Modeling interactions between turbulence induced mechanical vibrations and subsurface flow of water

A thesis submitted to the Delft Institute of Applied Mathematics in partial fulfillment of the requirements for the degree

MASTER OF SCIENCE in APPLIED MATHEMATICS

by

M. RAHRAH

Delft, The Netherlands October, 2012

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MSc thesis APPLIED MATHEMATICS

Modeling interactions between turbulence induced mechanical vibrations and subsurface flow of water

M. RAHRAH

Delft University of Technology

Daily supervisor
Dr.ir. F.J. Vermolen

Responsible professor
Prof.dr.ir. C. Vuik

Other thesis committee members
Ir. H.G.J.W.A. Wolfs

Dr. J.L.A. Dubbeldam

October, 2012

Delft, The Netherlands
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Preface

This master thesis has been written as the final part of my master programme Applied Mathematics at Delft University of Technology. It contains the details and results of a ten month research.

The MSc-project isadduced by Foundation O2DIT (Research and Development Sustainable Infiltration Techniques) and is carried out at Fugro GeoServices, which is one of the largest engineering consulting companies in the Netherlands and leader with services in the field of Geo-Research, Geo-Information and Geo-Consultancy.

I would firstly thank my daily supervisor at Delft University of Technology, Fred Vermolen, for his support, patience and trust.
I am also very grateful to my supervisors at Fugro GeoServices BV, Herman Wolfs and Barbara Snacken, for their help and supervision.
Furthermore, I would like to thank Kees Vuik and Johan Dubbeldam, who are part of my thesis committee.
Last but not least, lots of thanks are due to my family and my friends and fellow students, for their support and for the great time at Delft University of Technology.

Delft, October 2012
1 Introduction

1.1 Problem formulation

In the Netherlands the phreatic groundwater level is generally located shallow below ground surface. In order to achieve dry building sites with an excavation level below the groundwater level, the groundwater level should be lowered. Therefore drainage systems are installed.

A drainage system usually consists of vertical filters or deep wells of which the filters are installed in the sand layers (aquifers) from which groundwater is extracted. The extraction of groundwater is adjusted so that the groundwater level and / or hydraulic head is reduced to sufficient depth below the building site floor. Because of the extraction, the groundwater level on site and near the building site will be lowered. The drawdown is largest next to the building site and decreases with increasing distance.

Environmental regulations do not allow large amounts of extracted groundwater to be pumped into surface water systems like rivers, canals, ditches or neighboring lakes. Furthermore, the sewer is often not a solution given the (often) limited capacity. One solution is to return the extracted groundwater in the soil. The injection of groundwater, however, increases the groundwater level in and around return sources. Returning groundwater with a traditional drainage system is accompanied by a round pumping effect which increases the discharge if the injection field is located near the project site. The rebound effect can be reduced by increasing the distance between the injection field and the drainage point. To obtain sufficient reductions on the building site the extracted groundwater should be injected back into the soil at a relatively large distance from the building site. The ideal distance of the injection field depends on the geohydrological parameters, the amount of groundwater to be extracted and the percentage of the extracted groundwater that will have to be returned in the ground. For a small rebound effect of less than 25%, this could mean that the injection field should be arranged at several hundred meters away from the drainage.

Both in practice and in theory it is known that there are ‘injection points’ or preferential flow paths in the subsurface. At such points the soil absorbs significantly more water than at other depths on such a location. One of the reasons of this phenomenon is the heterogeneity of the subsurface. A significantly higher permeability is expected at these injection points or preferential flow paths.
The use of these ‘injection points’ for injecting / returning the extracted groundwater has shown that groundwater can be returned at a (very) short distance from the extraction means without the usual negative effects of an injection field including rebound effects (higher surplus water), and flooding by groundwater elevations. Field measurements also seem to indicate that as the injection means are installed closer to the extraction means, the drawdowns at the location of the building site are achieved more quickly and the area of influence of the extraction is further reduced. It is assumed that the ‘injection points’ and injection filters of the return sources are installed deeper than the extraction filters.

The above effect has already been established in pilot projects where the extraction and injection are combined into one source location. Groundwater is extracted with a filter in the upper part of the source and then injected into a filter in the lower part. The results indicate that using this setup, the groundwater / hydraulic head drawdowns can be significant.
1.2 Background and motivation

1.2.1 Research question

Injection of extracted groundwater in the soil (sand / gravel formations / aquifers) from which it is extracted, raises the groundwater level around the ‘injection point’. The classical hydrological models are based on calculations using the superposition principle and laminar groundwater flow according to the Darcy equation

$$\mathbf{v}(x, y, z) = -\frac{k}{\mu} [\nabla p + \rho g e_z].$$

At equal geo-hydrological parameterization and configuration of the extraction and infiltration means, the drawdown of the groundwater level / hydraulic head by the extraction is equal to the increase of the hydraulic head by an equal injection (principle of superposition). According to Darcy’s law, if groundwater is extracted and fully (100%) injected at the same location into the same aquifer, there will be neither drawdown nor raising of the hydraulic head.

Using a new injection technique, where water is injected (usually with high pressure) on a very limited filter length at an ‘injection point’, measurements differ significantly from the expected values based on the classical Darcy’s law. An ‘injection point’ characteristically absorbs more water than other levels in the subsurface. This implies a local higher porosity and thus a higher permeability.

Even if this local higher permeability is taken into account, the measured groundwater / hydraulic head drawdowns and elevations do not correspond to the results calculated using the Darcy equation. The premise is that the groundwater flow at the location of the ‘injection point’ is not laminar and therefore the classical equation of Darcy is not valid.

In this project, only the flow of water immediately after the injection is considered and the previously performed extraction is disregarded. It is known that water in porous media laminar flows, hence the Darcy’s law is used in this study.

1.2.2 Research approach

The first thing to examine is whether the effect of anisotropy will be incorporated into the permeability term where the permeability in the (normally) horizontal directed flow depends on the magnitude of the flow (in the tangential direction). This will hinder the vertically directed flow so that the injected water will be allowed to penetrate into the soil over a larger distance. A second task in this project is to quantify the relationship between the magnitude of the flow, change in the permeability tensor (especially the (normally) horizontal direction) and the distance over which water flows in (almost) horizontal direction.

Another focus will be that, especially in a turbulent flow, pressure variations occur in the porous medium. Sand / gravel layers (aquifers) are not rigid, but constitute more or less an elastic matrix. Variations in pressure due to the turbulent flowing water will evoke some vibration frequencies in the porous medium.
These vibrations would be of utmost interest at a certain frequency to explain / describe the measured phenomenon. Propagation of vibrations could also explain the relatively broad scope of the phenomenon around an ‘injection point’.

This task will involve a complicated, but also a physical approach of model adaptation. In this approach, we will scrutinize the use of Biot’s Theory for poro-elasticity, where flow in porous media is combined with mechanical deformations of the aquifer in which water is injected. Mechanical deformations of the porous media induce changes in the porosity, which will result into permeability changes. In this task, we will analyse whether poro-elasticity can give any contribution to understand the long-lasting horizontal flow pattern in the subsurface at an ‘injection point’. Insight into the flow patterns can be obtained by combining elasticity and poro-elasticity of the porous media in the models with Darcy’s formalism. This approach will need a careful discretization method since a (non-linear) saddle-point problem needs to be solved. The simulation software will have to be developed in a great extent.
1.3 Calculation area

For the calculation area, we consider a cylinder-shaped part from the ground, with a height $H$ and radius $R$, of which the upper surface corresponds with the ground level. In the center line of the cylinder we put a long tube having a length $H$ in which a filter is placed, the injected water flows from the filter into the ground (see Figure 1).

![Figure 1: A cylindrical part of the ground with height $H$ and radius $R$](image)

We assume that the solution of this problem is rotationally symmetric, so it is sufficient to determine the solution for a fixed polar angle $\hat{\theta}$. The calculation plane is a rectangular surface in 2D, with cylindrical coordinates $(r, z)$, as shown in Figure 2.

![Figure 2: The calculation area in 2D, with a filter on boundary $\Gamma_5$](image)
2 The Darcy equation

At the beginning we solve the problem in which the water flow in porous media is described by the Darcy equation. In addition to this equation we use the law of continuity as a result of the incompressibility of water

\[ \nabla \cdot \mathbf{v} = 0. \]

Let the total domain of calculation given by \( \Omega \) (as shown in Figure 2), we get the following system of equations for the flow velocity \( \mathbf{v} \)

\[
\begin{align*}
\mathbf{v} &= \frac{k}{\mu} [\nabla p + \rho g e_z], & \text{for} \ (x, y, z) & \in \Omega; \\
\nabla \cdot \mathbf{v} &= 0, & \text{for} \ (x, y, z) & \in \Omega,
\end{align*}
\]

(2)

where:
- \( \mathbf{v} \) is the superficial velocity through the porous medium in \( \text{m/s} \),
- \( k \) is the permeability of the porous medium in \( \text{m}^2 \),
- \( \mu \) is the dynamic viscosity of water in \( \text{Pa} \cdot \text{s} \),
- \( p \) is the pressure of water in \( \text{Pa} \),
- \( \rho \) is the density of water in \( \text{kg/m}^3 \),
- \( g \) is the gravitational acceleration in \( \text{m/s}^2 \).

After substitution, we get the following equation in \( \Omega \)

\[ \nabla \cdot \left[ \frac{k}{\mu} (\nabla p + \rho g e_z) \right] = 0. \]

(3)

In this project we assume that the ground is saturated with water, and that there is no outflow along the boundaries \( \Gamma_3 \) and \( \Gamma_4 \). The water is injected into the soil only through the filter, placed in boundary \( \Gamma_5 \).

Hence we have the following boundary conditions on the boundaries, by prescribing the injection velocity at boundary \( \Gamma_5 \),

\[
\begin{align*}
p &= 0, & \text{for} \ (x, y, z) & \in \Gamma_1; \\
p &= \rho g (H - z), & \text{for} \ (x, y, z) & \in \Gamma_2; \\
\mathbf{v} \cdot \mathbf{n} &= 0, & \text{for} \ (x, y, z) & \in \Gamma_3; \\
\mathbf{v} \cdot \mathbf{n} &= 0, & \text{for} \ (x, y, z) & \in \Gamma_4; \\
\mathbf{v} \cdot \mathbf{n} &= \frac{Q_{in}}{2 \pi r l (\Gamma_5)}, & \text{for} \ (x, y, z) & \in \Gamma_5,
\end{align*}
\]

(4)

in which \( H \) is the depth of the well in meters, \( l(\Gamma_5) \) is the filter length in meters and \( Q_{in} \) is the discharge in \( \text{m}^3/\text{s} \).
We can also prescribe the imposed pressure on boundary $\Gamma_5$ instead of the injection velocity, then we get the following boundary conditions

\[
\begin{align*}
  p &= 0, & \text{for } (x, y, z) \in \Gamma_1; \\
  p &= \rho g (H - z), & \text{for } (x, y, z) \in \Gamma_2; \\
  \mathbf{v} \cdot \mathbf{n} &= 0, & \text{for } (x, y, z) \in \Gamma_3; \\
  \mathbf{v} \cdot \mathbf{n} &= 0, & \text{for } (x, y, z) \in \Gamma_4; \\
  p &= \rho g (H - z) + p_{\text{pump}}, & \text{for } (x, y, z) \in \Gamma_5,
\end{align*}
\]

(5)

where $p_{\text{pump}}$ is the maximum capacity of the pump which is used to inject water into the ground.

We will now solve equation (3) with both boundary conditions.
2.1 Weak formulation

2.1.1 Weak formulation of the problem with given discharge

Let \( \varphi \in \Sigma_0 = \{ \varphi \in H^1(\Omega) : \varphi = 0 \text{ on } \Gamma_1 \cup \Gamma_2 \} \), then we multiply equation (3) by \( \varphi \) and integrate it over \( \Omega \) to get

\[
- \int_{\Omega} \nabla \cdot \left[ \frac{k}{\mu} (\nabla p + \rho g e_z) \right] \varphi \, d\Omega = 0 \quad \text{or} \quad \int_{\Omega} (\nabla \cdot \mathbf{v}) \varphi \, d\Omega = 0. \tag{6}
\]

Furthermore,

\[
\int_{\Omega} (\nabla \cdot \mathbf{v}) \varphi \, d\Omega = \int_{\Omega} \nabla \cdot (\mathbf{v} \varphi) \, d\Omega - \int_{\Omega} \mathbf{v} \cdot \nabla \varphi \, d\Omega.
\]

From equation (6) it follows

\[
\int_{\Omega} \nabla \cdot (\mathbf{v} \varphi) \, d\Omega - \int_{\Omega} \mathbf{v} \cdot \nabla \varphi \, d\Omega = 0.
\]

The Divergence Theorem of Gauss gives

\[
\int_{\Omega} \nabla \cdot (\mathbf{v} \varphi) \, d\Omega = \int_{\partial \Omega} \mathbf{v} \cdot \mathbf{n} \, \varphi \, d\Gamma,
\]

hence

\[
\int_{\partial \Omega} \mathbf{v} \cdot \mathbf{n} \, \varphi \, d\Gamma - \int_{\Omega} \mathbf{v} \cdot \nabla \varphi \, d\Omega = 0. \tag{7}
\]

From boundary conditions (4), it follows that we have Dirichlet boundary conditions on the boundaries \( \Gamma_1 \) and \( \Gamma_2 \), hence \( \varphi = 0 \), and thus

\[
\int_{\Gamma_1 \cup \Gamma_2} \mathbf{v} \cdot \mathbf{n} \, \varphi \, d\Gamma = 0.
\]

On the boundaries \( \Gamma_3 \) and \( \Gamma_4 \) we have \( \mathbf{v} \cdot \mathbf{n} = 0 \), from which it follows

\[
\int_{\Gamma_3 \cup \Gamma_4} \mathbf{v} \cdot \mathbf{n} \, \varphi \, d\Gamma = 0.
\]

And on boundary \( \Gamma_5 \) we have \( \mathbf{v} \cdot \mathbf{n} = -\frac{Q_{in}}{2\pi r l(\Gamma_5)} \), and thus

\[
\int_{\Gamma_5} \mathbf{v} \cdot \mathbf{n} \, \varphi \, d\Gamma = -\int_{\Gamma_5} \frac{Q_{in}}{2\pi r l(\Gamma_5)} \varphi \, d\Gamma,
\]

hence we get

\[
\int_{\partial \Omega} \mathbf{v} \cdot \mathbf{n} \, \varphi \, d\Gamma = -\int_{\Gamma_5} \frac{Q_{in}}{2\pi r l(\Gamma_5)} \varphi \, d\Gamma.
\]
From equation (7) it follows
\[- \int_{\Omega} \mathbf{v} \cdot \nabla \varphi \, d\Omega = \int_{\Gamma_5} \frac{Q_{\text{in}}}{2\pi r l(\Gamma_5)} \varphi \, d\Gamma.\]

The Darcy equation then gives
\[\int_{\Omega} \frac{k}{\mu} (\nabla p + \rho g \mathbf{e}_z) \cdot \nabla \varphi \, d\Omega = -\int_{\Omega} \frac{k}{\mu} \rho g \mathbf{e}_z \cdot \nabla \varphi \, d\Omega + \int_{\Gamma_5} \frac{Q_{\text{in}}}{2\pi r l(\Gamma_5)} \varphi \, d\Gamma, \tag{8}\]

from which it follows
\[\int_{\Omega} \frac{k}{\mu} \nabla p \cdot \nabla \varphi \, d\Omega = -\int_{\Omega} \frac{k}{\mu} \rho g \frac{\partial \varphi}{\partial z} \mathbf{e}_z \cdot \nabla \varphi \, d\Omega + \int_{\Gamma_5} \frac{Q_{\text{in}}}{2\pi r l(\Gamma_5)} \varphi \, d\Gamma.\]

Subsequently, we use the transformation to cylinder coordinates \((r,z)\), hence we get
\[\int_{\Omega} \frac{k}{\mu} \nabla p \cdot \nabla \varphi \, d\Omega_{rz} = -\int_{\Omega} \frac{k}{\mu} \rho g \frac{\partial \varphi}{\partial z} r \, d\Omega_{rz} + \int_{\Gamma_5} \frac{Q_{\text{in}}}{2\pi l(\Gamma_5)} \varphi \, d\Gamma_{rz}. \tag{9}\]

So the weak formulation of this problem states:

Find \( p \in \Sigma \) such that
\[\int_{\Omega} \frac{k}{\mu} \nabla p \cdot \nabla \varphi \, d\Omega_{rz} = -\int_{\Omega} \frac{k}{\mu} \rho g \frac{\partial \varphi}{\partial z} r \, d\Omega_{rz} + \int_{\Gamma_5} \frac{Q_{\text{in}}}{2\pi l(\Gamma_5)} \varphi \, d\Gamma_{rz}, \forall \varphi \in \Sigma_0,\]
where \( \Sigma = \{ p \in H^1(\Omega) : p|_{\Gamma_1} = 0 \text{ and } p|_{\Gamma_2} = \rho g (H - z) \} \).

### 2.1.2 Weak formulation of the problem with imposed pressure

For the problem with prescribed water pressure on boundary \( \Gamma_5 \), let \( \varphi \in \Sigma_0 = \{ \varphi \in H^1(\Omega) : \varphi = 0 \text{ on } \Gamma_1 \cup \Gamma_2 \cup \Gamma_5 \} \). Then it follows from Section 2.1.1, after applying the Divergence Theorem of Gauss,
\[\int_{\partial \Omega} \mathbf{v} \cdot \mathbf{n} \, d\Gamma - \int_{\Omega} \mathbf{v} \cdot \nabla \varphi \, d\Omega = 0.\]

On the boundaries \( \Gamma_1, \Gamma_2 \) and \( \Gamma_5 \) we have \( \varphi = 0 \), and on the boundaries \( \Gamma_3 \) and \( \Gamma_4 \) we have \( \mathbf{v} \cdot \mathbf{n} = 0 \), thus we get
\[- \int_{\Omega} \mathbf{v} \cdot \nabla \varphi \, d\Omega = 0.\]
The Darcy equation then gives

\[ \int_{\Omega} k \mu (\nabla p + \rho g \mathbf{e}_z) \cdot \nabla \phi \, d\Omega = 0, \]

from which it follows, after the transformation to cylinder coordinates \((r, z)\),

\[ \int_{\Omega} k \frac{\mu}{\rho} \nabla p \cdot \nabla \phi \, r \, d\Omega_{rz} = -\int_{\Omega} k \frac{\mu}{\rho} \frac{\partial \phi}{\partial z} \, r \, d\Omega_{rz}. \] (10)

So the weak formulation of this problem states:

Find \( p \in \tilde{\Sigma} \) such that

\[ \int_{\Omega} k \frac{\mu}{\rho} \nabla p \cdot \nabla \phi \, r \, d\Omega_{rz} = -\int_{\Omega} k \frac{\mu}{\rho} \frac{\partial \phi}{\partial z} \, r \, d\Omega_{rz}, \forall \phi \in \tilde{\Sigma}_0, \]

with

\[ \tilde{\Sigma} = \{ p \in H^1(\Omega) : p|_{\Gamma_1} = 0, p|_{\Gamma_2} = \rho g (H - z) \text{ and } p|_{\Gamma_5} = \rho g (H - z) + p_{pump} \}. \]
2.2 Method of finite elements applied to the pressure

2.2.1 Solving problem with given injection velocity

In order to solve this problem, the method of finite elements is used, where triangular elements and piecewise linear basis functions are used. To determine the element matrix and element vector we use the approximation

\[ p(r, z) \approx p_n(r, z) = \sum_{j=1}^{n} p_j \varphi_j(r, z) + \sum_{j=n+1}^{n+n_{ess}} p_B j(r, z), \]

where \( \varphi_j \) are the basis functions in \( \Sigma_0 \), \( n \) is the number of mesh points with unknown pressure values, \( n_{ess} \) is the number of mesh points on boundaries with Dirichlet boundary conditions and \( p_B j \), for \( j \in \{ n+1, \ldots, n+n_{ess} \} \), are the pressure values of these mesh points.

The testfunction \( \varphi \) is replaced with \( \varphi_i \), where \( i \in \{ 1, \ldots, n \} \). Hence we get from equation (9), for \( i = 1, \ldots, n \),

\[ \int_{\Omega} \frac{k}{\mu} \left[ \sum_{j=1}^{n} p_j \varphi_j(r, z) + \sum_{j=n+1}^{n+n_{ess}} p_B j \varphi_j(r, z) \right] \cdot \nabla \varphi_i \, r \, d\Omega_{rz} = - \int_{\Omega} \frac{k}{\mu} \rho g \frac{\partial \varphi_i}{\partial z} \, r \, d\Omega_{rz} + \int_{\Gamma_5} \frac{Q_{in}}{2\pi l(\Gamma_5)} \varphi_i \, d\Gamma_{rz}. \]

This gives for \( i = 1, \ldots, n \)

\[ \sum_{j=1}^{n} p_j \int_{\Omega} \frac{k}{\mu} \nabla \varphi_j \cdot \nabla \varphi_i \, r \, d\Omega_{rz} = - \sum_{j=n+1}^{n+n_{ess}} p_B j \int_{\Omega} \frac{k}{\mu} \nabla \varphi_j \cdot \nabla \varphi_i \, r \, d\Omega_{rz} - \int_{\Omega} \frac{k}{\mu} \rho g \frac{\partial \varphi_i}{\partial z} \, r \, d\Omega_{rz} + \int_{\Gamma_5} \frac{Q_{in}}{2\pi l(\Gamma_5)} \varphi_i \, d\Gamma_{rz}. \]

Let

\[ S_{ij} = \int_{\Omega} \frac{k}{\mu} \nabla \varphi_j \cdot \nabla \varphi_i \, r \, d\Omega_{rz}; \]

\[ f_i = - \sum_{j=n+1}^{n+n_{ess}} S_{ij} p_B j - \int_{\Omega} \frac{k}{\mu} \rho g \frac{\partial \varphi_i}{\partial z} \, r \, d\Omega_{rz} + \int_{\Gamma_5} \frac{Q_{in}}{2\pi l(\Gamma_5)} \varphi_i \, d\Gamma_{rz}. \]

Then we get in matrix-vector notation: \( \sum_{j=1}^{n} S_{ij} p_j = f_i \) or \( Sp = f \).
We now determine the corresponding element matrix and element vector. Let \( n_{el} \) the number of internal elements and \( n_{be} \) the number of boundary elements on boundary \( \Gamma_5 \), then applies

\[
S_{ij} = \sum_{l=1}^{n_{el}} S_{ij}^{el} \quad \text{and} \quad f_i = \sum_{l=1}^{n_{el}} f_i^{el} + \sum_{l=1}^{n_{be}} f_i^{be}. 
\]

For the matrix element we use piecewise linear basis functions

\[
\varphi_j = \alpha_j + \beta_j r + \gamma_j z, \quad \text{with} \quad \alpha_j, \beta_j, \gamma_j \in \mathbb{R} \quad \text{and} \quad j \in \{1, \ldots, n\}. 
\]

In order to approximate the integrals, linear interpolation and Newton-Côtes integration are applied. We number for this purpose the corner points of each element with the local numbering \( r_1, r_2, r_3 \).

A function \( g(r, z) \) on the element \( e_l \) is then approximated by linear interpolation

\[
g(r, z) \approx g(r_1, z_1) \varphi_1(r, z) + g(r_2, z_2) \varphi_2(r, z) + g(r_3, z_3) \varphi_3(r, z), 
\]

this gives

\[
\int_{e_l} g(r, z) \, d\Omega_{rz} \approx g(r_1, z_1) \int_{e_l} \varphi_1 \, d\Omega_{rz} + g(r_2, z_2) \int_{e_l} \varphi_2 \, d\Omega_{rz} + g(r_3, z_3) \int_{e_l} \varphi_3 \, d\Omega_{rz}. 
\]

Define \( |\Delta| = ||(r_1 - r_2) \times (r_1 - r_3)|| \), then it follows from Holand and Bell’s Theorem (see [5]) that

\[
\int_{e_l} \varphi_j \, d\Omega_{rz} = \frac{|\Delta|}{6}, \quad j = 1, 2, 3. 
\]

Hence we get

\[
\int_{e_l} g(r, z) \, d\Omega \approx \frac{|\Delta|}{6} \sum_{p=1}^{3} g(r_p, z_p). 
\]

If we now assume that \( \mu \) and \( \rho \) are constant and \( k = k(r, z) \), then we get

\[
S_{ij}^{el} = \int_{e_l} \frac{k}{\mu} \nabla \varphi_j \cdot \nabla \varphi_i \, r \, d\Omega_{rz} \approx \frac{1}{\mu} (\beta_j \beta_i + \gamma_j \gamma_i) \frac{|\Delta|}{6} \sum_{p=1}^{3} k(r_p, z_p) r_p. 
\]

Thus, the element matrix can be given by

\[
S_{ij}^{el} = \frac{1}{\mu} \frac{|\Delta|}{6} \sum_{p=1}^{3} k(r_p, z_p) r_p \left( \begin{array}{ccc}
\beta_1^2 + \gamma_1^2 & \beta_1 \beta_2 + \gamma_1 \gamma_2 & \beta_1 \beta_3 + \gamma_1 \gamma_3 \\
\beta_1 \beta_2 + \gamma_1 \gamma_2 & \beta_2^2 + \gamma_2^2 & \beta_2 \beta_3 + \gamma_2 \gamma_3 \\
\beta_1 \beta_3 + \gamma_1 \gamma_3 & \beta_2 \beta_3 + \gamma_2 \gamma_3 & \beta_3^2 + \gamma_3^2
\end{array} \right). 
\]
Furthermore, we have \( f_i = \sum_{l=1}^{n_{el}} f_i^e_l + \sum_{l=1}^{n_{bel}} f_i^{be_l} \), with

\[
\begin{align*}
    f_i^e_l &= -\sum_{j=n+1}^{n+n_{ax}} S_{ij}^e p_{B_j} - \int_{\Omega} \frac{k}{\mu \rho g} \frac{\partial \phi_i}{\partial z} r \ d\Omega_{rz} \approx \\
    &\approx -\sum_{j=n+1}^{n+n_{ax}} S_{ij}^e p_{B_j} - \frac{1}{\mu \rho g \gamma_i} \left| \frac{\Delta}{6} \sum_{p=1}^{3} k(r_p, z_p) r_p \right| \\
    &= -\frac{1}{\mu} \frac{|\Delta|}{6} \sum_{p=1}^{3} k(r_p, z_p) r_p \left[ \sum_{j=n+1}^{n+n_{ax}} \left( \beta_j \beta_i + \gamma_j \gamma_i \right) + \rho \rho g \gamma_i \right].
\end{align*}
\]

For boundary elements applies

\[
\int_{\Gamma_{rz}} h(r, z) \ d\Gamma_{rz} \approx \frac{||r_2 - r_1||}{2} [h(r_1, z_1) + h(r_2, z_2)],
\]

and this gives

\[
\begin{align*}
    f_i^{be_l} &= \int_{\Gamma_{5,be_l}} \frac{Q_{in}}{2\pi l(\Gamma_5)} \phi_i \ d\Gamma_{rz} \approx \\
    &\approx \frac{||r_2 - r_1||}{2} \frac{Q_{in}}{2\pi l(\Gamma_5)} [\phi_i(r_1, z_1) + \phi_i(r_2, z_2)] = \\
    &= \frac{||r_2 - r_1||}{2} \frac{Q_{in}}{2\pi l(\Gamma_5)}.
\end{align*}
\]

Thus, the element vector for an internal element is given by

\[
\begin{align*}
    r^e_i &= -\frac{1}{\mu} \frac{|\Delta|}{6} \sum_{p=1}^{3} k(r_p, z_p) r_p \left[ \sum_{j=n+1}^{n+n_{ax}} \left( \beta_j \left( \begin{array}{c} \beta_1 \\ \beta_2 \\ \beta_3 \end{array} \right) + \gamma_j \left( \begin{array}{c} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{array} \right) \right) p_{B_j} + \rho \rho g \gamma_i \right].
\end{align*}
\]

For a boundary element we get

\[
\begin{align*}
    r^{be_l} &= \frac{||r_2 - r_1||}{2} \frac{Q_{in}}{2\pi l(\Gamma_5)} \left( \begin{array}{c} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{array} \right).)
\end{align*}
\]
2.2.2 Solving problem with known pressure near the filter

Also for this problem we use the approximation

\[ p(r, z) \approx p_n(r, z) = \sum_{j=1}^{n} p_j \varphi_j(r, z) + \sum_{j=n+1}^{n+n_{ess}} p_B \varphi_j(r, z). \]

Note that we have a Dirichlet boundary condition also on boundary \( \Gamma_5 \) according to boundary conditions (5).

From equation (10), we get for \( i = 1, \ldots, n \)

\[
\sum_{j=1}^{n} p_j \int_{\Omega} \frac{k}{\mu} \nabla \varphi_j \cdot \nabla \varphi_i \ r \ d\Omega_r z = - \sum_{j=n+1}^{n+n_{ess}} p_B \int_{\Omega} \frac{k}{\mu} \nabla \varphi_j \cdot \nabla \varphi_i \ r \ d\Omega_r z - \int_{\Omega} \frac{k}{\mu} \frac{\partial \varphi_i}{\partial z} r \ d\Omega_r z.
\]

Let

\[
\begin{align*}
S_{ij} &= \int_{\Omega} \frac{k}{\mu} \nabla \varphi_j \cdot \nabla \varphi_i \ r \ d\Omega_r z; \\
\xi_i &= - \sum_{j=n+1}^{n+n_{ess}} S_{ij} p_j - \int_{\Omega} \frac{k}{\mu} \frac{\partial \varphi_i}{\partial z} r \ d\Omega_r z.
\end{align*}
\]

Then we get in matrix-vector notation:

\[
\sum_{j=1}^{n} S_{ij} p_j = \xi_i \text{ or } S p = \xi.
\]

Note that we recover the matrix \( S \) as defined in Section 2.2.1. For the element vector we have

\[
\xi_i = \sum_{l=1}^{n_{el}} \xi_{i}^{el},
\]

with

\[
\begin{align*}
\xi_{i}^{el} &= - \sum_{j=n+1}^{n+n_{ess}} S_{ij}^{el} p_j - \int_{\xi_i} \frac{k}{\mu} \frac{\partial \varphi_i}{\partial z} r \ d\Omega_r z \\
&\approx - \sum_{j=n+1}^{n+n_{ess}} S_{ij}^{el} p_B - \frac{1}{\mu} \rho g \gamma_i \sum_{p=1}^{3} k(r_p, z_p) r_p = \\
&= - \frac{1}{\mu} \frac{|\Delta|}{6} \sum_{p=1}^{3} k(r_p, z_p) r_p \left[ \sum_{j=n+1}^{n+n_{ess}} (\beta_j \beta_i + \gamma_j \gamma_i) p_B + \rho g \gamma_i \right].
\end{align*}
\]

Thus, the element vector for an internal element is given by

\[
\xi_{i}^{el} = - \frac{1}{\mu} \frac{|\Delta|}{6} \sum_{p=1}^{3} k(r_p, z_p) r_p \left[ \sum_{j=n+1}^{n+n_{ess}} \left( \beta_j \left( \begin{array}{c} \beta_1 \\ \beta_2 \\ \beta_3 \end{array} \right) + \gamma_j \left( \begin{array}{c} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{array} \right) \right) p_B + \rho g \left( \begin{array}{c} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{array} \right) \right].
\]
2.3 Exact solution

We hypothesize that the exact solution for the stationary problem without water injection \((Q_n = 0)\) can be written as

\[
\tilde{p} = \rho g(H - z), \text{ for } (x, y, z) \in \Omega.
\] (12)

To verify this assumption, we check whether this solution satisfies the differential equations and the boundary conditions. We start with the differential equations, from system (2) it follows that \(\tilde{p}\) should satisfy

\[
\nabla \cdot \left[-k \mu \left(\nabla \tilde{p} + \rho g e_z\right)\right] = 0.
\] (13)

In addition we get,

\[
\nabla \tilde{p} = \left(\frac{\partial \tilde{p}}{\partial x}, \frac{\partial \tilde{p}}{\partial y}, \frac{\partial \tilde{p}}{\partial z}\right)^T = (0, 0, -\rho g)^T,
\]

and it follows

\[
\nabla \cdot \left[-\frac{k}{\mu} \left(\nabla \tilde{p} + \rho g e_z\right)\right] = \nabla \cdot \left[-\frac{k}{\mu} \left(0, 0, -\rho g + \rho g\right)^T\right] = 0.
\]

Thus, solution (12) satisfies equation (13). We now check whether this solution satisfies the boundary conditions.

On boundary \(\Gamma_1\) we have \(z = H\) and applies the condition \(p = 0\). Furthermore,

\[
\tilde{p}(x, y, H) = \rho g(H - H) = 0.
\]

Hence solution (12) satisfies this boundary condition.

On boundary \(\Gamma_2\) we have the condition \(p = \rho g(H - z)\). So the solution satisfies also this boundary condition.

For \((x, y, z) \in \Gamma_3\) we have \(\mathbf{v} \cdot \mathbf{n} = 0\) and \(\mathbf{n} = (0, 0, -1)\), hence we get

\[
0 = \mathbf{v} \cdot \mathbf{n} = -\frac{k}{\mu} (\nabla p + \rho g e_z) \cdot \mathbf{n} = \frac{k}{\mu} (\frac{\partial p}{\partial z} + \rho g).
\]

Furthermore, \(\frac{\partial \tilde{p}}{\partial z} = -\rho g\), hence

\[
\tilde{\mathbf{v}} \cdot \mathbf{n} = \frac{k}{\mu} (\nabla \tilde{p} + \rho g) = \frac{k}{\mu} (-\rho g + \rho g) = 0,
\]

thus solution (12) satisfies also the boundary condition on boundary \(\Gamma_3\).
For \((x, y, z) \in \Gamma_4\) we have \(v \cdot n = 0\) and \(n = (-1, 0, 0)\), hence this gives

\[
0 = v \cdot n = -\frac{k}{\mu} (\nabla p + \rho g e_z) \cdot n = \frac{k}{\mu} \frac{\partial p}{\partial x}.
\]

Also we have \(\frac{\partial \tilde{\rho}}{\partial x} = 0\), from which it follows

\[
\tilde{v} \cdot n = \frac{k}{\mu} \frac{\partial \tilde{\rho}}{\partial x} = 0.
\]

This also applies for \((x, y, z) \in \Gamma_5\). Further, it follows from boundary conditions (4) for this boundary

\[
v \cdot n = -\frac{Q_{in}}{2\pi r l(\Gamma_5)} = 0,
\]

since \(Q_{in} = 0\) in this problem.

According to boundary conditions (5), we have for \((x, y, z) \in \Gamma_5\)

\[
p = \rho g (H - z) + p_{pump}.
\]

We assume that there is no water injection, hence we have \(p_{pump} = 0\), and thus solution (12) satisfies the boundary conditions on the boundaries \(\Gamma_4\) and \(\Gamma_5\) for both boundary conditions.

**Conclusion:** The exact solution for the problem without water injection, under \(Q_{in} = 0\), is

\[
\tilde{p} = \rho g (H - z), \text{ for } (x, y, z) \in \Omega.
\]

We can now verify the consistency between the exact solution of the problem without water injection and the numerical solution obtained by the method of finite elements.
2.4 Numerical solution of the problem without injection

To validate the method of finite elements, we determine the numerical solution of the problem without water injection, for both boundary conditions. In the beginning, we assume that the porous medium has a constant average permeability, $k(r, z) = k_0$ with $k_0 \in \mathbb{R}_{>0}$.

To this extent, the domain $\Omega$, with height $H = 20 m$ and radius $R = 100 m$, is divided into small triangular elements with width $\Delta r$ and height $\Delta z$. In total 8 241 mesh points are used (201 points in the $r$-direction and 41 points in the $z$-direction). In Figure 3, an example of a computational mesh is shown, for clarity only 200 points are used.

![Figure 3: The created mesh with triangular elements](image)

For the problem without water injection is in Figure 4 a 3D plot shown of the numerical solution. Here the following values are used

$$\mu = 1.518 \cdot 10^3 \text{ Pa} \cdot \text{s}, \rho = 1000 \text{ kg/m}^3, g = 9.81 \text{ m/s}^2,$$

$$k_0 = 2.3148 \cdot 10^{-4} \text{ m}^2, \Delta r = 0.5 \text{ m}, \Delta z = 0.5 \text{ m}$$

and $\Gamma_5 = \{(r, z) \in \mathbb{R}^2 \mid r = 0 \land 8 \leq z \leq 9\}$. 
Figure 4: Numerical solution of the problem without water injection

The solution shown in Figure 4 is obtained with both boundary conditions. We observe that we recover the exact solution $\tilde{p} = \rho g (H - z)$. Thus the method of calculation is consistent and can be used to determine the solution of the problem with water injection.
2.5 Numerical solution of the problem with injection

2.5.1 Solution for the pressure by given injection velocity

We will now determine the numerical solutions of the problem with water injection, where 50 liters of water per second are injected into the ground \(Q_{in} = 0.050 \, m^3/s\).

In Figure 5 the numerical solution of the problem with boundary conditions (4) is shown as a 3D plot.

![Figure 5: Numerical solution of the problem with water injection](image)

In this figure, we see a peak in the water pressure along the injection filter, this high pressure decreases with increasing distance from the inlet. After a distance of about 50 meters the hydrostatic pressure is restored.

To determine the effect of the injection of water, we also make a plot of the difference between the solution with and the solution without water injection (see Figure 6).
We see that around the injection filter the effect of water injection is greatest, this effect decreases as the distance from the filter increases. The injected water reaches in this case a maximum horizontal distance of 46.5 meters.

**Error estimation with Richardson’s extrapolation**

For the solution shown in Figure 5, a mesh is used with step size \( h \) where

\[
h := \Delta r = \Delta z.
\]

Let \( N(h) \) be the \( L_2 \)-norm of the numerical solution for the pressure obtained by using a mesh with step size \( h \) and let \( M \) the \( L_2 \)-norm of the exact solution \( p_{\text{exact}} \) for the water pressure.

Then we assume that we can approximate \( M \) by

\[
M = N(h) + Kh^s,
\]

where \( K \) and \( s \) are unknown constants (see [6]). This approach can also be used for different step sizes, such as

\[
M = N\left(\frac{h}{2}\right) + K \left(\frac{h}{2}\right)^s,
\]

\[
M = N\left(\frac{h}{4}\right) + K \left(\frac{h}{4}\right)^s.
\]

Through subtracting and dividing these expressions, we get

\[
\frac{N\left(\frac{h}{2}\right) - N(h)}{N\left(\frac{h}{4}\right) - N\left(\frac{h}{2}\right)} = 2^s. \tag{14}
\]

Table 1 shows the results for the \( L_2 \)-norm of the numerical solution of the pressure at different step sizes \( h \).
From these results and expression (14), it follows that \( s \approx 1.14 \). This means that this method is of first-order accuracy \( O(1) \). However, we know that the finite elements method is second-order accurate, the deviation in the order that we get with Richardson’s extrapolation may be caused by the singularity on the boundary with \( r = 0 \).

The choice of a step size of half a meter (\( \Delta r = \Delta z = 0.5 \) m) is based on the reasonable computational time required to solve this problem. A finer mesh is also not necessary for the purposes of this problem.

### 2.5.2 Solution for the pressure by known pressure near the filter

The numerical solution of the problem with boundary conditions (5) is shown in Figure 7, the maximum capacity of the pump is chosen equal to 1.6 bar.

![Figure 7: Numerical solution of the problem with water injection](image)

Also in this figure we see a peak in the water pressure along the injection filter, this high pressure decreases with increasing distance from the inlet. After a distance of about 60 meters, the hydrostatic pressure is restored. The injected water reaches in this case a maximum horizontal distance of 58.5 meters.

The water pressure values we get back by these models differ from the values measured at an ‘injection point’. The pressure is now higher than we expect, we will therefore investigate a number of hypotheses to test whether we can reproduce the measurements from this model.

<table>
<thead>
<tr>
<th>( h ) in m</th>
<th>( N(h) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>3.5870 \cdot 10^7</td>
</tr>
<tr>
<td>0.25</td>
<td>3.5859 \cdot 10^7</td>
</tr>
<tr>
<td>0.125</td>
<td>3.5854 \cdot 10^7</td>
</tr>
</tbody>
</table>

Table 1: \( L_2 \)-norm of the numerical solution at different step sizes \( h \)
2.6 Method of finite elements applied to the velocity

Once the water pressure \( p \) has been determined from system (2), with both boundary conditions (4) and (5), we can now determine the velocity vector \( \mathbf{v} \) from equation (1). From which it follows

\[
\mathbf{v} = -\frac{k}{\mu}\left[\nabla p + \rho g \mathbf{e}_z\right], \quad \text{in } \Omega.
\]

So we get for the components of vector \( \mathbf{v} \)

\[
\begin{align*}
v_r &= -\frac{k}{\mu} \frac{\partial p}{\partial r}, \\
v_z &= -\frac{k}{\mu} \left(\frac{\partial p}{\partial z} + \rho g\right).
\end{align*}
\]

(15) \hspace{1cm} (16)

We now determine the weak formulation for equation (15). To this extent, we multiply the equation by a testfunction \( \varphi \in H^1(\Omega) \) and integrate it over \( \Omega \) to get

\[
\int_{\Omega} v_r \varphi \, d\Omega = -\int_{\Omega} \frac{k}{\mu} \frac{\partial p}{\partial r} \varphi \, d\Omega.
\]

Using the transformation to cylinder coordinates \((r, z)\), we get

\[
\int_{\Omega} v_r \varphi \, r \, d\Omega_{rz} = -\int_{\Omega} \frac{k}{\mu} \frac{\partial p}{\partial r} \varphi \, r \, d\Omega_{rz}. \tag{17}
\]

So the weak formulation of this problem states:

Find \( v_r \in H^1(\Omega) \) such that

\[
\int_{\Omega} v_r \varphi \, r \, d\Omega_{rz} = -\int_{\Omega} \frac{k}{\mu} \frac{\partial p}{\partial r} \varphi \, r \, d\Omega_{rz}, \quad \forall \varphi \in H^1(\Omega).
\]

To solve this problem, we use the method of finite elements on the same mesh as described in Section 2.4. And we use the approximation

\[
v_r(r, z) \approx v_r^0(r, z) = \sum_{j=1}^{n+n_{\text{ess}}} v_j^r \varphi_j(r, z).
\]

The testfunction \( \varphi \) is replaced with \( \varphi_i \), where \( i \in \{1, \ldots, n+n_{\text{ess}}\} \). Hence we get from equation (17), for \( i = 1, \ldots, n+n_{\text{ess}} \),

\[
\int_{\Omega} \left[ \sum_{j=1}^{n+n_{\text{ess}}} v_j^r \varphi_j(r, z) \right] \varphi_i \, r \, d\Omega_{rz} = -\int_{\Omega} \frac{k}{\mu} \frac{\partial p}{\partial r} \varphi_i \, r \, d\Omega_{rz}.
\]

This gives for \( i = 1, \ldots, n+n_{\text{ess}} \)

\[
\sum_{j=1}^{n+n_{\text{ess}}} v_j^r \int_{\Omega} \varphi_j \varphi_i \, r \, d\Omega_{rz} = -\int_{\Omega} \frac{k}{\mu} \frac{\partial p}{\partial r} \varphi_i \, r \, d\Omega_{rz}.
\]
Let
\[
M_{ij} = \int_\Omega \phi_j \phi_i \, r \, d\Omega_{rz};
\]
\[
b_i = -\int_\Omega \frac{k}{\mu} \frac{\partial p}{\partial r} \phi_i \, r \, d\Omega_{rz}.
\]

Then we get in matrix-vector notation:
\[
\sum_{j=1}^{n+d_{n_{rez}}} M_{ij} v^j_r = b_i \text{ or } M v_r = b.
\]

To determine the corresponding element matrix and element vector, let \( n_{el} \) be the number of internal elements, then applies
\[
M_{ij} = \sum_{l=1}^{n_{el}} M^{el}_{ij} \text{ and } b_i = \sum_{l=1}^{n_{el}} b^{el}_i,
\]
where
\[
M^{el}_{ij} = \int_{\Omega_{el}} \phi_j \phi_i \, r \, d\Omega_{rz} \approx \frac{|\Delta|}{6} \sum_{p=1}^{3} [\phi_j(r_p, z_p)\phi_i(r_p, z_p)r_p] = \begin{cases} \frac{|\Delta|}{6} r_i & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}
\]

Thus, the element matrix is given by
\[
M^{el} = \frac{|\Delta|}{6} \begin{pmatrix} r_1 & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & r_3 \end{pmatrix}.
\]

Furthermore, we have
\[
b^{el}_i = \sum_{l=1}^{n_{el}} b^{el}_i, 
\]
with
\[
b^{el}_i = -\int_{\Omega_{el}} \frac{k}{\mu} \frac{\partial p}{\partial r} \phi_i \, r \, d\Omega_{rz} \approx -\frac{1}{\mu} \int_{\Omega_{el}} \frac{k}{\partial r} \left( \sum_{m=1}^{3} p_m \phi_m(r, z) \right) \phi_i \, r \, d\Omega_{rz} =
\]
\[
= -\frac{1}{\mu} \sum_{m=1}^{3} p_m \int_{\Omega_{el}} k \frac{\partial \phi_m}{\partial r} \phi_i \, d\Omega_{rz} \approx
\]
\[
= -\frac{1}{\mu} \left( \sum_{m=1}^{3} p_m \beta_m \right) \frac{|\Delta|}{6} \sum_{p=1}^{3} [k(r_p, z_p)\phi_i(r_p, z_p)r_p] =
\]
\[
= -\frac{1}{\mu} \left( \sum_{m=1}^{3} p_m \beta_m \right) \frac{|\Delta|}{6} k(r_i, z_i) r_i.
\]

Thus, the element vector for an internal element is given by
\[
b^{el} = -\frac{1}{\mu} \left( \sum_{m=1}^{3} p_m \beta_m \right) \frac{|\Delta|}{6} \begin{pmatrix} k(r_1, z_1) r_1 \\ k(r_2, z_2) r_2 \\ k(r_3, z_3) r_3 \end{pmatrix}.
\]
For equation (16), we get the following weak formulation:

Find \( v_z \in H^1(\Omega) \) such that

\[
\int_{\Omega} v_z \varphi r \, d\Omega_{rz} = - \int_{\Omega} \frac{k}{\mu} \left( \frac{\partial p}{\partial z} + \rho g \right) \varphi r \, d\Omega_{rz}, \quad \forall \varphi \in H^1(\Omega).
\]

To solve this problem, we use the approximation

\[
v_z(r, z) \approx v_z^n(r, z) = \sum_{j=1}^{n+n_{ess}} v_z^j \varphi_j(r, z).
\]

Hence, we get for \( i = 1, \ldots, n+n_{ess} \)

\[
\sum_{j=1}^{n+n_{ess}} v_z^j \int_{\Omega} \varphi_j \varphi_i r \, d\Omega_{rz} = - \int_{\Omega} \frac{k}{\mu} \left( \frac{\partial p}{\partial z} + \rho g \right) \varphi_i r \, d\Omega_{rz}.
\]

Note that we get back the elements \( M_{ij} \) of the equation for \( v_r \). Let

\[
\psi_i = - \int_{\Omega} \frac{k}{\mu} \left( \frac{\partial p}{\partial z} + \rho g \right) \varphi_i r \, d\Omega_{rz},
\]

then we get in matrix-vector notation: \( \sum_{j=1}^{n+n_{ess}} M_{ij} v_z^j = \psi_i \) or \( Mv_z = \psi \).

Furthermore, we have \( \psi_i = \sum_{l=1}^{n_{el}} \psi_{i}^{el} \), with

\[
\psi_{i}^{el} = - \int_{e_l} \frac{k}{\mu} \left( \frac{\partial p}{\partial z} + \rho g \right) \varphi_i r \, d\Omega_{rz} \approx

\approx - \frac{1}{\mu} \int_{e_l} k \left( \sum_{m=1}^{3} p_m \gamma_m(r, z) + \rho g \right) \varphi_i r \, d\Omega_{rz} \approx

\approx - \frac{1}{\mu} \left( \sum_{m=1}^{3} p_m \gamma_m + \rho g \right) \frac{|\Delta|}{6} \sum_{p=1}^{3} \left[ k(r, z_p) \varphi_i(r, z_p) r_p \right] =

= - \frac{1}{\mu} \left( \sum_{m=1}^{3} p_m \gamma_m + \rho g \right) \frac{|\Delta|}{6} k(r_i, z_i) r_i.
\]

Thus, the element vector for an internal element is given by

\[
\psi^{el} = - \frac{1}{\mu} \left( \sum_{m=1}^{3} p_m \gamma_m + \rho g \right) \frac{|\Delta|}{6} \begin{pmatrix} k(r_1, z_1) r_1 \\ k(r_2, z_2) r_2 \\ k(r_3, z_3) r_3 \end{pmatrix}.
\]

Note that a problem occurs in \( r = 0 \), by solving both equations. We know from the boundary conditions (4) and (5) that on boundary \( \Gamma_4 \) we have \( v_r = 0 \), but \( v_z \) is unknown at this boundary.
Further, holds for boundary $\Gamma_5$ according to boundary conditions (4)

$$v \cdot n = -\frac{Q_m}{2\pi r l(\Gamma_5)}.$$

It should be noted that the value of $v_r$ is not defined at boundary $\Gamma_5$ in this case, and that the flow velocity goes to infinity as we approach this boundary. Under boundary conditions (5) is the velocity vector at $\Gamma_5$ unknown.

To solve this problem, we use finite differences for the derivatives of the pressure $p$ to determine $v_z$ and $v_r$ at $r = 0$. From equation (15) we get

$$v_r = -\frac{k}{\mu} \frac{\partial p}{\partial r}.$$

Consider a point $(0, z)$ on boundary $\Gamma_4 \cup \Gamma_5$, then it follows from the finite difference method

$$v_r(0, z) = \frac{k(0, z)}{\mu} \frac{\partial p}{\partial r}(0, z) \approx -\frac{k(0, z)}{\mu} \left(\frac{p(\Delta r, z) - p(0, z)}{\Delta r}\right).$$

Consider now equation (16)

$$v_z = -\frac{k}{\mu} \left(\frac{\partial p}{\partial z} + \rho g\right),$$

then we get from the finite difference method

$$v_z(0, z) = -\frac{k(0, z)}{\mu} \left(\frac{\partial p}{\partial z}(0, z) + \rho g\right) \approx -\frac{k(0, z)}{\mu} \left(\frac{p(0, z + \Delta z) - p(0, z)}{\Delta z} + \rho g\right).$$

Note that if $z = H$ then $z + \Delta z$ is not defined in our mesh. To determine the value of $v_z(0, H)$ we have to use a ghost cell, so we assume that there exists a mesh point at position $(0, H + \Delta z)$.

The value of the pressure at position $(0, H - \Delta z)$ is determined and the value of $p(0, H)$ is equal to zero because the mesh point $(0, H)$ is on boundary $\Gamma_1$. From the midpoint formula it follows for the pressure value of the ghost cell

$$p(0, H + \Delta z) = -p(0, H - \Delta z).$$

Thus we get

$$v_z(0, H) \approx -\frac{k(0, H)}{\mu} \left(\frac{p(0, H + \Delta z) - p(0, H)}{\Delta z} + \rho g\right) = -\frac{k(0, H)}{\mu} \left(-p(0, H - \Delta z) - p(0, H) + \rho g\right).$$
2.7 Numerical solution for the velocity vector

We now determine the solution for the velocity vector \( \mathbf{v} \) for the problem with water injection, hereby is the same mesh used as in Section 2.4.

2.7.1 Solution for the velocity by given discharge

In Figure 8 a vector plot can be seen of the numerical solution of the velocity vector with boundary conditions (4), in this case 50 liters water per second are injected into the ground. For the clarity of the figure only the first five meters away from the filter are plotted.

![Figure 8: Numerical solution of the velocity vector](image)

In this figure, the arrows indicating the direction of the velocity vector \( \mathbf{v} \) are scaled in size, with a maximum speed of 2.45 \( cm/s \). Further, we see that the water speed is highest near the filter and decreases with increasing distance from the filter.

We now look at the behavior of the velocity vector along the boundaries. In Figure 9 a graph can be seen of the numerical solution of component \( v_z \) of the velocity vector on boundary \( \Gamma_1 \).
The component $v_z$ is positive on this boundary, from which it follows that water flows out of the soil to the surface.

The rate of outflow of water from the ground to the earth’s surface indicates how far water is flowing into the ground and it indicates the magnitude of flooding by groundwater elevations, so we want to determine the outflow rate of water on the upper surface of our area (shown in Figure 1).

Let $A$ be the upper surface of our area, then it follows for the discharge $Q_{uit}^A$ of the water flowing out of plane $A$

$$Q_{uit}^A = \int_A \mathbf{v}(x, y, H) \cdot \mathbf{n} \, dA = \int_0^{2\pi} \int_0^R \left[ \begin{array}{c} v_r(r, H) \\ v_z(r, H) \end{array} \right] \cdot \left[ \begin{array}{c} 0 \\ 1 \end{array} \right] r \, dr \, d\theta =$$

$$= 2\pi \int_0^R v_z(r, H) r \, dr \approx 2\pi \sum_{i \in \Gamma_1} v_z^i r_i \, dr.$$

By the injection of 50 liters of water per second, the outflow rate of water on the upper surface in this case is about 50 liters per second.

In order to get an idea of the distribution of the water flow over the surface $A$, we consider the area $A$ as a function of the radius $r$

$$A(r) = 2\pi r, \text{ for } 0 \leq r \leq R.$$

In Figure 10 a plot is shown of the distribution of the outflow of water as a function of the radius $r$. 

![Figure 9: Numerical solution of the component $v_z$ on boundary $\Gamma_1$](image-url)
We observe that the total outflow of water is larger than the inflow of 50 liters per second as a result of numerical errors, these errors become smaller if we use more mesh points, as we can see in Figure 10.

On boundary $\Gamma_2$, we get the numerical solution of component $v_r$ of the velocity vector shown in Figure 11.

We see that the water flows out of the calculation area also on the sides 100 meters from the injection filter. We note that the outflow of water along boundary $\Gamma_1$ is larger than the outflow along boundary $\Gamma_2$, as a result of the Dirichlet boundary condition on boundary $\Gamma_1$ where the pressure is equal to zero while on boundary $\Gamma_2$ the hydrostatic pressure prevails ($p|_{\Gamma_2} \geq 0$).
On boundary $\Gamma_3$ we get the vector plot shown in Figure 12.

![Figure 12: Numerical solution of the velocity vector on boundary $\Gamma_3$](image1.png)

On this boundary we have the condition $\mathbf{v} \cdot \mathbf{n} = 0$ and this means that there is no flow of water along this boundary. However, in Figure 12 we see that water is flowing along this boundary in the first 10 meters, with a maximum size of the vertical component of $1.12 \cdot 10^{-5} \text{ m/s}$. This is probably due to numerical errors in this method, because the values of the vertical components approach zero at a finer mesh, as we can see in Figure 13.

![Figure 13: Numerical solution of the component $v_z$ on boundary $\Gamma_3$](image2.png)
2.7.2 Solution for the velocity by known imposed pressure

The method described in Section 2.6 is now used to determine the numerical solution of the velocity vector with boundary conditions (5). In Figure 14 a vector plot is shown of the numerical solution of $v$. For the clarity of the figure only the first five meters away from the filter are plotted.

![Figure 14: Numerical solution of the velocity vector](image)

Also in this vector plot are the arrows, indicating the direction of the velocity vector $v$, scaled in size with a maximum speed of 4.21 cm/s. For the velocity vector on boundary $\Gamma_5$, the finite difference method is used.

The numerical solution along the boundaries is in this case similar to the solution obtained with boundary conditions (4).

To get a better idea of the magnitude of the velocity in this case, we determine the magnitude of the velocity vector $||v||$ (see Figure 15).

![Figure 15: Magnitude of the velocity vector as function of the position](image)
2.8 Stochastically determined permeability

In Section 2.4, we assumed that the soil has a constant permeability that is on average equal to $k_0$. In reality, the permeability is not the same everywhere in the soil but it depends on the depth and the geo-hydrological parameters of the soil.

The composition of the soil varies by location and is usually difficult to determine exactly. In order to get an idea of the effect of different compositions on the flow of water, we will try to investigate the impact of stochastic perturbations on the permeability. Here, we assume that the permeability is constant over each triangular element (see Figure 3). Hence, the following formula is used for the permeability in each element $e_l$

$$k^{e_l} = k_0 + \sigma \text{rand}(-1, 1),$$

where:

$\sigma$ is the maximum of the fluctuation,

$\text{rand}(a, b)$ is a random number generator based on the standard uniform distribution on the interval $(a, b)$.

We note that we get a discontinuous formula for the permeability and we want to show that we can easily handle with this discontinuity in the finite elements method.

From equation (8), we get

$$\int_{\Omega} \frac{k}{\mu} (\nabla p + \rho g e_z) \cdot \nabla \varphi \, d\Omega = \int_{\Gamma_5} \frac{Q_{in}}{2\pi r} l(\Gamma_5) \varphi \, d\Gamma. \quad (18)$$

Suppose now that we can divide the area $\Omega$ into two parts $\Omega_1$ and $\Omega_2$, with a discontinuity at the boundary $\Gamma_1 \cap \Gamma_2$. Then we get from equation (18)

$$\int_{\Omega_1} \frac{k_1}{\mu} (\nabla p + \rho g e_z) \cdot \nabla \varphi \, d\Omega + \int_{\Omega_2} \frac{k_2}{\mu} (\nabla p + \rho g e_z) \cdot \nabla \varphi \, d\Omega = \int_{\Gamma_5} \frac{Q_{in}}{2\pi r} l(\Gamma_5) \varphi \, d\Gamma. \quad (18)$$

From which it follows

$$\int_{\Omega_1} \nabla \cdot \left[ \frac{k_1}{\mu} (\nabla p + \rho g e_z) \varphi \right] \, d\Omega + \int_{\Omega_2} \nabla \cdot \left[ \frac{k_2}{\mu} (\nabla p + \rho g e_z) \varphi \right] \, d\Omega =$$

$$= \int_{\Omega_1} \nabla \cdot \left[ \frac{k_1}{\mu} (\nabla p + \rho g e_z) \right] \varphi \, d\Omega + \int_{\Omega_2} \nabla \cdot \left[ \frac{k_2}{\mu} (\nabla p + \rho g e_z) \right] \varphi \, d\Omega +$$

$$+ \int_{\Gamma_5} \frac{Q_{in}}{2\pi r} l(\Gamma_5) \varphi \, d\Gamma.$$

The Divergence Theorem of Gauss gives

$$\int_{\partial \Omega_1} \frac{k_1}{\mu} (\nabla p + \rho g e_z) \cdot \mathbf{n} \varphi \, d\Gamma + \int_{\partial \Omega_2} \frac{k_2}{\mu} (\nabla p + \rho g e_z) \cdot \mathbf{n} \varphi \, d\Gamma =$$

$$= \int_{\Omega_1} \nabla \cdot \left[ \frac{k_1}{\mu} (\nabla p + \rho g e_z) \right] \varphi \, d\Omega + \int_{\Omega_2} \nabla \cdot \left[ \frac{k_2}{\mu} (\nabla p + \rho g e_z) \right] \varphi \, d\Omega +$$

$$+ \int_{\Gamma_5} \frac{Q_{in}}{2\pi r} l(\Gamma_5) \varphi \, d\Gamma.$$
From the Darcy equation it follows

$$\nabla \cdot \left[ \frac{k}{\mu} (\nabla p + \rho g e_z) \right] = 0, \text{ in } \Omega,$$

thus we get

$$\int_{\partial \Omega_1} \frac{k_1}{\mu} (\nabla p + \rho g e_z) \cdot \mathbf{n} \, d\Gamma + \int_{\partial \Omega_2} \frac{k_2}{\mu} (\nabla p + \rho g e_z) \cdot \mathbf{n} \, d\Gamma = \int_{\Gamma_5} \frac{Q_{in}}{2\pi r \, l(\Gamma_5)} \varphi \, d\Gamma.$$

This can also be written as

$$\int_{\partial \Omega_1} k (\nabla p + \rho g e_z) \cdot \mathbf{n} \, d\Gamma + \int_{\partial \Omega_1 \cap \partial \Omega_2} k (\nabla p + \rho g e_z) \cdot \mathbf{n} \, d\Gamma = \int_{\Gamma_5} \frac{Q_{in}}{2\pi r \, l(\Gamma_5)} \varphi \, d\Gamma.$$

From Section 2.1.1 it follows that

$$\int_{\partial \Omega} \frac{k}{\mu} (\nabla p + \rho g e_z) \cdot \mathbf{n} \, d\Gamma = \int_{\Gamma_5} \frac{Q_{in}}{2\pi r \, l(\Gamma_5)} \varphi \, d\Gamma.$$

Hence, we get

$$\int_{\partial \Omega_1 \cap \partial \Omega_2} \frac{k}{\mu} (\nabla p + \rho g e_z) \cdot \mathbf{n} \, d\Gamma = 0.$$

We find that the discontinuity at the boundaries between the elements poses no problems in the weak formulation and will therefore not affect the final result.

We will now show the numerical results for different values of the maximum of the fluctuation $\sigma$ and for different compositions. The value of the maximum of the fluctuation $\sigma$ has to be estimated, and because the permeability is always positive, we use: $\sigma < k_0$.

### 2.8.1 Influence of different compositions of the soil

To investigate the influence of different compositions of the soil, we keep the maximum of the fluctuation constant and we generate different values from the random number generator for each element.

For boundary conditions (4), the injection velocity is prescribed, causing changes in the permeability will have a greater influence on the water pressure than on the flow velocity. However, for boundary conditions (5) the imposed pressure at the filter prescribed, this will make the variations in the permeability have more influence on the flow velocity than on the water pressure.

To investigate the effect of the different soil compositions, we will determine the water pressure using boundary conditions (4) and the flow velocity with boundary conditions (5).
First, we determine the numerical solution of the water pressure for the problem with prescribed water injection, where 50 liters of water per second are injected into the ground. Let $\sigma = 0.5k_0$, the numerical solutions with different outcomes from the random number generation (RND) are shown in Figure 16.

![Figure 16: Numerical solutions of the pressure for different soil compositions](image)

In this figure only the first five meters away from the filter are displayed, at further distances the hydrostatic pressure is more or less reached (see Figure 4).

We also determine the numerical solution of the magnitude of the flow velocity for the problem with prescribed pressure on $\Gamma_5$, equal to 1.6 bar. For this, we use the same permeability profile as for the calculations of the water pressure. The numerical solutions are shown in Figure 17.
We see that varying the composition of the soil can have a large impact on the water pressure and on the magnitude of the flow velocity.
We also investigate the influence of the maximum of the fluctuation $\sigma$. In Figure 18, the numerical solutions of the water pressure for various values of $\sigma$ are shown, these results are obtained with boundary conditions (4).

![Numerical solutions of the water pressure for different $\sigma$-values](image1.png)

(a) Pressure with $\sigma = 0.2k_0$  
(b) Pressure with $\sigma = 0.5k_0$  
(c) Pressure with $\sigma = 0.8k_0$

Figure 18: Numerical solutions of the pressure for different $\sigma$-values

These pressure profiles are obtained by averaging between five different results with five different outcomes of the random number generator. Analogous, we determine the magnitude of the velocity vector using boundary conditions (5). The numerical solutions of the velocity are shown in Figure 17.
From these results it can be concluded that the value of the maximum of the fluctuation $\sigma$ also has a great influence on the water pressure and on the magnitude of the flow velocity.

From these results we conclude that the composition of the soil can have a great influence on the water pressure and on the flow velocity. Unfortunately, this composition is difficult to determine and is strongly dependent on the location. In the remainder of this study, we therefore consider a porous medium whose permeability is assumed to be constant.
3 Varying permeability

A number of hypotheses on the permeability will be tested now and for each hypothesis the pressure and the velocity field is determined and compared to the already calculated standard model where the permeability is assumed to be constant.

The hypothesis that provides the lowest water pressure and the fastest flowing water is considered in further research.

3.1 Permeability oscillating in time

From literature (see [10]) is known that, especially in turbulent flow, large pressure variations occur in the porous medium. These variations will evoke some vibration frequencies and they may cause mechanical deformations in the porous medium. These mechanical deformations induce changes in the porosity of the porous medium, which will result into permeability changes. Because the porosity is not modeled in the current model, we vary the permeability in order to get an idea of the occurring pressure variations.

We note that the suspected permeability variations are merely phenomenological and that the changes do not necessarily reflect physical reality. The variations have been imposed to get some feeling of the impact of directional variations. The aim is to determine whether the available experimental (though limited in both quantity and reliability) can be explained by the imposing of spatial, temporal and directional variations of the permeability.

Let the permeability depend on the position coordinates \( r = (r, z) \) and the time \( t \) in seconds, so that we can simulate the behavior of the resulting vibrations. We assume that the porous medium at rest has an average permeability equal to \( k_0 \). As a result of the force of gravity and the upper layers, the porosity of the soil at rest will be minimal. The resulting vibrations by injecting will increase the porosity, so that the permeability also increases to a maximum \( k \)-value. In this case, it is chosen for a maximum \( k \)-value equal to \( 3k_0 \). Thus we use the following formula voor the permeability \( k \) oscillating in time

\[
k(r, z, t) = k_0 + k_0 e^{-\lambda(r^2 +(z-z_m)^2)}[1 - \cos(\omega t)], \quad r \in \Omega, \quad 0 \leq t,
\]

(19)

where:
- \( k_0 \) is the average permeability of the porous medium in \( m^2 \),
- \( \lambda \) is a damping factor of the mechanical deformations in \( m^{-2} \),
- \( z_m \) is the center of the filter in vertical direction in \( m \),
- \( \omega_t \) is the frequency of the oscillating permeability in time in \( s^{-1} \).
In Figure 20 is a 3D plot shown of the oscillating permeability at various times. The following values are used in formula (19)

\[ k_0 = 2.3148 \cdot 10^{-4} \, m^2, \lambda = 0.001 \, m^{-2}, z_m = 8.5 \, m \text{ and } \omega_t = 2\pi \, s^{-1}. \]

![Figure 20: Permeability at consecutive times as a function of the position (r, z)](image)

For the problem with water injection and varied permeability some 3D plots of the numerical solution at consecutive times are shown in Figure 21.
Figure 21: Numerical solutions for the water pressure at consecutive times

In this figure only the first five meters away from the filter are displayed, at further distances the hydrostatic pressure is reached more or less (see Figure 4). However, the injected water also reaches a maximum horizontal distance of 46.5 meters in this model, with the chosen values for the parameters in formula (19).
We notice that the water pressure is low at a high permeability and high for small values of the permeability $k$. At an ‘injection point’ where the permeability is high, we expect that the water pressure is not as high as the pressure that is obtained in the case of injecting water elsewhere in the ground. In order to check whether the modeled oscillations actually make us lose more water through an ‘injection point’ than elsewhere in the soil, we determine the difference between the time average of the pressure at the oscillating permeability, and the pressure determined by a constant permeability $k_0$ (see Section 2.5).

In Figure 22 is a 3D plot shown of the pressure difference $p_{\text{diff}}$ where

$$p_{\text{diff}}(r, z) = p_{\text{constant, } k}(r, z) - \frac{1}{T} \int_0^T p_{\text{oscillation}}(r, z, t) \, dt \approx$$

$$\approx p_{\text{constant, } k}(r, z) - \frac{1}{N} \sum_{i=1}^N p_{\text{oscillation}}(r, z, t_i), \quad (20)$$

with $0 = t_0 < t_1 < \ldots < t_N = T$, and $T = 1$ divided in $100$ time steps.

![Figure 22: The pressure difference in the first five meters around the filter](image)

In this figure it is clear to see that we get, with the modeled vibrations, a lower water pressure around the filter. Hence, we can deduce that we can lose more water in an ‘injection point’, than in other layers where these vibrations do not occur.

In addition to the water pressure, we can also determine the flow velocity by means of the Darcy equation, as described in Section 2.6. For this we use boundary conditions (5), so that we can best represent the influence of the oscillating permeability on the flow velocity. In Figure 23, some 3D plots can be seen of the numerical solution of the magnitude of the velocity vector at consecutive times.
Figure 23: Numerical solutions for the speed at consecutive times

For the clarity of this figures only the first five meters away from the filter are plotted. The time average of the numerical solution for the speed is shown in Figure 24.
Figure 24: The time average of the numerical solution for the speed

In comparison to the speed displayed in Figure 15, we see that the occurrence of vibrations yields higher flow velocities. Hence, we get a better understanding of the behavior of the flowing water in an ‘injection point’ if we consider a porous medium with permeability oscillating in time.
3.2 Permeability oscillating in time and space

We can also oscillate the permeability in time and space, to this extent we use the following formula for $r \in \Omega$ and $0 \leq t$

$$k(r, z, t) = k_0 + k_0 e^{-\lambda(r^2 + (z-z_m)^2)} [1 + \cos(\omega_p t) \cos(\omega_p \sqrt{r^2 + (z-z_m)^2})],$$

where:

$\omega_p$ is the frequency of the oscillating permeability in space in $m^{-1}$.

Note that the equilibrium position of this oscillation is the same as in formula (19), and is equal to $k_0 + k_0 e^{-\lambda(r^2 + (z-z_m)^2)}$. This oscillation varies also between the values $k_0$ and $3k_0$.

In Figure 25 some 3D plots are shown of the oscillating permeability at consecutive times. In these plots we chose for $\omega_p = 0.2\pi$ for illustrative reasons. However, in the calculations we use $\omega_p = \pi$, because in reality we expect high frequencies of the vibrations.

(a) Permeability at $t = 0$ s and $t = 1$ s  (b) Permeability at $t = 0.25$ s and $t = 0.75$ s

(c) Permeability at $t = 0.5$ s

Figure 25: Oscillating permeability in time and space at various times
For the problem with water injection and varied permeability in time and space, some 3D plots of the numerical solutions for the water pressure at consecutive times are shown in Figure 26.

![3D plots of water pressure](image)

(a) Solution at $t = 0$ s and $t = 1$ s  
(b) Solution at $t = 0.25$ s and $t = 0.75$ s  
(c) Solution at $t = 0.5$ s

Figure 26: Numerical solutions for the water pressure at various times

Also in this figure, only the first five meters in the $r$-direction are displayed, at further distances the hydrostatic pressure is reached more or less. The injected water reaches a maximum horizontal distance of 43 meters in this case.

In order to check whether we reach lower pressure values with this adaptation, we determine the difference between the time average of the pressure at the oscillating permeability, and the pressure determined by a constant permeability $k_0$. 
In Figure 27 a 3D plot of the pressure difference $p_{\text{diff}}(r, z)$, as defined in formula (20), is shown.

Figure 27: The pressure difference in the first five meters around the filter

Also now we get, with the modeled vibrations, a lower water pressure around the filter and we can deduce that we can lose more water in an ‘injection point’, than in other layers where these vibrations do not occur.

If we compare Figure 22 to Figure 27, then we observe that the average pressure in the case of a porous medium, with a permeability that is oscillating in the time and the space, is smaller in general than the average pressure in a porous medium with a permeability which is oscillating only in time. Although, water penetrates into the soil over a larger distance in the second case.

We determine also the flow velocity using boundary conditions (5). In Figure 28, some 3D plots can be seen of the numerical solution of the magnitude of the velocity vector at consecutive times.
Figure 28: Numerical solutions for the speed at consecutive times

For the clarity of this figures only the first five meters away from the filter are plotted. The time average of the numerical solution for the speed is shown in Figure 29.
In comparison to the speed displayed in Figure 15, we see that the occurrence of vibrations yields higher flow velocities.
3.3 Permeability depending on the velocity

Another way to incorporate the effect of anisotropy into the permeability term is to look at the effect of the velocity of the flowing water on the permeability. In this case, the permeability in the horizontal direction depends on the magnitude and direction of the flow and increases for higher horizontal flow velocities. This will hinder the vertically directed flow so that the injected water will be allowed to penetrate into the soil over a larger distance.

Hence the permeability is chosen to be dependent on the direction of the coordinates \((r, z)\) and on the velocity vector, therefore, it follows for the Darcy equation

\[
v = -\frac{1}{\mu} k(v_r) \circ \left[ \nabla p + \rho g e_z \right], \quad \text{for } (x, y, z) \in \Omega,
\]

where the symbol \(\circ\) indicates the Hadamard product or simply the entrywise multiplication of vectors. Thus we have to solve the following system

\[
\begin{align*}
\left\{ \begin{array}{l}
v = -\frac{1}{\mu} k(v_r) \circ \left[ \nabla p + \rho g e_z \right], \quad \text{for } (x, y, z) \in \Omega; \\
\nabla \cdot v = 0, & \quad \text{for } (x, y, z) \in \Omega,
\end{array} \right.
\end{align*}
\]

(22)

with the boundary conditions as given in (4).

The weak formulation of this problem, with boundary conditions (4), states (for the derivation see Section 2.1.1):

Find \(p \in \Sigma\) such that

\[
\begin{align*}
\int_{\Omega} \frac{1}{\mu} \left[ k(v_r) \circ \nabla p \right] \cdot \nabla \varphi \, r \, d\Omega_{rz} &= -\int_{\Omega} \frac{1}{\mu} \left[ k(v_r) \circ \rho g e_z \right] \cdot \nabla \varphi \, r \, d\Omega_{rz} + \\
&+ \int_{\Gamma} \frac{Q_{in}}{2\pi l(\Gamma_5)} \varphi \, d\Gamma_{rz},
\end{align*}
\]

\(\forall \varphi \in \Sigma_0,\) with \(\Sigma\) and \(\Sigma_0\) as defined in Section 2.1.1.

3.3.1 Method of finite elements applied to the pressure

First, we apply the method of finite elements to determine the element matrix and element vector in the prescribed injection velocity case. For \(p\) we still use the following approximation as defined in Section 2.2.1

\[
p(r, z) \approx p_n(r, z) = \sum_{j=1}^{n} p_j \varphi_j(r, z) + \sum_{j=n+1}^{n+n_{e,x}} p_{Bj} \varphi_j(r, z).
\]
From equation (23) it follows for $i = 1, \ldots, n$

$$\sum_{j=1}^{n} p_j \int_{\Omega} \frac{1}{\mu} (k(v_r) \circ \nabla \varphi_j) \cdot \nabla \varphi_i \, r \, d\Omega_{rz} =$$

$$= - \sum_{j=n+1}^{n+n_{ess}} p_{Bj} \int_{\Omega} \frac{1}{\mu} (k(v_r) \circ \nabla \varphi_j) \cdot \nabla \varphi_i \, r \, d\Omega_{rz} -$$

$$- \int_{\Omega} \frac{1}{\mu} k_z(v_r)(\rho g) \frac{\partial \varphi_i}{\partial z} \, r \, d\Omega_{rz} + \int_{\Gamma_5} \frac{Q_{in}}{2\pi \ell(\Gamma_5)} \varphi_i \, d\Gamma_{rz}.$$  

The resulting system of differential equations is not linear, because $v_r$ depends on $p$ according to the Darcy equation. Thus a fixed point iteration method is to be used to determine the solution. In this project we chose Picard’s method, then the method of finite elements is applied to determine the numerical solution.

Here it is assumed that an approximation $p^l(r, z)$ of the solution is known, from this value of the pressure an approximation of the speed $v^l_r(r, z)$ is determined using the Darcy equation. Subsequently, a better approximation $p^{l+1}(r, z)$ is sought by determining the values for $p^{l+1}_j$, with $j \in \{1, \ldots, n\}$, such that

$$\sum_{j=1}^{n} p^{l+1}_j \int_{\Omega} \frac{1}{\mu} (k(v^l_r) \circ \nabla \varphi_j) \cdot \nabla \varphi_i \, r \, d\Omega_{rz} =$$

$$= - \sum_{j=n+1}^{n+n_{ess}} p_{Bj} \int_{\Omega} \frac{1}{\mu} (k(v^l_r) \circ \nabla \varphi_j) \cdot \nabla \varphi_i \, r \, d\Omega_{rz} -$$

$$- \int_{\Omega} \frac{1}{\mu} k_z(v^l_r)(\rho g) \frac{\partial \varphi_i}{\partial z} \, r \, d\Omega_{rz} + \int_{\Gamma_5} \frac{Q_{in}}{2\pi \ell(\Gamma_5)} \varphi_i \, d\Gamma_{rz}.$$  

Let

$$F_{ij}(v^l_r) = \int_{\Omega} \frac{1}{\mu} (k(v^l_r) \circ \nabla \varphi_j) \cdot \nabla \varphi_i \, r \, d\Omega_{rz},$$

and

$$c_i(v^l_r) = - \sum_{j=n+1}^{n+n_{ess}} F_{ij}(v^l_r)p_{Bj} - \int_{\Omega} \frac{1}{\mu} k_z(v^l_r)(\rho g) \frac{\partial \varphi_i}{\partial z} \, r \, d\Omega_{rz} + \int_{\Gamma_5} \frac{Q_{in}}{2\pi \ell(\Gamma_5)} \varphi_i \, d\Gamma_{rz}.$$  

Then we get in matrix-vector notation: $\sum_{j=1}^{n} F_{ij}(v^l_r)p^{l+1}_j = c_i(v^l_r).$
Thus, the element vector for an internal element is given by

\[ \mathbf{c}_i(v_r^i) = \sum_{l=1}^{n_{el}} c_i^e(v_r^i) + \sum_{l=1}^{n_{bci}} c_i^{bc}(v_r^i), \]

where

\[ F_{ij}^e(v_r^i) = \int_{e_i} \frac{1}{\mu} (\mathbf{k}(v_r^i) \cdot \nabla \varphi_j) \cdot \nabla \varphi_i \, d\Omega_{rz} = \int_{e_i} \frac{1}{\mu} [k_r(v_r^i) \beta_j \beta_i + k_z(v_r^i) \gamma_j \gamma_i] \, d\Omega_{rz} \approx \frac{1}{\mu} |\Delta| \frac{3}{6} \sum_{p=1}^{3} [\beta_j \beta_k v_r^i(r_p, z_p) + \gamma_j \gamma_k v_r^i(r_p, z_p)] \, r_p. \]

Thus, the element matrix is given by

\[ F^e(v_r^i) = \frac{1}{\mu} |\Delta| \left[ \sum_{p=1}^{3} \Delta(r_p, z_p) r_p \left( \begin{array}{ccc} \beta_1^2 & \beta_1 \beta_2 & \beta_1 \beta_3 \\ \beta_1 \beta_2 & \beta_2^2 & \beta_2 \beta_3 \\ \beta_1 \beta_3 & \beta_2 \beta_3 & \beta_3^2 \end{array} \right) + \right. \]

\[ + \left. \sum_{p=1}^{3} k_z(v_r^i(r_p, z_p)) r_p \left( \begin{array}{ccc} \gamma_1^2 & \gamma_1 \gamma_2 & \gamma_1 \gamma_3 \\ \gamma_1 \gamma_2 & \gamma_2^2 & \gamma_2 \gamma_3 \\ \gamma_1 \gamma_3 & \gamma_2 \gamma_3 & \gamma_3^2 \end{array} \right) \right]. \]

Furthermore, we have

\[ c_i(v_r^i) = \sum_{l=1}^{n_{el}} c_i^e(v_r^i) + \sum_{l=1}^{n_{bci}} c_i^{bc}(v_r^i), \]

with

\[ c_i^e(v_r^i) = - \sum_{j=n+1}^{n+n_{el}} F_{ij}^e(v_r^i) p_{Bj} - \int_{e_i} \frac{1}{\mu} k_z(v_r^i) \rho g \frac{\partial \varphi_i}{\partial z} \, d\Omega_{rz} \approx \sum_{j=n+1}^{n+n_{el}} F_{ij}^e(v_r^i) p_{Bj} - \frac{1}{\mu} \rho g \gamma_i \sum_{p=1}^{3} k_z(v_r^i(r_p, z_p)) r_p, \]

and

\[ c_i^{bc} = \int_{\Gamma_{bc,el}} \frac{Q_m}{2\pi l(\Gamma_5)} \varphi_i \, d\Gamma_{rz} \approx \frac{||\mathbf{r}_2 - \mathbf{r}_1^e||}{2 \pi l(\Gamma_5)} Q_m. \]

Thus, the element vector for an internal element is given by

\[ \mathbf{c}_i(v_r^i) = - \sum_{j=n+1}^{n+n_{el}} \left( \begin{array}{c} F_{1j}^e(v_r^i) \\ F_{2j}^e(v_r^i) \\ F_{3j}^e(v_r^i) \end{array} \right) p_{Bj} - \frac{1}{\mu} \rho g \sum_{p=1}^{3} k_z(v_r^i(r_p, z_p)) r_p \left( \begin{array}{c} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{array} \right). \]
For a boundary element we get

$$c^{bc} = \frac{||r_2^{bc} - r_1^{bc}||}{2} \frac{Q_{in}}{2\pi l(\Gamma)} \left( \begin{array}{c} 1 \\ 1 \end{array} \right).$$

As an initial estimate $k(v^0_r)$, we take the constant value of $k_0$, and we assume that the permeability is everywhere equal in $\Omega$. With this initial estimate $k(v^0_r)$, we determine the initial approximation of the pressure $p^0(r, z)$ (see Figure 5). Using the water pressure, we then determine a first approximation of the velocity vector $v^0$ according to the Darcy equation.

### 3.3.2 Relationship between the permeability and the velocity

To solve the problem for the approximation $p^{l+1}(r, z)$, we must first determine the relationship between the permeability and the horizontally directed velocity.

We assume that the permeability in the horizontal direction depends on the magnitude of the horizontal component of the flow and increases by higher flow velocities. Hence we assume that the permeability in both directions is constant and equal to $k_0$ for low speeds, whereas if the velocity in the horizontal direction increases, the permeability in the horizontal direction increases to a maximum $k_r$-value, after which the permeability becomes constant. We take this maximum $k_r$-value equal to $3k_0$.

In Figure 30, the relationship between the permeability in the horizontal direction and the horizontally directed velocity is displayed.

![Figure 30: Relationship between horizontally directed permeability and velocity](image-url)
The changes in the permeability in the horizontal direction will hinder the vertically directed flow. This can be explained by the permeability in the vertical direction that decreases with increasing horizontally directed velocity, to a minimal small value near zero. In this project we chose for a minimal $k_z$-value equal to $0.2k_0$.

![Figure 31: Relationship between vertically directed permeability and velocity](image)

We determine the values of $\hat{v}_r$ and $v_r^*$ with the aid of the simplified Darcy equation in the horizontal direction, which states

$$v_r = \frac{k_r \rho g H \mu}{R},$$

where $H$ and $R$ are the height and the radius of our area respectively (see Figure 1). Thus we get

$$\hat{v}_r = \frac{k_0 \rho g H}{\mu R}$$ and $$v_r^* = \frac{3k_0 \rho g H}{\mu R}.$$ 

3.3.3 Method of finite elements applied to the velocity

In order to be able to use the relationship between the permeability and the velocity described in Section 3.3.2, we need to know the values of the speed $v_r^j(r, z)$, assuming that the approximation $p^j(r, z)$ is known.

First, we determine the velocity vector $\mathbf{v}^j$ from equation (21). Hence for the vector components of $\mathbf{v}^j$ we have to solve the equations

$$v_r^j = -\frac{k_r (v_r^{j-1})}{\mu} \frac{\partial p^j}{\partial r}, \quad (24)$$

$$v_z^j = -\frac{k_z (v_z^{j-1})}{\mu} \left( \frac{\partial p^j}{\partial z} + \rho g \right). \quad (25)$$

The weak formulation for solving the problem for component $v_r^j$ states (for derivation see Section 2.6):

Find $v_r^j \in H^1(\Omega)$ such that

$$\int_{\Omega} v_r^j \varphi \ r \ d\Omega_{rz} = -\int_{\Omega} \frac{k_r (v_r^{j-1})}{\mu} \frac{\partial p^j}{\partial r} \ \varphi \ r \ d\Omega_{rz}, \forall \varphi \in H^1(\Omega).$$
It appears that we obtain the same equation as equation (17), except that the permeability is now dependent on the speed in the horizontal direction \( v_x \). Using the approximation

\[
v_x(r, z) \approx v_x^{n}(r, z) = \sum_{j=1}^{n+n_{ess}} v_x^{j} \varphi_j(r, z),
\]

we can write the problem in matrix-vector notation as:

\[
\sum_{j=1}^{n+n_{ess}} \tilde{M}_{ij} v_x^{j} = \tilde{b}_i.
\]

For the element matrix it follows that

\[
\tilde{M}_{ij} = \frac{\mid \Delta \mid}{6} \begin{pmatrix}
    r_1 & 0 & 0 \\
    0 & r_2 & 0 \\
    0 & 0 & r_3
\end{pmatrix},
\]

analogous to what we derived in Section 2.6. For an internal element, the element vector is given by

\[
\tilde{\psi}^i = \frac{1}{\mu} \left( \sum_{m=1}^{3} p_m^l \beta_m \right) \frac{\mid \Delta \mid}{6} \begin{pmatrix}
    k_x(v^{l-1}_x(r_1, z_1))r_1 \\
    k_x(v^{l-1}_x(r_2, z_2))r_2 \\
    k_x(v^{l-1}_x(r_3, z_3))r_3
\end{pmatrix}.
\]

The weak formulation for component \( v_z \) states:

Find \( v_z^l \in H^1(\Omega) \) such that

\[
\int_{\Omega} v_z^l \varphi \ r \ d\Omega_{xz} = - \int_{\Omega} \frac{k_x(v^{l-1}_x)}{\mu} \left( \frac{\partial p^l}{\partial z} + \rho g \right) \varphi \ r \ d\Omega_{xz}, \forall \varphi \in H^1(\Omega).
\]

To solve this problem, we use the approximation

\[
v_z^l(r, z) \approx v_z^{n}(r, z) = \sum_{j=1}^{n+n_{ess}} v_z^{j} \varphi_j(r, z).
\]

Hence we get the system:

\[
\sum_{j=1}^{n+n_{ess}} \tilde{M}_{ij} v_z^{j} = \tilde{\psi}_i. \quad \text{In which the matrix } \tilde{M} \text{ is known from the previous problem.}
\]

The element vector for an internal element is given by

\[
\tilde{\psi}^i = \frac{1}{\mu} \left( \sum_{m=1}^{3} p_m^l \gamma_m + \rho g \right) \frac{\mid \Delta \mid}{6} \begin{pmatrix}
    k_x(v^{l-1}_x(r_1, z_1))r_1 \\
    k_x(v^{l-1}_x(r_2, z_2))r_2 \\
    k_x(v^{l-1}_x(r_3, z_3))r_3
\end{pmatrix}.
\]
3.3.4 Convergence

For convergence of Picard’s iteration scheme, we use the $H^1$-norm of the water pressure. Let $e^l = \Phi^{l+1} - \Phi^l$, then it follows

$$||e^l||_{H^1} = \left( \int_{\Omega} \left[ (e^l)^2 + \nabla e^l \cdot \nabla e^l \right] d\Omega \right)^{\frac{1}{2}} =$$

$$= \left( \int_{\Omega} \left[ (e^l)^2 + \left( \frac{\partial e^l}{\partial r} \right)^2 + \left( \frac{\partial e^l}{\partial z} \right)^2 \right] r d\Omega \right)^{\frac{1}{2}} =$$

$$= \left( \sum_{e_k} \int_{e_k} \left[ (e^l)^2 + \left( \frac{\partial e^l}{\partial r} \right)^2 + \left( \frac{\partial e^l}{\partial z} \right)^2 \right] r d\Omega \right)^{\frac{1}{2}}.$$

Furthermore, on an element $e_k$, we get

$$e^l = \Phi^{l+1} - \Phi^l \approx \sum_{m=1}^{3} (p^{l+1}_m - p^l_m) \varphi_m(r,z);$$

$$\frac{\partial e^l}{\partial r} = \frac{\partial (\Phi^{l+1} - \Phi^l)}{\partial r} = \frac{\partial}{\partial r} \left( \sum_{m=1}^{3} (p^{l+1}_m - p^l_m) \varphi_m(r,z) \right) \approx \sum_{m=1}^{3} (p^{l+1}_m - p^l_m) \beta_m;$$

$$\frac{\partial e^l}{\partial z} = \frac{\partial (\Phi^{l+1} - \Phi^l)}{\partial z} = \frac{\partial}{\partial z} \left( \sum_{m=1}^{3} (p^{l+1}_m - p^l_m) \varphi_m(r,z) \right) \approx \sum_{m=1}^{3} (p^{l+1}_m - p^l_m) \gamma_m.$$

Hence, we have

$$||e^l||_{H^1} \approx \left( \sum_{k=1}^{n_k} \left| \Delta i \right| \sum_{j=1}^{3} r_j \left[ (e^l)^2 + \left( \sum_{m=1}^{3} e^l_m \beta_m \right)^2 + \left( \sum_{m=1}^{3} e^l_m \gamma_m \right)^2 \right] \right)^{\frac{1}{2}}.$$

3.3.5 Numerical solution of the problem with water injection

We will now determine the numerical solution of the problem with water injection, where 50 liters of water per second are injected into the ground ($Q_{in} = 0.050 \text{ m}^3/\text{s}$). The numerical solution of the water pressure, after 44 iterations, is shown in Figure 32 as a 3D plot.
If we compare the pressure in this figure to the pressure in Figure 5, we see that the pressure values in the case of a porous medium having a constant permeability is higher than the pressure of the water in a porous medium with a permeability depending on the velocity.

The convergence plot of Picard’s iteration scheme can be seen in Figure 33, the convergence limit was set equal to one.
For the velocity vector $v$ we get after 44 iterations the numerical solution shown as a vector plot in Figure 34.

![Vector plot showing the numerical solution of the velocity vector $v$.](image)

**Figure 34: Numerical solution of the velocity vector**

In this vector plot are the arrows, indicating the direction of the velocity vector $v$, scaled in size with a maximum speed of 2.07 cm/s. From the vector plot, it can be seen that the vertically directed flow is indeed prevented, causing the water to predominantly flow horizontally. However, also in this case, the injected water penetrates up to a maximum distance of 46.5 meters away from the filter.
4 Biots Theory

We know that sand / gravel layers (aquifers) are not rigid, but constitute more or less an elastic matrix. To be able to determine the local deviation of the porous medium, we will use Biots Theory for poro-elasticity, where flow in porous media is combined with mechanical deformations of the aquifer in which water is injected.

Here insight in the flow patterns can be obtained by combining elasticity and poro-elasticity of the porous media in the models with Darcy’s formalism.

In this model, we have to deal with the following equations for the mechanical changes and the fluid flow respectively (see [7] and [9]), using $\mathbf{x} = (x, y, z)$ for a generic point in $\Omega$

\[
\begin{align*}
-\mu_L \Delta \mathbf{u} - (\lambda_L + \mu_L) \nabla (\nabla \cdot \mathbf{u}) + \nabla p &= \mathbf{0}, \quad \text{for } \mathbf{x} \in \Omega; \quad (26) \\
\frac{\partial}{\partial t} (n_f \beta p + \nabla \cdot \mathbf{u}) + \nabla \cdot \mathbf{v} &= f(\mathbf{x}, t), \quad \text{for } \mathbf{x} \in \Omega, t > 0, \quad (27)
\end{align*}
\]

where:

- $\mathbf{u}$ is the displacement vector in $m$,
- $\mu_L$ and $\lambda_L$ are the Lamé coefficients in $Pa$,
- $p$ is the pressure of water in $Pa$,
- $n_f$ is the porosity of the porous medium,
- $\beta$ is the compressibility coefficient of water in $Pa^{-1}$,
- $f(\mathbf{x}, t)$ is a source term,

and

\[
\tilde{\Delta} = \begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix}.
\]

First we consider only equation (27), where we assume that $\mathbf{u}$ is a given displacement vector. With the Darcy equation, we get the following system

\[
\begin{align*}
\frac{\partial}{\partial t} (n_f \beta p + \nabla \cdot \mathbf{u}) + \nabla \cdot \mathbf{v} &= f(\mathbf{x}, t), \quad \text{for } \mathbf{x} \in \Omega, t > 0; \\
\mathbf{v} &= -\frac{k}{\mu} [\nabla p + \rho g \mathbf{e}_z], \quad \text{for } \mathbf{x} \in \Omega,
\end{align*}
\]

(28)

with the boundary conditions as given in (4) and (5).

First of all, we assume that water is incompressible, hence $\beta = 0$. Furthermore, we assume that our source term is on boundary $\Gamma_5$, hence we have no source term in the inner region $\Omega$. Further, from this, it follows

\[
f(\mathbf{x}, t) = 0, \quad \forall \mathbf{x} \in \Omega \quad \text{and} \quad \forall t > 0.
\]
Hence we get the following equation
\[
\frac{\partial}{\partial t} (\nabla \cdot \mathbf{u}) + \nabla \cdot \mathbf{v} = 0, \text{ for } \mathbf{x} \in \Omega \text{ and } t > 0, \quad (29)
\]
where we assume that the displacement vector \( \mathbf{u} \) is known, and so the term \( \frac{\partial}{\partial t} (\nabla \cdot \mathbf{u}) \) is also known.

Note that we get back equation (3), if the displacement vector \( \mathbf{u} \) is a constant. In this model, we assume that the displacement is not constant, from which follows that the term \( \frac{\partial}{\partial t} (\nabla \cdot \mathbf{u}) \) is not equal to zero. Hence we get \( \nabla \cdot \mathbf{v} \neq 0 \).

This is fortunately not in conflict with the incompressibility of water, since we have mechanical deformations that provide volume changes of the pores through which water flows.
4.1 Solving Biot’s model with known displacement

4.1.1 The prescribed injection velocity case

First we determine the weak formulation for equation (29) with boundary conditions (4). For this we multiply the equation by $\varphi \in \Sigma_0$ (as defined in Section 2.1.1) and integrate it over $\Omega$ to get

$$
\int_{\Omega} (\nabla \cdot v) \, \varphi \, d\Omega = - \int_{\Omega} \frac{\partial}{\partial t} (\nabla \cdot u) \, \varphi \, d\Omega.
$$

After applying the Divergence Theorem of Gauss, we get

$$
\int_{\partial\Omega} v \cdot n \, \varphi \, d\Gamma - \int_{\Omega} v \cdot \nabla \varphi \, d\Omega = - \int_{\Omega} \frac{\partial}{\partial t} (\nabla \cdot u) \, \varphi \, d\Omega. \quad (30)
$$

We still have Dirichlet boundary conditions on the boundaries $\Gamma_1$ and $\Gamma_2$, and no flow ($v \cdot n = 0$) through the boundaries $\Gamma_3$ and $\Gamma_4$.

On boundary $\Gamma_5$ we have $v \cdot n = -\frac{Q_{in}}{2\pi r l(\Gamma_5)}$, from which it follows

$$
- \int_{\Omega} v \cdot \nabla \varphi \, d\Omega = \int_{\Gamma_5} \frac{Q_{in}}{2\pi r l(\Gamma_5)} \, \varphi \, d\Gamma - \int_{\Omega} \frac{\partial}{\partial t} (\nabla \cdot u) \, \varphi \, d\Omega.
$$

After transformation to cylinder coordinates $(r, z)$ and substituting, the Darcy equation gives

$$
\int_{\Omega} \frac{k}{\mu} \nabla p \cdot \nabla r \, d\Omega_{rz} = - \int_{\Omega} \frac{k}{\mu} \rho g \frac{\partial \varphi}{\partial z} \, r \, d\Omega_{rz} + \int_{\Gamma_5} \frac{Q_{in}}{2\pi r l(\Gamma_5)} \, \varphi \, d\Gamma_{rz} - \int_{\Omega} \frac{\partial}{\partial t} (\nabla \cdot u) \, \varphi \, d\Omega_{rz}. \quad (31)
$$

So the weak formulation of this problem states:

Find $p \in \Sigma$ such that

$$
\int_{\Omega} \frac{k}{\mu} \nabla p \cdot \nabla r \, d\Omega_{rz} = - \int_{\Omega} \frac{k}{\mu} \rho g \frac{\partial \varphi}{\partial z} \, r \, d\Omega_{rz} + \int_{\Gamma_5} \frac{Q_{in}}{2\pi r l(\Gamma_5)} \, \varphi \, d\Gamma_{rz} - \int_{\Omega} \frac{\partial}{\partial t} (\nabla \cdot u) \, \varphi \, d\Omega_{rz}, \forall \varphi \in \Sigma_0,
$$

where

$$
\Sigma_0 = \{ \varphi \in H^1(\Omega) : \varphi = 0 \text{ on } \Gamma_1 \cup \Gamma_2 \},
$$

and

$$
\Sigma = \{ p \in H^1(\Omega) : p|_{\Gamma_1} = 0 \text{ and } p|_{\Gamma_2} = \rho g(H - z) \}.
$$
In order to solve this problem, we use the mesh with triangular elements (see Figure 3) and piecewise linear basis functions. We apply the method of finite elements to determine the element matrix and element vector. For the pressure $p$ we still use the following approximation as defined in Section 2.2.1

$$p(r, z) \approx p_n(r, z) = \sum_{j=1}^{n} p_j \varphi_j(r, z) + \sum_{j=n+1}^{n+n_{ess}} p_{Bj} \varphi_j(r, z).$$

From equation (31) it follows for $i = 1, \ldots, n$

$$\int_{\Omega} k \frac{\nabla}{\mu} \left[ \sum_{j=1}^{n} p_j \varphi_j(r, z) + \sum_{j=n+1}^{n+n_{ess}} p_{Bj} \varphi_j(r, z) \right] \cdot \nabla \varphi_i \ r \ d\Omega_{rz} =$$

$$= - \int_{\Gamma_5} \frac{k}{\mu} \rho g \frac{\partial \varphi_i}{\partial z} \ r \ d\Omega_{rz} + \int_{\Gamma_5} \frac{Q_{in}}{2\pi l(\Gamma_5)} \varphi_i \ d\Gamma_{rz} -$$

$$- \int_{\Omega} \frac{\partial}{\partial t} (\nabla \cdot \mathbf{u}) \varphi_i \ r \ d\Omega_{rz}. $$

This gives for $i = 1, \ldots, n$

$$\sum_{j=1}^{n} p_j \int_{\Omega} k \frac{\nabla}{\mu} \varphi_j \cdot \nabla \varphi_i \ r \ d\Omega_{rz} = - \left( \sum_{j=n+1}^{n+n_{ess}} p_{Bj} \right) \int_{\Omega} k \frac{\nabla}{\mu} \varphi_j + \nabla \varphi_i \ r \ d\Omega_{rz} -$$

$$- \int_{\Omega} k \frac{\rho g}{\mu} \frac{\partial \varphi_i}{\partial z} \ r \ d\Omega_{rz} +$$

$$+ \int_{\Gamma_5} \frac{Q_{in}}{2\pi l(\Gamma_5)} \varphi_i \ d\Gamma_{rz} -$$

$$- \int_{\Omega} \frac{\partial}{\partial t} (\nabla \cdot \mathbf{u}) \varphi_i \ r \ d\Omega_{rz}.$$

Let $L_{ij} = \int_{\Omega} k \frac{\nabla}{\mu} \varphi_j \cdot \nabla \varphi_i \ r \ d\Omega_{rz}$, and

$$w_i = - \left( \sum_{j=n+1}^{n+n_{ess}} L_{ij} p_{Bj} \right) \int_{\Omega} k \frac{\rho g}{\mu} \frac{\partial \varphi_i}{\partial z} \ r \ d\Omega_{rz} + \int_{\Gamma_5} \frac{Q_{in}}{2\pi l(\Gamma_5)} \varphi_i \ d\Gamma_{rz} -$$

$$- \int_{\Omega} \frac{\partial}{\partial t} (\nabla \cdot \mathbf{u}) \varphi_i \ r \ d\Omega_{rz}.$$

Then we can write the system in matrix-vector notation as

$$\sum_{j=1}^{n} L_{ij} p_j = w_i \text{ or } Lp = w.$$
Note that we get $L = S$, with the matrix $S$ as defined in Section 2.2.1. Hence for the element matrix we have

$$L_{ei} = \frac{1}{\mu} \frac{\Delta}{6} \sum_{p=1}^{3} k(r_p, z_p) r_p \left( \begin{array}{ccc} \beta_1^2 + \gamma_1^2 & \beta_1 \beta_2 + \gamma_1 \gamma_2 & \beta_1 \beta_3 + \gamma_1 \gamma_3 \\ \beta_1 \beta_2 + \gamma_1 \gamma_2 & \beta_2^2 + \gamma_2^2 & \beta_2 \beta_3 + \gamma_2 \gamma_3 \\ \beta_1 \beta_3 + \gamma_1 \gamma_3 & \beta_2 \beta_3 + \gamma_2 \gamma_3 & \beta_3^2 + \gamma_3^2 \end{array} \right).$$

Furthermore, we have $w_i = \sum_{l=1}^{n_{el}} w_{ei} + \sum_{l=1}^{n_{bel}} w_{be}^i$, with

$$w_{ei} = - \sum_{j=n+1}^{n+n_{ex}} L_{ij} p_{B_j} - \int_{\Omega} \frac{k}{\mu} \frac{\partial \varphi_i}{\partial z} r d\Omega_{rz} - \int_{\Omega} \frac{\partial}{\partial t} (\nabla \cdot u) \varphi_i \ r d\Omega_{rz},$$

and

$$w_{be}^i = \int_{\Gamma_{5,be}} \frac{Q_{in}}{2\pi l(\Gamma_5)} \varphi_i \ d\Gamma_{rz}.$$

Thus, the element vector for an internal element is given by

$$w_{ei} = - \frac{1}{\mu} \frac{\Delta}{6} \sum_{p=1}^{3} k(r_p, z_p) r_p \sum_{j=n+1}^{n+n_{ex}} \left( \beta_j \left( \begin{array}{c} \beta_1 \\ \beta_2 \\ \beta_3 \end{array} \right) + \gamma_j \left( \begin{array}{c} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{array} \right) \right) p_{B_j} - \frac{\rho g}{\mu} \sum_{p=1}^{3} k(r_p, z_p) r_p \frac{\Delta}{6} \left( \begin{array}{c} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{array} \right) - \frac{|\Delta|}{6} \left( \frac{\partial (\nabla \cdot u)}{\partial t} \right) r_i.$$

For a boundary element, we get

$$w_{be} = \frac{|r_{i\Gamma_5}^b - r_{i\Gamma_5}^s|}{2} \frac{Q_{in}}{2\pi l(\Gamma_5)} \left( \begin{array}{c} 1 \\ 1 \end{array} \right).$$

### 4.1.2 The imposed pressure case

According to equation (30) and boundary conditions (5), we get for the problem with imposed pressure

$$- \int_{\Omega} \mathbf{v} \cdot \nabla \varphi \ d\Omega = - \int_{\Omega} \frac{\partial}{\partial t} (\nabla \cdot u) \varphi \ d\Omega,$$

where $\varphi \in \tilde{\Sigma}_0 = \{ \varphi \in H^1(\Omega) : \varphi = 0 \text{ on } \Gamma_1 \cup \Gamma_2 \cup \Gamma_5 \}$.

After transformation to cylinder coordinates $(r, z)$ and substituting, the Darcy equation gives

$$\int_{\Omega} \frac{k}{\mu} \nabla p \cdot \nabla \varphi \ r d\Omega_{rz} = - \int_{\Omega} \frac{k}{\mu} \rho g \frac{\partial \varphi}{\partial z} r d\Omega_{rz} - \int_{\Omega} \frac{\partial}{\partial t} (\nabla \cdot u) \varphi \ r d\Omega_{rz}. \quad (32)$$
So the weak formulation of this problem states:

Find \( p \in \tilde{\Sigma} \) such that

\[
\int_\Omega \frac{k}{\mu} \nabla p \cdot \nabla \varphi \: r \: d\Omega_{rz} = - \int_\Omega \frac{k}{\mu} \rho g \frac{\partial \varphi}{\partial z} \: r \: d\Omega_{rz} - \int_\Omega \frac{\partial}{\partial t} (\nabla \cdot u) \varphi \: r \: d\Omega_{rz}, \forall \varphi \in \tilde{\Sigma}_0,
\]

where

\[
\tilde{\Sigma} = \{ p \in H^1(\Omega) : p|_{\Gamma_1} = 0, p|_{\Gamma_2} = \rho g (H - z) \text{ and } p|_{\Gamma_3} = \rho g (H - z) + p_{\text{pump}} \}.
\]

Even for this problem we use the approximation

\[
p(r, z) \approx p_n(r, z) = \sum_{j=1}^{n} p_j \varphi_j(r, z) + \sum_{j=n+1}^{n+n_{\text{ess}}} p_{Bj} \varphi_j(r, z).
\]

From which it follows for \( i = 1, \ldots, n \)

\[
\sum_{j=1}^{n} p_j \int_\Omega \frac{k}{\mu} \nabla \varphi_j \cdot \nabla \varphi_i \: r \: d\Omega_{rz} = - \sum_{j=n+1}^{n+n_{\text{ess}}} p_{Bj} \int_\Omega \frac{k}{\mu} \nabla \varphi_j \cdot \nabla \varphi_i \: r \: d\Omega_{rz} - \int_\Omega \frac{k}{\mu} \rho g \frac{\partial \varphi_i}{\partial z} \: r \: d\Omega_{rz} - \int_\Omega \frac{\partial}{\partial t} (\nabla \cdot u) \varphi_i \: r \: d\Omega_{rz}.
\]

For the element matrix we get matrix \( L \) as defined in Section 4.1.1. Let

\[
\chi_i = - \sum_{j=n+1}^{n+n_{\text{ess}}} L_{ij} p_{Bj} - \int_\Omega \frac{k}{\mu} \rho g \frac{\partial \varphi_i}{\partial z} \: r \: d\Omega_{rz} - \int_\Omega \frac{\partial}{\partial t} (\nabla \cdot u) \varphi_i \: r \: d\Omega_{rz}.
\]

Then we can write the system in matrix-vector notation as

\[
\sum_{j=1}^{n} L_{ij} p_j = \chi_i \text{ or } Lp = \chi.
\]

For the element vector, we have \( \chi_i = \sum_{l=1}^{n_{\text{el}}} \chi_{i,l} \), with

\[
\chi_{i,l} = - \sum_{j=n+1}^{n+n_{\text{ess}}} L_{ij}^{el} p_{Bj} - \int_\Omega \frac{k}{\mu} \rho g \frac{\partial \varphi_i}{\partial z} \: r \: d\Omega_{rz} - \int_\Omega \frac{\partial}{\partial t} (\nabla \cdot u) \varphi_i \: r \: d\Omega_{rz}.
\]

Thus, the element vector for an internal element is given by

\[
\chi_{i,l} = - \frac{1}{\mu} \frac{\partial |\Delta|}{6} \sum_{p=1}^{3} k(r_p, z_p) r_p \sum_{j=n+1}^{n+n_{\text{ess}}} \left( \beta_j \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} + \gamma_j \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{pmatrix} \right) p_{Bj} - \frac{\rho g}{\mu} \frac{3}{6} \sum_{p=1}^{3} k(r_p, z_p) r_p \frac{|\Delta|}{6} \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{pmatrix} - \frac{\partial |\Delta|}{6} \frac{\partial (\nabla \cdot u)}{\partial t} (r_i, z_i) r_i.
\]
4.1.3 Dilatation and Kozeny-Carman equation

To solve this problem we need to prescribe the displacement vector \( \textbf{u} \) and to define the relationship between the displacement and the permeability.

We assume that the displacement vector has an oscillating behavior, so we use the following formulas for the components \( u_r \) and \( u_z \) of the displacement vector \( \textbf{u} \) for \( \textbf{x} \in \Omega \) and \( t \geq 0 \):

\[
\begin{align*}
    u_r(r, z, t) &= u_0^r e^{-\lambda(r^2 + (z-z_m)^2)[1 - \cos(w t)]}; \\
    u_z(r, z, t) &= u_0^z e^{-\lambda(r^2 + (z-z_m)^2)[1 - \cos(w t)]}.
\end{align*}
\]

Thus we get for the dilatation \( \epsilon \) in cylinder coordinates \((r, z)\)

\[
\begin{align*}
    \epsilon &= \nabla \cdot \textbf{u} = \frac{1}{r} \frac{\partial (ru_r)}{\partial r} + \frac{\partial u_z}{\partial z} \\
    &= \left[ \frac{u_0^r}{r} - 2\lambda(u_0^r r + u_0^z(z-z_m)) \right] e^{-\lambda(r^2 + (z-z_m)^2)[1 - \cos(w t)]},
\end{align*}
\]

then it follows that

\[
\frac{\partial \epsilon}{\partial t} = \left[ \frac{u_0^r}{r} - 2\lambda(u_0^r r + u_0^z(z-z_m)) \right] e^{-\lambda(r^2 + (z-z_m)^2)} w t \sin(w t). \tag{35}
\]

Note that we get a problem on boundary \( \Gamma_4 \cup \Gamma_5 \), because on this boundary we have \( r = 0 \), so in equation (35) we devide by zero. The dilatation appears in the element vector for an internal element in the following form

\[
-\frac{|\Delta|}{6} \frac{\partial \epsilon}{\partial t}(r_i, z_i) r_i.
\]

The problem on boundary \( \Gamma_4 \cup \Gamma_5 \) is solved by the multiplication by \( r_i \).

In addition to the dilatation, we also want to know the relationship between the permeability and the dilatation. To this extent, we use the Kozeny-Carman equation (see [8])

\[
k(r, z) = \frac{d_m^2}{\alpha} \frac{\theta^3(r, z)}{(1 - \theta(r, z))^2}, \tag{36}
\]

where:

- \( d_m \) is the mean particle size of the soil in m,
- \( \alpha \) is a constant depending on the shape of the particles,
- \( \theta \) is the porosity of the soil with a value between 0 and 1.

To determine the value of \( \alpha \) in this equation, we use the following values known from the field

\[
k_0 = 2.3148 \cdot 10^{-4} \text{ m}^2, \theta_0 = 0.3 \text{ and } d_m = 200 \text{ \mu m},
\]

where \( \theta_0 \) is the porosity at \( t = 0 \). Thus we get \( \alpha \approx 9.5 \cdot 10^{-6} \).
Further, the relationship between the porosity and the dilatation is well known (see [7]) and is given by

$$\frac{\partial \epsilon}{\partial t} = \frac{1}{1 - \theta} \frac{\partial \theta}{\partial t}.$$ 

Hence we get the formula \( \epsilon = \ln\left(\frac{1 - \theta_0}{1 - \theta}\right) \), from which it follows

$$\theta = 1 - \frac{1 - \theta_0}{e^\epsilon}.$$ 

With this equation, we can now determine the permeability from the dilatation using the Kozeny-Carman equation (36).

Note that in this function, we use formula (34), in which we divide by \( r \). Therefore, we also get a problem on boundary \( \Gamma_4 \cup \Gamma_5 \) in this function, because we divide by zero then. To solve this problem, we use the translation

$$r|_{\Gamma_4 \cup \Gamma_5} \rightarrow (r + \delta)|_{\Gamma_4 \cup \Gamma_5},$$

with \( \delta > 0 \) having a small value near zero. We choose also a small value for \( u^0_r \) to prevent that \( \epsilon \) goes to infinity.

### 4.1.4 Numerical solution with prescribed injection velocity

For the problem with water injection, where 50 liters of water per second are injected into the ground (\( Q_{in} = 0.050 \text{ m}^3/\text{s} \)), some 3D plots of the numerical solutions for the pressure at consecutive times are shown in Figure 35. Here the following values are used

\( u^0_r = 1 \cdot 10^{-4} \text{ m and } u^0_z = 1 \cdot 10^{-4} \text{ m.} \)
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In this figure only the first forty meters away from the filter are displayed, at further distances the hydrostatic pressure is reached more or less (see Figure 4).

Figure 35: Numerical solutions for the water pressure at consecutive times

(a) Numerical solution at $t = 0$ s
(b) Numerical solution at $t = 0.25$ s
(c) Numerical solution at $t = 0.5$ s
(d) Numerical solution at $t = 0.75$ s
(e) Numerical solution at $t = 1$ s
At time $t = 0.25$ s, we get a negative pressure with respect to the reference hydrostatic pressure $p = \rho g (H - z)$. In Figure 36 the pressure at $t = 0.25$ s is plotted against the hydrostatic pressure, on boundary $\Gamma_4 \cup \Gamma_5$.

This behavior may be caused by the emergence of a suction force, due to an under-pressure, near the filter.

At time $t = 0.75$ s a positive water pressure can be seen with respect to the hydrostatic pressure, in Figure 37 the pressure values on boundary $\Gamma_4 \cup \Gamma_5$ are shown.
These results can be explained by the oscillation of the displacement vector which leads to an alternating behavior of the time derivative of the dilatation.

At time $t = 0.25$ s, this derivative is positive near the filter, from equation (29) it follows that the net-outflow of water ($\nabla \cdot \mathbf{v}$) is negative in that area. However, this derivative is negative at time $t = 0.75$ s, hence the net-outflow of water is positive near the filter. This explains the low water pressure at time $t = 0.25$ s and the high pressure at time $t = 0.75$ s.

To get a better idea of this behavior, we determine the velocity vector $\mathbf{v}$ according to the method described in Section 2.6. In Figure 38 vector plots of the numerical solution for the velocity vector at consecutive times are shown. For the clarity of the figure only the first five meters away from the filter are plotted.
Figure 38: Numerical solutions for the velocity vector at consecutive times
In this figure, the arrows indicating the direction of the velocity vector $\mathbf{v}$ are scaled in size. Further, we see that the water speed is maximal near the filter and decreases with increasing distance from the filter. At time $t = 0.25$ s, the arrows in the layers above and below the ‘injection point’ are directed to the left. In Figure 39 the velocity vector in the upper layer (of ten meters) of the soil is shown.

![Figure 39: Velocity vector in the upper layer of the soil](image)

In this figure, we see that a vortex develops caused by a suction force that allows the inclusion of more water.

### 4.1.5 Numerical solution with imposed pressure

For the problem with water injection, where a maximum pump pressure of 1.6 bar is imposed ($p_{\text{pump}} = 1.6$ bar), some 3D plots of the numerical solutions for the pressure at consecutive times are shown in Figure 40.
Figure 40: Numerical solutions for the water pressure at consecutive times

In this figure only the first forty meters away from the filter are displayed, at further distances the hydrostatic pressure is reached more or less (see Figure 4).
Also in this case, we get a negative pressure with respect to the reference hydrostatic pressure \( p = \rho g (H - z) \) at time \( t = 0.25 \) s. This behavior may be caused by the emergence of a suction force, due to an under-pressure, near the filter. At time \( t = 0.75 \) s we get a positive water pressure with respect to the hydrostatic pressure.

To get a better idea of this behavior, we determine the magnitude of the velocity vector \( \mathbf{v} \). In Figure 41, some 3D plots can be seen of the numerical solution for the magnitude of the velocity vector at consecutive times. For the clarity of the figure only the first five meters away from the filter are plotted.

![3D plots](image.png)

(a) Speed at \( t = 0, t = 0.5 \) and \( t = 1 \) s

(b) Speed at \( t = 0.25 \) s

(c) Speed at \( t = 0.75 \) s

Figure 41: Numerical solutions for the speed at consecutive times

In this figure, we see that the flow is highest at time \( t = 0.25 \) s. We also see that the velocity at time \( t = 0.75 \) s not as low as we would expect from the high water pressure, shown in Figure 40(d).
Conclusion

In this master thesis, an important phenomenon in the field of hydrology, and in particular in the dewatering applications, is investigated.

In practice and in theory it is known that there are ‘injection points’ or preferential flow paths in the subsurface. At such points the soil absorbs significantly more water than at other depths on such a location. Through mathematical approaches is attempted to get sense of the behavior of the soil grains around an ‘injection point’ and of the behavior of the injected water that flows into the ground at such a point (Section 1.1).

It is known that water flows laminar in porous media, hence to describe the water flow the Darcy’s law is used. First, the water pressure and the flow velocity were determined, in a porous medium with a constant average permeability (Section 2).

Subsequently, the influence of a diverse soil composition on the water pressure and the flow velocity is examined. In order to model this variation, use is made of stochastic perturbations on the permeability (Section 2.8). It has been found that a varied composition of the soil, may affect the flow of water, depending on the random variables that effect may be small or large.

In Section 3, we brought variations in permeability. From literature is known that pressure variations in a porous medium will evoke some vibration frequencies, that cause mechanical deformations. To model these vibrations, use is made of oscillations in the permeability. From the obtained profile for the water pressure and the flow velocity, with this model, has revealed that by the occurrence of these vibrations, which probably arise around an ‘injection point’, the soil can absorb more water. The flow is also accelerated by these vibrations.

Another variation on the permeability was the dependence of this parameter of the flow velocity (Section 3.3). Here we assumed that the flow allows, the horizontally directed component of the permeability, to increase. Causing the vertically directed component of the permeability decrease. In solving this problem, an iteration process is used. From the resulting water pressure and flow velocity follows that this adaptation of the permeability hinder the vertically directed flow, causing the water to predominantly flow horizontally.

At the end of this study, the Biot’s theory for poro-elasticity is considered, where flow in porous media is combined with mechanical deformations of the aquifer in which water is injected (Section 4). We have partially solved this model for our problem, here we assumed the displacement vector, which reflects the local deviation of the porous medium, given by an oscillating behavior. And for the first time, we saw in the results the appearance of a negative pressure, which indicates the presence of suction forces which ensure more absorption of water by the soil.
In summary, in this study a number of hypotheses were tested, which may explain the more absorption of water by the soil around an ‘injection point’. Since we don’t have many good and useful measurements, we can not draw conclusions from the obtained results of these hypotheses. This research is, however, a good orientation study that can be used for further investigations about this phenomenon.
References


