The Effect of Initial Imperfections on Shell Stability
- a Review

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LIST OF SYMBOLS

\(a\)  
first postbuckling coefficient (see Eq. 4.1)

\(A_{ij}\)  
extensional stiffness matrix (see Eq. A.7)

\(A^*_{ij}\)  
semi-inverted extensional stiffness matrix (see Eq. A.9)

\(\bar{A}^*_{ij}\)  
nondimensional \(A^*_{ij}\) \((\bar{A}^*_{ij} = Et_{ij})\)

\(b\)  
second postbuckling coefficient (see Eq. 4.1)

\(B_{ij}\)  
bending-stretching coupling matrix (see Eq. A.7)

\(B^*_{ij}\)  
semi-inverted bending-stretching coupling matrix (see Eq. A.9)

\(\bar{B}^*_{ij}\)  
nondimensional \(B^*_{ij}\) \((\bar{B}^*_{ij} = (2\pi c)^2 B^*_{ij})\)

\(c\)  
\(-\sqrt{3(1-v^2)}\)

\(D_{ij}\)  
flexural stiffness matrix (see Eq. A.7)

\(D^*_{ij}\)  
semi-inverted flexural stiffness matrix (see Eq. A.11)

\(\bar{D}^*_{ij}\)  
nondimensional \(D^*_{ij}\) \((\bar{D}^*_{ij} = (4c^2/Ei^3)D^*_{ij})\)

\(E\)  
arbitrarily chosen reference Young's modulus

\(F\)  
Airy stress function

\(F^{(0)}, F^{(1)}, F^{(2)}\)  
zeroth order, first order, second order fields, respectively

\(h_k\)  
thickness of the \(k^{th}\) layer

\(K\)  
slope of the variable load vs generalized displacement curve just before buckling

\(K^*\)  
slope of the variable load vs generalized displacement curve just after buckling

\(L_{A*}\)  
linear operator defined by Eq. (1.34)

\(L_{B*}\)  
linear operator defined by Eq. (1.35)

\(L_{D*}\)  
linear operator defined by Eq. (1.36)

\(L_{NL}\)  
nonlinear operator defined by Eq. (1.37)

\(M_x, M_y, M_{xy}\)  
moment resultants

\(n\)  
number of full waves in the circumferential direction

\(N_x, N_y, N_{xy}\)  
stress resultants

\(\bar{N}\)  
variable applied stress resultant (see Eq. 5.1)

\(\alpha, \beta\)  
external pressure

\(\bar{p}\)  
nondimensional external pressure \((\bar{p} = (cR^2/Ei^2)p)\)

\(P_c\)  
buckling load of the 'perfect' structure

\(q\)  
aloxid load eccentricity measured from the midsurface of the shell wall - positive inward
\( \bar{q} \) - nondimensional load eccentricity (\( \bar{q} = 4cRq/\bar{t}^2 \))

\( Q_{ij} \) - specially orthotropic laminar stiffness matrix (see Eq. A.4)

\( \bar{Q}_{ij} \) - generally orthotropic laminar stiffness matrix (see Eq. A.4)

\( R \) - shell radius

\( t \) - shell wall-thickness

\( u, v \) - displacement components in the x and y directions, respectively

\( W \) - radial displacement (positive inward)

\( W^{(0)}, W^{(1)}, W^{(2)} \) - zeroth order, first order, second order fields, respectively

\( \bar{W} \) - initial radial imperfection (\( \bar{W} = \bar{x} \bar{W} \))

\( \hat{W} \) - shape of the initial radial imperfection

\( W_v \) - axial Poisson's effect (\( W_v = \bar{A}_{12} \bar{p} / \bar{c} \))

\( W_p \) - radial Poisson's effect (\( W_p = \bar{A}_{22} \bar{p} / \bar{c} \))

\( W_t \) - circumferential Poisson's effect (\( W_t = \bar{A}_{26} \bar{\tau} / \bar{c} \))

\( x, y \) - axial and circumferential coordinates on the middle surface of the shell, respectively

\( \bar{x}, \bar{y} \) - nondimensional coordinates (\( \bar{x} = x/R, \bar{y} = y/R \))

\( z \) - coordinate normal to the middle surface of the shell (positive inward)

\( \bar{Z} \) - modified Batdorf parameter (\( \bar{Z} = L^2 / R \))

\( \alpha \) - first imperfection form factor (see Eq. 4.11)

\( \beta \) - second imperfection form factor (see Eq. 4.11)

\( \gamma_{xy} \) - shearing strain

\( \Delta \) - generalized displacement

\( \varepsilon_{cl} \) - membrane strain which corresponds to the applied variable load (see Eq. 5.10)

\( \varepsilon_{\alpha\beta} \) - linear strain which corresponds to the variable applied stress resultant (see Eq. 5.2)

\( \varepsilon_{x'}, \varepsilon_{y} \) - normal strains

\( E_{\alpha\beta} \) - nonlinear strain (see Eq. 5.2)

\( \Delta \) - generalized displacement (see Eq. 5.1)

\( \theta \) - circumferential coordinate (\( \theta = y/R \))

\( \bar{\theta} \) - angle of the fundamental path (\( \bar{\theta} = \tan^{-1} (K_c) \))

\( \bar{\theta}^* \) - angle of the initial slope just after buckling (\( \bar{\theta}^* = \tan^{-1} (K_c^*) \))

\( \kappa_{x'}, \kappa_{y}, \kappa_{xy} \) - curvature components
\( \lambda \)  
nondimensional axial load parameter \((\lambda = (cR/EI)^2) N_0)\)

\( \Lambda \)  
nondimensional variable load factor

\( \Lambda_c \)  
nondimensional variable load factor evaluated at the bifurcation point

\( \Lambda_s \)  
nondimensional variable load factor evaluated at the limit point

\( \nu \)  
arbitrarily chosen reference Poisson's ratio

\( \xi \)  
perturbation parameter (see Eq. 4.1)

\( \xi_c \)  
amplitude of the initial imperfection (see Eq. 6.1)

\( \rho_s \)  
normalized variable load factor \((\rho_s = \Lambda_s / \Lambda_c)\)

\( \sigma_x, \sigma_y \)  
normal stresses

\( \tau \)  
nondimensional torque parameter \((\tau = (cR/EI)^2) N_{xy}\)

- positive counter-clockwise

\( \tau_{xy} \)  
shering stress
THE EFFECT OF INITIAL IMPERFECTIONS ON SHELL STABILITY - A REVIEW

by

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Abstract - The development of "DISDECO", the Delft Interactive Shell Design Code is described. The purpose of this project is to make the accumulated theoretical, numerical and practical knowledge of the last 20 years readily accessible to users interested in the analysis of buckling sensitive structures. With this open ended, hierarchical, interactive computer code the user can access from his work-station successively programs of increasing complexity. The initial level consists of semi-analytical solutions for the buckling load of perfect anisotropic circular cylindrical shells under axial compression, internal or external lateral pressure and torsion. Also included are modules that contain Koiter's imperfection sensitivity theory extended to anisotropic shell structures under combined loading. The nonlinear Donnell-type shell equations in terms of the radial displacement $W$ and the Airy stress function $F$ are used. Using a procedure that is equivalent to an approximate minimization of the second variation of the potential energy by the Rayleigh-Ritz method simple eigenvalue equations are obtained in terms of the integers $m$ and $n$, representing the number of half-waves in the axial and the number of full waves in the circumferential direction, and Khot's skewedness parameter $\tau_K^*$. Results of the search for the minimum eigenvalue and its imperfection sensitivity are displayed in convenient graphical form.

INTRODUCTION

In modern designs, which are often obtained by one of the structural optimization codes and which are made out of high strength materials (read advanced composites), it happens frequently that the stability behavior dictates the choice of some of the critical dimensions of the structures. This implies that one has to investigate the different loading cases quite accurately by carrying out extensive numerical calculations and/or experimental verifications.

Twenty five years ago it was so that numerical results were looked upon with a certain degree of distrust and they were only accepted if supported by some other facts. Now-a-days, as the older generation of engineers (the ones who have gotten their degrees before the advent of computers) is retiring and the younger ones with extensive training in the ever-so-popular finite element techniques take over, one begins to encounter in technical discussion a new mentality; the insight of how structures behave under loading of the older generation is being more and more replaced by the nearly religious faith of the younger ones in the predictions of their favorite computer codes.
Actually what one needs is not more of one and less of the other (and the reader is free to associate his preference with one or the other), but an optimal combination of both, namely insight into structural behavior and familiarity with the appropriate numerical techniques. To provide the means for such an approach the development of DISDECO, the Delft Interactive Shell Design Code has been initiated. When finished this open ended, hierarchical, interactive computer code will provide for easy access to the theoretical knowledge, that has been accumulated by the many scientists who have been active in the field of shell stability, via the advanced interactive and computational facilities offered by the modern high-speed 32 bit personal workstations. Great care is being taken to present the results in a unified form so as to make it easy for the user to proceed step-by-step from the simpler approaches used by the early investigators to the more sophisticated analytical and numerical methods used presently.

**Development of Level-1 of DISDECO**

The central part of the whole system is the "Command and Control Processor". Its function is to control and direct the activities of the system. It starts-up and winds-down the design system, processes user input through execution of modules, creates a working environment and in general is the working partner of the user.

The link from the user to the command and control processor passes through the "Man-Machine Interface". Assuming that the user employs a terminal device or workstation which supports graphics, the man-machine interface controls the input stream from the user, analyses it, checks it for correct syntax, validates the commands and passes it in an interpretable format to the command and control processor. In a similar manner the output stream from the design and analysis system is converted to a meaningful output for the user.

The real work of DISDECO is done by a number of dedicated "Analysis Modules". Supplementary modules are required for pre- and postprocessing functions, remote batch processing of analysis and general utility functions.

In DISDECO the lowest hierarchical level consists of semi-analytical solutions for the buckling load of perfect anisotropic circular cylindrical shells under axial compressions, internal or external lateral pressure and torsion. Also included are modules that contain Koiter's imperfections sensitivity theory [1,2,3] extended to anisotropic shell structures.
1. DERIVATION OF THE NONLINEAR EQUILIBRIUM EQUATIONS

The nonlinear equilibrium equations can be derived from the principle of stationary potential energy. A loaded shell is in equilibrium if its total potential energy $\pi$ is stationary, and $\pi$ is stationary if the integrand in the expression for $\pi$ satisfies the Euler equations of the calculus of variations.

The total potential energy of an anisotropic shell subjected to axial compression, external pressure (dead load) and torsion is

$$\pi = U_m + U_b + \Omega \quad (1.1)$$

where

$$U_m = \text{membrane strain energy}$$

$$= \frac{1}{2} \int_0^L \int_0^L (N_x \varepsilon_x + N_y \varepsilon_y + N_{xy} \gamma_{xy}) dx \, dy \quad (1.2)$$

$$U_b = \text{bending strain energy}$$

$$= \frac{1}{2} \int_0^L \int_0^L (M_x \kappa_x + M_y \kappa_y + \frac{M_{xy} + M_{yx}}{2} \kappa_{xy}) dx \, dy \quad (1.3)$$

$$\Omega = \text{potential energy of the applied loads}$$

$$= - \int_0^L \int_0^L \{ N_x \left[ u_{xx} - qW_{xx} \right] + p_{xx} W + N_{xy} \left[ v_{xx} \right] \} dx \, dy \quad (1.4)$$

For the notation and sign convention used see Fig. 1. The constitutive equations for an anisotropic shell can be written in matrix notation as

$$[N] = [A] \begin{bmatrix} [\varepsilon] \end{bmatrix} + [B] \begin{bmatrix} [\kappa] \end{bmatrix} \quad (1.5a)$$

$$[M] = [B] \begin{bmatrix} [\varepsilon] \end{bmatrix} + [D] \begin{bmatrix} [\kappa] \end{bmatrix} \quad (1.5b)$$

where

$$[N] = \begin{bmatrix} N_x & N_y & N_{xy} \end{bmatrix}^T \quad [\varepsilon] = \begin{bmatrix} \varepsilon_x & \varepsilon_y & \gamma_{xy} \end{bmatrix}^T$$

$$[M] = \begin{bmatrix} M_{xy} + M_{yx} \end{bmatrix}^T \quad [\kappa] = \begin{bmatrix} \kappa_x & \kappa_y & \kappa_{xy} \end{bmatrix}^T \quad (1.6a)$$
and

$$
[A] = \begin{bmatrix}
A_{11} & A_{12} & A_{16} \\
A_{12} & A_{22} & A_{26} \\
A_{16} & A_{26} & A_{66}
\end{bmatrix}
$$

$$
[B] = \begin{bmatrix}
B_{11} & B_{12} & B_{16} \\
B_{12} & B_{22} & B_{26} \\
B_{16} & B_{26} & B_{66}
\end{bmatrix}
$$

(1.7)

$$
[D] = \begin{bmatrix}
D_{11} & D_{12} & D_{16} \\
D_{12} & D_{22} & D_{26} \\
D_{16} & D_{26} & D_{66}
\end{bmatrix}
$$

The coefficients $A_{ij}$, $B_{ij}$, and $D_{ij}$ are given in Appendix A. For the definition of stress- and moment resultants see Fig. 2.

Using a Donnell type theory the strain-displacement relations are for $W$ and $\overline{W}$ positive inward

$$
\varepsilon_x = u_x + \frac{1}{2} W_x^2 + \overline{W}_x W_x \\
\gamma_{xy} = u_y + v_x + W_x W_y + \overline{W}_x W_y + \overline{W}_y W_x \\
\kappa_x = -W_{xx} \\
\kappa_y = -W_{yy} \\
\kappa_{xy} = -2W_{xy}
$$

(1.8)

Letting now $N_{x|x=L} = -N_o$ and $N_{xy|x=L} = T_o$, then the variational statement of equilibrium for an anisotropic circular cylindrical shell under the specified external loading is

$$
\delta \pi = \delta \int [F(u,u_x,u_y,v,v_x,v_y;W,W_x,W_y,W_{xx},W_{yy},W_{xy})] dx dy = 0
$$

(1.9)

where

$$
F = \frac{1}{2} \left\{ A_{11} \varepsilon_x^2 + 2A_{12} \varepsilon_x \varepsilon_y + 2A_{16} \varepsilon_x \gamma_{xy} + 2A_{26} \varepsilon_y \gamma_{xy} + A_{22} \varepsilon_y^2 + A_{66} \gamma_{xy}^2 \\
+ 2B_{11} \kappa_x^2 + 2B_{12} (\varepsilon_x \kappa_x + \varepsilon_y \kappa_y) + 2B_{16} (\varepsilon_x \gamma_{xy} + \varepsilon_y \kappa_y) \\
+ 2B_{26} (\varepsilon_y \gamma_{xy} + \kappa_x \gamma_{xy}) + 2B_{22} \kappa_y^2 + 2B_{66} \gamma_{xy}^2 \\
+ D_{11} \kappa_x^2 + 2D_{12} \kappa_x \kappa_y + 2D_{16} \kappa_x \gamma_{xy} + 2D_{26} \kappa_y \gamma_{xy} + D_{22} \kappa_y^2 + D_{66} \gamma_{xy}^2 \\
+ 2N_o (u_x - qW_{xx}) - 2p W - 2T_o v_x \right\}
$$

(1.10)
Thus, in this case

$$\delta \pi = \frac{\partial \pi}{\partial u} \delta u + \frac{\partial \pi}{\partial v} \delta v + \frac{\partial \pi}{\partial W} \delta W = 0$$  \hspace{1cm} (1.11)$$

implies the 3 conditions

$$\frac{\partial \pi}{\partial u} = \frac{\partial \pi}{\partial v} = \frac{\partial \pi}{\partial W} = 0$$  \hspace{1cm} (1.12)$$

Carrying out the indicated variations and integrating by parts wherever necessary one obtains the expressions

$$\frac{\partial \pi}{\partial u} = \int \int \left\{ \frac{\partial F}{\partial u} - \frac{\partial}{\partial x} \frac{\partial F}{\partial u_x} - \frac{\partial}{\partial y} \frac{\partial F}{\partial u_y} \right\} \delta u \, dx \, dy + \oint \frac{\partial F}{\partial u_x} \delta u_x \, dy + \oint \frac{\partial F}{\partial u_y} \delta u_y \, dx = 0$$  \hspace{1cm} (1.13)$$

$$\frac{\partial \pi}{\partial v} = \int \int \left\{ \frac{\partial F}{\partial v} - \frac{\partial}{\partial x} \frac{\partial F}{\partial v_x} - \frac{\partial}{\partial y} \frac{\partial F}{\partial v_y} \right\} \delta v \, dx \, dy + \oint \frac{\partial F}{\partial v_x} \delta v_x \, dy + \oint \frac{\partial F}{\partial v_y} \delta v_y \, dx = 0$$  \hspace{1cm} (1.14)$$

$$\frac{\partial \pi}{\partial W} = \int \int \left\{ \frac{\partial F}{\partial W} - \frac{\partial}{\partial x} \frac{\partial F}{\partial W_x} - \frac{\partial}{\partial y} \frac{\partial F}{\partial W_y} \right\} \delta W \, dx \, dy + \oint \left( \frac{\partial F}{\partial W_x} - \frac{\partial}{\partial x} \frac{\partial F}{\partial W_{xx}} - \frac{\partial}{\partial y} \frac{\partial F}{\partial W_{xy}} \right) \delta W_x \, dy + \oint \left( \frac{\partial F}{\partial W_y} - \frac{\partial}{\partial x} \frac{\partial F}{\partial W_{yx}} - \frac{\partial}{\partial y} \frac{\partial F}{\partial W_{yy}} \right) \delta W_y \, dx = 0$$  \hspace{1cm} (1.15)$$

For this to be true for arbitrary nonzero variations \( \delta u \), \( \delta v \) and \( \delta W \) one must have

$$\frac{\partial^2 F}{\partial u \partial x} - \frac{\partial}{\partial x} \frac{\partial F}{\partial u_x} - \frac{\partial}{\partial y} \frac{\partial F}{\partial u_y} = 0$$  \hspace{1cm} (1.16)$$

and at \( x = 0,L \) either \( \frac{\partial F}{\partial u_x} = 0 \) or \( \delta u = 0 \)  \hspace{1cm} (1.17)$$

$$\frac{\partial^2 F}{\partial v \partial x} - \frac{\partial}{\partial x} \frac{\partial F}{\partial v_x} - \frac{\partial}{\partial y} \frac{\partial F}{\partial v_y} = 0$$  \hspace{1cm} (1.18)$$
and at $x = 0, L$ either $\frac{\partial F}{\partial v_x} = 0$ or $\delta v = 0$ (1.19)

\[
\frac{\partial F}{\partial W_x} - \frac{\partial}{\partial x} \frac{\partial F}{\partial W_{xx}} - \frac{\partial}{\partial y} \frac{\partial F}{\partial W_{xy}} + \frac{\partial^2 F}{\partial x^2 \partial W_{xx}} + \frac{\partial^2 F}{\partial x \partial y \partial W_{xy}} + \frac{\partial^2 F}{\partial y^2 \partial W_{yy}} = 0
\] (1.20)

and at $x = 0, L$ either $\frac{\partial F}{\partial W_{xx}} - \frac{\partial}{\partial x} \frac{\partial F}{\partial W_{xx}} - \frac{\partial}{\partial y} \frac{\partial F}{\partial W_{xy}} = 0$ or $\delta W = 0$ (1.21)

and either $\frac{\partial F}{\partial W_{xx}} = 0$ or $\frac{\partial}{\partial x} \delta W = 0$ (1.22)

Notice that at $x = \text{constant} dx = 0$.

Carrying out the indicated differentiations, substituting and regrouping yields the following boundary value problem

\[ N_{x,x} + N_{xy,y} = 0 \quad \text{at} \quad x = 0, L \quad \text{either} \quad N_{x} = -N_{0} \quad \text{or} \quad \delta u = 0 \] (1.23)

\[ N_{xy,x} + N_{y,y} = 0 \quad \text{at} \quad x = 0, L \quad \text{either} \quad N_{xy} = T_{0} \quad \text{or} \quad \delta v = 0 \] (1.24)

\[ M_{x,xx} + (M_{xy} + M_{yx})_{x} + M_{y,yy} \]

\[ + \frac{1}{R} N_{y} + N_{x} (W_{xx} + \overline{W}_{xx}) + 2N_{xy} (W_{xy} + \overline{W}_{xy}) + N_{y} (W_{yy} + \overline{W}_{yy}) = -p_{e} \]

at $x = 0, L$ either $M_{x,x} + (M_{xy} + M_{yx})_{y} + N_{x} (W_{xx} + \overline{W}_{xx}) + N_{xy} (W_{xy} + \overline{W}_{xy}) = 0$ (1.26)

or $\delta W = 0$

and either $M_{x} = -N_{0}q$ or $\frac{\partial}{\partial x} \delta W = 0$ (1.27)

Further, if one introduces the Airy stress function $F$ such that

\[ N_{x} = F_{yy} ; \quad N_{y} = F_{xx} ; \quad N_{xy} = -F_{xy} \] (1.28)
then the in-plane equilibrium equations (1.23) and (1.24) are identically satisfied. Inverting the first constitutive equation (1.5a) one obtains

\[ [\varepsilon] = [A^*][N] + [B^*][\kappa] \]  

(1.29)

where

\[ [A^*] = [A^{-1}] \quad ; \quad [B^*] = -[A^{-1}][B] \]  

(1.30)

Substituting the inverted first equation (1.29) into the second constitutive equation (1.5b) yields

\[ [M] = [C^*][N] + [D^*][\kappa] \]  

(1.31)

where

\[ [C^*] = [B][A^{-1}] \quad ; \quad [D^*] = [D] - [B][A^{-1}][B] \]  

(1.32)

Notice that since [A] is symmetric, therefore also [A^*] is symmetric, and since further [B] is also symmetric, therefore [C^*] = -[B^*]^T. Finally since [D] is symmetric, therefore also [D^*] is symmetric.

Substituting now into the out-of-plane equilibrium equation (1.25) one obtains a single equation in the 2 unknowns W and F. A second equation involving the unknowns W and F is obtained by deriving a compatibility condition. This is done by eliminating u and v from the strain-displacement relations (1.8) yielding

\[ \varepsilon_{x,yy} + \varepsilon_{y,xx} - \gamma_{x,y} = -\frac{1}{R}W_{,xx} - \frac{1}{2}\{ W_{,xx}(W_{,yy} + 2\overline{W}_{,yy}) \}
\]

\[ - 2W_{,xy}(W_{,xy} + 2\overline{W}_{,xy}) + W_{,yy}(W_{,yy} + 2\overline{W}_{,yy}) \} \]

(1.33)

Substituting for \( \varepsilon_x^* \), \( \varepsilon_y \) and \( \gamma_{xy} \) yields a second equation in the unknowns W and F. Introducing the following linear operators

\[ L_{A^*}(.) = A^*_{22}(.)_{xxxx} - 2A^*_{26}(.)_{xyyy} + (2A^*_{12} + A^*_{66})(.)_{xyyy} - 2A^*_{16}(.)_{xyyy} + A^*_{11}(.)_{yyyy} \]

(1.34)
\[ L_B^* (t) = B_{21}^* (t),_{xxxx} + (2B_{26}^* - B_{61}^*) (t),_{xyxy} + (B_{11}^* + B_{22}^* - 2B_{66}^*) (t),_{xyyy} \]
\[ + (2B_{16}^* - B_{62}^*) (t),_{xyyy} + B_{12}^* (t),_{yyyy} \]  
\[ L_D^* = D_{11}^* (t),_{xxxx} + 4D_{16}^* (t),_{xyxy} + 2(D_{12}^* + 2D_{66}^*) (t),_{xyyy} + 4D_{26}^* (t),_{xyyy} + D_{22}^* (t),_{yyyy} \]  

and the following nonlinear operator

\[ L_{NL} (S,T) = S_{xx},_{yy} - 2S_{xy},_{xy} + S_{yy},_{xx} \]  

one can write the governing nonlinear partial differential equations as follows:

\[ L_A^* (F) - L_B^* (W) = - \frac{1}{R} W_{xx} - \frac{1}{2} L_{NL} (W,W+2\overline{W}) \] - compatibility equation  
\[ L_B^* (F) + L_D^* (W) = \frac{1}{R} F_{xx} + L_{NL} (F,W+\overline{W}) + p_e \] - out-of-plane equilibrium equation  

These equations, together with the appropriate boundary conditions, govern the behavior of circular cylindrical shells.

1. In the prebuckling stress and deformation state.
2. At the limit point or bifurcation point (if there is one).
3. In the postbuckling stress and deformation state.
2. CLASSIC LINEARIZED SMALL DEFLECTION THEORY FOR A PERFECT SHELL

For a perfect shell $\bar{W} = 0$. If we let

$$W = W^{(0)} + W^{(1)} , \quad F = F^{(0)} + F^{(1)}$$  \hspace{1cm} (2.1)

where $W^{(0)}, F^{(0)}$ represent the prebuckling solutions and $W^{(1)}, F^{(1)}$ represent small perturbations at buckling, then a direct substitution into Eqs. (1.38) and (1.39) and deletion of products of the perturbation quantities yields a set of nonlinear governing equations for the prebuckling quantities

$$L_{A^*}(F^{(0)}) - L_{B^*}(W^{(0)}) = -\frac{1}{R} W^{(0)} \cdot \frac{1}{2} L_{NL}(W^{(0)}, W^{(0)})$$  \hspace{1cm} (2.2)

$$L_{B^*}(F^{(0)}) + L_{D^*}(W^{(0)}) = \frac{1}{R} F^{(0)} \cdot \frac{1}{2} L_{NL}(F^{(0)}, W^{(0)}) + p_e$$  \hspace{1cm} (2.3)

and a set of linearized stability equations governing the perturbation quantities,

$$L_{A^*}(F^{(1)}) - L_{B^*}(W^{(1)}) = -\frac{1}{R} W^{(1)} \cdot \frac{1}{2} L_{NL}(W^{(0)}, W^{(1)})$$  \hspace{1cm} (2.4)

$$L_{B^*}(F^{(1)}) + L_{D^*}(W^{(1)}) = \frac{1}{R} F^{(1)} \cdot \frac{1}{2} L_{NL}(F^{(0)}, W^{(1)}) + L_{NL}(F^{(1)}, W^{(0)})$$  \hspace{1cm} (2.5)

2.1 Axial compression and internal pressure ($p_e = -p_i$)

If one assumes the following 'membrane' solution to represent the prebuckling stress and deformation state

$$W^{(0)} = tW_v + tW_{\hat{p}_i}$$  \hspace{1cm} (2.6)

$$F^{(0)} = \frac{Et}{cR} \left( -\frac{1}{2} \lambda y^2 + \frac{1}{2} \hat{p}_i x^2 \right)$$  \hspace{1cm} (2.7)

where

$$\hat{\lambda} = \lambda - \frac{1}{2} \frac{p_i}{\hat{p}_i} ; \quad \lambda = \frac{\sigma_c}{\sigma_{cl}} = \frac{N_x}{N_{cl}} ; \quad \hat{p}_i = \frac{p_i}{\frac{1}{R} \sigma_c} ; \quad \sigma_{cl} = \frac{Et}{cR}$$  \hspace{1cm} (2.8)

and $c = \sqrt{3(1-v^2)}$
then the equations governing the prebuckling state are identically satisfied and the linearized stability equations reduce to

\[
L_A(F^{(1)}) - L_B(W^{(1)}) = -\frac{1}{R} W_{xx}^{(1)}
\]

(2.9)

\[
L_B(F^{(1)}) + L_D(W^{(1)}) = \frac{1}{R} c^2 \frac{2}{c R} \frac{E}{\rho_i} W_{yy}^{(1)} - \frac{2}{c R} \lambda W_{xx}^{(1)}
\]

(2.10)

Assuming that the radial displacement component of the classic buckling mode can be written as

\[
W^{(1)} = t \sin \frac{\pi x}{L} \cos \frac{n}{R} (\gamma - \tau_K x) + \frac{t}{2} \left\{ \sin \left( \frac{\ell}{m} \frac{x - l}{n} y \right) + \sin \left( \frac{\ell}{p} \frac{x + l}{n} y \right) \right\}
\]

(2.11)

where

\[
l_m = m \frac{\pi}{L} + \frac{n}{R} \tau_K \quad \quad l_p = m \frac{\pi}{L} - \frac{n}{R} \tau_K \quad \quad l_n = \frac{n}{R}
\]

(2.12)

and Khot's skewness parameter \( \tau_K^{[4]} \) is introduced in order to account for the possibility of torsional coupling.

Substituting into the compatibility equation yields a linear, inhomogeneous partial differential equation for \( F^{(1)} \). Using the Method of Undetermined Coefficients one can obtain the following particular integral

\[
F^{(1)} = \frac{E}{4c} \frac{3}{T_{5,m,n}} \left( \frac{\bar{T}_{3,m,n}}{T_{5,m,n}} \sin \left( \frac{\ell}{m} \frac{x - l}{n} y \right) + \frac{\bar{T}_{4,p,n}}{T_{6,p,n}} \sin \left( \frac{\ell}{p} \frac{x + l}{n} y \right) \right)
\]

(2.13)

where the coefficients \( \bar{T}_{3,m,n} \bar{T}_{4,p,n} ... \) are defined in Appendix B. An approximate solution of the equilibrium equation by Galerkin's procedure then yields the following eigenvalue

\[
\hat{\lambda}_{mn} = \lambda - \frac{1}{2} \frac{2 - \beta^2}{\alpha^2} \left( \frac{\bar{T}_{1,m,n}}{T_{5,m,n}} + \frac{\bar{T}_{2,p,n}}{T_{6,p,n}} \right) + \frac{2 \beta^2}{\alpha^2} \frac{\bar{T}_{3,m,n}}{T_{5,m,n}} + \frac{\bar{T}_{4,p,n}}{T_{6,p,n}}
\]

(2.14)

Notice that the eigenvalue \( \hat{\lambda}_{mn} \) depends on the wave numbers \( m \) and \( n \) and on Khot's parameter \( \tau_K \). The classic buckling load parameter \( \hat{\lambda}_c \) is the lowest of all eigenvalues.

The quantities \( \hat{W}_v \) and \( \hat{W}_p \) are evaluated by enforcing the circumferential periodicity condition.
\[ \int_0^{2\pi R} v_y \, dy = 0 \]  

(2.15)

where for a perfect shell

\[ v_y = \frac{W}{R} \left( -\frac{1}{2} y^2 + \frac{1}{2} R^2 - 1 \right) + F^{(1)} \]  

(2.19)

Substitution and deletion of squares of the perturbation quantities yields

\[ \int_0^{2\pi R} \left\{ -A_{12}^* \frac{E t}{c R} \hat{A} + A_{22}^* \frac{E t}{c R} \hat{P}_i + A_{12}^* F^{(1)}_{yy} + A_{22}^* F^{(1)}_{xx} - A_{26}^* F^{(1)}_{xy} - B_{21}^* W_{1x}^{(1)} + B_{22}^* W_{1y}^{(1)} - 2B_{26}^* W_{1y}^{(1)} + \frac{t}{R} W_v + \frac{t}{R} W_p + \frac{W}{R} \right\} \, dy = 0 \]  

(2.20)

Substituting for \( W^{(1)} \) and \( F^{(1)} \) from Eqs. (2.11) and (2.13) and carrying out the integrals one obtains

\[ \left\{ -A_{12}^* \frac{E t}{c R} \hat{A} + A_{22}^* \frac{E t}{c R} \hat{P}_i + \frac{t}{R} W_v + \frac{t}{R} W_p \right\} (2\pi R) = 0 \]  

(2.21)

Thus if

\[ \hat{W}_v = \frac{E t}{c} A_{12}^* \hat{A} = \frac{A_{12}^*}{12} \frac{\hat{A}}{c} \quad ; \quad W_p = -\frac{E t}{c} A_{22}^* \hat{P}_i = -\frac{A_{22}^*}{22} \frac{\hat{P}_i}{c} \]  

(2.22)

then the circumferential periodicity condition is identically satisfied.
2.2 External pressure and axial compression

If one assumes the following 'membrane' solution to represent the prebuckling stress- and deformation state

\[ W^{(0)} = t W_v + t W_{pe} \]  \hfill (2.23)

\[ F^{(0)} = \frac{E t}{cR} \left( -\frac{1}{2} R \frac{p_e}{\sigma_{ct}} \left[ x^2 - \frac{1}{2} \lambda y^2 \right] \right) \]  \hfill (2.24)

where

\[ \frac{p_e}{\sigma_{ct}} = \frac{1}{R} \sigma_{ct} \]  \hfill (2.25)

then the equations governing the prebuckling state are identically satisfied and the linearized stability equations reduce to

\[ L_A^*(F^{(1)}) - L_B^*(W^{(1)}) = -\frac{1}{R} W_{xx}^{(1)} \]  \hfill (2.26)

\[ L_B^*(F^{(1)}) + L_D^*(W^{(1)}) = \frac{1}{R} F_{xx}^{(1)} - \frac{E t}{c R} \frac{p_e}{\sigma_{ct}} W_{yy}^{(1)} - \frac{E t}{c R} \lambda W_{xx}^{(1)} \]  \hfill (2.27)

By assuming that the radial displacement component of the classic buckling mode can be written as

\[ W^{(1)} = t \sin \frac{\pi m}{L} x \cos \frac{n}{R} (y - \tau_k x) \]  \hfill (2.28)

an approximate solution of the stability equations can be obtained as in the preceding case by Galerkin's method yielding the following eigenvalue

\[ \lambda_{mn\tau} = \frac{1}{2(\alpha_m^2 + \alpha_n^2)} \left( T_{1,m,n} + T_{2,p,n} + T_{3,m,n}^2 + T_{4,p,n}^2 + T_{5,m,n}^2 + T_{6,p,n}^2 \right) \frac{2p_e}{\sigma_{ct}} \frac{\beta_n^2}{\rho} \]  \hfill (2.29)

Introducing the nondimensional ratio \( \tilde{R} = \lambda/\tilde{p_e} \), then the eigenvalue can be written as
\[
\tilde{P}_{mn\tau} = \frac{1}{2} \left( \frac{1}{R} \alpha \frac{2}{m} + \frac{2}{p} \right) \left( \tilde{T}_{1,m,n} + \tilde{T}_{2,p,n} + \tilde{T}_{3,m,n} + \tilde{T}_{4,p,n} + \tilde{T}_{5,m,n} + \tilde{T}_{6,p,n} \right)
\]  

(2.30)

Notice that the eigenvalue \( \tilde{P}_{mn\tau} \) depends on the wave numbers \( m \) and \( n \) and on Khot's parameter \( \tau_{K} \). The classic buckling load parameter \( \tilde{P}_{c} \) is the lowest of all eigenvalues.

The quantities \( W_{V} \) and \( W_{P_{e}} \) are evaluated by enforcing the circumferential periodicity condition yielding

\[
W_{V} = \tilde{A}_{12c} \frac{\lambda}{c} ; \quad W_{P_{e}} = \tilde{A}_{22c} \frac{\tilde{P}_{e}}{c}
\]

(2.31)

2.3 Torsion and axial compression

If one assumes the following 'membrane' solution to represent the prebuckling stress- and deformation state

\[
W^{(0)} = tW_{V} + tW_{1}
\]

(2.32)

\[
F^{(0)} = \frac{Et}{cR} \left( -\tilde{T}_{1,xy} - \frac{1}{2} \lambda y^{2} \right)
\]

(2.33)

where

\[
\tilde{T}_{1,xy} = \frac{T_{xy}}{G_{c1}} ; \quad N_{xy} = \frac{T_{0}}{2\pi R^{2}}
\]

(2.34)

and \( T_{0} \) is the applied torque, then the equations governing the prebuckling state are identically satisfied and the linearized stability equations reduce to

\[
L_{A^{*}}(F^{(1)}) - L_{B^{*}}(W^{(1)}) = -\frac{1}{R} W^{(1)}_{xx}
\]

(2.35)

\[
L_{B^{*}}(F^{(1)}) + L_{D}(W^{(1)}) = \frac{1}{R} F^{(1)}_{xx} + 2 \frac{E}{cR} \tilde{T}_{1} W^{(1)}_{xy} - \frac{E}{cR} \lambda W^{(1)}_{xx}
\]

(2.36)

Under torsional loading the buckling deformation of a cylindrical shell consists of a number of circumferential waves that spiral around the cylinder from one end to the other. Such waves can be represented by the following deflection function
\[ W^{(1)} = t \sin m \pi \frac{x}{L} \cos n \frac{y}{R} (y \cdot \tau_K x) \]  

(2.37)

Proceeding as in the previous 2 cases one can obtain an approximate solution of the stability equations yielding the following eigenvalue

\[ \tau_{mn} = \frac{1}{4(\alpha_m - \alpha_p)\beta} \left( \frac{\tau_{1,m,n}^2}{1,m,n} + \frac{\tau_{2,p,n}^2}{2,p,n} + \frac{\tau_{4,p,n}^2}{4,p,n} + \frac{\tau_{6,p,n}^2}{5,p,n} \right) \frac{\lambda}{2(\alpha_m - \alpha_p)\beta} \]  

(2.38)

Notice that the eigenvalue \( \tau_{mn} \) depends on the wave numbers \( m \) and \( n \) and on Khot's parameter \( \tau_K \). The classic buckling load \( \tau_c \) is the lowest of all eigenvalues.

Notice also that if \( \tau_K = 0 \) then \( \alpha_m = \alpha_p \). Thus the above formula is only valid if \( \tau_K \) is not equal to 0.

The quantities \( W_v \) and \( W_t \) can be evaluated by enforcing the circumferential periodicity condition yielding

\[ W_v = \tilde{A}_{12} \lambda \; ; \; W_t = -\frac{1}{26} \tau_t \]  

(2.39)

To obtain a solution valid when \( \tau_K = 0 \) one must assume the following buckling mode

\[ W^{(1)} = t \sin (m \pi \frac{x}{L} - n \frac{y}{R}) \]  

(2.40)

Proceeding as before one can obtain an approximate solution of the stability equations yielding the following eigenvalue

\[ \tau_{mn} = \frac{1}{4\alpha_0 \beta_n} \left( \frac{\tau_{1}^{(o)} (x)}{1} + \frac{\tau_{3}^{(o)} (x)}{3} + \frac{\tau_{4}^{(o)} (x)}{5} - 2\alpha_0^2 \frac{\lambda}{\beta_n} \right) \]  

(2.41)

where the coefficients \( \alpha_0 \), \( \tau_{1}^{(o)} \), \( \tau_{3}^{(o)} \), and \( \tau_{5}^{(o)} \) are defined in Appendix B.
3. EFFECT OF AXISYMMETRIC IMPERFECTION

If we let

\[ W = W^{(0)} + W^{(1)} \quad , \quad F = F^{(0)} + F^{(1)} \]  \hspace{1cm} (3.1)

where \( W^{(0)} \), \( F^{(0)} \) represent the prebuckling solutions and \( W^{(1)}, F^{(1)} \) represent small perturbations at buckling, then a direct substitution into Eqs. (1.38) and (1.39) and deletion of products of the perturbation quantities yields a set of nonlinear governing equations for the prebuckling quantities

\[ L_{A*}(F^{(0)}) - L_{B*}(W^{(0)}) = \frac{1}{R} W^{(0)}_{xx} - \frac{1}{2} L_{NL}(W^{(0)} + 2\bar{W}W^{(0)}) \]  \hspace{1cm} (3.2)

\[ L_{B*}(F^{(0)}) + L_{D*}(W^{(0)}) = \frac{1}{R} F^{(0)}_{xx} + L_{NL}(F^{(0)}, W^{(0)} + \bar{W}) + p_e \]  \hspace{1cm} (3.3)

and a set of linearized equations governing the perturbation quantities

\[ L_{A*}(F^{(1)}) - L_{B*}(W^{(1)}) = \frac{1}{R} W^{(1)}_{xx} - L_{NL}(W^{(0)} + \bar{W}, W^{(1)}) \]  \hspace{1cm} (3.4)

\[ L_{B*}(F^{(1)}) + L_{D*}(W^{(1)}) = \frac{1}{R} F^{(1)}_{xx} + L_{NL}(F^{(0)}, W^{(1)}) + L_{NL}(F^{(1)}, W^{(0)} + \bar{W}) \]  \hspace{1cm} (3.5)

3.1 Axial compression and internal pressure \( (p_e = p_i) \)

If the initial imperfection is axisymmetric, e.g.,

\[ \bar{W} = t \bar{\xi}_1 \cos \pi \frac{x}{L} \]  \hspace{1cm} (3.6)

then the prebuckling solution will also be axisymmetric, namely

\[ W^{(0)} = t \dot{W} + tW_{P_i} + w_0(x) \]  \hspace{1cm} (3.7)

\[ F^{(0)} = \frac{Et^2}{cR} \left( -\frac{1}{2} \dot{\gamma}^2 + \frac{1}{2} \dot{P}_i^2 \right) + f_0(x) \]  \hspace{1cm} (3.8)

Substitution into the equations governing the prebuckling state (Eqs. 3.2 - 3.3) yields
\[ A_{22} w_{o,xxxx} - B_{21}^* w_{o,xxxx} = \frac{1}{R} w_{o,xx} \]  
\[ B_{21}^* f_{o,xxxx} + D_{11}^* w_{o,xxxx} = \frac{1}{R} f_{o,xx} \cdot \frac{E_l}{cR} \lambda^2 w_{o,xx} + \frac{E_l}{cR} \left( \frac{1}{\xi_1} \right)^2 \xi_1 \lambda \cos x \frac{x}{L} \]  

Neglecting the effect of boundary conditions the (particular) solution of these equations is

\[ w_o(x) = \frac{\lambda}{\lambda_{c_i} - \lambda} \xi_1 \cos x \frac{x}{L} \]  
\[ f_o(x) = \frac{E_l}{c} \frac{\lambda}{\lambda_{c_i} - \lambda} \frac{(1+\bar{B}_{21}^* \alpha_i^2)}{2 \alpha_i^2 A_{22}^*} \xi_1 \cos x \frac{x}{L} \]  

where

\[ \lambda_{c_i} = \frac{1}{2} \left\{ \alpha_i^2 \bar{B}_{11}^* + \frac{(1+\bar{B}_{21}^* \alpha_i^2)^2}{\alpha_i^2 A_{22}^*} \right\} \]  

is the classic axisymmetric buckling load for axial compression only.

For the axisymmetric imperfection and prebuckling state the linearized stability equations reduce to

\[ L_{A*}(F^{(1)}) - L_{B*}(W^{(1)}) = \frac{1}{R} W_{xx}^{(1)} + \frac{2c}{R} \alpha_i^2 (\xi_1 + \bar{A}) \cos x \frac{x}{L} W_{yy}^{(1)} \]  
\[ L_{B*}(F^{(1)}) + L_{D*}(W^{(1)}) = \frac{1}{R} F_{xx}^{(1)} + \frac{E_l}{cR} \left( \frac{p_i - 2c \alpha_i^2 \bar{B}}{\xi_1 + \bar{A}} \right) \cos x \frac{x}{L} W_{yy}^{(1)} \]  

where

\[ \bar{A} = \frac{\lambda}{\lambda_{c_i} - \lambda} \xi_1 : \bar{B} = \frac{\lambda}{\lambda_{c_i} - \lambda} \frac{(1+\bar{B}_{21}^* \alpha_i^2)}{2 \alpha_i^2 A_{22}^*} \xi_1 \]  

Assuming that the radial displacement component of the classic buckling mode can be written as
\[ W^{(1)} = \hat{\lambda}^2 \sin \frac{\pi}{L} \cos \frac{n}{R} (y - \tau_k x) \]  

(3.17)

then substitution into the compatibility equation (3.14) yields a linear, inhomogenous partial differential equation for \( F^{(1)} \). By the Method of Undetermined Coefficients one can obtain the following particular integral

\[
F^{(1)} = \hat{A}_1 \sin \left( \frac{\pi}{L} x + \frac{n}{R} y \right) + \hat{A}_2 \sin \left( \frac{\pi}{L} x + \frac{n}{R} y \right) + \hat{A}_3 \sin \left( \frac{\pi}{L} x - \frac{n}{R} y \right) + \hat{B}_1 \cos \left( \frac{\pi}{L} x + \frac{n}{R} y \right) + \hat{B}_2 \cos \left( \frac{\pi}{L} x - \frac{n}{R} y \right) + \hat{B}_3 \cos \left( \frac{\pi}{L} x + \frac{n}{R} y \right) + \hat{B}_4 \cos \left( \frac{\pi}{L} x - \frac{n}{R} y \right) + \hat{B}_5 \cos \left( \frac{\pi}{L} x - \frac{n}{R} y \right)
\]  

(3.18)

where

\[
l_{i+m} = (i+m) \frac{\pi}{L} + \frac{n}{R} \tau_K
\]

(3.19)

and the constants \( \hat{A}_1, \hat{A}_2, ..., \hat{B}_1, ..., \hat{B}_5 \) are defined in Appendix C.

An approximate solution of the equilibrium equation (3.15) by Galerkin’s procedure then yields the following cubic polynomial for the eigenvalue \( \hat{\lambda} \):

\[ \hat{\lambda}^3 - \left\{ \lambda_{mn} + 2 \lambda_c + \hat{C}_1 \delta_{i=2m} \right\} \hat{\lambda}^2 \]

\[ + \left\{ 2 \lambda_{mn} + \lambda_c + \hat{C}_2 \delta_{i=2m} \right\} \lambda_c \hat{\lambda} - \left\{ \lambda_{mn} + \hat{C}_2 \delta_{i=2m} + \hat{C}_3 \delta_{i=m} + \hat{C}_4 \delta_{i=0} \right\} \delta_{i=2m} = 0 \]  

(3.20)

where

\[
\delta_{i=2m} = 1 \text{ if } i = 2m, \quad \delta_{i=m} = 1 \text{ if } i = m, \quad \delta_{i=0} = 1 \text{ if } \tau_k = 0
\]

(3.21)

and the constants \( \hat{C}_1, ..., \hat{C}_4 \) are defined in Appendix C.
3.2 External pressure and axial compression

If the initial imperfection is axisymmetric, e.g.,

\[
\overline{W} = t \xi_1 \cos \pi \frac{x}{L}
\]  

(3.22)

then the prebuckling solution will also be axisymmetric, namely

\[
W^{(0)} = tW_c + w_0(x)
\]  

(3.23)

\[
F^{(0)} = \frac{Et^2}{cR} \left( -\frac{1}{2} P_e \right) \left( x^2 + Rx^2 \right) + f_0(x)
\]  

(3.24)

where

\[
W_c = W_v + \frac{P_e}{c} \left( \overline{A}_{22}^* + R \overline{A}_{12}^* \right) ; \quad \overline{R} = \frac{\lambda}{P_e}
\]

A substitution into the equations governing the prebuckling state (Eqs. 3.2 and 3.3) yields

\[
A_{22}^* f_{0,xxxx} - B_{21}^* w_{0,xxxx} = -\frac{1}{R} w_{0,xx}
\]  

(3.25)

\[
B_{21}^* f_{0,xxxx} + D_{11}^* w_{0,xxxx} = \frac{1}{R} f_{0,xx} + \frac{Et^2}{cR} P_e \overline{w}_{0,xx} + \frac{Et^3}{cR} \left( \frac{\pi}{L} \right)^2 \xi_1 P_e \cos \pi \frac{x}{L}
\]  

(3.26)

Neglecting the effect of boundary conditions, the (particular) solution of these equations is

\[
w_0(x) = t \frac{P_e}{\lambda c_i - \overline{R} P_e} \xi_1 \cos \pi \frac{x}{L}
\]  

(3.27)

\[
f_0(x) = \frac{Et^3}{c} \frac{P_e}{\lambda c_i - \overline{R} P_e} \left( 1 + \overline{B}_{21} \frac{\alpha^2}{2} \right) \xi_1 \cos \pi \frac{x}{L}
\]  

(3.28)

where
\[
\lambda_{c_1} = \frac{1}{2} \left\{ \alpha_i^2 \tilde{D}_{11}^* + \frac{(1+\tilde{E}_{21}^* \alpha_i^2)}{\alpha_1^2 \tilde{A}_{22}^*} \right\}
\]

is the classic axisymmetric buckling load for axial compression only.

For the axisymmetric imperfection and prebuckling state the linearized stability equations reduce to

\[
L_{A^*}(f^{(1)}) - L_{B^*}(W^{(1)}) = -\frac{1}{R} W_x^{(1)} + \frac{2c}{R} \alpha_1^2 (\tilde{\xi}_1 + \tilde{A}) \cos \pi \frac{x}{L} W_y^{(1)}
\]

\[
L_{B^*}(f^{(1)}) + L_{D^*}(W^{(1)}) = \frac{1}{R} F_x^{(1)} - \frac{Et^2}{cR} \frac{\tilde{P}_e}{2\alpha_1^2 \tilde{B}} \cos \pi \frac{x}{L} W_y^{(1)}
\]

\[
\frac{Et^2}{cR} \frac{\tilde{P}_e}{2\alpha_1^2 \tilde{B}} W_x^{(1)} + \frac{2c}{R} \alpha_1^2 (\tilde{\xi}_1 + \tilde{A}) \cos \pi \frac{x}{L} f_x^{(1)}
\]

where

\[
\tilde{A} = \frac{\tilde{R}_{p_e}}{\alpha_i^2 \tilde{A}_{22}^*} \tilde{\xi}_1, \quad \tilde{B} = \frac{(1+\tilde{E}_{21}^* \alpha_i^2)}{\alpha_1^2 \tilde{A}_{22}^*} \tilde{\xi}_1
\]

By assuming that the radial displacement component of the classic buckling mode can be written as

\[
W^{(1)} = t \frac{\pi x}{L} \cos \pi \frac{x}{R} (y - \tau_y \xi)
\]

an approximate solution of the stability equations can be obtained as in the preceding case yielding the following cubic polynomial for the eigenvalue \( \tilde{P}_e \):

\[
\tilde{R}^{3} \tilde{P}_e^2 - \left( \tilde{P}_e \tilde{D}_{11}^* \tilde{R}^2 + 2\lambda \tilde{R}_{c_1} \tilde{D}_{11}^* \tilde{R}^2 \delta_{i=2m} \right) \tilde{P}_e + \left( 2\tilde{P}_e \tilde{D}_{11}^* \tilde{R}^2 + \tilde{D}_{11}^* \tilde{R}^2 \delta_{i=2m} + \tilde{D}_{11}^* \tilde{R}^2 \delta_{i=2m} + \tilde{D}_{11}^* \tilde{R}^2 \delta_{i=2m} + \tilde{D}_{11}^* \tilde{R}^2 \delta_{i=2m} \right) \tilde{P}_e^2 = 0
\]

Thus for \( \tilde{R} = 0 \) \( \tilde{P}_e \rightarrow \) external pressure case

\[
\tilde{P}_e = \tilde{P}_e \tilde{D}_{11}^* \tilde{R}^2 + \tilde{D}_{11}^* \tilde{R}^2 \delta_{i=2m} + \tilde{D}_{11}^* \tilde{R}^2 \delta_{i=2m}
\]
whereas for $R > 0$ \rightarrow \text{combined loading case}

\[
\begin{align*}
P_e^{-3} \cdot \left\{ \frac{p_{mn\tau}}{R} + 2\lambda_{c_i} \frac{\hat{R}}{R} \cdot D_1 \cdot t_{i=2m} \cdot \delta \right\}P_e^{-2} + (1/R) \left\{ 2p_{mn\tau} + \lambda_{c_i} \frac{\hat{R}}{R} + (D_2 + D_4) \xi_i \cdot \delta_{i=2m} \right\} \lambda_{c_i} P_e \times \left( \frac{\hat{R}}{R} \right)^2 \left\{ \frac{p_{mn\tau}}{R} + D_2 \cdot t_{i=2m} + (D_3 - D_4) \cdot \delta \cdot \tau \cdot \xi \right\} \lambda_{c_i}^2 & = 0
\end{align*}
\]

The constants $\hat{D}_1, \ldots, \hat{D}_4$ are listed in Appendix C.
4. EFFECT OF ASYMMETRIC IMPERFECTION (B-FACTOR METHOD)

Koiter\[1\] has shown that the imperfection sensitivity of shell structures is closely related to their initial postbuckling behavior. That is, one is interested in the variation of $\Lambda$ with $\xi$ in the vicinity of the bifurcation point $\Lambda = \Lambda_c$, where $\Lambda$ is the loading parameter and $\xi$ is the amplitude of the buckling mode normalized with respect to the wall thickness $t$. If the shell structure possesses a unique buckling mode associated with the lowest buckling load, then its buckling and initial postbuckling behavior can be expressed as

$$\Lambda = \Lambda_c + a\xi + b\xi^2 + ...$$  \hspace{1cm} (4.1)

Figure 3a illustrates the case of asymmetric imperfection for $a < 0$, whereas Figures 3b and 3c show cases of symmetric bifurcation with $a = 0$ and $b > 0$ and $b < 0$, respectively.

As can be seen from Fig. 3 the shape of the postbuckling equilibrium path plays a central role in determining the influence of the initial imperfections. When the initial portion of the postbuckling path has a positive curvature (see Fig. 3b) the structure can develop considerable postbuckling strength, and loss of stability of the primary path does not result in structural collapse. However, when the initial portion of the postbuckling path has a negative curvature (see Fig. 3c), then in most cases buckling will occur violently and the magnitude of the critical load is subject to the degrading influence of initial imperfections.

Thus, in order to describe the expected behavior of real (read imperfect) shell structures, one is interested in the variation of $\Lambda(\xi, \xi)$ with $\xi$ for an imperfect shell ($\xi \neq 0$) in the vicinity of $\Lambda = \Lambda_c$, so that the expressions used for perfect shells are still useful. In the following a small initial stress free imperfection $\bar{W} = \xi \hat{W}$ will be assumed, where $\hat{W}$ is a suitably normalized continuously differentiable radial displacement field and $\xi$ is its amplitude. If only the lowest order terms in $\xi$ are retained then an asymptotic expansion of the following form

$$(\Lambda - \Lambda_c)\xi = a\Lambda_c\xi + b\Lambda_c\xi^2 + ... - \alpha\Lambda_c\xi - \beta(\Lambda - \Lambda_c)\xi - ...$$  \hspace{1cm} (4.2)

becomes appropriate. Notice that

$$\lim_{\xi \to 0} \{ \lim_{\xi \to 0} \Lambda \} = \Lambda_c \hspace{1cm} \text{but for } \xi \neq 0 \hspace{1cm} \lim_{\xi \to 0} \Lambda = 0$$  \hspace{1cm} (4.3)
The relation between the limit loads $\Lambda_s$ of the imperfect structure and the bifurcation load $\lambda_c$ of the perfect structure can now be found from Eq. (4.2) and the condition

$$\frac{d\Lambda}{d\xi} \bigg|_{\Lambda = \Lambda_s} = 0$$

(4.4)

For $a = 0$, one obtains after some algebraic manipulation the formula

$$\left(1 - \frac{\Lambda_s}{\Lambda_c}\right)^{3/2} = \frac{3}{2} \sqrt{-3\alpha^2 \beta \xi} \left\{1 - \frac{\beta}{\alpha} \left(1 - \frac{\Lambda_s}{\Lambda_c}\right)\right\}$$

(4.5)

Using the following perturbation expansion

$$W = W^{(0)}(\lambda) + \xi W^{(1)} + \xi^2 W^{(2)} + ...$$

(4.6)

$$F = F^{(0)}(\lambda) + \xi F^{(1)} + \xi^2 F^{(2)} + ...$$

(4.7)

and membrane prebuckling solutions, formulas for the postbuckling coefficients "a" and "b" have been derived by Budiansky and Hutchinson[2] yielding

$$a = - (3/2) F^{(1)*} (W^{(1)}, W^{(1)}) / (\Lambda_c \hat{\Delta})$$

(4.8)

$$b = - \{2F^{(1)*} (W^{(1)}, W^{(2)}) + F^{(2)*} (W^{(1)}, W^{(1)})\} / (\Lambda_c \hat{\Delta})$$

(4.9)

where

$$\hat{\Delta} = F^{*} (W^{(1)}, W^{(1)}) \quad ; \quad (\hat{\Delta}) = \frac{\partial}{\partial \Lambda} (\Delta)$$

(4.10)

and the subscript ( )$_c$ denotes the fact that the prebuckling solution is evaluated at the bifurcation point.

Using the same perturbation expansion, the formulas for the first and second imperfection form factors $\alpha$ and $\beta$ published by Cohen[3] become for membrane prebuckling solutions

$$\alpha = F^{*} (\hat{\Delta}, W^{(1)}) / (\Lambda_c \hat{\Delta}) \quad ; \quad \beta = (1/\Delta) F^{*} (\hat{\Delta}, W^{(1)})$$

(4.11)
In these formulas the following shorthand notations has been used

\[
A^*(B,C) = \int \int_0^L \{ A_{xx} B_x C_y - A_{xy} (B_x C_y + B_y C_x) + A_{yy} B_y C_x \} \, dx \, dy
\]  

(4.12)

Thus the calculation of the postbuckling and the imperfection form coefficients involves the solution of the zeroth, the first and the second order states. The equations governing these states can be derived as follows:

The governing equations in terms of \( W \) and \( F \) for a perfect shell (\( W=0 \)), made out of general anisotropic material are (with \( W \) positive inward)

\[
L_{A^*}(F) - L_{B^*}(W) = -\frac{1}{R} W_{xx} - \frac{1}{2} L_{NL}(W,W)
\]

(1.38)

\[
L_{B^*}(F) + L_{D^*}(W) = \frac{1}{R} F_{xx} + L_{NL}(F,W) + P_e
\]

(1.39)

A formal substitution of the asymptotic expansion for \( W \) and \( F \) (Eqs. 4.6 and 4.7) and regrouping according to powers of \( \xi \) generates the following sequence of linear equations for the functions appearing in the expansion.

For the zeroth order state:

\[
L_{A^*}(F^{(0)}) - L_{B^*}(W^{(0)}) = -\frac{1}{R} W_{xx} - \frac{1}{2} L_{NL}(W^{(0)}, W^{(0)})
\]

(4.13)

\[
L_{B^*}(F^{(0)}) + L_{D^*}(W^{(0)}) = \frac{1}{R} F_{xx} + L_{NL}(F^{(0)}, W^{(0)}) + P_e
\]

(4.14)

For the first order state:

\[
L_{A^*}(F^{(1)}) - L_{B^*}(W^{(1)}) = -\frac{1}{R} W_{xx} - L_{NL}(W^{(0)}, W^{(0)})
\]

(4.15)

\[
L_{B^*}(F^{(1)}) + L_{D^*}(W^{(1)}) = \frac{1}{R} F_{xx} + L_{NL}(F^{(0)}, W^{(1)}) + L_{NL}(F^{(1)}, W^{(0)})
\]

(4.16)

For the second order state:

\[
L_{A^*}(F^{(2)}) - L_{B^*}(W^{(2)}) = -\frac{1}{R} W_{xx} - L_{NL}(W^{(0)}, W^{(2)}) - \frac{1}{2} L_{NL}(W^{(1)}, W^{(1)})
\]

(4.17)
\[ L_{B^*}(F^{(2)}) + L_{D^*}(W^{(2)}) = \frac{1}{R} F^{(2)}_{xx} + L_{NL}(F^{(0)}, W^{(2)}) + L_{NL}(F^{(2)}, W^{(0)}) + L_{NL}(F^{(1)}, W^{(1)}) \] (4.18)

4.1 Axial compression and internal pressure \((p_e = -p_i)\)

The membrane prebuckling solution

\[ W^{(0)} = t W_i \] (4.19)

\[ F^{(0)} = \frac{E t}{c R} \left( -\frac{1}{2} \lambda \gamma^2 + \frac{1}{2} \lambda x^2 \right) \] (4.20)

where

\[ W_i = \hat{W}_v + W_i = \frac{1}{c} \left( \hat{\lambda}_1 - \hat{\lambda}_2 \right) \] : \[ \hat{\lambda} = \lambda + \frac{1}{2} \pi \] (4.21)

satisfies the governing equations of the zeroth order state identically, and reduces the governing equations of the first- and second order state to

\[ L_{A^*}(F^{(1)}) - L_{B^*}(W^{(1)}) = -\frac{1}{R} W_{xx}^{(1)} \] (4.22)

\[ L_{B^*}(F^{(1)}) + L_{D^*}(W^{(1)}) = \frac{1}{R} F^{(1)}_{xx} + \frac{E t}{c R} \pi W_{yy}^{(1)} - \frac{E t}{c R} \hat{\lambda} W_{xx}^{(1)} \] (4.23)

and

\[ L_{A^*}(F^{(2)}) - L_{B^*}(W^{(2)}) = -\frac{1}{R} W_{xx}^{(2)} + W_{xy}^{(1)} W_{xy}^{(1)} - W_{xx}^{(1)} W_{yy}^{(1)} \] (4.24)

\[ L_{B^*}(F^{(2)}) + L_{D^*}(W^{(2)}) = \frac{1}{R} F^{(2)}_{xx} + \frac{E t}{c R} \pi W_{yy}^{(2)} - \frac{E t}{c R} \hat{\lambda} W_{xx}^{(2)} \] (4.25)

\[ + F^{(1)}_{xx} W_{yy}^{(1)} - 2 F^{(1)}_{xy} W_{xy}^{(1)} + F^{(1)}_{yy} W_{xx}^{(1)} \]

In this case the set of equations for \(W^{(1)} \) and \(F^{(1)} \) represent the classic eigenvalue problem (see Eq. 2.9, 2.10), the solution of which is for \( \hat{\lambda} = \lambda_{mn} \).
\[ W^{(1)} = t \sin \frac{\pi x}{L} \cos \frac{n}{R} (y - \tau_K x) = \frac{1}{2} \{ \sin (\frac{l_i}{m} x - \frac{l_i}{n} y) + \sin (\frac{l_i}{p} x + \frac{l_i}{n} y) \} \]  
(4.26)

\[ F^{(1)} = \frac{E_l^3}{4c} \left\{ \frac{T_{3,m,n}}{T_{5,m,n}} \sin (\frac{l_i}{m} x - \frac{l_i}{n} y) + \frac{T_{4,p,n}}{T_{6,p,n}} \sin (\frac{l_i}{p} x + \frac{l_i}{n} y) \right\} \]  
(4.27)

where

\[ l_i = m \frac{\pi}{L} + n \frac{n}{R} \tau_K, \quad l_i = m \frac{\pi}{L} - n \frac{n}{R} \tau_K, \quad l_i = n \frac{n}{R} \]  
(4.28)

Substituting for \( W^{(1)} \) and \( F^{(1)} \), the governing equations for the second order state become

\[ L_{A_1}(F^{(2)}_i) - L_{B_1}(W^{(2)}_i) = \frac{1}{R} W^{(2)}_{ixx} - \frac{c^2}{R^2} (\alpha_m + \alpha_p)^2 \beta_n^2 \sin (\frac{l_i}{m} x - \frac{l_i}{n} y) \sin (\frac{l_i}{p} x + \frac{l_i}{n} y) \]  
(4.29)

\[ L_{B_2}(F^{(2)}_i) + L_{D_2}(W^{(2)}_i) = \frac{1}{R} F^{(2)}_{ixx} + \frac{E_l^2}{cR} W^{(2)}_{yy} + \frac{E_l^2 \lambda_{mn}}{cR} W^{(2)}_{xx} \]  
(4.30)

\[ + \frac{E_l^2}{2R^2} c (\alpha_m + \alpha_p)^2 \beta_n^2 \left\{ \frac{T_{3,m,n}}{T_{5,m,n}} + \frac{T_{4,p,n}}{T_{6,p,n}} \right\} \sin (\frac{l_i}{m} x - \frac{l_i}{n} y) \sin (\frac{l_i}{p} x + \frac{l_i}{n} y) \]

These equations admit separable solutions of the following form

\[ W^{(2)}_i = t \left\{ \sum_{i=1,3,..}^\infty A_i \sin (\frac{l_i}{m} x) + \frac{1}{2} \sum_{i=1,3,..}^\infty [B_i \sin (\frac{l_i}{m} x - \frac{l_i}{n} y) + C_i \sin (\frac{l_i}{p} x + \frac{l_i}{n} y)] \right\} \]  
(4.31)

\[ F^{(2)}_i = \frac{E_l^3}{2c} \left\{ \sum_{i=1,3,..}^\infty D_i \sin (\frac{l_i}{m} x) + \frac{1}{2} \sum_{i=1,3,..}^\infty [E_i \sin (\frac{l_i}{m} x - \frac{l_i}{n} y) + F_i \sin (\frac{l_i}{p} x + \frac{l_i}{n} y)] \right\} \]  
(4.32)

where

\[ l_i = i \frac{\pi}{L} \quad m_i = i \frac{\pi}{L} + 2 \frac{n}{R} \tau_K \quad p_i = i \frac{\pi}{L} - 2 \frac{n}{R} \tau_K \quad 2n = 2 \frac{n}{R} \]  
(4.33)

and the summation sign over \( i \) implies summation over odd integers only.

The coefficients are readily determined by Galerkin's procedure and are listed in Appendix D.

Finally, the postbuckling coefficients "a" and "b" are calculated by evaluating the integrals indicated by Eqs. (4.8) and (4.9). In this case

\[ a = 0 \]  
(4.34)

and
\[ b = \frac{c \beta^2}{\pi (\alpha_m^2 + \alpha_p^2)} \frac{1}{\lambda_{nm}} \Sigma_1 \]  

(4.35)

where

\[ \Sigma_1 = 8 \alpha_m (\alpha_m^2 + \alpha_p^2) \left( \frac{\Sigma 3}{\Sigma 5} \right) \Sigma A_i \frac{1}{4m-i} + 4 \frac{\Sigma 3}{\Sigma 5} \alpha_m^2 \Sigma B_i \frac{m}{2m+i} \]  

\[ + \Sigma B_i \alpha_m \left( \frac{\Sigma 3}{\Sigma 5} \frac{\alpha_m^3}{2m-i} \right) + \frac{\Sigma 4}{\Sigma 6} \alpha_p \left( \frac{\alpha_m^3}{2m+i} \right) + 4 \frac{\Sigma 4}{\Sigma 6} \alpha_p^2 \Sigma B_i \frac{m}{2m+i} \]  

(4.36)

\[ + 2 \frac{\Sigma 4}{\Sigma 6} \alpha_m \Sigma B_i \frac{\alpha_p^2}{2m + \alpha_p} \]  

(4.37)

\[ + \frac{\Sigma 4}{\Sigma 6} \alpha_p \Sigma B_i \frac{\alpha_m^2}{2m + \alpha_m} \]  

(4.38)

\[ + \frac{\Sigma 4}{\Sigma 6} \frac{\alpha_m^2}{2m+i} \Sigma C_i \frac{m}{2m+i} \]  

(4.39)

\[ + \frac{\Sigma 4}{\Sigma 6} \alpha_p \Sigma C_i \frac{\alpha_m^2}{2m+i} \]  

(4.40)

\[ + 2 \frac{\Sigma 4}{\Sigma 6} \alpha_m \frac{\alpha_p}{2m+i} \]  

(4.41)

\[ + 2 \frac{\Sigma 4}{\Sigma 6} \alpha_p \frac{\alpha_m}{2m+i} \]  

(4.42)

\[ - 16 \alpha_m^2 \Sigma D_i \frac{1}{4m-i} + 2 \Sigma E_i \left( \frac{\alpha_m^2}{2m+i} + \frac{\alpha_p^2}{i+2m} \right) + 4 \Sigma \frac{\alpha_m^2}{\alpha_m} \frac{m^2}{2m+i} \]  

(4.43)

\[ - 2 \Sigma \alpha_m E_i \left( \frac{\alpha_m^3}{2m+i} + \frac{\alpha_p^2}{i+2m} \right) + 2 \Sigma \alpha_p F_i \left( \frac{\alpha_m^3}{i+2m} + \frac{\alpha_p^2}{i+2m} \right) \]  

(4.44)

\[ + 4 \Sigma \left( \frac{\alpha_p^2}{i+2m} \right) - 2 \Sigma \alpha_p F_i \left( \frac{\alpha_m^3}{i+2m} + \frac{\alpha_p^2}{i+2m} \right) \]  

(4.45)
Notice that the following short-hand notation has been used

\[ \bar{T}_3 = \bar{T}_{3,m,n} ; \quad \bar{T}_4 = \bar{T}_{4,p,n} \]
\[ \bar{T}_5 = \bar{T}_{5,m,n} ; \quad \bar{T}_6 = \bar{T}_{6,p,n} \] (4.37)

4.2 External pressure and axial compression

The membrane prebuckling solution

\[ W^{(0)} = tW_c \] (4.38)
\[ F^{(0)} = \frac{Et}{cR} \left( \frac{1}{2} \bar{P}_e \right) (x^2 + Ry^2) \] (4.39)

where

\[ W_c = W_v + \frac{\bar{P}_e}{c} \left( \bar{A}_{22} + \bar{R} \bar{A}_{12} \right) ; \quad \bar{R} = \frac{\lambda}{\bar{P}_e} \] (4.40)

satisfies the governing equations of the zeroth order state identically and reduces the governing equations of the first- and second order state to

\[ L_{A^*}(F^{(1)}) - L_{B^*}(W^{(1)}) = -\frac{1}{R} W^{(1)}_{xx} \] (4.41)
\[ L_{B^*}(F^{(1)}) + L_{D^*}(W^{(1)}) = \frac{1}{R} F^{(1)}_{xx} - \frac{Et}{cR} \bar{P}_e W^{(1)}_{yy} - \frac{Et}{cR} \bar{R}_p \bar{P}_e W^{(1)}_{xx} \] (4.42)

and

\[ L_{A^*}(F^{(2)}) + L_{B^*}(W^{(2)}) = -\frac{1}{R} W^{(2)}_{xx} + W^{(1)}_{xy} W^{(1)}_{yy} - W^{(1)}_{xx} W^{(1)}_{yy} \] (4.43)
\[ L_{B^*}(F^{(2)}) + L_{D^*}(W^{(2)}) = \frac{1}{R} F^{(2)}_{xx} - \frac{Et}{cR} \bar{P}_e W^{(2)}_{yy} - \frac{Et}{cR} \bar{R}_p \bar{P}_e W^{(2)}_{xx} \]
\[ + F^{(1)}_{xx} W^{(1)}_{yy} - 2 F^{(1)}_{xy} W^{(1)}_{xy} + F^{(1)}_{yy} W^{(1)}_{xx} \] (4.44)
In this case the set of equations of the first order state represent the classic eigenvalue problem (see Eqs. 2.26 and 2.27), the solution of which is for $\bar{p}_e = \bar{p}_{mn\pi}$

$$W^{(1)} = t \sin \frac{m\pi}{L} \cos \frac{n\pi}{R} (y - \tau_K) = \frac{1}{2} \{ \sin (\frac{m}{L} x + \frac{n}{R} y) + \sin (\frac{p}{L} x + \frac{l}{n} y) \}$$  \hspace{1cm} (4.45)

$$F^{(1)} = \frac{Et}{4c} \left\{ \frac{T_{3,m,n}}{T_{5,m,n}} \sin (\frac{m}{L} x - \frac{n}{R} y) + \frac{T_{4,p,n}}{T_{6,p,n}} \sin (\frac{p}{L} x + \frac{l}{n} y) \right\}$$  \hspace{1cm} (4.46)

where

$$l_m = \frac{m\pi}{L} + \frac{n\pi}{R} \tau_K \quad ; \quad p = \frac{m\pi}{L} + \frac{n\pi}{R} \tau_K \quad ; \quad n = \frac{n}{R}$$  \hspace{1cm} (4.47)

Substituting for $W^{(1)}$ and $F^{(1)}$ the governing equations for the second order state become

$$L_{A^*}(F^{(2)}) - L_{B^*}(W^{(2)}) = -\frac{1}{R} W^{(2)}_{xx} - \frac{2}{R^2} (\alpha_m + \alpha_p) \sin (\frac{m}{L} x - \frac{n}{R} y) \sin (\frac{p}{L} x + \frac{l}{n} y)$$  \hspace{1cm} (4.48)

$$L_{B^*}(F^{(2)}) + L_{D^*}(W^{(2)}) = \frac{1}{R} F^{(2)}_{xx} - \frac{Et}{cR} W^{(2)}_{yy} - \frac{\alpha_m}{R} W^{(2)}_{xy} + \frac{2}{R^2} \left( \frac{\alpha_m}{R} + \frac{\alpha_p}{R} \right) \sin (\frac{m}{L} x - \frac{n}{R} y) \sin (\frac{p}{L} x + \frac{l}{n} y)$$  \hspace{1cm} (4.49)

These equations admit separable solutions of the following form

$$W^{(2)} = t \left\{ \sum_{i=1,3,\ldots}^{\infty} \tilde{\alpha}_i \sin \frac{m_i}{L} x + \frac{1}{2} \sum_{i=1,3,\ldots}^{\infty} [\tilde{\beta}_i \sin (\frac{m_i}{L} x - 2\frac{n_i}{R} y) + \tilde{\gamma}_i \sin (\frac{p_i}{L} x + \frac{l_i}{n_i} y)] \right\}$$  \hspace{1cm} (4.50)

$$F^{(2)} = \frac{Et}{2c} \left\{ \sum_{i=1,3,\ldots}^{\infty} \tilde{D}_i \sin \frac{m_i}{L} x + \frac{1}{2} \sum_{i=1,3,\ldots}^{\infty} [\tilde{E}_i \sin (\frac{m_i}{L} x - 2\frac{n_i}{R} y) + \tilde{F}_i \sin (\frac{p_i}{L} x + \frac{l_i}{n_i} y)] \right\}$$  \hspace{1cm} (4.51)

where

$$l_i = \frac{i\pi}{L} \quad ; \quad m_i = \frac{i\pi}{L} + 2\frac{n}{R} \tau_K \quad ; \quad p_i = \frac{i\pi}{L} - 2\frac{n}{R} \tau_K \quad ; \quad 2n_i = 2\frac{n}{R}$$  \hspace{1cm} (4.52)

and the summation sign over i implies summation over odd integers only.
The coefficients are readily determined by Galerkin's procedure and are listed in Appendix D. Finally, the postbuckling coefficients "a" and "b" are calculated by evaluating the integrals indicated by Eqs. (4.8) and (4.9) yielding.

\[ a = 0 \]  (4.53)

and

\[ b = \frac{c}{\pi} \frac{\beta_n^2}{\left[ 2\beta_n^2 + R(\alpha_m^2 + \alpha_p^2) \right]} \frac{1}{\tilde{\Sigma}_1} \]  (4.54)

where

\[ \tilde{\Sigma}_1 = \Sigma_1 \text{ with the constants } A_i, B_i, \ldots, F_i \text{ replaced by } \tilde{A}_i, \tilde{B}_i, \ldots, \tilde{F}_i. \]

All the constants are listed in Appendix D.
5. GENERALIZED "LOAD-SHORTENING RELATION"

Information concerning the extent to which buckling can be expected to be gradual or sudden can be obtained from the postbuckling variation of the applied variable load \( \Delta \) with the generalized displacement \( \Delta \). Notice that \( \Delta \cdot \Delta \) represents the decrease in potential energy of the applied variable load. Thus

\[
\Delta \cdot \Delta = \int \int S_{\alpha \beta} \varepsilon_{\alpha \beta} \, dx \, dy
\]

(5.1)

where \( S_{\alpha \beta} \) is the variable applied stress resultant and \( \varepsilon_{\alpha \beta} \) is the corresponding linear strain. Notice the \( \varepsilon_{\alpha \beta} \) can be obtained from the nonlinear stress-strain relations

\[
E_{\alpha \beta} = \varepsilon_{\alpha \beta} + \frac{1}{2} W_{\alpha \beta} \varepsilon_{\alpha \beta}
\]

(5.2)

5.1 Axial compression and internal pressure \( (p_e = p_i) \)

In this case the initial slope of the load versus end-shortening curve evaluated at the bifurcation point (see Fig. 4) yields further information about the extent to which buckling can be expected to be gradual or sudden. By definition, the (average) axial end-shortening is

\[
\varepsilon = -\frac{1}{2\pi RL} \int_0^L \int_0 (u_{xx} - qW_{xx}) \, dx \, dy
\]

(5.3)

where

\[
\begin{align*}
\varepsilon_{xx} &= \varepsilon_{xx}^{1/2} W_{xx} \quad (x_{11}^{F} F_{yy} + A_{12}^{*} F_{xx} - A_{12}^{*} F_{xx} - 2 B_{11}^{*} W_{xx} - 2 B_{12}^{*} W_{xx} - 2 B_{11}^{*} W_{xx} - 2 B_{12}^{*} W_{xx} - 2 B_{12}^{*} W_{xx} - 2 B_{12}^{*} W_{xx} - 2 B_{12}^{*} W_{xx})
\end{align*}
\]

(5.4)

\( q = \) load eccentricity (positive inward)

Introducing the perturbation expansion for \( W \) and \( F \) (Eqs. 4.6 and 4.7) and regrouping according to powers of \( \xi \) yields
\[ u_x = A_{11}^* F_{yy}^{(0)} + A_{12}^* F_{xx}^{(0)} - A_{16}^* F_{yy}^{(0)} - B_{11}^* W_{yy}^{(0)} - B_{12}^* W_{xx}^{(0)} - 2B_{16}^* W_{xy}^{(0)} \cdot \frac{1}{2} W_x^{(0)} W_x^{(0)} \]  
\[ + \xi \{ A_{11}^* F_{yy}^{(1)} + A_{12}^* F_{xx}^{(1)} - A_{16}^* F_{yy}^{(1)} - B_{11}^* W_{yy}^{(1)} - B_{12}^* W_{xx}^{(1)} - 2B_{16}^* W_{xy}^{(1)} \cdot W_x^{(0)} W_x^{(1)} \} \]  
\[ + \xi^2 \{ A_{11}^* F_{yy}^{(2)} + A_{12}^* F_{xx}^{(2)} - A_{16}^* F_{yy}^{(2)} - B_{11}^* W_{yy}^{(2)} - B_{12}^* W_{xx}^{(2)} - 2B_{16}^* W_{xy}^{(2)} \cdot W_x^{(0)} W_x^{(2)} \cdot \frac{1}{2} W_x^{(1)} W_x^{(1)} \} \]  
\[ + O(\xi^3) \]

Expanding the prebuckling variables in the usual Taylor series one gets

\[ W^{(0)} = W_x + \xi a \lambda_c \dot{W}_c + \xi^2 \{ b \lambda_c \ddot{W}_c + \frac{1}{2} (a \lambda_c)^2 \dddot{W}_c \} + O(\xi^3) \]  
\[ F^{(0)} = F_c + \xi a \lambda_c \dot{F}_c + \xi^2 \{ b \lambda_c \ddot{F}_c + \frac{1}{2} (a \lambda_c)^2 \dddot{F}_c \} + O(\xi^3) \]

where (\cdot) = \frac{\partial}{\partial \lambda} and (\cdot)_c indicates that the corresponding variable is evaluated at the bifurcation point.

Notice that for a membrane prebuckling state where

\[ W^{(0)} = \frac{1}{c} (A_{12} \lambda - A_{22} \lambda) \]  
\[ F^{(0)} = \frac{Et^2}{cR} (\lambda \frac{1}{2} + \frac{1}{2} \frac{1}{I' R}) \]

one obtains

\[ \ddot{W}_c = -\frac{1}{c} \dddot{A}_{12} \]  
\[ \ddot{F}_c = -\frac{1}{2} \frac{Et^2}{cR} \gamma^2 \]

Introducing the normalized axial end-shortening \( \delta = \varepsilon / \varepsilon_c \) and assuming that the first-postbuckling coefficient \( a = 0 \), then Eq. (5.3) becomes upon substitution and regrouping
\[
\delta = \frac{\varepsilon}{\varepsilon_{cL}} = \delta_0 + \xi \left\{ \frac{1}{\varepsilon_{cL}} \frac{1}{2\pi RL} \int_0^L \int_0^L \left( A_{11}^{*} F_{yy}^{(1)} + A_{12}^{*} F_{xx}^{(1)} - A_{16}^{*} F_{xy}^{(1)} - B_{12}^{*} W_{11}^{(1)} - B_{16}^{*} W_{16}^{(1)} \right) dx dy \right\} \\
- B_{12}^{*} W_{1y}^{(1)} - 2B_{16}^{*} W_{1x}^{(1)} - W_{c,x}^{(1)} W_{c,x}^{(1)} q W_{xx}^{(1)} dx dy \}
+
\xi^2 \left\{ \frac{1}{\varepsilon_{cL}} \frac{1}{2\pi RL} \int_0^L \int_0^L \left( A_{11}^{*} F_{yy}^{(2)} + A_{12}^{*} F_{xx}^{(2)} - A_{16}^{*} F_{xy}^{(2)} - B_{11}^{*} W_{11}^{(2)} - B_{12}^{*} W_{12}^{(2)} \right) \right\} \\
- 2B_{16}^{*} W_{1x}^{(2)} W_{c,x}^{(2)} - q W_{xx}^{(2)} W_{c,x}^{(1)} - 2W_{c,x}^{(1)} W_{c,x}^{(1)} + b^\lambda_c A_{11}^{*} \xi \hat{\varepsilon}_{c,y} dx dy \}
+
O(\xi^3)
\]

and where the end-shortening along the prebuckling path \( \delta_0 \) evaluated at the bifurcation point \( \hat{\lambda} = \hat{\lambda}_{mn} \) is given by

\[
\delta_c = \frac{1}{\varepsilon_{cL}} \frac{1}{2\pi RL} \int_0^L \int_0^L \left( A_{11}^{*} F_{yy}^{(1)} + A_{12}^{*} F_{xx}^{(1)} - A_{16}^{*} F_{xy}^{(1)} - B_{12}^{*} W_{11}^{(1)} \right) dx dy
- B_{12}^{*} W_{c,y}^{(1)} - 2B_{16}^{*} W_{c,x}^{(1)} - 12 W_{c,x}^{(1)} W_{c,x}^{(1)} - q W_{c,xx}^{(1)} ) dx dy \]

Further, by definition

\[
\varepsilon_{cL} = \left| \varepsilon_x^{(0)} \right|_{\hat{\lambda} = \hat{\lambda}_{mn}} = \left| \frac{1}{cR} (A_{11}^{*} \hat{\lambda} - A_{12}^{*} \hat{\lambda}_{12}^{*} R) \right| \]

Notice also that since the buckling modes \( F^{(1)}, W^{(1)} \) and their derivatives are periodic functions in the circumferential direction, whereas the prebuckling solution and its derivatives are axisymmetric, therefore

\[
\frac{2\pi R}{L} \int_0^L \int_0^L \left( A_{11}^{*} F_{yy}^{(1)} + A_{12}^{*} F_{xx}^{(1)} - A_{16}^{*} F_{xy}^{(1)} - B_{12}^{*} W_{11}^{(1)} - B_{16}^{*} W_{16}^{(1)} - 2B_{16}^{*} W_{1x}^{(1)} - B_{12}^{*} W_{11}^{(1)} - B_{16}^{*} W_{16}^{(1)} \right) dx dy = 0
\]
Recalling now the asymptotic expansion

\begin{equation}
(\hat{\lambda} - \hat{\lambda}_c) \xi = a\hat{\lambda}_c \xi^2 + b\hat{\lambda}_c \xi^3 + ... - \alpha\hat{\lambda}_c \xi + (\hat{\lambda} - \hat{\lambda}_c) \xi + ...
\end{equation}

(4.2)

But for a perfect shell $\xi = 0$ and since here $a = 0$ then

\begin{equation}
\xi^2 = \frac{1}{b\hat{\lambda}_c} \left( \frac{\hat{\lambda}}{\hat{\lambda}_c} - 1 \right) \rho = \frac{\hat{\lambda}}{\hat{\lambda}_c} = \rho\hat{\lambda}_c
\end{equation}

(5.12)

where

\begin{equation}
\rho = \frac{\hat{\lambda}}{\hat{\lambda}_c} = \text{a new load parameter normalized so that if } \hat{\lambda} = \hat{\lambda}_c = \hat{\lambda}_m \text{ then } \rho = 1.
\end{equation}

(5.13)

Hence the end-shortening after buckling can be written as

\begin{equation}
\delta = \delta_c + \frac{1}{b} (\rho - 1) \left\{ \frac{1}{2\pi RL} \int_0^L \int_0^L \left[ (A_{11}^* F_{yy}^{(2)} + A_{12}^* F_{xx}^{(2)} - A_{16}^* F_{xy}^{(2)}) \right] \, dx \, dy \right\} + O(\xi^3)
\end{equation}

(5.14)

Substituting for $W^{(0)}$ and $F^{(0)}$ from Eq. (4.19) and (4.20) into Eq. (5.9), one can calculate the end-shortening along the prebuckling path $\delta_0$

\begin{equation}
\delta_0 = \frac{1}{\epsilon c R} \left( \frac{1}{A_{11}^* \hat{\rho} - A_{12}^* \hat{p}_i} \right) = \frac{1}{\epsilon c R} \left( \frac{1}{A_{11}^* \hat{\rho} - A_{12}^* \hat{p}_i} \right) = \frac{1}{\epsilon c R} \left( \frac{1}{A_{11}^* \hat{\rho} - A_{12}^* \hat{p}_i} \right)
\end{equation}

(5.15)

Similarly, substituting for $W^{(1)}$ and $W^{(2)}$, $F^{(2)}$ from Eqs. (4.26), (4.31) and (4.32) into Eq. (5.14) and evaluating the integrals, one obtains the end-shortening after buckling $\delta$ as

\begin{equation}
\delta = \delta_c + \frac{1}{b\hat{\lambda}_c} (\rho - 1) \left\{ \frac{1}{A_{11}^* A_{12}^*} \frac{1}{\hat{\lambda}_c} \right\} \left\{ \hat{A}_{11}^* b\hat{\lambda}_c \hat{\lambda}_m \hat{\lambda}_n + \hat{\Sigma}_2 \right\}
\end{equation}

(5.16)

where
\[ \Sigma_2 = \pi l \frac{R}{L L} \left\{ \bar{A}^*_{12} \sum_{i=1,3,\ldots} (i A_i + (\frac{1}{2} R \frac{a}{l}) \sum_{i=1,3,\ldots} (i A_i)) + \frac{c^2}{8} (\alpha_m^2 + \alpha_p^2) + O(\xi^3) \right\} \]  \hfill (5.17)

Notice that the summation sign over \( i \) implies summation over odd integers only.

From Eqs. (5.15) and (5.16)

\[ \frac{\partial}{\partial \rho} \delta_{\rho} = \frac{\bar{A}_{11}^*}{\bar{A}^*_{11} - \bar{A}^*_{12} \rho_i / \lambda_{mn}} \]  \hfill (5.18)

\[ \frac{\partial}{\partial \rho} \delta = \frac{1}{b \lambda_{mn}} \frac{1}{\bar{A}^*_{11} \bar{A}^*_{12} \rho_i / \lambda_{mn}} \left\{ \bar{A}_{11}^* b \lambda_{mn} \rho_i / \lambda_{mn} + \Sigma_2 \right\} \]  \hfill (5.19)

Hence the slope of the load vs end-shortening curve along the prebuckling path becomes

\[ K_c = \frac{\partial \rho}{\partial \delta} = 1 - \frac{\bar{A}_{12}^* \rho_i}{\bar{A}_{11}^* \lambda_{mn}} \]  \hfill (5.20)

Whereas the slope of the load vs end-shortening curve just after buckling is

\[ K^* = \frac{\partial \rho}{\partial \delta} = \frac{\bar{A}_{11}^* - \bar{A}_{12}^* \rho_i / \lambda_{mn}}{\bar{A}_{11}^* + (1/b \lambda_{mn}) \Sigma_2} \]  \hfill (5.21)

and both quantities are evaluated at the bifurcation point \( \lambda_c = \lambda_{mn} \).

As can be seen from Fig. 4, the angle of the initial slope just after buckling \( \beta^* \), where

\[ \beta^* = \tan^{-1}(K_c) \]  \hfill (5.22)

is indeed helpful in determining whether the buckling will be gradual or catastrophic. Notice that by normalizing the generalized displacement \( \Delta \) by the appropriate membrane strain \( \varepsilon_{cl} \), for membrane prebuckling (if \( \rho_i = 0 \)) the angle of the fundamental path \( \beta = \tan^{-1}(K_c) = 45^\circ \).
5.2 External pressure and axial compression

If external pressure is the variable load (that is \( \Lambda = \bar{p}_e \)) then the appropriate
generalized displacement \( \Delta \) is the average normal displacement defined as

\[
W_{\text{ave}} = \frac{1}{2\pi RL} \int_0^L \int_0^W dx dy
\]

(5.23)

where now

\[
W = W^{(0)} + \xi W^{(1)} + \xi W^{(2)} + \ldots
\]

(5.24a)

\[
W^{(0)} = W_c + (\bar{p}_e - \bar{p}_c) \dot{W}_c + \frac{1}{2} (\bar{p}_e - \bar{p}_c)^2 \ddot{W}_c + \ldots
\]

(5.24b)

\[
\bar{p}_e - \bar{p}_c = \alpha \bar{P}_c \xi + \beta \bar{p}_c \xi^2 + \ldots
\]

(5.24c)

and \( \dot{} = \partial / \partial \bar{p}_e \). The subscript c indicates that the corresponding variable is evaluated at the
bifurcation point.

Notice that for a membrane prebuckling state where

\[
W^{(0)} = \frac{1}{c} (\bar{A}_{12} \dot{R} + \bar{A}_{22} \dot{R}) \bar{P}_e
\]

(2.25)

\[
F^{(0)} = \frac{E}{cR} \left( \frac{1}{2} \bar{P}_e \xi^2 + \frac{1}{2} \bar{P}_e \xi^2 \right)
\]

(2.26)

and \( \bar{R} = \lambda / \bar{P}_e \) one obtains

\[
\dot{W}_c = \frac{1}{c} (\bar{A}_{12} \dot{R} + \bar{A}_{22} \dot{R}) \quad ; \quad \ddot{W}_c = 0
\]

(5.27a)

\[
\dot{F}_c = \frac{E}{cR} \left( \frac{1}{2} \bar{P}_e \xi^2 + \frac{1}{2} \bar{P}_e \xi^2 \right) \quad ; \quad \ddot{F}_c = 0
\]

(5.27b)

Introducing the nondimensional average normal displacement \( \bar{W}_{\text{ave}} = W_{\text{ave}} / W_{\text{Cl}} \), assuming that the
first-postbuckling coefficient \( a = 0 \), then Eq. (5.23) becomes upon substitution of the
appropriate perturbation expansions (Eqs. (5.24)) and regrouping according to powers of \( \xi \)
\[ \bar{W}_{\text{ave}} = \frac{W_{\text{ave}}}{W_{c1}} = \bar{W}_c + \xi \left\{ \frac{1}{W_{c1}} \int_{0}^{L} \int_{0}^{L} W^{(1)} \, dx \, dy \right\} + \xi^2 \left\{ \frac{1}{W_{c1}} \int_{0}^{L} \int_{0}^{L} (W^{(2)} + b \bar{p}_c W_c) \, dx \, dy \right\} + O(\xi^3) \]  

(5.28)

Notice that the nondimensional average normal displacement along the prebuckling path \( \bar{W}_o \) evaluated at the bifurcation point \( \bar{p}_e = \bar{p}_c = \bar{p}_{mn_1} \) is given by

\[ \bar{W}_c = \frac{1}{W_{c1}} \frac{2\pi R}{2\pi RL} \int_{0}^{L} \int_{0}^{L} W_c \, dx \, dy \]  

(5.29)

Further, by definition

\[ W_{c1} = t(W_v + W_{pe}) \bigg| \frac{\bar{p}_e = \bar{p}_c = \bar{p}_{mn_1}}{C_{12R} + C_{22}} \]  

(5.30)

Notice also that since the buckling mode \( W^{(1)} \) is a periodic function in the circumferential direction

\[ \int_{0}^{L} \int_{0}^{L} W^{(1)} \, dx \, dy = 0 \]  

(5.31)

Recalling now the asymptotic expansion Eq. (5.24c) one sees that for \( a = 0 \)

\[ \xi^2 = \frac{1}{b \bar{p}_c} (\bar{p}_e - \bar{p}_c) = \frac{1}{b} (\bar{p} - 1) \]  

(5.32)

Where now

\[ \bar{p} = \frac{\bar{p}_e}{\bar{p}_c} \text{ a new load parameter normalized so that if } \bar{p}_e = \bar{p}_c = \bar{p}_{mn_1} \text{ then } \bar{p} = 1 \]  

(5.33)

Hence the nondimensional average normal displacement after buckling can be written as

\[ \bar{W}_{\text{ave}} = \frac{\bar{W}_c}{W_{c1}} + \frac{1}{b} (\bar{p} - 1) \left\{ \frac{1}{W_{c1}} \int_{0}^{L} \int_{0}^{L} (W^{(2)} + b \bar{p}_c W_c) \, dx \, dy \right\} \]  

(5.34)
Substituting for \( W^{(0)} \) from Eq. (4.38) into Eq. (5.29) one can calculate the nondimensional average normal displacement along the prebuckling path \( \bar{W}_o \) as

\[
\bar{W}_o = \frac{\bar{p}}{W_{c1}} \frac{1}{c} (\bar{A}_{12}^* \bar{R} + \bar{A}_{22}^*) \bar{p} = \bar{p}
\]  

(5.35)

Similarly, substituting for \( W^{(2)} \) from Eq. (4.50) into Eq. (5.34) and evaluating the integrals, one obtains the nondimensional average normal displacement after buckling \( \bar{W}_{ave} \) as

\[
\bar{W}_{ave} = \bar{W}_c + \frac{1}{bp_{mn\tau}} (\bar{R} - 1) \frac{1}{\bar{A}_{12}^* \bar{R} + \bar{A}_{22}^*} \{ \bar{A}_{12}^* \bar{R} + \bar{A}_{22}^* \} bp_{mn\tau} + \Sigma_3 \}
\]  

(5.36)

where

\[
\Sigma_3 = \frac{2c}{\pi} \sum_{i=1,3,\ldots} (\frac{1}{\bar{A}_i})
\]  

(5.37)

Notice that the summation sign over \( i \) implies summation over odd integers only.

From Eqs. (5.35) and (5.36)

\[
\frac{\partial}{\partial \bar{p}} \bar{W}_o = 1
\]  

(5.38)

\[
\frac{\partial}{\partial \bar{p}} \bar{W}_{ave} = \frac{1}{bp_{mn\tau}} \frac{1}{\bar{A}_{12}^* \bar{R} + \bar{A}_{22}^*} \{ \bar{A}_{12}^* \bar{R} + \bar{A}_{22}^* \} bp_{mn\tau} + \Sigma_3 \}
\]  

(5.39)

Hence the slope of the external pressure vs average normal displacement curve along the prebuckling path becomes

\[
K_c = \frac{\partial \bar{p}}{\partial \bar{W}_o} = 1
\]  

(5.40)

Whereas the slope of the external pressure vs average normal displacement curve just after buckling is
\[ K^*_c = \frac{\partial \hat{p}}{\partial \hat{W}_{\text{ave}}} = \frac{1}{1 + \frac{1}{b \bar{p}_{mn} (\bar{A}_{12}^* \hat{R} + \bar{A}_{22}^*) \Sigma_3}} \]  \hspace{1cm} (5.41)

Notice that due to the appropriate choice of the generalized displacement \( \Delta \) (see Eq. (5.23)) also in this case, where external pressure \( \bar{p}_c \) is the variable load, the angle of the fundamental path \( \hat{\theta} = \tan^{-1}(K^*_c) = 45^0 \) in the generalized load vs displacement plane.
6. NUMERICAL RESULTS FOR ORTHOTROPIC SHELLS

Thanks to extensive NASA sponsored research programs carried out in the sixties and in the early seventies, much is known about the buckling behavior of thin-walled shell structures. The degrading effect of initial geometric imperfections has been extensively investigated using Koiter's general elastic postbuckling theory\textsuperscript{[1]}. Though it was found that besides boundary conditions\textsuperscript{[5]} and nonlinear modal interaction\textsuperscript{[6]} also the use of a rigorous prebuckling analysis\textsuperscript{[7]} can affect the predicted imperfection sensitivity of the buckling loads, it is customary, because of the overall view it provides to rely on membrane prebuckling analysis for the initial computations. Thus in the Level-1 computational modules of DISDECO, described in this report, all computations are carried out assuming a membrane prebuckling state. The accuracy of the final predictions can then be verified by Level-2 computational modules such as ANILISA\textsuperscript{[8]}, which includes both a rigorous prebuckling analysis and a rigorous satisfaction of the prescribed boundary conditions.

To test the accuracy and reliability of the Level-1 computational modules of DISDECO, initially an attempt will be made to reproduce the results published by Hutchinson and Amazigo\textsuperscript{[9]} in 1967. In all cases first the figure out of the original publication will be presented followed by hard copies of the graphic output of DISDECO run on a SUN IPC workstation.

6.1 Classical buckling and imperfection sensitivity of stringer stiffened cylinders under axial compression

The results of the original paper (Ref. [9], Fig. 3) are here reproduced as Fig. 5 (for light stiffening), Fig. 8 (for medium stiffening) and as Fig. 11 (for heavy stiffening). Notice that for light stiffening from the nondimensional parameters given in Fig. 5 and from the geometric properties listed in Table 1 one can easily calculate the quantities $A_s$, $I_s$ and $e_s$. Using these values with Eq. (2.14) yields the curves for the classical normalized buckling load $\lambda_c$ as a function of Batdorf's Z-parameter displayed in Fig. 6.

Once for a given Z-value the lowest buckling load $\lambda_c$ and the corresponding wave numbers $m$ and $n$ have been determined, one can use Eq. (4.35) to evaluate the b-factors displayed in Fig. 7. The reader is reminded that according to Koiter's theory a negative b-factor is a measure of the expected imperfection sensitivity. For a given negative b-factor one can use Eq. (4.5) to calculate the decrease in buckling load as a function of the amplitude $\xi$ of an asymmetric imperfection which has the shape of the classical buckling mode

$$\bar{W} = \xi \hat{W} = \xi t \sin \frac{m}{L} \cos \frac{n}{R}$$  \hspace{1cm} (6.1)
For medium stiffening, the results obtained by DISDECO are displayed in Figures 9 and 10, whereas for heavy stiffening the DISDECO results are shown in Figures 12 and 13.

In general the agreement between the present results and those of Reference\textsuperscript{[9]} are very good, whereby it must be remembered that Hutchinson and Amazigo minimized their eigenvalues by considering $\tilde{n} = nL/\pi R$ as a continuous variable, whereas in DISDECO minimization is carried out using the wave numbers $m$ and $n$ as integers.

6.2 Classical buckling and imperfection sensitivity of stringer stiffened cylinders under hydrostatic pressure

The results of the original paper (Ref. [9], Fig. 5) are here reproduced as Fig. 14. Notice that from the nondimensional parameters given in Fig. 14 and from the geometric properties listed in Table 1 once again one can easily calculate the quantities $A_s$, $I_s$, and $e_s$. Using these values with Eq. (2.30) yields the curves for the classical normalized buckling pressure $\bar{P}_c$ as a function of Batdorf's Z-parameter displayed in Fig. 15 for light stiffening, in Fig. 17 for medium stiffening and in Fig. 19 for heavy stiffening, respectively. For hydrostatic pressure $\bar{R} = \lambda/\bar{P}_e = 0.5$ is used.

Once for a given Z-value the lowest buckling pressure $\bar{P}_c$ and the corresponding wave numbers $m$ and $n$ have been found, one can use Eq. (4.54) to evaluate the b-factors displayed in Fig. 16 for light stiffening, in Fig. 18 for medium stiffening and in Fig. 20 for heavy stiffening, respectively.

Notice that once again the results obtained with DISDECO agree very well with those published in Ref. [9].

6.3 Classical buckling and imperfection sensitivity of ring stiffened cylinders under hydrostatic pressure

The results of the original paper (Ref. [9], Fig. 8) are here reproduced as Fig. 21. Notice that from the nondimensional parameters given in Fig. 21 and from the geometric properties listed in Table 2 one can easily calculate the quantities $A_r$, $I_r$, and $e_r$. Using these values with Eq. (2.30) yields the curves for the classical normalized buckling pressure $\bar{P}_c$ as a function of Batdorf's Z-parameter shown in Fig. 22.

Once for a given Z-value the lowest buckling pressure $\bar{P}_c$ and the corresponding wave numbers $m$ and $n$ have been found, one can use Eq. (4.54) to evaluate the b-factors displayed in Fig. 23.

The lightly stiffened cylinder just discussed buckles into a mode with one half-wave in the axial direction and several full waves in the circumferential direction. As pointed out in Ref. [9], if the amount of
stiffening is increased the number of axial half-waves will not necessarily be one and, in fact may be very large depending on the stiffening and the value of $Z$.

Results for three other degrees of stiffening from the original paper (Ref. [9], Fig. 9) are here reproduced as Fig. 24. Notice that from the nondimensional parameters given in Fig. 24 and from the geometric parameters listed in Table 2 one can easily calculate the quantities $A_r^+, l_r$ and $e_r^+$. Using these values with Eq. (2.30) yields the curves for the classical normalized buckling pressure $\bar{p}_c^*$ as a function of Batdorf's $Z$-parameter displayed in Fig. 25 for light stiffening, in Fig. 26 for medium stiffening and in Fig. 27 for heavy stiffening, respectively.

Notice that the agreement between the results obtained with DISDECO and those published in Ref. [9] is very good.

6.4 Effect of axisymmetric imperfections

As pointed out by Koiter[1] and by others[2,9] whenever the lowest buckling load is axisymmetric, the postbuckling coefficient $b$ is identically equal to zero. In this case, once the classical buckling load $\Lambda_c$ is attained the perfect shell deflects in its axisymmetric buckling mode under constant load $\Lambda_c$ until an asymmetric bifurcation from the axisymmetric state of deformation occurs. Following bifurcation the load falls below $\Lambda_c$ with increasing axisymmetric and nonaxisymmetric deflections.

Axially compressed cylinders with outside axial stiffeners

Hutchinson and Amazigo[9] have carried out calculations to determine the effect of an imperfection in the shape of the axisymmetric buckling mode

$$\bar{W} = \bar{\delta} \sin \pi \frac{X}{L}$$

(6.2)

on the bifurcation load $\lambda_s$. The results of this original paper (Ref. [9], Fig. 4) are here reproduced as Fig. 28. Notice that in Ref. [9] $W$ is positive outward, thus Eq. (6.2) represents an inward imperfection. From the nondimensional parameters given in Fig. 28 and the geometric properties listed in Table 1 one can calculate the quantities $A_s^-, l_s^- e_s^-$ needed in Eq. (3.20) to calculate the bifurcation load $\lambda_s$ of the imperfect shell.
Results of calculations using Eq. (3.20) as the characteristic equation did not agree at all with those obtained in Ref. [9]. This is due to the fact that in the derivation of Eq. (3.20) the following half-wave cosine axisymmetric imperfection

$$W = t_2 \bar{e}_1 \cos i \pi \frac{x}{L} \quad (3.6)$$

is used. One can argue that when $i$ is large enough (for long shells thus) it must not matter whether one uses half-wave sine of half-wave cosine representation. For $i=1$, however, the two representations are not equivalent.

In order to investigate this disagreement further in Appendix E the bifurcation buckling problem of a cylindrical shell with axisymmetric imperfection was solved again using

$$W = t_2 \bar{e}_1 \sin i \pi \frac{x}{L} \quad (3.7)$$

also a half-wave sine axisymmetric imperfection. The results of the calculations using Eq. (E.15) as the characteristic equation are shown in Fig. 29 for $Z = 60$, $Z = 80$ and $Z = 100$. In all cases the asymmetric buckling mode has many waves in the circumferential direction (between 40 and 65). Although at the first sight these results appear to agree quite well with those of Ref. [9] shown in Fig. 28, a closer examination reveals that in some cases the present results are up to about 35% lower.

The reason for this discrepancy is unknown at the present time.

Axially compressed cylinders with ring stiffeners

The results of the original paper (Ref. [9], Fig. 7) are here reproduced as Fig. 30. Using the non-dimensional parameters given in Fig. 30 and the geometric properties listed in Table 2 one can calculate the quantities $A_{\nu}, I_{\nu}$ and $e_{\nu}$ needed for further analysis. For a length of $L = 10.16$ cm (= 4.0 in) both for the light and the medium rings placed on the outside the lowest buckling load is axisymmetric with $i=13$, respectively $i=14$ half-waves in the axial direction. Using the following half-wave cosine axisymmetric imperfection

$$W = t_2 \bar{e}_1 \cos 14 \pi \frac{x}{L} \quad (6.3)$$
and Eq. (3.20) as the characteristic equation the results shown in Figs. 31 and 32, for light respectively medium stiffeners are obtained. When computing these results special care must be taken to satisfy the strong coupling condition $i=2m$ used in Eq. (3.20).

Comparison of the results indicates an excellent agreement between the present analysis and that of Ref. [9].

**Hydrostatic buckling of ring stiffened cylinders in the Z-independent range**

The results of the original paper (Ref. [9], Fig. 10) are here reproduced as Fig. 33. Using the non-dimensional parameters given in Fig. 33 and the geometric properties listed in Table 2 one can calculate the quantities $A_r$, $L$, and $e_r$ needed for further analysis. To remain in the Z-independent range for the lightly stiffened shells a length of $L = 3.556$ cm (≈ 1.4 in) and for the shells with medium rings a length of $L = 12.7$ cm (≈ 5.0 in) is used. Then both for the light and the medium rings placed on the outside the lowest buckling load is axisymmetric with $i=5$, respectively $i=18$ half-waves in the axial direction. In order to be able to satisfy the strong coupling condition $i=2m$ in the characteristic equation (3.33), for shells with light rings the following half-wave cosine axisymmetric imperfection

$$\bar{W} = t \xi_1 \cos 4\pi \frac{x}{L}$$  \hspace{1cm} (6.4)

is used. The results obtained for outside and inside stiffeners is shown in Fig. 34.

For shells with medium rings the following half-wave cosine axisymmetric imperfection

$$\bar{W} = t \xi_1 \cos 18\pi \frac{x}{L}$$

must be used. If the strong coupling condition $i=2m$ is satisfied then the characteristic equation (3.33) yields the results displayed in Fig. 35. Notice that in all cases the eigenvalues are normalized by the lowest eigenvalue of the corresponding perfect shell. The numerical values used for normalization are printed on each of the figures.

Once again a comparison of the results obtained by DISDECO and those of Ref. [9] reveal a very good agreement.
7. NUMERICAL RESULTS FOR LAYERED COMPOSITE SHELLS

To demonstrate the accuracy and reliability of the Level-1 computational modules of DISDECO when calculating the classical buckling load and the imperfection sensitivity of layered composite shells, initially an attempt will be made to reproduce the results published by Khot and Venkayya in 1970. They used shells of either glass-epoxy or boron-epoxy composites. The geometric properties of the shells are given in Tables 3 and 4, respectively. To illustrate the effect of fiber orientation on the buckling load and on the initial postbuckling behavior two different layups will be used. The fiber orientations in outer, middle and inner layers are for

\[ \text{case 1: } -\theta, \theta, 0^\circ \quad \text{and} \quad \text{case 2: } -\theta, \theta, 90^\circ. \]

The value of $\theta$ is varied from $0^\circ$ to $90^\circ$ at ten degree increments. The convention used in defining the angle $\theta$ is shown in Fig. 1. Notice that when $\theta$ is $0^\circ$ all fibers are oriented axially and when $\theta$ is $90^\circ$ all fibers are oriented circumferentially.

The results of the buckling load and imperfection sensitivity calculations are listed in Tables 5 through 8 and Tables 9 through 12, respectively. To obtain the buckling loads Eq. (2.14) is used, whereby one must remember that the eigenvalue $\lambda_{mn}$ must not only be minimized with respect to $m$ and $n$ (the number of axial half-waves and the number of circumferential full waves, respectively), but also with respect to the real number $\tau_K$. As can be seen from Eq. (2.11) $\tau_K$ is a measure of the skewedness of the buckling pattern. It was first introduced by Khot and Venkayya in Ref. [4].

Once for a given configuration the minimum buckling load $\lambda_c = \lambda_{\text{min}}$ with the corresponding wave numbers $m$ and $n$ and skewedness parameter $\tau_K$ is found, one can proceed and investigate the expected postbuckling behavior. This is done by evaluating the $b$-factor using Eq. (4.35) and by calculating the angle of the initial slope just after buckling $\tilde{\theta}^*$ from Eqs. (5.21) and (5.22).

When one compares the results obtained with DISDECO (present work) with the results published in Ref. [4], it appears that the agreement is better when in the present analysis one is using a $\tau_K$ which is close to zero. This apparent discrepancy will be discussed in more detail later.

The results of the original paper (Ref. [4], Fig. 3) showing the effect of fiber orientation on the $b$-factor is here reproduced as Fig. 36. Notice that the curves for glass-epoxy and boron-epoxy are shown on the same figure to illustrate the effect of elastic properties of the material. As mentioned earlier a negative $b$-factor can be used as an indicator for the degree of imperfection sensitivity of a structure. From Fig. 36 one can see that the magnitude of the $b$-factor is greater for glass-epoxy than for boron-epoxy shells. This indicates that the glass-epoxy shells are more imperfection sensitive than the boron-epoxy shells.
Figures 37 through 44 show the results obtained by DISDECO for the same cases, whereby on each figure the effect of fiber orientation on both the buckling load $\lambda_c$ and the postbuckling coefficient $b$ is displayed. Notice that for each case there are two figures presented. In the first figure $\tau_K$ is varied such that for each configuration the minimum buckling load $\lambda_c$ is found. For this case an increase of the classical buckling load with fiber orientation is not necessarily accompanied by an increase in imperfection sensitivity. In the second figure $\tau_K$ is set equal to zero. Notice that for this case an increase of the classical buckling load with fiber orientation is well accompanied by an increase in imperfection sensitivity.

To elucidate this question further, in Fig. 45 the variation of the buckling load $\lambda_{\text{mnt}}$ with Khot’s skewedness parameter $\tau_K$ for a specific configuration (glass-epoxy, case 1: -40°, 40°, 0°) is plotted. As one can see there are 3 distinct local minima at $\tau_K = -0.76$ (with $\lambda_c = 0.540706$ and $n=12$), at $\tau_K = -0.12$ (with $\lambda_{6,12,\tau} = 0.561107$) and at $\tau_K = +0.76$ (with $\lambda_{1,11,\tau} = 0.593021$). A look at the contour plots of the corresponding buckling modes shown in Figures 46, 49 and 52, respectively, confirms the existence of these nearly simultaneous, but distinct eigenvalues. By the way, it appears that in most cases Khot and Venkayya found only the local minimum closest to $\tau_K = 0$ in Reference [9], which explains why there is a better agreement with the present results in Tables 5 through 12 if a $\tau_K$ close to zero is used.

If one wants to compare the results obtained with the Level-1 computational modules presented in this report with similar results obtained with the more accurate Level-2 computational module ANILISA[8], it is helpful to have information available about the shape of the buckling and postbuckling modes belonging to the different buckling loads calculated. Figures 47 through 48, 50 through 51 and 53 through 54 display this information for the 3 nearly simultaneous buckling loads discussed earlier.

The remaining figures 55 through 84 display similar information for representative configurations of the other cases investigated. Thus Fig. 55 shows the variation of the buckling load $\lambda_{\text{mnt}}$ with Khot’s skewedness parameter $\tau_K$ for glass-epoxy, case 2: -40°, 40°, 90°, Fig. 65 the same for boron-epoxy, case 1: -40°, 40°, 0° and Fig. 75 the same for boron-epoxy, case 2: -40°, 40°, 90°. Finally, it must be mentioned that all these figures are hard copies of the graphic output of DISDECO run on a SUN IPC workstation.
8. CONCLUSIONS

It is by-now widely accepted that Koiter's General Theory of Elastic Stability\textsuperscript{[1]} has greatly contributed to our understanding of the sometimes perplexing stability behavior of thin-walled structures. However, due to its mathematical complexity it is not always easy for the practicing structural engineer to find the information he wants for the particular structure at hand from the many publications that are available. What he needs is a computational module that enables him to obtain the desired information readily.

It has been shown that within the context of Koiter's initial postbuckling theory the Level-1 computational modules described in this report can be used successfully to investigate the imperfection sensitivity of the buckling loads of isotropic, orthotropic and of fully anisotropic cylindrical shells under combined axial compression and external or internal pressure taking into account the effect of different initial imperfection shapes.

As a "building block" of the hierarchical design and analysis system "DISDECO" these computational modules make the first step towards acquiring of detailed understanding of the expected instability behavior of different cylindrical shell configurations feasible. This knowledge is a prerequisite for the development of discrete nonlinear computational models which can reliably predict the load carrying capability of the structure.

It must be stressed that the predictions of the Level-1 computational modules provide only a first indication of the expected nonlinear behavior and all its findings must be evaluated within the context of the fundamental assumptions involved in the theory.

Thus, as mentioned earlier, it is known that also boundary conditions\textsuperscript{[5]} and nonlinear modal interactions\textsuperscript{[6]} can have a profound effect on the imperfection sensitivity prediction. Further, it has been shown by Tennyson et al.\textsuperscript{[7]} that with short anisotropic shells for reliable prediction of the postbuckling behavior one must use a rigorous prebuckling analysis.

To take also these effects into account, the user must switch to other more advanced computational modules available within "DISDECO" such as ANILISA\textsuperscript{[8]}, which can handle the above mentioned effects.

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REFERENCES


APPENDIX A: The laminate constitutive equations

In deriving the constitutive equations for the shell wall made out of filamentary laminae, both unidirectional or woven, it is assumed that the laminae are homogeneous orthotropic layers in a plane stress state. The constitutive equations of the individual lamina in turn are used to derive the stress-strain relations for the laminate.

Normally, as can be seen from Fig. 1, the lamina principal axis (1,2) do not coincide with the reference axes of the shell wall. When this occurs, the constitutive relations for each individual lamina must be transformed to the shell wall reference axes in order to determine the shell wall constitutive relations.

Following Ashton et al.\textsuperscript{[10]}, one obtains the stress-strain relationship for the \(k^{th}\)-lamina in terms of the midsurface strain- and curvature tensors of the shell wall as

\[
[\sigma]_k = [\bar{Q}]_k [\varepsilon] + z[\bar{Q}]_k [\kappa]
\]  
(A.1)

where

\[
[\bar{Q}]_k = \begin{bmatrix}
\bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\
\bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\
\bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66}
\end{bmatrix}
\]  
(A.2)

and

\[
[\sigma]_k = \begin{bmatrix}
\sigma_x & \sigma_y & \tau_{xy}
\end{bmatrix}_k^T
\]

\[
[\varepsilon] = \begin{bmatrix}
\varepsilon_x & \varepsilon_y & \gamma_{xy}
\end{bmatrix}_k^T
\]

\[
[\kappa] = \begin{bmatrix}
\kappa_x & \kappa_y & \kappa_{xy}
\end{bmatrix}_k^T
\]

with

\[
\bar{Q}_{11} = Q_{11} c^4 + 2(Q_{12} + 2Q_{66}) c^2 s^2 + Q_{22} s^4
\]

\[
\bar{Q}_{12} = (Q_{11} + Q_{22} - 4Q_{66}) c^2 s^2 + Q_{12} (c^4 + s^4)
\]

\[
\bar{Q}_{22} = Q_{11} s^4 + 2(Q_{12} + 2Q_{66}) c^2 s^2 + Q_{22} c^4
\]

\[
\bar{Q}_{66} = (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66}) c^2 s^2 + Q_{66} (c^4 + s^4)
\]

(A.4)
\[ \overline{Q}_{16} = (Q_{11} - Q_{12} - 2Q_{66})c^3 s + (Q_{12} - Q_{22} + 2Q_{66})c^3 s \]

\[ \overline{Q}_{26} = (Q_{11} - Q_{12} - 2Q_{66})c^3 s + (Q_{12} - Q_{22} + 2Q_{66})c^3 s \]

and

\[ Q_{11} = \frac{E_{11}}{1 - \nu_{12} \nu_{21}} \quad ; \quad Q_{22} = \frac{E_{22}}{1 - \nu_{12} \nu_{21}} \quad ; \quad c = \cos \theta_K \quad ; \quad s = \sin \theta_K \quad (A.5) \]

\[ Q_{12} = \frac{\nu_{21} E_{11}}{1 - \nu_{12} \nu_{21}} = \frac{\nu_{12} E_{22}}{1 - \nu_{12} \nu_{21}} \quad ; \quad Q_{66} = G_{12} \]

Notice that the factor (1/2) which appears in front of the shearing strain \( \gamma_{xy} \) in the strain tensor has been incorporated into the \( \overline{Q} \) matrix.

The laminate constitutive equations are obtained by integrating the constitutive equation for a laminate, Eq. (A.1), over the thickness \( t \) of the laminate (see Fig. 1) yielding

\[ [N] = [A] [\varepsilon] + [B] [\kappa] \quad (A.6) \]

\[ [M] = [B] [\varepsilon] + [D] [\kappa] \]

where

\[ A_{ij} = \sum_{k=1}^{N} (\overline{Q}_{ij})_k (h_k - h_{k-1}) \quad ; \quad \overline{A}_{ij} = \frac{1}{E_t} A_{ij} \]

\[ B_{ij} = \frac{1}{2} \sum_{k=1}^{N} (\overline{Q}_{ij})_k (h_k^2 - h_{k-1}^2) \quad ; \quad \overline{B}_{ij} = \frac{2c}{E_t} B_{ij} \quad (A.7) \]

\[ D_{ij} = \frac{1}{3} \sum_{k=1}^{N} (\overline{Q}_{ij})_k (h_k^3 - h_{k-1}^3) \quad ; \quad \overline{D}_{ij} = \frac{4c^2}{E_t^3} D_{ij} \quad \text{for } i, j = 1, 2, 6 \]

and

\[ E = \text{reference modulus} \]

\[ \nu = \text{reference Poisson's ratio} \]

\[ t = \text{laminate thickness} = \sum_{k=1}^{N} (h_k - h_{k-1}) \]

\[ c = \sqrt{3(1-\nu^2)} \]
Inverting the first constitutive equation one gets

\[
\begin{bmatrix} \varepsilon \end{bmatrix} = \begin{bmatrix} A^* \end{bmatrix} \begin{bmatrix} N \end{bmatrix} + \begin{bmatrix} B^* \end{bmatrix} \begin{bmatrix} \kappa \end{bmatrix}
\]

(A.8)

where

\[
\begin{bmatrix} A^* \end{bmatrix} = \begin{bmatrix} A^{-1} \end{bmatrix}; \quad \begin{bmatrix} B^* \end{bmatrix} = -\begin{bmatrix} A^{-1} \end{bmatrix} \begin{bmatrix} B \end{bmatrix}
\]

(A.9)

Substituting the above equation into the second constitutive equation (A.6) yields

\[
\begin{bmatrix} M \end{bmatrix} = \begin{bmatrix} C^* \end{bmatrix} \begin{bmatrix} N \end{bmatrix} + \begin{bmatrix} D^* \end{bmatrix} \begin{bmatrix} \kappa \end{bmatrix}
\]

(A.10)

where

\[
\begin{bmatrix} C^* \end{bmatrix} = \begin{bmatrix} B \end{bmatrix} \begin{bmatrix} A^{-1} \end{bmatrix} = - \begin{bmatrix} B^* \end{bmatrix}^T; \quad \begin{bmatrix} D^* \end{bmatrix} = \begin{bmatrix} D \end{bmatrix} - \begin{bmatrix} B \end{bmatrix} \begin{bmatrix} A^{-1} \end{bmatrix} \begin{bmatrix} B \end{bmatrix}
\]

(A.11)

Finally

\[
\begin{array}{l}
\bar{A}^*_{ij} = Et \bar{A}^*_{ij}; \quad \bar{B}^*_{ij} = (2c/\ell) \bar{B}^*_{ij}; \quad \bar{D}^*_{ij} = (4c^2/Et) \bar{D}^*_{ij}
\end{array}
\]

(A.12)
APPENDIX B: Definition of the coefficients used in Eqs. (2.14), (2.30), (2.38) and (2.41)

Recalling that

\[
\begin{align*}
\alpha^2_m &= (m \pi \frac{n}{L} \tau^2 K) \frac{2 \text{Rt}}{2c} \\
\beta^2_n &= n \frac{2 \frac{\text{Rt}}{2c} \left(\frac{1}{\text{R}}\right)^2}
\end{align*}
\]  \quad (B.1)

\[
\begin{align*}
\alpha^2_p &= (m \pi \frac{n}{R} \tau^2 K) \frac{2 \text{Rt}}{2c} \\
c &= \sqrt{3(1-v^2)}
\end{align*}
\]

where

\[
\begin{align*}
\text{R} &= \text{radius of the cylinder's reference surface} \\
\text{L} &= \text{length of the shell} \\
\text{t} &= \text{thickness of the laminate} \\
\text{E} &= \text{reference modulus} \\
\nu &= \text{reference Poisson's ratio} \\
\tau^2 K &= \text{Khot's}[^4] \text{coupling parameter}
\end{align*}
\]

then one has

\[
\begin{align*}
\overline{\gamma}^e_{A^*,m,n} &= \overline{\alpha}^4 \frac{\alpha^2 m}{22} + (2 \overline{\alpha}^*_{_{12}} + \overline{\alpha}^*_{_{66}}) \alpha^2 \frac{\beta^2 n}{m} + \overline{\alpha}^*_{_{11}} \frac{\beta^4 n}{m} \\
\overline{\gamma}^0_{A^*,m,n} &= 2 \overline{\alpha}^*_{26} \alpha^2 \frac{\beta^3 n}{m} + 2 \overline{\alpha}^*_{16} \alpha^2 \frac{\beta^3 n}{m} \\
\overline{\gamma}^e_{A^*,p,n} &= \overline{\alpha}^4 \frac{\alpha^2 p}{22} + (2 \overline{\alpha}^*_{_{12}} + \overline{\alpha}^*_{_{66}}) \alpha^2 \frac{\beta^2 p}{n} + \overline{\alpha}^*_{_{11}} \frac{\beta^4 p}{n} \\
\overline{\gamma}^0_{A^*,p,n} &= 2 \overline{\alpha}^*_{26} \alpha^2 \frac{\beta^3 p}{n} + 2 \overline{\alpha}^*_{16} \alpha^2 \frac{\beta^3 p}{n} \\
\overline{\gamma}^e_{B^*,m,n} &= \overline{\alpha}^4 \frac{\alpha^2 m}{21} + (\overline{\alpha}^*_{_{11}} + \overline{\alpha}^*_{_{66}} - 2 \overline{\alpha}^*_{_{26}}) \alpha^2 \frac{\beta^2 n}{m} + \overline{\alpha}^*_{_{12}} \frac{\beta^4 n}{m} \\
\overline{\gamma}^0_{B^*,m,n} &= (2 \overline{\alpha}^*_{_{61}} - \overline{\alpha}^*_{_{12}}) \alpha^3 \frac{\beta m}{n} + (2 \overline{\alpha}^*_{_{16}} - \overline{\alpha}^*_{_{62}}) \alpha^3 \frac{\beta m}{n} \\
\overline{\gamma}^e_{B^*,p,n} &= \overline{\alpha}^4 \frac{\alpha^4 p}{21} + (\overline{\alpha}^*_{_{11}} + \overline{\alpha}^*_{_{66}} - 2 \overline{\alpha}^*_{_{26}}) \alpha^2 \frac{\beta^2 p}{n} + \overline{\alpha}^*_{_{12}} \frac{\beta^4 p}{n} \\
\overline{\gamma}^0_{B^*,p,n} &= (2 \overline{\alpha}^*_{_{61}} - \overline{\alpha}^*_{_{12}}) \alpha^3 \frac{\beta p}{n} + (2 \overline{\alpha}^*_{_{16}} - \overline{\alpha}^*_{_{62}}) \alpha^3 \frac{\beta p}{n}
\end{align*}
\]  \quad (B.2-9)
\[ \bar{\gamma}^{e}_{D^*,m,n} = \bar{D}^{*}_{11} m^4_n + 2 (\bar{D}^{*}_{12} + 2 \bar{D}^{*}_{66}) \alpha^2_{m} \beta^2_n + \bar{D}_2^{*} \beta^4_n \]  
(B.10)

\[ \bar{\gamma}^{o}_{D^*,m,n} = 4 \bar{D}^{*}_{16} \alpha^3 m^3_n + 4 \bar{D}^{*}_{26} \alpha \beta^3_n \]  
(B.11)

\[ \bar{\gamma}^{e}_{D^*,p,n} = \bar{D}^{*}_{11} \alpha^4_p + 2 (\bar{D}^{*}_{12} + 2 \bar{D}^{*}_{66}) \alpha^2 \beta^2_p + \bar{D}_2^{*} \beta^4_n \]  
(B.12)

\[ \bar{\gamma}^{o}_{D^*,p,n} = 4 \bar{D}^{*}_{16} \alpha^3 \beta^3_p + 4 \bar{D}^{*}_{26} \alpha \beta^3_n \]  
(B.13)

\[ \bar{T}_{1,m,n} = \bar{\gamma}^{e}_{D^*,m,n} + \bar{\gamma}^{o}_{D^*,m,n} ; \quad \bar{T}_{4,p,n} = \bar{\gamma}^{e}_{B^*,p,n} + \bar{\gamma}^{o}_{B^*,p,n} + \alpha^2_p \]  
(B.14)

\[ \bar{T}_{2,p,n} = \bar{\gamma}^{e}_{D^*,p,n} + \bar{\gamma}^{o}_{D^*,p,n} ; \quad \bar{T}_{5,m,n} = \bar{\gamma}^{e}_{A^*,m,n} + \bar{\gamma}^{o}_{A^*,m,n} \]  
(B.15)

\[ \bar{T}_{3,m,n} = \bar{\gamma}^{e}_{B^*,m,n} + \bar{\gamma}^{o}_{B^*,m,n} + \alpha^2_m ; \quad \bar{T}_{6,p,n} = \bar{\gamma}^{e}_{A^*,p,n} + \bar{\gamma}^{o}_{A^*,p,n} \]  

where

\[ \bar{A}^{*}_{ij} = \frac{\pi}{L} \frac{2 R}{2c} \]  
(B.16)

In the case that \( \tau_K = 0 \) one gets

\[ \alpha^2_{m_o} = \left( \frac{m}{L} \right)^2 \frac{2 R}{2c} \]  
(B.17)

and hence

\[ \bar{T}^{(o)}_{1} = \bar{D}^{*}_{11} \alpha^4 m_o + 4 \bar{D}^{*}_{16} \alpha^3 \beta^2 m_o \beta^2_n + 2 (\bar{D}^{*}_{12} + 2 \bar{D}^{*}_{66}) \alpha^2 \beta^2 m_o \beta^2_n + 4 \bar{D}^{*}_{26} \alpha m_o \beta^3_n + \bar{D}_2^{*} \beta^4_n \]  
(B.18)

\[ \bar{T}^{(o)}_{3} = \bar{B}^{*}_{21} \alpha^4 m_o + (2 \bar{B}^{*}_{26} + \bar{B}^{*}_{61}) \alpha^3 \beta^2 m_o \beta^2_n + (\bar{B}^{*}_{11} + \bar{B}^{*}_{22} - 2 \bar{B}^{*}_{66}) \alpha^2 \beta^2 m_o \beta^2_n \]  
(B.19)
APPENDIX C: Coefficients for calculating the effect of axisymmetric imperfection

Using the previously defined stiffness and wave number parameters one gets for the case of axial compression and internal pressure in Eq. (3.18) the following constants

\[
\begin{align*}
\bar{A}_1 &= \frac{E}{4c} \frac{3}{\xi} \bar{A}_{1i} \quad \bar{B}_1 = \frac{E}{4c} \frac{3}{\xi} \bar{B}_{1i} \\
&\text{for } i = 1, 2, \ldots 5
\end{align*}
\]

where

\[
\bar{A}_1 = \frac{T_{3,m,n}}{T_{5,m,n}} \quad \bar{B}_1 = \frac{T_{4,p,n}}{T_{6,p,n}}
\]

\[
\bar{A}_2 = -c_a \bar{A}_1 \beta^2 \frac{\xi + \bar{A}}{\gamma_{A^*,i,m,n}^{+} - \gamma_{A^*,i,m,n}^{-}} \quad \bar{B}_2 = -c_a \bar{A}_1 \beta^2 \frac{\xi + \bar{A}}{\gamma_{A^*,i+p,n}^{+} - \gamma_{A^*,i+p,n}^{-}}
\]

\[
\bar{A}_3 = c_a \bar{A}_1 \beta^2 \frac{\xi - \bar{A}}{\gamma_{A^*,i-m,n}^{-} - \gamma_{A^*,i-m,n}^{+}} \quad \bar{B}_3 = c_a \bar{A}_1 \beta^2 \frac{\xi - \bar{A}}{\gamma_{A^*,i-p,n}^{-} - \gamma_{A^*,i-p,n}^{+}}
\]

\[
\bar{A}_4 = -\bar{A}_2 \quad \bar{B}_4 = -\bar{B}_2
\]

\[
\bar{A}_5 = \bar{A}_3 \quad \bar{B}_5 = -\bar{B}_3
\]

and

\[
\gamma_{A^*,i,m,n}^{+} = \bar{A}_5 \alpha^4_{22} + (2\bar{A}_5 + \bar{A}_6) \alpha^2_{16} \beta^2 + \bar{A}_1 \beta^4
\]

\[
\gamma_{A^*,i,m,n}^{-} = \bar{A}_5 \alpha^4_{26} + \bar{A}_1 \alpha^2_{16} \beta^2 + \bar{A}_1 \beta^4
\]

\[
\gamma_{A^*,i-m,n}^{+} = \bar{A}_5 \alpha^4_{22} + (2\bar{A}_5 + \bar{A}_6) \alpha^2_{16} \beta^2 + \bar{A}_1 \beta^4
\]

\[
\gamma_{A^*,i-m,n}^{-} = \bar{A}_5 \alpha^4_{26} + \bar{A}_1 \alpha^2_{16} \beta^2 + \bar{A}_1 \beta^4
\]

\[
\gamma_{A^*,i+p,n}^{+} = \bar{A}_5 \alpha^4_{22} + (2\bar{A}_5 + \bar{A}_6) \alpha^2_{16} \beta^2 + \bar{A}_1 \beta^4
\]

\[
\gamma_{A^*,i+p,n}^{-} = \bar{A}_5 \alpha^4_{26} + \bar{A}_1 \alpha^2_{16} \beta^2 + \bar{A}_1 \beta^4
\]
\(\gamma_{A^*,i-p,n} = \tilde{\alpha}_{22}^e \alpha_{i-p}^4 + (2\tilde{\alpha}_{12}^* + \tilde{\alpha}_{66}^*) \alpha_{i-p}^2 \beta_{n}^2 + \tilde{\alpha}_{11}^* \beta_{n}^4\)  
(C.13)

\(\gamma_{A^*,i-p,n}^0 = 2\tilde{\alpha}_{26}^* \alpha_{i-p}^2 \beta_{n}^2 + 2\tilde{\alpha}_{16}^* \alpha_{i-p} \beta_{n}^3\)  
(C.14)

where
\[
\alpha_{i+m}^2 = [(i+m) \frac{\pi}{L} + \frac{n}{R} \tau_{K}]^2 \frac{2 \Omega_t}{2c}  
(C.15)
\]
\[
\alpha_{i-m}^2 = [(i-m) \frac{\pi}{L} - \frac{n}{R} \tau_{K}]^2 \frac{2 \Omega_t}{2c} 
(C.16)
\]
\[
\alpha_{i+p}^2 = [(i+m) \frac{\pi}{L} + \frac{n}{R} \tau_{K}]^2 \frac{2 \Omega_t}{2c} 
(C.17)
\]
\[
\alpha_{i-p}^2 = [(i-m) \frac{\pi}{L} - \frac{n}{R} \tau_{K}]^2 \frac{2 \Omega_t}{2c} 
(C.18)
\]

Further in Eq. (3.20) the following constants are used

\[
\hat{C}_1 = \frac{c \beta_{n}^2}{(\alpha_{m}^2 + \alpha_{p}^2)} \left(1 + \frac{\tilde{B}_{21}^* \alpha_{i}^2}{\tilde{\alpha}_{22}^*}\right) 
(C.19)
\]

\[
\hat{C}_2 = \frac{c}{2} \frac{\alpha_{i}^2 \beta_{n}^2}{(\alpha_{m}^2 + \alpha_{p}^2)} \left\{\frac{\tilde{T}_{3,i-p,n}}{\tilde{T}_{5,i-p,n}} + \frac{\tilde{T}_{4,i-m,n}^*}{\tilde{T}_{6,i-m,n}} + \frac{\tilde{T}_{3,m,n}^*}{\tilde{T}_{5,m,n}} + \frac{\tilde{T}_{4,p,n}^*}{\tilde{T}_{6,p,n}}\right\} 
(C.20)
\]

\[
\hat{C}_3 = \frac{c^2}{2} \frac{\alpha_{i}^4 \beta_{n}^4}{(\alpha_{m}^2 + \alpha_{p}^2)} \left\{\frac{1}{\tilde{T}_{5,i-m,n}} + \frac{1}{\tilde{T}_{6,i-m,n}} + \frac{1}{\tilde{T}_{5,i-p,n}} + \frac{1}{\tilde{T}_{6,i-p,n}}\right\} 
(C.21)
\]

\[
\hat{C}_4 = \frac{c^2}{2} \frac{\alpha_{i}^4 \beta_{n}^4}{(\alpha_{m}^2 + \alpha_{p}^2)} \left\{\frac{1}{\tilde{T}_{5,i-m,n}} + \frac{1}{\tilde{T}_{6,i-m,n}}\right\} 
(C.22)
\]

where
\[
\tilde{T}_{3,i-p,n} = \gamma_{B^*,i-p,n}^e - \gamma_{B^*,i-p,n}^0 + \alpha_{i-p}^2 
(C.23)
\]
\[
\tilde{T}_{4,i-m,n} = \gamma_{B^*,i-m,n}^e + \gamma_{B^*,i-m,n}^0 + \alpha_{i-m}^2 
(C.24)
\]
\[
\tilde{T}_{5,i-m,n} = \gamma_{A^*,i+m,n}^e + \gamma_{A^*,i+m,n}^0 
(C.25)
\]
\[ \hat{T}_{5,i-p,n} = \gamma^e_{A^*,i-p,n} + \gamma^o_{A^*,i-p,n} \] (C.26)

\[ \hat{T}_{6,i-m,n} = \gamma^e_{A^*,i-m,n} + \gamma^o_{A^*,i-m,n} \] (C.27)

\[ \hat{T}_{6,i+p,n} = \gamma^e_{A^*,i+p,n} + \gamma^o_{A^*,i+p,n} \] (C.28)

and

\[ \gamma^e_{B^*,i-m,n} = \beta^e_{21} \alpha^4_{i-m,n} + (\bar{\beta}^e_{11} + \bar{\beta}^e_{22} - 2\bar{\beta}^e_{66}) \alpha^2_{i-m,n} \beta^2_{n} + \bar{\beta}^e_{12} \beta^4_{n} \] (C.29)

\[ \gamma^o_{B^*,i-m,n} = (2\bar{\beta}^e_{26} - \bar{\beta}^e_{61}) \alpha^3_{i-m,n} + (2\bar{\beta}^e_{16} - \bar{\beta}^e_{62}) \alpha^2_{i-m,n} \beta^3_{n} \] (C.30)

\[ \gamma^e_{B^*,i-p,n} = \beta^e_{21} \alpha^2_{i-p,n} + (\bar{\beta}^e_{11} + \bar{\beta}^e_{22} - 2\bar{\beta}^e_{66}) \alpha^2_{i-p,n} \beta^2_{n} + \bar{\beta}^e_{12} \beta^4_{n} \] (C.31)

\[ \gamma^o_{B^*,i-p,n} = (2\bar{\beta}^e_{26} - \bar{\beta}^e_{61}) \alpha^3_{i-p,n} + (2\bar{\beta}^e_{16} - \bar{\beta}^e_{62}) \alpha^2_{i-p,n} \beta^3_{n} \] (C.32)

Finally for the case of external pressure and axial compression in Eq. (3.33) the following constants are used

\[ \hat{D}_1 = \frac{c \beta^2_{21} \alpha^2_{21,1}}{\Delta_{mn\tau}} \] (C.33)

\[ \hat{D}_2 = \frac{c \beta^2_{21} \alpha^2_{1}}{\Delta_{mn\tau}} \left\{ \frac{\hat{T}_{3,i-p,n}}{\Delta_{mn\tau}} + \frac{\hat{T}_{4,i-m,n}}{\Delta_{mn\tau}} + \frac{\hat{T}_{5,m,n}}{\Delta_{mn\tau}} + \frac{\hat{T}_{6,p,n}}{\Delta_{mn\tau}} \right\} \] (C.34)

\[ \hat{D}_3 = \frac{c \beta^4_{21} \alpha^4_{i-p,n}}{\Delta_{mn\tau}} \left\{ \frac{1}{\hat{T}_{5,i-m,n}} + \frac{1}{\hat{T}_{6,i-m,n}} + \frac{1}{\hat{T}_{5,i-p,n}} + \frac{1}{\hat{T}_{6,i+p,n}} \right\} \] (C.35)

\[ \hat{D}_4 = \frac{c \beta^4_{21} \alpha^4_{i-p,n}}{\Delta_{mn\tau}} \left\{ \frac{1}{\hat{T}_{6,i-m,n}} + \frac{1}{\hat{T}_{5,i-p,n}} \right\} \] (C.36)

\[ \Delta_{mn\tau} = 2\beta^2_{n} + \hat{R} (\alpha^2_{m} + \alpha^2_{p}) \] (C.37)

\[ \hat{R} = \frac{\lambda}{\bar{P}_e} \] (C.38)
APPENDIX D: Coefficients for the computation of the b-factor

Using the previously defined stiffness and wave number parameters one gets for the case of axial compression and internal pressure in Eqs. (4.31) and (4.32) the following constants

\[
A_i = \frac{c}{\pi} \frac{\alpha_m + \alpha_p}{\alpha_i^2} \frac{2\beta_n^2}{\alpha_i^2} \frac{\left(1+\alpha_i^2 \hat{B}_{11}^*\right) + 2F_1^2 \hat{A}_{122}^*}{\left(1+\alpha_i^2 \hat{B}_{11}^*\right) + \left(\alpha_i^2 \hat{B}_{12}^* - 2\lambda_{mn}\right) \alpha_i^2 \hat{A}_{22}^*} \left(\frac{i}{i^2 - 4m^2}\right) \theta_i \quad (D.1)
\]

\[
B_i = \frac{c}{\pi} \frac{\alpha_m + \alpha_p}{\alpha_i^2} \frac{2\beta_n^2}{\alpha_i^2} \frac{T_3, m_i, 2n + 2F T_5, m_i, 2n}{T_3, m_i, 2n + \left(T_{1,1}, m_i, 2n - 2\alpha_i^2 \hat{A}_{11}^* \lambda_{mn} + 2\beta_i^2 \hat{B}_{11}^*\right) T_5, m_i, 2n} \theta_i \quad (D.2)
\]

\[
C_i = \frac{c}{\pi} \frac{\alpha_m + \alpha_p}{\alpha_i^2} \frac{2\beta_n^2}{\alpha_i^2} \frac{T_4, p_i, 2n + 2F T_6, p_i, 2n}{T_4, p_i, 2n + \left(T_{2,1}, p_i, 2n - 2\alpha_i^2 \hat{A}_{11}^* \lambda_{mn} + 2\beta_i^2 \hat{B}_{11}^*\right) T_6, p_i, 2n} \theta_i \quad (D.3)
\]

\[
D_i = \frac{c}{\pi} \frac{\alpha_m + \alpha_p}{\alpha_i^2} \frac{2\beta_n^2}{\alpha_i^2} \frac{2F\left(1+\alpha_i^2 \hat{B}_{11}^*\right) - \left(\alpha_i^2 \hat{D}_{11}^* - 2\lambda_{mn}\right)}{\left(1+\alpha_i^2 \hat{B}_{11}^*\right) + \left(\alpha_i^2 \hat{D}_{11}^* - 2\lambda_{mn}\right) \alpha_i^2 \hat{A}_{22}^*} \left(\frac{i}{i^2 - 4m^2}\right) \theta_i \quad (D.4)
\]

\[
E_i = \frac{c}{\pi} \frac{\alpha_m + \alpha_p}{\alpha_i^2} \frac{2\beta_n^2}{\alpha_i^2} \frac{T_3, m_i, 2n - \left(T_{1,1}, m_i, 2n - 2\alpha_i^2 \hat{A}_{11}^* \lambda_{mn} + 2\beta_i^2 \hat{B}_{11}^*\right) T_5, m_i, 2n}{T_3, m_i, 2n + \left(T_{1,1}, m_i, 2n - 2\alpha_i^2 \hat{A}_{11}^* \lambda_{mn} + 2\beta_i^2 \hat{B}_{11}^*\right) T_5, m_i, 2n} \theta_i \quad (D.5)
\]

\[
F_i = \frac{c}{\pi} \frac{\alpha_m + \alpha_p}{\alpha_i^2} \frac{2\beta_n^2}{\alpha_i^2} \frac{T_4, p_i, 2n - \left(T_{2,1}, p_i, 2n - 2\alpha_i^2 \hat{A}_{11}^* \lambda_{mn} + 2\beta_i^2 \hat{B}_{11}^*\right) T_6, p_i, 2n}{T_4, p_i, 2n + \left(T_{2,1}, p_i, 2n - 2\alpha_i^2 \hat{A}_{11}^* \lambda_{mn} + 2\beta_i^2 \hat{B}_{11}^*\right) T_6, p_i, 2n} \theta_i \quad (D.6)
\]

where

\[
2F = \frac{T_3, m_n + T_4, p_n}{T_5, m_n + T_6, p_n} \quad (D.7)
\]

\[\theta_i = \text{ODD-ness operator} = 1 \text{ if } i = \text{ odd integer}\]

\[= 0 \text{ otherwise}\]

and
\( \bar{T}_{1,m_1,2n} = \gamma^e_{D^*,m_1,2n} - \gamma^o_{D^*,m_1,2n} \)  
(D.8)

\( \bar{T}_{2,p_1,2n} = \gamma^e_{D^*,p_1,2n} + \gamma^o_{D^*,p_1,2n} \)  
(D.9)

\( \bar{T}_{3,m_1,2n} = \gamma^e_{B^*,m_1,2n} - \gamma^o_{B^*,m_1,2n} + \alpha_m^2 \)  
(D.10)

\( \bar{T}_{4,p_1,2n} = \gamma^e_{B^*,p_1,2n} + \gamma^o_{B^*,p_1,2n} + \alpha_p^2 \)  
(D.11)

\( \bar{T}_{5,m_1,2n} = \gamma^e_{A^*,m_1,2n} + \gamma^o_{A^*,m_1,2n} \)  
(D.12)

\( \bar{T}_{6,p_1,2n} = \gamma^e_{A^*,p_1,2n} - \gamma^o_{A^*,p_1,2n} \)  
(D.13)

\( \gamma^e_{A^*,m_1,2n} = \bar{A}_2 \alpha_{m_1}^4 + (2\bar{A}_{12} + \bar{A}_{66}) \alpha_{m_1} \beta_{2n} + \bar{A}_{11} \beta_{2n}^4 \)  
(D.14)

\( \gamma^o_{A^*,m_1,2n} = 2\bar{A}_2 \alpha_{m_1}^3 \beta_{2n} + 2\bar{A}_{16} \alpha_{m_1} \beta_{2n} \)  
(D.15)

\( \gamma^e_{A^*,p_1,2n} = \bar{A}_2 \alpha_{p_1}^4 + (2\bar{A}_{12} + \bar{A}_{66}) \alpha_{p_1} \beta_{2n} + \bar{A}_{11} \beta_{2n}^4 \)  
(D.16)

\( \gamma^o_{A^*,p_1,2n} = 2\bar{A}_2 \alpha_{p_1}^3 \beta_{2n} + 2\bar{A}_{16} \alpha_{p_1} \beta_{2n} \)  
(D.17)

\( \gamma^e_{B^*,m_1,2n} = \bar{B}_1 \alpha_{m_1}^4 + (\bar{B}_{11} + \bar{B}_{22} + 2\bar{B}_{66}) \alpha_{m_1} \beta_{2n} + \bar{B}_{12} \beta_{2n}^4 \)  
(D.18)

\( \gamma^o_{B^*,m_1,2n} = (2\bar{B}_{26} - \bar{B}_{66}) \alpha_{m_1} \beta_{2n} + (2\bar{B}_{16} \bar{B}_{62}) \alpha_{m_1} \beta_{2n} \)  
(D.19)

\( \gamma^e_{B^*,p_1,2n} = \bar{B}_1 \alpha_{p_1}^4 + (\bar{B}_{11} + \bar{B}_{22} + 2\bar{B}_{66}) \alpha_{p_1} \beta_{2n} + \bar{B}_{12} \beta_{2n}^4 \)  
(D.20)

\( \gamma^o_{B^*,p_1,2n} = (2\bar{B}_{26} - \bar{B}_{66}) \alpha_{p_1} \beta_{2n} + (2\bar{B}_{16} \bar{B}_{62}) \alpha_{p_1} \beta_{2n} \)  
(D.21)

\[ \alpha_m^2 = \left( \frac{\pi}{L} + 2 \frac{n}{R} \frac{\tau}{K} \right) \frac{2Rt}{2c} \]  
(D.22)
\[ \alpha_{p_i}^2 = (\frac{2\pi}{L} - 2\frac{n}{R} + \frac{\tau}{K})^2 \frac{2R_t}{2c} \]  
(D.23)

\[ \beta_{2n}^2 = (2n)^2 \frac{2R_t}{2c} \left( \frac{1}{R} \right)^2 \]  
(D.24)

Finally, for the case of external pressure and axial compression in Eqs. (4.70) and (4.71) the following constants are used

\[ \tilde{\alpha}_i = \frac{c}{\pi} \frac{(\alpha_m + \alpha_p)^2 \beta_n^2}{\alpha_i^2} \frac{1}{1 + \alpha_i^2 B_{21}^*} \left( \frac{2F \alpha_i^2 A_{12}^*}{\alpha_i^2 A_{12}^*} \right) \]  
(D.25)

\[ \tilde{\beta}_i = \frac{c}{\pi} \frac{(\alpha_m + \alpha_p)^2 \beta_n^2}{\alpha_i^2} \frac{1}{1 + \alpha_i^2 B_{21}^*} \left( \frac{2F \tilde{\alpha}_i^2 A_{12}^*}{\alpha_i^2 A_{12}^*} \right) \]  
(D.26)

\[ \tilde{\gamma}_i = \frac{c}{\pi} \frac{(\alpha_m + \alpha_p)^2 \beta_n^2}{\alpha_i^2} \frac{1}{1 + \alpha_i^2 B_{21}^*} \left( \frac{2F \tilde{\alpha}_i^2 A_{12}^*}{\alpha_i^2 A_{12}^*} \right) \]  
(D.27)

\[ \tilde{\delta}_i = \frac{c}{\pi} \frac{(\alpha_m + \alpha_p)^2 \beta_n^2}{\alpha_i^2} \frac{1}{1 + \alpha_i^2 B_{21}^*} \left( \frac{2F \tilde{\alpha}_i^2 A_{12}^*}{\alpha_i^2 A_{12}^*} \right) \]  
(D.28)

\[ \tilde{\epsilon}_i = \frac{c}{\pi} \frac{(\alpha_m + \alpha_p)^2 \beta_n^2}{\alpha_i^2} \frac{1}{1 + \alpha_i^2 B_{21}^*} \left( \frac{2F \tilde{\alpha}_i^2 A_{12}^*}{\alpha_i^2 A_{12}^*} \right) \]  
(D.29)

\[ \tilde{\zeta}_i = \frac{c}{\pi} \frac{(\alpha_m + \alpha_p)^2 \beta_n^2}{\alpha_i^2} \frac{1}{1 + \alpha_i^2 B_{21}^*} \left( \frac{2F \tilde{\alpha}_i^2 A_{12}^*}{\alpha_i^2 A_{12}^*} \right) \]  
(D.30)
APPENDIX E: Effect of half-wave sine axisymmetric imperfection

E.1 Axial compression and internal pressure \((p_e = -p_i)\)

If the initial imperfection is axisymmetric, e.g.,

\[
\overline{W} = t\xi_1 \sin \pi \frac{x}{L}
\]  
(E.1)

then the prebuckling solution will also be axisymmetric, namely

\[
W^{(o)} = tW_y + tW_{p_i} + w_o(x)
\]  
(E.2)

\[
F^{(o)} = \frac{Et}{cR} \left( -\frac{1}{2} \alpha y^2 + \frac{1}{2} p_i x^2 \right) + f_o(x)
\]  
(E.3)

Substitution into the equations governing the prebuckling state (Eqs. 3.2 - 3.3) yields

\[
A_{22}^* f_{o,xxxx}^{(o)} - B_{21}^* w_o,x_{xxx} = -\frac{1}{R} w_o,xx
\]  
(E.4)

\[
B_{21}^* f_{o,xxxx}^{(o)} + D_{11}^* w_o,x_{xxx} = \frac{1}{R} f_{o,xx}^{(o)} + \frac{Et}{cR} \lambda w_o,xx + \frac{Et}{cR} \left( \frac{\pi}{L} \right)^2 \xi_1 \lambda \sin \pi \frac{x}{L}
\]  
(E.5)

Neglecting the effect of boundary conditions the (particular) solution of these equations is

\[
w_o(x) = t \frac{\lambda}{\lambda_c} \xi_1 \sin \pi \frac{x}{L}
\]  
(E.6)

\[
f_o(x) = \frac{Et}{c} \frac{\lambda}{\lambda_c} \left( \frac{1}{\lambda_{21}} \alpha_1^2 \right) \xi_1 \sin \pi \frac{x}{L}
\]  
(E.7)

where

\[
\lambda_{c_i} = \frac{1}{2} \left\{ \alpha_1^2 \frac{B_{11}^*}{A_{11}} + \frac{(1+\alpha_1^2) \alpha_1^2}{\alpha_1^2 \frac{A_{22}^*}{\lambda_{11}}} \right\}
\]  
(E.8)

is the classic axisymmetric buckling load for axial compression only.

For the axisymmetric imperfection and prebuckling state the linearized stability equations reduce to
\[ L_{A^*}(F^{(1)}) - L_{B^*}(W^{(1)}) = \frac{1}{R} W^{(1)}_{xx} + \frac{2c}{R} \alpha_i^2 (\xi_i + \tilde{A}) \sin \pi \frac{x}{L} W^{(1)}_{yy} \]  
\[ (E.9) \]

\[ L_{B^*}(F^{(1)}) + L_{D^*}(W^{(1)}) = \frac{1}{R} F^{(1)}_{xx} + \frac{Et^2}{cR} (\tilde{p}_i - 2c \alpha_i^2 \tilde{B} \sin \pi \frac{x}{L}) W^{(1)}_{yy} \]  
\[ (E.10) \]

\[ - \frac{Et^2}{cR} \lambda \frac{w^{(1)}}{xx} - 2c \alpha_i^2 \xi_i + \tilde{A}) \sin \pi \frac{x}{L} F^{(1)}_{yy} \]

where

\[ \tilde{A} = \frac{\lambda}{\lambda_c - \lambda} \xi_1 ; \quad \tilde{B} = \frac{\lambda}{\lambda_c - \lambda} \frac{(1 + \tilde{B}^*) \alpha_i^2}{2 \alpha_i^2 \tilde{A}^*} \xi_1 \]  
\[ (E.11) \]

Assuming that the radial displacement component of the classic buckling mode can be written as

\[ W^{(1)} = t^2 \sin \frac{\pi}{L} \cos \frac{n}{R} (y - \tau_K x) \]  
\[ (E.12) \]

then substitution into the compatibility equation (E.9) yields a linear, inhomogenous partial differential equation for \( F^{(1)} \). By the Method of Undetermined Coefficients one can obtain the following particular integral

\[ F^{(1)} = \hat{A}_1 \sin i \pi x - \hat{B}_1 \sin \frac{n}{R} \pi y \]  
\[ + \hat{A}_2 \sin i+m \pi x \cos \frac{n}{R} \pi y \]  
\[ + \hat{A}_3 \sin i-m \pi x \sin \frac{n}{R} \pi y \]  
\[ + \hat{A}_4 \cos i \pi x \cos \frac{n}{R} \pi y \]  
\[ + \hat{A}_5 \cos i-p \pi x \cos \frac{n}{R} \pi y \]  
\[ (E.13) \]

where

\[ l_{i+m} = (i+m) \frac{\pi}{L} + \frac{n}{R} \pi_K \]  
\[ l_{i-p} = (i-p) \frac{\pi}{L} + \frac{n}{R} \pi_K \]  
\[ (E.14) \]

and the constants \( \hat{A}_1, \hat{A}_2, ..., \hat{B}_1, ..., \hat{B}_5 \) are defined at the end of this appendix.
An approximate solution of the equilibrium equation (E.10) by Galerkin’s procedure then yields the following cubic polynomial for the eigenvalue $\hat{\lambda}$:

\begin{equation}
\hat{\lambda}^3 - \left\{ \hat{A}_{mn\tau} + 2\hat{A}_{c_1} + \hat{C}_1 \xi_1 \frac{16}{\pi (4m^2 - i^2)i} \right\} \hat{\lambda}^2 \\
+ \left\{ 2\hat{A}_{mn\tau} + \hat{A}_{c_1} + \hat{C}_1 \xi_1 \frac{16}{\pi (4m^2 - i^2)i} \theta_i - \left[ \hat{C}_2 \frac{m}{(2m-i)i} + \hat{C}_3 \frac{m}{(2m+i)i} \right] \xi_1 \frac{4}{\pi} \theta_i \right\} \hat{\lambda} \\
- \left\{ \hat{A}_{mn\tau} - \left[ \hat{C}_2 \frac{m}{(2m-i)i} + \hat{C}_3 \frac{m}{(2m+i)i} \right] \xi_1 \frac{4}{\pi} \theta_i + (\hat{C}_4 \hat{C}_5 \delta_{i m} \delta_{\tau o} \xi_1 \xi_2) \right\} \hat{\lambda}^0 = 0
\end{equation}

(E.15)

where

\begin{align*}
\delta_{i m} &= 1 \text{ if } i = m \\
&= 0 \text{ otherwise} \\
\delta_{\tau o} &= 1 \text{ if } \tau = 0 \\
&= 0 \text{ otherwise} \\
\theta_i &= 1 \text{ if } i = \text{ odd integer} \\
&= 0 \text{ otherwise}
\end{align*}

(E.16)

and the constants $\hat{C}_1, \ldots, \hat{C}_5$ are defined further on.

### E.2 External pressure and axial compression

If the initial imperfection is axisymmetric, e.g.,

\begin{equation}
\bar{W} = t \bar{\xi}_1 \sin \pi \frac{x}{L}
\end{equation}

(E.17)

then the prebuckling solution will also be axisymmetric, namely

\begin{equation}
W^{(0)} = tW_c + W_0(x)
\end{equation}

(E.18)

\begin{equation}
P^{(0)} = \frac{E}{\pi R} \left( -\frac{1}{2} \frac{1}{p_e} \right) (x^2 + R^2) + f_0(x)
\end{equation}

(E.19)

where

\begin{equation}
W_c = W_v + W_p = \frac{p_c}{c} (\bar{A}^{*}_{22} + \bar{A}^{*}_{12}) \\
\hat{R} = \frac{\lambda}{p_e}
\end{equation}

A substitution into the equations governing the prebuckling state (Eqs. 3.2 and 3.3) yields.
\[ A_{22}^* f_{o_{xxxx}} - B_{21}^* w_{o_{xxxx}} = \frac{1}{R} w_{o_{xx}} \]  
(E.20)

\[ B_{21}^* f_{o_{xxxx}} + D_{11}^* w_{o_{xxxx}} = R f_{o_{xx}} - \frac{Et}{cR} \frac{2π}{ξ_1} \frac{Rp_e}{\lambda_{c_i}} \sin \frac{iπx}{L} \]  
(E.21)

Neglecting the effect of boundary conditions, the (particular) solution of these equations is

\[ w_o(x) = t \frac{\hat{R}_{pe}}{\lambda_{c_i} - \hat{R}_{pe}} \xi_1 \sin \frac{iπx}{L} \]  
(E.22)

\[ f_o(x) = \frac{Et}{c} \frac{\hat{R}_{pe}}{\lambda_{c_i} - \hat{R}_{pe}} \frac{(1+\bar{B}_{21}^*\alpha_i^2)}{2\alpha_i^2 \bar{A}_{22}^*} \xi_1 \sin \frac{iπx}{L} \]  
(E.23)

where

\[ \lambda_{c_i} = \frac{1}{2} \left\{ \alpha_i^2 \bar{D}_{11}^* + \frac{(1+\bar{B}_{21}^*\alpha_i^2)}{\alpha_i^2 \bar{A}_{22}^*} \right\} \]  
(E.8)

is the classic axisymmetric buckling load for axial compression only.

For the axisymmetric imperfection and prebuckling state the linearized stability equations reduce to

\[ L_A(f^{(1)}) - L_B^*(w^{(1)}) = \frac{1}{R} w^{(1)}_{xx} + \frac{2c}{R} \alpha_i^2 (\xi_1 + \bar{A}) \sin \frac{iπx}{L} w^{(1)}_{yy} \]  
(E.24)

\[ L_B(f^{(1)}) + L_D^*(w^{(1)}) = \frac{1}{R} f^{(1)}_{xx} - \frac{Et}{cR} \frac{2π}{ξ_1} \frac{Rp_e}{\lambda_{c_i}} \bar{A}^* \sin \frac{iπx}{L} w^{(1)}_{yy} \]  
(E.25)

\[ \frac{Et}{cR} \frac{2π}{ξ_1} \frac{Rp_e}{\lambda_{c_i}} w^{(1)}_{xx} \bar{A}^* (\xi_1 + \bar{A}) \sin \frac{iπx}{L} f^{(1)}_{yy} \]

where

\[ \bar{A} = \frac{\hat{R}_{pe}}{\lambda_{c_i} - \hat{R}_{pe}} \xi_1 \]  
\[ \bar{B} = \frac{\hat{R}_{pe}}{\lambda_{c_i} - \hat{R}_{pe}} \frac{(1+\bar{B}_{21}^*\alpha_i^2)}{2\alpha_i^2 \bar{A}_{22}^*} \xi_1 \]  
(E.26)
By assuming that the radial displacement component of the classic buckling mode can be written as

$$W^{(1)} = t \xi \sin \frac{n \pi}{L} \sin \frac{n}{R} (ySh_x)$$

(E.12)

an approximate solution of the stability equations can be obtained as in the preceding case yielding the following cubic polynomial for the eigenvalue $\bar{p}_e$:

$$R^3 \bar{p}_e^3 - \{ \bar{p}_{mn} \hat{R}^2 + 2\lambda c_i \hat{R} + \hat{D}_1 \xi_1 \hat{R}^2 \frac{16}{\pi} \frac{m^2}{(4m^2 - 1)} \theta \} \bar{p}_e^2$$

$$+ \{ 2\bar{p}_{mn} \hat{R} + \hat{D}_1 \xi_1 \hat{R} \frac{16}{\pi} \frac{m^2}{(4m^2 - 1)} \theta \} \bar{p}_e^2$$

$$- \{ \bar{p}_{mn} \hat{D}_2 \frac{m}{3(2m+1)} \xi_1 \frac{4}{\pi} \theta + \hat{D}_4 \hat{D}_5 \delta \tau \xi_1 \frac{4}{\pi} \theta \} \lambda c_i$$

(E.27)

Thus for $\hat{R} = 0$ \rightarrow external pressure case

$$\bar{p}_e = \bar{p}_{mn} \left[ \frac{m}{3(2m+1)} \xi_1 \frac{4}{\pi} \theta + \hat{D}_4 \hat{D}_5 \delta \tau \xi_1 \frac{4}{\pi} \theta \right] = 0$$

(E.28)

whereas for $\hat{R} > 0$ \rightarrow combined loading case

$$\bar{p}_e^3 - \{ \bar{p}_{mn} \hat{R} + \hat{D}_1 \xi_1 \hat{R} \frac{16}{\pi} \frac{m^2}{(4m^2 - 1)} \theta \} \bar{p}_e^2$$

$$+ (1/\hat{R}) \{ 2\bar{p}_{mn} \hat{R} + \hat{D}_1 \xi_1 \hat{R} \frac{16}{\pi} \frac{m^2}{(4m^2 - 1)} \theta \} \bar{p}_e^2$$

$$- (1/\hat{R}) \{ \bar{p}_{mn} \hat{D}_2 \frac{m}{3(2m+1)} \xi_1 \frac{4}{\pi} \theta + \hat{D}_4 \hat{D}_5 \delta \tau \xi_1 \frac{4}{\pi} \theta \} \lambda c_i$$

(E.29)

The constants $\hat{D}_1, ..., \hat{D}_4$ are listed below.

The constants used in Eq. E.13) are as follows:

$$\hat{A}_i = \frac{E_t}{4c} \xi_i; \hat{B}_i = \frac{E_t}{4c} \xi_i$$

for $i = 1, 2, ..., 5$
where

\[
\bar{A}_1 = \frac{T_{3,m,n}}{T_{5,m,n}} \quad \bar{B}_1 = \frac{T_{4,p,n}}{T_{6,p,n}}
\]

(E.31)

\[
\bar{A}_2 = c \alpha_i \beta_n^2 \gamma_{A^*,i+m,n} \gamma_{A^*,i+m,n}^o \quad \bar{B}_2 = -c \alpha_i \beta_n^2 \gamma_{A^*,i+p,n}^o \gamma_{A^*,i+p,n}^o
\]

(E.32)

\[
\bar{A}_3 = c \alpha_i \beta_n^2 \gamma_{A^*,i-m,n} \gamma_{A^*,i-m,n}^o \quad \bar{B}_3 = -c \alpha_i \beta_n^2 \gamma_{A^*,i-p,n}^o \gamma_{A^*,i-p,n}^o
\]

(E.33)

\[
\bar{A}_4 = \bar{A}_2 \quad \bar{B}_4 = \bar{B}_2
\]

(E.34)

\[
\bar{A}_5 = -\bar{A}_3 \quad \bar{B}_5 = \bar{B}_3
\]

(E.35)

Further in Eq. (E.15) the following constants are used

\[
\hat{C}_1 = \frac{c \beta_n^2}{(\alpha_m^2 + \alpha_p^2)} \frac{(1 + \bar{B}_2^* \alpha_i^2)}{\bar{A}_2^2}
\]

(E.36)

\[
\hat{C}_2 = \frac{c \alpha_i \beta_n^2}{2 (\alpha_m^2 + \alpha_p^2)} \left\{ \frac{T_{3,i-p,n}}{T_{5,i-p,n}} + \frac{T_{4,i-m,n}}{T_{6,i-m,n}} + \frac{T_{3,m,n}}{T_{5,m,n}} + \frac{T_{4,p,n}}{T_{6,p,n}} \right\}
\]

(E.37)

\[
\hat{C}_3 = \frac{c \alpha_i \beta_n^2}{2 (\alpha_m^2 + \alpha_p^2)} \left\{ \frac{T_{3,i+m,n}}{T_{5,i+m,n}} + \frac{T_{4,i+p,n}}{T_{6,i+p,n}} + \frac{T_{3,m,n}}{T_{5,m,n}} + \frac{T_{4,p,n}}{T_{6,p,n}} \right\}
\]

(E.38)

\[
\hat{C}_4 = \frac{c^2 \alpha_i \beta_n^4}{2 (\alpha_m^2 + \alpha_p^2)} \left\{ \frac{1}{T_{5,i+m,n}} + \frac{1}{T_{6,i-m,n}} + \frac{1}{T_{5,i-p,n}} + \frac{1}{T_{6,i+p,n}} \right\}
\]

(E.39)

\[
\hat{C}_5 = \frac{c^2 \alpha_i \beta_n^4}{2 (\alpha_m^2 + \alpha_p^2)} \left\{ \frac{1}{T_{5,i+m,n}} + \frac{1}{T_{6,i-m,n}} \right\}
\]

(E.40)
Finally for the case of external pressure and axial compression in Eq. (E.27) the following constants are used:

\[ \hat{D}_1 = \frac{c \beta_n^2 (1 + B_{21}^* \alpha_i^2)}{\Delta_{mn\tau} \overline{A}_{22}} \]  
(E.41)

\[ \hat{D}_2 = \frac{c \alpha_i^2 \beta_n^2}{2 \Delta_{mn\tau}} \left\{ \frac{T_{3,i-p,n} + T_{4,i-m,n} + T_{5,m,n} + T_{6,p,n}}{T_{5,i-p,n} + T_{6,i-m,n} + T_{5,m,n} + T_{6,p,n}} \right\} \]  
(E.42)

\[ \hat{D}_3 = \frac{c \alpha_i^2 \beta_n^2}{2 \Delta_{mn\tau}} \left\{ \frac{T_{3,i+m,n} + T_{4,i+p,n} + T_{5,m,n} + T_{6,p,n}}{T_{5,i+m,n} + T_{6,i+p,n} + T_{5,m,n} + T_{6,p,n}} \right\} \]  
(E.43)

\[ \hat{D}_4 = \frac{c^2 \alpha_i^4 \beta_n^4}{2 \Delta_{mn\tau}} \left\{ \frac{1}{T_{5,i+m,n}} + \frac{1}{T_{6,i-m,n}} + \frac{1}{T_{5,m,n}} + \frac{1}{T_{6,i+p,n}} \right\} \]  
(E.44)

\[ \hat{D}_5 = \frac{c^2 \alpha_i^4 \beta_n^4}{2 \Delta_{mn\tau}} \left\{ \frac{1}{T_{6,i-m,n}} + \frac{1}{T_{5,i+p,n}} \right\} \]  
(E.45)

\[ \Delta_{mn\tau} = 2\beta_n^2 + \hat{R} (\alpha_{m}^2 + \alpha_{p}^2) \]  
(E.46)

\[ \hat{R} = \frac{\lambda}{P_e} \]  
(E.47)
Table 1. Geometric properties of stringer stiffened shells

<table>
<thead>
<tr>
<th>Symbol</th>
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<th>Unit</th>
<th>Conversion</th>
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<td>R</td>
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<td>cm</td>
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<td>cm</td>
<td>0.3163 in</td>
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<td>variable</td>
<td>cm</td>
<td></td>
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<tr>
<td>$A_s$</td>
<td>variable</td>
<td>cm²</td>
<td></td>
</tr>
<tr>
<td>$l_{11}$</td>
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<td>N/cm²</td>
<td>10 x 10⁶ psi</td>
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Table 2. Geometric properties of ring stiffened shells

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<th>Conversion</th>
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<td>R</td>
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<td>cm</td>
<td>4.0 in</td>
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<td>cm</td>
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<tr>
<td>$e_r$</td>
<td>variable</td>
<td>cm</td>
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</tr>
<tr>
<td>$A_r$</td>
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<td>cm²</td>
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<tr>
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<td>cm⁴</td>
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</tr>
<tr>
<td>$E$</td>
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<td>N/cm²</td>
<td>10 x 10⁶ psi</td>
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Table 3. Geometric properties of Khot's\[^4\] glass-epoxy shells

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<th>Property</th>
<th>Value</th>
<th>Unit</th>
<th>Conversion</th>
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</thead>
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<tr>
<td>$t_{\text{total}}$</td>
<td>9.14400 x 10^{-2}</td>
<td>cm</td>
<td>(= 0.036 in)</td>
</tr>
<tr>
<td>L</td>
<td>31.75</td>
<td>cm</td>
<td>(= 12.5 in)</td>
</tr>
<tr>
<td>R</td>
<td>15.24</td>
<td>cm</td>
<td>(= 6.0 in)</td>
</tr>
<tr>
<td>$E_{11}$</td>
<td>5.17104 x 10^6</td>
<td>N/cm(^2)</td>
<td>(= 7.5 x 10^6 psi)</td>
</tr>
<tr>
<td>$E_{22}$</td>
<td>2.41315 x 10^6</td>
<td>N/cm(^2)</td>
<td>(= 3.5 x 10^6 psi)</td>
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<tr>
<td>$\nu_{12}$</td>
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<td></td>
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<tr>
<td>G</td>
<td>8.61840 x 10^5</td>
<td>N/cm(^2)</td>
<td>(= 1.25 x 10^6 psi)</td>
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</tbody>
</table>

Note: $\nu_{21} = \nu_{12} E_{22} / E_{11}$

Cylindrical shells with 3 layers where the thickness of each layer is 0.012 inches.

---

Table 4. Geometric properties of Khot's\[^4\] boron-epoxy shells

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<tr>
<th>Property</th>
<th>Value</th>
<th>Unit</th>
<th>Conversion</th>
</tr>
</thead>
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<tr>
<td>$t_{\text{total}}$</td>
<td>9.144400 x 10^{-2}</td>
<td>cm</td>
<td>(= 0.036 in)</td>
</tr>
<tr>
<td>L</td>
<td>31.75</td>
<td>cm</td>
<td>(= 12.5 in)</td>
</tr>
<tr>
<td>R</td>
<td>15.24</td>
<td>cm</td>
<td>(= 6.0 in)</td>
</tr>
<tr>
<td>$E_{11}$</td>
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<td>(= 40 x 10^6 psi)</td>
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<tr>
<td>$E_{22}$</td>
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<td>(= 4.5 x 10^6 psi)</td>
</tr>
<tr>
<td>$\nu_{12}$</td>
<td>0.25</td>
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<td></td>
</tr>
<tr>
<td>G</td>
<td>1.03421 x 10^6</td>
<td>N/cm(^2)</td>
<td>(= 1.5 x 10^6 psi)</td>
</tr>
</tbody>
</table>

Note: $\nu_{21} = \nu_{12} E_{22} / E_{11}$

Cylindrical shells with 3 layers where the thickness of each layer is 0.012 inches.
Table 5. Comparison of buckling loads using Khot's\textsuperscript{4} glass-epoxy shell, case 1: $\theta$, $0$, $0^\circ$

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$N_x$ (lb/in)</th>
<th>$\tau_K$</th>
<th>$m$</th>
<th>$n$</th>
<th>$N_x$ (lb/in)</th>
<th>$\tau_K$</th>
<th>$m$</th>
<th>$n$</th>
<th>$N_x$ (lb/in)</th>
<th>$\tau_K$</th>
<th>$m$</th>
<th>$n$</th>
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<td>6</td>
<td>11</td>
<td>-491.599</td>
<td>0.</td>
<td>6</td>
<td>11</td>
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<td>6</td>
<td>11</td>
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<td>12</td>
<td>-499.162</td>
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<td>11</td>
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<tr>
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<td>0.0062</td>
<td>7</td>
<td>12</td>
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<td>12</td>
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<td>12</td>
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<td>-0.76</td>
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<tr>
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<td>0.</td>
<td>6</td>
<td>11</td>
<td>-480.818</td>
<td>0.</td>
<td>6</td>
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</table>
Table 6. Comparison of buckling loads using Khot's\textsuperscript{[4]} glass-epoxy shell, case 2: -θ, θ, 90°

<table>
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<tr>
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<th>$N_x$ (lb/in)</th>
<th>$\tau_K$</th>
<th>m</th>
<th>n</th>
<th>$N_x$ (lb/in)</th>
<th>$\tau_K$</th>
<th>m</th>
<th>n</th>
<th>$N_x$ (lb/in)</th>
<th>$\tau_K$</th>
<th>m</th>
<th>n</th>
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Table 8. Comparison of buckling loads using Khol'is\textsuperscript{[4]} boron-epoxy shell, case 2: \( \theta, 90^\circ \)

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>( N_x ) (lb/in)</th>
<th>( \tau_K )</th>
<th>m</th>
<th>n</th>
<th>( N_x ) (lb/in)</th>
<th>( \tau_K )</th>
<th>m</th>
<th>n</th>
<th>( N_x ) (lb/in)</th>
<th>( \tau_K )</th>
<th>m</th>
<th>n</th>
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<td>-776.067</td>
<td>0.</td>
<td>7</td>
<td>11</td>
<td>-776.067</td>
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<td>14</td>
<td>12</td>
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<td>11</td>
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Table 9. Comparison of postbuckling behavior using Khot's\textsuperscript{[4]} glass-epoxy shell, case 1: $-\theta$, $\theta$, $0^\circ$

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<tr>
<th>$\theta$</th>
<th>$b$</th>
<th>$K$</th>
<th>$\tilde{\theta}^*$</th>
<th>$b$</th>
<th>$K$</th>
<th>$\tilde{\theta}^*$</th>
<th>$b$</th>
<th>$K$</th>
<th>$\tilde{\theta}^*$</th>
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Note: $\tan \tilde{\theta}^* = K/(K+1)$
Table 10. Comparison of postbuckling behavior using Khot’s \(^4\) glass-epoxy, case 2: \(-\theta, \theta, 90^\circ\)

<table>
<thead>
<tr>
<th>(\theta)</th>
<th>(b)</th>
<th>(K)</th>
<th>(\tilde{\theta}^*)</th>
<th>(b)</th>
<th>(K)</th>
<th>(\tilde{\theta}^*)</th>
<th>(b)</th>
<th>(K)</th>
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Note: \(\tan \tilde{\theta}^* = \frac{K}{K+1}\)
Table 11. Comparison of postbuckling behavior using Khot’s \cite{4} boron-epoxy shell, case 1: -θ, θ, 0°

<table>
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<th>K</th>
<th>θ*</th>
<th>b</th>
<th>K</th>
<th>θ*</th>
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Note: \( \tan \theta^* = \frac{K}{K+1} \)
Table 12. Comparison of postbuckling behavior using Khot’s [4] boron-epoxy shell, case 2: -θ, θ, 90°

<table>
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<th>Reference [4]</th>
<th>Present work (using $\lambda_{\text{min}}$)</th>
<th>Present work ($\tau_K = 0.$)</th>
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Note: $\tan \ddot{\theta}^* = K/(K+1)$
Fig. 1 Notation and sign convention
Fig. 2 Definition of stress- and moment resultants
Fig. 3  Nondimensional load versus additional displacement relationships
Fig. 4  Generalized "Load-Shortening" curve
Fig. 5  Buckling and imperfection-sensitivity of simply supported, axially stiffened cylinders under axial compression[5] (light stiffening)
Fig. 6  Buckling of simply supported, axially stiffened cylinders under axial compression (light stiffening)
Fig. 7. Imperfection sensitivity of simply supported, axially stiffened cylinders under axial compression (light stiffening)
Fig. 8 Buckling and imperfection-sensitivity of simply supported, axially stiffened cylinders under axial compression [5] (medium stiffening)
Fig. 9  Buckling of simply supported, axially stiffened cylinders under axial compression (medium stiffening)
Fig. 10 Imperfection sensitivity of simply supported, axially stiffened cylinders under axial compression (medium stiffening)
Fig. 11 Buckling and imperfection sensitivity of simply supported, axially stiffened cylinders under axial compression [5] (heavy stiffening)
Fig. 12  Buckling of simply supported, axially stiffened cylinders under axial compression (heavy stiffening)
Fig. 13 Imperfection sensitivity of simply supported, axially stiffened cylinders under axial compression (heavy stiffening)
Fig. 14  Buckling and imperfection-sensitivity of simply supported, axially stiffened cylinders under hydrostatic pressure[5]
Fig. 15  Buckling of simply supported, axially stiffened cylinders under hydrostatic pressure (light stiffening)
Fig. 16  Imperfection sensitivity of simply supported, axially stiffened cylinders under hydrostatic pressure (light stiffening)
Fig. 17  Buckling of simply supported, axially stiffened cylinders under hydrostatic pressure (medium stiffening)
Fig. 18 Imperfection sensitivity of simply supported, axially stiffened cylinders under hydrostatic pressure (medium stiffening)
Fig. 19  Buckling of simply supported, axially stiffened cylinders under hydrostatic pressure (heavy stiffening)
Fig. 20 Imperfection sensitivity of simply supported, axially stiffened cylinders under hydrostatic pressure (heavy stiffening)
Fig. 21 Buckling and imperfection sensitivity of simply supported, ring stiffened cylinders under hydrostatic pressure (light stiffening)
Fig. 22  Buckling of simply supported, ring stiffened cylinders under hydrostatic pressure (light stiffening)
Fig. 23  Imperfection sensitivity of simply supported, ring stiffened cylinders under hydrostatic pressure (light stiffening)
Fig. 24  Classical buckling of simply supported, ring stiffened cylinders under hydrostatic pressure [5]
Fig. 25 Classical buckling of simply supported, ring stiffened cylinders under hydrostatic pressure (light stiffening)
Fig. 26 Classical buckling of simply supported, ring stiffened cylinders under hydrostatic pressure (medium stiffening)
Fig. 27  Classical buckling of simply supported, ring stiffened cylinders under hydrostatic pressure (heavy stiffening)
Fig. 28 Effect of axisymmetric imperfections on axially compressed cylinders with outside axial stiffeners\cite{5}
Fig. 29 Effect of axisymmetric imperfections on axially compressed cylinders with outside axial stiffeners.
Fig. 30 Effect of axisymmetric imperfections on axial buckling of ring stiffened cylinders [5]
Fig. 31  Effect of axisymmetric imperfections on axially compressed ring stiffened cylinders (light stiffening)
Fig. 32 Effect of axisymmetric imperfections on axially compressed ring stiffened cylinders (medium stiffening)
Fig. 33 Effect of axisymmetric imperfections on hydrostatic buckling of ring stiffened cylinders in the Z-independent range [5]
Fig. 34  Effect of axisymmetric imperfections on hydrostatic buckling of ring stiffened cylinders in the Z-independent range (light stiffening)
Fig. 35 Effect of axisymmetric imperfections on hydrostatic buckling of ring stiffened cylinders in the Z-independent range (medium stiffening)
Fig. 36  Effect of fiber orientation on the postbuckling coefficient $b$ [4]
Fig. 37  Effect of fiber orientation on the buckling load $\lambda_C$ and on the postbuckling coefficient $b$.
(Glass epoxy; case 1: $-\theta, \theta, 0^\circ$; $\tau_K$ is variable)
Fig. 38  Effect of fiber orientation on the buckling load $\lambda_c$ and on the postbuckling coefficient $b_c$.
(Glass epoxy; case 1: $-\theta, \theta, 0^\circ; \tau_K = 0$)
Fig. 39  Effect of fiber orientation on the buckling load $\lambda$ and on the postbuckling coefficient $b$. (Glass epoxy; case 2: $-\theta, \theta, 90^\circ$; $\tau_k$ is variable)
Fig. 40  Effect of fiber orientation on the buckling load $\lambda_c$ and on the postbuckling coefficient $b$.
(Glass epoxy; case 2: $\theta$, $0^\circ$, $90^\circ$; $\tau_K = 0$)
Fig. 41 Effect of fiber orientation on the buckling load $\lambda$, and on the postbuckling coefficient $b$. (Boron epoxy; case 1: $\theta$, $\theta$, $0^\circ$; $\tau_k$ is variable).
Fig. 42  Effect of fiber orientation on the buckling load \( \lambda \) and on the postbuckling coefficient \( b \).
(Boron epoxy; case 1: \(-\theta, \theta, 0^\circ; \tau_K = 0\))
Fig. 43  Effect of fiber orientation on the buckling load $\lambda$, and the postbuckling coefficient $b$. (Boron epoxy; case 2: $\theta, 90^\circ, \tau_K$ is variable)
Fig. 44 Effect of fiber orientation on the buckling load $\lambda_c$ and on the postbuckling coefficient b. (Boron epoxy; case 2: $-\theta$, $\theta$, $90^\circ$; $\tau_K = 0$)
Khot's tau vs lambda (perfect shell)

\[ \text{min norm load} = 0.54 \text{ for } k=1, l=12 \text{ and } \tau = -0.76 \]

Fig. 45 Variation of the buckling load \( \lambda_{mn\tau} \) with Khot's skewness parameter \( \tau_K \) for \( \theta = 40^\circ \).

(Glass epoxy; case 1: -40°, 40°, 0°)
Fig. 46  Contour plot of the lowest buckling mode ($\lambda_c = 0.540706$ at $\tau_K = -0.76, n=12$)  
(Glass epoxy; case 1: $-40^\circ, 40^\circ, 0^\circ$)
components of buckling mode, axial trace at y=0.00 deg
- shell with 1 asymmetric imperfection -

buckling mode, axial trace at y=0.00 deg
- shell with 1 asymmetric imperfection -

Fig. 47  Axial traces of the lowest buckling mode ($\lambda_c = 0.540706$ at $\tau_k = -0.76$, n=12)
(Glass epoxy; case 1: $-40^\circ, 40^\circ, 0^\circ$)
components of post-buckling mode, axial trace at \( y = 0.00 \) deg
- shell with 1 asymmetric imperfection -

\[
\begin{align*}
W_2(x, y) & \\
\end{align*}
\]

post-buckling mode, axial trace at \( y = 0.00 \) deg
- shell with 1 asymmetric imperfection -

\[
\begin{align*}
\end{align*}
\]

Fig. 48 Axial traces of the postbuckling modes (\( \lambda_c = 0.540706 \) at \( \tau_K = -0.76, n=12 \))
(Glass epoxy; case 1: -40°, 40°, 0°)
Fig. 49 Contour plot of the buckling mode at $\tau_K = -0.12$ ($\lambda_{6,12}, \tau = 0.561107$)
(Glass epoxy, case 1: $-40^\circ, 40^\circ, 0^\circ$)
components of buckling mode, axial trace at $y=0.00$ deg
- shell with 1 asymmetric imperfection -

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![Graph of components of buckling mode](image)

buckling mode, axial trace at $y=0.00$ deg
- shell with 1 asymmetric imperfection -

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<th>W(x,y)</th>
</tr>
</thead>
</table>
```

![Graph of buckling mode](image)

Fig. 50  Axial traces of the buckling mode at $\tau = -0.12 (\lambda_{6,12,\tau} = 0.561107)$
(Glass epoxy; case 1: $-40^\circ, 40^\circ, 0^\circ$)
Fig. 51  Axial traces of the postbuckling modes at $\tau_{K} = -0.12$ ($\lambda_{6,12,\tau} = 0.561107$)  
(Glass epoxy; case 1: -40°, 40°, 0°)
Contour plot of the buckling mode at $\tau_r = +0.76$ ($\lambda_{11}, \tau = 0.593021$)
(Glass epoxy; case 1: $-40^\circ, 40^\circ, 0^\circ$)
components of buckling mode, axial trace at y=0.00 deg
shell with 1 asymmetric imperfection

buckling mode, axial trace at y=0.00 deg
shell with 1 asymmetric imperfection

Fig. 53  Axial traces of the buckling mode at $\tau_K = +76$ ($\lambda_{1,11}, \tau = 0.593021$)
(Glass epoxy; case 1: $-40^\circ, 40^\circ, 0^\circ$)
components of post–buckling mode, axial trace at $y=0.00$ deg
- shell with 1 asymmetric imperfection -

post–buckling mode, axial trace at $y=0.00$ deg
- shell with 1 asymmetric imperfection -

Fig. 54  Axial traces of the postbuckling modes at $\tau_K = +0.76$ ($\lambda_{1,11,\tau} = 0.593021$)
(Glass epoxy; case 1: -40°, 40°, 0°)
Fig. 55  Variation of the buckling load $\lambda_{mn}$ with Khot's skewedness parameter $\tau_K$ for $\theta = 40^\circ$.

(Glass Epoxy; case 2: -40°, 40°, 90°)
iso-contour plot of buckling mode
- shell with 1 asymmetric imperfection -

Contour plot of the lowest buckling mode, $\alpha = 0.59830$ at $T = 1.28$, $m=12$

Axial coordinate $x$, C[0.0, 12.50]

Circumferential coordinate $y$, $[0.0 \leq y \leq 2\pi R]$

Contour levels:
- $1 = -1.00$
- $2 = -0.80$
- $3 = -0.60$
- $4 = -0.40$
- $5 = -0.20$
- $6 = 0.00$
- $7 = 0.20$
- $8 = 0.40$
- $9 = 0.60$
- $10 = 0.80$
- $11 = 1.00$
components of buckling mode, axial trace at $y=0.00$ deg
shell with 1 asymmetric imperfection

--- $w_1 / 1.99$ --- $w_2 / 1.98$

buckling mode, axial trace at $y=0.00$ deg
shell with 1 asymmetric imperfection

--- $w_1(0,y)$ ---

Fig. 57 Axial traces of the lowest buckling mode ($\lambda_c = 0.539920$ at $\tau_K = -1.28$, $n=12$)
(Glass epoxy; case 2: $-40^\circ, 40^\circ, 90^\circ$)
Fig. 58 Axial traces of the postbuckling modes ($\lambda_c = 0.539920$ at $\tau_K = -1.28$, $n=12$) (Glass epoxy; case 2: $-40^\circ$, $40^\circ$, $90^\circ$)
iso-contour for buckling mode at k=10, l=12 and tau=-0.16
- shell with 1 asymmetric imperfection -

contour levels
1 -1.00
2 -0.80
3 -0.60
4 -0.40
5 -0.20
6 0.00
7 0.20
8 0.40
9 0.60
10 0.80
11 1.00

Fig. 59
Contour plot of the buckling mode at k=0.16, l=10, z=0.555850

circ. coordinate y [0. <= y <= 2 pi R]

axial coordinate x

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components of buckling mode, axial trace at $y=0.00$ deg
- shell with 1 asymmetric imperfection -

Fig. 60  Axial traces of the buckling mode at $\tau = -0.16$ ($\lambda_{10,12,1} = 0.555810$)
(Glass epoxy; case 2: -40°, 40°, 90°)
components of post-buckling mode, axial trace at $y=0.00$ deg
- shell with 1 asymmetric imperfection -

\[ \begin{align*}
\text{axial coord} x & \quad [0 \leq x \leq L] \\
\text{graph showing} & \\
\end{align*} \]

post-buckling mode, axial trace at $y=0.00$ deg
- shell with 1 asymmetric imperfection -

\[ \begin{align*}
\text{axial coord} x & \quad [0 \leq x \leq L] \\
\text{graph showing} & \\
\end{align*} \]

Fig. 61 Axial traces of the postbuckling modes at $\tau_0 = -0.16$ ($\lambda_{10,12} = 0.555810$)
(Glass epoxy; case 2: $-40^\circ, 40^\circ, 90^\circ$)
Contour plot of the buckling mode at $\tau = 0.6$ ($\lambda_{8,10,\tau} = 0.572088$) (Glass epoxy; case 2: $-40^\circ, 40^\circ, 90^\circ$)
components of buckling mode, axial trace at $y=0.00$ deg
- shell with 1 asymmetric imperfection -

---

Fig. 63 Axial traces of the buckling mode at $\tau_K = +0.6$ ($\lambda_{8,10,1} = 0.572088$)
(Glass Epoxy; case 2: -40°, 40°, 90°)
components of post-buckling mode, axial trace at $y=0.00$ deg
- shell with 1 asymmetric imperfection -

post-buckling mode, axial trace at $y=0.00$ deg
- shell with 1 asymmetric imperfection -

Fig. 64  Axial traces of the postbuckling modes at $\tau_0 = +0.6, (\lambda_{8,10,\tau} = 0.572088)$
(Glass Epoxy; case 2: $-40^\circ, 40^\circ, 90^\circ$)
Fig. 65. Variation of the buckling load $\lambda_{mnt}$ with Khot’s skewedness parameter $\tau_K$ for $\theta = 40^\circ$. (Boron epoxy; case 1: $-40^\circ$, $40^\circ$, $0^\circ$)
Fig. 66  Contour plot of the lowest buckling load ($\lambda_c = 0.232429$ at $\tau_k = -0.68$, $n=15$)
(Boron epoxy; case 1: -40°, 40°, 0°)
components of buckling mode, axial trace at $y=0.00$ deg
- shell with 1 asymmetric imperfection -

buckling mode, axial trace at $y=0.00$ deg
- shell with 1 asymmetric imperfection -

Fig. 67 Axial traces of the lowest buckling mode ($\lambda_c = 0.232429$ at $\tau_K = -0.68$, n=15)
(Boron epoxy; case 1: $-40^\circ$, $40^\circ$, $0^\circ$)
components of post-buckling mode, axial trace at \( y=0.00 \) deg
- shell with 1 asymmetric imperfection -

post-buckling mode, axial trace at \( y=0.00 \) deg
- shell with 1 asymmetric imperfection -

Fig. 68  Axial traces of the postbuckling modes (\( \lambda_c = 0.232429 \) at \( \tau_K = -0.68, n=15 \))
(Boron epoxy; case 1: \(-40^\circ, 40^\circ, 0^\circ\))
Fig. 69  Contour plot of the buckling mode at $\tau_K = -0.08$ ($\lambda_{5,13}, \tau = 0.253818$) 
(Boron epoxy; case 1: $-40^\circ, 40^\circ, 0^\circ$)
Fig. 70  Axial traces of the buckling mode at $\tau_0 = -0.08$ ($\lambda_{5,13,\tau} = 0.253818$) 
(Boron epoxy; case 1: $-40^\circ, 40^\circ, 0^\circ$)
components of post-buckling mode, axial trace at $y=0.00$ deg
- shell with 1 asymmetric imperfection -

post-buckling mode, axial trace at $y=0.00$ deg
- shell with 1 asymmetric imperfection -

Fig. 71  Axial traces of the postbuckling modes at $\tau_k = -0.08$ ($\lambda_{5,13,\tau} = 0.253818$)
(Boron epoxy; case 1: $-40^\circ, 40^\circ, 0^\circ$)
Fig. 72  Contour plot of the buckling mode at \( \tau_\nu = +0.56 \) (\( \lambda_{1,12}, \tau = 0.282014 \))
(Boron epoxy; case 1: \( -40^\circ, 40^\circ, 0^\circ \))
components of buckling mode, axial trace at \( y = 0.00 \) deg
- shell with 1 asymmetric imperfection -

\[
\begin{align*}
\text{axial coord } x & \quad [0. \leq x \leq L] \\
\end{align*}
\]

buckling mode, axial trace at \( y = 0.00 \) deg
- shell with 1 asymmetric imperfection -

\[
\begin{align*}
\text{axial coord } x & \quad [0. \leq x \leq L] \\
\end{align*}
\]

Fig. 73  Axial traces of the buckling mode at \( \tau = +0.56 \) \( (\lambda, \tau) = 0.282014 \)
(Boron epoxy; case 1: \( -40^\circ, 40^\circ, 0^\circ \))
components of post-buckling mode, axial trace at $y=0.00$ deg
- shell with 1 asymmetric imperfection -

post-buckling mode, axial trace at $y=0.00$ deg
- shell with 1 asymmetric imperfection -

Fig. 74 Axial traces of the postbuckling modes at $\tau = 0.56$ ($\lambda_{1,12,1} = 0.282014$)
(Boron epoxy; case 1: $-40^\circ$, $40^\circ$, $0^\circ$)
Khot's tau vs lambda (perfect shell)

min norm load = 0.23 for k=1, l=11 and tau=1.92

Fig. 75 Variation of the buckling load $\lambda_{mn\theta}$ with Khot's skewedness parameter $\tau_K$ for $\theta = 40^\circ$

(Boron epoxy; case 2: -40°, 40°, 90°)
iso-contour for buckling mode at $k=1$, $l=11$ and $\tau=1.92$
- perfect shell -

Contour plot of the lowest buckling load ($\lambda_1 = 0.227115$ at $k=4.92, n=1$)

Contour levels
1 - 1.00
2 - 0.80
3 - 0.60
4 - 0.40
5 - 0.20
6 0.00
7 0.20
8 0.40
9 0.60
10 0.80
11 1.00

circ. coordinate $y$ [$0. \leq y \leq 2\pi R$]
components of buckling mode, axial trace at $y=0.00$ deg  
- shell with 1 asymmetric imperfection -

buckling mode, axial trace at $y=0.00$ deg  
- shell with 1 asymmetric imperfection -

Fig. 77 Axial traces of the lowest buckling mode ($\lambda_c = 0.227115$ at $\tau_K = +1.92$, $n=11$)  
(Boron epoxy; case 2: $-40^\circ, 40^\circ, 90^\circ$)
components of post-buckling mode, axial trace at \( y = 0.00 \) deg
- shell with 1 asymmetric imperfection -

post-buckling mode, axial trace at \( y = 0.00 \) deg
- shell with 1 asymmetric imperfection -

Fig. 78 Axial traces of the postbuckling modes (\( \lambda_c = 0.227115 \) at \( \tau_K = +1.92 \), \( n=11 \))
(Boron epoxy; case 2: -40°, 40°, 90°)
iso-contour for buckling mode at $k=1$, $l=15$ and $\tau = -1.28$
perfect shell

Contour levels
1 -1.00
2 -0.80
3 -0.60
4 -0.40
5 -0.20
6 0.00
7 0.20
8 0.40
9 0.60
10 0.80
11 1.00

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components of buckling mode, axial trace at $y=0.00$ deg
- shell with 1 asymmetric imperfection -

Fig. 80 Axial traces of the buckling mode at $\tau_\nu = -1.28$ ($\lambda_{1,15} = 0.228279$)
(Boron epoxy; case 2: $-40^\circ, 40^\circ, 90^\circ$)
components of post-buckling mode, axial trace at $y=0.00$ deg
- shell with 1 asymmetric imperfection -

---

post-buckling mode, axial trace at $y=0.00$ deg
- shell with 1 asymmetric imperfection -

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Fig. 81  Axial traces of the postbuckling modes at $\tau_1 = -1.28$ ($\lambda_{1,15}, \tau = 0.228279$)
(Boron epoxy; case 2: $-40^\circ, 40^\circ, 90^\circ$)
Fig. 82  Contour plot of the buckling mode at $\tau_1 = +0.04$ ($\lambda_{13,13}, \tau = 0.236147$)
(Boron epoxy; case 2: $-40^\circ, 40^\circ, 90^\circ$)
components of buckling mode, axial trace at $y=0.00$ deg
- shell with 1 asymmetric imperfection -

Fig. 83  Axial traces of the buckling mode at $\tau_k = +0.04$ \((\lambda_{13,13}, \tau = 0.236147)\)
(Boron epoxy; case 2: -40°, 40°, 90°K)
components of post–buckling mode, axial trace at $y=0.00$ deg
– shell with 1 asymmetric imperfection –

post–buckling mode, axial trace at $y=0.00$ deg
– shell with 1 asymmetric imperfection –

Fig. 84 Axial traces of the postbuckling modes at $\tau_K = +0.04$ ($\lambda_{13,13}, \tau = 0.236147$)
(Boron epoxy; case 2: $-40^\circ, 40^\circ, 90^\circ$)