix 1.1)

$$dM = Re \left\{ (z - z_0) \rho \, dz \right\}$$  \hspace{1cm} (3.9)

and

$$M = Re \left\{ \int_S (z - z_0) \rho \, dz \right\}$$  \hspace{1cm} (3.10)

where Re denotes the real part of a complex value, and the integration is performed
along the "airfoil surface".

The force and moment may be evaluated also in terms of the conjugate
complex velocity as follows. Applying the Bernoulli equation,

$$p = p_i - \frac{1}{2} \rho |w_a|^2$$  \hspace{1cm} (3.11)

or in its non-dimensional form

$$c_p = 1 - \frac{|w_a|^2}{V_a^2} = 1 - |w_a|^2$$  \hspace{1cm} (3.12)

where the free stream velocity has been set to unity, and noting that on a
streamline we have (Appendix 1.2)

$$|w_a|^2 \, dz = w_a^2 \, dz$$  \hspace{1cm} (3.13)

equation (3.8) may be written as
\[ c_F = i \int_{s}^{c} dz - i \int_{s}^{C} w_s^2 \frac{d\overline{z}}{c} \]  \hspace{2cm} (3.14) \\

or

\[ \overline{c_F} \cdot c = -i \int_{s}^{c} d\overline{z} + i \int_{s}^{s} \overline{w}_s^2 dz \]  \hspace{2cm} (3.15) \\

When the path \( s \) of the integration makes a closed contour, the first term on the right hand side will be zero. In this case, the famous Blasius theorem is re-obtained.

Equation (3.15) may be evaluated as follows. By writing \((z_{TE, I} - z_{TE, u}) = Z_{TE} = \Delta_{TE} e^{i\alpha}\) (figure 3.5) the first term of equation (3.15) results in

\[ -i \int_{s}^{c} d\overline{z} = -i \Delta_{TE} e^{i\alpha} \]  \hspace{2cm} (3.16) \\

The second term on the right hand side of equation (3.15) will be evaluated by substituting the variables of the \( z \)-domain with the corresponding variables in the \( \zeta \)-domain. The complex conjugate velocities of both domains are related by

\[ \overline{w}_s = \overline{w}_c \frac{f'}{f} \]  \hspace{2cm} (3.17) \\

The conjugate complex velocity in the circle plane, at an angle of attack \( \alpha \), is obtained form equation (2.13), chapter 2, as

\[ \overline{w}_c = e^{-i\alpha} + \frac{i(\Gamma / 2\pi)}{\zeta} - \frac{R^2 e^{i\alpha}}{\zeta^2} \]  \hspace{2cm} (3.18) \\

where \( R \) is the radius of the circle, and the circulation \( \Gamma \) is a real number. In general, the derivative of the mapping function is given by equation (1.15) of chapter 1 (where \( C_1 \) is replaced by \(-b/2\pi\)),

\[ f' = 1 - \left( \frac{b / 2\pi}{\zeta} \right) + \frac{C_2}{\zeta^2} + \frac{C_3}{\zeta^3} + \ldots \]  \hspace{2cm} (3.19) \\

where \( b \) is given by equation (3.3). Substituting equations (3.17), (3.18), (3.19) into the second term of equation (3.15) will give (Appendix 1.3),
\[ i\int \bar{w}^2 \, dz = \int \bar{w}^2 \, d\zeta \]
\[ = \int \left[ e^{-i\alpha} + \left[ be^{-i2\alpha} - i2\Gamma e^{-i\alpha} \right] / (2\pi \zeta) + \frac{d}{\zeta^2} + \ldots \right] d\zeta \]
\[ = -\left[ be^{-i2\alpha} - i2\Gamma e^{-i\alpha} \right] \]
\[ = -i\Delta_{TE} e^{-i\alpha} e^{-i2\alpha} + i2\Gamma e^{-i\alpha} \]

Combining this result with (3.16) gives
\[ \bar{c}_F \cdot c = e^{-i\alpha} \left[ 2\Gamma - i\Delta_{TE} \left( e^{i(\gamma + \alpha)} + e^{-i(\gamma + \alpha)} \right) \right] \]

or
\[ \bar{c}_F \cdot c = e^{-i\alpha} \left[ 2\Gamma - i2\Delta_{TE} \cos(\gamma + \alpha) \right] \] (3.21)

It is seen that equation (3.21) reduces to the well-known Kutta-Joukowsky theorem when,
- magnitude of the trailing edge gap \( \Delta_{TE} \) reduces to zero, or
- when \( \cos(\gamma + \alpha) = 0 \).

When \( \cos(\gamma + \alpha) = 0 \), the trailing edge base surface is perpendicular to the free stream flow. A similar conclusion was obtained, see Thwaites [30], in the case of an airfoil (with a sharp trailing edge) in a real flow, where a wake develops behind the airfoil. In such a case, the Kutta-Joukowsky theorem is valid when the circuit on which the circulation is computed cuts the wake at a large distance from the airfoil and perpendicularly to the free stream direction.

It should be observed that this analysis does not yield the drag associated with the trailing edge thickness. This matter will be discussed in chapter 7.

The moment on the airfoil may be determined as follows. Equation (3.9) may be re-written in non-dimensional form as
\[ dc_{m_{\alpha}} = \text{Re} \left[ (z - z_0) c_p d\bar{z} / c^2 \right] \] (3.22)

Substituting equation (3.15) into (3.22) yields
\[ dc_{m_{\alpha}} = \text{Re} \left[ (z - z_0) d\bar{z} - |w_a|^2 (z - z_0) d\bar{z} / c^2 \right] \]

\[ = \text{Re} \left[ (z - z_0) d\bar{z} - w_a \bar{w}_a (z - z_0) d\bar{z} / c^2 \right] \] (3.23)

Since on a stream line we have
\[ w_a \, d\bar{z} = \overline{w}_a \, dz \]

we obtain

\[ dc_{M_0} = \text{Re} \left[ (z - z_0) \, d\bar{z} - \overline{w}_a^2 (z - z_0) \, dz \right] / c^2 \]  \hspace{1cm} (3.24)

This equation can be integrated to obtain

\[ c_{M_0} \cdot c^2 = \text{Re} \left[ \frac{1}{2} z \bar{z} - z_0 \bar{z} \right]_{z_0}^{z_{TE}} - \text{Re} \left[ \int_{z_0}^{z_{TE}} \overline{w}_a^2 (z - z_0) \, dz \right] \]  \hspace{1cm} (3.25)

We recover the second Blasius theorem when the integration path s is a closed contour. Using the result given in Appendix 1.4 for the integration of the second term, equation (3.25) gives

\[ c_{M_0} \cdot c^2 = \text{Re} \left[ \frac{1}{2} z \bar{z} - z_0 \bar{z} \right]_{z_0}^{z_{TE}} - \text{Im} \left[ 4\pi (-C_2 e^{-i\alpha} + R^2) - \frac{1}{2\pi} + \frac{b e^{-i\alpha}}{2\pi} \{ (b e^{-i\alpha}) + i2\Gamma \} \right] \]  \hspace{1cm} (3.26)

where \( b = i(z_{TE, l} - z_{TE, u}) = i\Delta_{TE} e^{i\gamma} \). Since \( R \) and \( \Gamma \) are real values, equation (3.26) reduces to

\[ c_{M_0} \cdot c^2 = \text{Re} \left[ \frac{1}{2} z \bar{z} - z_0 \bar{z} \right]_{z_0}^{z_{TE}} - \text{Im} \left[ -4\pi C_2 e^{-i\alpha} + \frac{b e^{-i\alpha}}{2\pi} \{ (b e^{-i\alpha}) + i2\Gamma \} \right] \]  \hspace{1cm} (3.27)

These formulae are used in chapter 7, in the calculation of the airfoil lift and moment, where they are also compared with certain experiments, showing that they agree satisfactorily.
CHAPTER 4
COMPATIBILITY CONDITIONS BETWEEN THE Z- AND ζ- PLANES
IN THE DIRECT AND INVERSE PROBLEMS

4.1. Introduction

It has been discussed in chapter 2 that Timman's conditions can be obtained from the requirement that the z- and ζ-planes are identical at infinity and that the airfoil forms a closed contour in the z-plane. The last condition requires \( \oint dz = \oint \frac{dz}{dζ} dζ = \oint \frac{\tilde{w}_c}{\tilde{w}_c} dζ = 0 \). As an airfoil is not necessarily closed, \( \oint \frac{\tilde{w}_c}{\tilde{w}_c} dζ \) has not necessarily to be zero. It is therefore that the conditions attributed to Lighthill or Timman will in the present thesis be referred to as the particular compatibility conditions since they deal only with closed trailing edge airfoils. In this chapter, more general conditions, referred to as the general compatibility conditions, will be derived. The basic idea is that the transformation is made using a Laurent series with in our case a log term added. Since this series relates both the geometry and the flow, there will be a certain compatibility conditions to be observed when deriving algorithms for direct and inverse problems. The compatibility conditions can be used to determine the flow in the circle plane corresponding to a specified speed distribution \( V_s(s) \). However, it must be realized that the compatibility conditions do not give any hint as to how to avoid a physically not realistic airfoil, e.g. a self intersecting airfoil.

4.2. Recapitulation of basic relations

In chapter 1 we have seen that the existence of a mapping function

\[
    z = f(ζ)
\]

which controls the correspondence between the z- and the ζ-domain, is assured. While equation (4.1) gives point to point relations, both complex domains can be related by the line element relation,

\[
    dz = f'(ζ)dζ
\]
The complex potential in the two domains being taken equal in corresponding points, the conjugate of the complex velocities in both planes can be related by

\[ \overline{w}_a \, dz = \overline{w}_c \, d\zeta \quad (4.3) \]

where \(a\) and \(c\) stand respectively for airfoil and circle planes. These are the fundamental equations from which the following equations will be derived.

### 4.3. The general compatibility condition

Consider now a (piece-wise) continuous speed distribution \(V_a(s)\). When expressed in \(\zeta\) it can be expanded into

\[ \overline{w}_a = 1 + \frac{B_1}{\zeta} + \frac{B_2}{\zeta^2} + \frac{B_3}{\zeta^3} + \ldots \quad (4.4) \]

where the free stream speed has been taken as unity. The function \(1/\overline{w}_a\) can be written as

\[ \frac{1}{\overline{w}_a} = 1 - \frac{B_1}{\zeta} - \frac{B_2}{\zeta^2} - \ldots \quad (4.5) \]

The conjugate complex velocity in the circle plane is known from equation (2.13), which can be written as

\[ \overline{w}_c = 1 + \frac{E_1}{\zeta} + \frac{E_2}{\zeta^2} \quad (4.6) \]

The product of the last two equations gives

\[ \frac{\overline{w}_c}{\overline{w}_a} = 1 + \frac{E_1 - B_1}{\zeta} + \frac{D_2}{\zeta^2} + \frac{D_3}{\zeta^3} + \ldots \quad (4.7) \]

From equation (1.15) of chapter 1 we have,

\[ \frac{\overline{w}_c}{\overline{w}_a} = f'(\zeta) = 1 + \frac{C_1}{\zeta} + \frac{C_2}{\zeta^2} + \frac{C_3}{\zeta^3} + \ldots \quad (4.8) \]

From equations (4.7) and (4.8) follows that,

\[ B_1 + C_1 = E_1 \quad (4.9) \]
From equations (4.4) and (4.6) we observe that the circulation in the airfoil plane is contained in the coefficient $B_1$ only, while in the circle plane the circulation is contained in the coefficient $E_1$ only. Hence, from equation (4.9) follows that the component of the trailing edge gap perpendicular to the free stream flow contributes to the circulation in the airfoil plane (as obtained in Chapter 3, equation (3.21)).

Circulation is defined as a line integral of velocity along a closed contour. We are now facing a line integral of velocity around a possibly unclosed contour a-b-c. For the magnitude of the line integral of the velocity distribution on the airfoil surface, we will borrow the term "circulation" and denote it by $\Gamma^*$ as can be obtained from

$$\Gamma^* = \int V_a \, ds$$

(4.10)

Summarized below are the generalized compatibility equations,

along corresponding line elements:

$$\bar{w}_a dz = \bar{w}_c d\zeta$$

integrated along corresponding arcs:

$$\int \bar{w}_a \, dz = \int \bar{w}_c \, d\zeta$$

(4.11)

$$\frac{\bar{w}_c}{\bar{w}_a} = f'(\zeta) = 1 + \frac{C_1}{\zeta} + \frac{C_2}{\zeta^2} + \frac{C_3}{\zeta^3} + \ldots \quad (|\zeta| \geq R)$$

These equations are referred to as the general compatibility conditions.

It was shown in this chapter that the compatibility conditions (and so are the Lighthill or Timman conditions) are equations to be satisfied to determine the solution in the circle domain to correspond to a given speed distribution in the airfoil domain. The practical applications will be demonstrated in the following chapters.

Since in the computer programs we will develop in Chapter 5 the algorithms are based on equations (4.11), the compatibility conditions which are inherent to these equations will automatically be satisfied within the accuracy of the combination of the numerical formulations and the computer used. Since the series of the conformal transformation will be used with a finite number of terms there may be an additional constraint. Of course the number of terms will be taken large enough not to let this influence the practical application.
CHAPTER 5
IMPLEMENTATION OF THE PROPOSED METHOD FOR
THE SOLUTION OF DIRECT AND INVERSE PROBLEMS

We will now discuss the development of the procedure to analyze the flow field around a given arbitrary airfoil (direct problem) and to obtain the airfoil from the prescribed speed distribution along a (yet unknown) surface (inverse problem) using the theory previously presented. The direct problem will be dealt with first, the inverse problem will be discussed subsequently, and the mixed problem will be discussed in Chapter 6.

5.1. Direct problem

A given airfoil will be transformed (using equation (5.1)) to a circle of radius R. This transformation is merely a geometrical mapping and hence, it should not be connected to a particular angle of attack of the oncoming flow. The airfoil, generally given in a tabular form, will be preserved through the transformation process without stretching or rotation. The transformation process (shown schematically in figure 5.1) begins with an initial guess of the radius of the circle and the distribution of the points on it corresponding to the given points on the airfoil contour. With this, (the first guess of) the series coefficients can then be determined. Since, for a given airfoil, the corresponding points on the circle are not arbitrarily located, they are relocated using the current series coefficients. This process is repeated until a convergence criterion is satisfied. The angular position of a point on the circle corresponding to the airfoil trailing edge is taken as a measure for the convergence. It is clear, that the final corresponding points on the circle will, in general, not be equally spaced and the point corresponding to the airfoil trailing edge need not lie on the real axis.

If, however, the trailing edge of the airfoil was forced to correspond to a point (1,0) of a unit circle and the points on the circle are equally spaced, the given airfoil should be continuously rescaled, rotated, and interpolated at each iteration step. The final result would then be a scaled airfoil with interpolated coordinate points positioned at zero lift attitude. An example of this is the procedure as given by van Ingen [29].
Figure 5.1. The flow diagram for the direct problem.
In the present procedure, the following equation is employed

\[ z = \zeta + C_1 \log(\zeta) + C_0 - \sum_{n=2}^{N} \frac{C_n}{(n-1)\zeta^{n-1}} \]  \hspace{1cm} (5.1)

where \( z \) is known (generally in a tabular form). The coefficient of the first term on the right hand side - earlier denoted by \( C_1 \) - has been set to unity to ensure that the free stream flows in the \( z \)- and \( \zeta \)-planes are identical. The coefficient of the log term can be determined from the trailing edge gap,

\[ C_1 = -i \frac{Z_{TE}}{2\pi} \]  \hspace{1cm} (5.2)

where \( Z_{TE} \) is a vector from the upper corner of the trailing edge pointing to the lower corner of the trailing edge (figure 2.2 of chapter 2). The crux of the direct problem is the determination of points \( (\zeta) \) on the circle corresponding to the known points \( (z) \). When the corresponding points on the circle are known, the series coefficients \( (C_n) \) are determined using residue analysis,

\[ \oint (z - C_1 \log(\zeta))\zeta^{n-2} d\zeta = \oint \left( \zeta + C_0 - \sum_{n=2}^{N} \frac{C_n}{(n-1)\zeta^{n-1}} \right)\zeta^{n-2} d\zeta = 2\pi i \frac{C_n}{n-1} \]

or

\[ C_n = \frac{n-1}{2\pi i} \oint (z - C_1 \log(\zeta))\zeta^{n-2} d\zeta \]  \hspace{1cm} (5.3)

where the sign \( \oint \) expresses the integration around the circle, and \( n = 2, 3, \ldots \). The right hand side of this equation can be evaluated numerically. Since the constant \( C_0 \) does not affect the shape of the geometry, it will be determined after each iteration step is performed. It is obtained by translating the reconstructed airfoil such that one of its nodal points (e.g. the trailing edge upper corner) coincides with the same nodal point of the given airfoil.

5.1.1. The radius of the corresponding circle

Initially the radius of the circle is guessed such that the perimeter of the given airfoil (SA) is equal to \((2\pi+8)/2\) times the radius of the corresponding circle. (The numbers \( 2\pi \) and 8 represent respectively a very thick airfoil, in fact a circle, and a flat plate.) Or, the radius of the circle is \( R = 2 \text{ SA} / (2\pi+8) \) as a first guess.
(Note that $R$ is contained in $\zeta$ in (5.3).) A better initial value of $R$, for example by using a von Kármán-Trefftz mapping, has been tried, it does not show a significant advantage however. The crude initial value reaches the converged value as fast as the better initial value does. At each step, the perimeter of the intermediate airfoil (SI) may be computed from

$$ds = |dz| = |f'(\zeta)| \, |d\zeta| = R \, |f'(\zeta)| \, d\theta$$

and on integration gives

$$SI = R \int_{\theta_{re}}^{\theta_{re} + 2\pi} |f'(\zeta)| \, d\theta$$

(5.4)

Since SI should be equal to SA, the radius of the new circle is given by

$$R_{new} = \frac{SA}{\int_{\theta_{re}}^{\theta_{re} + 2\pi} |f'(\zeta)| \, d\theta}$$

(5.5)

where $f'(\zeta)$, the derivative of equation (5.1), is evaluated at each step. A typical convergence history of the circle radius can be seen in the following figure 5.2.

![Figure 5.2. A typical convergence history of the circle radius](image-url)
5.1.2. The location of the corresponding points on the circle

At the critical point, $\zeta_{TE}$, the derivative of the mapping function yields zero, $f'(\zeta_{TE}) = 0$. The root of the function $f'(\zeta) = 0$, is approximated successively using Newton's method, $\zeta_{TE,i+1} = \zeta_{TE,i} - \frac{f'(\zeta)}{f''(\zeta)}$, to result in the point $\zeta_{TE}$ or the nearest point to it. With the point $\zeta_{TE}$ (or its angular position $\theta_{TE}$) known, the line integration on the circle will begin from this point. This is an obvious choice since the trailing edge of the airfoil should correspond to $\zeta_{TE}$. If the derivative of $f(\zeta)$ is known at any other coordinate point, number $j$ say, the integration can also begin from this point. As a first guess, $\theta_{TE}$ may be located on the positive real axis, $\theta_{TE} = 0$. In what follows $\zeta_{TE}$ and $\theta_{TE}$ are respectively interchangeable with $\zeta_i$ and $\theta_i$ as also indicated in figure 5.1.

As a first guess, the points $z_i$ to $z_{j+1}$ on the given airfoil are respectively related to points $\zeta_i$, $j = 1, J+1$ equally spaced on the circle. This approximate distribution of points $\zeta_i$ on the circle will be corrected after the series coefficients $C_n$ are evaluated from this current correspondence. The current series coefficients $C_n$ form the function $f'(\zeta)$ from which the intermediate airfoil can be obtained by integration. Upon scaling the perimeter of the intermediate to the given airfoil, two results can be obtained, 1) a new estimate of the radius of the circle as discussed in the previous section, and 2) a new distribution of the corresponding points $\zeta_i$ on the circle. In the present thesis, the new distribution is obtained by a "systematic trial". The "systematic trial" routine can be formulated as

$$S_i = R_{new} \int_{\theta_{TE}}^{\theta_{TE}+\theta_i} \left| f'(\zeta) \right| d\theta$$

where $S_i$ is the contour length from the trailing edge to the point $z_i$ of the given airfoil, $f'(\zeta)$ is the derivative of the current mapping function, and $\theta_i$ is to be evaluated. Figure 5.3 shows a typical convergence history of the angular position of the corresponding points on the circle.

It can be seen that the series coefficients together with the corresponding points on the circle are the end results of the iteration. Nevertheless only the series coefficients, the circle radius, and the $\theta_{TE}$ are required, since the airfoil and its related flow field can be represented by the series.
Figure 5.3. A typical convergence history of the angular position of the corresponding points on the circle.
5.2. Inverse problem

In inverse problems one prescribes the distribution of the speed \( V_0/V_\infty \) (\( V_\infty \) will be made equal to unity) as a function of \( s \), the length measured along a contour, scaled to an arbitrary reference length \( L \), and the trailing edge gap (of the airfoil to be obtained). The mathematical formulation of the problem is to obtain the (Laurent) series representation of the ratio of the complex conjugate velocities \( \bar{w}_c/\bar{w}_a \) \((-f')\) where the subscripts \( c \) and \( a \) stand for circle and airfoil respectively. From the given \( V_a(s) \), the corresponding points on the circle and the related complex conjugate velocity \( \bar{w}_c \) can be obtained (by using the compatibility conditions discussed in chapter 4). The series representation to be obtained is the following:

\[
\frac{\bar{w}_c}{\bar{w}_a} = f' = 1 + \frac{C_1}{\zeta} + \frac{C_2}{\zeta^2} + \frac{C_3}{\zeta^3} + \ldots \tag{5.7}
\]

where the left hand side can be expressed as

\[
\frac{\bar{w}_c}{\bar{w}_a} = \frac{\bar{w}_c}{V_a} e^{i\omega}
\]

where \( \omega \) is the surface angle on the airfoil.

Now the crux of an inverse problem is the determination of \( \omega \). Initially, \( \omega \) will be guessed, e.g. to be that of a flat plate lying on the horizontal axis. Assuming an initial left hand side of equation (5.7), the series coefficients \( C_n \) can be obtained, e.g. using residue analysis. The coefficient \( C_1 \) is determined by the choice of the trailing edge gap. Using the current series coefficients, a better function of \( \omega \) can then be determined, from \( \omega = \arg(f') + \theta - \pi/2 \). Now we have, hopefully, a better function in the left hand side of equation (5.7) with which the process can be repeated, until some convergence criterion is satisfied. This process of the inverse problem is shown schematically in figure 5.4. Finally, the airfoil geometry can be constructed using equation (5.1).

5.2.1. Angular position of the rear stagnation point \( \theta_{\text{TE}} \)

From the speed distribution along the airfoil surface, the "elemental" and the total circulation \( \Gamma^* \) around the airfoil can be computed. ("Elemental circulation" is to denote the line integral of the speed along a piece wise contour.) This distribution of the "elemental circulation" must be identical with that in the \( \zeta \) plane around the circle. On the other hand, as the total circulation is fixed, as soon as \( V_a(s) \) is given, independent of the shape of the airfoil, it will correspond to a circle with a fixed
Figure 5.4. The flow diagram for the inverse problem
speed distribution. This fact enables us to compute the angular location of the stagnation point $\theta_{\text{TE}}$ on the circle as discussed below.

From the given $V_\infty(s)$ the following values can be computed:

the total circulation $\Gamma^*$,

$$\Gamma^* = \Gamma^*_u + \Gamma^*_l$$  \hspace{1cm} (5.8)

with, along the upper surface,

$$\Gamma^*_u = \int_{\text{TE}}^\text{SP} V_\infty \, ds_u$$  \hspace{1cm} (5.9)

and along the lower surface,

$$\Gamma^*_l = \int_{\text{TE}}^\text{SP} V_\infty \, ds_l$$  \hspace{1cm} (5.10)

where the subscripts $u$ and $l$ stand for upper and lower part of the airfoil surface, TE for the trailing edge, and SP for the forward stagnation point.

Since the pressure distribution around a circle is a function of the total circulation, where the total circulation is given by $\Gamma = 4\pi R \, V_\infty \, \sin(\alpha - \theta_{\text{TE}})$, with a given $\alpha$, it is also unique as to $\theta_{\text{TE}}$. The "elemental circulation" on the circle surface, from a point $a$ to $b$, can be evaluated from

$$\Gamma^b_a = \int_{\theta_a}^{\theta_b} d\Gamma = R \int_{\theta_a}^{\theta_b} V_\infty \, d\theta = R \int_{\theta_a}^{\theta_b} 2V_\infty (\sin(\theta - \alpha) + \sin(\alpha - \theta_{\text{TE}})) \, d\theta$$

$$= 2V_\infty R (\sin(\alpha - \theta_{\text{TE}}) - \cos(\theta - \alpha)) \bigg|_{\theta_a}^{\theta_b}$$  \hspace{1cm} (5.11)

The upper and the lower contributions to the circulation on the circle can be obtained by selecting the proper end points of the integration,

$$\Gamma_u = \Gamma_{\theta_{\text{TE}}}^{\pi - \theta_{\text{TE}} + 2\alpha} = 2V_\infty R (\sin(\alpha - \theta_{\text{TE}}) - \cos(\theta - \alpha)) \bigg|_{\theta_{\text{TE}}}^{\pi - \theta_{\text{TE}} + 2\alpha}$$

$$= 2V_\infty R [\pi - 2\theta_{\text{TE}} + 2\alpha) \sin(\alpha - \theta_{\text{TE}}) + 2 \cos(\theta_{\text{TE}} - \alpha)]$$  \hspace{1cm} (5.12)

and

$$\Gamma_l = \Gamma_{\pi + \theta_{\text{TE}}}^{2\pi - \theta_{\text{TE}} + 2\alpha} = 2V_\infty R (\sin(\alpha - \theta_{\text{TE}}) - \cos(\theta - \alpha)) \bigg|_{\pi - \theta_{\text{TE}} + 2\alpha}^{2\pi + \theta_{\text{TE}}}$$

$$= 2V_\infty R [\pi + 2\theta_{\text{TE}} - 2\alpha) \sin(\alpha - \theta_{\text{TE}}) - 2 \cos(\theta_{\text{TE}} - \alpha)]$$  \hspace{1cm} (5.13)
Since \( \Gamma^* \) on the upper and lower surfaces can be determined, the following relation may be obtained for \( A = \frac{\Gamma^*_u}{\Gamma^*_l} \)

\[
A = \frac{\Gamma^*_u}{\Gamma^*_l} = \frac{\Gamma_u}{\Gamma_l} = \frac{(\pi - 2\theta_{\text{TE}} + 2\alpha) \sin(\alpha - \theta_{\text{TE}}) + 2\cos(\theta_{\text{TE}} - \alpha)}{(\pi + 2\theta_{\text{TE}} - 2\alpha) \sin(\alpha - \theta_{\text{TE}}) - 2\cos(\theta_{\text{TE}} - \alpha)}
\] (5.14)

This equation can be re-arranged as

\[
\tan(\theta_{\text{TE}} - \alpha) = \frac{-2(A + 1)}{A(\pi + 2\theta_{\text{TE}} - 2\alpha) - (\pi - 2\theta_{\text{TE}} + 2\alpha)}
\] (5.15)

This equation can be solved for \( \theta_{\text{TE}} \) when \( A \) and \( \alpha \) are known. Since, at a certain \( \Gamma^* \), the angles \( \alpha \) and \( \theta_{\text{TE}} \) are not independent, in the present procedure \( \alpha \) is taken always zero. Setting \( \alpha \) to zero leads to

\[
\tan(\theta_{\text{TE}}) = \frac{-1}{\frac{\pi(A - 1)}{2(A + 1)} + \theta_{\text{TE}}}
\] (5.16)

This equation can be solved, since \( A \) is known, for \( \theta_{\text{TE}} \) e.g. using the Newton iteration method. Setting \( \alpha \) to zero (the free stream flow is along the horizontal axis), means that the amount of the circulation in the circle plane is controlled by \( \theta_{\text{TE}} \) only and finally the resulting airfoil will be at the appropriate angle of attack.

5.2.2. Radius of the corresponding circle

Once \( \theta_{\text{TE}} \) is known, the radius of the corresponding circle can also be determined from

\[
\Gamma = 4\pi RV_{\infty} \sin(-\theta_{\text{TE}})
\] (5.17)

or

\[
R = \frac{\Gamma}{4\pi V_{\infty} \sin(-\theta_{\text{TE}})}
\] (5.18)

where \( \alpha \) has been set to zero, and \( \Gamma = \Gamma^* \) as given in equation (5.8). However, when the speed distribution is such that \( \Gamma^*_u = \Gamma^*_l \), hence \( \theta_{\text{TE}} = 0 \), equation (5.18) can not be applied, alternatively \( R \) should be found from equation (5.12), which for \( \theta_{\text{TE}} = 0 \) reduces to

\[
R = \frac{4}{V_{\infty}}
\] (5.19)
where $\Gamma_u$ equal to $\Gamma^*_u$ as given by equation (5.9).

### 5.2.3. Distribution of the corresponding points around the circle

At this point sufficient information is available to determine the angular location of the corresponding coordinates on the circle. Referring to equation (5.11), the left hand side is known and in the right hand side only $\theta$ is unknown and therefore may be solved for. On the upper surface, the circulation from the trailing edge to point $a'$ on the airfoil is

$$\Gamma^*_a = \int_{\text{TE}}^{a'} V_a \, ds$$  \hspace{1cm} (5.20)

and on the corresponding element on the circle is given by equation (5.11)

$$\Gamma_a = 2V_a \, R \left[ \theta \sin(\theta_{TE}) - \cos(\theta) \right] \Bigr|_{0}^{\theta_{a'}}$$  \hspace{1cm} (5.21)

where $\Gamma_a = \Gamma^*_a$ is known. The corresponding points ($\theta$) on the circle upper surface can then be evaluated from equation (5.21), in the present procedure we use a "systematic trial" routine as discussed in the direct problem, section 5.1.2. When a "systematic trial" routine is used, a continuous and monotonic function will be easier to handle than a non-monotonic function. Since the $\int V_a \, ds$ will reach its maximum value at the front stagnation point, the search for $\theta$ on the circle is sub-divided into the upper and lower paths. On the lower surface, the circulation is counted from the front stagnation point, equation (5.21) is still applied by interchanging $\theta_{\text{TE}}$ and $\theta_{sp}$.

### 5.2.4 The direction of the velocity on the airfoil

Initially, the airfoil may be assumed as a flat plate on the horizontal axis so that the flow on its surface may be expressed as $\bar{w}_a(s) = V_a(s) \, e^{-\omega(s)}$ with $\omega(s) = 0$ on the upper surface and $\omega(s) = 2\pi$ on the lower surface. Recalling equation (5.7),

$$f^* = \frac{\bar{w}_c}{\bar{w}_a} = 1 + \frac{C_1}{\zeta} + \frac{C_2}{\zeta^2} + \frac{C_3}{\zeta^3} + \ldots$$  \hspace{1cm} (5.7)

where
\[ \zeta = R e^{i\theta} \]

\[ \bar{W}_c(\theta) = V_c(\theta) i e^{i\theta} \]

\[ V_c(\theta) = 2V_w (\sin(\theta) + \sin(-\theta)) \]

with R and \( \theta \) known from equations (5.18) and (5.21). Inserting \( \bar{W}_c \) and \( \bar{W}_a \) into equation (5.7) we have

\[ \frac{V_c e^{-i\theta}}{V_a e^{-i\omega}} = \frac{V_c}{V_a} e^{i\tau} = 1 + \frac{C_1}{\zeta} + \frac{C_2}{\zeta^2} + \frac{C_3}{\zeta^3} + \ldots \] (5.22)

where \( \tau = \pi/2 - \theta + \omega \). The argument \( \omega \) is first assumed, then \( \tau \) is evaluated at the first iteration step and so on. Equation (5.22) can be solved for the residue \( C_n \),

\[ C_n = \frac{1}{2\pi i} \oint \left( \frac{V_c}{V_a} e^{ik} \right) \zeta^{n-1} d\zeta \] (5.23)

In the next iteration, \( \tau \) follows from

\[ \tau_{new} = \arg \left( 1 + \frac{C_1}{\zeta} + \frac{C_2}{\zeta^2} + \frac{C_3}{\zeta^3} + \ldots \right) \] (5.24)

It should be recalled that for a given pressure distribution there can be many solutions since the geometry depends on the value of \( C_1 \) (see section 5.3.3).

5.3. Examples

Computer programs were developed in FORTRAN to verify the procedures presented in this chapter. Most of the algebraic operations are made in complex variables. Integration and differentiation procedures are carried out using cubic-spline interpolation. To begin with, the procedures presented here are verified for direct problems involving thick and thin airfoils, secondly inverse problems are presented and followed by a third case discussing the robustness of the procedures.
5.3.1. Direct problem: comparison for Kármán-Trefftz airfoil and some further examples

Figure 5.5 shows a von Kármán-Trefftz airfoil generated from a circle centered at \((\xi, \eta)\) where \(\xi=-0.15\), with zero lift angle of -5 degrees, and 10 degrees trailing edge included angle. For this airfoil, the following items will be discussed:

- the convergence history (the number of terms of the series),
- the convergence history (the number of iteration cycles),
- comparisons between the input and the reconstructed contours,
- detailed plot near the trailing edge region,
- comparisons between the exact and the predicted pressure distributions.

![Figure 5.5. Kármán-Trefftz airfoil used as exact test case](image)

Figure 5.6 shows the convergence history with the increasing number of terms. The vertical axis, labeled as \(|DZ_{avg}|\), is the average error over all the nodal points. The error at a nodal point measures the distance between two points, which respectively belong to the input and the reconstructed coordinates. The reconstructed contour is translated such that the trailing edge always coincides with that of the input contour.
Figure 5.6. Kármán-Trefftz airfoil, convergence history as measured with the average error of coordinates with increasing number of terms in the series for 20 iteration cycles.

The point wise error is shown in figure 5.7 for the case of 20 series terms. It should be noted that the von Kármán-Trefftz mapping function will have an infinite number of terms when represented in a Laurent series where the circle centers at the origin of the coordinate system. It can be seen from figure 5.6 that increasing the number of series terms beyond about 20 does not improve the accuracy. As will be found also in the following examples, the optimum number 20 will correspond to about 1/5 of the number of input coordinate points.

According to Lanczos [31], the "smoothing" parameter J/N, associated with Fourier series representation where J is the number of input data and N is the number of series terms, should be at least 2, preferably 4 to 5. If J/N is smaller than 1, this means that the series "is aiming at" a higher order approximation than the data to be represented allow. If J/N = 1, the order of the series is the same as the order of the data, we have just the minimum number of input points for the determination of f(ζ) and nothing is left over for "smoothing".
Figure 5.7. Kármán-Trefftz airfoil, distribution of local error for 100 input points with 20 series terms and 20 iteration cycles.

Let us consider a case where the airfoil is described exactly by a series with a finite number of series terms \( N = m \). In the absence of noise, the series coefficients \( C_1, C_2, C_3, \ldots C_m \) would have certain values but would be zero beyond \( C_m \). In the presence of noise, \( C_{m+1}, C_{m+2}, \ldots \) are no longer zero nor have they the tendency to diminish. Ideally, such a noise would show a random distribution of error, which has no preference for any frequency. In the present case, although the data points can be generated theoretically free of noise, the overall procedure allows itself to introduce noise. The main sources of the noise are - the truncation of the series, and - the numerical imperfection of the differentiation and integration routines.

Figure 5.7 shows the distribution of the local error in a reconstruction of an exact von Kármán-Trefftz airfoil. It shows that the noise is not quite random and the reconstruction still misses certain low harmonic contributions. This is a problem inherent to the present procedure. Since the corresponding points on the circle are determined using equation (5.6) successively, starting from the rear stagnation point, the errors are transmitted forwards. Figure 5.8 shows the convergence with
increasing the number of iteration cycles. It can be seen that the iteration will converge in about 16 cycles.

![Graph](image)

**Figure 5.8. Kármán-Trefftz airfoil, convergence history as measured with the average error of coordinates with increasing number of iteration cycles.**

Let us now observe the reconstruction quality by comparing the contour plot. Mostly, conformal mappings are done by contour fitting as discussed in chapter 2 rather than by "systematic point tracking" as the present procedure. Therefore, point to point comparison as is given here is rarely found in the literature. Figure 5.9 shows contour comparison for 100 input coordinate points reconstructed using 20 series terms and cubic spline integration, where the average error is 0.003 % as shown in figures 5.6 and 5.8. Figure 5.10 shows contour comparison using 10 series terms and trapezoidal rule integration where the average error is 0.22 %. It can be seen that, by graphical comparison, the reconstruction shown in figure 5.10 is as good as that in figure 5.9.
Figure 5.9. Kármán-Trefftz airfoil, contour reconstruction with 20 terms for 100 input coordinate points, average error is 0.003 %

Figure 5.10. Kármán-Trefftz airfoil, contour reconstruction with 10 terms for 100 input coordinate points, average error is 0.220 % (integration with trapezoidal rule).
Figures 5.11a, b, c, and d show the contour comparison in the neighborhood of the trailing edge using the same mapping function (equation (5.1)) as in figure 5.9.

Figure 5.11. Kármán-Trefftz airfoil, contour reconstruction with 20 terms, in the trailing edge region, with increasing detail.
It can be expected that equation (5.1) will produce a round trailing edge as discussed in chapter 1. If it is needed to improve the quality of the contour in the neighborhood of the trailing edge, equation (5.16) of chapter 1 should be used instead. Results using equation (5.16) of chapter 1 are shown in figures 5.12a and b. However, we should observe that the rounding of the contour in figure 5.11c and d is significant within 0.01 % and 0.001 % chord in the trailing edge region, respectively. It is unlikely to introduce a significant effect in the pressure distribution away from this very small region. In engineering practice we can disregard the trailing edge rounding by arguing on the good quality of the pressure distribution in the major part of the airfoil as will be shown in the following figures. Furthermore, it should be realized that equation (5.16) of chapter 1 will involve more numerical approximation which will reduce the overall accuracy (figure 5.13) and increase the computation time.

![Figure 5.12. Kármán-Trefftz airfoil, contour reconstruction with 20 terms, in the trailing edge region, using equation (5.16) of chapter 1 with increasing detail.](image-url)
Figure 5.13. Distribution of local error using equation (5.16) of chapter 1.

It should be observed that the angular position $\theta_{TE}$ on the circle corresponding to the airfoil trailing edge was located by finding the zero of $f'(\zeta)$. Figure 5.14 shows the typical values of $f'(\zeta)$ around the circle. The solution for $\theta_{TE}$ of $f'(\zeta) = 0$ was obtained by Newton iteration. In this example the given airfoil has a zero lift angle (equal to the $\theta_{TE}$ on the circle) of -5 degrees, while the present procedure results in a converged $\theta_{TE} = -5.0325$ degrees. This difference of the zero lift angle leads to a difference in the lift coefficient, at an angle of attack, of about 0.0036.

Figure 5.14. Typical values of $f'(\zeta)$ for the airfoil defined in figure 5.5.
Figures 5.15 and 5.16 show speed distributions corresponding to the reconstruction shown in figures 5.9 and 5.10 respectively. It can be seen that the speed distribution in figure 5.16 is as good as that in figure 5.15 except in the region very close to the trailing edge, i.e. between the last four points.

![Figure 5.15. Kármán-Trefftz airfoil, speed distribution corresponding to figure 5.9.](image)

![Figure 5.16. Kármán-Trefftz airfoil, speed distribution corresponding to figure 5.10](image)
Figure 5.17 shows the convergence history of the LS(1)-0417(mod) airfoil with increasing the number of terms with 100 input coordinate points. Figure 5.18 shows the distribution of the local error.

![Convergence History Diagram](image)

**Figure 5.17.** LS(1)-417 (mod) airfoil, convergence history as measured with the average error of coordinates with increasing number of terms.

![Distribution of Local Error Diagram](image)

**Figure 5.18.** LS(1)-417 (mod) airfoil, distribution of local error for 100 input points with 20 terms.
Figure 5.19 shows the convergence history in the number of iteration cycles. Figure 5.20 shows the comparison between the reconstructed and the input airfoils, using 20 terms. This graphical comparison shows that an arbitrary airfoil can be described with 20 terms with confidence. However, figure 5.17 suggests that 10 terms might result in a better reconstruction accuracy.

Figure 5.19. LS(1)-417 (mod) airfoil, convergence history as measured with the average error of coordinates with increasing number of iteration cycles.

Figure 5.20. LS(1)-0417 (mod) airfoil, contour reconstruction with 20 terms for 100 input coordinate points.
Figures 5.21, 5.22, 5.23, and 5.24 show similar results in the case of a thin airfoil. The airfoil is a single sheet arc (figure 5.24) having the maximum camber at about 25% chord while the arc portion between 50 and 100% chord is a flat plate.

Figure 5.21. Single sheet arc, convergence history as measured with the average error of coordinates with increasing number of terms.

Figure 5.22. Single sheet arc, distribution of local error.
Figure 5.23. Single sheet arc, convergence history as measured with the average error of coordinates with increasing number of iteration cycles.

Figure 5.24. Single sheet arc, contour reconstruction with 20 terms for 100 input coordinate points.

Figures 5.21 and 5.22 show that the arc shown in figure 5.24 can be represented by a Laurent series with 20 terms which can be obtained after about 28 iteration cycles.
5.3.2. Inverse problem: Reconstruction of conventional airfoils and an airfoil with a "slot"

Four examples of the inverse problem are given below. An inverse problem is here referred to as a design problem where the pressure distribution is given for the whole surface. A mixed problem will be treated in chapter 6. Figure 5.25b shows the reconstruction of the Kennedy-Marsden airfoil [34] from its zero degree angle of attack pressure distribution shown in figure 5.25a. Figure 5.26a and b show the speed distribution at 8 degrees angle of attack and its reconstruction respectively. When the data of 5.26b is rotated 8 degrees counter clock wise and overlaid on the data of 5.25b, both will give the same contour as shown in figure 5.27. Figure 5.25 demonstrates that the present method is capable to reconstruct an airfoil from its pressure distribution. Further, from figures 5.26 and 5.27, it can be seen that a given pressure distribution carries a "hidden" information of the angle of attack of the corresponding airfoil. The angle of attack, an angle between the free stream flow and the longest airfoil chord, in the present method, can be determined after the airfoil reconstruction is completed. Figure 5.28 shows the convergence history with increasing the number of iteration cycles as measured by $\tau$ (defined in eq.(5.24)) at the leading edge. It shows that the converged solution is obtained after about 16 iteration cycles.
Figure 5.25. Kennedy-Marsden airfoil, reconstruction from its 0 degree angle of attack speed distribution, with 20 terms.
Figure 5.26. Kennedy-Marsden airfoil, reconstruction from its 8 degree angle of attack speed distribution, with 20 terms.
Figure 5.27. Airfoils from figures 5.25 and 5.26 (rotated 8° ccw).

Figure 5.28. The convergence history related to figures 5.25 and 5.26 as measured with the changing of $\tau_{LE}$. 
Figure 5.29. Reconstruction of an arc from its speed distribution shown in the top figure.
Figure 5.29a shows a speed distribution for a circular arc at ideal angle of attack. For this pressure distribution, the upper and lower surfaces of the arc obtained from the inverse procedure are shown in figure 5.29b, where this is compared with the corresponding original arc. In the inset the arc is shown with an increased y scale to visualize more clearly the quality of the reconstruction. It is seen that an arc can be obtained with a reasonable accuracy in an inverse problem. Nevertheless, the leading and trailing edge of this reconstructed arc will not be sharp as a consequence of using the truncated Laurent series.

Figure 5.30 shows the geometry solution of a speed distribution corresponding to the Kennedy-Marsden airfoil at zero angle of attack (as shown in figure 5.25a) with a requirement of having 0.7% chord trailing edge gap. The original airfoil has a very thin trailing edge. The fact that the trailing edge gap can be specified, will enable us to modify an airfoil with too thin a trailing edge, such as this example, to have a thicker one with very nearly the same speed distribution.

Figure 5.30. An other solution from the given speed distribution shown in figure 5.25a, with trailing edge gap $\Delta y = 0.7\%$ chord and $\Delta x = 0$.

Figure 5.31b shows an airfoil, with a specified trailing edge gap, reconstructed from the speed distribution (drawn by hand) shown in figure 5.31a. The speed distribution in figure 5.31a shows more or less a discontinuity as would occur for a suction slot.
Figure 5.31 a) A specified speed distribution with a strong jump at about 85% chord, b) the airfoil solution.

For this type of speed distribution, the region near the "slot" needs to be specified with an increased number of nodal points. It needs a large number of terms to be able to cope with the speed "jump". Figure 5.32 shows the off-design solution of the airfoil using 96 terms. Figure 5.33 shows the solution using 30 terms. For this relatively small number of series terms, it gives wavy solutions as can be seen. It is a known phenomenon in representing a "discontinuous" curve with a Fourier series.
5. Implementation of the proposed method

Figure 5.32. The speed distribution at various angles of attack computed using 96 terms.
Figure 5.33. The speed distribution at various angles of attack computed using 30 terms.
5.3.3 The freedom to specify $C_1$

In figures 5.30 and 5.31 two examples have been given for the inverse problem where the trailing edge gap, represented by the coefficient $C_1$ in the series for $dz/d\zeta$, was prescribed and not determined from the residue analysis as the other terms. One could argue that $C_1$ should or could not be prescribed but that indeed it should follow from the residue analysis. This leads again to the question, whether specifying $C_1$ and hence the trailing edge gap would lead to a constraint on the prescribed pressure distribution. In the present section we will give some examples in which for one and the same prescribed pressure distribution three different airfoils are derived by prescribing three different values for $C_1$, corresponding to trailing thickness of 0%, and about 0.7%, and 7.0% chord. The prescribed pressure distribution is taken from a computation for the LS(1)-0417(mod) airfoil with a 0.7% trailing edge thickness. In all cases the series for $dz/d\zeta$ has been used up to and including $C_{11}$.

![Figure 5.34](image)

Figure 5.34 a) Prescribed and computed speed distributions  
b) the reconstructed airfoils with different $C_1$. 
Appendix A.5 gives the values for the $C_n$ corresponding to the three airfoils after 60 iterations. Of course it follows that changing the prescribed $C_1$ leads to a different set of coefficients $C_2$ through $C_{11}$. The numerical procedure tries to approximate the prescribed pressure distribution as close as possible.

Figure 5.34a shows that, despite the differences in $C_1$ (and hence in the trailing edge gap), the obtained pressure distributions are very nearly always the same and equal to the prescribed one except in the leading- and trailing edge regions. The resulting airfoils are remarkably different (figure 5.34b). Hence we conclude that, at least for the engineering practice, there is a substantial freedom to specify $C_1$. In all cases, at each iteration, $C_1$ has also been determined from residue analysis. Figure 5.35 shows that the calculated $C_1$ approaches the prescribed value in about 12 iterations. It should be emphasized that $C_1$, as determined from the residue analysis, was only obtained as an intermediate result and was not allowed to influence further iterations. During the iterative procedure itself $C_1$ was kept fixed at the prescribed value.

Figure 5.36 shows the development of $C_1$ in the case that it is not fixed, but left free to establish itself from the residue analysis. It was found that from iteration number 12 on, the obtained pressure distribution, as plotted for a representative point, did not change (bottom figure 5.36). According to figure 5.36 $C_1$ is still changing on a scale comparable to trailing edge thicknesses of the order of a few percent. It should be remembered that the speed distribution, which was prescribed in these examples, was taken from an airfoil with a 0.7% trailing edge gap, corresponding to $C_{1 \text{ real}} = -0.0051906940$ and $C_{1 \text{ im}} = 0$. Then the question arises whether in case $C_1$ is left free, it should not have converged to the value mentioned above. From the difference in the graphically observed convergence of $C_1$ and the pressure distribution it follows that we are here confronted with an intricate problem. Also we can expect that possible constraints may be affected by small details of the pressure distribution and the geometry near the trailing edge. Solving this problem would require a much more detailed mathematical analysis and/or a much more refined numerical treatment and a larger number of terms in the series, than has been within the scope of the present thesis. For the time being the author has to be satisfied with the good engineering method which has been obtained. A more refined analysis will be left for a later research project.
Figure 5.35 Development of the computed $C_1$ compared to the specified $C_1$.

Figure 5.36 Development of $C_1$ as computed from residue analysis.
5.3.4. The robustness of the procedures

The following examples demonstrate the robustness of the present procedures. A prescribed pressure distribution may lead to unrealistic geometry solutions, which is useless in engineering. Nevertheless, an inverse procedure that has a capability to result in contours, whatever the shape, will be a valuable tool to the designer since he or she will see the direction to where to move. Specifying a trailing edge gap may help to obtain a realistic airfoil shape, but in an other case, the prescribed pressure distribution need to be modified to result in a realistic airfoil shape. Figure 5.37 shows an example, not intended for engineering purposes, where a trailing edge gap helps to obtain a realistic airfoil shape.

Figure 5.37 a) A speed distribution to serve as an example (not intended for practical use), b) the airfoil solutions with the same speed distribution.
Figure 5.38 shows an example where a pressure distribution needs to be modified (in combination with a trailing edge gap specification) to result in a realistic airfoil. As in the case for figure 5.37, figure 5.38 is presented to serve merely as an example without any engineering intentions.

![Graph](image)

Figure 5.38 a) A speed distribution to serve as an example (not intended for practical use), b) The airfoil solution.

However, a designer must have a sufficient knowledge to recognize the region of the speed distribution which needs to be modified to obtain a physically acceptable airfoil. In the case shown in figure 5.38, introduction of a trailing edge gap and or speed modification in the rear region will not help to obtain a physically acceptable
airfoil. A sufficiently thick nose, hence a sufficiently accelerated flow, is needed to obtain a physically thick airfoil.

Figure 5.39 shows an example where the front stagnation point of the prescribed speed distribution moved rearward approaching the rear stagnation point. In the circle plane, this speed distribution corresponds to the speed distribution on the circle having a "super circulation". Hence, the resulting airfoils are at angles of attack where the front and rear stagnation points are at the trailing edge.

Figure 5.39 a) A speed distribution to serve as an example (not intended for practical use), b) the airfoil solution.
CHAPTER 6
SOLUTION TO MIXED PROBLEMS

When the complete geometry is given, the problem is referred to as "direct", an inverse problem is a situation where a complete speed distribution is specified. A mixed problem occurs in a situation where a part of the airfoil geometry is known while over the rest of the region, the speed distribution is specified. Before proceeding with the solution of mixed problems, the "unification" of the governing equation in the direct and inverse problems will be discussed.

6.1. Unification of the governing equation

The series coefficients are the targets in both direct and inverse problems. Being unknown in the direct problem are the points on the circle corresponding to the airfoil points. In the inverse problem, the velocity directions along the (yet unknown) airfoil are unknown. Both the direct and the inverse problem can be handled with the same expansion, i.e. the mapping function for the derivative. For the direct problem, we have (eqs. 1.14 and 1.15)

\[
\frac{dz}{d\zeta} = 1 + \frac{C_1}{\zeta} + \frac{C_2}{\zeta^2} + \frac{C_3}{\zeta^3} + \ldots \quad (6.1)
\]

with \( dz, \; d\zeta, \) and \( C_1 \) known and for the inverse problem, the left hand side is expressed as (eqs. 1.14 and 1.15)

\[
\frac{V_o}{V_a} e^{i\tau} = 1 + \frac{C_1}{\zeta} + \frac{C_2}{\zeta^2} + \frac{C_3}{\zeta^3} + \ldots \quad (6.2)
\]

with \( V_o/V_a, \; \zeta, \; \theta \) and \( C_1 \) prescribed and \( \tau = \pi/2 - \theta + \omega \).

It can be shown now that equations (6.1) and (6.2) can be used together so that it will be possible to solve a mixed problem. The coefficients \( C_n \) can be evaluated by residue analysis from equation (6.1),

\[
C_n = \frac{1}{2\pi i} \oint \frac{dz}{d\zeta} \zeta^{n-1} d\zeta \quad (6.3)
\]
or from equation (6.2),

$$ C_n = \frac{1}{2\pi i} \oint \frac{V_c}{V_a} e^{i\pi n-1} d\zeta $$  \hspace{1cm} (6.4) $$

The line integration of equations (6.3) and (6.4) around the cylinder in the \( \zeta \)-plane can be split into a number of surface elements,

$$ C_n = \frac{1}{2\pi i} \left[ \int_{\text{part 1}} \frac{dz}{d\zeta} \zeta^{n-1} d\zeta + \int_{\text{part 2}} \frac{dz}{d\zeta} \zeta^{n-1} d\zeta \right] $$

and

$$ C_n = \frac{1}{2\pi i} \left[ \int_{\text{part 1}} \frac{V_c}{V_a} e^{i\pi} \zeta^{n-1} d\zeta + \int_{\text{part 2}} \frac{V_c}{V_a} e^{i\pi} \zeta^{n-1} d\zeta \right] $$

The last two equations can be mixed to result in

$$ C_n = \frac{1}{2\pi i} \left[ \int_{\text{part 1}} \frac{dz}{d\zeta} \zeta^{n-1} d\zeta + \int_{\text{part 2}} \frac{V_c}{V_a} e^{i\pi} \zeta^{n-1} d\zeta \right] $$  \hspace{1cm} (6.5) $$

This equation provides a way of computing \( C_n \) for mixed paths of partly specified geometry (part 1) and partly specified speed distribution (part 2). Of course more than two parts of the contour can be handled in the same way.

6.2. Local modification of speed distribution

One of the many problems in aerodynamic design of wings is a local modification of an airfoil. The requirement for the airfoil modification can be either preserving the rest of the geometry or the rest of the pressure distribution. When both the rest of the geometry and the related pressure distribution should be maintained it has no solution other than the original one. When a new pressure distribution is specified in a certain region and the rest of the pressure distribution is to be maintained, the requirement is the determination of the corresponding airfoil contour and the problem reduces to a pure inverse problem. When a new pressure distribution is specified in a certain region and the rest of the geometry is to be maintained, the situation is more restricted than a pure inverse. Here, the restrictions are that the end points are fixed, the end tangents are fixed, and the surface length where the pressure distribution is specified cannot be less than the shortest distance between the two end points. Such a case has never been tried by
the author. A relatively less restricted situation, while retaining a very interesting application, is the modeling of a separated streamline of a stalled airfoil as will be discussed in the following section.

6.2.1. Mapping including a separated region

An airfoil with a separated region can be considered as a mixed problem. The region wherein the flow remains attached is given by the airfoil geometry and the separated flow region is represented by the speed distribution. The point where the stream line separates from the surface and the corresponding speed are the needed information. This information must be separately given (from experimental data or from a boundary layer calculation) since an inviscid flow model does not give any hints to find it. For real practical applications, the separation point and the pressure are parts of a boundary layer calculation (chapter 7). The pressure distribution in the separated region may be assumed to follow from an empirical observation, for example by stating that "the pressure within the separated region is constant and equal to the pressure at the separation point". The corresponding speed distribution can be obtained using Bernoulli’s equation.

6.2.2. Mathematical procedures

As was discussed in chapter 5, the free stream flow direction is set at zero degree with respect to the x axis. An angle of attack is provided by rotating the airfoil physically. The rear stagnation point on the circle will also be determined by solving \( f'(\zeta) = 0 \) and finding the corresponding \( \theta_{TE} \). For the first guess, \( \theta_{TE} \) is chosen equal to that from a direct solution for the given airfoil at a certain angle of attack without separation (figure 6.1). Since the lift of the stalled airfoil would be different (lower) as compared to the unstalled one, the stagnation points on the circle would move closer to the horizontal axis. Not only the stagnation points but also all the corresponding points on the circle must be reevaluated at every step as is done in the direct problem. The determination of the corresponding points on the circle in the region corresponding to the separated flow can be done in the same way as in the inverse problem. Similarly, the corresponding points on the circle in the region corresponding to the attached flow can be done in the same way as in the direct problem. The radius of the circle must be determined first before "tracking" the corresponding points. The radius of the circle may be computed in the same way as
in the direct problem by assuming that the total surface length of the (to be found equivalent) airfoil remains un-changed.

![Diagram of prescribed geometry and V(s) = const.]

Figure 6.1. Attached flow as a starting correspondence.

The series coefficients are computed using equations (6.3) and (6.4) respectively for the region of specified geometry and the region of specified pressure distribution. Having determined the series coefficients at each step, the corresponding geometry of the separated stream line can be determined. The separated stream line added to the given airfoil will update the coefficient of the logarithmic term in the series.

The pressure distribution around the equivalent airfoil finally can be computed. The given airfoil is assumed to sense the same pressure at the corresponding point on the equivalent airfoil. The force and moment are then computed on the given airfoil.

6.2.3. Illustration

Figure 6.2 shows an example for the case of airfoil LS(1)-0417(mod). The airfoil has trailing edge separation at about 8 degrees angle of attack. The separation point moves forward gradually with increasing angle of attack. Beyond
this angle of attack, the lift coefficient cannot be computed accurately without including the separation effect [32]. From the experimental results [33] it is found that at 16 degrees angle of attack the boundary layer separates at about 65% chord distance and gives a speed level $V/V_\infty$ about 1.01 on the separated streamline. Figure 6.2 shows the free streamline shape attached to the airfoil together with the speed distribution corresponding to the attached flow case and the stalled airfoil case.

Figure 6.2. LS(1)-0417(mod) airfoil at 16 degree angle of attack.
Figure 6.3 shows the results for the same airfoil at 20 degree angle of attack where the flow separates at about 40% chord and the speed level on the separated streamline is about 1.2. It should be observed that the boundary layer effect in the attached region is not taken into account. However, a good agreement between the measurement and the present prediction is found.

Figure 6.3. LS(1)-0417(mod) airfoil at 20 degree angle of attack.
Figures 6.4 and 6.5 show the case of an airfoil (NLR7702) with a spoiler. In this case, the end point of the spoiler is assumed to be the separation point. The spoiler, length 15% of the airfoil chord, is attached to the airfoil at 53% chord distance. The measurements were conducted in the low speed wind tunnel of the Faculty of Aerospace Engineering of the Delft University of Technology [37]. Figure 6.4 shows the case with the spoiler at 10 degree deflection angle and the airfoil at zero angle of attack.

Figure 6.4. NLR7702 airfoil at 0 degree angle of attack with spoiler, at 53 % chord, at 10 degrees deflection angle.
Figure 6.5 shows similar results for the case with the spoiler at 20 degrees deflection angle and the airfoil at zero angle of attack. These figures show a good agreement between the measurement and the present method. A relatively remarkable disagreement on the lower surface can be attributed to the fact that the boundary layer effect is not taken into account.

Figure 6.5. NLR7702 airfoil at 0 degree angle of attack with spoiler, at 53% chord, at 20 degree deflection angle
7.1. Introduction

Inviscid flow analysis must always be coupled with a boundary layer calculation when computing the aerodynamic characteristics of airfoils. The drag cannot be estimated without accounting for the viscous effects (which can be approximated by some boundary layer models). Using the boundary layer representation, the viscous drag can be estimated to a reasonable accuracy with the inviscid (uncorrected) pressure distribution. For thin airfoils, in the range where the boundary layer is not separated and not too thick, the lift and pitching moment coefficients, can be reasonably predicted by the inviscid pressure distribution. For thick airfoils, the pressure distribution should be corrected to estimate the lift and moment coefficients.

The effect of the boundary layer on the pressure distribution, using the mapping formulation proposed in the present thesis, may be accounted for by adding the boundary layer displacement thickness to the airfoil surface. When there is no turbulent separation, applying one time the correction for the surface displacement is enough to approximate the correct pressure distribution.

When the boundary layer is separated from the body surface the effective displacement of the surface becomes very large. Boundary layer methods usually are not designed to deal with this situation so that there is no easy procedure to compute the effective displacement. The mapping formulation proposed in this thesis is capable to handle this situation provided the boundary layer calculation could give sufficient information i.e. the separation point and the pressure (speed) at this point. As was discussed in chapter 6, the mixed mode may be used to deal with this situation.

7.2. Boundary layer
7.2.1. Displacement effect

Due to the viscous nature of the fluid and the condition of no slip to be satisfied on the surface, the velocity will vary from zero at the surface to $u_a$ at a
distance $\delta$ normal to the surface. When the flow is inviscid, the distribution will be $u_i(y)$ which need not be uniform and will slip at the surface. The mass flow through the space from $y=0$ to $y=\delta$ in a real (viscous) flow is equal to the mass flow through the space from a "displacement body surface" $y=\delta_{1,B}$ to $y=\delta$ in the inviscid incompressible flow i.e.,

$$\int_{\delta_{1,B}}^{\delta} u_i dy = \int_{0}^{\delta} u dy \tag{7.1}$$

where the subscript $i$ denotes the inviscid case. A further discussion on $\delta_{1,B}$ may be found in [4], there it is shown that $\delta_{1,B}(s)$ is a stream line of an "equivalent inviscid flow". Hence $\delta_{1,B}$ may be considered as the displacement thickness which can be added to the physical body, as the pressure distribution is to be obtained from the "equivalent" inviscid flow. When we assume that $u_i = u_e = \text{constant}$, i.e. when the inviscid flow is assumed to be uniform near the surface, equation (7.1) reduces to

$$\delta_i = \int_{0}^{\delta} \left(1 - \frac{u_i}{u_e}\right) dy \tag{7.2}$$

The displacement thickness is added (normal) to the airfoil surface to obtain the "equivalent airfoil" shape. This "new" airfoil shape will result in a "new", hopefully better, displacement thickness distribution. This process is repeated iteratively until a converged result is obtained. However, at the early stage of the iteration, only a fraction of the displacement thickness is taken into account (under relaxation). In the present work, the airfoil drag coefficient is computed only in the first iteration, while the lift and moment coefficients are obtained after the iteration is completed.

### 7.2.2. Boundary layer evaluation

Laminar boundary layer approximate methods can be derived from the Prandtl boundary layer equations,

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{dp}{dx} + \frac{1}{\rho} \frac{\partial \tau}{\partial y} \tag{7.3}$$

$$\tau = \mu \frac{\partial u}{\partial y} \tag{7.4}$$
\[
\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0
\] (7.5)

with the boundary conditions,

\[
y = 0 : \ u = v = 0, \quad y = \delta : \ u = u_e
\] (7.6)

In the turbulent boundary layer the equations for the time averaged quantities (indicated by an overbar) will appear as

\[
\bar{u} \frac{\partial \bar{u}}{\partial x} + \bar{v} \frac{\partial \bar{u}}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \frac{1}{\rho} \frac{\partial \tau}{\partial y}
\] (7.7)

\[
\tau = \mu \frac{\partial \bar{u}}{\partial y} - \rho \bar{u} \bar{v}^i
\] (8)

\[
\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0
\] (9)

with the boundary conditions,

\[
y = 0 : \ \bar{u} = \bar{v} = 0, \quad y = \delta : \ \bar{u} = u_e
\] (7.10)

These equations are derived from the Navier-Stokes equations with the assumptions (valid at high Reynolds number):

- the boundary layer thickness, \( \delta \) is very small relative to body length scale \( L \)
- \( v \ll u \) in the boundary layer
- the body surface is essentially flat on the scale of the boundary layer so that the normal pressure gradient, \( \partial p/\partial y \) may be neglected.

It may be expected that a boundary layer solution in the region where the radius of curvature of the surface is small (e.g. near the nose) and in the region where \( v \) is as large as \( u \) (e.g. in the separated region) will be less accurate.

The boundary layer equations above may be combined (by adding \( u \) times equation (7.5) to equation (7.3) and similarly for (7.9) and (7.7)) to obtain, by integration, the momentum integral equation

\[
\frac{d\delta_2}{dx} + \frac{\delta_2}{u_e}(2+H) \frac{du_e}{dx} = \frac{1}{2} c_i
\] (7.11)

where \( c_i \) is the surface friction coefficient and

\[
H = \frac{\delta_1}{\delta_2}
\] (7.12)
where the boundary layer displacement thickness $\delta_1$ is given by equation (7.2), and

$$
\delta_1 = \int_0^\delta u \left(1 - \frac{u}{u_e}\right) dy
$$

(7.13)

is the momentum loss thickness. It should be noted that equation (7.11) is valid for laminar boundary layers as well as for turbulent boundary layers.

7.2.3. Laminar boundary layer

The Thwaites' approximate method for laminar boundary layers is used in the present study. This method is based on the momentum integral equation (7.11). Unknowns in equation (7.11) are $\delta_2$, $H$ and $c_i$. To solve this equation, some relations among these unknowns must be provided. Thwaites [38] determined these relations from known numerical solutions of the boundary layer equation. Non dimensional quantities are introduced as follow

$$
Re_{\delta_2} = \frac{\rho u_e \delta_2}{\mu}
$$

(7.14)

where $C$ is the airfoil chord

$$
Re_{\delta_2} = \frac{\rho u_e \delta_2}{\mu}
$$

(7.15)

$H$ is already dimensionless. Since $c_i$ is a rather strong function of the Reynolds number, a new parameter $l$ is introduced,

$$
l = \frac{1}{2} Re_{\delta_2} c_i
$$

(7.16)

Now equation (7.11) can be rearranged to read

$$
\frac{\rho u_e \delta_2}{\mu} \frac{d\delta_2}{dx} + (2 + H) \frac{\rho \delta_2^2}{\mu} \frac{du_e}{dx} = l
$$

(7.17)

By writing

$$
\lambda_e = \frac{\rho \delta_2^2}{\mu} \frac{du_e}{dx}
$$

(7.18)

equation (7.17) reduces to
\[
\frac{\rho u_e \frac{d\delta_2^2}{dx}}{\mu} = 2 \left[ I - (2 + H) \lambda \right] 
\]

(7.19)

Following Thwaites, the right hand side of (7.19) can be approximated by a linear expression so that (7.19) can be written as

\[
\frac{\rho u_e \frac{d\delta_2^2}{dx}}{\mu} = 0.45 - 6 \lambda
\]

(7.20)

Substituting (7.18) into (7.19) leads to

\[
\frac{\rho u_e \frac{d\delta_2^2}{dx}}{\mu} = 0.45 - 6 \frac{\rho \delta_2^2}{\mu} \frac{du_e}{dx}
\]

(7.21)

Multiplying through by \(u_e^6\) results in

\[
\frac{\rho}{\mu} \frac{d}{dx} \left( \delta_2^2 u_e^6 \right) = 0.45 u_e^6
\]

(7.22)

This equation can be integrated to solve for \(\delta_2(x)\) when \(u_e(x)\) is given and the starting value of \(\delta_2\) is supplemented. The other boundary layer quantities may be found from the following relations [39]

\[
I = 0.22 + 1.57 \lambda - 1.8 \lambda^2
\]

\[
H = 2.61 - 3.75 \lambda + 5.24 \lambda^2
\]

for \(0 < \lambda < 0.1\)

(7.23)

or

\[
I = 0.22 + 1402 \lambda + \frac{0.018 \lambda}{\lambda + 0.107}
\]

\[
H = 2.088 + \frac{0.0731}{\lambda + 0.14}
\]

for \(-0.1 < \lambda < 0\)

(7.24)

Solving the momentum integral equation in the region near the stagnation point for \(\delta_2\) provides the starting value of \(\delta_2(x)\). Since near a stagnation point \(u_e(x)\) may be approximated as \(u_e(x) = x(du_e/dx)\), solving equation (7.22) for \(\delta_2\) gives [39]

\[
\delta_2(0) = \sqrt{\frac{0.075 \mu}{\rho \frac{du_e}{dx}}}
\]

(7.25)

The calculation of the laminar boundary layer is terminated when separation or transition is detected. Laminar separation is encountered when the friction
coefficient $c_t$ becomes equal to zero, or the parameter $l=0$, or the pressure gradient parameter $\lambda=-0.0898$.

### 7.2.4. Laminar separation bubble

The method due to Schmidt [40] is adopted here to analyze the laminar separation bubble. The method was developed in analogy between the separated shear layer and the free shear layer produced by the interaction of a wide, uniform jet with a region of stagnant fluid. Velocity profiles in the free shear layer are self-similar and can be determined in an analogous manner to the solution for the flow along a flat plate. It is assumed that the edge speed in the interval between the separation and the transition points is constant. The other parameters can be solved from the momentum integral equation, equation (7.11), and from the following energy integral equation,

$$
\frac{d(u^3 \delta_3)}{dx} = c_d u_e^3
$$

(7.26)

where

$$
\delta_3 = \int_0^s \frac{u}{u_e} \left( 1 - \left( \frac{u}{u_e} \right)^2 \right) dy
$$

(7.27)

is the energy thickness, and

$$
c_d = \frac{2}{\rho u_e^3} \int_0^s \tau \left( \frac{\partial u}{\partial y} \right) dy
$$

(7.28)

is the dissipation coefficient. Solving equation (7.11) for $\delta_2$ results in [40]

$$
\delta_2 = 1.241 \sqrt{\frac{v \chi}{u_e}}
$$

(7.29)

When referred to quantities at the separation point, this equation can be rearranged to yield

$$
\frac{\delta_2(x)}{\delta_{2,s}} = \sqrt{1 + (1241)^2 \left( x - x_s \right) \left( \delta_{2,s} \text{Re}_{\delta_{2,s}} \right)}
$$

(7.30)

where subscript $s$ refers to the separation point.
The calculation of the laminar part of the separation bubble is terminated at the transition point. The transition criterion within the laminar separation bubble is defined as [40]

\[ \text{Re}_{\text{lam}} = 2175 \text{Re}_{0.515}^{0.2} \]  

(7.31)

where \( \text{Re}_{\text{lam}} \) is the Reynolds number referred to the length of the laminar part of the separation bubble.

The development of the momentum loss thickness in the turbulent part of the separation bubble is determined by integrating equation (7.26) from transition to reattachment by assuming constant \( c_d \) and \( H_{32} \), where \( H_{32} = \delta_3 / \delta_2 \). The resulting formula is

\[ \delta_{2T} = \delta_{2T} \left( \frac{u_T}{u_R} \right)^3 + \left( \frac{c_d}{H_{32} \delta_m} \right) \left[ 1 + \left( \frac{u_T}{u_R} \right)^2 \right] \frac{l_{\text{tur}}}{4} \]  

(7.32)

where subscript \( T \) stands for transition, \( R \) for reattachment, \( (c_d/H_{32})_{\delta_m} = 0.0182/1.5 \) (the Horton value), and \( l_{\text{tur}} \) is the length of the turbulent part of the separation bubble. The length of this turbulent part can be found after the reattachment point is determined. The separation bubble is assumed to reattach when \( \delta_2 du_w / dx / u_w \) reaches a certain value, which in [40] is suggested to be -0.0075. However, in the present study, after examining a number of airfoils with Reynolds number above 0.7 million, this reattachment criterion was found to produce rather long reattachment region. It appeared necessary to change this number into -0.0075/12. The speed gradient \( du_w / dx \) is taken to be \( (u_{in} - u_0) / l_{\text{tur}} \). The momentum thickness at the reattachment point will serve as the starting value for the turbulent boundary layer calculation.

7.2.5. Laminar / Turbulent transition

The prediction of transition from the laminar to the turbulent boundary layers follows the "extended Granville criterion" [41]. The idea of the Granville method is to relate the distance between the calculated point of instability (critical point) and the observed point of transition to a parameter representing the boundary layer development. The distance between the critical point and the transition point depends on the rate of amplification of the unstable disturbances and on the intensity of turbulence in the free stream. The rate of amplification is influenced by
the pressure gradient. The distance between the two points may be represented in the form of the difference between the Reynolds numbers formed using the momentum thickness at the two points, $Re_{52,T} - Re_{52,cr}$, where T and cr stand respectively for transition and critical points. The mean Pohlhausen parameter

$$\bar{K} = \frac{1}{X_T - X_{cr}} \int_{X_T}^{X_{cr}} K \, dx = \frac{1}{X_T - X_{cr}} \int_{X_T}^{X_{cr}} \frac{\delta^2}{\nu} \frac{du}{dx} \, dx$$

(7.33)

where

$$K = \frac{\delta^2}{\nu} \frac{du}{dx}$$

(7.34)

the second Pohlhausen parameter, was chosen as the parameter to which the Reynolds number difference may be related, $Re_{52,T} - Re_{52,cr} = f(\bar{K})$. Such a correlation is found in [42] and shown in figure 7.1. This method can be applied to non self-similar boundary layers since K is allowed to vary with x. It may be seen in figure 7.1 that the free stream turbulence is not taken into account.

Michel et al [41] extended the method of Granville to take into account the influence of the free stream turbulence, $Tu$. The extended relation was assumed to be of the form $Re_{52,T} - Re_{52,cr} = f(\bar{K}, Tu)$. The relation was put in an analytic form by Michel as

$$Re_{52,T} - Re_{52,cr} = -206 \exp(25.7 \bar{K_T}) \log(16.8 Tu) - 2.77 \bar{K_T}$$

(7.35)

Figure 7.1. Granville's transition criterion [42]
In the present study it is found, after examining a number of airfoils, that a factor of -149 fits better than -206 to the experimental results. This criterion coincides practically with the Granville method for $0.05 \% < T_u < 0.1 \%$ as can be seen in figure 7.2 (note the reversed scale for $\bar{K}$).

![Graph showing $Re_{\infty, T} - Re_{\infty, cr}$ vs $\bar{K}$ with $T_u = 0.1 \%$ and $0.05 \%$.](image)

**Figure 7.2. Two-dimensional transition criterion [41].**

(dotted line according to figure 7.1)

The free stream turbulence ($T_u$) may be related to the factor $n$ of the so-called "e" method", obtained empirically from experimental results on a flat plate,

$$n = -8.43 - 2.4 \log(T_u)$$  \hspace{1cm} (7.36)

This relation is assumed to remain valid for flows with a pressure gradient.

The local Reynolds number at the critical point, $Re_{\infty, cr}$ can be determined from the Pohlhausen approximate method. This method provides the laminar velocity profiles which can be analyzed for stability. Stability calculations were carried out by Schlichting and Ulrich [41].

Figure 7.3 shows the critical Reynolds number based on displacement thickness (the local Reynolds number below which no disturbance amplification is possible) plotted against the first Pohlhausen parameter $\Lambda$, where

$$\Lambda = \frac{\delta^2}{v} \frac{d u_*}{d x}$$  \hspace{1cm} (7.37)

In developing the computer program, the graph in figure 7.3 was put in the following polynomial equation
\[
\log_{10}(Re_{b,r}) = a + b \Lambda + c \Lambda^2 + d \Lambda^3 + e \Lambda^4 + f \Lambda^5 + g \Lambda^6
\] (7.38)

where the polynomial coefficients are obtained from a regression.

![Graph showing the relationship between Re_{b,r} and \Lambda](image)

**Figure 7.3.** Critical Reynolds number of boundary layer velocity profiles with pressure gradient as a function of \(\Lambda\) [42].

The first Pohlhausen parameter, \(\Lambda\), is related to the second parameter \(K\) through [42]

\[
K = \left( \frac{37}{315} - \frac{\Lambda}{945} - \frac{\Lambda^2}{9072} \right)^2 \Lambda
\] (7.39)

This equation may be inverted and approximated by the following polynomial equation,

\[
\Lambda = a + bK + cK^2 + dK^3 + eK^4 + fK^5 + gK^6
\] (7.40)

From the Thwaites' approximate method, \(K\) may be readily found and then \(\Lambda\) may be computed using equation (7.40) so that the local Reynolds number at the critical point may also be computed easily from equation (7.38).

### 7.2.6. Turbulent boundary layer

The lag entrainment method due to Green et al. [43] is used in the present study. The method makes use of the momentum integral equation, equation (7.11),
and the entrainment equation. The entrainment equation may be found from \( Q \), where

\[
Q = \int_{0}^{\delta} u \, dy = u_e (\delta - \delta_1)
\]

is the volume flow across the turbulent boundary layer. The entrainment equation is represented non dimensionally in the following form

\[
C_E = \frac{1}{u_e} \frac{dQ}{dx} = \frac{1}{u_e} \frac{d}{dx} \left( u_e (\delta - \delta_1) \right)
\]

\[
= \frac{1}{u_e} \frac{d}{dx} \left( u_e \delta_2 H_1 \right)
\]

(7.42)

where \( H_1 = (\delta - \delta_1)/\delta_2 \). This equation is further rearranged to read

\[
\delta_2 \frac{dH}{dx} = \frac{dH}{dH_1} \left( C_E - H_1 \frac{d}{dx} \left( \frac{u_e \delta_2}{u_e} \right) \right)
\]

(7.43)

Substituting equation (7.11) into the above equation results in

\[
\delta_2 \frac{dH}{dx} = \frac{dH}{dH_1} \left[ C_E - H_1 \left( c_t - \frac{2(1/2) \delta_2}{u_e} \frac{du_e}{dx} \right) \right]
\]

(7.44)

Now we have additional unknowns, \( C_E \) and \( H_1 \), in addition to \( H \), \( \delta_2 \), and \( c_t \) from the momentum integral equation. Some additional equations must be devised to make the system solvable. The following lag equation was derived from the turbulent kinetic energy equation and some empirical relations,

\[
\delta_2 (H_1 + H) \frac{dC_E}{dx} = \frac{C_E (C_E + 2) + \frac{0.8 c_{10}}{3}}{C_E + 0.01} \left[ f(C_E, c_{10}) - vf(C_E, c_{10}) \right]
\]

\[+ \left( \frac{\delta_2}{u_e} \frac{du_e}{dx} \right) \left( \frac{\delta_2}{u_e} \frac{du_e}{dx} \right)_{eq}
\]

(7.45)

where \( v \) is a factor, which in the present thesis is taken as 1.0 but may be different from 1.0 to take into account the secondary effects such as the surface curvature or factors affecting the dissipation process. The subscript eq stands for the "equilibrium boundary layer". The function \( f \) in the square brackets is given as
\[ f(C_e, c_{10}) = 2.8 \left( 0.32 c_{10} + 0.024 C_e + 12 C_e^2 \right)^{\frac{1}{2}} \]  

(7.46)

and \( c_{10} \) is the skin-friction coefficient for a flat-plate with zero pressure gradient. Following Winter and Gaudet [43]

\[ c_{10} = \left( \frac{0.01013}{\log_{10}(Re_{e}) - 102} \right) - 0.00075 \]  

(7.47)

The skin friction coefficient \( c_r \) is given by the Swafford and Whitfield relation

\[ c_r = \frac{0.3 \exp(-1.33H)}{\log_{10}(Re)^{(1.74 + 0.31H)}} + 10^{-4} \left[ \tan h(4 - 1.14H) - 1 \right] \]  

(7.48)

The relation between \( H \) and \( H_1 \) follows from

\[ H_1 = 2 + 1.5 \left( \frac{112}{H - 1} \right)^{1.093} + 0.5 \left( \frac{H - 1}{112} \right)^{1.093} \]  

\( H < 4 \)  

\[ H_1 = 4 + \frac{1}{3} (H - 4) \]  

\( 4 < H < 12 \)  

(7.49)

Some formulae related to equilibrium flows are given as

\[ \left( \delta_2 \frac{d u_e}{dx} \right)_{eq} = \frac{125}{H} \left[ \frac{c_r}{2} - 0.0242 \left( \frac{H - 1}{H} \right)^2 \right] \]  

(7.50)

and

\[ C_{e \, eq} = H_1 \left[ 0.0302 \left( \frac{H + 1}{H} \right)^2 - c_r \left( \frac{0.25 + 125}{H} \right) \right] \]  

(7.51)

Now, equation (7.11), (7.42), and (7.44) to (7.51) can be solved numerically and simultaneously to calculate the development of \( \delta_2 \), \( c_r \), \( H \), and \( \delta_1 \). The starting value for \( \delta_2 \) and \( H \) are taken equal to those at the transition or reattachment point. Nevertheless, when the transition or reattachment calculation results in \( H < 1.1 \) (or \( H > 1.4 \)), the starting value for \( H \) is taken to be 1.1 (or 1.4). When \( c_r \) is equal to or less than zero is detected, the boundary layer is assumed to separate, the boundary layer calculation is terminated and the airfoil analysis is continued in the mixed mode as discussed in chapter 6.
In the wake, the formulations and procedure are still valid, with $c_f=0$. The $v$ can be maintained equal to 1.0 when the wake upper side is evaluated separately from the lower side.

### 7.2.7. Airfoil drag

At a large distance downstream from the trailing edge, the wake can be assumed to have a uniform static pressure equal to the free stream static pressure. Through a small strip with height $dy$ in the wake, the mass flow is $\rho u dy$. This mass has a velocity $u_\infty$ far upstream. Hence the momentum loss is equal to $(u_\infty - u) \rho u dy$.

After integration across the wake this gives the drag as,

$$D = \int_{\text{wake}} \rho u (u_\infty - u) dy$$

(7.52)

In coefficient form it can be written as

$$C_D = \frac{D}{\frac{1}{2} \rho u_\infty^2 c} = \frac{2}{c} \int_{\text{wake}} \frac{u}{u_\infty} \left(1 - \frac{u}{u_\infty}\right) dy$$

(7.53)

where the integral will result in the momentum loss thickness $\delta_{2,\infty}$, hence the drag coefficient can be written as

$$c_d = 2 \frac{\delta_{2,\infty}}{c}$$

(7.54)

Now we see that we need to determine the momentum loss thickness at infinity downstream in order to obtain the airfoil drag. In principle, the momentum loss thickness along the wake can be computed from the momentum integral equation as we did on the airfoil surfaces. However, here the friction coefficient along the center line of the wake is equal to zero. It should be realized however, that in practice the computation of the wake development will have to be terminated at a finite distance downstream of the trailing edge. The following sub-section will discuss this matter.

Another problem arises when the airfoil being computed has a blunt trailing edge, or when the boundary layer separates from the surface. Then a base drag, associated with the trailing edge thickness, should be taken into account. In chapter 3 we have discussed the aerodynamic force and moment of an airfoil. We have found that an airfoil with a thick trailing edge, or an equivalent airfoil (with the displacement thickness or with the separated stream line) will not give a drag force
as it treated in an inviscid flow. The base drag hence, should be associated with the local deviation of the pressure from the inviscid solution. In the present study, the base drag is approximated by

\[ c_{d,\text{base}} = C_{p,\text{TE}} \Delta_{\text{TE}} \sin(\gamma) \cos(\alpha) \]  

(7.55)

where \( \gamma \) is defined in figure 3.5, and \( \alpha \) is the angle of attack.

7.2.7.1. Squire - Young approximation to the drag

In the preceding section the drag coefficient was related to the momentum loss thickness at infinity downstream. Practically, the boundary layer and wake calculations will be terminated at a certain distance downstream of the trailing edge. A simple method in relating the momentum thickness at the trailing edge (or at a certain station downstream of the trailing edge) to that at infinity downstream was proposed by Squire and Young [44]. It uses the momentum integral equation where now the friction coefficient can be put equal to zero. Then equation (7.11) takes the following form

\[ \frac{d\delta_2}{dx} + \frac{\delta_2}{u_\infty} (2 + H) \frac{du_\infty}{dx} = 0 \]

This equation may be rearranged as follows,

\[ \frac{\delta_{2,\infty}}{\delta_2} \frac{d}{dx} \left( \frac{\delta_2}{\delta_{2,\infty}} \right) + \frac{H+2}{u_\infty / u_\infty} \frac{d}{dx} \left( \frac{u_\infty}{u_\infty} \right) = 0 \]  

(7.56)

or

\[ \frac{d}{dx} \left[ \log_e \left( \frac{\delta_2}{\delta_{2,\infty}} \right) \right] + \frac{d}{dx} \left[ (H+2) \log_e \left( \frac{u_\infty}{u_\infty} \right) \right] = \log_e \left( \frac{u_\infty}{u_\infty} \right) \frac{dH}{dx} \]

(7.57)

Integrating this equation from the trailing edge (TE) to \( x = \infty \) results in

\[ \log_e \left( \frac{\delta_{2,\infty}}{\delta_{2,\infty}} \right) + (H_{\text{TE}} + 2) \log_e \left( \frac{u_\infty}{u_\infty} \right) = \int_{H_{\text{TE}}}^{H_{\infty}} \log_e \left( \frac{u_\infty}{u_\infty} \right) \, dH \]  

(7.58)

or

\[ \delta_{2,\infty} = \delta_{2,\text{TE}} \left( \frac{u_\infty}{u_\infty} \right)^{H_{\text{TE}}+2} \exp \left[ \int_{H_{\infty}}^{H_{\text{TE}}} \log_e \left( \frac{u_\infty}{u_\infty} \right) \, dH \right] \]  

(7.59)
In this result $H_\infty$ has been taken equal to 1 since at infinity $\delta_1$ and $\delta_2$ become equal due to the small defect $u_a - u$. Squire and Young observed that experiments suggested a relation between $u_a$ and $H$ as follows,

$$\log_e \left( \frac{u_a}{u_\infty} \right) = \text{constant} \cdot (H - 1) \quad (7.60)$$

or

$$\log_e \left( \frac{u_a}{u_\infty} \right) \left( \frac{u_a}{u_{a,TE}} \right) = \frac{H - 1}{H_{TE} - 1} \quad (7.61)$$

Substituting this equation into equation (7.59) results in

$$\delta_{2,\infty} = \delta_{2,TE} \left( \frac{u_{a,TE}}{u_\infty} \right)^{H_{TE}+2} \exp \left[ \int_1^{H_{TE}} \log_e \left( \frac{u_a}{u_{a,TE}} \right) \left( \frac{H - 1}{H_{TE} - 1} \right) \, dH \right]$$

$$= \delta_{2,TE} \left( \frac{u_{a,TE}}{u_\infty} \right)^{H_{TE}+2} \left( \frac{u_a}{u_{a,TE}} \right)^{H_{TE}+1/2}$$

or

$$\delta_{2,\infty} = \delta_{2,TE} \left( \frac{u_{a,TE}}{u_\infty} \right)^{(H_{TE}+5)/2} \quad (7.62)$$

Finally the airfoil drag (associated with the momentum loss) may be found as

$$c_d = \frac{2}{C} \delta_{2,TE} \left( \frac{u_{a,TE}}{u_\infty} \right)^{(H_{TE}+5)/2} \quad (7.63)$$

It should be noted that the (associated) inviscid flow will produce, in general, zero velocity at the trailing edge except for airfoils where the trailing edge upper surface is parallel to the lower surface. For equation (7.63) to be useful, the velocity at the trailing edge may be obtained by extrapolation from a region upstream of it where the value of $u_a$ is still changing slowly. This is the velocity with which equation (7.55) will be evaluated also. The total drag hence is the sum of the contribution from equations (7.55) and (7.63),

$$C_D = c_d + C_{d,base} \quad (7.64)$$
7.3. Airfoil lift and moment

Since the equivalent airfoil is treated as in an inviscid flow, the force and moment will be taken from the results obtained in chapter 3. For convenient reading, the necessary formulae will be rewritten here,

\[ c_{\text{p}} = e^{-i\theta} \left[ 2\Gamma - 2\Delta_{TE} \cos(\gamma + \alpha) \right] \]  \hspace{1cm} (7.65)

\[ \Gamma = 4\pi R \omega \sin(\alpha - \theta_{\text{TE}}) \]  \hspace{1cm} (7.66)

where \( \Delta_{TE} \) is the trailing edge thickness, \( c \) the airfoil chord, \( R \) the circle radius, and \( \theta_{\text{TE}} \) the rear stagnation point on the circle. Since the force associated with the viscous friction acts essentially along the airfoil chord, the airfoil lift hence is obtained from

\[ c_{\text{l}} = c_{\text{f}} - c_{\text{D}} \sin(\alpha) \]  \hspace{1cm} (7.67)

The moment coefficient is given by (eq. (3.26) of chapter 3),

\[ c_{m_{\text{c}}} = \text{Re} \left[ \frac{1}{2} \left( z\bar{z} - z_0 \bar{z} \right) \right]_{\Delta_{TE}} \]  \hspace{1cm} (7.68)

\[ \text{Im} \left[ -4\pi C_2 e^{-i2\alpha} + i \frac{\Delta_{TE}}{2\pi} e^{-i(\alpha + \gamma)} \left\{ i\Delta_{TE} e^{-i(\alpha + \gamma)} + i2\Gamma \right\} \right] \]

7.4. Practical examples

This section presents computational results for the boundary layer development and its effects on the pressure distribution and the forces and moment. Airfoil LS(1)-0417(mod) will be used as an example. Figure 7.4 shows the airfoil geometry and its pressure distributions at some angles of attack. Figure 7.5 shows the development of the displacement thickness and the momentum loss thickness at zero angle of attack. Figure 7.6 shows the pressure distributions at zero degree angle of attack with and without adding the boundary layer displacement thickness to the airfoil. Figure 7.7 shows the inviscid pressure distribution at 14 degrees angle of attack and the pressure distribution computed on the effective airfoil (airfoil with the displacement thickness and the separated streamline added to it). Figure 7.8 shows the lift, moment, and drag coefficients computed using the present method. These results are compared with the experimental results in chapter 8.
Figure 7.4. LS(1)-0417 (mod) airfoil and the inviscid pressure distributions.

Figure 7.5. The displacement thickness and the momentum loss thickness development of the LS(1)-0417 (mod) airfoil at zero angle of attack at 2 million Reynolds number.
Figure 7.6. LS(1)-0417 (mod) airfoil, the inviscid and viscous pressure distributions at zero angle of attack, Re=2x10^6.

Figure 7.7. LS(1)-0417 (mod) airfoil, the inviscid and viscous pressure distributions at 14 degrees angle of attack, Re=2x10^6.
Figure 7.8. Lift, moment, and drag coefficients of the LS(1)-0417 (mod) airfoil, at 2 million Reynolds number.
CHAPTER 8
COMPARISON WITH EXPERIMENTAL RESULTS

Experimental data are presented in this chapter to check the quality of the computational method discussed in the previous chapters. Only pressure distributions and the coefficients of the lift, moment, and drag will be compared. It would be better if the displacement thickness and the separated stream line data were available, such that the "effective" airfoil can also be compared. However, these are not readily available and hence the pressure distributions are used as an indication of the effective airfoil shape.

8.1. The models and the windtunnels

Most of the experimental data presented in this chapter were obtained at the Low Speed Wind Tunnel of IPTN (NLST). The airfoils measured at the NLST are:

1. LS(1)-0417 (mod) This is an airfoil with a thick trailing edge of about 6% chord. The airfoil thickness is 17% chord.
2. LS(1)-0417 (mod1) This airfoil has thickness and trailing edge thickness similar to the LS(1)-0417 (mod) airfoil.
3. NACA 653-218 (mod) This is a laminar airfoil with thickness of 18% chord, it has a thin - sharp trailing edge.

In addition, measured data of the following airfoils are available from measurements in the Low Speed Tunnel (LST) of The Delft University of Technology:

4. NACA 64A-010 This is a symmetrical airfoil with thickness of 10% chord, it has a thin - sharp trailing edge.
5. DU-W21 This airfoil, designed for windmills, has a maximum thickness of about 21% and a thick trailing edge.

The NLST has a rectangular test section with dimensions of 1474 mm wide and 1100 mm high. The maximum air speed in the test section is about 80 m/s. Typically, airfoils measured in this wind tunnel have chords of about 400 mm, hence the Reynolds number is about 1.1 million. The LST-TU Delft has an octagonal test section where the distance between the floor and the ceiling is 1250 mm and the
distance between the side walls is 1800 mm. This wind tunnel has a very low turbulence level, of about 0.06%. 2-dimensional measurements in this wind tunnel are typically done at Reynolds numbers from about 0.7 up to 2.5 million.

Measurements on airfoils in the NLST, as well as in the LST-TU Delft, are done by measuring the static pressure on the airfoil surface and the total and static pressures in the wake. Data presented in this section were measured using well accepted standard equipment and procedures. From the wake measurement, the airfoil drag has been estimated using the Jones method. The lift was computed from the measured static pressures on the airfoil surface and from the estimated drag using the formulae as presented in chapter 7. Corrections due to the presence of the walls, i.e. the body blockage, wake blockage, and the camber effect are estimated using a so called image method as described by Allen & Vincenti [45].

8.2. LS(1)-0417 (mod) airfoil

The agreement between the computed and the measured pressure distributions is influenced among others by the accuracy of the computation of the boundary layer displacement thickness and the separated stream line (whenever separation occurs). This accuracy is determined by the boundary layer methods including the predictions of the transition location and the separation point. Since the prediction of the boundary layer development is not at the center of the study presented in this thesis, we select the best and relevant methods and procedures, known to the author, and accept the results. The computer program is arranged such that if better methods and procedures in predicting the boundary layer development become available, the subroutine containing this prediction can be easily replaced. Nevertheless, the method and procedures selected, result in a reasonable agreement with the experimental data as presented below.

Figures 1 and 2 show respectively the pressure distributions of the LS(1)-0417 (mod) airfoil at zero and 8 degrees angle of attack. Both pressure distributions are within the angle of attack range where there is no separation. It can be seen in the figures that the separation bubble location can also be predicted reasonably well.

One should be aware that the pressure distribution of an airfoil is very sensitive to geometry changes in the trailing edge region. As the displacement thickness grows strongly near the trailing edge this may result in a significant influence on the pressure distribution since the effective airfoil is obtained by adding the displacement thickness to the airfoil surface.
Figure 8.1. Pressure distribution of the LS(1)-0417(mod) airfoil at zero angle of attack at 1.1 million Reynolds number.

Figure 8.2. Pressure distribution of the LS(1)-0417(mod) airfoil at 8 degrees angle of attack at 1.1 million Reynolds number.
Figure 8.3. Pressure distribution of the LS(1)-0417(mod) airfoil at 12 degrees angle of attack at 1.1 million Reynolds number.

Figure 8.4. Pressure distribution of the LS(1)-0417(mod) airfoil at 16 degrees angle of attack at 1.1 million Reynolds number.
It is thought that it might be better to replace the last part of the displacement thickness distribution (obtained from the boundary layer computation) with a distribution obtained by extrapolation from the region upstream of the trailing edge. However, this idea has not been further pursued in this thesis and the author leaves it for a future project which will need also a more detailed observation of the boundary layer development near the trailing edge.

Figures 8.3 and 8.4 show the pressure distributions at angles of attack 12 of and 14 degrees respectively, where substantial separated regions are present. It can be observed that the computed results shown in these figures are less accurate than the examples shown in figures 2 and 3 of chapter 6. The separation points and the speed in the separated region in the examples given in chapter 6 were specified to be equal to that observed in the experiment. In the examples shown in figures 8.3 and 8.4, the computed separation point and the speed in the separated region do not converge to the experimental value. This fact indicates that the interactive computation of the inviscid and the viscous cases still needs to be refined.

\[ \text{Figure 8.5a. Lift and moment coefficients against angle of attack of the LS(1)-0417 (mod) airfoil at 1.1 million Reynolds number} \]
Figure 8.5b Drag against lift coefficients of the LS(1)-0417(mod) airfoil at 1.1 million Reynolds number.

Figure 8.5 shows the comparison between the computed and measured \( c_l \), \( c_m \) (at the quarter chord point), and \( c_d \). The characteristics of this turbulent airfoil are predicted satisfactorily in the range where the boundary layer remains attached. However, the interactive procedure still fails to handle the forward movement of the boundary layer separation.

8.3. NACA 64A-010 airfoil

Figures 8.6 shows lift, moment, and drag coefficients, in the range where the boundary layers remain attached, for the NACA 64A-010 airfoil at Reynolds number 2 million. At about 2.5 degrees angle of attack the transition location on the upper surface moves very rapidly forward close to the nose, while on the lower surface, the transition location moves a little bit downstream. The upper surface is effectively occupied by a turbulent boundary layer which grows more rapidly than the laminar boundary layer on the lower surface.
Figure 8.6a. Lift and moment coefficients against angle of attack of the NACA 64A-010 airfoil at 2 million Reynolds number

Figure 8.6b Drag against lift coefficient of the NACA 64A-010 airfoil at 2 million Reynolds number
This situation leads to a significant difference in boundary layer displacement thickness, where on the upper surface it is much thicker than on the lower surface, hence creating effectively a negative camber. Furthermore, the rapid increase of the boundary layer displacement thickness in the trailing edge region leads to a more pronounced decambering effect for a thin airfoil as compare to a thick airfoil. The boundary layer methods and the interactive procedures presented in this thesis fail to predict the effective shape of this thin airfoil through the $\alpha$-range where the transition location moves rapidly. Beyond four degrees angle of attack, experimentally, the transition point occurs in a small laminar separation bubble near the leading edge. Downstream of a laminar separation bubble the turbulent boundary layer becomes thicker, than if there were no laminar separation bubble. The present method still underpredicts the growth of the boundary layer thickness through the bubble, which then leads to an overprediction of the lift coefficient.

8.4. DU-W21 airfoil

A wedge was attached to the lower surface of the trailing edge of the DU-W21 airfoil. The wedge was meant to increase the maximum lift coefficient while keeping the lift to drag ratio low. Figure 8.7 shows the prediction of the effect of the wedge on the pressure distribution at zero angle of attack. It can be seen that the wedge, located at the last 5% chord with a thickness of less than 1.5% chord, significantly increases the camber. At a given angle of attack, the wedge increases the circulation so that it reduces the pressure over the upper surface and increases the pressure over the lower surface. This theoretical prediction is confirmed experimentally as can be seen in figure 8.8. Figure 8.8 shows the comparison between the computed and the measured pressure distributions. Physically, the wedge can be less effective because it is partly immersed in the boundary layer. Nevertheless, the deviation of the theoretical prediction of the pressure distribution from the experimental results is, to a large extent, caused by the inaccuracy of the boundary layer calculation and the interactive procedure. Note that two pressure orifices were covered due to the application of the wedge on this model.

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1 This airfoil was tested recently in the Low Turbulence Tunnel of Delft University of Technology. The author is indebted to N.Timmer and R. van Rooy, from the Institute for Wind Energy, for providing this data.
Figure 8.7. Theoretical prediction of the effect of a wedge on the pressure distribution of the DU-W21 airfoil

Figure 8.9 shows the computed and measured lift, moment, and drag coefficients of the airfoil. The computed drag is unsatisfactorily below the measured value (figure 8.9b). The inaccuracy of the computed base drag contributes strongly to the inaccurate total drag. Base drag is present because the pressure at the trailing edge base surface deviates from the inviscid value. The author leaves this problem for future research, which will need detailed flow measurements in the trailing edge region to improve the prediction quality of the base drag.
Figure 8.8. Pressure distributions at Re = 2 million, with and without wedge.
Figure 8.9. Lift, moment, and drag coefficients at RE = 2 million, with and without wedge.
8.5. LS(1)-0417(mod1) airfoil

A number of wedges of various sizes were developed for this airfoil and tested in the windtunnel. The investigation is meant to improve the airfoil performance, i.e. by increasing the lift coefficient with a minimum increase of the drag coefficient, and with a minimum modification to the geometry. The testing was carried out in the NLST windtunnel. Figure 8.10 shows the dimensions of the wedges relative to the airfoil ($t_w = 0.7\%$ chord). Figure 8.11 shows the inviscid pressure distributions with the various wedges.

<table>
<thead>
<tr>
<th>wedge nr.</th>
<th>h/c</th>
<th>l/c</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>2</td>
<td>0.5</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>0.5</td>
<td>1.5</td>
</tr>
<tr>
<td>4</td>
<td>0.5</td>
<td>2</td>
</tr>
<tr>
<td>5</td>
<td>0.25</td>
<td>0.5</td>
</tr>
</tbody>
</table>

Figure 8.10. The wedge dimensions relative to the airfoil.

Figure 8.11. Inviscid pressure distributions at zero angle of attack.
Figure 8.12 shows the comparison between the computed (with the boundary layer effects included) and the measured pressure distribution at 1.1 million Reynolds number for the airfoil with wedge number 3. It follows that, the experimental results show a more pronounced effect of the wedge than the theoretical prediction. The theoretical prediction provides only indications for selecting the wedge that results in a good airfoil improvement.

![Graph showing pressure distribution](image)

**Figure 8.12.** The effect of wedge number 3 on the pressure distribution at Re=1.1 million.

Figure 8.13 shows the global effects of the wedges on the lift, moment, and drag coefficients. A significant increase in lift is obtained, and it can be seen that the smaller the length of the wedge the larger the increase in lift. Nevertheless the moment becomes more negative and the drag increases also as can be seen in the figure. The increase in moment cannot be avoided as the wedge increases the effective camber of the airfoil. The larger the base thickness, the larger the base drag. However, the effectiveness of the trailing edge shape will depend on the boundary layer thickness which further depends on the Reynolds number. The flow pattern downstream of the trailing edge base will determine also the base drag.
Figure 8.13. The effect of the wedges on the lift, moment, and drag coefficients at $RE = 1.1$ million.

Figure 8.14 shows the curve for the lift to drag ratio. It can be seen that the wedges do improve the climb performance of the airfoil (at $c_r$ about 1.0). It can be identified that wedge number 3 provides the best performance improvement.
8.6. NACA 65\,218 (mod) airfoil

A research program was developed with the aim to improve the climb performance of an existing wing model with a minimum geometry modification. A number of wedges of various sizes were theoretically developed for this airfoil and tested in the NLST wind tunnel as was done for the previous airfoil. The two-dimensional testing was conducted to select a proper wedge size for further testing of a three-dimensional wing model in the ILST wind tunnel.

Figure 8.15 shows the dimensions of the wedges relative to the airfoil and the pressure distributions of the airfoil with the various wedges without taking into account the viscous effects. Figure 8.16 shows the comparison between the computed (with the boundary layer effects included) and the measured pressure distribution at 1.1 million Reynolds number for the airfoil with wedge number 2. The results shown in figures 8.15 and 8.16 also show that the wedge effects are more evident in the experiment than in the theoretical prediction with the present method. This is caused by the less accurate interactive procedures, i.e. in taking the boundary layer thickness into account. Figure 8.17 shows the global effect of wedge number 2 on the lift, moment, and drag coefficients. Figure 8.18 shows the curve for lift to drag ratio. Figure 8.19 is presented here, although less relevant for the present discussion, to show that the wedge can be very effective on the three-dimensional model tested in the wind tunnel.
Figure 8.15. The wedge dimensions relative to the airfoil and the inviscid pressure distributions at zero angle of attack.

Figure 8.16. The effect of wedge number 2 to the pressure distribution measured at Re=1.1 million.
Figure 8.17. The effects of the wedge number 2 on the lift, moment, and drag coefficients measured at \( RE = 1.1 \) million,
Figure 8.18. The effects of wedge number 2 on the lift to drag ratio measured at RE = 1.1 million.

Figure 8.19. The effects of wedge number 3 on the lift to drag ratio of a 3D-wing model measured at RE = 1.1 million.
CHAPTER 9
CONCLUSIONS

9.1. General

The present research contributes to the study of airfoil problems in:
- Presenting an alternative solution method, in conformal mapping, using a Laurent series with a logarithmic term. An airfoil is finally represented by the series.
- Enhancing the conformal mapping method is able to deal with airfoils with physically thick trailing edges or airfoils thickened by boundary layer displacement thickness, and airfoils with boundary layer separation.
- Introducing the general compatibility conditions relating the specified speed distribution in the airfoil plane to the corresponding speed and the corresponding points on the circle. The general compatibility conditions encompass the Lighthill (or Timman) closing conditions.
- Introducing the trailing edge gap as a parameter to select a particular solution in inverse problems. The trailing edge thickness is not an output that depends on the specified speed distribution but becomes part of the design specification. Whether or not the output airfoil is realistic can only be seen after the process is completed.
- Allowing to specify a sufficiently arbitrary speed distribution along a surface (of the yet unknown airfoil). The aerodynamic characteristics then can be estimated before knowing the airfoil shape.

9.2. Physical and mathematical modeling

The rear stagnation stream line in the circle plane is considered as two coinciding stream lines. One should not cross this stream line in going from the upper surface to the lower surface. The upper surface of the stream line, in the circle plane, has $2\pi$ angular distance from the lower surface. This modeling allows to transform an airfoil with non zero trailing edge gap or airfoils thickened by the boundary layer displacement thickness, and airfoils with boundary layer separation
onto a circle. The stream line in the circle plane can be transformed onto two
stream lines trailing from the upper and lower corners of the airfoil trailing edge.
This is achieved by having a logarithmic term in a Laurent series as the mapping
function.

The coefficients of the Laurent series can be determined by residue analysis
in the circle plane when the relation between the circle- and the airfoil-plane is
provided. In the direct problems, the angular positions of the points on the circle are
initially assumed. Then, in the intermediate steps, they are determined from the
series. The coefficient of the logarithmic term is determined from the trailing edge
gap. In the inverse problems, the corresponding points on the circle are determined
by using the compatibility relations between the airfoil- and the circle-planes. The
direction of the velocity vector in the airfoil plane (the surface tangents of the airfoil)
can be obtained from the series. The coefficient of the logarithmic term is in general
specified.

The Kutta condition can be applied without modification, that is the rear
stagnation point on the circle is made to correspond to the trailing edge upper and
lower corners.

9.3. The freedom in inverse problems

A sufficiently arbitrary speed distribution specified in the airfoil plane can be
made to correspond to the speed distribution on a circle. The specified speed
distribution must not have more than two stagnation points, however both points
may coincide. Speed distributions with strong jumps are not excluded, however
when these distributions will be approximated by the Laurent series, this may result
in wiggles close to the jump. The correspondence between the airfoil- and the circle
planes is achieved by using the compatibility conditions, that is the elemental line
integrals of the specified speed distributions are set equal to that in the circle plane.
The total line integrals of the specified speed distribution will determine the
circulation (in both planes) and finally will determine the aerodynamic angle of
attack of the airfoil. The resulting computer program has been shown to be a
valuable design tool. However, more research is needed to obtain a more detailed
insight in the possible constraint on the pressure distributions to be specified.
9.4. The viscous-inviscid interactive procedure

The inaccuracy of the prediction of the growth of the boundary layer thickness contributes significantly to the inaccuracy of the pressure distribution prediction. As shown in the examples, when the separation point and the pressure within the separated flow region obtained from the experiment are given, the present method can reasonably duplicate the experimental pressure distributions. It should be understood that in the case of an airfoil with turbulent boundary layer separation, the displacement effect of the attached part of the boundary layer is relatively small. Nevertheless, when the boundary layers remain attached up to the trailing edge, particularly for thin airfoils, the boundary layer displacement thickness growth shows a more pronounced effect on the pressure distribution. In such a case, the accuracy of the transition prediction also becomes important.

With regard to the drag prediction for thick trailing edge airfoils, the accuracy of the base drag prediction is essential. To improve the base drag prediction more detailed measurements of the flow in the region upstream and downstream of the trailing edge are needed. From such measurements, hopefully, the pressure recovery at the trailing edge can be modeled properly.

9.5. The experimental results

With regard to the trailing edge thickness it should be observed that a thick trailing edge is not only important for manufacturing / construction matters, it also shows significant aerodynamic effects. Thick divergent trailing edges may have a large effect on the circulation. This emphasizes the importance of the trailing edge gap to provide more freedom in specifying a speed distribution in inverse problems.

9.6. Computer code

The procedures outlined in chapters 5 and 6 are coded into a FORTRAN program with two modes, the inverse mode and the analysis mode. In a 486 PC, the program takes a few seconds for one angle of attack in the analysis mode, while the inverse mode takes much less time than the analysis mode. The executable files amount to only about 400 kb, run under DOS. The program can be made available to the reader on request.
9.7. Comments on existing methods

The author believes that the existing methods in 2-dimensional inverse problems (not restricted to conformal mapping, also not restricted to incompressible flow) can be "renovated" to extend their freedom in prescribing speed distributions by including some ideas, presented in this thesis. Methods incorporating mapping in the procedure, e.g. to provide a computational grid, can be improved by adopting the mapping function proposed in this thesis, to be able to deal with a non zero trailing edge thickness as an additional parameter in the design. In panel methods, the trailing edge gap can be dealt with in inverse problems.
A.1. Determination of moment

Consider figure 1 below,

![Diagram of airfoil surface with forces and moments](image)

Figure 1. Elemental force on the airfoil surface

The moment about \( z_0 \) due to the force \( d\mathbf{F} = d\mathbf{X} + id\mathbf{Y} \) at point \( z \) may be expressed as

\[
M_{z_0} = -\text{Im}(d\mathbf{F}) \text{ Re}(z-z_0) + \text{Re}(d\mathbf{F}) \text{ Im}(z-z_0)
\]  

(1)

where the sign of the moment is taken positive in the direction of increasing polar angle (counter clockwise). On the other hand we have

\[
(z-z_0) d\mathbf{F} = [\text{Re}(z-z_0) + i\text{Im}(z-z_0)] [\text{Re}(d\mathbf{F}) - i\text{Im}(d\mathbf{F})]
\]

\[
= [\text{Re}(z-z_0) \text{ Re}(d\mathbf{F}) + \text{Im}(z-z_0) \text{ Im}(d\mathbf{F})]
\]

\[
+ i [-\text{Im}(d\mathbf{F}) \text{ Re}(z-z_0) + \text{Re}(d\mathbf{F}) \text{ Im}(z-z_0)]
\]  

(2)

Comparing equations 1 and 2 we have

\[
M_{z_0} = \text{Im}[(z-z_0) d\mathbf{F}]
\]  

(3)

Using \( d\mathbf{F} = i \rho \, dz \), we obtain

\[
dM_{z_0} = \text{Im}[(z-z_0) i \rho \, d\bar{z}] = \text{Re}[(z-z_0) \rho \, d\bar{z}]
\]  

(4)
A.2. The identity \( w_a^2 \, d\bar{z} = |w_a|^2 \, dz \)

Consider figure 2 below,

![Airfoil surface diagram](image)

**Figure 2. Complex velocity \( w \) on the airfoil surface**

We readily obtain

\[
dz = |dz| \, e^{i\alpha} \quad \text{and} \quad d\bar{z} = |dz| \, e^{-i\alpha}
\]

and it follows that

\[
w_a = -(-w_a) = -(|w_a| \, e^{i\alpha}) = -|w_a| \, e^{i\alpha}
\]

\[
w_a^2 \, d\bar{z} = |w_a|^2 \, |dz| \, e^{i\alpha} = |w_a|^2 \, dz
\]

It must be realized that this identity is valid only when the vectors \( w \) and \( dz \) are in the same line as they are on the contour.

A.3. Evaluation of \( \frac{w_c^2}{f'} \)

Using equations (3.18) and (3.19) of chapter 3, the following relations may be derived,

\[
w_c^2 = e^{-i2\alpha} + \frac{i2\Gamma e^{-i\alpha}}{2\pi \zeta} + \left( \frac{\Gamma}{2\pi} \right)^2 \frac{1}{\zeta^2} \left( \frac{i2\Gamma R^2 e^{i\alpha}}{2\pi \zeta^3} + \frac{R^4 e^{2i\alpha}}{\zeta^4} \right)
\]

\[
f' = 1 + \frac{b_0}{2\pi \zeta} + \left[ -C_2 + \left( \frac{\zeta}{2\pi} \right)^2 \right] \frac{1}{\zeta^2} + ...
\]
\[
\frac{-2}{w_c^2} \begin{align*}
\frac{1}{f} & = e^{-i2\alpha} + \left[ b e^{-i2\alpha} + 2i r e^{-i\alpha} \right] \frac{1}{2\pi \zeta} + \frac{d_2}{\zeta^2} + \frac{d_3}{\zeta^3} + \ldots
\end{align*}
\]  
\tag{6}

where R is the radius of the circle, b is given by equation 3 of chapter 3, and

\[
d_2 = \left[ \frac{(2\pi)^2 (C_2 e^{-i2\alpha} - 2 R^2)}{r^2} \right] + \left( b e^{-i\alpha} \right)^2 + i2\Gamma \left( b e^{-i\alpha} \right) \right] \frac{1}{(2\pi)^2}
\]  
\tag{7}

A.4. Evaluation of \( \int \frac{-2}{w_a^2} (z - z_0) dz \)

The integrand \( \frac{-2}{w_a^2} (z - z_0) dz \) may be written as

\[
\frac{-2}{w_a^2} (z - z_0) dz = \frac{-2}{f^2} f' f \frac{d\zeta}{d\zeta} = \frac{-2}{f'^2} f d\zeta
\]  
\tag{8}

where \( z - z_0 = f \). Using equation (6) and the mapping function

\[
f = \zeta - \left( \frac{b}{2\pi} \right) \log(\zeta) - \frac{C_2}{\zeta} - \frac{1}{2} \frac{C_3}{\zeta^2} - \ldots
\]

one may write,

\[
\frac{-2}{w_c^2} \frac{f'}{f''} = \frac{-C_2 e^{-i2\alpha} + d_2}{\zeta} + \frac{E_1}{\zeta^2} + \frac{E_2}{\zeta^3} + \ldots + E_0 + \zeta e^{-i2\alpha}
\]

Upon integration around the circle, only the first term contributes to the integration, which will result in

\[
\int \frac{-2}{w_c^2} f' f \frac{d\zeta}{d\zeta} = 2\pi i \left( -C_2 e^{-i2\alpha} + d_2 \right)
\]  
\tag{9}

where \( d_2 \) is given by equation 7. Finally we obtain,

\[
\int \frac{-2}{w_c^2} f' f \frac{d\zeta}{d\zeta} = \left[ 4\pi \left( C_2 e^{i2\alpha} - R^2 \right) \frac{1}{\zeta} + \frac{b e^{-i\alpha}}{2\pi} \left( b e^{-i\alpha} + i2\Gamma \right) \right]
\]  
\tag{10}
### A.5. Coefficients of the Laurent series discussed in section 5.3.3

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<th>(Real)</th>
<th>(Imaginary)</th>
<th>Description</th>
</tr>
</thead>
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<td>( C_1 ) = -0.0051906940</td>
<td>0.0000000000</td>
<td>Coefficients obtained by residue analysis from the LS(1)-0417 airfoil, with a trailing edge thickness 0.7% chord.</td>
</tr>
<tr>
<td>( C_2 ) = 0.7344007000</td>
<td>-0.0098270350</td>
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<tr>
<td>( C_3 ) = 0.0781057200</td>
<td>-0.0200175700</td>
<td></td>
</tr>
<tr>
<td>( C_4 ) = 0.0173965200</td>
<td>-0.0015042820</td>
<td>These coefficients are used to generate a pressure distribution discussed in section 5.3.3</td>
</tr>
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<td>( C_5 ) = 0.0083699410</td>
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<tr>
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<td>( C_{11} ) = -0.0001491269</td>
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EEN UITGEBREIDE CONFORME AFBEELDINGSMETHODE
VOOR HET ONTWERP EN DE ANALYSE
VAN VLEUGEL PROFIELEN

SAMENVATTING

Dit proefschrift beschrijft een nieuwe methode van conforme transformatie voor de oplossing van het directe en inverse probleem van de onsamendrukbare stroming om twee-dimensionale vleugelprofielen. Door een logarithmische term toe te voegen aan de klassieke transformatie formules kan een willekeurige eindige staardikte worden verkregen.

Voor het inverse probleem wordt een algemene compatibiliteitsvoorwaarde afgeleid, waarvan de bekende Lighthill sluitingsvoorwaarde voor staardikte nul een bijzonder geval is. In de gepresenteerde methode kan een willekeurige staardikte worden voorgeschreven. De methode biedt een ontwerpprocedure met een grotere vrijheid bij het kiezen van de gewenste drukverdeling dan bij de gebruikelijke methoden. De voorgeschreven drukverdeling behoeft, bij de praktische uitvoering, niet te worden aangepast bij de keuze van de staardikte. Of de resulterende geometrie realistisch is of niet, dient te worden gecontroleerd nadat de oplossing is verkregen.

Het is ook mogelijk gemengde problemen te behandelen, waar een deel van de oplossing geometrisch wordt voorgeschreven en een ander deel wordt bepaald door eisen gesteld aan de stroming. Als voorbeeld kan worden genoemd een profiel met een groot gebied met losgelaten stroming.

De methode van Thwaites and Green's "lag entrainment" methode worden gebruikt voor het in rekening brengen van de effecten van de laminaire respectievelijk turbulente grenslaag. Als omslag kriterium wordt de "nieuwe Michel" of "uitgebreide Granville" methode gebruikt. Laminaire losluchtblasen worden beschreven met de Gordon-Schmidt methode.

Voorbeelden van directe, inverse en gemengde problemen worden gegeven. Ook experimentele resultaten worden gepresenteerd om de bruikbaarheid van de methode vast te stellen en het effect van het verdikken van de staart te illustreren.
About the author

Born in Deles 25 km from Klaten, Indonesia on July 1, 1958.

1971 - 1973 : Staying in the town Boyolali, 80 km from his parents, for studying at the Junior High School. Diploma obtained with honors.

1974 - 1976 : Studying at the Senior High School in Boyolali, Diploma obtained with honors.

1977 - 1982 : Studying at the Institut Teknologi Bandung (ITB) in the sub Department of Aeronautical Engineering under the Department of Mechanical Engineering.

1983 - 1984 : Performing his Msc. thesis at the Delft University of Technology under the guidance of Prof. Van Ingen. An additional 3 months were spent in the Netherlands to do practical work at the Low Speed Laboratory of the NLR at the North East Polder, the Netherlands.

1984, March : Graduated at ITB with honors.

1984 - 1988 : Staff member at ITB at the sub Department of Aeronautical Engineering. In the mean time he worked also on a part time basis at the Indonesian Aircraft Industries (PT IPTN).

1989 - 1993 : Performing Ph.D. research at Delft University of Technology in the Low Speed Laboratory. The results of the research are described in this thesis.

1993 - 1995 : Continuing his service at ITB and at PT IPTN.

1995 - : Leads a contract research granted by the government of the Republic of Indonesia, for a period of 3 years, on “Wing Aerodynamics Technology”, with participation of ITB, IPTN, and LAGG.

1995 - : Coordinator for Wing Technology of the N2130 Technology Program at PT IPTN.