A MIXED FINITE ELEMENT METHOD FOR COMPUTING THREE-DIMENSIONAL TIME-DOMAIN ELECTROMAGNETIC FIELDS IN STRONGLY INHOMOGENEOUS MEDIA

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Abstract

A mixed finite-element method is presented for the computer modeling of three-dimensional time-dependent (transient) electromagnetic fields in electrically and/or magnetically strongly inhomogeneous time-invariant media. In the spatial domain of computation both Edge and Cartesian elements are used for the electric field strength as well as the magnetic field strength and, for both expansions, the corresponding program decides locally what type of element to be used for obtaining the best results both as regards computational efficiency and final accuracy. Both implicit and explicit methods can be used for carrying out the integration along the time axis.

Introduction

In the present paper an accurate finite-element method for solving the time-dependent electromagnetic field equations in strongly inhomogeneous media is presented. By formulating the problem in terms of the electric and/or the magnetic field strength in stead of (vector)potentials we can avoid the loss of accuracy that results from the differentiations required for computing the electric or the magnetic field strength from these potentials. This approach has the additional advantage of avoiding problems due to multiple connectedness and the choice of gauges for ensuring uniqueness, that are met when using potentials [1,2,3]. In our method the electric and the magnetic field strengths are computed simultaneously, methods of this type are referred to as mixed methods [4]. This method was chosen to be able to treat the electric and the magnetic field strength in a symmetric way for obtaining the best overall accuracy. When using the finite-element method for the computer modeling of three-dimensional electromagnetic fields in strongly inhomogeneous media, it is necessary to use elements that automatically account for the continuity of the tangential field components across interfaces where the relevant constitutive coefficients jump, and allow for a jump in the normal components [5,6]. Elements satisfying these conditions are referred to as Edge elements, the classical elements that make all field components continuous across interfaces, and that are accurate only in homogeneous media, are referred to as Cartesian elements. In [6] a finite-element method for computing time-harmonic electromagnetic fields in strongly inhomogeneous media was presented that uses both Edge and Cartesian elements for obtaining optimum results both as regards computational efficiency and as regards accuracy.

As regards the discretization in space, the method presented in this paper was developed along similar lines as the one described in [6]. It uses computationally optimum combinations of Cartesian and Edge elements for the expansion of both the electric and the magnetic field strengths. The optimum expansions are obtained by letting the program decide, for each combination of two adjacent domains and for both the expansion of the electric field strength and the expansion of the magnetic field strength, which type of element will, locally, be used. For small differences in electromagnetic medium properties it will use Cartesian elements. For large differences in the electrical properties of the media, Edge elements will be used for the expansion of the electric field strength, similarly, for large differences in the magnetic prop-

erties of the media, Edge elements will be used for the expansion of the magnetic field strength. The degree of inhomogeneity above which Edge elements are used, i.e. a strong inhomogeneity, is user-defined and depends on the final accuracy the user aims at.

Using the method of weighted residuals, systems of coupled first-order ordinary differential equations are derived that can be used for computing the evolution in time of the expansion coefficients by either using an implicit or an explicit method.

Cartesian expansion functions and Edge expansion functions

The geometrical domain in which the finite-element method is applied, is subdivided into adjoining tetrahedra. To specify a position in space, we employ the coordinates $\{x, y, z\}$ with respect to the Cartesian reference frame with origin 0 and three mutually perpendicular base vectors $\{i_x=i_1, i_y=i_2, i_z=i_3\}$ of unit length each. The time coordinate is denoted by t .

The position vectors of the vertices of a particular tetrahedron T are $\{r_0, r_1, r_2, r_3\}$; the outwardly directed vectorial areas of the faces of T are $\{A_0, A_1, A_2, A_3\}$ [5] and the volume of T is denoted by V . Let r_b be the position vector of the barycenter of T , then the linear functions ϕ_i that equal unity when $r = r_i$ ($i=0,1,2,3$) and are zero at the remaining three vertices of T can be expressed as

$$\phi_i(r) = 1/4 - (r - r_b) \cdot A_i / 3V. \quad (1)$$

Introducing the vectors $V_j^{(C)}$ and $V_j^{(E)}$ through $V_j^{(C)} = i_j$ ($j=1,2,3$), and $V_j^{(E)} = -A_j / 3V$ ($j=0,1,2,3$), the

local Cartesian expansion functions $W_{i,j}^{(C)}$ are given by

$$W_{i,j}^{(C)}(r) = \phi_i(r) V_j^{(C)}, \quad i=0,1,2,3, \quad j=1,2,3, \quad (2)$$

and local the Edge expansion functions $W_{i,j}^{(E)}$ are given by

$$W_{i,j}^{(E)}(r) = \phi_i(r) a_{i,j} V_j^{(E)}, \quad i,j=0,1,2,3, \quad i \neq j, \quad (3)$$

where the factor $a_{i,j} = |r_i - r_j|$, denoting the length of the edge joining r_i and r_j , was introduced and for

making $W_{i,j}^{(E)}$ dimensionless and for normalizing it for improving the condition of the final system of equations. Note that both the Cartesian and the Edge expansion functions are consistently linear functions of all spatial coordinates. Consistently linear expansion functions are preferred to expansion functions that are not consistently linear [7,8] because they yield more accurate results with less computational effort [5].

When $r \in T$, the electric field strength E and the magnetic field strength H are locally expanded as

$$E(\mathbf{r}, t) = \sum_{i=0}^3 \sum_j e_{i,j}(t) \mathbf{w}_{i,j}^{(C,E)}(\mathbf{r}), \quad (4)$$

$$H(\mathbf{r}, t) = \sum_{i=0}^3 \sum_j h_{i,j}(t) \mathbf{w}_{i,j}^{(C,E)}(\mathbf{r}), \quad (5)$$

where $e_{i,j}(t)$ and $h_{i,j}(t)$ are time dependent expansion coefficients. In (4) and (5) the summation over j and j' runs over values that depend on the type of element. Since, in general, the materials in a given configuration will, as regards their electric and magnetic properties, not have the same degree, or kind, of inhomogeneity, the distribution of Cartesian and Edge elements used for the expansion of the electric field strength will differ from the distribution used for the expansion of the magnetic field strength. In our notation we have stressed this difference by using primed subscripts for the expansion of the magnetic field strength. Note, however, that both expansions are related to the same subdivision of the domain of computation into tetrahedra.

The expansion functions given above are all of the local type, i.e. they apply to one tetrahedron only. We now introduce the global expansion functions as the functions that consist of all local expansion functions that are related to the same expansion coefficient. Global Cartesian expansion functions therefore consist of all local Cartesian expansion functions that are related to a Cartesian component in a common vertex of a number of adjoining tetrahedra (the simplicial star [9] of this vertex). Global Edge expansion functions consist of all local Edge expansion functions that are related to an Edge expansion coefficient related to a common edge of a number of adjoining tetrahedra (the simplicial star of this edge). From this it follows that the simplicial star (the support) of global Cartesian expansion functions is much larger than the simplicial star of global Edge expansion functions; this fact causes Edge expansion functions to yield more unknowns which makes them computationally less efficient. This larger number of unknowns may, however, be compensated by the increased sparsity of the system matrices, which is another consequence of difference in the support of the global expansion functions.

The system of differential equations in the expansion coefficients

The time-domain electromagnetic field equations are written in the form

$$\partial_t \epsilon E + \sigma E - \nabla \times H = -J^e, \quad (6)$$

$$\partial_t \mu H + \nabla \times E = -K^e, \quad (7)$$

where J^e and K^e denote external sources of electric and magnetic current, respectively. The system of differential equations in the expansion coefficients is obtained by applying to (6) and (7) the method of weighted residuals, where the weighting functions

$\mathbf{w}_{p,q}^{(C,E)}(\mathbf{r})$, that are used for weighting (6), and

$\mathbf{w}_{p',q'}^{(C,E)}(\mathbf{r})$, that are used for weighting (7), are taken equal to the corresponding expansion functions for E and H , respectively. In this way we obtain

$$\sum_{\alpha} \{ \partial_t (e_{\alpha} \int_D \mathbf{w}_{\beta} \cdot \epsilon \mathbf{w}_{\alpha} dV) + e_{\alpha} \int_D \mathbf{w}_{\beta} \cdot \sigma \mathbf{w}_{\alpha} dV \} - \sum_{\alpha} h_{\alpha} \cdot \left\{ \int_D \mathbf{w}_{\beta} \cdot \nabla \times \mathbf{w}_{\alpha} dV \right\} = - \int_D \mathbf{w}_{\beta} \cdot J^e dV \quad \forall \beta, \quad (8)$$

$$\sum_{\alpha} \{ \partial_t (h_{\alpha} \cdot \int_D \mathbf{w}_{\beta} \cdot \mu \mathbf{w}_{\alpha} dV) \} + \sum_{\alpha} e_{\alpha} \left\{ \int_D \mathbf{w}_{\beta} \cdot \nabla \times \mathbf{w}_{\alpha} dV \right\} = - \int_D \mathbf{w}_{\beta} \cdot K^e dV \quad \forall \beta', \quad (9)$$

where all double summations have, for brevity, been replaced by a single summation sign, and where the double subscripts, indicating the relevant local expansion, have been replaced by a single greek subscript related to the global expansions (α and α') and the global weighting (β and β'). Assuming the medium properties to be constant in the interior of each tetrahedron and assuming the external source distributions to vary linearly in the interior of each tetrahedron, all spatial integrations implied in (8) and (9) can be carried out analytically [10]. For linear media, the resulting system of equations can, in matrix form, be written as

$$\mathbf{M}_{\epsilon} \partial_t \mathbf{e} + \mathbf{M}_{\sigma} \mathbf{e} - \mathbf{M}_H \mathbf{h} = -\mathbf{j}^e, \quad (10)$$

$$\mathbf{M}_{\mu} \partial_t \mathbf{h} + \mathbf{M}_E \mathbf{e} = -\mathbf{k}^e. \quad (11)$$

Note that $\mathbf{M}_{\epsilon}$, $\mathbf{M}_{\sigma}$ and $\mathbf{M}_{\mu}$ are square matrices, whereas $\mathbf{M}_H$ and $\mathbf{M}_E$ are, in general, not square. Together with the appropriate initial and boundary conditions, (10) and (11) are a coupled system of ordinary linear differential equations from which the evolution in time of the expansion coefficients can be obtained by, for instance, using a single step time marching scheme [11]. In a first attempt to solve the electromagnetic-field problem, however, we have replaced the above fully implicit formulation by a simpler explicit one.

Explicit method of solution

For obtaining an explicit formulation of the problem it is general practice to use a so-called lumping procedure [12]. Lumping implies choosing a lower-order integration rule for evaluating the spatial integrations so as to obtain approximate matrices that are diagonal because of which the formulation becomes explicit (in our case a diagonalization of the matrices $\mathbf{M}_{\epsilon}$, $\mathbf{M}_{\sigma}$ and $\mathbf{M}_{\mu}$ is required). Rather than choosing lower-order integration rules we will obtain the diagonalization by using modified weighting functions

$\mathbf{w}_{i,j}^{(C,E)}(\mathbf{r})$. These modified weighting functions are

chosen as $\mathbf{w}_{i,j}^{(C)} = \delta(\mathbf{r}_i) \mathbf{v}_j^{(C)}$ and $\mathbf{w}_{i,j}^{(E)} = \delta(\mathbf{r}_i) (\mathbf{r}_i - \mathbf{r}_j) / a_{i,j}$, where $\delta(\mathbf{r}_i)$ denotes a spatial Dirac delta function and where $\mathbf{r}_i$ is taken as the limit towards

the vertex i , taken from the inside of the relevant tetrahedron. The integrations in (8) and (9) can be evaluated exactly by using the definition of the spatial Dirac delta function and an explicit formulation of the problem is obtained. Our approach in deriving an explicit formulation can be interpreted as replacing the Galerkin-type weighting of the approximated form of (6) and (7) over the relevant simplicial stars by collocation, i.e. by requiring the approximated form of these equations to be satisfied exactly at, or in the immediate vicinity of, the relevant vertices. The explicit integration in time is carried out by evaluating $\mathbf{e}$ at $t = t_0 + n\Delta t$, $n = 1, 2, \dots$ and evaluating $\mathbf{h}$ at $t = t_0 + (n-1/2)\Delta t$. The differentiations with respect to time can be carried out as central differences and we obtain a result that can be written as

$$\mathbf{e}_{n+1} = (\mathbf{M}_e + \Delta t \mathbf{M}_e/2)^{-1} ((\mathbf{M}_e - \Delta t \mathbf{M}_e/2) \mathbf{e}_n + \Delta t (\mathbf{M}_h \mathbf{h}_{n+1/2} - \mathbf{j}_{n+1/2}^e)), \quad (12)$$

$$\mathbf{h}_{n+1/2} = \mathbf{h}_{n-1/2} - \Delta t \mathbf{M}_h^{-1} (\mathbf{M}_e \mathbf{e}_n + \mathbf{k}_n^e). \quad (13)$$

Note that the initial conditions $\mathbf{e}_0$ and $\mathbf{h}_{-1/2}$ are required at $t=t_0$ and $t=t_0-\Delta t$, respectively.

Numerical results

To demonstrate the accuracy of our method, we have applied the FEMAXT code, developed by us using the methods described in the present paper, to a problem for which the solution is known analytically. This problem was first solved by using (12) and (13) under the condition that Edge expansion functions are used for the expansion of the electric field strength when the relative contrast in the numerical values of ϵ and/or σ in two adjacent tetrahedra exceeds 10%. Similarly, Edge expansion functions are used for the expansion of the magnetic field strength when the relative contrast in the numerical values of μ in two adjacent tetrahedra exceeds 10%. Cartesian elements are used when contrasts less than 10% are found. For comparison, the same problem is also solved by using Cartesian elements only. The configuration used for testing the method is the cubic, source-free, region D $\{-0.5 < x < 0.5\text{m}, 0 < y < 1\text{m}, 0 < z < 1\text{m}\}$ consisting of two homogeneous parts with different medium properties viz. D_1 $\{-0.5 < x < 0\text{m}, 0 < y < 1\text{m}, 0 < z < 1\text{m}\}$ with the properties of a vacuum $\{\epsilon_1=\epsilon_0, \mu_1=\mu_0, \sigma_1=0\}$ and $D_2=D \setminus D_1$ with the medium properties $\{\epsilon_2=10\epsilon_0, \mu_2=\mu_0, \sigma_2=0.001\text{S/m}\}$ (this choice applies to the realistic case of a plane interface between a lossy dielectric and a vacuum). The magnetic field strength in the configuration is chosen as a time-harmonic plane wave in D_1 , that has its magnetic field polarized along the z -axis. Because of the discontinuity at $x=0$, we also have a reflected wave in D_1 and a transmitted wave in D_2 . A field of this type can, using complex arithmetic, be represented as

$$\begin{aligned} H_x &= 0, \quad H_y = 0, \\ H_z &= \text{Re}(\exp(j\omega t - \gamma^i \cdot \mathbf{r}) + R \exp(j\omega t + \gamma^r \cdot \mathbf{r})) \\ &\text{when } x < 0, \text{ and} \\ H_z &= \text{Re}(T \exp(j\omega t - \gamma^t \cdot \mathbf{r})) \\ &\text{when } x \geq 0, \end{aligned} \quad (14)$$

where $\text{Re}(\)$ indicates that the real part has to be taken. Writing $\gamma^i = \gamma^i \mathbf{s}^i$, where $\mathbf{s}^i$ denotes the unit vector in the direction of propagation of the incident wave, we have chosen $\mathbf{s}_x^i = \mathbf{s}_y^i = 2^{-1/2}$ and $\mathbf{s}_z^i = 0$. With this choice, (14) represents a plane wave in D_1 travelling in the x - y plane and having an angle of incidence of 45° on the interface. Using (6) and (7) together with the continuity conditions at $x=0$, expressions for the unknown quantities in (14) can be obtained easily. In the same way expressions for E_x and E_y can be obtained, $E_z = 0$. Due to the discontinuity in the medium properties E_x will be discontinuous at $x=0$, the remaining field components being continuous throughout the domain of computation. The field is assumed to have a frequency $f = 5 \cdot 10^7 \text{ Hz}$ ($\omega = 2\pi f$).

In all computations the mesh consists of $8 \times 8 \times 8$ cubes of equal size that are each subdivided into 6 tetrahedra. The computations are carried out for the time interval $t_0 = 0 \leq t \leq 2.5 \cdot 10^{-9} \text{ s} = t_{\text{end}}$, using 50

steps in time ($\Delta t = 0.5 \cdot 10^{-10} \text{ s}$). The initial conditions for the magnetic field strength at $t = -\Delta t/2$ are taken from (14), the initial conditions for the electric field strength at $t = 0$ are computed using analytical expressions for the electric field strength. The latter expressions are also used for prescribing the boundary conditions for the electric field strength at the outer boundary of the domain of computation, the magnetic field strength at the outer boundary remains unspecified.

In Fig. 1 the local relative error in the solution at $t = t_{\text{end}}$ is plotted for a given cross-section ($z = 0.5001$) of the configuration, and for the case Edge elements are used along the plane of discontinuity. In Fig. 2 results are plotted for exactly the same problem that was now solved using Cartesian elements only. In the results presented, the local relative error in the solution is defined as the absolute value of the error at a given point in the domain of computation and at a certain moment of time, divided by the maximum absolute value of the exact solution anywhere in the domain of computation and at the same moment of time. In Table 1 the root mean square (RMS) error of all field components are presented for the same problems. The RMS error of, for instance, E_x is defined as

$$\text{RMS}(E_x) = \left(\frac{\int_S |E_{x,\text{num}} - E_{x,\text{exact}}|^2 dA}{\int_S |E_{x,\text{exact}}|^2 dA} \right)^{1/2} \cdot 100\%, \quad (15)$$

where S denotes the cross-section of the configuration mentioned above. As regards the computation times for the two problems, we note that they were about the same, the effects of the larger number of unknowns that are required when using Edge elements being eliminated by the effects of the increased sparsity of the matrices. From the plots it is observed that, contrary to Edge elements, Cartesian elements produce extremely large errors in a region near the interface of discontinuity. From other results it was observed that the width of the region in which the results show large errors increases in time. The fact that the use of Edge elements greatly improves the accuracy is confirmed by the results for the RMS-errors given in Table 1. Regarding the stability of the present explicit method we note that it is not unconditionally stable. Numerical experiments show that Edge elements are more sensitive to instabilities than Cartesian elements. When problems have to be solved for large intervals of time, it will be necessary to use an implicit method for solving (10) and (11), in order to cope with the stability problem.

All computations have been carried out at a VAX-Station 2000, using the SEPRAN finite element package [13].

Table 1. Relative errors in % when using Cartesian elements together with Edge elements or using Cartesian elements only.

Elements used	RMS-error in %					
	E_x	E_y	E_z	H_x	H_y	H_z
Cart.+ Edge	2.20	3.38	0.38	0.45	0.59	3.30
Cart. only	40.7	4.07	0.04	0.12	0.03	3.98

Conclusion

We have shown that a mixed finite element method that uses Edge elements together with Cartesian elements for computing three-dimensional time-domain electromagnetic fields in strongly inhomogeneous media, yields results that are, as regards accuracy, superior to results that are obtained using Cartesian elements only. This conclusion does not only apply to the simple configuration for which we have tested the method, but also to configurations containing arbitrary inhomogeneities. The reason for this is that in the method a decision is made, for each combination of adjacent tetrahedra, what type of elements should be used for obtaining accurate results. Thus the ability of the method to model arbitrary inhomogeneities is limited only by the ability of the mesh-generator used to accurately model the configuration geometrically. As regards stability an implicit method is required for being able to solve problems of this type for large intervals of time, for relatively short intervals of time the explicit method can be employed. Finally we note that our method for computing electric and/or magnetic field strengths is more accurate than the often used (vector)potential methods because, for computing the electric and/or the magnetic field strength from these potentials, the latter require a numerical differentiation that causes a large loss of accuracy in the final results.

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Fig.1. Contour plot of the local relative error in E_x , Edge elements, $RMS(E_x) = 2.2\%$, 4137 unknowns.

Fig.2. Contour plot of the local relative error in E_x , Cartesian elements, $RMS(E_x) = 40.7\%$, 3510 unknowns.