A hydromechanically coupled flowline model for simulating glacier surge behaviour

Land-terminating, hard-bedded, temperate glaciers

C.A. van Geffen
A HYDROMECHANICALLY COUPLED FLOWLINE MODEL FOR SIMULATING GLACIER SURGE BEHAVIOUR

LAND-TERMINATING, HARD-BEDDED, TEMPERATE GLACIERS

by

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Glacier surge behaviour is characterized by multiyear, quasi-periodic changes in glacier geometry and ice flow. Various attempts have been done on modelling glacier surge behaviour. The focus of this research will be on land-terminating, hard-bedded, temperate glaciers (LHT-glaciers). Whereas the drainage system at the base of the glacier is thought to be the key element in the surge behaviour of those glaciers, developed models do not include a basal drainage model.

The surges of LHT-glaciers are characterized by a sudden initiation and a sudden termination of the surge. This sudden on- and offset is assigned to switches in basal drainage system. Therefore, a description of the transformation of the basal drainage system during a surge cycle of LHT-glaciers is proposed. The proposed transformation is based on observations of the 1982-1983 surge of Variegated Glacier, Alaska. Despite the fact that models that are capable of simulating quasi-periodic behaviour related to basal frictional lubrication do not include a basal drainage model, building blocks of a basal drainage model for surging glaciers have been proposed for a long time. By means of translating the proposed transformation of the basal drainage system into those building blocks, a basal drainage model is constructed.

The constructed basal drainage model is coupled to the ice mass continuity equation. By means of implementing the coupled model, it is shown that the model simulates the characteristic features of the surge cycle of LHT-glaciers.
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1

Introduction

1.1. Land-Terminating Glaciers

Land-terminating glaciers form where accumulation of snow exceeds ablation of ice, i.e. the combination of melting and sublimation of ice, for a long period. In the upper part of a glacier, the snow that fell during winter does not entirely melt during the successive meltseason. In this zone yearly accumulation is larger than yearly ablation, hence this zone is referred to as the accumulation zone of the glacier. The lower part of a glacier is characterized by a yearly ablation of ice that exceeds the yearly snow fall, hence this zone is referred to as the ablation zone. The accumulation zone and ablation zone are separated by the so-called equilibrium line. This line represents the altitude at which yearly snow fall equals yearly ablation. The two zones are interlinked through ice flowing downslope under the force of gravity, see Fig. 1.1.

Figure 1.1: Schematic representation of the accumulation zone and ablation zone of land-terminating glaciers. The two zones are interlinked through ice flowing downslope under the force of gravity.

In addition to ice, meltwater flows downglacier. This meltwater can be produced (i) at the surface, (ii) englacially through strain heating, and (iii) at the base of the glacier through basal frictional heating and through a geothermal heat flux from the bed towards the ice. The produced meltwater is transported downglacier by supraglacial streams and englacial and subglacial channels, see Fig. 1.2. (Oerlemans, 2013)
1.2. CHARACTERISTICS OF GLACIER SURGE BEHAVIOUR

Glacier surge behaviour is characterized by multiyear, quasi-periodic changes in geometry and ice flow, see Fig. 1.3. The changes in geometry and ice flow are irrespective of variations in climate, but instead indicate a relaxation process. So, glaciers that exhibit surge behaviour do not represent changes in climate accurately.

Meier and Post (1969) suggested that two distinct phases make up the surge cycle: the quiescent phase and the surge phase. The quiescent phase can be defined as a relatively long period ($10^1$-$10^2$y) of low ice velocity. It is accompanied by gradual thickening of the reservoir area and gradual thinning of the receiving area. In addition, a gradual retreat of the glacier can be observed. Fig. 1.4(a) presents the elevation change of Variegated Glacier, Alaska, during a documented quiescent phase. It displays the characteristic thickening of the reservoir area and the characteristic thinning of the receiving area and the resulting steepening of the glacier. Fig. 1.4(b) shows the gradual changes in winter surface velocity during a quiescent phase. The
1.2. Characteristics of glacier surge behaviour

The quiescent phase is followed by the surge phase. This phase can be described as a relatively short period (several years) of ice flowing at speeds exceeding the ice velocity during the quiescent phase by at least one order of magnitude, see Fig. 1.5. It is characterized by large thinning (10^1-10^2 m) of the reservoir area and large thickening of the receiving area of the glacier. In addition, the surge phase is marked by a sudden glacier advance, and the termination of this phase is accompanied by a flood discharge in the terminal streams. Fig. 1.6 shows two photos taken from the same vantage in the receiving area of Variegated Glacier. As a result of high surface velocities during the surge, it reveals a severely cracked glacier surface. (Meier and Post, 1969)
Figure 1.5: Time evolution of the surface velocity of Variegated Glacier, Alaska at the time of the initiation of its 1982-1983 surge. (Raymond, 1987)

Figure 1.6: Photos taken from the same vantage in the receiving area of Variegated Glacier, Alaska before a surge (a) and during a surge (b). The position of the ice surface during the surge, as seen in (b), is marked with dashed lines in (a). (Raymond and Harrison, 1988)
1.3. BASAL DRAINAGE SYSTEMS OF LAND-TERMINATING, HARD-BEDDED, TEMPERATE GLACIERS

Various glaciers, concentrated in some glaciated regions around the world, show surge behaviour, see Fig. 1.7. The focus of this research will be on land-terminating, hard-bedded, temperate glaciers (LHT-glaciers), i.e. glaciers without calving, with hard bed rock, and ice at pressure melting point throughout the glacier, located in Western North America.

The large ice velocities that characterize the surge phase of surging glaciers are assigned to large sliding velocities. Because of the hard bed of LHT-glaciers, bed deformation does not contribute to the downslope movement of the ice, hence sliding is caused solely by sliding of the ice over bed. For LHT-glaciers the mechanism that causes the large sliding that characterize the surge phase is slippage (Oerlemans, 2013). Slippage is driven by a decrease in basal friction due to the presence of water at the ice-bed rock interface. Therefore, the drainage system at the base of the glacier is thought to be the key element in surge behaviour of LHT-glaciers.

Two basal drainage systems can be distinguished for LHT-glaciers: a channel system and a cavity system. A channel system consists of a network of channels covering only a limited part of the bed. This system is very efficient at draining water (Rothlisberger, 1972). Fig. 1.8 pictures a tunnel at the terminus of a glacier. A cavity system on the other hand is far less efficient at draining water. The system is composed of cavities that are interlinked by small channels called orifices, see Fig. 1.9 (Kamb et al., 1985).

The surges of LHT-glaciers are characterized by a sudden initiation and a sudden termination of the surge. Kamb et al. (1985) suggests that the sudden initiation of the surge is caused by a switch from channel system to cavity system. The cavity system that drains the water during the surge, is far less efficient at draining water. Consequently, the amount of water at the base of the glacier increases quickly. This reduces basal friction and
Figure 1.9: Schematic representation of a cavity system at the base of a glacier. The cavities are interlinked by small channels called orifices. (Fountain and Walder, 1998)

results in an increase in sliding velocity. Because a channel system is far more efficient at draining water than a cavity system, and because the termination of the surge is observed to be accompanied by a flood discharge at the terminus of the glacier, the sudden termination of the surge is thought to be caused by a switch back from cavity system to channel system (Kamb et al., 1985).

1.4. **Modelling Surge Behaviour of Land-Terminating, Hard-Bedded, Temperate Glaciers**

Various attempts have been done on modelling glacier surge behaviour. One of the reasons that modelling surge behaviour is relevant is that the effect of surge behaviour on glacier dynamics interferes with the effect of changes in climate on glacier dynamics. Therefore, the role of surge behaviour should be taken into account in for example the interpretation of glacier records in an attempt to get insight in the effects of climate change on glacier geometry. Modelling surge behaviour is also desirable because a rapid glacier advance characterizing the surge phase of surging glaciers can demolish infrastructure or can form an ice-dammed lake that breaks when the glacier retreats during the subsequent quiescent phase.

The state-of-the-art model that is capable of simulating quasi-periodic behaviour related to basal frictional lubrication was presented in Budd (1975). This model parameterizes basal lubrication by means of a friction lubrication factor. However, this model does not include a basal drainage model. Therefore, the model does not capture the switches in basal drainage system suggested for surging LHT-glaciers. Furthermore, the surge front, i.e. a large jump in ice thickness with a large difference in ice velocity over the jump, see Fig. 1.10, propagates solely through longitudinal stresses, i.e. the ice is pushed downslope, whereas the propagation of this front through the drainage of water at the base of the glacier is not taken into account. Because the drainage system at the base of a glacier is thought to be the key element in surge behaviour, the inclusion of a realistic basal drainage model seems essential in simulating surge behaviour more accurately. For a long time, building blocks of a basal drainage model for surging glaciers have been proposed, see for example Fowler (1987). However, until now these components have not been coupled to a glacier dynamics
1.5. RESEARCH QUESTIONS

The goal of this research is to construct a simple basal drainage model from the building blocks proposed in literature, and to couple this model to a glacier dynamics model in order to simulate surge behaviour of LHT-glaciers. To achieve this goal, the following research questions are formulated:

- How does the basal drainage system of land-terminating, hard-bedded, temperate glaciers transform during a surge cycle?
- How can the proposed transformation during the quiescent phase be represented in a simple basal drainage model?
- How can the proposed transformation during the surge phase be represented in a simple basal drainage model?
- How can the initiation and termination of the surge be represented in a simple basal drainage model?

Those questions will be answered starting in Chapter 4. However, first a chapter will be dedicated to glacier dynamics. It focusses on the ice mass continuity equation and on two widely applied approximations to the momentum conservation equations, because those equations will be referred to in subsequent chapters. In Chapter 3 an attempt will be made on validating the capability of the glacier flowline model presented in Budd (1975) to simulate surge behaviour.
2

GLACIER DYNAMICS

In this chapter equations describing glacier dynamics will be discussed using the concept of continuum mechanics. Continuum mechanics approaches a physical object as a continuous distribution of matter, and determines the values of collective properties of large numbers of particles, as density and temperature. The treatment of a physical object as a continuous distribution of matter, momentum and energy is realistic as long as our spatial scale of interest is large compared to the distance between individual particles.

In the following paragraphs the equations relevant for this MSc Thesis will be derived. The conservation laws of mass and linear momentum for ice flow will be discussed. Also, two approximations to the momentum conservation equations commonly applied in glaciology will be treated. In addition, a relation between forces and deformation, a so-called constitutive relation, is presented. For temperate glaciers there is no thermodynamic coupling, i.e. the energy balance equation is decoupled from the momentum conservation equations. Therefore, the conservation law of energy will not be discussed. The three dimensional conservation equations will be transformed into equations integrated vertically and with only one horizontal dimension. (Van der Veen, 2013)

2.1. ICE MASS CONTINUITY EQUATION

In continuum mechanics the mass continuity equation in vector notation is given by

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \quad (2.1)$$

with $\rho$ the density of the continuum and $\mathbf{u}$ the velocity vector. Eq. 2.1 states that the rate of change of the density of the mass body changes by divergence of mass flux. For ice the assumption of an incompressible continuum is appropriate, i.e. the density of ice can be thought to be constant during motion. Hence, it holds that

$$\frac{D\rho}{Dt} = 0. \quad (2.2)$$

Therefore, Eq. 2.1 can be written as

$$\nabla \cdot \mathbf{u} = 0. \quad (2.3)$$

When a Cartesian coordinate system is applied, and in $y$-direction an uniform bed profile is assumed and the effect of side drag can be neglected, this equation reduces to

$$\nabla \cdot \mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \quad (2.4)$$

with $u$ the horizontal velocity in x-direction, and $w$ the vertical velocity. Integrating this equation from the base of the glacier, $z = b$, to the surface, $z = h$, see Fig. 2.1 and applying Leibniz’s rule in order to interchange the operations of integration and partial differentiation, results in

$$w(h) - w(b) = - \int_b^h \frac{\partial u}{\partial x} \, dz = u(h) \frac{\partial b}{\partial x} - u(b) \frac{\partial h}{\partial x} + \frac{\partial}{\partial x} \int_b^h u \, dz. \quad (2.5)$$
In this equation the vertical material velocities at the base and surface of the glacier appear. These velocities are given by

\[
w(h) = \frac{dh}{dt} - \dot{b} = \frac{\partial h}{\partial t} + u(h) \frac{\partial h}{\partial x} - \dot{b}
\]  
(2.6)

\[
w(b) = \frac{db}{dt} - \dot{b}_b = \frac{\partial b}{\partial t} + u(b) \frac{\partial b}{\partial x} - \dot{b}_b
\]  
(2.7)

with \(\dot{b}\) the specific balance rate at the surface of the glacier and \(\dot{b}_b\) the specific balance rate at the base of the glacier. Combining Eq. 2.5-2.7 gives

\[
\frac{\partial (h - b)}{\partial t} + \dot{b}_b - \dot{b} = -\frac{\partial}{\partial x} \int_b^h u \, dz.
\]  
(2.8)

With the definition of the ice thickness, \(H = h - b\), the vertically integrated ice mass continuity equation becomes

\[
\frac{\partial H}{\partial t} = -\frac{\partial (U H)}{\partial x} + \dot{b} - \dot{b}_b
\]  
(2.9)

with \(U\) the mean horizontal velocity over an ice column. It is customary to align the remaining horizontal direction, the x-direction, to the main direction of ice motion, i.e. along the flowline, see Fig. 2.2.
2.2. **Constitutive relation**

Constitutive relations describe the relation between forces and deformation. A commonly used constitutive relation in glaciology is Glen’s flow law

\[ \dot{\epsilon}_{ij} = A r^{n-1} \tau_{ij} \]  

(2.10)

with \( \epsilon_{ij} \) the deformation tensor and \( \tau_{ij} \) the deviatoric stress tensor, whose nine components represent the deviatoric stresses. Eq. 2.10 implicates that pressure is taken positive concerning stresses. The relation between this deviatoric stress tensor \( \tau_{ij} \) and the stress tensor \( \sigma_{ij} \) is given by

\[ \sigma_{ij} = \tau_{ij} + \sigma_{kk} \delta_{ij} \]  

(2.11)

with \( \delta_{ij} \) the Kronecker delta. Fig. 2.3 shows the sign convention for the stress tensor \( \sigma_{ij} \). In Eq. 2.10 \( \tau \) represents the effective stress defined as

\[ \tau = \sqrt{\frac{1}{2} \tau_{ij} \tau_{ji}}. \]  

(2.12)

It represents the second invariant of the deviatoric stress tensor. For steady motion the net force is equal to zero. Therefore, to ensure the body does not rotate, \( \tau_{ij} = \tau_{ji} \). In a Cartesian coordinate system with only one horizontal dimension, Eq. 2.12 can then be written as

\[ 2\tau^2 = \tau_{xx}^2 + \tau_{zz}^2 + 2\tau_{xz}^2. \]  

(2.13)

The so-called rate-factor \( A \) in Eq. 2.10 represents the ‘softness’ of the ice. The rate-factor depends strongly on temperature. However, for a temperate glacier \( A = \text{constant} \) is a reasonable approximation.

![Figure 2.3: The nine components of the stress tensor acting on an infinitesimal block.](http://www.globalspec.com/reference/70300/203279/chapter-1-stress-and-strain)

2.3. **Momentum conservation equations**

In continuum mechanics conservation of linear momentum is given by the Navier-Stokes equations, which in summation convention are given by

\[ \rho \frac{du_i}{dt} = F_i + \frac{\partial \sigma_{ij}}{\partial x_j} \]  

(2.14)

in which the first term on the right-hand side represents body force and the second term embodies the nine components of the stress gradient tensor. Because motion of glaciers evolves slowly with time, the motion can be thought of as being in a state of quasi-static equilibrium, i.e. zero net force and no accelerations. Therefore, the inertial terms on the left-hand side of Eq. 2.14 can be neglected. In addition, the stress tensor can be decomposed in a deviatoric and a spherical part, see Eq. 2.11. Defining the pressure \( p \) as \( p := -\frac{\sigma_{kk} \delta_{ij}}{3} \) and again considering only one horizontal dimension, leads to the following two equation
\[ \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} = \frac{\partial p}{\partial x} \quad (2.15) \]

\[ \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zz}}{\partial z} = \frac{\partial p}{\partial z} + \rho g. \quad (2.16) \]

Also in the case of surge behaviour in which the motion is evolving relatively rapidly in time, the motion can be considered to be in a state of quasi-static equilibrium (Raymond, 1987). Consequently, Eq. 2.15 and 2.16 also hold for surging glaciers.

### 2.3.1. Shallow Ice Approximation

In practice, often approximations to Eq. 2.15 and 2.16 in which not all terms are kept, are used. One widely applied approximation is the so-called Shallow Ice Approximation (SIA). This approximation holds for glaciers with basal velocities that add significantly to the surface velocity. For these glaciers the so-called slip ratio, defined as the ratio between basal velocity and internal deformation velocity \( \frac{u_b}{u_i} \), is large. The SIA is based on the following assumptions:

- Horizontal length scale of motion much larger than ice thickness.
- Vertical shear stresses small compared to pressure.
- Horizontal deviatoric stresses small compared to vertical shear stresses.

These assumptions lead to the following set of momentum conservation equations

\[ \frac{\partial \tau_{xz}}{\partial z} = \frac{\partial p}{\partial x} \quad (2.17) \]

\[ 0 = \frac{\partial p}{\partial z} + \rho g. \quad (2.18) \]

Eq. 2.17 and 2.18 can be rewritten into the following equations

\[ \tau_{xz} = \int \frac{\partial p}{\partial x} \, dz \quad (2.19) \]

\[ p(z) = \rho g (h - z). \quad (2.20) \]

Eq. 2.20 reveals that in the SIA the vertical pressure is hydrostatic. Together, Eq. 2.20 and 2.19 lead to an equation for the internal shearing

\[ \tau_{xz} (z) = -\rho g (h - z) \sin(\frac{\partial h}{\partial x}) \quad (2.21) \]

which for small surface slopes can be written as

\[ \tau_{xz} (z) = -\rho g (h - z) \frac{\partial h}{\partial x}. \quad (2.22) \]

This equation shows that in the SIA the shear stress increases linearly with depth, as depicted in Fig. 2.4(a). Thereby, the basal stress \( \tau_b \) is equal to the shear stress at the base of the glacier, which is equal to the driving stress \( \tau_d = -\rho g H \frac{\partial h}{\partial x} \).

In the SIA, only \( \tau_{xz} \) is not negligible, so that Glen’s flow law, Eq. 2.10, becomes

\[ \dot{\epsilon}_{xz} = \frac{\partial u}{\partial z} = 2A \tau_{xz}^n. \quad (2.23) \]

Combining this with Eq. 2.22, the vertical gradient of the horizontal velocity yields

\[ \frac{\partial u}{\partial z} = 2A \left( -\rho g (h - z) \frac{\partial h}{\partial x} \right)^n. \quad (2.24) \]

Integrating this equation from the base of the glacier, \( z = b \), to a height \( z \) gives
2.3. MOMENTUM CONSERVATION EQUATIONS

\[ u(z) - u(b) = -\frac{2A}{n+1} \left( -\rho g \frac{\partial h}{\partial x} \right)^n \left( (H-z)^{n+1} - H^{n+1} \right) \]  

(2.25)

which with zero basal sliding velocity, \( u(b) = 0 \), results in

\[ u(z) = -\frac{2A}{n+1} \left( -\rho g \frac{\partial h}{\partial x} \right)^n \left( (H-z)^{n+1} - H^{n+1} \right). \]  

(2.26)

From this equation it can be concluded that the internal deformation velocity is a diagnostic output of the SIA, i.e. it can be determined locally from the glacier geometry. The resulting velocity profile is depicted in Fig. 2.4(b).

The vertically mean velocity \( U \) can be determined with

\[ U = \frac{1}{H} \int_b^h u \, dz \]  

(2.27)

and with Eq. 2.26 this mean internal deformation velocity becomes

\[ U = \frac{2AH}{n+2} \tau_b^n. \]  

(2.28)

2.3.2. SHALLOW SHELF APPROXIMATION

Another widely applied approximation to the momentum continuity equations Eq. 2.15 and 2.16 is the so called Shallow Shelf Approximation (SSA). This approximation can be used when basal lubrication, and therefore basal velocity, adds significantly to the surface velocity. In this case longitudinal stretching has to be taken into account. The SSA is based on the following assumptions:

- Horizontal length scale of motion much larger than ice thickness.
- Vertical shear stresses smaller than all other stresses.

These assumptions result in the following set of momentum continuity equations

\[ \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} = \frac{\partial p}{\partial x} \]  

(2.29)

\[ \frac{\partial \tau_{zz}}{\partial z} = \frac{\partial p}{\partial z} + \rho g. \]  

(2.30)

From these two equations it can be concluded that in the SSA the vertical stress is lithostatic, i.e. \( \sigma_{zz} = -\rho g (h-z) \), but the pressure is not hydrostatic, i.e. \( p(z) \neq \rho g (h-z) \).

Vertical integration of Eq. 2.30 and applying the mass continuity equation for an incompressible continuum, Eq. 2.9, the stress condition at the surface, \( \tau_{xz}(h) = 0 \), and the stress condition at the base, \( \tau_{xz}(b) = \tau_b \) gives (see Appendix A)
\[ \frac{\partial}{\partial x} (2H \tau_{xx}) - \tau_b = \rho g H \frac{\partial h}{\partial x}. \]  

(2.31)

This is the equation for longitudinal stress equilibrium for small surface slopes in the SSA. From this equation it can be concluded that in the SSA the driving stress is balanced by a combination of basal shearing and a gradient in the strain rate, i.e. compression or expansion of ice in longitudinal direction, see Fig. 2.5. Furthermore, from this equation is can be seen that the SSA results in a non-local problem for the velocity. The resulting velocity is constant over the ice column.

![Figure 2.5: Schematization of the stress balance in the SSA.](image)

The resulting equations, Eq. 2.9, Eq. 2.10, Eq. 2.28 and Eq. 2.31 will be applied in the upcoming chapters.
The goal of this chapter is to validate the capability of the model presented in Budd (1975) to simulate surge behaviour. Budd (1975) starts with the ice mass continuity equation along the central flowline of a glacier. In order to be able to simulate quasi-periodic behaviour, a friction lubrication factor is added, see Fig. 3.1. However, because no basal drainage model is included, the model does not capture the switches in basal drainage system suggested for LHT-glaciers, and the model does not capture the propagation of the surge front through the drainage of water at the base of the glacier.

**Figure 3.1**: Schematic representation of the model presented in Budd (1975). In this model basal lubrication is represented by a friction lubrication factor.

### 3.1. System of Equations

Fig. 3.2 shows the system of equations applied in Budd (1975). Table 3.1 provides an explanation of the symbols used in these equations. The bed elevation \( b \) and specific balance rate \( \dot{b} \), i.e. the rate at which mass is added to, or removed from a glacier, along the glacier, together with the flow properties \( k, n, \eta, s, \rho, g \), and the friction lubrication factor \( \varphi \), form the input of the model.

The starting point of this system of equations is the vertically integrated mass continuity equation for an incompressible continuum Eq. 2.9

\[
\frac{\partial H}{\partial t} = - \frac{\partial (U H)}{\partial x} + \dot{b} - \dot{b}_b \tag{3.1}
\]

which forms the outer loop in Fig. 3.2. Budd (1975) assumes that the specific balance rate at the base of the glacier \( \dot{b}_b \) is negligible compared to the specific balance rate at the surface of the glacier \( \dot{b} \). Therefore, \( \dot{b}_b \) is set to zero. The vertical mean total velocity \( U \) in the first term on the right-hand side of Eq. 3.1, is a combination of the sliding velocity, which is constant throughout the ice column, and the internal deformation velocity averaged over the ice column. The latter is calculated from the Shallow Ice Approximation Eq. 2.28
Table 3.1: List of symbols used in the system of equations applied in Budd (1975).

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>SI unit</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>α</td>
<td>surface slope</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>ϵ</td>
<td>threshold value</td>
<td>–</td>
<td>0.001</td>
</tr>
<tr>
<td>η</td>
<td>viscosity</td>
<td>Pas</td>
<td>$3.2 \cdot 10^{13}$</td>
</tr>
<tr>
<td>$\rho_i$</td>
<td>ice density</td>
<td>kgm$^{-3}$</td>
<td>910</td>
</tr>
<tr>
<td>$\tau_b$</td>
<td>basal stress</td>
<td>Pa</td>
<td>–</td>
</tr>
<tr>
<td>$\tau_d$</td>
<td>driving stress</td>
<td>Pa</td>
<td>–</td>
</tr>
<tr>
<td>$\bar{\tau}_d$</td>
<td>average driving stress along the flowline</td>
<td>Pa</td>
<td>–</td>
</tr>
<tr>
<td>$\tau^*_d$</td>
<td>lubrication-lowered stress</td>
<td>Pa</td>
<td>–</td>
</tr>
<tr>
<td>$\bar{\tau}^*_d$</td>
<td>average lubrication-lowered stress along the flowline</td>
<td>Pa</td>
<td>–</td>
</tr>
<tr>
<td>$\phi$</td>
<td>friction lubrication factor</td>
<td>Pa$^{-1}$ m$^{-1}$ s</td>
<td>–</td>
</tr>
<tr>
<td>$b$</td>
<td>bed elevation</td>
<td>m</td>
<td>–</td>
</tr>
<tr>
<td>$\dot{b}$</td>
<td>mass balance rate</td>
<td>ms$^{-1}$</td>
<td>–</td>
</tr>
<tr>
<td>$g$</td>
<td>gravitational acceleration</td>
<td>ms$^{-2}$</td>
<td>9.81</td>
</tr>
<tr>
<td>$H$</td>
<td>ice thickness</td>
<td>m</td>
<td>–</td>
</tr>
<tr>
<td>$k$</td>
<td>constant</td>
<td>Pa$^{-2}$ s$^{-1}$</td>
<td>$4.8 \cdot 10^{-19}$</td>
</tr>
<tr>
<td>$L$</td>
<td>glacier length</td>
<td>m</td>
<td>–</td>
</tr>
<tr>
<td>$n$</td>
<td>constant</td>
<td>–</td>
<td>2</td>
</tr>
<tr>
<td>$s$</td>
<td>shape factor representing the glacier cross-section</td>
<td>–</td>
<td>1</td>
</tr>
<tr>
<td>$u$</td>
<td>total velocity</td>
<td>ms$^{-1}$</td>
<td>–</td>
</tr>
<tr>
<td>$u_b$</td>
<td>sliding velocity</td>
<td>ms$^{-1}$</td>
<td>–</td>
</tr>
<tr>
<td>$u_i$</td>
<td>average deformation velocity over the ice column</td>
<td>ms$^{-1}$</td>
<td>–</td>
</tr>
</tbody>
</table>

Figure 3.2: System of equations applied in Budd (1975).

$$U = \frac{2AH}{n + 2} \tau_b^n.$$  \hspace{1cm} (3.2)

Hence, the internal deformation velocity is determined locally from the glacier geometry. The sliding velocity on the other hand, is determined from the set of equations in the inner loop in Fig. 3.2. The remaining part
of this section focuses on the inner loop, which represents the basal lubrication.

The second equation in the inner loop, i.e. the equation for the lubrication-lowered stress $\tau_{d}^*$, assumes that the amount of water at the base of the glacier is proportional to the total rate of energy dissipation $\tau_{d}u$, and indicates a local decrease in basal friction due to the presence of water at the ice-bed interface. After the lubrication-lowered stress, the basal stress is determined. The equation for the basal stress

$$\tau_{b} = \tau_{d} + (\tau_{d}^{*} - \tau_{d}^{*})$$  \hspace{1cm} (3.3)

is a result of the prescription that static equilibrium of the glacier as a whole should be preserved, i.e. $\tau_{b} = \tau_{d}$. Thereafter, the sliding velocity is calculated from the basal stress using the equation for longitudinal stress equilibrium from the Shallow Shelf Approximation Eq. 2.31

$$-2 \frac{\partial H \tau_{xx}}{\partial x} = \tau_{d} - \tau_{b}. \hspace{1cm} (3.4)$$

With the appropriate form of Glen’s flow law

$$2\eta \dot{\epsilon}_{x} = \tau_{xx} \hspace{1cm} (3.5)$$

and the equation emanating from the assumption that changes in longitudinal strain-rate are predominantly related to changes in sliding velocity

$$\dot{\epsilon}_{x} = \frac{\partial u_{b}}{\partial x} \hspace{1cm} (3.6)$$

the gradient in sliding velocity appears in Eq. 3.4. Combining the resulting equation for longitudinal stress equilibrium with: (i) the downstream boundary condition

$$\frac{\partial u_{b}}{\partial x} = 0 \hspace{0.5cm} \text{at} \hspace{0.5cm} x = L \hspace{1cm} (3.7)$$

which implies that near the terminus the ice mass moves as one block; and (ii) the upstream boundary condition

$$u_{b} = 0 \hspace{0.5cm} \text{at} \hspace{0.5cm} x = 0 \hspace{1cm} (3.8)$$

i.e. the sliding velocity at the ice divide is equal to zero, results in the following equations for the sliding velocity and the gradient in sliding velocity (see Appendix B)

$$\frac{\partial u_{b}}{\partial x} = -\frac{1}{4H\eta} \int_{L}^{x} (\tau_{d} - \tau_{b}) \, dx \hspace{1cm} (3.9)$$

$$u_{b} = \int_{0}^{x} \frac{\partial u_{b}}{\partial x} \, dx. \hspace{1cm} (3.10)$$

From these equations it can be concluded that this model does not contain a local relation between basal stress $\tau_{b}$ and sliding velocity $u_{b}$, as both are determined from properties of the glacier as a whole.

It has to be noted that Eq. 3.9 is similar to the equation for the gradient in sliding velocity in Budd (1975) except for a minus sign. Besides from the derivation of this equation in Appendix B, also from a physical point of view it can be concluded that this minus sign should be included: when near the terminus of the glacier the driving stress $\tau_{d}$ is smaller than the basal stress $\tau_{b}$, the gradient in sliding velocity near the terminus should be negative. This requires a minus sign in front of the integral on the right-hand side of Eq. 3.9.

### 3.2. Numerical Implementation

In order to test the model presented in Budd (1975), the system of equations in Fig. 3.2 has to be implemented numerically. Because there is no information on the grid and numerical scheme in Budd (1975), an appropriate grid and numerical scheme have to be selected. For the remainder of this section Zijlema (2012) was used.
3. Verification of the Results of the Model Presented in Budd (1975)

3.2.1. Grid

The glacier considered in Budd (1975) is schematized as a one-dimensional flow of ice with variables like ice thickness and surface velocity calculated along the central flowline of the glacier. The maximum length of the considered glacier is approximately 58 km. A somewhat larger domain (60 km) is applied, so that the simulated glacier length will not be limited by the domain length. The schematization is discretized with a grid spacing of 200 m. This grid spacing resulted in an acceptable time step restriction, see one of the following subsections, and with this grid spacing it is still possible to accurately simulate the characteristic changes in glacier length during a surge cycle, which are of the order of several kilometers. Because of the application of a fixed grid, a discrete length change instead of a continuous length change will be simulated.

A staggered grid arrangement is applied with scalar variables as ice thickness defined in the center of the grid cells and vector variables as ice velocity defined at the faces of the grid cells, see Fig. 3.3. By applying a staggered grid, odd-even decoupling between the ice thickness and ice velocity that can arise when a colocated grid arrangement is used, will be avoided.

3.2.2. Numerical Approximation of the Ice Mass Continuity Equation

The vertically integrated ice mass continuity equation used in Budd (1975)

$$\delta H = \left( \dot{b} - \frac{\partial}{\partial x} (uH) \right) \delta t \tag{3.11}$$

will be approximated numerically by applying in succession (i) an approximation of the spatial derivative by means of space discretization, (ii) an approximation of the resulting ordinary differential equation (ODE) by means of time integration.

**Space discretization**

Eq. 3.11 is an advection equation that prescribes the propagation of disturbances in the ice thickness \(H\). The solution of this advection equation can be written as \(H(x, t) = f(x - ut)\) with \(f\) any function. For the numerical approximation of the spatial derivative it is relevant to notice that Eq. 3.11 is non-linear, i.e. \(f\) depends on \(H\) itself. Preservation of monotonicity is desirable, hence an asymmetric scheme has to be chosen. A first order upwind scheme, which introduces numerical diffusion so that the generation of wiggles in the numerical solution is prevented, seems a suitable option. Fig. 3.4 shows the stencil of upwinding combined with forward Euler, i.e. the so-called Forward in Time, Backward in Space (FTBS) scheme. Eq. 3.12 gives the resulting ODE for first order upwinding.

![Figure 3.3: Staggered grid with scalar variables as ice thickness \(H\) defined in the center of the grid cells and vector variables as ice velocity \(u\) defined at the faces of the grid cells.](image)

![Figure 3.4: Stencil of the FTBS numerical scheme.](image)
This first order upwinding scheme is consistent up to first order in space with Eq. 3.11.

In theory, implementing a first order upwind scheme, introduces a virtual grid point at the upper boundary of the domain. However, in the case of the glacier from Budd (1975), the glacier head represents an ice divide, hence no mass is flowing in at the upper boundary. Therefore, the upper boundary of the domain can be schematized as a closed boundary, i.e.

\[ u_{H} = 0 \text{ at } x = 0. \] (3.13)

Consequently, the gradient in mass flux at the first grid point does not have to be determined with the help of a virtual grid point.

**TIME INTEGRATION**

The approximation of the spatial derivative led to the semi discretization Eq. 3.12. The resulting ODE has to be approximated by means of time integration. Applying a forward Euler approximation that is consistent up to first order in time with partial differential equation 3.11 should suffice. Implementing this approximation results in the following numerical scheme

\[
\frac{dH_m}{dt} = b_m - \frac{u_{m+\frac{1}{2}}H_m - u_{m-\frac{1}{2}}H_{m-1}}{\Delta x} \Delta t \text{ for } u_{m+\frac{1}{2}} > 0, u_{m-\frac{1}{2}} > 0.
\] (3.12)

3.2.3. **CONVERGENCE OF THE ICE MASS CONTINUITY EQUATION**

For the numerical implementation of the ice mass continuity equation, a simple explicit finite difference scheme was proposed, see Eq. 3.14. The implementation of the proposed scheme entails a time stepping restriction, which will be encountered in this subsection. In addition, Richardson extrapolation will be used to verify the correct implementation of the numerical scheme.

**TIME STEPPING RESTRICTION**

The numerical scheme depicted in Eq. 3.14, is conditionally stable, which means that a condition has to be fulfilled in order to preclude linear instabilities. The FTBS scheme is linearly stable if the following Courant–Friedrichs–Lewy (CFL) condition is met:

\[ \Delta t \leq \frac{\Delta x}{\max|u_b|}. \] (3.15)

This stability condition is a necessary stability condition. However, together with the preservation of monotonicity it is a sufficient stability condition.

According to Budd (1975), the maximum velocity is of the order of \(10^4\) m/yr. With the applied grid spacing of 200 m, the CFL condition 3.15 then restricts the time step to the order of days. As the period of the surge cycle is in the order of tens of years, this is considered to be a reasonable time step. So, from stability considerations it is not necessary to adopt a semi-implicit scheme. By adopting adaptive time stepping, every timestep \(\Delta t\) is taken to be the largest timestep that fulfills the above CFL condition.

**VERIFICATION OF THE CORRECT IMPLEMENTATION OF THE NUMERICAL SCHEME**

By comparing the order of accuracy of the numerical solution with the available theoretical value for the order of accuracy of the applied numerical scheme, it can be checked if the numerical scheme is implemented correctly. For the determination of the order of accuracy of the numerical solution, the following theory is used (Ferziger and Perić, 2002): the numerical approximation \(H_m\) is said to converge to the exact solution \(H\) if there exists a constant \(C\) for which

\[ |H_m - H| < C\Delta x^p \text{ for all } m. \] (3.16)

In this equation, the parameter \(p\) represents the order of accuracy with which the numerical solution converges to the exact solution. When the exact solution \(H\) is not known, one has to turn to a slightly different form of this inequality, which reads...
3. Verification of the results of the model presented in Budd (1975)

Figure 3.5: The rate of convergence (a) and the difference in ice thickness for a grid spacing of $\Delta x = 200$ m and a grid spacing of $\Delta x = 100$ m, i.e. the numerator in Eq. 3.17 (b), for the steady state solution of $\varphi = 0$.

$$\frac{|H_{\Delta x} - H_{\Delta x}|}{|H_{\Delta x} - H_{\Delta x}|} = 2^p + O(\Delta x)$$

and is called Richardson extrapolation. With Eq. 3.17 the rate of convergence $p$ can be determined from the numerical solutions for three different values of the grid spacing $\Delta x$. For the considered numerical approximation of the ice mass continuity equation, Eq. 3.14, Fig. 3.5(a) depicts the rate of convergence $p$ obtained with Eq. 3.17.

The graph in Fig. 3.5(a) mainly shows an almost horizontal line between $p = 1$ and $p = 2$. This value for the order of accuracy roughly corresponds to the available theoretical value for the FTBS scheme, $p_{\text{theory}} = 1$. However, besides this almost horizontal line, Fig. 3.5(a) displays a discontinuity near the middle of the glacier. The reason for this discontinuity can be deduced from Fig. 3.5(b). This figure shows that the difference in numerical solution for a grid spacing of $\Delta x = 200$ m and a grid spacing of $\Delta x = 100$ m, i.e. $H_{\Delta x} - H_{\Delta x}$, passes zero near the middle of the glacier, i.e. applying a smaller grid size results in a somewhat smaller surface slope. Locally, this results in a large rate of convergence. Despite this discontinuity, it can be concluded that because the determined value for the order of accuracy $p$ is approximately equal to the theoretical order of accuracy, the numerical scheme is implemented correctly.

3.2.4. Stop criterium for the inner loop

The sliding velocity $u_b$ has to be determined from the set of equations within the inner loop in Fig. 3.2. For each time step, the sliding velocity $u_b$ is determined with the Picard iteration method (Ferziger and Perić, 2002). For this iteration method, a stop criterium has to be prescribed. The following criterium is implemented: the program exits the inner loop as soon as

$$\left|\frac{u_{n+1} - u_{n}}{u_n} \right| \leq \epsilon,$$

holds for all grid points.

3.3. Results

For the purpose of verifying the results presented in Budd (1975), the same values for the input variables are applied. The bed profile and specific balance rate profile are obtained by fitting a polynomial to the digitized profiles from Budd (1975), see Fig. 3.6 for the result. It should be mentioned that the applied mass balance rate profile is constant in time and lacks mass-balance/height feedback.

First, the results of a run without basal lubrication will be discussed as for this case the numerical solution can be checked with an available analytical solution. Subsequently, results for two different non-zero values of the friction lubrication factor $\varphi$ will be studied, one that leads to a new steady state, and one that leads to oscillatory behaviour according to Budd (1975). This last run should validate the capability of the model to simulate surge behaviour.
3.3. Results

3.3.1. Results of the run without basal lubrication

When the friction lubrication factor $\varphi$ is set to zero, the basal stress equals the driving stress as in the SIA, see subsection 2.3.1. Consequently, the sliding velocity is uniform and equal to zero along the glacier. The glacier will grow to steady state, i.e.

$$\frac{\partial H}{\partial t} = 0 \quad \text{for all } x. \quad (3.18)$$

Fig. 3.7 shows the numerical results for a friction lubrication factor equal to zero. It displays the surface profile and the ice velocity profiles in steady state and presents the time evolution of the glacier length and the time evolution of the maximum vertical mean ice velocity. Fig. 3.7(c) shows that for $\varphi = 0$ the sliding velocity in steady state is indeed equal to zero along the glacier. The numerical results in Fig. 3.7 agree with the results published in Budd (1975).

The numerical results presented in Fig. 3.7 can also be checked with available analytical solutions. Namely, for a mass balance rate profile that is constant in time and lacks mass-balance/height feedback, the glacier length $L$ and the maximum ice flux $F_m$ in steady state can be determined analytically from

$$\int_0^L b \, dx = 0 \quad (3.19)$$

$$F_m = \frac{1}{2} \int_0^L |b| \, dx. \quad (3.20)$$

For the specific mass balance profile depicted in Fig. 3.6(b), these equations designate a steady state glacier length of approximately 51.8 km and a maximum ice flux of approximately $52 \, \text{m}^2 \, \text{s}^{-1}$. Thereby, the location of the maximum flux coincides with the location of the equilibrium line. The numerical results in Fig. 3.7 agree with these available analytical solutions.
Figure 3.7: Results of the run without basal lubrication, i.e. $\varphi = 0$. (a): Evolution of the glacier length in time. (b): Surface profile in steady state. (c): Ice velocity profiles in steady state. (d): Evolution of the maximum velocity along the glacier in time.

3.3.2. RESULTS OF THE RUNS WITH BASAL LUBRICATION

According to Budd (1975), the value of the friction lubrication factor dictates whether a glacier evolves to a new steady state or whether it shows quasi-oscillatory behaviour. Results for two non-zero values of the friction lubrication factor are discussed: (i) results for a small non-zero value that induced a new steady state, (ii) results for a larger non-zero value that induced quasi-oscillatory behaviour.

NEW STEADY STATE SOLUTION

According to Budd (1975), small non-zero values of the friction lubrication factor $\varphi$ induce a new steady state solution. Fig. 3.9 shows the numerical results for a friction lubrication factor $\varphi = 9.5 \times 10^{-3}$ Pa$^{-1}$ m$^{-1}$ s. In comparison with the case $\varphi = 0$, at the location of maximum ice flux the total velocity $u$ is slightly larger and the ice thickness $H$ is slightly smaller, which corresponds to the results presented in Budd (1975). However, Fig. 3.9(c) depicts a physically incorrect result: for the applied small non-zero value of $\varphi$, the sliding velocity in the lower part of the glacier becomes negative. The remainder of this subsection focusses on the origin of this physically incorrect result.

For small non-zero values of the friction lubrication factor $\varphi$, the lubrication-lowered stress $\tau^*_d$ is slightly less than the driving stress $\tau_d$, see the relevant equation in Fig. 3.2. The lubrication-lowered stress $\tau^*_d$ is smallest in a zone in the upper part of the glacier, because there the rate of energy dissipation $\tau_d u$ is largest, see Fig. 3.8(a). Due to the requirement of preservation of gross static equilibrium, i.e. $\tau_b = \tau_d$, in this zone in the upper part of the glacier the basal stress $\tau_b$ is slightly smaller than the driving stress $\tau_d$. On the contrary, in the lower part of the glacier and in the most upper part of the glacier the rate of energy dissipation is relatively small, hence for these parts of the glacier preservation of gross static equilibrium dictates a basal stress that is slightly larger than the driving stress, see Fig. 3.8(b). Eq. 3.9 then results in a positive gradient in sliding velocity $\frac{\partial u_b}{\partial x}$ in the upper part of the glacier, and a negative gradient in sliding velocity in the lower part of the
Figure 3.8: Typical results for the rate of energy dissipation (a), the difference between the driving stress and the basal stress (b), and the gradient in sliding velocity (c) for small non-zero values of the friction lubrication factor $\varphi$.

glacier, as depicted in Fig. 3.8(c). This is physically correct: in the upper part of the glacier the gradient in sliding velocity should be positive as this corresponds to extension, whereas in the lower part of the glacier the gradient should be negative as this corresponds to compression. Combining this profile in sliding velocity gradient with the upstream boundary condition for the sliding velocity, Eq. 3.8, it can be concluded that a small non-zero value of the friction lubrication factor $\varphi$ results in a positive basal sliding velocity in the upper part of the glacier, and a small or even negative basal sliding velocity in the lower part of the glacier. So, in order to preserve gross equilibrium, Budd’s model needs to allow for negative basal velocities.

That Budd’s model needs to allow for negative basal velocities can also be concluded from a closer examination of the equality that holds in steady state

$$\frac{\partial H}{\partial t} = \dot{b} - \frac{\partial}{\partial x} (uH) = 0 \quad \text{for all } x. \quad (3.21)$$

From this equality, it can be seen that in the lower part of the glacier where the specific balance rate $\dot{b}$ is negative, the gradient in ice flux $\frac{\partial}{\partial x} (uH)$ needs to be negative. This restriction can be rewritten to read

$$\frac{\partial}{\partial x} (uH) = u \frac{\partial H}{\partial x} + H \frac{\partial u}{\partial x} < 0. \quad (3.22)$$

Because near the terminus of the glacier it holds that $\frac{\partial H}{\partial x} < 0$, $H > 0$, and $\frac{\partial u}{\partial x} < 0$, also with this analysis it can be concluded that $u$ should be either small, or negative near the terminus. For the larger values in the range of values of $\varphi$ that lead to a new steady state, the sliding velocity in the lower part of the glacier needs to be negative.
Figure 3.9: Results of the run for a small non-zero value of the friction lubrication factor $\varphi$. (a): Evolution of the glacier length in time. (b): Surface profile in steady state. (c): Ice velocity profiles in steady state. (d): Evolution of the maximum velocity along the glacier in time.

OSCILLATORY BEHAVIOUR

According to Budd (1975), for larger values of the friction lubrication factor $\varphi$, the considered glacier does not grow to a new steady state, but instead shows quasi-oscillatory behaviour. For large values of $\varphi$, a zone with very large velocities characterizing the surge phase results from a positive feedback mechanism in the inner loop in the system of equations depicted in Fig. 3.2. The feedback mechanism in the inner loop dictates that locally a small basal stress results in an increase in the sliding velocity. Therefore, the local rate of energy dissipation also increases, hence the local basal stress decreases further. This positive feedback mechanism is essential in giving rise to a zone of very large velocities.

Unfortunately, for values of $\varphi$ for which the glacier shows quasi-oscillatory behaviour according to Budd (1975), the inner loop, which entails the essential positive feedback mechanism, does not converge. Therefore, the results in Budd (1975) for values of $\varphi$ that should have led to quasi-oscillatory behaviour, could not be verified with the applied numerical implementation. However, because the applied parametrization leads to physically incorrect results, no further attempt is made in verifying the results in Budd (1975). In the next chapter, the non-convergent inner loop in Fig. 3.2 that represents basal lubrication and determines the sliding velocity, will be replaced by a basal drainage model.
The model reconstructed in Chapter 3 does not include a basal drainage model. Therefore, the model does not capture the switches in basal drainage system suggested for surging LHT-glaciers. Furthermore, the surge front propagates solely through longitudinal stresses, i.e. the ice is pushed downslope, whereas the propagation of this front through the drainage of water at the base of the glacier is not taken into account. Because the drainage system at the base of a glacier is thought to be the key element in surge behaviour, the inclusion of a realistic basal drainage model seems essential in simulating surge behaviour more accurately. Therefore, the parameterization of frictional lubrication applied in Budd (1975) will be replaced by a simple basal drainage model. This drainage model will be coupled to the ice mass continuity equation by means of a sliding law, see Fig. 4.1. The result will be a Hydromechanically Coupled Model (HMCM).

The basal drainage model will be based on the transformation of the basal drainage system during a surge cycle as proposed in Section 4.2. In Section 4.3 building blocks for a basal drainage model proposed in literature will be discussed. Successively, the proposed transformation of the basal drainage system is translated into these building blocks by means of assumptions that simplify the equations. The result is a basal drainage model for surging LHT-glaciers, see Fig. 4.2. Table 4.1 provides an explanation of the symbols used in the remainder of this chapter.

4.1. SLIDING LAW

The sliding law forms the coupling between the ice mass continuity equation and the basal drainage model, see Fig. 4.1. A sliding law expresses basal sliding velocity as a function of basal stress. For this research, a sliding law of the ‘Weertman-type’ is adopted. This law is given by
with \( c \) a coefficient depending on the general physical characteristics of the bed; \( n \) an exponent taken to be equal to 3 (Budd et al., 1979); and \( N \) the effective pressure at the base of the glacier. This effective pressure is defined by \( N = p_o - p_w \), i.e. overburden pressure minus water pressure, with the overburden pressure given by \( p_o = \rho_i g H \). Eq. 4.1 states that the basal velocity is inversely related to the effective pressure at the base of the glacier, i.e. a larger water pressures at the base of the glacier results in larger basal velocities. However, a limiting value has to be set on the basal velocity for \( N \to 0 \). By involving the effective pressure, a non-unique relation between the basal velocity \( u_b \) and the basal stress \( \tau_b \) is obtained. As a simplification, the basal shear stress \( \tau_b \) will be replaced by the driving stress \( \tau_d \). (Oerlemans, 2013)

**4.2. Transformation of the Basal Drainage System During a Surge Cycle**

Fig. 4.3 shows a schematic depiction of the transformation of the basal drainage system during a surge cycle of a LHT-glacier, as suggested by this MSc Thesis. The depiction encompasses the quiescent phase, represented by green text boxes, and the surge phase, represented by black text boxes, and will be elucidated in the remainder of this section.

The surge cycle is made up by two distinct phases, the quiescent phase and the surge phase (Meier and Post, 1969). The most essential feature of the drainage system during the quiescent phase is the storage of water in growing cavities. During the quiescent phase, the cavities grow, see Fig. 4.4. As soon as the cavities drown the bed, see Fig. 4.5, water will no longer flow through discrete channels, see Fig. 4.8(a), but instead will flow between the ice and bed, see Fig. 4.14. This will be referred to as the switch in basal drainage system from channel system to cavity system. The surge starts.

During the surge phase, the glacier is thought to be divided in three distinguishable zones, see Fig. 4.6. In the upstream zone and the downstream zone, the glacier bed is not (yet) drowned. However, in the surge zone the glacier bed is drowned. In this zone, the water is drained over the whole width of the glacier, so that the drained water does not enter the discrete channels in the downstream zone, but instead fills the cavities located just downstream of the surge front. A consequence of this assumption is that cavities located just downstream of the surge front fill faster, which represents the desired propagation of the surge front through the transport of water. In addition, it can be seen that the velocity gradient over the surge front is large, which agrees with observations (Kamb et al., 1985). This large velocity gradient results in a bump of ice moving down with the surge front, see Fig. 1.10.

Where the cavities drown the bed the drainage of water becomes far less efficient, which is confirmed by Kamb et al. (1985). The flow of water in the layer between the ice and bed is instable, i.e. the flow wants to concentrate in discrete channels (Weertman, 1972). This will cause the switch back from cavity system to
4.2. Transformation of the Basal Drainage System During a Surge Cycle

The water in the layer between the ice and bed is released. **The surge terminates.**

The surge front reaches the terminus of the glacier.

The water flow between the ice and bed is unstable and wants to concentrate in discrete channels.

The water flowing out of the surge zone fills the cavities located just downstream of the surge front.

Cavities drown the bed.

Water no longer flows through discrete channels, but instead flows between the ice and bed. **The surge starts.**

Figure 4.3: Schematic rendition of the detailed transformation of the basal drainage system during surge cycles of LHT-glaciers proposed by this MSc Thesis.

Figure 4.4: Schematic representation of the increase of the amount of water stored in cavities, from (a) to (b), during the quiescent phase. The white area represents ice; the blue area water; and the grey area bed rock.

Figure 4.5: Schematic representation of cavities that drown the glacier bed, which defines the start of the surge phase.

channel system. Hence, also the surge rear moves downslope during the surge. When the surge front reaches the terminus of the glacier, the surge phase terminates and the water in the layer between the ice and bed is released. This causes the drop in glacier surface and the flood discharge observed during the termination of the surge (Kamb et al., 1985). The surge cycle now restarts.

This proposed transformation of the basal drainage system during surge cycles of LHT-glaciers capture the switches in basal drainage system and includes the essential role of water transport in the propagation of the surge front. This description will form the starting point for the basal drainage model constructed in the next sections.
4.3. **Building Blocks of the Basal Drainage Model**

The basal drainage model will be built out of the following building blocks: a water input model; a water balance model; a water transport model; a model for the capacity of the basal drainage system; and a model for the englacial storage rate, see Fig. 4.7. With the combination of these models, the effective pressure $N$, needed as input for the sliding law, can be determined. The building blocks have been proposed in literature and will be discussed in this section. One elemental assumption is applied:

(i) The channel and cavity system are not interlinked, i.e. they coexist.

This entails the simplification that the water pressures in the two separated drainage systems are not coupled. In addition, no term accounting for exchange of water between the two systems has to be included in the water balance models.

![Schematic representation of the building blocks of the basal drainage model and the applied connection to ice mass continuity equation.](image)
### 4.3. Building Blocks of the Basal Drainage Model

Table 4.1: List of symbols used in the building blocks of the basal drainage model. (Hewitt et al., 2012) (Schoof et al., 2014) (Bueler and Brown, 2009) (Schoof et al., 2012) (Budd et al., 1979)

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>SI Unit</th>
<th>Value</th>
</tr>
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<tbody>
<tr>
<td>α</td>
<td>parameter in Darcy-Weisbach law</td>
<td>–</td>
<td>5/4</td>
</tr>
<tr>
<td>β</td>
<td>parameter in Darcy-Weisbach law</td>
<td>–</td>
<td>3/2</td>
</tr>
<tr>
<td>μ</td>
<td>englacial porosity</td>
<td>–</td>
<td>0.01</td>
</tr>
<tr>
<td>ρ_i</td>
<td>ice density</td>
<td>kg m⁻³</td>
<td>910</td>
</tr>
<tr>
<td>ρ_w</td>
<td>water density</td>
<td>kg m⁻³</td>
<td>1000</td>
</tr>
<tr>
<td>τ_b</td>
<td>basal stress</td>
<td>Pa</td>
<td>–</td>
</tr>
<tr>
<td>φ</td>
<td>hydraulic potential</td>
<td>Pa</td>
<td>–</td>
</tr>
<tr>
<td>a</td>
<td>channel aspect ratio</td>
<td>–</td>
<td>0.061</td>
</tr>
<tr>
<td>A</td>
<td>fluidity coefficient Glen’s flow law</td>
<td>Pa⁻¹ s⁻¹</td>
<td>4.5 · 10⁻²⁴</td>
</tr>
<tr>
<td>c</td>
<td>coefficient in sliding law</td>
<td>m s⁻¹ Pa⁻²</td>
<td>5.1 · 10⁻²³</td>
</tr>
<tr>
<td>c₁</td>
<td>cavitation coefficient</td>
<td>m⁻¹</td>
<td>0.5</td>
</tr>
<tr>
<td>c₂</td>
<td>creep closure coefficient cavity system</td>
<td>–</td>
<td>0.158</td>
</tr>
<tr>
<td>c₃</td>
<td>conductivity coefficient channel system</td>
<td>m⁷/₄ Pa⁻¹/₂ s⁻¹</td>
<td>6.3 · 10⁻²³</td>
</tr>
<tr>
<td>c₄</td>
<td>wall melting coefficient</td>
<td>–</td>
<td>3.3 · 10⁻¹⁹</td>
</tr>
<tr>
<td>c₅</td>
<td>creep closure coefficient channel system</td>
<td>Pa⁻³ s⁻¹</td>
<td>2.6 · 10⁻²⁴</td>
</tr>
<tr>
<td>c.rel</td>
<td>scale coefficient cavity system</td>
<td>–</td>
<td>0.1</td>
</tr>
<tr>
<td>d</td>
<td>representative layer thickness capacity cavities</td>
<td>m</td>
<td>–</td>
</tr>
<tr>
<td>d.en</td>
<td>englacial stored water around cavities</td>
<td>m</td>
<td>–</td>
</tr>
<tr>
<td>d_water</td>
<td>representative water layer thickness in cavities</td>
<td>m</td>
<td>–</td>
</tr>
<tr>
<td>d.ar</td>
<td>representative layer thickness cavities linking orifices</td>
<td>m</td>
<td>–</td>
</tr>
<tr>
<td>dₑ</td>
<td>mean roughness scale basal topography</td>
<td>m</td>
<td>0.25</td>
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<tr>
<td>f</td>
<td>friction factor</td>
<td>m</td>
<td>0.5</td>
</tr>
<tr>
<td>g</td>
<td>gravitational acceleration</td>
<td>m s⁻²</td>
<td>9.81</td>
</tr>
<tr>
<td>H</td>
<td>ice thickness</td>
<td>m</td>
<td>–</td>
</tr>
<tr>
<td>k</td>
<td>conductivity coefficient cavity system</td>
<td>m⁷/₄ kg¹/₂</td>
<td>0.01</td>
</tr>
<tr>
<td>Lf</td>
<td>latent heat of fusion for ice</td>
<td>J kg⁻¹</td>
<td>3.35 · 10⁻⁶</td>
</tr>
<tr>
<td>m</td>
<td>total water input</td>
<td>m s⁻¹</td>
<td>–</td>
</tr>
<tr>
<td>m_ba</td>
<td>basal water input</td>
<td>m s⁻¹</td>
<td>–</td>
</tr>
<tr>
<td>m.ca</td>
<td>water input cavity system</td>
<td>m s⁻¹</td>
<td>–</td>
</tr>
<tr>
<td>m.ch</td>
<td>water input channels</td>
<td>m s⁻¹</td>
<td>–</td>
</tr>
<tr>
<td>m.en</td>
<td>englacial water input</td>
<td>m s⁻¹</td>
<td>–</td>
</tr>
<tr>
<td>n</td>
<td>exponent Glen’s flow law</td>
<td>–</td>
<td>3</td>
</tr>
<tr>
<td>p</td>
<td>percentage of basal meltwater stored in cavity system</td>
<td>–</td>
<td>20%</td>
</tr>
<tr>
<td>p_o</td>
<td>overburden pressure</td>
<td>Pa</td>
<td>–</td>
</tr>
<tr>
<td>p_w</td>
<td>water pressure drainage system</td>
<td>Pa</td>
<td>–</td>
</tr>
<tr>
<td>Q</td>
<td>water discharge through drainage system</td>
<td>m³ s⁻¹</td>
<td>–</td>
</tr>
<tr>
<td>q.ca</td>
<td>water discharge through cavity system</td>
<td>m² s⁻¹</td>
<td>–</td>
</tr>
<tr>
<td>Q.ch</td>
<td>water discharge through channels</td>
<td>m³ s⁻¹</td>
<td>–</td>
</tr>
<tr>
<td>S</td>
<td>cross-sectional area subglacial channels</td>
<td>m²</td>
<td>–</td>
</tr>
<tr>
<td>S.en</td>
<td>englacial stored water around channels</td>
<td>m²</td>
<td>–</td>
</tr>
<tr>
<td>S_water</td>
<td>amount of water in channels</td>
<td>m²</td>
<td>–</td>
</tr>
<tr>
<td>u_b</td>
<td>sliding velocity</td>
<td>m s⁻¹</td>
<td>–</td>
</tr>
<tr>
<td>u_b,max</td>
<td>limitation sliding velocity</td>
<td>m s⁻¹</td>
<td>1.3 · 10⁻⁴</td>
</tr>
<tr>
<td>W.ch</td>
<td>lateral distance between channels</td>
<td>m</td>
<td>200</td>
</tr>
<tr>
<td>W.en</td>
<td>width exchange water between channel and ice</td>
<td>m</td>
<td>10</td>
</tr>
</tbody>
</table>
4.3.1. **WATER INPUT MODEL**

With the water input model, the meltwater that enters the basal drainage system is determined. It has to be pointed out that for temperate glaciers it holds that no heat is used for increasing the temperature of the ice to pressure melting point, i.e. all available heat is used for melting ice. In order to simplify the model, various assumptions are made.

(i) **Surface meltwater does not reach the base of the glacier.**

As a consequence of this assumption, the associated seasonal variation in meltwater input is not captured. However, because the timescale of a surge cycle, $10^{-1} - 10^2$ y, is at least one order of magnitude larger than the timescale of seasonal variation in surface meltwater input, it can be concluded that this seasonal variation does not play an essential role in the initiation and termination of the surge phase. Therefore, assumption (ii) seems appropriate.

(ii) **The meltwater input from wall melting of the channels is negligible.**

Whereas the timescale of changes in geothermal heat flux are at least one order of magnitude larger than the timescale of a surge cycle, it can be concluded that also the geothermal heat flux does not play a determinant role in the initiation and termination of the surge phase. To avoid the necessity of including this heat flux anyway, basal melt due to geothermal heat flux from bed towards ice is not taken into account. The only meltwater inputs that are left and taken into account are englacially produced meltwater through strain heating and meltwater produced at the base of the glacier through basal frictional heating at the ice-bed interface.

(iii) **All englacially produced meltwater is transported vertically through the ice column, i.e. the at a specific location along the glacier produced englacial meltwater enters the basal drainage system at the same location.**

This assumption is underpinned by the fact that the vertical gradient in velocity $\frac{\partial u}{\partial z}$ is largest in the lower part of the ice column, see Eq. 2.24, so that strain heating is largest in this lower part.

(iv) **The timescale of vertical englacial transport of water is small compared to the applied time step $\Delta t$, i.e. the in a time step produced englacial meltwater enters the basal drainage system in the same time step.**

In conclusion, no englacial drainage model has to be included.

With these assumptions, the water input model is then given by (Benn and Evans, 2014)

$$m = m_{en} + m_{ba} = \frac{2AH}{\rho_w L_f (n+2)} r_{d}^{n+1} + \frac{\tau_b \mu_b}{\rho_w L_f}, \quad (4.2)$$

Where the first term represents englacial produced meltwater through strain heating, and the second term represents meltwater produced at the base of the glacier through basal frictional heating at the ice-bed interface. Again, as a simplification, the basal shear stress $\tau_b$ will be replaced by the driving stress $\tau_d$. Because of the assumption that both drainage systems are not interlinked, it has to be prescribed which part of the total melt input enters the channel system $m_{ch}$ and which part enters the cavity system $m_{ca}$.

4.3.2. **WATER BALANCE MODEL**

The second building block of the basal drainage model concerns the water balance model. Following Bueler and Van Pelt (2014), water balance models for channel systems and cavity systems are proposed.

**Channel system**

Channels are characterized by a direct relation between effective pressure and water discharge, i.e. $\frac{\partial N}{\partial Q_{ch}} > 0$ (Nye, 1976). Because water flows from high to low hydraulic potential $\phi$, defined by $\phi = p_w + \rho_w gb$, a linear stability analysis shows that the direct relation between effective pressure and water discharge entails the growth of one channel at the expense of nearby ones (Nye, 1976). So, the resulting channels will be located at some distance, and will cover only a limited part of the bed. Therefore, a discrete treatment of a channel system seems naturally, see Fig. 4.8(a).

The water balance of an individual channel is then given by
4.3. BUILDING BLOCKS OF THE BASAL DRAINAGE MODEL

Figure 4.8: Schematic representation of the discrete treatment of the channel system (a) and the continuum treatment of the cavity system (b).

\[
\frac{\partial S_{\text{water}}}{\partial t} = -\nabla Q_{ch} + m_{ch} W_{ch} \tag{4.3}
\]

with \( S_{\text{water}} \) the amount of water in the channel and \( W_{ch} \) the average lateral distance between two channels. Eq. 4.3 states that the difference between the local water input and gradient in discharge equals the rate of change of the amount of water in the channel.

**CAVITY SYSTEM**

The interlinked cavities that make up the cavity system are characterized by an inverse relation between effective pressure and water discharge, i.e. \( \frac{\partial p}{\partial q} < 0 \) (Kamb, 1987), resulting in the coexistence of many cavities. Therefore, for a cavity system it seems natural to apply a continuum treatment, see Fig. 4.8(b) with \( d_{\text{water}} \) as the representative water layer thickness in the cavities. The water balance model of a cavity system can then be written as

\[
\frac{\partial d_{\text{water}}}{\partial t} = -\nabla q_{ca} + m_{ca}. \tag{4.4}
\]

**4.3.3. WATER TRANSPORT MODEL**

For both drainage systems, the equation for the transport of water is based on a Darcy-Weisbach law. Thereby, in contrast to the flow of ice, the flow of water is assumed to be turbulent.

**CHANNEL SYSTEM**

The transport through a channel can be determined with (Schoof et al., 2014)

\[
Q_{ch} = -c_3 S_{\text{water}}^{\alpha} |\nabla \phi_{ch}|^{\beta} - 2 |\nabla \phi_{ch}|. \tag{4.5}
\]

Herein, the gradient in hydraulic potential forms the driving force, whereas \( c_3 S_{\text{fill}}^{\alpha} \) represents friction. Following Schoof et al. (2014), the channels are assumed to have a low aspect ratio \( a \), i.e. the channels are assumed to be more flat than semi-circular conduits. In addition, it is assumed that the tortuosity of the channels is negligible and that the channels are directed downslope.

**CAVITY SYSTEM**

The transport through a cavity system can be written as (Schoof et al., 2012)

\[
q_{ca} = -k d_{or}^{\alpha} |\nabla \phi_{ca}|^{\beta} - 2 |\nabla \phi_{ca}|. \tag{4.6}
\]

The flux of water through a cavity system \( q_{ca} \) is controlled by the smallest connections, the so-called orifices. Following Schoof et al. (2012), it is assumed that the representative layer thickness of the orifices \( d_{or} \) scales with the representative water layer thickness in the cavities \( d_{water} \): 

\[
d_{or} = e_{or} d_{water}. \tag{4.7}
\]
4.3.4. Model for the Capacity of the Drainage System

The rate of change of the capacity of a drainage system is determined by the difference between opening and closure of the system.

Channel System

Channels open by wall melting and close by viscous creep, see Fig. 4.9. These mechanisms are captured by the following equation for the rate of change of the cross-sectional area of a channel (Schoof et al., 2014)

\[
\frac{\partial S}{\partial t} = -c_4 Q_{ch} \nabla \phi_{ch} - c_5 (p_o - p_{w, ch})^n S
\]  

(4.8)

with \( S \) the cross-sectional area of the channels. In Eq. 4.8, the first term represents opening and the second term represents closure.

Cavity System

Cavities close by the same mechanism as channels do, see the similarity between the second term in Eq. 4.8 and the second term in Eq. 4.9. However, cavities open by a different mechanism: they form in the lee of bed perturbations due to sliding of the ice over these perturbations, see Fig. 4.10. These mechanisms are captured by the following equation for the rate of change of the representative layer thickness of the capacity of the cavities (Schoof et al., 2012)

\[
\frac{\partial d}{\partial t} = c_1 u_b (d_r - d) - c_2 A (p_o - p_{w, ca})^n d
\]  

(4.9)

with \( d_r \) the mean roughness height of the bed protrusions.

4.3.5. Model for the Englacial Storage Rate

The combination of Eq. 4.4 and 4.9 results in the following equation for the water pressure evolution:

\[
0 = -\nabla q_{ca} + m_{ca} - c_1 u_b (d_r - d) - c_2 A (p_o - p_{w, ca})^n d.
\]  

(4.10)

This equation is stiff, and this stiffness is caused by the incompressibility of water and the low distensibility, defined as \( \frac{1}{V} \frac{\partial V}{\partial P} \), of ice. Moreover, Eq. 4.10 is elliptic, and therefore hard to implement numerically. However, Qin (2014) shows that Eq. 4.10 can be transformed into a parabolic equation by means of: (i) including the rate of change of englacially stored water \( \frac{\partial d_{w, ca}}{\partial t} \) in the water balance model, i.e.

\[
\frac{\partial d_{water}}{\partial t} + \frac{\partial d_{en}}{\partial t} = -\nabla q_{ca} + m_{ca}
\]  

(4.11)
and (ii) assuming that the englacial system of cracks is connected efficiently to the basal drainage system, the amount of englacially stored water is equivalent to the subglacial water pressure, i.e.

\[ \frac{\partial d_{en}}{\partial t} = \frac{\mu}{\rho_w g} \frac{\partial p_{w,ca}}{\partial t}. \]  

Combining these equations with Eq. 4.9, the following equation for the rate of change of water pressure results

\[ \frac{\mu}{\rho_w g} \frac{\partial p_{w,ca}}{\partial t} = -\nabla q_{ca} + m_{ca} - c_1 u_b (d_r - d) + c_2 A (p_o - p_{w,ca})^n d \]  

which is parabolic instead of elliptic.

In Eq. 4.13 the porosity of ice \( \mu \) plays a determinative role: it regulates how rapidly the water pressure responds to changes in melt input and changes in the gradient in discharge. It is important to emphasize that including the rate of change of englacially stored water does not alter the steady state solution. The only effect of a large value for the porosity \( \mu \) is a reduction in the rate of change in water pressure and a reduction in the range of the change in water pressure.

For channel systems the following equation for the rate of change of englacially stored water will be included

\[ \frac{\partial S_{en}}{\partial t} = \frac{\mu W_{en}}{\rho_w g} \frac{\partial p_{w,ch}}{\partial t} \]  

with \( W_{en} \) the width over which water is exchanged efficiently between channels and ice.

![Figure 4.11: Schematic representation of only partly filled cavities.](image)

**4.3.6. CLOSURE OF THE SYSTEM OF EQUATIONS**

Regarding cavity systems, Eq. 4.11, 4.9, and 4.12 together form a set of 3 equations for the 4 unknowns \( d_{water}, d_{en}, d, \) and \( p_{w,ca} \). To close this set of equations, it is assumed that the cavities are completely filled at every moment in time (Bueler and Van Pelt, 2014), hence the situation depicted in Fig. 4.11 does not occur. Therefore, the representative water layer thickness in the cavities \( d_{water} \) equals the representative layer thickness of the capacity of the cavities \( d \). The resulting system of equations for cavity systems is given by

\[ \frac{\partial d}{\partial t} + \frac{\partial d_{en}}{\partial t} = -\nabla q_{ca} + m_{ca} \]  

\[ \frac{\partial d}{\partial t} = c_1 u_b (d_r - d) - c_2 A (p_o - p_{w,ca})^n d \]  

\[ \frac{\partial d_{en}}{\partial t} = \frac{\mu}{\rho_w g} \frac{\partial p_{w,ca}}{\partial t}. \]

Alike, the resulting system of equations for channel systems is given by

\[ \frac{\partial S}{\partial t} + \frac{\partial S_{en}}{\partial t} = -\nabla Q_{ch} + m_{ch} W_{ch} \]  

\[ \frac{\partial S}{\partial t} = -c_4 Q_{ch} \nabla \phi_{ch} - c_5 (p_o - p_{w,ch})^n S \]
\[
\frac{\partial S_{en}}{\partial t} = \frac{\mu W_{en}}{\rho_w g} \frac{\partial p_{w, ch}}{\partial t}.
\]  

(4.20)

4.4. A BASAL DRAINAGE MODEL FOR SURGING, LAND-TERMINATING, HARD-BEDDED, TEMPERATE GLACIERS

In this section, basal drainage models for the two distinct phases of the surge cycle will be constructed. The models will be based on the transformation of the basal drainage system proposed in Section 4.2, and will be composed of the building blocks presented in Section 4.3, see Fig. 4.2. Additionally, schematic conditions that prescribe the sudden initiation and the sudden termination of the surge phase will be discussed.

4.4.1. QUIESCENT PHASE

In order to construct a simple basal drainage model representative for the quiescent phase, the following assumptions are made.

(i) During the quiescent phase drainage of water through the cavity system is negligible, i.e. \( q_{ca} = 0 \).

Therefore, during the quiescent phase the cavity system is used only for storage of water.

(ii) All englacially produced meltwater \( m_{en} \) finds its way to the subglacial channels.

(iii) A certain part of the meltwater produced at the base of the glacier, \( p m_{ba} \), is stored in not-interlinked cavities. The remaining part, \( (1 - p) m_{ba} \), finds its way to the subglacial channels.

(iv) At the start of the quiescent phase, the representative layer thickness of the cavities and the water pressure in the cavities are equal to zero, i.e. \( d = 0 \) and \( p_{w, ca} = 0 \) for all \( x \).

During the quiescent phase, the cavities grow to cover an increasing part of the bed, see Fig. 4.4. However, during a large part of the quiescent phase this coverage is limited, i.e. large parts of the ice-bed interface are not lubricated. This limited coverage has to be accounted for in a representative value for the width-averaged effective pressure \( N \). Fowler (1987) proposes a shadowing function which determines what part of the bed is overlain by cavities. However, in order to keep the basal drainage model as simple as possible, in this research the following assumption is made:

(v) During the quiescent phase the coverage of glacier bed by cavities is small.

As a result, the effective pressure can be approximated by \( N = p_o \), hence the basal velocity does not depend on the water pressure at the base of the glacier, see Eq. 4.1. This assumption is supported by observations revealing that the change in surface velocity during the quiescent phase can largely be assigned to the change in internal deformation velocity, hence to changes in ice profile, i.e. changes in ice thickness and surface slope.

(vi) The rate of change of englacially stored water is negligible compared to the rate of change of the capacity of the drainage system, i.e. \( \frac{\partial S_{en}}{\partial t} \ll \frac{\partial S}{\partial t} \) and \( \frac{\partial d_{en}}{\partial t} \ll \frac{\partial d}{\partial t} \).

hence the water balance models Eq. 4.15 and 4.18 can be approximated by

\[
\frac{\partial S}{\partial t} = -\nabla Q_{ch} + m_{ch} W_{ch}
\]  

(4.21)

\[
\frac{\partial d}{\partial t} = -\nabla q_{ca} + m_{ca}
\]  

(4.22)

so that equations for the capacities of the drainage systems result. Those equations for the capacities of the drainage systems are preferred over the equations presented in subsection 4.3.4

\[
\frac{\partial S}{\partial t} = -c_4 Q_{ch} \nabla \phi_{ch} - c_5 (p_o - p_{w, ch})^n S
\]  

(4.23)

\[
\frac{\partial d}{\partial t} = c_1 u_b (d_r - d) - c_2 A (p_o - p_{w, ca})^n d
\]  

(4.24)
4.4. A BASAL DRAINAGE MODEL FOR SURGING, LAND-TERMINATING, HARD-BEDDED, TEMPERATE GLACIERS

because Eq. 4.23 is hard to implement numerically, for the opening term, which is the first term on the right-hand side, is equal to zero when no channels are present yet.

These six assumptions applied to the building blocks presented in Section 4.3 result in the equations for the four unknowns \( S, p_{w, ch}, d \) and \( p_{w, ca} \) displayed in Table 4.2 and 4.3.

4.4.2. INITIATION OF THE SURGE

When, during the quiescent phase, the cavities grow, their coverage of the bed increases. Lastly, the assumption that the coverage of glacier bed by cavities is small, is no longer met. The effective pressure then has to be determined from \( N = p_0 - p_{w, ca} \). By means of the sliding law Eq. 4.1, this change locally results in a large increase in basal velocity. The applied condition for this local change is given by

\[
d(x) = d_r.
\]  

As soon as the representative layer thickness of the cavity system \( d \) equals the mean roughness scale of the basal topography \( d_r \), the cavities do not grow any longer. Meanwhile, the water pressure in the cavities \( p_{w, ca} \) is still lower than the local overburden pressure \( p_0 \). However, due to the closure term in Eq. 4.13 the water pressure keeps increasing. As soon as the water pressure in the cavities equals the local overburden pressure, the ice is lifted and water will no longer flow through discrete channels, but instead will flow between the ice and bed. The time lag between \( d = d_r \) and \( p_{w, ca} = p_0 \) depends on the applied value for the ice porosity \( \mu \). Because a switch criterion that is independent of the applied value for the ice porosity is indispensable, criterion 4.25 is taken to prescribe the switch from channel system to cavity system.

4.4.3. SURGE PHASE

During the surge phase a zone with large surface velocities moves downslope. This divides the glacier in three distinguishable zones: an upstream zone; the surge zone; and a downstream zone, see Fig. 4.12. Each zone will be treated separately.

<table>
<thead>
<tr>
<th>Model component</th>
<th>Channel system</th>
</tr>
</thead>
<tbody>
<tr>
<td>Water input model</td>
<td>( m_{ch} = (1 - p) m_{ba} + m_{en} )</td>
</tr>
<tr>
<td>Water balance model</td>
<td>( \frac{dS}{dt} = -\nabla Q_{ch} + m_{ch} W_{ch} )</td>
</tr>
<tr>
<td>Water transport model</td>
<td>( Q_{ch} = -c_3 S^2 \left</td>
</tr>
<tr>
<td>Water pressure model</td>
<td>( W_{en} \mu \frac{\partial p_{w, ch}}{\partial t} = -\nabla Q_{ch} + m_{ch} W_{ch} + c_4 Q_{ch} \nabla \phi_{ch} + c_5 \left( p_0 - p_{w, ch} \right)^n S )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Model component</th>
<th>Cavity system</th>
</tr>
</thead>
<tbody>
<tr>
<td>Water input model</td>
<td>( m_{ca} = p m_{ba} )</td>
</tr>
<tr>
<td>Water balance model</td>
<td>( \frac{dd}{dt} = m_{ca} )</td>
</tr>
<tr>
<td>Water transport model</td>
<td>( q_{ca} = 0 )</td>
</tr>
<tr>
<td>Water pressure model</td>
<td>( \mu \frac{\partial p_{w, ca}}{\partial t} = m_{ca} - c_1 u_b (d_r - d) + c_2 A (p_0 - p_{w, ca})^n d )</td>
</tr>
</tbody>
</table>

Figure 4.12: Schematic representation of the three distinguishable zones during the surge phase.
UPSTREAM ZONE

In the zone upstream of the large surface velocities nothing changed with respect to the quiescent phase. Therefore, this zone can be represented by the equations in Table 4.2 and 4.3.

SURGE ZONE

In the surge zone, water no longer flows through discrete channels, but instead flows between the ice and bed. Because solving the water flow between the ice and bed \( Q_{\text{ice-bed}} \) with a Darcy-Weisbach law would require the inclusion of a poorly-defined coefficient that is not set in literature, this flow and the thickness of the water film between the ice and bed will not be resolved. Instead, it is assumed that

(i) no matter the length of the surge zone, the time water needs to cross the surge zone is small compared to the applied time step \( \Delta t \). i.e. friction can be neglected.

In Section 4.2 it was proposed that water in the surge zone flows over the entire width. Therefore, the drained water does not enter the discrete channels in the downstream zone, but instead fills the cavities located just downstream of the surge front. It is assumed that

(ii) all the water reaching the surge front, can flow into these cavities.

This assumption implies that no water is stored in the layer between the ice and bed. The two assumptions together imply that the surge zone can be treated as one large grid cell with the following water balance model, see Fig. 4.13

\[
Q_{\text{out}} = Q_{\text{in}} + \int_{x=x_{\text{begin}}}^{x=x_{\text{end}}} (m) \, dx. \tag{4.26}
\]

Herein, \( Q_{\text{in}} \) is equal to the discharge through the channels just upstream of the surge zone. This treatment allows for the determination of \( Q_{\text{out}} \) without solving the water flow between the ice and bed \( Q_{\text{ice-bed}} \).

![Figure 4.13: Schematic representation of the treatment of the surge zone as one large grid cell.](image)

Because in the surge zone the cavities do not grow and the channels are lying on top of the water film between the ice and bed, see Fig. 4.14, the amount of water \( m \) entering the water layer is equal to the total water input given by Eq. 4.2. A remark regarding water input due to basal frictional heating is in place. Namely, in the surge zone, potential energy is not converted to heat through friction between the ice and bed, but instead is converted to kinetic energy. However, the compression of ice in the lower part of the surge zone ensures that this kinetic energy will be converted to heat within the surge zone. Because of the assumption that all heat is used for melting ice and because of the treatment of the surge zone as one large grid cell, it makes sense that Eq. 4.2 still holds for the surge zone as a whole.

Table 4.4 and 4.5 display the applied equations in the surge zone. For the channel system it is assumed that in the surge zone the channels are in steady state, i.e. \( \frac{\partial S}{\partial t} = 0 \) and \( \frac{\partial p_{\text{w,ca}}}{\partial t} = 0 \). To add, the size of the cavities in the surge zone remains equal to the mean roughness scale of the basal topography \( d_r \), and the water pressure \( p_{\text{w,ca}} \) increases to become and remain equal to the local overburden pressure. The discharge through orifices linking the cavities is assumed to be negligible compared to the flow of water between the ice and bed \( Q_{\text{ice-bed}} \).
Table 4.4: System of equations for the channel system in the surge zone during the surge phase.

<table>
<thead>
<tr>
<th>Model component</th>
<th>Channel system</th>
</tr>
</thead>
<tbody>
<tr>
<td>Water input model</td>
<td>$m_{ch} = 0$</td>
</tr>
<tr>
<td>Water balance model</td>
<td>$\frac{\partial S}{\partial t} = 0$</td>
</tr>
<tr>
<td>Water transport model</td>
<td>$Q_{ch} = 0$</td>
</tr>
<tr>
<td>Water pressure model</td>
<td>$\frac{\mu}{\rho_w g} \frac{\partial p_{w,ch}}{\partial t} = 0$</td>
</tr>
</tbody>
</table>

Table 4.5: System of equations for the cavity system in the surge zone during the surge phase.

<table>
<thead>
<tr>
<th>Model component</th>
<th>Cavity system</th>
</tr>
</thead>
<tbody>
<tr>
<td>Water input model</td>
<td>$m_{ca} = 0$</td>
</tr>
<tr>
<td>Water balance model</td>
<td>$d = d_r$</td>
</tr>
<tr>
<td>Water transport model</td>
<td>$q_{ca} = 0$</td>
</tr>
<tr>
<td>Water pressure model</td>
<td>$\frac{\mu}{\rho_w g} \frac{\partial p_{w,ca}}{\partial t} = c_2 A (p_o - p_{w,ca})^n$</td>
</tr>
</tbody>
</table>

**Downstream zone**

It was assumed that during the surge phase water in the surge zone flows over the entire width. Therefore, this water will not enter the discrete channels, but will instead flow into the cavities located just downstream of the surge front. An important consequence of this assumption is that downstream of the surge zone the channels are only partly filled, hence the building blocks for channel systems presented in subsection 4.3.6 are no longer valid. Instead it is assumed that the size of the channels in the downstream zone does not change in time, i.e. $\frac{\partial S}{\partial t} = 0$, and that the water pressure in the channels in the downstream zone is equal to the atmospheric pressure, i.e. $p_{w,ch} = 0$. Table 4.6 and 4.7 display the applied equations in the downstream zone.

Table 4.6: System of equations for the channel system in the downstream zone during the surge phase.

<table>
<thead>
<tr>
<th>Model component</th>
<th>Channel system</th>
</tr>
</thead>
<tbody>
<tr>
<td>Water input model</td>
<td>$m_{ch} = (1-p) m_{ba} + m_{en}$</td>
</tr>
<tr>
<td>Water balance model</td>
<td>$\frac{\partial S}{\partial t} = 0$</td>
</tr>
<tr>
<td>Water transport model</td>
<td>$Q_{ch} = \int_{x=x_{begin}}^{x=x_{end}} (m_{ch} W_{ch}) , dx$</td>
</tr>
<tr>
<td>Water pressure model</td>
<td>$p_{w,ch} = 0$</td>
</tr>
</tbody>
</table>

Table 4.7: System of equations for the cavity system in the downstream zone during the surge phase.

<table>
<thead>
<tr>
<th>Model component</th>
<th>Cavity system</th>
</tr>
</thead>
<tbody>
<tr>
<td>Water input model</td>
<td>$m_{ca} = p m_{ba}$</td>
</tr>
<tr>
<td>Water balance model</td>
<td>$\frac{\partial d}{\partial t} = m_{ca}$</td>
</tr>
<tr>
<td>Water transport model</td>
<td>$q_{ca} = 0$</td>
</tr>
<tr>
<td>Water pressure model</td>
<td>$\frac{\mu}{\rho_w g} \frac{\partial p_{w,ca}}{\partial t} = m_{ca} - c_1 u_b (d_r - d) + c_2 A (p_o - p_{w,ca})^n , d$</td>
</tr>
</tbody>
</table>
4.4.4. Termination of the Surge
The flow of water in the layer between the ice and bed is unstable, i.e. the flow wants to concentrate in discrete channels. This will cause the switch back from cavity system to channel system. Hence, also the surge rear moves downslope during the surge. However, the criterion for the switch back from cavity system to channel system is not obvious within the framework of the constructed schematization of the transformation of the basal drainage model during the surge phase. Because of the limited time assigned to this research, the draft of this criterion was left to future studies. When the surge front reaches the terminus of the glacier, the surge phase terminates and the water in the layer between the ice and bed is released.

4.5. Numerical Implementation of the Basal Drainage Model
The constructed basal drainage model will be coupled to the ice mass continuity equation by means of a sliding law, see Fig. 4.1. The numerical implementation of the ice mass continuity equation is already elaborated. Appropriate numerical schemes for the building blocks of the basal drainage model have yet to be selected. From a closer look at Table 4.2-4.7, it can be concluded that the numerical approximation of the channel system during the quiescent phase is most complex. Therefore, this numerical approximation will be elaborated in the remainder of this chapter. It must be added that in a similar, but less comprehensive way, appropriate numerical approximations for the surge phase and the cavity system were obtained.

4.5.1. Numerical Approximation
In the upcoming sections the numerical approximation of consecutively the water input model, water balance model and water pressure model for a channel system during the quiescent phase, see Table 4.2, will be discussed. First, explicit numerical schemes were considered. However, the normative time stepping restriction resulted in a maximum time step in the order of seconds (see Appendix C). For practical computation time, this was decided to be far too restrictive as the period of the surge cycle is in the order of tens of years. Therefore, a semi-implicit scheme was implemented. In this semi-implicit scheme, the advective and diffusive terms are taken implicitly. The other terms are source terms and are approximated using Ferziger and Perić (2002). This paper puts forward that generally, any source term \( S \) can be taken explicitly whenever \( S(p_{w,ch,m}^n) > 0 \), but that it has to be replaced by \( S(p_{w,ch,m}^n) p_{w,ch,m}^{n+1} \) whenever \( S(p_{w,ch,m}^n) < 0 \). Hence, whenever \( S(p_{w,ch,m}^n) < 0 \) this term is lumped into the main diagonal element of \( p_{w,ch,m}^{n+1} \), which enhances the numerical stability.

Because of the presence of time-dependent source terms in the equations presented in Table 4.2, these equations are stiff. Therefore, applying \( \theta = 1 \) in the \( \theta \)-method is most appropriate (Zijlema, 2012). Now, the maximum time step for which the basal drainage model is stable, is determined by the time scale of variation in the source terms, which turned out to be in the order of hours/days. This is still one order of magnitude smaller than the in Chapter 3 obtained time stepping restriction for the ice mass continuity equation, but was decided to be large enough.

Water Input Model
The water input model does not contain a time derivative or a spatial derivative of one of the unknowns in the basal drainage model, see Eq. 4.2. In addition, the source terms on the right-hand side of this equation, are positive for every \( x \). Therefore, at every time step the meltwater that enters the channel system can be determined explicitly and locally with

\[
m_{ch,m}^{n+1} = \frac{2AH_m^n}{\rho_w L_f (n+2)} r_{d,m}^{n+1} + (1 - p) \frac{r_{b,m}^n u_{b,m}^n}{\rho_w L_f}.
\]

(4.27)

Water Balance Model
For a channel system in the quiescent phase of a surge cycle, the following water balance model was proposed

\[
\frac{\partial S}{\partial t} = -\nabla Q_{ch} + m_{ch} W_{ch}.
\]

(4.28)

This water balance model can be rewritten in an advection-diffusion equation (see Appendix D)

\[
\frac{\partial S}{\partial t} = -\nabla (u_{ch} S) + \nabla (D_{ch} \nabla S) + m_{ch} W_{ch}
\]

(4.29)
with

\[ u_{ch} = -c_3 S^{n-1} |\nabla (p_w + \rho_w g b)|^{\beta-2} \nabla (p_w + \rho_w g b) \]  

(4.30)

\[ D_{ch} = \rho_w g c_3 S^{n-1} |\nabla (p_w + \rho_w g b)|^{\beta-2} S. \]  

(4.31)

The first term on the right-hand side of Eq. 4.29 is the advective term; the second term the diffusive term; and the third term a source term. From the first model runs, it was observed that diffusion of the cross-sectional area of channels is negligible compared to advection of this quantity. Therefore, it seemed reasonable to apply the following simplification

\[ \frac{\partial S}{\partial t} = -\nabla (u_{ch} S) + m_{ch} W_{ch}. \]  

(4.32)

This resulting equation is an advection equation.

**Discretization** Following Chapter 3, the first term on the right-hand side of Eq. 4.32, the advective term, will be approximated by a first order upwind schemes, whereas the third term can be determined locally. With noting that \( m_{ch,m} W_{ch} > 0 \) for all \( m \), this term can be taken explicitly for all \( m \), and Eq. 4.32 can be approximated numerically by

\[ \left( \frac{1}{\Delta t} + \frac{u_{ch,m}^{n+1} - u_{ch,m}^n}{\Delta x} \right) S_{m+1}^n - \frac{u_{ch,m}^n - u_{ch,m-1}^n}{\Delta x} S_{m-1}^n = \frac{1}{\Delta t} S_m^n + m_{ch,m}^n W_{ch}. \]  

(4.33)

So, because of the adoption of a semi-implicit scheme, a matrix results that can be solved easily in one sweep (Zijlema, 2013).

**Virtual grid points** Implementing Eq. 4.33 as a numerical approximation for the water balance model of a channel system during the quiescent phase, in theory does introduce a virtual grid point at the upper boundary. However, the glacier head represents an ice divide, hence no water is flowing in at the upper boundary. Therefore, the cross-sectional area of the channels at the upper boundary can be schematized as

\[ S = 0 \quad \text{at} \quad x = 0 \]  

(4.34)

so that no virtual grid point has to be introduced.

**Water pressure model**

For a channel system in the quiescent phase of a surge cycle, the following water pressure model was proposed

\[ W_{ch} \frac{\mu}{\rho_w g} \frac{\partial p_{w, ch}}{\partial t} = -\nabla Q_{ch} + m_{ch} W_{ch} + c_4 Q_{ch} \nabla \phi_{ch} + c_5 (p_o - p_{w, ch})^n S. \]  

(4.35)

This water pressure model can be rewritten in a diffusion equation (see Appendix D)

\[ W_{ch} \frac{\mu}{\rho_w g} \frac{\partial p_{w, ch}}{\partial t} = \frac{1}{\rho_w g} \nabla \left( D_{ch} \nabla p_w \right) + \nabla \left( D_{ch} \nabla \left( b + \sqrt{a S} \right) \right) + m_{ch} W_{ch} + c_4 Q_{ch} \nabla \phi_{ch} + c_5 (p_o - p_{w, ch})^n S. \]  

(4.36)

**Discretization** Again following Chapter 3, the first term on the right-hand side of Eq. 4.32, the diffusive term, will be approximated by central differences. The other terms on the right-hand side of this equation, are source terms. Two of those terms are positive for all \( m, m_{ch,m} W_{ch} > 0 \) and \( c_5 \left( p_{w, ch}^n - p_{w, ch,m}^n \right) \left( S_{m+1}^n \right) > 0 \), and therefore can be taken explicitly for all \( m \). However for some \( m \), the source term that represents opening of the channels is negative, i.e. \( c_4 Q_{ch,m}^n \nabla \phi_{ch,m}^n < 0 \) for some \( m \). Therefore, this source term has to be approximated by
4. HYDROMECHANICALLY COUPLED MODEL

\[ c_4 Q_{ch,m}^n \nabla \phi_{ch,m}^n \quad \text{if} \ c_4 Q_{ch,m}^n \nabla \phi_{ch,m}^n > 0 \]
\[ c_4 Q_{ch,m}^n \nabla \phi_{ch,m}^n \quad \frac{p_{w,ch,m}^{n+1}}{p_{w,ch,m}^n} \quad \text{if} \ c_4 Q_{ch,m}^n \nabla \phi_{ch,m}^n < 0. \quad (4.37) \]

The same holds for the remaining source term \( \nabla \left( D_{ch,m}^n \nabla \left( b_m + \sqrt{aS_{m+1}^n} \right) \right) \). Whenever \( c_4 Q_{ch,m}^n \nabla \phi_{ch,m}^n < 0 \) and \( \nabla \left( D_{ch,m}^n \nabla \left( b_m + \sqrt{aS_{m+1}^n} \right) \right) < 0 \), Eq. 4.36 can be approximated numerically by

\[ A p_{w,ch,m+1}^{n+1} + B p_{w,ch,m}^{n+1} + C p_{w,ch,m-1}^{n+1} = \frac{1}{\Delta t} p_{w,ch,m}^n + \frac{\rho_w g}{W_{en} \mu} \left( m_{ch,m}^n \right) W_{ch} + c_3 \left( p_{o,m}^n - p_{w,ch,m}^n \right) S_{m+1}^n \quad (4.38) \]

with

\[ A = -\frac{D_{ch,m+1}^{1/2}}{W_{en} \mu \Delta x^2} \quad (4.39) \]
\[ B = \frac{1}{\Delta t} + \frac{D_{ch,m+1}^{1/2}}{W_{en} \mu \Delta x^2} + \frac{D_{ch,m-1}^{1/2}}{W_{en} \mu \Delta x^2} - \frac{\rho_w g c_4 Q_{ch,m}^n \nabla \phi_{ch,m}^n}{W_{en} \mu \Delta x p_{w,ch,m}^n} - \frac{\rho_w g}{W_{en} \mu \Delta x p_{w,ch,m}^n} \]
\[ \left( D_{ch,m+1}^{1/2} \frac{b_{m+1} + \sqrt{aS_{m+1}} - b_m}{\Delta x} - D_{ch,m-1}^{1/2} \frac{b_m + \sqrt{aS_{m}} - b_{m-1} + \sqrt{aS_{m+1}}}{\Delta x} \right) \quad (4.40) \]
\[ C = -\frac{D_{ch,m-1}^{1/2}}{W_{en} \mu \Delta x^2} \quad (4.41) \]

the elements of the input vectors for the to be solved tri-diagonal matrix. Matrices of this form, can be solved easily with the so-called double sweep method (Zijlema, 2013). If \( c_4 Q_{ch,m}^n \nabla \phi_{ch,m}^n > 0 \) and/or \( \nabla \left( D_{ch,m}^n \nabla \left( b_m + \sqrt{aS_{m+1}^n} \right) \right) > 0 \), a similar tri-diagonal matrix results. Note that the cross-sectional area of the channels appearing in some source terms in Eq. 4.38 is taken at the new time step because the cross-sectional area at the new time step is determined prior to the water pressure at the new time step.

**Virtual grid points** Implementing Eq. 4.38 as a numerical approximation for the water pressure model of a channel system during the quiescent phase, in theory does introduce two virtual grid points, one at the upper boundary and one at the lower boundary. However, the water pressure is known to be equal to zero at both the upper and the lower boundary, i.e.

\[ p_{w,ch} = 0 \quad \text{at} \ x = 0 \quad (4.42) \]
\[ p_{w,ch} = 0 \quad \text{at} \ x = L \quad (4.43) \]

so that again no virtual grid points have to be introduced.

Summarizing, in this chapter building blocks for a basal drainage model proposed in literature were presented. Based on the transformation of the basal drainage system during a surge cycle proposed by this MSc Thesis, see Fig. 4.3, various assumptions were applied to these building blocks. The result was a basal drainage model for the quiescent phase and a basal drainage model for the surge phase of LHT-glaciers. By implementing the numerical approximation to these models, it will be tested if the constructed HMCM is capable of simulating the characteristic features of the quiescent phase and the surge phase.
By means of implementing the HMCM constructed in the previous chapter, an attempt is made in simulating surge behaviour of LHT-glaciers. It is explored whether the constructed HMCM is able to capture the characteristic features of surge behaviour, like the gradual thickening of the reservoir area and the gradual thinning of the receiving area during the quiescent phase. However, first the results of a run without coupling between the ice mass continuity equation and the basal drainage model will be discussed. Comparing the results of the coupled model with the results of this decoupled model, provides insight into the effect of drainage of water at the base of the glacier on the evolution of the glacier. After having concluded that the HMCM displays the characteristic features of the distinct phases of the surge cycle, the output of the model will be tested using available observations in order to say something meaningful about the plausibility of the basal drainage model.

Table 4.1 presents the applied values for the ice and bed properties that appear in the basal drainage model. Because the spatial variability of these properties is unknown, uniform values are applied. One subsection in this chapter is dedicated to a sensitivity analysis of the output with respect to the values assigned to the unknown ice and bed properties.

5.1. DECOPLED MODEL

First, the results of a run without coupling between the ice mass continuity equation and the basal drainage model will be discussed. The decoupling was established by approximating the effective pressure \(N\) in the sliding law Eq. 4.1 by \(N = p_o\). In the run without coupling, initially the surface elevation at the end of the surge phase as depicted in Budd (1975) is imposed. Fig. 5.1 shows the results for the decoupled model. Because the ice mass continuity equation and the basal drainage model are not coupled, the presence of water at the base of the glacier does not influence the ice dynamics. From Fig. 5.1(a) it can be concluded that the glacier returns to its steady state. The resulting surface elevation profile is depicted in Fig. 5.1(b) and agrees with the steady state profile pictured in Budd (1975).

In this decoupled model, the produced meltwater is drained through subglacial channels. For the resulting steady state situation, the variables of the channel system \(S\) and \(p_{w,ch}\) can be determined from the following forms of the water balance model Eq. 4.3, the water transport model Eq. 4.5, and the model for the cross-sectional area of the channels Eq. 4.8:

\[
0 = -\nabla Q_{ch} + m_{ch} W_{ch} \tag{5.1}
\]

\[
Q_{ch} = -c_3 S^\alpha |\nabla \phi_{ch}|^{\beta-2} \nabla \phi_{ch} \tag{5.2}
\]

\[
0 = -c_4 Q_{ch} \nabla \phi_{ch} - c_5 (p_o - p_{w,ch})^n S. \tag{5.3}
\]

Solving this set of equations by means of Richardson extrapolation, gives the resulting steady state solution. Fig. 5.1(c)-5.1(e) show the overlap of the run without coupling and the steady state solution obtained by means of Richardson extrapolation. Fig. 5.1(e) shows a small difference in the water discharge near the terminus of the glacier. This difference can be attributed to the fact that the building blocks of the basal drainage
model presented in Section 4.3 do not hold near the terminus of the glacier, as the assumption of completely filled channels is not valid there, see Fig. 1.8. Despite this difference in water discharge near the terminus of the glacier, it can be concluded from Fig. 5.1(c)-5.1(e) that the basal drainage model is implemented correctly.

5.1.1. TESTING OF THE BASAL DRAINAGE MODEL
The water outflow at the terminus of the glacier, see Fig. 5.1(e), is some order of magnitudes smaller than observed outflows (Kamb et al., 1985). This discrepancy is caused partly by the assumption that surface melt-
5.2. COUPLED MODEL: THE QUIESCENT PHASE

Regarding the quiescent phase, the HMCM should be able to capture the gradual retreat of the glacier; the gradual thickening of the reservoir area; and the gradual thinning of the receiving area. Fig. 5.3(a) and 5.3(b) show that the constructed HMCM captures these characteristic features of the quiescent phase.

From Fig. 5.3(a) it should be noticed that a discrete instead of a continuous length change is simulated. The cause of the discrete length change is the application of a fixed grid. Furthermore, it should be noticed that the length change in Fig. 5.3(a) resembles the first part of the length change in Fig. 5.1(a). This resemblance is caused by the fact that also during the quiescent phase the effective pressure \( N \) is approximated by \( N = p_o \), for it was assumed that during the quiescent phase the coverage of glacier bed by cavities is small.

![Figure 5.3: Results for the quiescent phase.](image)

**Figure 5.3:** Results for the quiescent phase. (a): Evolution of the length of the glacier in time. (b): Glacier profiles at various moments in time during the quiescent phase.

5.2.1. TESTING OF THE BASAL DRAINAGE MODEL

After having concluded that the constructed HMCM displays the characteristic features of the quiescent phase, it still has to be explored whether the included basal drainage model is plausible. This will be done...
using published observations on the location of the initiation of the surge: for Variegated Glacier, Alaska it was observed that a zone with large surface velocities forms in the upper part of the glacier (Raymond, 1987).

The following criterion was taken to prescribe the start of the surge phase

\[ d(x) = d_r \quad \text{for any } x \]  

(5.4)

i.e. the surge phase starts when the representative layer thickness of the cavities \( d \) locally equals the mean roughness scale of the basal topography \( d_r \). The graph in Fig. 5.4(a) shows that the implemented basal drainage model suggests that criterion 5.4 is first met at approximately 10km from the ice divide, which is in the upper part of the glacier. Therefore, it is concluded that, as regards the location of the initiation of the surge, the basal drainage model is plausible.

5.2.2. **Sensitivity analysis**

This subsection explores the sensitivity of the output with respect to the values assigned to two unknown ice and bed properties: the sensitivity of the output with respect to the value assigned to the englacial porosity \( \mu \); and the sensitivity of the output with respect to the value assigned to the percentage of basal meltwater that is stored in the cavity system \( p \).
5.3. Coupled model: the surge phase

Englacial porosity
The englacial porosity $\mu$ was inserted because its inclusion significantly simplified numerical implementation. However, the applied value for $\mu$ affects the time lag between the representative layer thickness of the cavity system $d$ and the water pressure in the cavities $p_{w,ca}$, and the time lag between the cross-sectional area of the channels $S$ and the water pressure in the channels $p_{w,cb}$.

Fig. 5.5 shows the pressure at the ice-bed interface at the end of the quiescent phase for a smaller value for the englacial porosity $\mu$, Fig. 5.5(a), and for a larger value for the englacial porosity $\mu$, Fig. 5.5(b), with $0 \leq \mu < 1$ (Bueler and Van Pelt, 2014). A comparison between Fig. 5.4(b) and Fig. 5.5(a) proves that a smaller value for $\mu$ results in a larger water pressure in the cavities $p_{w,ca}$ at the time of the initiation of the surge phase. Nonetheless, this larger water pressure does not alter the evolution of the glacier during the surge phase significantly. Fig. 5.5(b) reveals that for very large values for the englacial porosity, the time lag between the cross-sectional area of the channels $S$ and the water pressure in the channels $p_{w,cb}$ becomes so large that, because those two variables are interlinked, the solution of the draining channel system becomes unstable. Concluding, the applied value for $\mu$ should be $\mu = 0.01$ or smaller in order to keep the solution stable.

Figure 5.6: Evolution of the length of the glacier in time for $m_{ca} = 0.50m_{ba}$, i.e. during the quiescent phase 50% of the meltwater produced at the base of the glacier is stored in cavities.

Percentage of the meltwater that is stored in cavities
The value assigned to the percentage of basal frictional meltwater stored in the cavity system $p$ affects the duration of the quiescent phase, as can be concluded from a combination of Eq. 4.25 and Table 4.3. Up till now, a value of 20% was applied, because this value resulted in a duration of the quiescent phase comparable with this duration as simulated by Budd (1975). Fig. 5.6 depicts the evolution of the length of the glacier for a larger value of $p$. A comparison of Fig. 5.6 and 5.3(a) affirms that a larger value for the percentage of basal frictional meltwater stored in cavities $p$ results in a shorter quiescent phase. Because during the quiescent phase the effective pressure $N$ is approximated by $N = p_o$, the ice mass continuity equation and the basal drainage model are not coupled. Therefore the shorter quiescent phase simply leads to a larger glacier length at the end of the quiescent phase.

5.3. Coupled model: the surge phase
Regarding the surge phase, the HMCM should be able to capture the rapid advance of the glacier; the rapid thinning of the reservoir area; and the rapid thickening of the receiving area. Fig. 5.7(a) and 5.7(b) reveal that the constructed HCM captures these characteristic features of the surge phase.

Nevertheless, it should be noticed that the surface elevation profile at the end of the surge phase as depicted in Fig. 5.7(b) is not realistic: the thinning of the reservoir area is too local. This unrealistic profile is a result of the fact that the surge rear does not propagate downslope, for no criterion for the switch back from a cavity system to a channel system is included. Another consequence of not including a criterion for the switch back is an unrealistic length of the surge zone at the end of the surge phase: the simulated length is significantly larger than observed lengths of surge zones (Kamb et al., 1985).

Furthermore, the peculiar profile of the ice velocity profile in Fig. 5.7(c) requires some explanation. In the surge zone, the assumption that the coverage of glacier bed by cavities is small is no longer valid. Therefore,
in this zone the effective pressure has to be determined from $N = p_o - p_{w,c}$. However, as soon as the water pressure in the cavities equals the local overburden pressure, the effective pressure becomes equal to zero, and a limitation on the basal velocity is indispensable, see Eq. 4.1. The results in Budd (1975) suggest that $u_{b,max} = 11 \text{ m} \text{d}^{-1}$ is suitable.

Fig. 5.7(d) displays the cross-sectional area of the channels at the end of the surge phase. For the channels it was assumed that during the surge phase the cross-sectional area of the channels in the surge zone and the cross-sectional area of the channels in the downstream zone does not change in time, i.e. $\frac{\partial S}{\partial t} = 0$. Therefore, over a large part of the glacier, the cross-sectional area of the channels did not change during the surge phase, see Fig. 5.7(d). However, in the upstream zone, the rapid thinning of the reservoir area did increase the surface slope, hence the meltwater input, and therefore the cross-sectional area of the channels draining the water in this zone.

Fig. 5.7(e) displays the water outflow at the terminus of the glacier. It was assumed that in the surge zone water flows over the entire width. Therefore, this water does not enter the discrete channels, but instead flows into the cavities located just downstream of the surge front. As a result, the amount of water flowing out at the terminus of the glacier decreases as long as the surge front propagates downslope. However, as soon as the surge front reaches the terminus, the amount of water flowing out is equal to the amount of water entering the surge zone, see Eq. 4.26.
5.3. COUPLED MODEL: THE SURGE PHASE

Figure 5.7: Results for the surge phase. The limiting value on the basal velocity for \( N \to 0 \) was set to be equal to 11 m d\(^{-1}\). (a): Evolution of the length of the glacier in time. (b): Glacier surface profile at various moments in time during the surge phase. In addition, the ice velocity (c) and the cross-sectional area of the channels (d) at the end of the surge phase are depicted. (e): Evolution of the water flux at the terminus of the glacier.

5.3.1. TESTING OF THE BASAL DRAINAGE MODEL

After having concluded that the constructed HMCM captures the characteristic features of the surge phase, it still has to be explored whether the included basal drainage model is plausible. This will be done using published observations on the time it takes the surge front to reach the terminus of the glacier; published observations on water speeds at the base of the glacier; and published observations on the flood discharge during the termination of the surge.

Observations show that the downward propagation of the surge front takes the whole surge phase (Ray-
mond et al., 1987). However, the discontinuity in Fig. 5.7(e) reveals that the simulated surge front reached the terminus of the glacier after only one month, whereas the surge phase lasted approximately 1.5y. Hence, it can be concluded that the propagation speed of the surge front is far too large. A consequence of this far too large propagation speed of the surge front is that the glacier does not display a bulge of ice moving downslope. Therefore, this feature can not be observed in Fig. 5.7(b) that displays glacier profiles at various moments in time during the surge phase. Whereas the propagation of the surge front is imposed by the transport of water, it can be concluded that the implemented basal drainage model is not realistic.

Observed water velocities during the surge of 1982-1983 of Variegated Glacier, Alaska can help verify the assumption that the time water needs to cross the surge zone is small compared to the applied time step $\Delta t$. The observed average water speed is $0.02 \text{m s}^{-1}$ (Kamb et al., 1985). Therefore, it takes water a couple of days to flow through an observed length of the surge zone of approximately 17 km (Kamb et al., 1985). The applied time step is determined by the time it takes the representative layer thickness $d$ of the cavities located just downstream of the surge zone to grow to the value for the mean roughness scale of the basal topography $d_r$. This process takes only seconds. Therefore, it has to be concluded that the assumption that the time water needs to cross the surge zone is small compared to the applied time step $\Delta t$ is not valid. As a consequence, in future research the water flow between the ice and bed $Q_{\text{ice-bed}}$ and the thickness of the water film between the ice and bed have to be resolved. It should be noted that the necessary inclusion of friction in future research will entail the desired lowering of the propagation speed of the surge front.

Observations reveal a flood discharge at the terminus of the glacier during the termination of the surge. This flood discharge is assigned to the release of water stored in the layer between the ice and bed as this flood discharge comes together with a significant lowering of the glacier surface near the terminus of the glacier (Kamb et al., 1985). However, in Section 4.4, it was assumed that all the water reaching the surge front, can flow into the cavities located downstream of this front, which implies that no water is stored in the layer between the ice and bed. Due to this contradiction, it has to be concluded that also the assumption that all the water reaching the surge front can flow into the cavities located downstream of this front is not valid. It should be noted that also the necessary inclusion of storage of water in the layer between the ice and bed in future research will entail the desired lowering of the propagation speed of the surge front.

In summary, from observations on the time it takes the surge front to reach the terminus of the glacier; observations on water speeds at the base of the glacier; and observations on the flood discharge during the termination of the surge, it can be concluded that the implemented basal drainage model is not realistic.

### 5.4. Surge Cycle

Fig. 5.8 shows the evolution of the length of the glacier in time for successively a quiescent phase, a surge phase, and another quiescent phase. For the termination of the surge phase a simple criterion was invoked: the surge phase ends as soon as the glacier advanced to its original length. Thus, the duration of the surge phase is forced instead of determined by the relaxation process. Therefore, it makes no sense to compare the resulting duration of the surge phase with the duration obtained by Budd (1975). The successive quiescent phase is started by setting the representative layer thickness of the cavities and the water pressure in the cavities to zero, i.e. $d = 0$ and $p_{w,ca} = 0$ for all $x$.

![Figure 5.8: Evolution of the length of the glacier in time for successively a quiescent phase, a surge phase, and another quiescent phase.](image-url)
6

TESTING OF ASSUMPTIONS APPLIED IN THE BASAL DRAINAGE MODEL

In Chapter 5 it was concluded that the basal drainage model constructed for the surge phase is not realistic. This may be caused by the applied building blocks; by the proposed transformation of the basal drainage system during a surge cycle; or by the translation of this transformation into the building blocks by means of assumptions that simplify the relevant equations, see Fig. 4.2. In this chapter, first three important assumptions that were applied in order to simplify the relevant equations are tested. Afterwards, it is tested if the assumption that underlies the equation for the closure of the set of equations resulting from the building blocks holds.

6.1. THE MELTWATER INPUT FROM OPENING OF THE CHANNELS IS NEGLIGIBLE

In Section 4.3, it was assumed that the water input from wall melting of channels is negligible compared to the water input from englacial melting and the water input from melting at the base of the glacier. Because the discharge through the subglacial channels $Q_{ch}$ is largest during the quiescent phase, this phase is normative concerning the wall melting of channels, see Eq. 4.8. Fig. 6.1 shows the relevant meltwater inputs at a representative moment during the quiescent phase. From this figure, it can be concluded that indeed the water input from wall melting of channels is negligible compared to the water input from englacial melting and the water input from melting at the base of the glacier, i.e. the water from wall melting of channels does not have to be included in the water input model Eq. 4.2. However, wall melting of channels is crucial in keeping the channels open. Therefore, in the model for the capacity of the channel system Eq. 4.8 the wall melting of channels still has to be included.

6.2. THE RATE OF CHANGE OF ENGLACIALLY STORED WATER IS NEGLIGIBLE

The equations for the capacities of the basal drainage systems as proposed in literature read

$$\frac{\partial S}{\partial t} = -c_4 Q_{ch} \nabla \phi_{ch} - c_5 \left( p_o - p_{w, ch} \right)^n S$$  (6.1)

$$\frac{\partial d}{\partial t} = c_1 u_b \left( d_r - d \right) - c_2 A \left( p_o - p_{w, ca} \right)^n d.$$  (6.2)

However, Eq. 6.1 is hard to implement numerically, for the opening term, which is the first term on the right-hand side, is equal to zero when no channels are present yet. To overcome this difficulty, it was assumed that the rate of change of englacially stored water is negligible compared to the rate of change of the capacity of the drainage system, i.e. $\frac{\partial S_{en}}{\partial t} \ll \frac{\partial S}{\partial t}$ and $\frac{\partial d_{en}}{\partial t} \ll \frac{\partial d}{\partial t}$. Therefore, instead of Eq. 6.1 and Eq. 6.2 the following equations could be used to determine the rate of change of the capacity of the drainage systems

$$\frac{\partial S}{\partial t} = -\nabla Q_{ch} + m_{ch} W_{ch}$$  (6.3)
Fig. 6.1: Testing of the assumption that the water input from wall melting of channels is negligible. The englacially produced meltwater and the at the base produced meltwater are depicted together with the water input from opening of the channels.

\[ \frac{\partial d}{\partial t} = -\nabla q_{ca} + m_{ca}. \] (6.4)

Fig. 6.2 shows the rate of change of englacially stored water and the rate of change of the representative layer thickness of the cavities determined with the simplified water balance model Eq. 6.4 during the first time step of the quiescent phase. From this figure, it can be concluded that the assumption that the rate of change of englacially stored water is negligible compared to the rate of change of the capacity of the cavity system, i.e. \( \frac{\partial d_{en}}{\partial t} \approx \frac{\partial d}{\partial t} \), is not valid. Besides, applying Eq. 6.1 and 6.2 instead of Eq. 6.3 and 6.4 has one important advantage: Eq. 6.2 precludes a value of the water pressure \( p_{w,ca} \) equal to the local overburden pressure \( p_o \), so that it is not necessary to set a limiting value on the basal velocity \( u_b \) for \( N \to 0 \). Therefore, it can be concluded that it is desirable to apply Eq. 6.1 and 6.2 for the determination of the capacity of the drainage systems.

It should be noted that the numerical implementation of Eq. 6.1 can be simplified by adding an extra opening term to this equation. This term should be significant when the cross-sectional area of the channels \( S \) is small, and should become negligible relative to the term representing opening through wall melting when the discharge through the channels is significant. Schoof (2010) applies

\[ \frac{\partial S}{\partial t} = -c_4 Q_{ch} \nabla \phi_{ch} + c_6 u_b d_r - c_5 (p_o - p_{w,ca})^n S \] (6.5)

in which the second term on the right-hand side is the extra opening term.

Fig. 6.2: The rate of change of englacially stored water and the rate of change of the representative layer thickness of the cavities determined with the simplified water balance model Eq. 6.4 during the first time step of the quiescent phase.
6.3. The Basal Shear Stress Can Be Replaced by the Driving Stress

As a simplification, the basal shear stress $\tau_b$ in the applied sliding law

$$u_b = c \frac{\tau_b}{N} \quad (6.6)$$

and the basal shear stress $\tau_b$ in the applied equation for the meltwater produced at the base of the glacier through basal frictional heating

$$m_{ba} = \frac{\tau_b u_b}{\rho_w L_f} \quad (6.7)$$

were replaced by the driving stress $\tau_d$. However, from the equation for longitudinal stress equilibrium in the SSA written as

$$-4\eta \frac{\partial}{\partial x} \left( H \frac{\partial u_b}{\partial x} \right) = \tau_d - \tau_b \quad (6.8)$$

it can be concluded that when the gradient in sliding velocity $u_b$ varies strongly along the flowline of the glacier, i.e. when $\frac{\partial}{\partial x} \left( H \frac{\partial u_b}{\partial x} \right)$ is large, the applied simplification may lead to inaccurate solutions. During the surge phase, locally this gradient is large, as can be deduced from Fig. 1.5. Fig. 6.3 shows the driving stress $\tau_d$ and the basal shear stress $\tau_b$ at a specific moment in time during the surge phase. From this figure it can be concluded that at the boundaries of the surge zone, replacing the basal shear stress $\tau_b$ by the driving stress $\tau_d$ does not yield a realistic schematization.

Figure 6.3: As a simplification, in the sliding law and in the equation for the meltwater production at the base of the glacier, the basal shear stress $\tau_b$ was replaced by the driving stress $\tau_d$. However, when the gradient in sliding velocity $u_b$ varies strongly along the flowline of the glacier, the applied simplification may lead to inaccurate solutions. In order to verify this simplification, the driving stress and the basal shear stress at a specific moment in time during the surge phase are depicted.
6.4. **The cavities and channels are completely filled at every moment in time**

In subsection 4.3.6 it was assumed that the cavities and channels are completely filled at every moment in time. This assumption implies that the representative water layer thickness in the cavities $d_{\text{water}}$ equals the representative layer thickness of the capacity of the cavities $d$, and implies that the amount of water in the channels $S_{\text{water}}$ equals the cross-sectional area of the channels $S$. However, Fig. 6.4 depicts a rate of change of the layer thickness determined with the simplified water balance model Eq. 6.4 that is significantly smaller than the same rate of change determined with the model for the capacity of the cavities Eq. 6.2, i.e. $-\nabla q_{ca} + m_{ca} \ll c_1 u_b (d_r - d) - c_2 A (p_a - p_{w,ca})^{n} d$. This corresponds to only partly filled cavities, hence the system of equations in Table 4.3 is not valid. Therefore, it has to be concluded that the applied initial condition $d(x) = 0$ for all $x$ is not suitable for the proposed basal drainage models depicted in Table 4.2 and 4.3. So for the quiescent phase, either the applied initial condition, or the constructed basal drainage model, or both have to be adapted.

![Figure 6.4](image-url)
CONCLUSIONS

The goal of this research was to construct a realistic basal drainage model for land-terminating, hard-bedded, temperate glaciers (LHT-glaciers), and to couple this model to a glacier dynamics model in order to simulate surge behaviour of those glaciers. For that purpose, first the transformation of the basal drainage system of LHT-glaciers during a surge cycle had to be described. The proposed transformation focuses on the increasing storage of meltwater in cavities during the quiescent phase. As soon as the cavities drown the bed, water will no longer flow through discrete channels, but instead will flow between the ice and bed. This was referred to as the switch in basal drainage system from channel system to cavity system. The surge starts. During the surge phase, the surge front propagates downslope following the displacement of water at the base of the glacier. When the switch from channel system to cavity system reaches the terminus of the glacier, the surge terminates and the water in the layer between the ice and bed is released.

By means of translating the proposed transformation of the basal drainage system into building blocks known from literature, a basal drainage model was constructed. The applied translation of the transformation during the quiescent phase and the applied translation of the transformation during the surge phase are based on the following assumptions:

(i) During the quiescent phase, a certain part of the meltwater produced at the base of the glacier, \( p_m \), is stored in not-interlinked cavities.

(ii) At the start of the quiescent phase, the representative layer thickness of the cavities and the water pressure in the cavities are equal to zero, i.e. \( d = 0 \) and \( p_{w,c} = 0 \) for all \( x \).

(iii) No matter the length of the surge zone, the time water needs to cross the surge zone is small compared to the applied time step \( \Delta t \). i.e. friction can be neglected.

(iv) All the water reaching the surge front, can flow into the cavities located just downstream of this front.

In addition, the sudden initiation and sudden termination of the surge are captured by the following criteria:

(i) The surge starts as soon as the representative layer thickness of the cavity system \( d \) equals the mean roughness scale of the basal topography \( d_r \) somewhere along the glacier.

(ii) The surge terminates when the surge front reaches the terminus of the glacier.

The resulting model proved to be capable of showing the characteristic features of the quiescent phase, which are the progressive retreat of the glacier; the progressive thickening of the reservoir area; and the progressive thinning of the receiving area. In addition, the model showed the rapid advance of the glacier; the rapid thinning of the reservoir area; and the rapid thickening of the receiving area that characterize the surge phase.

However, in order to say something meaningful about the plausibility of the constructed basal drainage model the output of the model was tested by means of more specific observations on surge behaviour. Concerning the quiescent phase, model output suggested that the surge starts in the upper part of the glacier, which is in agreement with observations. However, other model output revealed that the model simulated partly filled cavities, whereas the applied building blocks are only valid for completely filled drainage systems. Therefore, it is concluded that concerning the quiescent phase the constructed basal drainage model has to be revised.

Concerning the surge phase, observations show that the propagation of the surge front lasts as long as the surge phase itself. However, the model output showed a surge front that reached the terminus of the glacier after only a short time after the start of the surge. In addition, observations on water velocities at the base of the glacier revealed that the time that water needs to cross the surge zone is not negligible. Furthermore, observations on a flood discharge during the termination of a surge indicate that not all the water reaching the surge front can flow into the cavities located downstream of this front. Therefore, it is concluded that also concerning the surge phase the constructed basal drainage model has to be revised.
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Sharon van Geffen


Oerlemans, J. (2013). Lecture notes ice and climate. MSc Meteorology, Physical Oceanography and Climate at Utrecht University.


DEVIATION OF THE EQUATION FOR LONGITUDINAL STRESS EQUILIBRIUM IN THE SHALLOW SHELF APPROXIMATION

The starting points for the derivation of the equation for longitudinal stress equilibrium in the Shallow Shelf Approximation (SSA) are:

- The momentum conservation equations in the SSA

\[
\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} = \frac{\partial p}{\partial x} \tag{A.1}
\]

\[
\frac{\partial \tau_{zz}}{\partial z} = \frac{\partial p}{\partial z} + \rho g \tag{A.2}
\]

- the mass continuity equation for an incompressible continuum

\[
\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \Leftrightarrow \dot{\epsilon}_{xx} + \dot{\epsilon}_{zz} = 0 \tag{A.3}
\]

- and Glen's flow law

\[
\dot{\epsilon}_{ij} = A\tau^{n-1} \tau_{ij}. \tag{A.4}
\]

First, Eq. A.1 is manipulated:

\[
\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} = \frac{\partial p}{\partial x} \Leftrightarrow \tag{A.5}
\]

\[
\int_b^h \frac{\partial \tau_{xx}}{\partial x} \, dz + \int_b^h \frac{\partial \tau_{xz}}{\partial z} \, dz = \int_b^h \frac{\partial p}{\partial x} \, dz \Leftrightarrow \tag{A.6}
\]

\[
\frac{\partial}{\partial x} \int_b^h \tau_{xx} \, dz + \tau_{xz}(h) - \tau_{xz}(b) = \int_b^h \frac{\partial p}{\partial x} \, dz. \tag{A.7}
\]

With the boundary conditions \(\tau_{xz}(h) = 0\) and \(\tau_{xz}(b) = \tau_b\), this equation can be written as

\[
\frac{\partial}{\partial x} (H\tau_{xx}) - \tau_b = \int_b^h \frac{\partial p}{\partial x} \, dz. \tag{A.8}
\]

Second, it is noticed that a combination of Eq. A.3 and A.4 leads to
\( \tau_{xx} + \tau_{zz} = 0 \Leftrightarrow \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{zz}}{\partial z} = 0. \) \hfill (A.9)

Third, Eq. A.2 is manipulated:

\[ \frac{\partial \tau_{zz}}{\partial z} = \frac{\partial p}{\partial z} + \rho g \Leftrightarrow \] (A.10)

\[ \int_{b}^{h} \frac{\partial \tau_{zz}}{\partial z} \, dz = \int_{b}^{h} \frac{\partial p}{\partial z} \, dz + \int_{b}^{h} \frac{\partial}{\partial z} (\rho g) \, dz \Leftrightarrow \] (A.11)

\[ -\tau_{zz} = -p + \rho g (h - z) \Leftrightarrow \] (A.12)

\[ \frac{\partial p}{\partial x} = \frac{\partial \tau_{zz}}{\partial x} + \rho g \frac{\partial h}{\partial x}. \] (A.13)

Together with Eq. A.9 this results in

\[ \int_{b}^{h} \frac{\partial p}{\partial x} \, dz = -\int_{b}^{h} \frac{\partial \tau_{xx}}{\partial x} \, dz + \int_{b}^{h} \rho g \frac{\partial h}{\partial x} \, dz \Leftrightarrow \] (A.14)

\[ \int_{b}^{h} \frac{\partial p}{\partial x} \, dz = -\frac{\partial}{\partial x} (H \tau_{xx}) + \rho g H \frac{\partial h}{\partial x}. \] (A.15)

Combining Eq. A.8 and A.15 then gives the final result

\[ \frac{\partial}{\partial x} (2H \tau_{xx}) - \tau_b = \rho g H \frac{\partial h}{\partial x}. \] (A.16)
The starting point for the derivation of the equation for the gradient in basal velocity is the equation for longitudinal stress equilibrium

\[-2 \frac{\partial}{\partial x} (H \tau_{xx}) = \tau_d - \tau_b. \tag{B.1}\]

Combining the appropriate form of Glen's flow law, i.e.

\[2 \eta \dot{\epsilon}_x = \tau_{xx}, \tag{B.2}\]

with the equation emanating from the assumption that changes in longitudinal strain-rate are predominantly related to changes in sliding velocity, i.e.

\[\dot{\epsilon}_x = \frac{\partial u_b}{\partial x}, \tag{B.3}\]

results in

\[2 \eta \frac{\partial u_b}{\partial x} = \tau_{xx}. \tag{B.4}\]

Eq. B.1 and B.4 together give

\[-4 \eta \frac{\partial}{\partial x} \left( H \frac{\partial u_b}{\partial x} \right) = \tau_d - \tau_b. \tag{B.5}\]

Integrating Eq. B.5 from \(x\) to \(x = L\), i.e.

\[\int_x^L \frac{\partial}{\partial x} \left( H \frac{\partial u_b}{\partial x} \right) \, dx = -\frac{1}{4 \eta} \int_x^L (\tau_d - \tau_b) \, dx, \tag{B.6}\]

and applying the downstream boundary condition

\[\frac{\partial u_b}{\partial x} = 0 \quad \text{at} \ x = L \tag{B.7}\]

results in

\[-H \frac{\partial u_b}{\partial x} = -\frac{1}{4 \eta} \int_x^L (\tau_d - \tau_b) \, dx, \tag{B.8}\]

which can be rewritten to give
\[ \frac{\partial u_b}{\partial x} = -\frac{1}{4\eta H} \int_L^x (\tau_d - \tau_b) \, dx. \quad (B.9) \]
TIME STEPPING RESTRICTIONS FOR THE NUMERICAL APPROXIMATION OF THE BASAL DRAINAGE MODEL

For the numerical approximation of the water balance model and the water pressure model for a channel system during the quiescent phase, an explicit numerical scheme was considered. The proposed water balance model, see Table 4.2, can be written as an advection-diffusion equation reading

$$\frac{\partial S}{\partial t} = -\nabla (u_{ch} S) + \nabla (D_{ch} \nabla S) + m_{ch} W_{ch}$$  \hspace{1cm} (C.1)

see Eq. D.10. From the first model runs, it was observed that the diffusion of the cross-sectional area of channels $S$ is negligible compared to the advection of this quantity. Therefore, it seemed reasonable to apply the following simplification

$$\frac{\partial S}{\partial t} = -\nabla (u_{ch} S) + m_{ch} W_{ch}.$$  \hspace{1cm} (C.2)

This resulting equation is an advection equation, and was approximated numerically by a FTBS scheme

$$S_{m+1}^{n} = S_{m}^{n} + \left( \frac{u_{ch,m+\frac{1}{2}}}{\Delta x} \left[ S_{m}^{n} - \frac{u_{ch,m-\frac{1}{2}}}{\Delta x} S_{m-1}^{n} \right] + m_{ch,m}^{n} W_{ch} \right) \Delta t.$$  \hspace{1cm} (C.3)

This numerical scheme is conditionally stable, i.e. a condition has to be fulfilled in order to get a stable numerical solution. The following stability requirement should be met

$$\Delta t_{adv} \leq \frac{\Delta x}{\max |u_{ch,m}|}.$$  \hspace{1cm} (C.4)

The proposed water pressure model, see Table 4.2, can be written as a diffusion equation reading

$$W_{en} \frac{\mu}{\rho_{w} g} \frac{\partial p_{w,ch}}{\partial t} = -\nabla \left( -\frac{1}{\rho_{w} g} D_{ch} \nabla p_{w} - D_{ch} \nabla \left( b + \sqrt{aS} \right) \right) + m_{ch} W_{ch} + c_{q} Q_{ch} \nabla \phi_{ch} + c_{p} \left( p_{o} - p_{w,ch} \right) S_{n}$$  \hspace{1cm} (C.5)

see Appendix D. This equation was approximated numerically by a FTCS scheme, which is conditionally stable. The following stability requirement should be met

$$\Delta t_{diff} \leq \frac{1}{2} \mu \Delta x^{2} W_{en} \frac{\max D_{ch}}{\max D_{ch}}.$$  \hspace{1cm} (C.6)

By adopting adaptive time stepping, every timestep $\Delta t$ is taken to be the largest timestep that fulfills these two restrictions. The normative restriction turned out to be the time stepping restriction for the water pressure...
equation, i.e. restriction C.6. Increasing the value of the porosity \( \mu \) results in a less restrictive time step, however the maximum time step remains in the order of seconds. For practical computation time, this is thought to be far too small as the period of the surge cycle is in the order of tens of years. Therefore, a semi-implicit scheme has to be implemented.
Rewriting of the Water Balance Model and the Water Pressure Model for Channel Systems

D.1. Water Balance Model

The water balance model applied to channel systems is given by

\[
\frac{\partial S}{\partial t} = - \nabla Q_{ch} + m_{ch} W_{ch}. \tag{D.1}
\]

The discharge through the channels \( Q_{ch} \), can be determined from the water transport model

\[
Q_{ch} = -c_3 S^a |\nabla \phi_{ch}|^{\beta-2} \nabla \phi_{ch}. \tag{D.2}
\]

The hydraulic potential \( \phi_{ch} \) can be defined by (Schoof et al., 2014)

\[
\phi_{ch} = p_w + \rho_w g (b + \sqrt{aS}) \tag{D.3}
\]

with \( \sqrt{aS} \) the flow depth of a low aspect ratio channel (Schoof et al., 2014). Following Bueler and Van Pelt (2014), a simplification can be introduced: inside the absolute value signs in Eq. D.2, the flow depth of a low aspect ratio channel \( \sqrt{aS} \) will be assumed to be small, so that

\[
|\nabla \phi_{ch}| \approx |\nabla (p_w + \rho_w g b)|. \tag{D.4}
\]

Applying this simplification to Eq. D.2 leads to

\[
Q_{ch} = -c_3 S^a |\nabla (p_w + \rho_w g b)|^{\beta-2} \nabla (p_w + \rho_w g b) - \rho_w g c_3 S^a |\nabla (p_w + \rho_w g b)|^{\beta-2} \nabla \sqrt{aS}. \tag{D.5}
\]

By introducing the following three definitions

\[
K_{ch} = c_3 S^{a-1} |\nabla (p_w + \rho_w g b)|^{\beta-2} \tag{D.6}
\]

\[
u_{ch} = -K_{ch} \nabla (p_w + \rho_w g b) \tag{D.7}
\]

\[
D_{ch} = \rho_w g KS \tag{D.8}
\]

Eq. D.2 can be written as

\[
Q_{ch} = u_{ch} S - D_{ch} \nabla S. \tag{D.9}
\]
Herein, $K_{ch}$ can be thought of as an effective hydraulic conductivity, $u_{ch}$ as a velocity, and $D_{ch}$ as a diffusivity. Substituting this expression for $Q_{ch}$ in Eq. D.1, an advection-diffusion equation for the water balance model results:

$$\frac{\partial S}{\partial t} = -\nabla (u_{ch} S) + \nabla (D_{ch} \nabla S) + m_{ch} W_{ch}. \quad (D.10)$$

### D.2. WATER PRESSURE MODEL

The water pressure model applied to channel systems is given by

$$W_{en} \frac{\mu}{\rho_w g} \frac{\partial p_{w,ch}}{\partial t} = -\nabla Q_{ch} + m_{ch} W_{ch} + c_4 Q_{ch} \nabla \phi_{ch} + c_5 (p_o - p_{w,ch})^n S. \quad (D.11)$$

With a similar procedure as applied in Section D.1, the discharge through the channels $Q_{ch}$ can be rewritten to read

$$Q_{ch} = -\frac{1}{\rho_w g} D_{ch} \nabla p_w - D_{ch} \nabla \left( b + \sqrt{a S} \right) \quad (D.12)$$

with $\sqrt{a S}$ the flow depth of a low aspect ratio channel (Schoof et al., 2014). Inserting Eq. D.12 in Eq. D.11, results in a diffusion equation for the water pressure model:

$$W_{en} \frac{\mu}{\rho_w g} \frac{\partial p_{w,ch}}{\partial t} = \frac{1}{\rho_w g} \nabla \left( D_{ch} \nabla p_w \right) + \nabla \left( D_{ch} \nabla \left( b + \sqrt{a S} \right) \right) + m_{ch} W_{ch} + c_4 Q_{ch} \nabla \phi_{ch} + c_5 \left( p_o - p_{w,ch} \right)^n S \quad (D.13)$$

with $\nabla \left( D_{ch} \nabla \left( b + \sqrt{a S} \right) \right)$ an extra source term emerging from the rewritten equation for the discharge through the channels $Q_{ch}$ Eq. D.12.