Stellingen
behorende bij het proefschrift van Paul J.Th. Venhovens getiteld:
"Optimal Control of Vehicle Suspensions"

1. De benaming lineaire optimale regeltheorie wekt ten onrechte de suggestie dat de berekende regelaar optimaal is. De ontworpen regelaar is alleen optimaal met betrekking tot de vooraf gekozen weegfactoren. Een verkeerde keuze van weegfactoren hoeft dus niet tot een optimale regelaar te leiden.

2. De relatie tussen dynamische bandkrachtvariaties en het wegedrag van een voertuig op een oneffen wegdek is niet zo eenduidig zoals vaak in de literatuur wordt aangenomen.

3. Actieve vering zoals dit tot voor kort in menig formule 1 auto werd toegepast, wordt vaak gezien als een technisch hoogstandje. Deze veronderstelling is echter onterecht daar het systeem slechts een constante bodemvrijheid beoogt en dus als een relatief eenvoudige hoogteregeling van de carrosserie gezien kan worden.

4. Voertuigmodellen met veel graden van vrijheid (> 20) welke vaak op een eenvoudige wijze met behulp van multi-body software gemodelleerd kunnen worden, zijn slechts van beperkte praktische betekenis daar het vrijwel onmogelijk is het model te valideren en alle parameters te identificeren.

5. De kwaliteit van presentaties op wetenschappelijke symposia is vaak omgekeerd evenredig met het aantal getoonde formules en grafieken.
6. 'Lean production' van personenauto's zoals dit bij uitstek in Japan gebeurt, wordt vaak geassocieerd met een grote produktvariëteit en korte ontwerp-cycli. De toenemende onderlinge gelijkenis van Japanse middenklasse auto's spreekt deze associatie tegen.

7. Het feit dat de meeste auto's voor de Europese markt niet aan de strenge Amerikaanse normen voor passieve veiligheid voldoen, geeft aan dat de Europese Gemeenschap het niet zo nauw neemt met de veiligheid van haar burgers op de weg.

8. Verstevigingsbalken in de portieren van een auto zijn vaak meer van een marketing strategisch belang dan dat ze bijdragen tot de veiligheid bij een zijdelingse aanrijding. De balken in de portieren vervullen geen enkele functie als de totale carrosseriestructuur niet opgewassen is tegen een zijdelingse aanrijding.

9. De objectiviteit van autotests in programma's van commerciële omroepen moet men in twijfel trekken zolang het betreffende autoprogramma gesponsord wordt door de importeurs van de testvoertuigen.

10. De verkeersborden langs de Nederlandse autosnelwegen welke weggebruikers attenderen op hun snelheid, brengen de verkeersveiligheid in gevaar daar ze de bestuurders afleiden mede door het kleine formaat van de letters en het geringe onderscheid in de gebruikte kleuren.

11. De prijs van magnetische opslagmedia (zoals harddisks) zou meer dan evenredig met de opslagcapaciteit moeten zijn om ervoor te zorgen dat software producenten meer effort steken in het minimaliseren van hun pakketten.
OPTIMAL CONTROL OF VEHICLE SUSPENSIONS

Paul J.Th. Venhovens
OPTIMAL CONTROL OF VEHICLE SUSPENSIONS

Proefschrift

ter verkrijging van de graad van doctor aan de Technische Universiteit Delft, op gezag van de Rector Magnificus, prof.ir. K.F. Wakker, in het openbaar te verdedigen ten overstaan van een commissie aangewezen door het College van Dekanen op 7 februari 1994 te 14:00 uur

door

Paulus Johannes Theodorus Venhovens,

geboren te Roosendaal,
werktuigbouwkundig ingenieur.
Optimal Control of Vehicle Suspensions /
Paulus Johannes Theodorus Venhovens.
Delft: Delft University of Technology, Faculty of Mechanical Engineering
and Marine Technology. -Ill.
Thesis Technische Universiteit Delft. - With ref. - With summary in Dutch.
Subjects headings: vehicle suspension ; optimal control.

Copyright © by Paul Venhovens, 1993.

All rights reserved. No part of this thesis may be reproduced, stored in a retrieval
system, or transmitted, in any form or by any means, electronic, mechanical,
photocopying, recording, or otherwise, without the prior written permission of the
author.

The author makes no warranty, that the methods, calculations and data in this book
are free from error. The application of the methods and results is at the user's risk
and the author disclaims all liability for damages, whether direct, incidental or
consequential, arising from such application or from any other use of this book.
To my mother
In memory of my father
ACKNOWLEDGEMENTS

First of all I would like to express my gratitude to NedCar Engineering and Development B.V. for their scientific and financial support during the preparation of my doctorate thesis. Although their position changed in an unfavorable way due to the economic recession, I appreciate it very much that the people of NedCar and in particular Hans Fun maintained supporting my doctorate study.

During this study I had the pleasure and privilege of interacting with many persons from the Delft University of Technology. It is a pleasure to thank Doctor Ton van der Weiden and Professor Okko Bosgra from the Laboratory of Measurement and Control for their advice in the field of control system engineering. Many thanks also to Professor Hans Pacejka and Doctor Arvin
Acknowledgments

Savkoor from the Vehicle Research Laboratory, each of them has given me much useful criticism and suggestions. I had also a very good feedback from my fellow post doctorate student Albert van der Knaap. I appreciated his help very much when he checked the equations of motion of the full-vehicle model and tested the preliminary versions of my simulation software. Together we were also engaged in the practical realization of a newly developed active suspension system. Despite the many disappointments due to hardware failures, we both enjoyed the trail runs in our laboratory and on the road very much. I do not want to express my gratefulness to Mr. Murphy. During the test work I have been challenged by his famous laws too often. Fortunately, my perseverance got the better of his laws.

I am also very grateful to all the members of the scientific staff, the post graduate students and other employes of the Vehicle Research Laboratory who have supported me during the preparation of my thesis. Their thoughtful criticisms, suggestions and proof-reading have added greatly to this thesis and to the pleasure writing it. Although I attempted to be self supporting as much as I could, occasionally, I had to make an appeal to their knowledge and skills.

Finally, I would like to thank my mother and brother for their understanding and non-technical support. I have appreciated their patience and encouragements very much when I spent many evenings and weekends with typing this thesis and programming controllers on my notebook computer.

Delft, a chilly winter day, December 6, 1993

Paul Venhovens
## CONTENTS

Notation ................................................................................................................... vii
Introduction ............................................................................................................... 1

ONE DIMENSIONAL RIDE MODEL

1 Modeling a Quarter-Car System ........................................................................... 15
   1.1 Vehicle Model .................................................................................................. 16
   1.2 Road Model .................................................................................................... 18
2 Suspension Design Criteria ..................................................................................... 23
   2.1 Stochastic responses ....................................................................................... 24
   2.2 Transient responses ......................................................................................... 33
3 Active Suspension Control using Linear Gaussian Control ......................37
   3.1 Full-State Feedback Control ........................................44
   3.2 Limited State Feedback Control ...................................48
       3.2.1 Active Damping ...............................................50
       3.2.2 Passive Damping ..............................................54
   3.3 Closed-Loop Stability Analysis .....................................56
   3.4 Full-state versus Limited State Feedback Control .................59
4 Semi-Active Suspension Control ..................................................67
   4.1 Ride Comfort Control ...............................................69
   4.2 Road Holding Control ...............................................89
   4.3 The Influence of the Damper Range on the Performance ..........94
5 Suspension Control with Adaptive Properties .................................103
6 Implementation of Suspension Control using a Kalman filter .............115
   6.1 An Introduction to Discrete-Time Systems .............................117
   6.2 Kalman Filter Design for an Active Suspension System ...........118
   6.3 Kalman Filter Design for a Semi-Active Suspension System .......121
   6.4 Kalman Filter Design using Multi-Model Techniques ...............124
   6.5 Tuning the Kalman Filter ...........................................126

THREE DIMENSIONAL RIDE AND HANDLING MODEL

7 Modeling a Full-Vehicle System ...............................................137
   7.1 Vehicle Model .......................................................138
       7.1.1 The Suspension ...............................................140
       7.1.2 The Powertrain ...............................................142
       7.1.3 The Tires .....................................................143
   7.2 Road Model ..........................................................153
       7.2.1 Two Track Road Model .......................................153
       7.2.2 Pure Time Delay Approximation ...............................158
   7.3 Driver Model ........................................................164
   7.4 Natural Frequencies and Mode Shapes ................................167
8 Additional Criteria for Full-Vehicle Suspension Design ....................177
   8.1 Perception of ride ..................................................178
       8.1.1 Evaluation of the Passive Vehicle Ride ....................181
   8.2 Assessment of handling ............................................188
       8.2.1 Evaluation of the Passive Vehicle Handling ...............193
9 Active Suspension Control using Linear Optimal Control

9.1 State Feedback Control ........................................... 207
  9.1.1 Identical Left and Right Road Track .................... 210
    9.1.1.1 Identical Front and Rear Wheel Input .............. 211
    9.1.1.2 Uncorrelated Front and Rear Wheel Input .......... 212
    9.1.1.3 Delayed Rear Wheel Input .......................... 213
  9.1.2 Correlated Left and Right Road Track .................. 213
    9.1.2.1 Uncorrelated Front and Rear Wheel Input .......... 214
    9.1.2.2 Delayed Rear Wheel Input ........................... 215
  9.1.3 Uncorrelated Left and Right Road Track ................. 216
  9.1.4 Evaluation of the Active Vehicle Ride and Handling ... 216

9.2 Output Feedback Control ....................................... 238
  9.2.1 Active Damping .............................................. 240
  9.2.2 Partially Coupled Active Damping ....................... 243
  9.2.3 Decoupled Active Damping .................................. 246
  9.2.4 Passive Damping ............................................ 250
  9.2.5 Evaluation of the Active Vehicle Ride and Handling ... 252

10 Suspension Control with Adaptive Properties ................. 271

11 Kalman filter Design for an Active Suspension System ........ 279
  11.1 Selection of the Internal Model ......................... 280
  11.2 Selection of the Measurement Signals .................... 288
  11.3 Discrete-Time Kalman Filter .............................. 294
  11.4 Tuning the Kalman Filter ................................... 296

Conclusions and Recommendations .................................. 325

Appendix A: Equations of Motion of a Full-Vehicle Model ........ 331

Bibliography .................................................................. 357

Summary ......................................................................... 363

Samenvatting .................................................................. 369

Biography ....................................................................... 377
# NOTATION

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Unit</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>[rad/m]</td>
<td>Road unevenness parameter</td>
</tr>
<tr>
<td>$A$</td>
<td></td>
<td>Continuous-time system matrix</td>
</tr>
<tr>
<td>$B$</td>
<td></td>
<td>Continuous-time control input matrix</td>
</tr>
<tr>
<td>$c$</td>
<td>[N/m]</td>
<td>Stiffness</td>
</tr>
<tr>
<td>$C$</td>
<td></td>
<td>Output matrix</td>
</tr>
<tr>
<td>$C_{pa}$</td>
<td>[N/deg]</td>
<td>Cornering stiffness</td>
</tr>
<tr>
<td>$C_{xx}$</td>
<td></td>
<td>Auto covariance function of $x$</td>
</tr>
<tr>
<td>$d$</td>
<td>[m]</td>
<td>Deformation</td>
</tr>
<tr>
<td>$D$</td>
<td></td>
<td>Direct transmission matrix</td>
</tr>
<tr>
<td>$e$</td>
<td></td>
<td>Error</td>
</tr>
<tr>
<td>$E$</td>
<td></td>
<td>Expectation</td>
</tr>
<tr>
<td>$f$</td>
<td>[1/s]</td>
<td>Frequency</td>
</tr>
<tr>
<td>Notation</td>
<td>Symbol</td>
<td>Unit</td>
</tr>
<tr>
<td>----------</td>
<td>--------</td>
<td>------</td>
</tr>
<tr>
<td>$F$</td>
<td>$[N]$</td>
<td></td>
</tr>
<tr>
<td>$g$</td>
<td>$[m/s^2]$</td>
<td></td>
</tr>
<tr>
<td>$G$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$h$</td>
<td>$[m]$</td>
<td></td>
</tr>
<tr>
<td>$H_{xy}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$I$</td>
<td>$[kgm^2]$</td>
<td></td>
</tr>
<tr>
<td>$J$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$k$</td>
<td>$[Ns/m]$</td>
<td></td>
</tr>
<tr>
<td>$K$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$l$</td>
<td>$[m]$</td>
<td></td>
</tr>
<tr>
<td>$m$</td>
<td>$[kg]$</td>
<td></td>
</tr>
<tr>
<td>$N$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$q$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$Q$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$R$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$s$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$S$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$S_{xx}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$S_{xy}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$t$</td>
<td>$[s]$</td>
<td></td>
</tr>
<tr>
<td>$T$</td>
<td>$[kgm^2/s^2]$</td>
<td></td>
</tr>
<tr>
<td>$u$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$v$</td>
<td>$[m/s]$</td>
<td></td>
</tr>
<tr>
<td>$v$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$V$</td>
<td>$[kgm^2/s^2]$</td>
<td></td>
</tr>
<tr>
<td>$w$</td>
<td>$[m]$</td>
<td></td>
</tr>
<tr>
<td>$w$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$W$</td>
<td>$[kgm^2/s^2]$</td>
<td></td>
</tr>
<tr>
<td>$W_c$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$W_o$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$x$</td>
<td>$[m]$</td>
<td></td>
</tr>
<tr>
<td>$x$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$y$</td>
<td>$[m]$</td>
<td></td>
</tr>
<tr>
<td>$y$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$z$</td>
<td>$[m]$</td>
<td></td>
</tr>
<tr>
<td>$\alpha$</td>
<td>$[rad]$</td>
<td></td>
</tr>
<tr>
<td>$\beta$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\gamma$</td>
<td>$[rad]$</td>
<td></td>
</tr>
<tr>
<td>$\Gamma$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
\( \delta \) \quad \text{Dirac delta function, Steering angle}

\( \delta t \) [s] \quad \text{Sampling period}

\( \varepsilon \) \quad \text{Error}

\( \kappa \) \quad \text{Damping ratio}

\( \lambda \) \quad \text{Eigenvalue}

\( \mu \) \quad \text{Mean value}

\( \sigma^2 \) \quad \text{Variance}

\( \phi \) [rad] \quad \text{Roll angle}

\( \Phi \) \quad \text{Discrete-time system matrix}

\( \theta \) [rad] \quad \text{Pitch angle}

\( \psi \) [rad] \quad \text{Yaw angle}

\( \Psi \) \quad \text{Discrete-time disturbance input matrix}

\( \tau \) [s] \quad \text{Time constant}

\( \omega \) [rad/s] \quad \text{Radial frequency}

\( \Omega \) [rad/m] \quad \text{Angular frequency / wave number}

\textbf{Indices}

\( a \) \quad \text{Axle, Actuator}

\( a_1 \) \quad \text{Front axle}

\( a_2 \) \quad \text{Rear axle}

\( act \) \quad \text{Actuator}

\( adp \) \quad \text{Adaptive}

\( b \) \quad \text{Body}

\( c \) \quad \text{Control}

\( cd \) \quad \text{Discrete control}

\( d \) \quad \text{Driver}

\( dyn \) \quad \text{Dynamic}

\( e \) \quad \text{Engine}

\( f \) \quad \text{Kalman filter}

\( fd \) \quad \text{Discrete Kalman filter}

\( i \) \quad \text{Input}

\( o \) \quad \text{Output}

\( p \) \quad \text{Panhard rod}

\( r \) \quad \text{Road}

\( s \) \quad \text{Suspension}

\( s_1 \) \quad \text{Front suspension}

\( s_2 \) \quad \text{Rear suspension}

\( sad \) \quad \text{Semi-active damper}

\( sky \) \quad \text{Skyhook}
Notation

\[\begin{array}{ll}
stat & \text{Static} \\
 t & \text{Tire} \\
v & \text{Vehicle} \\
1R & \text{Front axle, right} \\
1L & \text{Front axle, left} \\
2R & \text{Rear axle, right} \\
2L & \text{Rear axle, left}
\end{array}\]

Abbreviations

- CSAD: Continuously variable Semi-Active Damper
- det: Determinant
- diag: Vector containing the diagonal elements of a matrix
- DOF: Degrees Of Freedom
- DSAD: Discretely variable Semi-Active Damper
- DLQOFB: Discrete Linear Quadratic Output Feedback
- FFT: Fast Fourier Transform
- ISO: International Organization for Standardization
- LQE: Linear Quadratic Estimator
- LQG: Linear Quadratic Gaussian
- LQR: Linear Quadratic Regulator
- LQOFB: Linear Quadratic Output Feedback
- MS: Mean Square
- RMS: Root Mean Square
- SAD: Semi-Active Damping
- tr: Trace
INTRODUCTION

The design of automotive suspension systems involves a number of compromises. In its simplest terms, the conflict is between a suspension that must appear soft to obtain a good level of ride comfort and one that must appear firm to control vehicle attitude changes and maintain good tire to ground contact. Contemporary vehicle suspensions predominantly contain passive elements, such as springs and dampers. They have evolved to the point at which it seems reasonable to suppose that they will not improve much without principle changes. Recently, such changes in principle have become available commercially with advances in force generators, sensors and micro-processors.
Introduction

A good suspension design for automotive application requires an effective compromise between the following quantifiable performance issues:

- Passenger ride comfort,
- Tire/ground contact force variations,
- Suspension working space,
- Vehicle attitude control.

Furthermore capital cost, power consumption, reliability, maintainability, durability, component weight and noise transmission play an important role in the design of a suspension system. The attitude aspect (roll and pitch behavior) is of key importance in indicating the handling behavior. A soft suspension which provides good isolation also allows significant body roll which is undesirable for good handling. Since most road vehicles do not operate with a single well-defined roadway input, real passive suspensions are rarely optimal for any particular road and speed. On smooth roads, the suspension will be stiffer than necessary and on rough roads softer than desirable.

The demand for improving the vehicle ride comfort can be dated back to the fifties, when the first application of active suspension control was applied by Féderspiel-Labrosse [10]. Since then, many vehicle and suspension component manufactures have been and are still being involved in developing advanced suspension systems. Unfortunately, much of these activities have not yet been made public. The aim of these kinds of advanced suspensions is mainly focussed on improving ride comfort of the passengers without harming the road holding too much. Mostly, these systems involve some kind of adaptation based on monitoring the different driving conditions. Many of these suspensions are based on adjustable shock absorber systems, sometimes combined with automatic leveling. However, systems like a fully active suspension are currently not produced in large series mainly due to technical, theoretical and economical problems. A few Japanese car manufactures offer a slow active suspension system as an option to some of their domestic top-models.

Recently, another application of advanced suspensions can be observed in the field of racing cars. For racing cars, aerodynamic downforces, that depend on maintaining small clearances between the underside of the car and the road surface, are crucial with respect to the generation of longitudinal and lateral tire forces. The aerodynamic downforces depend on the speed of travel and can range up to three times the total weight of the vehicle. To maintain a constant clearance, passive suspensions would have to be very stiff in order to be able

- 2 -
to cope with the downforces. However, a stiff setup is very unfavorable for the
driver ride comfort and the control of the wheel-load variations due to road
unevennesses. Mostly, these kinds of vehicles with a very stiff suspension tend
to bounce heavily while crossing road irregularities. In this application the
active suspensions have a lot to offer. Basically, the actuator serves merely as
a leveling device to compensate for the suspension deflections due to
downforces. The passive suspension can now be tuned for optimal road holding
rather than ground clearance control. However, the driver comfort is still of
minor importance in the racing car context.

Passenger and racing cars are not the only example in this field of application
of advanced suspension systems. Other areas of application are heavy trucks,
cab suspensions, driver seats and even suspensions of railway coaches.
Generally, the aim of applying suspension control coincides with the aim in
the automotive application. In the heavy truck application the reduction of the
road damage due to excessive tire load variations is also of considerable
interest.

For the reader's sake the various types of suspension systems may be
classified as follows:

- **Passive Suspension.** The conventional suspensions involving passive
  spring and damper elements with linear, non-variable rates are well
  known. In practice, with a conventional non-adjustable passive system,
  the stiffness and damping parameters are chosen on the basis of a
  compromise between the ride comfort and the road holding over a wide
  range of road/speed conditions. A possible addition is a self-levelling
  system which involves time delays of many seconds and which is
designed to compensate for variations in static load only.

- **Adaptive Suspension.** Adaptive systems are characterized by a slow
  change of the control parameters (e.g. damping, stiffness) according to
  changing circumstances, like road condition, loading or vehicle speed.
  For a given road and at a given vehicle speed, suspension parameters
  may be changed from those optimizing road holding, in order to enhance
  ride comfort, thereby permitting a slight worsening of the road holding
  capability. This may be done in several ways. It is possible to modify
  both damping and elastic elements. Changing the dampers does not
  require a new suspension design and is not attended with high costs;
  actually conventional springs should be used in order to ensure
  adequate attitude control. The modification of elastic elements implying
  for example air spring adaptation is more complex. During acceleration,
braking and cornering the safety of the car becomes more critical. Under these circumstances road holding can be improved at the cost of ride comfort to ensure the highest degree of safety.

- **Active Suspension.** The fully active suspension system involves replacing the conventional suspension elements with a (hydraulic) actuator that is controlled by a high frequency response servo valve. The force demand signal, typically generated in a microprocessor, is governed by a control law that is normally obtained by application of various forms of the optimal control theory. In order to obtain good performance it is necessary that the actuator control bandwidth extends to substantially beyond the wheel-hop natural frequency (typically 10-15 Hz). Achieving a reasonable bandwidth is not easy and the higher the bandwidth is made the more the power consumption of the system becomes. Beyond the actuator bandwidth noise transmission is likely to be a problem unless some significant flexibility is added in series with the strut. Required force levels, and consequently either the actuator size or the hydraulic supply pressure, can be reduced by relieving the strut of the static load of the car body. This can be done by incorporating a spring parallel to the actuator. Practical active suspension systems are likely to be electro-hydraulically powered. These systems are inherently costly, involving a number of precision engineering components, and they are likely to be noisy. Slow-active suspension systems will be thought of as active suspension systems as described before. In such a system an actuator acts in series with a passive spring. The crucial, practical implication of this design is that the actuator need to have only a limited frequency response capability - typically up to 3 Hz. This control bandwidth must embrace the normal range of sprung mass resonant frequencies in bounce, pitch and roll, but does not extend as far as the wheel-hop natural frequency. Generally, the actuator is placed in series with a spring and parallel to a damper. It requires to track a force demand signal within the limitations of its frequency response capabilities. In the higher frequency range the actuator itself becomes virtually rigid and the passive elements control the suspension. The force levels demanded from the actuator in this design are high because the static weight of the vehicle must be supported by the actuator. In most cases the actuator is integrated with various forms of air or hydropneumatic suspension systems.

- **Semi-Active Suspension.** Semi-active suspensions fill the gap between fully active and purely passive systems. The idea of the fully active system can be modified so that the actuator is only capable of
dissipating power rather than supplying it as well. The semi-active suspension system is characterized by a rapidly adjustable damper parallel with a spring which supports the static load. Strictly speaking the semi-active damper is a passive device, since it can only dissipate energy. Nevertheless semi-active systems may offer performances that are nearly as good as the vibrational properties that may be achieved with fully active systems. The alteration of the damping can be achieved by controlling the oil flow path area in a conventional shock absorber. Hardware requirements are considerable less with respect to those of fully active systems with clear savings in terms of hydraulic pumps, accumulators, filters, pipework, oil reservoirs and coolers. The semi-active suspension requires virtually no external power but instead provides high bandwidth control over damping forces which are generated passively. The only required external energy source is one associated with valve activation. It is in general very small in comparison with the total energy dissipated by the damper or the energy needed for a fully active system. The adjustable damper system must be contrasted with adaptive (damper) systems that are currently available in several production vehicles. These systems typically involve strategies such as switching to a soft damper setting for straight (and smooth/moderate roughness) road running and switching rapidly to firm for handling maneuvers and possibly rough roads or discrete events like potholes or speed humps. The semi-active damper would probably also need to include additional features such as switching to its firm setting for handling maneuvers. From a practical viewpoint, however, the semi-active system places a bigger demand on the damper hardware by virtue of its much greater number of switching cycles.

The major task of the automotive suspension engineers is to design a suspension system that provides optimal ride comfort and road holding for every given operating condition of the vehicle. This goal can not be pursued by suspensions with fixed spring and damper characteristics because of the conflicting properties. The potential for an intelligent suspension which can do better in relation to this compromise, is now clear. Besides a better performance for a given road and speed, the controlled suspension is able to adapt to the particular running condition, which offers another potential source of improvement.

This thesis will not deal with the design of hardware required for advanced suspension control. Various kinds of force actuators are currently available
ranging from hydraulic actuators to electric motors and adjustable shock absorbers. This thesis is mainly focussed on the design of controllers for active and semi-active suspension systems for a passenger car application. However, the theory of advanced suspension control as dealt with in this thesis is not only restricted to a passenger car application. It can also be used for other applications as mentioned above.

The different types of controllers known from the literature and used in intelligent suspension systems can roughly be divided into:

- **Linear Optimal Control.** Linear Quadratic Gaussian control (LQG) has been widely used for active suspension control. The theory requires that the system input is white noise, that the performance index to be minimized by the optimal control will be of a quadratic form, and, in its simplest form, that all the state variables are measurable. Typically the performance index will be a weighted sum of mean square values of output variables including body accelerations (ride comfort), wheel to body displacement, dynamic tire load (road holding) and actuator force. Many different laws can be derived by changing the performance index weighting constants. Finding the optimal control depends on the solution of the non-linear matrix Riccati equation. In the mid-seventies Thompson [35] examined as one of the first researchers the application of linear optimal control in combination with full-state feedback and a suitable choice of weighting constants. Once the system and penalty function are derived it is very easy to examine the performance of the closed-loop system as a function of the weighting factors. The main objection of LQG control is that the solution depends entirely on the selection of the weighting factors. Results from the literature are therefore not easily comparable because different authors use dissimilar systems with different parameters and various weighting factors.

Since the use of linear characteristics such as an LQG solution calculated for one operating condition is quite restrictive, one might consider adjusting parameters in the feedback loop. In [13] an alternative to parameter adaptation has been proposed. Gordon et al. introduced a non-linear law that is based on the optimization of the control force input with respect to a non-quadratic cost function, followed by least-square fitting of polynomial coefficients. The non-quadratic cost function arises from the idea that e.g. suspension deflections under extreme situations (e.g. rough roads) should be weighted stronger than in case of a moderate road condition. If the suspension deflection reaches one of the limits, the non-linear LQG
controller will adjust the feedback gains in such a way that the suspension travel is bounded by virtually stiffening the suspension. The non-linear LQG method requires tremendous calculation effort and performance calculations can only be based on time-domain simulations.

- **Preview Control.** Track preview is based on measuring the road profile ahead of the front wheels. Preview acts upon a sudden road bump instead of re-acting only after the bump has passed the tires [11]. Preview control can also be applied to active suspensions at the rear wheels of a vehicle [16]. While the front actuators work as a feedback controller, the rear actuators provide both feedback and preview control actions. Using the look-ahead preview principle does not appear realistic for cars because the recognition of obstacles ahead of the vehicle is not clear. How can a (solid) brick be distinguished from a (compressible) milk package?

- **Predictive Control.** Predictive controllers are based on the prediction of the future behavior of the process to be controlled. These predictions are based on a model of the process that is assumed to be available. In [44] the theory of predictive control has been applied to a simplified vehicle model, better known as the quarter-car model. Unlike linear optimal control, predictive control is based on minimization of a cost function over a finite time. The control structure is based on the comparison of the predicted output of the vehicle model for a finite time with a desired trajectory. The controller that results from the minimization of the cost function is valid for a finite time. The research carried out by Ywema [44] showed that the suspension based on predictive control behaves like a discrete-time LQG controller when a stochastic road input is assumed. Tuning a predictive controller is rather time consuming because besides choosing the weighting factors in the cost function, also the prediction, control and minimum cost horizon must be chosen.

- **Adaptive Control.** The term adaptive system control in control theory represents a control system that monitors the performance of a system and adjusts the parameters of the controller in the direction of better performance. Adaptation of passive suspensions has been worked out by many authors [9,32]. A supervisor controls the passive stiffness and damping parameters as a function of external signals such as steering, braking or throttle changes and different vehicle speed and road conditions.
Adaptive control can also be used for linear optimal control type of controllers. The feedback gains are calculated off-line for various velocities and types of road inputs and are stored as a lock-up table in an on-board computer. The control gains are evaluated by optimal control assuming the vehicle as linear time-invariant. External signals are used for switching to different modes by the micro-processor. This approach assumes that the vehicle parameters are completely known. The switching rules known from the literature do not seem to form a part of any consistent underlying framework.

The second type of adaptation is based in the concept that a priori knowledge of complete system dynamics along with operating conditions are not known. The desired performance corresponding to an optimal internal reference model is represented as a desired control input. This input is made to follow a desired optimal control force. Model reference adaptive control (MRAC) is one of the categories in which most of these adaptive controllers come under [6,9]. MRAC is based on a primary controller (inner loop) to obtain a suitable closed-loop behavior as in non-adaptive control schemes. However, when the process parameters vary with time (e.g. the road condition), a fixed parameter setting for the primary controller, such that the closed-loop behavior is acceptable under all circumstances, cannot be found. In the MRAC technique, the desired process response to a command signal is specified by means of a parametrically defined reference model. An adaptation mechanism (outer feedback loop) keeps track of the process output and reference model output and calculates a suitable parameter setting such that the difference between these outputs tends to zero.

**Robust Control.** As an alternative to adaptive control, robust controller design is an increasingly important research area. Robust control is based on the design of a fixed controller which provides acceptable closed-loop behavior under all operating conditions of the vehicle. One important class of controllers uses minimization of frequency domain criteria as in the $H_{\infty}$ approach. Palkovics [29] examined the performance of an active suspension system based on robust control in combination with an uncertainty in the tire stiffness. The performance of advanced suspensions based on robust controllers seem to be poor to some extend compared with LQG and adaptive controllers. This is mainly caused by the rather conservative setup of the controller that must guarantee stability under many different operating conditions of the vehicle.
For automotive application it seemed useful to conduct a study on advanced suspension control. In this thesis the problem addressed is how to control a suspension system as the vehicle operates over a wide range of running conditions, i.e. speeds and road conditions. The main question is whether the various types of controllers as described before can enhance the performance of an automotive active suspension. In order to gain insight into the principles of advanced suspension control it is necessary to understand the shortcomings of the passive suspension system. After that the operation of the controlled suspension needs to be understood if the vehicle operates on a particular road with a constant speed. Finally, the operation needs to be evaluated if the speed and road condition vary.

It has been chosen to base the control system design on the theory of linear optimal control. Linear optimal control (LQG) offers the possibility to emphasize quantifiable issues like ride comfort or road holding very easily by altering the weighting factor of a quadratic criterion. Furthermore, LQG control requires no extensive computational power or time consuming simulations because all calculations can be made in an analytical way as long as the system is linear.

During the last few years many scientific reports dealing with suspension design have been published. Especially attention has been devoted to the theoretical study of the dynamic behavior of controlled (active and semi-active) suspension systems. It has been generally concluded that suspension system control may noticeably enhance performance. However, the effort required to implement such an intelligent suspension is often involved with tremendous computational power and hardware costs. Simple vehicle models have been exploited to establish an understanding of the relationship between design criteria and performance. However, several aspects of performance remain to be understood. One of them is the selection of the weighting factors and evaluation of performance of an active suspension system with a controller based on the theory of linear optimal control. Now, the following questions arise with respect to the application of LQG control in the vehicle suspension application. What is really the gain in performance of an active suspension system compared with a well-tuned passive suspension? What does it involve when the LQG controller is implemented in a real vehicle? Can the controller be simplified without too much loss in performance? Can the performance of adjustable damper systems compete with the quality of real active suspension systems?
Another issue of examination is the design and working of adaptation strategies when the road surface quality is changing. The application of an adaptation might boost the performance noticeably. Unlike the switching rules known from the literature, the adaptation as introduced in this thesis is based on the vertical tire load variations, since the dynamic tire load is strongly related to the handling potential. Further work appears to be justified to examine the relationship between road holding and handling. No results from the literature are yet available on the relation between the degradation of handling and the increase of tire/ground contact forces while the vehicle operates on an uneven road. Furthermore, bridging the gap between theory and practice is still an area of major importance and interest. Many studies from the literature are not feasible at this moment or even in the (near) future because the design of the proposed intelligent suspension is based on assumptions that can never be realized.

The analysis concerning suspension control has been spilt in two parts. The first part deals with suspension control based on a quarter-car model; in the second part the control theory has been applied to a full-vehicle model. The quarter-car analysis is very useful to gain insight into the problems related to suspension control. It makes the analysis and synthesis of the full-vehicle much easier to conduct. The quarter-car analysis is covered by chapter 1 to 6. It starts with a discussion of modeling aspects related to the vehicle and the road surface. In chapter 2 the performance criteria of a quarter-car with respect to suspension control will be treated in detail. Both stochastic and deterministic performance parameters are defined and illustrated. Chapter 3 deals with the control design based on the theory of linear optimal control. The design and evaluation of performance are carried out both for control using full-state feedback and that using only partial state feedback. The analysis is based on the assumption of ideal force generators. With the help of the insight gained in chapter 3, control strategies for semi-active suspension systems will be derived (chapter 4). The semi-active system is characterized by the possibility to control the passive suspension damper force. Separate control algorithms are proposed to improve road holding or ride comfort. Chapter 5 deals with an adaptive control structure. The algorithms derived previously for the controlled suspension system will be extended with an adaptive loop in order to be able to respond to changing speed and road conditions. The study of the quarter-car model ends with a discussion of ways of estimating unknown or not easily measurable states that are necessary for adaptive suspension control. The design of a Kalman filter will be illustrated on the basis of a semi-active suspension system.
The second part covers the full-vehicle application. Basically, all the aspects of the quarter-car analysis are repeated for the full-vehicle application: modeling (chapter 7), definition of additional design criteria (chapter 8), control system design based on LQG control (chapter 9), adding adaptive properties (chapter 10) and Kalman filter design (chapter 11). Throughout these chapters, special attention will be paid to the influence of suspension control on handling. In chapter 11 (Kalman filter design) the influence of the acceleration due to gravity on the distortion of the quality of estimation will be examined. Finally, some conclusions are drawn.
ONE DIMENSIONAL RIDE MODEL
Chapter 1 Modeling a Quarter-Car System

MODELING A QUARTER-CAR SYSTEM

The analysis based on a simplified vehicle will start with the modeling aspects. As stated in the introduction, the analysis based on the simplified vehicle model is very useful to gain insight into the problems related to suspension control. The first task is to find a suitable mathematical model of the car that represents all dynamic effects to be investigated and has as few undesirable side effects as possible. The equations of motion of this so-called quarter-car model form the basis for all further investigations. In the second section special attention will be paid to the modeling aspects with relation to stochastic road irregularities. A mathematical description of the road profile will be established.
1.1 VEHICLE MODEL

The most general and useful automotive suspension system design information derives from a single wheel station or quarter car. This quarter car model contains no representation of geometric effects of having four wheels and offers no possibility of studying longitudinal interconnections, the use of front suspension state information to improve the performance at the rear or resonant excitation of the body roll and pitch motion. It can not describe problems related to handling. However, it does appear to contain the most basic features of the real problem and gives rise to design thinking which accords with experience. Important features of the quarter car model are that it includes a proper representation of the problem of controlling wheel load variations and suspension working space. It is the simplest model which has these features and possesses particular advantages over more complex models in terms of

- Being described by few design parameters,
- Having few performance parameters,
- Having only a single input,
- Providing the ease of understanding the relationship between design and performance.

![Quarter-car model diagram](image)

**Figure 1.1** Quarter-car model.

Although the name 'quarter-car' suggests that any of the four corners of a vehicle can be represented well enough by a two-mass quarter-car model, this is not completely true. The dynamics of the front part of the vehicle can only be described partially with a two-mass model. Due to the presence of a powertrain suspended on elastic mounts, a three-mass quarter-car model would fit better to the front section. Therefore, the two-mass model should only be associated with the rear side of a vehicle (with exception of mid- and rear-engine vehicles).
The quarter-car model contains two vertical degrees of freedom: (1) the displacement of the unsprung mass or axle \( z_a \) and (2) the displacement of the sprung mass or body \( z_b \). The road input is denoted by \( z_r \) and the dynamic force in the tire by \( P_{\text{dyn}} \). The differential equations of a 2 degrees of freedom (DOF) suspension model of a quarter car according to figure 1.1 are given by

\[
\begin{align*}
    m_a \ddot{z}_a - u - c_s (z_b - z_a) + c_t (z_a - z_r) &= 0 \\
    m_b \ddot{z}_b + u + c_s (z_b - z_a) &= 0
\end{align*}
\]  

(1.1)

where \( m_a \) represents the unsprung or axle mass, \( m_b \) is the sprung or body mass, \( c_s \) is the suspension stiffness and \( c_t \) represents the tire stiffness. The symbol \( u \) denotes a force that may be generated by a passive damper, a semi-active damper or an active force generator. For instance, if the system is totally passive, then

\[
u = k_s (\dot{z}_b - \dot{z}_a)
\]

(1.2)

where \( k_s \) is the damping constant in (Ns/m).

The tire is represented by a spring. Since damping in the rolling tire is typically very small, it is neglected in this analysis. It is assumed that the tire behaves as a point-contact follower that is in contact with the road all the time. In reality the tire contacts the road over a finite length and envelops short wavelength irregularities. Enveloping will provide a significant filtering of the road profile input. In the speed and frequency ranges of interest it will be considered adequate to think of the tire making contact with the road at a single point.

In chapter 3 and 6 it will appear that a state-space representation of the two second-order differential equations is necessary. The equations of motion in matrix form are given by

\[
\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u} + G\mathbf{z}_r
\]

(1.3)

where \( \mathbf{x} \) is the state vector containing the four state variables

\[
\mathbf{x}^T = [z_a \ z_b \ \dot{z}_a \ \dot{z}_b]
\]

Matrix \( A \) represents the state-matrix, vector \( \mathbf{B} \) represents the input vector for the actuator force \( u \) and vector \( \mathbf{G} \) is the input for road displacement \( z_r \). They are given by
The numerical values of the parameters are included in table 1.1.

<table>
<thead>
<tr>
<th>description</th>
<th>symbol</th>
<th>unit</th>
<th>value</th>
</tr>
</thead>
<tbody>
<tr>
<td>axle mass</td>
<td>$m_a$</td>
<td>kg</td>
<td>33</td>
</tr>
<tr>
<td>body mass</td>
<td>$m_b$</td>
<td>kg</td>
<td>200</td>
</tr>
<tr>
<td>suspension stiffness</td>
<td>$c_s$</td>
<td>N/m</td>
<td>9,000</td>
</tr>
<tr>
<td>tire stiffness</td>
<td>$c_t$</td>
<td>N/m</td>
<td>200,000</td>
</tr>
</tbody>
</table>

Table 1.1 Quarter car model parameters.

The parameters correspond with the rear suspension of an experimental vehicle equipped with a hydropneumatic suspension system. The suspension stiffness has been reduced to about 50% compared with a similar vehicle with a suspension based on coil springs. Load leveling forms a part of the hydropneumatic suspension system.

1.2 ROAD MODEL

A vehicle is subjected to many disturbances. In this section the road irregularities are considered. Two types of road disturbances can be distinguished: stochastic (random) irregularities and deterministic disturbances. The random road unevennesses are characterized by their stochastic properties and can be regarded as disturbances of long or infinite duration. The deterministic road disturbances are mostly related to a single excitation of short duration such as kerbs, speed humps and potholes. An accurate representation of the stochastic road irregularities is required in order to predict vehicle responses to road excitations. The vertical road input $z_r$ can be modelled as colored noise resulting from the application of a first-order shaping filter to a white noise signal $w$. This process is given by the following differential equation

$$\ddot{z}_r + avz_r = w$$

(1.4)
where \( a \) (rad/m) is a coefficient depending on the shape of the road irregularities and \( v \) (m/s) the forward vehicle speed. The model defined by equation (1.4) has finite power, a virtue that a simple model of integrated white noise lacks. Measurements of road profiles have shown that the vertical road irregularities are approximately Gaussian distributed.

The Laplace transformation of the applied first-order filter leads to

\[
H_r(s) = \frac{1}{av + s}
\]

(1.5)

where \( s \) is the Laplace variable. The frequency response function of \( H_r(s) \) can be obtained by substituting \( s = j\omega \). The square magnitude is given by

\[
|H_r(\omega)|^2 = \frac{1}{a^2 \omega^2 + \omega^2}
\]

(1.6)

White noise \( w \) has a flat double sided power spectral density \( S_{ww}(\omega) \) (m²rad/s) for \( \omega \) from \(-\infty\) to \(+\infty\). \( S_{ww}(\omega) \) is defined by

\[
S_{ww}(\omega) = \frac{\sigma_w^2 av}{\pi}
\]

(1.7)

where \( \sigma_w^2 \) is the variance or mean square value of random variable \( z_r \) with zero mean \( (\mu_{z_r} = 0) \). White noise process \( w \) has zero mean \( (\mu_w = 0) \) and the intensity function \( C_{ww} \) is given by

\[
C_{ww}(\tau) = \mathbb{E}\left[ (w(t) - \mu_w) (w(t + \tau) - \mu_w) \right] = \int_{-\infty}^{\infty} S_{ww}(\omega)e^{j\omega \tau} d\omega = 2\pi S_{ww}(\omega) \delta(\tau) = 2\sigma_w^2 av \delta(\tau)
\]

(1.8)

where \( \delta \) denotes the dirac function. The double sided power spectral density function \( S_{z_r z_r}(\omega) \) (m²s/rad) of colored noise \( z_r \) is now given by

\[
S_{z_r z_r}(\omega) = |H_r(\omega)|^2 \cdot S_{ww}(\omega) = \frac{\sigma_r^2 av}{\pi} \cdot \frac{1}{a^2 v^2 + \omega^2}
\]

(1.9)

The variance of \( z_r \) can also be calculated by integrating the power spectral density \( S_{z_r z_r}(\omega) \) for \( \omega \) from \(-\infty\) to \(+\infty\)

\[
\sigma_r^2 = \mathbb{E}\left[ (z_r(t) - \mu_{z_r})^2 \right] = \int_{-\infty}^{\infty} S_{z_r z_r}(\omega)d\omega = \frac{\sigma_r^2}{\pi} \arctan \left( \frac{\omega}{av} \right)_{-\infty}^{+\infty} = \sigma_r^2
\]

(1.10)

It can be seen that the power spectral density as formulated in equation (1.9) depends on the vehicle speed \( v \). A velocity independent spectral density can be
established by introducing the angular frequency or wave number $\Omega$ with dimension (rad/m). $\Omega$ and $S(\Omega)$ (m$^3$/rad) are related to $\omega$ and $S(\omega)$ by

$$\Omega = \frac{\omega}{v} = \frac{2\pi f}{v} = \frac{2\pi}{L}$$

(1.11a)

$$S(\Omega) = S(\omega) \cdot v$$

(1.11b)

where $L$ represents the wavelength in (m). The velocity independent double sided power spectral density can be calculated by combining equations (1.9) and (1.11) to

$$S_{z_z}(\Omega) = \frac{\sigma^2 a}{\pi} \cdot \frac{1}{a^2 + \Omega^2}$$

(1.12)

Figure 1.2 shows how well the first-order approximation fits with measured road data.

**Figure 1.2** Road spectra; measurements and analytical approximation [27].

Characteristic is the slope of the power spectral density (PSD) for medium and high frequencies. In case of an integrator or first-order filter this slope of the
PSD is -2. Recent roads tend more to a slope of -2.5 or -3. However, these slopes can not be obtained by the application of frequency shaping filters because the required differential equations are not of a full order. The stochastic description of road surfaces is useful for calculating average performance parameters, but such calculations do not account for the deterministic effect of road damage, obstacles and potholes, or for the high degree of regularity in road profiles that occasionally are present due to terrain features, road building practices, or repeated traversal by vehicles.

Table 1.2 gives for three types of road surfaces the parameters needed for the considered road description.

<table>
<thead>
<tr>
<th>road type</th>
<th>a (rad/m)</th>
<th>σ_r (m)</th>
<th>v (m/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>asphalt</td>
<td>0.15</td>
<td>0.0033</td>
<td>40</td>
</tr>
<tr>
<td>concrete</td>
<td>0.20</td>
<td>0.0056</td>
<td>30</td>
</tr>
<tr>
<td>rough</td>
<td>0.40</td>
<td>0.0120</td>
<td>15</td>
</tr>
</tbody>
</table>

Table 1.2 Road model parameters.

A new set of equations of motion in matrix form can be obtained by augmenting the previously derived vehicle related equations of motion (1.3) with the first-order differential equation of the road description (1.4)

$$\ddot{x} = Ax + Bu + Gw$$ (1.13)

where $x$ is the state vector containing now five state variables

$$\dot{x}^T = [z_r, z_a, z_h, \dot{z}_a, \dot{z}_h]$$

Matrices $A$, $B$ and $G$ read

$$A = \begin{bmatrix} -av & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \frac{c_t}{m_a} & -\frac{c_a - c_t}{m_a} & \frac{c_s}{m_a} & 0 & 0 \\ 0 & \frac{c_s}{m_b} & -\frac{c_s}{m_b} & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \quad G = \begin{bmatrix} 0 \\ 0 \\ 0 \\ m_a \\ -m_b \end{bmatrix}$$

Finally, figure 1.3 presents equation (1.13) in a block diagram.
Figure 1.3 Block diagram of the state-space representation.

The quarter-car model is now ready to be used for further analysis. In the next chapter the definition of performance criteria will be entered into detail. These design criteria are of major importance in the tuning phase of suspension systems.
For the successful design of both standard and controlled wheel suspension systems, it is important that the performance objectives are established first. A vehicle is subjected to many disturbances. In this stage only the road irregularities are considered. In section 1.2 two types of road disturbances were distinguished: stochastic (random) irregularities and deterministic disturbances. Consequently, the responses of a vehicle to road disturbances can be divided into stochastic and deterministic types. The stochastic responses of the suspension system to random road excitations are discussed in section 2.1 and the transient responses to deterministic road excitations are studied in section 2.2.
2.1 STOCHASTIC RESPONSES

The performance of a suspension system in terms of stochastic responses can be assessed quantitatively in terms of four parameters:

- Ride comfort.
- Tire load variations.
- Suspension working space.
- Sprung mass motion.

Other important criteria are static and dynamic attitude control, and especially for active systems, actuator force levels and power consumption. All these criteria can be calculated using the simple quarter-car model.

Ride Comfort

Much research effort has gone into the objective measurement of (dis)comfort. The main conclusion of all these studies was that the root mean square (RMS) vertical acceleration of the vehicle sprung mass was a good measure of discomfort. The model described in section 1.1 has only two degrees of freedom and correctly describes a vehicle with hard seats. This choice was made to avoid inclusion of the influence of seats in car performance evaluation. Furthermore, the quarter-car model is not able to describe the influence of the wheelbase filtering mechanism on ride comfort. A real vehicle is excited by four road unevennesses where the two rear wheel road excitations are just a delayed version of the front wheel excitations. In that case the level of discomfort strongly depends on the position of the seats in the vehicle and the speed of travel. Thus, the quarter-car model used here gives a comfort computation for a hard seat positioned over an axle.

Passenger ride comfort is difficult to determine due to its subjective nature. The human body is more sensitive to vibrations in some frequency ranges than in other ranges. Ride comfort can be determined by the accelerations of the vehicle body. These accelerations ($\ddot{z}_b$) must be frequency weighted as a result of the sensitivity of the human body to give a new response from which the RMS value can be calculated. The weighting function follows the recommendation in ISO 2631/1,3 [18] and is mathematically expressed as a function of frequency $f$ (Hz) by

$$H_{iso}(f) = \begin{cases} 0.5 \sqrt{f} & \text{if } 1.0 \leq f \leq 4.0 \\ 1.0 & \text{if } 4.0 \leq f \leq 8.0 \\ 8.0/f & \text{if } 8.0 \leq f \leq 80.0 \end{cases}$$ (2.1)
The weighting function is only defined between 1 and 80 Hz. From the graphical representation of the weighting function as shown in figure 2.1, it can be seen that the weighting function has a large value in the frequency range from 4 to 8 Hz where the human body has its lowest natural frequencies and is therefore most sensitive to vibrations.

![Figure 2.1 ISO 2631 weighting criterion for vertical vibrations.](image)

The weighting function of equation (2.1) gives together with the vertical body acceleration response $\ddot{z}_b$ a new random process

$$\ddot{z}_{b,iso}(j\omega) = H_{iso}(j\omega) \cdot \ddot{z}_b(j\omega)$$

(2.2)

Its mean square spectral density is given by

$$S_{\ddot{z}_{b,iso}\ddot{z}_{b,iso}}(\omega) = |H_{iso}(j\omega)|^2 \cdot S_{\ddot{z}_b\ddot{z}_b}(\omega)$$

(2.3)

By integrating the power spectral density (PSD) we obtain the ISO weighted mean square value (MS) of the sprung mass accelerations

$$MS(\ddot{z}_{b,iso}) = 2 \int_{2\pi/1}^{2\pi/80} S_{\ddot{z}_{b,iso}\ddot{z}_{b,iso}}(\omega) d\omega$$

(2.4)

Figure 2.2 shows an illustration of the human body built up of rigid bodies coupled together with spring/damper elements. Especially the bowels and the shoulders are very sensitive to vertical vibrations in the range from 4 to 8 Hz. Although it is possible to model the human body as a multi-body system, it is not recommendable to do this in the quarter-car analysis because of the many degrees of freedom and the large number of unknown parameters. Figure 2.2 serves only as an example that the human body has several natural frequencies from which experiments have proven that vertical vibrations in the range of 4 to 8 Hz can be regarded as the most unpleasant vibrations.
Figure 2.2 The human body depicted as a multi-body system [30].

Dynamic Tire Load Variation

The suspension system should be designed to enable the vehicle to remain on the track under all circumstances. To maintain control during maneuvers the wheel must have sufficient contact with the ground for the transmission of both lateral and longitudinal forces at any moment. The frictional forces transmitted by each wheel are related to the vertical contact force between tire and ground. Figure 2.3 shows a wheel at a side slip angle $\alpha$. This slip angle is the angle between the plane of symmetry of the wheel and speed vector $V$. Lateral force $F_y$ results from the presence of this slip angle. This force, which is generated in the tire contact area, enables a car to corner.

Leaving the coefficient of friction between tire and road, the camber angle and the longitudinal forces (driving and braking) out of consideration, side force $F_y$ in (N) depends on slip angle $\alpha$ in (deg) and wheel load $F_z$ in (N) according to the following non-linear relation

$$F_y = f(\alpha, F_z)$$  \hspace{1cm} (2.5)
Figure 2.4 shows the course of side force $F_y$ as a function of slip angle $\alpha$ and vertical load $F_z$. As can be seen from this figure, for small slip angles an approximate linear relationship between side force and slip angle may be used according to

$$F_y = C_{Fa}(F_z) \cdot \alpha$$  \hspace{1cm} (2.6)

with $C_{Fa}$ the wheel load ($F_z$) dependent cornering stiffness in (N/deg). Figure 2.5 shows the (non-linear) course of the cornering stiffness $C_{Fa}$ as a function of vertical load $F_z$. For low and medium wheel loads an almost linear relationship exists between the cornering stiffness and vertical load. This means that at a given (small) slip angle, a variation of the wheel load (road holding) changes the side force and consequently influences the lateral behavior (handling ability) of a car. At larger wheel loads a maximum may be reached. The wheel load is made up of a static component, due to gravity, and a dynamic component, due to the road unevennesses and the vertical motions of both the sprung and unsprung masses. In order to reduce variations of the side force during cornering, it is necessary that the dynamic tire load component is kept as small as possible.

The above considerations are based on the steady-state cornering behavior of tires. Consequently, the theory is only valid for small and slow changes of the vertical wheel load. For rapid and large wheel load variations a much more complicated (non-linear) model for transient tire behavior is necessary. This model will be discussed in chapter 7 and is used in the study on the influence of road unevennesses on the handling of a three-dimensional full-vehicle model.

Figure 2.3 A wheel moving at a slip angle.
Figure 2.4 Side force as a function of slip angle and vertical tire load.

Figure 2.5 Cornering stiffness as a function of vertical tire load.

The road holding is related to the relative displacement between road and unsprung mass: the smaller $|z_r - z_a|$ the smaller will be the pneumatic tire load variation $P_{dyn} = C_s(z_r - z_a)$ from its static value $P_{stat}$. The RMS value of the dynamic tire load variation can be used as a measure of wheel load control. The mean square value can be found by integrating the spectral density function over the frequency range of interest.

Figure 2.6 Tire load variations as a Gaussian process.

Linear systems produce Gaussian distributed outputs in response to a Gaussian input. The tire vertical force can often be interpreted as a Gaussian variable, exceeding two standard deviations 4.66% of the time and three standard deviations 0.27% of the time, when zero mean is assumed. With three standard deviations ($3\sigma$) the tire will 99.7% of the time be in contact
with the road as depicted in figure 2.6. With this assumption it is possible to define a maximum RMS value of the dynamic tire load variation depending on the static tire load \( P_{\text{stat}} \) being

\[
\text{RMS}\left( P_{\text{dyn}} \right)_{\text{max}} = \frac{P_{\text{stat}}}{3} = \frac{(m_a + m_b)g}{3} = 762 \text{ N}
\]  

(2.7)

**Suspension Working Space**

The working space is defined as the relative displacement between sprung and unsprung mass: \( z_a - z_b \). Working space may affect directional stability because of particular suspension geometries: especially on rough roads, the influence of working space on camber angle may be more important than loss in road holding for handling stability. Like dynamic tire load variations, working space can be expressed by its RMS value or standard deviation \( \sigma \) assuming that the suspension working space is a Gaussian distributed process. The suspension will use 99.7% of the time a space between -3\( \sigma \) and +3\( \sigma \). In practice there is about 0.1 m space available between the two bump stops. Furthermore, it is assumed that the equilibrium position at static load is located in the middle of the available space. Hence, with the 3\( \sigma \) criterion the maximum admissible standard deviation of the suspension working space amounts

\[
\text{RMS}(z_a - z_b)_{\text{max}} = \frac{0.1}{2 \cdot 3} = 0.0167 \text{ m}
\]  

(2.8)

It is well known that passive suspensions designed especially for good roads will run out of working space on rough roads at higher vehicle speeds. However, designing the system for the roughest conditions likely to be encountered will lead to very poor performance on smoother roads, because the system will use only a fraction of its working space. Finally, bump and rebound buffers will make the springing non-linear.

**Sprung Mass Motion**

Besides the three previously mentioned performance criteria (ride comfort, dynamic tire load variation and suspension working space) it is important to look at frequency responses too; particularly the amplitude ratio of sprung mass displacement \( z_b \) to road input \( z_r \) is of interest.

The reason of using this criterion apart from the other ones is the observation that an excellent damping of the sprung mass motion leads to a good
sensation of driving stability. A well-damped sprung mass mode will cause the driver and passengers to be visually less bothered with a continuous shift of the horizon due to the motion of the vehicle body. The driver will also be able to estimate the road condition better if the vehicle body motions are not magnified in the resonant zone. A properly damped body natural frequency enhances the correlation between the visual observed road input and the sensed body movements, which is assumed to improve the sensation of controllability towards the driver. Therefore, a perfect vibration isolation of the vehicle body to road disturbances is preferred. Passive suspension systems yield a resonance peak near the natural frequency of the sprung mass (typically 1–2 Hz). The value of the undamped sprung mass natural frequency can be approximated by

\[ f_0^b = \frac{1}{2\pi} \sqrt{\frac{c_s}{m_b}} = 1.068 \text{ Hz} \]  \hspace{1cm} (2.9)

The height of the resonance peak strongly depends on the damping ratio \( \kappa \) which is defined as

\[ \kappa = \frac{k_s}{2\sqrt{m_b c_s}} \]  \hspace{1cm} (2.10)

In the natural frequency range of the sprung mass, the damper will amplify rather than attenuate sprung mass motions due to road disturbances.

Figure 2.7 depicts the magnitude of the frequency response function \( z_0/z_r \). From this figure it can be seen that a large damping coefficient \( \kappa \) leads to a small resonance peak near the sprung mass natural frequency and therefore is preferable in terms of vibration magnification. On the other hand the figure shows that in the frequency range from 2 to 12 Hz and beyond the wheel-hop frequency a low damping coefficient provides a much better vibration isolation.

The undamped natural frequency related to the wheel-hop mode can be approximated using

\[ f_0^a = \frac{1}{2\pi} \sqrt{\frac{c_s + c_t}{m_a}} = 12.666 \text{ Hz} \]  \hspace{1cm} (2.11)

Interesting is also the so-called invariant point in the modules of the frequency response function of figure 2.7 near 1.445 Hz. Independent of any value of the passive damping constant, the frequency response curves will always cross that particular point. In practice this means that vibrations can
only be attenuated if the frequency is higher than the frequency belonging to this point. All vibrations with a frequency contents up to 1.445 Hz will be amplified depending on the value of damping constant \( k_a \).

Many studies [15] have shown that it is impossible to simultaneously reduce each one of these performance parameters, even with the most sophisticated control system (for example in combination with an active suspension system).

![Graph](image)

**Figure 2.7** Frequency responses of a passive suspension system.  
(The direction of the arrow denotes increasing damping)

**Normalized Performance Parameters**

It is well known that passive suspension systems have two optimal damping coefficients, one for ride comfort (\( z_b \) minimal) and one for road holding (\( P_{dyn} \) minimal). For the suspension working space there is no requirement to minimize \( z_a - z_b \). Obviously, as more damping is added to the passive system, the RMS working space will be reduced.

In order to be able to compare the performance of various suspension systems with different control strategies it is recommendable to introduce normalized performance parameters. A normalized RMS performance parameter is the
ratio of the RMS value to the smallest RMS value achievable by a passive suspension system with a fixed damping constant. For linear systems these normalized results are independent of the amplitude of the random white noise \( w \) (as long as product \( a \cdot v \) remains constant). Table 2.1 shows the analytically determined results for a passive suspension system for one road condition using covariance calculations (explained in chapter 3). The normalization values are printed in boldface.

\[
\begin{array}{|c|c|c|c|c|}
\hline
\kappa & P_{dyn} & \ddot{z}_b & ISO \ddot{z}_b & z_a - z_b \\
      & (N)     & (m / s^2) & (m / s^2) & (mm)     \\
\hline
0.07 & \min(ISO \ddot{z}_b) & 1098 & 0.684 & 0.423 & 12.1 \\
0.09 & \min(\ddot{z}_b)    & 971  & \textbf{0.674} & 0.428 & 10.7 \\
0.80 & \min(P_{dyn})      & 460  & 1.409 & 1.129 & 3.9 \\
1.00 & \ddot{z}_a - \ddot{z}_b & 465 & 1.571 & 1.290 & \textbf{3.6} \\
\hline
\end{array}
\]

Table 2.1 RMS performance values of a passive suspension system (analytical results, concrete road, \( v = 30 \text{ m/s} \)).

As already mentioned the suspension working space in only minimal for infinite damping. For this reason the working space is normalized with the RMS value which corresponds to a dimensionless damping coefficient equal to 1.0.

The normalized RMS performance parameters are defined as

\[
\text{normalized RMS}(P_{dyn}) = \frac{\text{RMS}(P_{dyn})}{460} \tag{2.12a}
\]

\[
\text{normalized RMS}(\ddot{z}_b) = \frac{\text{RMS}(\ddot{z}_b)}{0.674} \tag{2.12b}
\]

\[
\text{normalized RMS}(\ddot{z}_{b,iso}) = \frac{\text{RMS}(\ddot{z}_{b,iso})}{0.423} \tag{2.12c}
\]

\[
\text{normalized RMS}(z_a - z_b) = \frac{\text{RMS}(z_a - z_b)}{3.6e - 3} \tag{2.12d}
\]

The lower the normalized performance values are, the better the performance. A value smaller than '1' indicates that a performance level is achieved which is better than that reachable by adjusting the damping of a passive system (with exception of the suspension working space).
In accordance with the statements made before concerning the maximum dynamic tire load variation, the maximum admissible normalized dynamic tire load variation reads

\[
\text{normalized RMS}(P_{\text{dyn}})_{\text{max}} = \frac{762}{460} = 1.658
\]  

(2.13a)

The same can be done for the maximum admissible normalized suspension working space

\[
\text{normalized RMS}(z_a - z_b)_{\text{max}} = \frac{1.7e^{-2}}{3.6e^{-3}} = 4.662
\]

(2.13b)

### 2.2 Transient Responses

In the performance criteria developed in the previous section, time is the independent variable. It is also of interest to study the transient behavior or time response of a system when subjected by command inputs and/or initial conditions [8]. The command inputs are usually deterministic functions of time and commonly consist of a step function, an impulse function, a ramp function, or a sinusoidal function. The step input is the most realistic and useful function for studying the transient behavior of automotive suspension systems.

**Step Response**

Typical performance criteria that are used to characterize the transient response to a step input include delay time and rise time for the initial speed of response, maximum overshoot for the deviation, and settling time necessary for the response to settle within certain limits of its final steady-state value. These criteria are illustrated in figure 2.8 and are defined in the following.

**Rise time:**

The rise time \(T_r\) is defined as the time required for the step response to rise from 10% to 90% of its final value. It is a measure of the speed of the response.

**Delay time:**

The delay time \(T_d\) is defined as the time required for the step response to reach 50% of its final value.
Chapter 2 Suspension Design Criteria

**Maximum overshoot:**
The maximum overshoot is the maximum deviation of the output above its steady-state final value. The maximum overshoot may also be defined as the percentage of the final steady-state value of the response

\[
\% \text{ maximum overshoot} = \frac{\text{peak value} - \text{final value}}{\text{final value}} \times 100\% \tag{2.14}
\]

**Settling time:**
The settling time \( T_s \) is defined as the time required for the response to settle within a certain percentage of its final value. The commonly specified value is 5%.

![Figure 2.8 Transient performance criteria based on a step response.](image)

These four quantities are relatively easy to obtain when the step response is plotted. However, the selection of the control structure to satisfy the specifications is not straightforward for systems of order greater than two. In a second-order system, decreasing the damping coefficient decreases the rise time, but the maximum overshoot and the settling time increase. For vehicle suspension design, typically, sprung and unsprung mass motions as a result of
a step road input are examined. Most important are the maximum sprung and unsprung mass overshoot. Finally, for linear time-invariant systems the peak magnification of a frequency response function is related to the maximum overshoot and settling time of the transient response.

**A Tire Model for Step Responses**

In the field of control engineering the step input is often used to determine the transient behavior of a dynamic system. Mostly, the starting-point of such an analysis is a ‘true’, more or less synthetic step input with infinite velocity at the point of transition. The main drawback of the application of such a synthetic step input in the study concerning the transient behavior of the quarter-car model is that this step function is not realistic: the response is independent of the speed of travel and only single-point contact between tire and ground is considered. Therefore, it is more realistic to describe what really happens if a car crosses a kerb. Especially the contact between tire and kerb is of interest. Although this is much more difficult than the simple ‘synthetic’ step input, more realistic results to attach value to can be expected. Figure 2.9 presents a quarter-car model with a tire making contact at two points. The tire can be modeled as a circular brush with flexible tread elements. While crossing the kerb a transition from single-point contact to two-point contact to again single-point contact takes place. The influence of the speed of travel is introduced by the resolution of the horizontal tire contact force. No longitudinal interaction is considered (speed of travel remains constant).

![Figure 2.9 A tire model for step inputs.](image)

The definition of the performance criteria is now ready. Chapter 3 goes into the control system design of an active suspension system based on the theory of linear optimal control. The performance criteria as defined in this chapter will be used to define a cost function that will be minimized by the optimal
control algorithm. The design of the controller goes on with the evaluation of the closed-loop performance. Both stochastic and deterministic responses will be examined.
Chapter 3 Active Suspension Control using Linear Quadratic Gaussian Control

ACTIVE SUSPENSION CONTROL USING LINEAR QUADRATIC GAUSSIAN CONTROL

In this chapter the application of the Linear Quadratic Gaussian (LQG) approach to the design of an active automotive suspension system will be explored. The control system design is based on the theory of linear optimal control because LQG offers the possibility to emphasize quantifiable issues like ride comfort or road holding very easily by altering the weighting factor of a quadratic criterion. The theory used assumes that the plant (quarter-car + road unevenness model) is excited by white noise that is Gaussian distributed. Furthermore, it is assumed that the plant is linear and time-invariant. Time-invariancy is not a requirement for LQG control. However, with time-invariant systems the control law can be calculated offline while with time-variant systems the feedback structure must be
calculated every time-step (on-line). The term quadratic is related to a quadratic performance index. Minimization of this quadratic penalty function results in a feedback control law. In section 3.1 the results for full-state feedback will be discussed and in section 3.2 the same is done for limited state feedback where a predetermined structure of the feedback law is assumed. Limited state feedback will also be used for optimizing stiffness and damping constants of passive suspension systems under given weighting factors. Limited state feedback has the advantage that fewer sensors are required if the controller has to be implemented in a real vehicle. The closed-loop stability of an actively suspended quarter-car model will be entered into detail in section 3.3. The performance of both full and partial state feedback controllers will be compared in section 3.4. RMS values as well as step responses will be considered.

The object of linear optimal control theory is to specify an input vector \( u \) (in our situation \( u \) is a scalar being the actuator force) which drives a system to a specified target state in such a way that, during the process, a defined quadratic cost function \( J \) is minimized [23]. Minimization of the performance criterion yields an optimal feedback law compromising control effort (actuator power) and control quality (ride comfort and road holding ability).

The theory of linear optimal control can be applied to systems excited by Gaussian distributed white noise as well as to systems excited by a deterministic step input. In the first case the term linear quadratic Gaussian control (LQG) is used. The latter case is often referred to as the method of the linear quadratic regulator (LQR). The optimal control solution for the system driven by stochastic random road unevennesses (LQG) is also appropriate for the deterministic step input if the term \( a \cdot v \) (break frequency of the first-order road shaping filter) in state matrix \( A \) becomes zero (LQR). In this case the random road excitations will be represented as integrated white noise with a power spectral density consisting of the characteristic slope -2. The power spectral density of a step function exhibits also the same shape. In this case also step responses may be 'tuned' by means of adjusting the weighting factors of the cost function.

The starting-point for the control system design based on LQG control is the state-space representation of the quarter-car model as given by

\[ \dot{x} = Ax + Bu + Gw \]  

(3.1)

where \( x \) is the state vector containing six state variables.

- 38 -
\[ x^T = [z_r, z_a, z_h, \dot{z}_r, \dot{z}_a, \dot{z}_h] \]

Matrix \( A \) is the state matrix, \( B \) represents the input vector for the control effort \( u \) and \( G \) the input vector for the white noise vector \( w \). The matrices look as follows

\[
A = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
-\omega_1 \omega_2 & 0 & 0 & -\omega_1 - \omega_2 & 0 & 0 \\
\frac{c_t}{m_a} & -\frac{c_s - c_t}{m_a} & \frac{c_s}{m_a} & 0 & 0 & 0 \\
0 & \frac{c_s}{m_b} & -\frac{c_s}{m_b} & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
\frac{1}{m_a} \\
-1 \\
\end{bmatrix}
\]

\[
G = \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
\end{bmatrix}
\]

An extra state variable has been introduced being the vertical velocity \( \dot{z}_r \) of the road. The reason for that will be explained later on. Since there is no mathematical relationship between the quarter-car model and \( \dot{z}_r \) (no tire damping) and \( \dot{z}_r \), can not be determined using the first-order road description (\( \dot{w} \) unknown) this state variable has been added using an additional first-order filter with break frequency \( \omega_2 \). The transfer function of the road model shaping filter follows from

\[
\frac{z_r(s)}{u(s)} = \frac{1}{\frac{\omega_1}{s^2 + \left(\frac{\omega_1 + \omega_2}{\omega_1 \omega_2}\right)s + \frac{\omega_2}{\omega_1 \omega_2}} + 1} = \frac{\omega_2}{s^2 + \left(\frac{\omega_1 + \omega_2}{\omega_1 \omega_2}\right)s + \omega_1 \omega_2}
\]

in which \( \omega_1 \) equals the break frequency of the first-order road shaping filter (1.5). The shaping filter parameters are given by

\[
\omega_1 = a \cdot v \quad \omega_2 = 10000 \cdot \omega_1
\]

\( \omega_2 \) is the additionally introduced second break frequency which is chosen 10000 times larger than \( \omega_1 \). This means that the characteristic slope -1 (\( \omega > \omega_1 \)) of the first-order road description according to equation (1.4) will change in a slope -2 for \( \omega > \omega_2 \). Since the power spectral density is very small beyond \( \omega_2 \), performance parameters of the quarter-car model will not change much due to this approximation.
The state-space model of the second-order equation (3.2) is given by

\[
\begin{bmatrix}
\dot{z}_r \\
\ddot{z}_r
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
-\omega_1 \omega_2 & -\omega_1 - \omega_2
\end{bmatrix}
\begin{bmatrix}
z_r \\
\dot{z}_r
\end{bmatrix} +
\begin{bmatrix}
0 \\
-\omega_2
\end{bmatrix} w
\]

(3.4)

To meet the ride quality and road holding objectives in the design of the active suspension system, the following performance index \( J \) will be used

\[
J = \lim_{T \to \infty} \frac{1}{T} \int_0^T \left( q_1 \{z_r - z_a\}^2 + q_2 \{\dot{z}_r\}^2 + q_3 \{z_a - z_b\}^2 \right) dt
\]

(3.5)

As can be seen, \( J \) is a weighted quadratic sum of dynamic tire load variations, the sprung mass accelerations (without ISO weighting) and suspension working space with weighting factors \( q_1, q_2 \) and \( q_3 \). The ISO weighting criterion has not been included in the performance index because due to the shape of the weighting function (figure 2.1) it is hardly possible to describe this function in a differential equation expression and afterwards include it in the state-space formulation. Furthermore, the phase relation between input and output of the ISO weighting criterion is not known.

Many different control laws can be derived by changing the weighting constants. For example, dynamic tire load variations can be restricted by choosing \( q_1 \) very large. Other characteristic combinations are

- \( q_1 \) large, \( q_2 \) and \( q_3 \) small \( \rightarrow \) emphasizing road holding.
- \( q_2 \) large, \( q_1 \) and \( q_3 \) small \( \rightarrow \) emphasizing ride comfort.
- \( q_3 \) large, \( q_1 \) and \( q_2 \) small \( \rightarrow \) emphasizing suspension working space.

Using optimal control, only the relative ratio of the weighting factors is important, not the absolute values. Thus, dividing or multiplying the complete set of weighting factors by whatever value will result in the same control feedback matrix.

The stochastic cost function \( J \) can also be expressed in a matrix form according to

\[
J = \lim_{T \to \infty} E\left\{ \int_0^T \left( x^T Q_c x + 2 x^T N_c u + u^T R_c u \right) dt \right\}
\]

(3.6a)

or

\[
J = \lim_{T \to \infty} E\left\{ \int_0^T \begin{bmatrix} x^T & u^T \end{bmatrix} \begin{bmatrix} Q_c & N_c \\ N_c^T & R_c \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} dt \right\}
\]

(3.6b)
where matrices $Q_c$, $R_c$, $N_c$ are constant weighting matrices. They can be calculated by rearranging equation (3.5). The weighting matrices can also be determined using the output equation

$$\bar{y}_w = C_w \bar{x} + D_w u$$

(3.7)

where $\bar{y}_w$ is a vector containing the weighted variables. In accordance with the criterion as defined by equation (3.5) vector $\bar{y}_w$ and matrices $C_w$ and $D_w$ are given by

$$\bar{y}_w = \begin{bmatrix} c_t (z_r - z_a) \\ \bar{z}_b \\ z_a - z_b \end{bmatrix}$$

$$C_w = \begin{bmatrix} c_t & -c_t & 0 & 0 & 0 \\ 0 & \frac{c_s}{m_b} & -\frac{c_s}{m_b} & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \end{bmatrix}$$

$$D_w = \begin{bmatrix} 0 \\ \frac{1}{m_b} \\ 0 \end{bmatrix}$$

A diagonal weighting matrix $q$ can be defined according to

$$q = \begin{bmatrix} q_1 & 0 & 0 \\ 0 & q_2 & 0 \\ 0 & 0 & q_3 \end{bmatrix}$$

(3.8)

Weighting matrix $Q_c$ must be a positive semi-definite symmetric matrix ($Q_c = Q_c^T \geq 0$) and is defined as

$$Q_c = C_w^T \cdot q \cdot C_w$$

(3.9)

Weighting matrix $R_c$ is a positive definite symmetric matrix ($R_c = R_c^T > 0$) and is defined as

$$R_c = D_w^T \cdot q \cdot D_w$$

(3.10)

The diagonal elements of matrix $R_c$ must at least be larger than zero preventing in this way infinite control power. From equation (3.5) it can be seen that control effort $u$ was not weighted separately in the quadratic performance index. Nevertheless, weighting matrix $R_c$ is not equal to zero. The reason is that the actuator force is related to the sprung mass acceleration (equation (1.1)) and it is therefore not necessary to weight $u$ individually. Cross-product weighting matrix $N_c$ is given by

$$N_c = C_w^T \cdot q \cdot D_w$$

(3.11)

As can be seen from equation (3.5), performance index $J$ consists of three terms with different dimensions. In order to have a clear insight into the effect of different values of weighting factors $q_1 \ldots q_3$ on the closed-loop system
performance, a new set of normalized weighting factors \( \overline{q}_1 \ldots \overline{q}_3 \) is introduced providing a dimensionless or normalized index \( J \). With the help of the formerly defined normalized performance values (chapter 2.1) it is now possible to define a new set of normalized weighting factors \( \overline{q}_1 \ldots \overline{q}_3 \)

\[
\overline{q}_1 = q_1 \cdot (460)^2 \\
\overline{q}_2 = q_2 \cdot (0.674)^2 \\
\overline{q}_3 = q_3 \cdot (3.6e - 3)^2
\] (3.12a)(3.12b)(3.12c)

With the performance index we are now capable to search for a state feedback by matrix \( K_c \) that provides a minimum \( J \) at the given weighting factors. Actuator force \( u \) can is structured as a linear combination of \( x \) by the state feedback matrix \( K_c \) (proportional control)

\[
u = -K_c x
\] (3.13)

The closed-loop system is now described by

\[
\dot{x} = \left( A - BK_c \right)x + Gw
\] (3.14)

Feedback matrix \( K_c \) can be calculated with the help of the matrix Riccati equation.

If any modes of the open-loop system are unstable, they can be stabilized with a feedback using linear quadratic control. A necessary condition for solving the controller problem is that the system is controllable. This means that all the modes can be excited or controlled by the input \( u \). In terms of the state-space matrices, the condition of controllability can be shown by constructing the controllability matrix \( W_c \) defined by

\[
W_c = \begin{bmatrix}
B & AB & A^2B & \ldots & A^{m-1}B
\end{bmatrix}
\] (3.15)

Controllability is ensured when the rank of \( W_c \) equals \( m \), where \( m \) is the order of the system (here: 6). The controllability matrix shows that two states are not controllable \( (z_r, \dot{z}_r) \). This is not unexpected since the road profile can not be influenced by the actuator. In spite of the non-controllability, the algorithm solving the algebraic Riccati equation will solve the controller problem resulting in a state feedback matrix \( K_c \). However, the eigenvalues of the closed-loop system will not depend on the gains related to states \( z_r \) and \( \dot{z}_r \) in matrix \( K_c \). These particular gains may have any value. The performance of the closed-loop suspension system will very much depend on these particular two gains. They can be considered as a kind of feed-forward gains.
After having solved the controller problem it is important to evaluate the closed-loop performance for both stochastic and deterministic road inputs. The average behavior of dynamic systems can be described by the covariance matrix $C_{\dot{x}x}$ of state vector $\dot{x}$

$$C_{\dot{x}x}(\tau) = \text{cov}\{\dot{x}(t)\} = E\{\dot{x}(t)\dot{x}^T(t + \tau)\} \quad (3.16)$$

It is assumed that the mean value of $\dot{x}$ equals zero. In that case the covariance matrix and the variance matrix coincide. Matrix $C_{xx}$ is constant in a steady-state and can be found as a solution of the Lyapunov equation

$$(A - BK_c)C_{xx}^T + C_{xx}(A - BK_c)^T + G \cdot C_{ww} \cdot G^T = 0 \quad (3.17)$$

Where $C_{ww}$ is the intensity matrix of the white noise serving as an input for the stochastic road model according to equation (1.8). The performance indices representing the mean square (MS) values of the dynamic tire load variations, suspension working space and sprung mass acceleration are given by

$$\text{diag}\left[C_c \cdot C_{\dot{x}x} \cdot C_w^T\right] = [J_1 \quad J_2 \quad J_3] \quad (3.18)$$

where $J_1$, $J_2$ and $J_3$ are the variances of respectively the dynamic tire load variations, the sprung mass acceleration and the suspension working space

$$J_1 = E\left[c_i^2(z_r - z_a)^2\right]$$

$$J_2 = E\left[\dot{x}_h^2\right]$$

$$J_3 = E\left[(z_a - z_h)^2\right]$$

and $C_w^c$ represents the closed-loop output matrix

$$C_w^c = C_w - D_w K_c \quad (3.19)$$

The quadratic performance index $J$ can also be written in the form

$$J = \text{tr}\left[q \cdot C_w^c \cdot C_{\dot{x}x} \cdot C_w^c^T\right] = q_1 J_1 + q_2 J_2 + q_3 J_3 \quad (3.20)$$

where 'tr' denotes trace (sum of all diagonal elements).

After having determined the control gain matrix $K_c$ and the state covariance matrix $C_{xx}$ it is now easy to find all terms of the performance index from relations (3.18) and (3.19).
3.1 FULL-STATE FEEDBACK CONTROL

With full-state feedback it is assumed that a perfect knowledge of the complete system state is available. The way these states are obtained is of no importance right now. The optimization problem may be defined as the task of finding the minimum value of the quadratic performance index $J$ defined by equation (3.5) such that, an optimal control vector $u$ is determined as a function of $x$ by the state feedback matrix $K_c$ according to

$$u = -K_c x = -k_{c1} z_r - k_{c2} z_a - k_{c3} z_b - k_{c4} \dot{z}_r - k_{c5} \dot{z}_a - k_{c6} \dot{z}_b$$

(3.21)

Figure 3.1 shows the state feedback structure of the closed-loop system.

![Figure 3.1 Full-state feedback control.](image)

The determination of the control variable $u$ for the system described by equation (3.1) which minimizes the performance index is called the linear optimal regulator problem and its solution is given by

$$u = -K_c x = -R^{-1}_c \left( N_c^T + B^T S \right) x$$

(3.22)

in which $K_c$ is a matrix of gains and where $S$ is the symmetric, positive definite solution of the algebraic Riccati equation according to

$$S(A - B R^{-1}_c N_c^T) + (A - B R^{-1}_c N_c^T)^T S - S B R^{-1}_c B^T S + Q_c - N_c R^{-1}_c N_c^T = 0$$

(3.23)

Once the solution of the Riccati equation is known, performance index $J$ of the controlled system can be found from

$$J = \text{tr} [S \cdot G \cdot C_{wv}^T \cdot G^T]$$

(3.24)
where $C_{ww}$ is the intensity of white noise $w$ given by equation (1.8).

Table 3.1 gives the results for the feedback gains for seven combinations of weighting factors, each emphasizing a part of the performance index $J$. Table 3.2 shows the corresponding results in terms of the normalized performance parameters.

<table>
<thead>
<tr>
<th>normalized weighting factors</th>
<th>$\tilde{q}_1$</th>
<th>$\tilde{q}_2$</th>
<th>$\tilde{q}_3$</th>
<th>$k_1$</th>
<th>$k_2$</th>
<th>$k_3$</th>
<th>$k_4$</th>
<th>$k_5$</th>
<th>$k_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1.0E+10</td>
<td>1.0</td>
<td>1.0</td>
<td>5.86E+9</td>
<td>5.86E+9</td>
<td>-5.76E+4</td>
<td>-9.41E+4</td>
<td>4.21E+5</td>
<td>-1.22E+6</td>
</tr>
<tr>
<td></td>
<td>1.0</td>
<td>1.0E+10</td>
<td>1.0</td>
<td>5.86E-2</td>
<td>-9.00E+3</td>
<td>9.00E+3</td>
<td>9.61E-7</td>
<td>9.41E-3</td>
<td>-1.23E+1</td>
</tr>
<tr>
<td></td>
<td>1.0</td>
<td>1.0E+10</td>
<td>1.0E+10</td>
<td>1.72E+5</td>
<td>3.77E+9</td>
<td>-3.77E+9</td>
<td>2.19E+0</td>
<td>4.62E+5</td>
<td>-4.62E+5</td>
</tr>
<tr>
<td></td>
<td>1.0E+10</td>
<td>1.0E+10</td>
<td>1.0</td>
<td>-4.06E+3</td>
<td>-5.37E+2</td>
<td>9.00E+3</td>
<td>-6.90E-2</td>
<td>7.45E+2</td>
<td>-1.26E+1</td>
</tr>
<tr>
<td></td>
<td>1.0E+10</td>
<td>1.0E+10</td>
<td>1.0E+10</td>
<td>-4.89E+9</td>
<td>6.95E+9</td>
<td>-2.34E+9</td>
<td>-7.80E+4</td>
<td>-1.76E+7</td>
<td>-1.11E+8</td>
</tr>
<tr>
<td></td>
<td>1.0</td>
<td>1.0E+10</td>
<td>1.0E+10</td>
<td>1.13E+2</td>
<td>-8.46E+3</td>
<td>4.19E+2</td>
<td>1.87E-3</td>
<td>2.21E+0</td>
<td>-7.21E+4</td>
</tr>
<tr>
<td></td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>6.52E-2</td>
<td>1.68E+4</td>
<td>-2.87E+4</td>
<td>9.46E-3</td>
<td>8.32E+2</td>
<td>-3.77E+3</td>
</tr>
</tbody>
</table>

Table 3.1 Full-state feedback gains.

<table>
<thead>
<tr>
<th>normalized weighting factors</th>
<th>$\tilde{q}_1$</th>
<th>$\tilde{q}_2$</th>
<th>$\tilde{q}_3$</th>
<th>$P_{d}$</th>
<th>$\tilde{z}_b$</th>
<th>$z_a - z_b$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1.0E+10</td>
<td>1.0</td>
<td>1.0</td>
<td>6.3325E-2</td>
<td>4.3047E+3</td>
<td>7.3268E+0</td>
</tr>
<tr>
<td></td>
<td>1.0</td>
<td>1.0E+10</td>
<td>1.0</td>
<td>3.5262E+2</td>
<td>4.4030E-3</td>
<td>2.2663E+2</td>
</tr>
<tr>
<td></td>
<td>1.0</td>
<td>1.0E+10</td>
<td>1.0E+10</td>
<td>1.3421E+3</td>
<td>3.9295E+3</td>
<td>6.8311E-4</td>
</tr>
<tr>
<td></td>
<td>1.0E+10</td>
<td>1.0E+10</td>
<td>1.0</td>
<td>1.2775E+0</td>
<td>1.2265E+0</td>
<td>1.2938E+3</td>
</tr>
<tr>
<td></td>
<td>1.0E+10</td>
<td>1.0E+10</td>
<td>1.0E+10</td>
<td>4.9477E-1</td>
<td>3.3846E+3</td>
<td>1.0575E+0</td>
</tr>
<tr>
<td></td>
<td>1.0</td>
<td>1.0E+10</td>
<td>1.0E+10</td>
<td>2.7534E+1</td>
<td>5.6645E-2</td>
<td>1.7764E+1</td>
</tr>
<tr>
<td></td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.1784E+0</td>
<td>1.4146E+0</td>
<td>1.2354E+0</td>
</tr>
</tbody>
</table>

Table 3.2 Normalized performance parameters.

Most trivial are the results when ride comfort is emphasized. Here the actuator acts as a linear spring with a negative stiffness of 9000 N/m (note gains $k_c_2$ and $k_c_3$) and therefore cancels the spring mounted between axle and body. With this control adjustment a complete separation of both bodies is established. The result is a system with zero overall stiffness and can therefore not be used in practice.
Chapter 3 Active Suspension Control using Linear Quadratic Gaussian Control

A significant reduction of suspension working space can be achieved if the actuator operates like a linear stiff spring parallel with a linear firm damper. Again, this solution is trivial and in agreement with a physical approach.

Dynamic tire load variations can be reduced significantly by using optimal control and full-state feedback. Most striking are the large, equal and opposite gains concerning state variables $z_r$ and $z_o$. This indicates that a large tire stiffness is preferred. In reality these two gains do of course not represent a tire stiffness since the actuator force acts on both sprung and unsprung masses.

As can be seen from the normalized performance values, weighting only one term of the cost function is not so realistic since the performance of the non-weighted terms degrades drastically. Therefore, it is more realistic to weight several terms of the quadratic cost function. Emphasizing dynamic tire load variations as well as ride comfort results in an inadmissible increase of the suspension working space. The same applies to ride comfort when dynamic tire load and suspension working space are emphasized. Only the omission of the contribution of dynamic tire load variations to the cost function results in more realistic performance values. Interesting is the situation where all three weighting factors have an equal value. This situation will be discussed in section 3.4 where different control structures will be compared with each other.

The frequency responses as shown in figure 3.2 are also meaningful. In order to restrict the number of plots only the equal weighting combination $(\tilde{q}_1 = \tilde{q}_2 = \tilde{q}_3 = 1.0)$ has been plotted. All active responses are compared with those of a passive system with a fixed damping coefficient $\zeta = 0.4579$. The motivation of the choice of this damping coefficient will be explained in section 3.2 where passive damping has been optimized using LQG control.

It is observed that the active system shows a little increase of both dynamic tire load variations (tire deflection function multiplied with tire stiffness $c_t$) and suspension working space near the wheel-hop natural frequency (12.6 Hz). The benefit of the active system is a small improvement of sprung mass accelerations and the elimination of the resonant peak near the sprung mass natural frequency (1.0 Hz) in graph (d).

The steady-state error of transfer function (d) of the sprung mass displacement to the road input is caused by the feedback of absolute state
variables. For $\omega = 0$ the modulus does not equal 1.0. This implies that, after crossing a kerb, the vehicle body has come a bit closer to the axle. Therefore, crossing a mountain is impossible. If the feedback gains concerning the displacements do not meet a special condition a steady-state error will result. More about that in section 3.4.

Figure 3.2 Frequency responses of the full-state feedback system $(q_1 = q_2 = q_3 = 1.0)$.
(a) tire deflection  
(b) sprung mass acceleration
(c) suspension working space  
(d) sprung mass displacement
(e) actuator/damper force
3.2 LIMITED STATE FEEDBACK CONTROL

In a real active or semi-active suspension system it will be of interest to examine the performance of the system under the assumption that only velocity signals are available for the feedback. So, in this section the minimization of the linear quadratic performance index will be considered with respect to a predetermined structure of the feedback law. Linear Quadratic Output Feedback (LQOFB) [33] will be used to determine the control feedback matrix. Limited state feedback is often used to minimize the number of states to be determined by measurements. Sometimes states like e.g. the road profile $z$, are very difficult to measure and are therefore omitted in the control structure using limited state feedback [36,43].

The predetermined control structure can be selected through the output equation

$$y = Cx$$  \hspace{1cm} (3.25)

Input vector $u$ is proportionally related to the output vector $y$ by matrix $K_o$

$$u = K_o y = K_o C x$$  \hspace{1cm} (3.26)

The combination of feedback law (3.26) and the quarter-car system gives the closed-loop state equation according to

$$\dot{x} = (A + BK_o C)x + Gw$$  \hspace{1cm} (3.27)

Figure 3.3 shows the output feedback structure of the closed-loop system.
The performance index as defined by (3.5) remains the same. The design of an optimal constant gain output feedback matrix $K_o$ involves the selection of the weighting factors $q_1...q_3$, the choice of an initial stabilizing feedback law and the selection of an initial condition. $K_o$ can be found as follows

$$K_o = -R_c^{-1}(N_c^T + B^TP)SCT(CSC^T)^{-1} \quad (3.28)$$

where $P = P^T \geq 0$ is the solution of the Lyapunov equation given by

$$P(A+BK_oC)+(A+BK_oC)^TP+Q_c+N_cK_oC+C^TK_o^T(N_c^T+R_cK_oC) = 0 \quad (3.29)$$

and where $S = S^T \geq 0$ is the solution of the Lyapunov equation according to

$$S(A+BK_oC)^T + (A+BK_oC)S + X = 0 \quad (3.30)$$

In case of a deterministic control problem $X$ represents the contribution of an initial state condition $x_0$

$$X = x_0 \cdot x_0^T \quad (3.31)$$

For a stochastic situation $X$ symbolizes the road noise intensity matrix

$$X = G \cdot C_{ww} \cdot G^T \quad (3.32)$$

with $C_{ww}$ being the white noise intensity matrix given by equation (1.8)

The performance criterion can be expressed as

$$J = \text{tr}[P \cdot X] \quad (3.33)$$

The set of equations (3.28) up to (3.30) is implicit: the solution $K_o$ must be found using an iterative algorithm using a gradient search technique. The main drawback of this method, except that it requires longer calculation time, is that gain matrix $K_o$ is dependent on the type of excitation. For deterministic excitations the feedback gain will depend on the initial conditions of the states, for stochastic excitations the solution will depend on the intensity of white noise $w$. Another problem is that there is no guarantee that solution $K_o$ is unique. More minimizing solutions may exist in the limited state feedback case. Therefore, it is recommendable to solve the feedback problem using several different initial stabilizing feedback laws. If all initial laws lead to the same solution then it is likely that this solution is the unique minimizing solution.

The next two sections deal with the design of limited state feedback controllers for an actively suspended quarter-car model. The first structure to
be examined is based on a feedback of velocity related states. The systems based on velocity feedback are denoted by the term active damping. Limited state feedback can also be applied to determine the 'optimal' damping constant of a passive suspension system with fixed spring and damper rates. This analysis will be carried out in section 3.2.2.

3.2.1 Active Damping

In this part the feedback law will be derived if only the absolute vertical velocities of sprung and unsprung mass are available. Output matrix $C$ is given by

$$ C = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.34} $$

Control input $u$ is proportionally related to the output vector $y$ by $K_o$

$$ u = K_o y = K_o C x = k_{o1} \dot{z}_a + k_{o2} \dot{z}_b \tag{3.35} $$

Table 3.3 gives the results for the feedback gains in combination with this limited state feedback for seven combinations of weighting factors.

<table>
<thead>
<tr>
<th>normalized weighting factors</th>
<th>gain $k_{o1}$</th>
<th>gain $k_{o2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q_1$</td>
<td>$q_2$</td>
<td>$q_3$</td>
</tr>
<tr>
<td>1.0E+10</td>
<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td>1.0</td>
<td>1.0E+10</td>
<td>1.0</td>
</tr>
<tr>
<td>1.0</td>
<td>1.0</td>
<td>1.0E+10</td>
</tr>
<tr>
<td>1.0E+10</td>
<td>1.0E+10</td>
<td>1.0</td>
</tr>
<tr>
<td>1.0E+10</td>
<td>1.0</td>
<td>1.0E+10</td>
</tr>
<tr>
<td>1.0</td>
<td>1.0E+10</td>
<td>1.0E+10</td>
</tr>
<tr>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
</tr>
</tbody>
</table>

Table 3.3 Limited state feedback gains, active damping (I).

Table 3.4 shows that variations of the dynamic tire load can hardly be improved using active damping. A maximum gain of about 3% compared with a firm passive damper ($\kappa = 0.8$) is achievable. Since $k_{o2}$ is small compared to the magnitude of $k_{o1}$ it seems to be obvious that increasing $k_{o1}$ will reduce the dynamic tire load variations. Unfortunately, increasing gain $k_{o1}$ without using the Riccati minimization will not always lead to a stable closed-loop behavior. At a given value of $k_{o2}$ there is a maximum value of $k_{o1}$. This stability problem will be discussed in section 3.3.
Table 3.4 Normalized performance parameters, active damping (I).

<table>
<thead>
<tr>
<th>normalized weighting factors</th>
<th>normalized performance values</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q_1$</td>
<td>$q_2$</td>
</tr>
<tr>
<td>1.0E+10</td>
<td>1.0</td>
</tr>
<tr>
<td>1.0</td>
<td>1.0E+10</td>
</tr>
<tr>
<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td>1.0E+10</td>
<td>1.0E+10</td>
</tr>
<tr>
<td>1.0E+10</td>
<td>1.0</td>
</tr>
<tr>
<td>1.0</td>
<td>1.0E+10</td>
</tr>
<tr>
<td>1.0</td>
<td>1.0</td>
</tr>
</tbody>
</table>

Ride comfort can be improved significantly by the application of limited state feedback. In the case of heavily weighting ride comfort, the actuator force becomes proportionally related to the absolute velocity of the vehicle body. In the literature this configuration has been called the skyhook damper system [22]. Closed-loop stability is always ensured in this case. The larger gain $k_{o2}$ gets the more ride comfort can be improved. Thus, feedback gain $k_{o2}$ presented in table 3.3 provides not a minimum value of sprung mass accelerations. The suspension working space can be reduced using a firm passive damper. Weighting more than one term of the cost function results all in admissible solutions. Neglecting the contribution of the ride comfort part tends to an uncomfortable firm suspension.

Figure 3.4 shows the relevant frequency response functions of the limited state feedback system with equal weighting coefficients ($\tilde{q}_1 = \tilde{q}_2 = \tilde{q}_3 = 1.0$). Similar to the full-state feedback configuration (figure 3.2) the partial state feedback system tends to increase in dynamic tire load variations and suspension working space near the wheel-hop natural frequency. The benefit is a better ride comfort (over the entire frequency range) and no amplification of the modulus of displacement transfer function (d) near 1.0 Hz. The actuator force is only larger in the low frequency range compared with the passive damper force.

The energy consumption of this active suspension configuration can be cut down when a passive damper is installed parallel to the actuator (graph f). In this case, the passive damping constant is chosen equal to $k_{o1}$ (966 Ns/m) and the actuator is only responsible for $u_a$.
\[ u = u_p + u_a \]

\[ u_p = k_0(\dot{z}_a - \dot{z}_b) \]

\[ u_a = (k_{01} + k_{02})\dot{z}_b \]

**Figure 3.4** Frequency responses of the limited state feedback system

\( \bar{q}_1 = \bar{q}_2 = \bar{q}_3 = 1.0 \).

(a) tire deflection  
(b) sprung mass acceleration  
(c) suspension working space  
(d) sprung mass displacement  
(e) actuator force  
(f) damper/actuator force
Chapter 3 Active Suspension Control using Linear Quadratic Gaussian Control

Besides the vertical velocity of sprung and unsprung mass, also the vertical velocity of the road profile may be considered. Output matrix $C$ is then given by

$$
C = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
$$

(3.37)

Control input vector $u$ is proportionally related to the output vector $y$ by $K_o$

$$
u = K_o y = K_o C x = k_{o1} \dot{z}_r + k_{o2} \dot{z}_a + k_{o3} \dot{z}_b
$$

(3.38)

Table 3.5 gives the results for the feedback gains in combination with this kind of limited state feedback for seven combinations of weighting factors.

<table>
<thead>
<tr>
<th>normalized weighting factors</th>
<th>gain $k_{o1}$ ($\dot{z}_r$)</th>
<th>gain $k_{o2}$ ($\dot{z}_a$)</th>
<th>gain $k_{o3}$ ($\dot{z}_b$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tilde{q}_1$</td>
<td>$\tilde{q}_2$</td>
<td>$\tilde{q}_3$</td>
<td>$\tilde{q}_1$</td>
</tr>
<tr>
<td>1.0E+10</td>
<td>1.0</td>
<td>1.0</td>
<td>2.85E+5</td>
</tr>
<tr>
<td>1.0</td>
<td>1.0E+10</td>
<td>1.0</td>
<td>-1.11E+0</td>
</tr>
<tr>
<td>1.0</td>
<td>1.0E+10</td>
<td>1.0E+10</td>
<td>-3.91E+3</td>
</tr>
<tr>
<td>1.0E+10</td>
<td>1.0E+10</td>
<td>1.0</td>
<td>2.59E-1</td>
</tr>
<tr>
<td>1.0E+10</td>
<td>1.0</td>
<td>1.0E+10</td>
<td>6.96E+3</td>
</tr>
<tr>
<td>1.0</td>
<td>1.0E+10</td>
<td>1.0E+10</td>
<td>-6.19E+1</td>
</tr>
<tr>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>-1.43E+1</td>
</tr>
</tbody>
</table>

Table 3.5 Limited state feedback gains, active damping (II).

<table>
<thead>
<tr>
<th>normalized weighting factors</th>
<th>normalized performance values</th>
<th>$P_{dyn}$</th>
<th>$\dot{z}_b$</th>
<th>$z_a - z_b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tilde{q}_1$</td>
<td>$\tilde{q}_2$</td>
<td>$\tilde{q}_3$</td>
<td>$\tilde{q}_1$</td>
<td>$\tilde{q}_2$</td>
</tr>
<tr>
<td>1.0E+10</td>
<td>1.0</td>
<td>1.0</td>
<td>$6.1091E-2$</td>
<td>6.6528E+3</td>
</tr>
<tr>
<td>1.0</td>
<td>1.0E+10</td>
<td>1.0</td>
<td>3.6805E+0</td>
<td>$4.6237E-1$</td>
</tr>
<tr>
<td>1.0</td>
<td>1.0</td>
<td>1.0E+10</td>
<td>1.6930E+2</td>
<td>8.9237E+1</td>
</tr>
<tr>
<td>1.0E+10</td>
<td>1.0E+10</td>
<td>1.0</td>
<td>$1.3100E+0$</td>
<td>$1.2325E+0$</td>
</tr>
<tr>
<td>1.0E+10</td>
<td>1.0</td>
<td>1.0E+10</td>
<td>$6.7731E-1$</td>
<td>1.7323E+2</td>
</tr>
<tr>
<td>1.0</td>
<td>1.0E+10</td>
<td>1.0E+10</td>
<td>1.4167E+0</td>
<td>$1.1193E+0$</td>
</tr>
<tr>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>$1.1826E+0$</td>
<td>$1.4080E+0$</td>
</tr>
</tbody>
</table>

Table 3.6 Normalized performance parameters, active damping (II).

While comparing tables 3.3/3.4 with 3.5/3.6, one important difference may be noticed. With the help of the vertical velocity of the road profile it seems to be
possible to decrease the dynamic tire load variations significantly if the gains concerning $\dot{z}_r$ and $\dot{z}_a$ are given a large value. This is the reason why $\dot{z}_r$ has been included in the state-space formulation as described in the beginning of this chapter. This indicates that a firm damping parallel to the tire is preferred. In reality these two gains do, of course, not represent damping in the tire. Similar to the formerly discussed output feedback structure one might imagine that the larger gains $k_{o1}$ and $k_{o2}$ get the lesser the dynamic tire load variations become. But, because state $\dot{z}_r$ is uncontrollable and can therefore not influence stability, the closed-loop stability will depend on the values of gain $k_{o2}$ and $k_{o3}$ only. More about this in section 3.3. All the other weighting combinations give similar results as in the case where $\dot{z}_r$ was not included in the output equation.

3.2.2 Passive Damping

Using output feedback it is also possible to optimize passive systems under given weighting conditions. It is for example possible to determine the 'optimal' passive damping of the quarter-car model.

Assume therefore that output matrix $C$ is given by

$$C = [0 \ 0 \ 0 \ 0 \ 1 \ -1] \quad (3.39)$$

Output vector $y$ (which is a scalar in this case) represents now the relative velocity or deflection rate between sprung and unsprung mass. Control input $u$ is proportionally related to output vector $y$ by $K_o$

$$u = K_o y = K_o C x = k_{o1} (\dot{z}_a - \dot{z}_b) \quad (3.40)$$

Table 3.7 gives the results for the feedback gains (= -damping constant) for seven combinations of weighting factors, each emphasizing a part of the performance index $J$. Table 3.8 shows the corresponding results in terms of the normalized performance parameters.

Most obvious are the results when each time one part of the performance index is emphasized. Here the two optimal damping coefficients (one for ride comfort and one for road holding) meet with the former statements made in section 2.1. A reduction of suspension working space can be achieved when more damping is added. There is no minimum value.

Weighting more than one term of the cost function is more realistic since the performance of the non-weighted terms can degrade significantly. Emphasizing dynamic tire load variations as well as ride comfort result in an
admissible increase of the suspension working space. The same applies to
dynamic tire load variations when ride comfort and suspension working space
are emphasized. Only the omission of the contribution of ride comfort in the
cost function results in an uncomfortable firm damping.

<table>
<thead>
<tr>
<th>normalized weighting factors</th>
<th>gain $k_{v_1}$</th>
<th>damping ratio $k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q_1$ $q_2$ $q_3$</td>
<td>($z_a - z_b$)</td>
<td></td>
</tr>
<tr>
<td>1.0E+10 1.0 1.0</td>
<td>-2.14E+3</td>
<td>7.96E-1</td>
</tr>
<tr>
<td>1.0      1.0E+10 1.0</td>
<td>-2.42E+2</td>
<td>9.03E-2</td>
</tr>
<tr>
<td>1.0      1.0 1.0E+10</td>
<td>-1.24E+8</td>
<td>4.63E+4</td>
</tr>
<tr>
<td>1.0E+10 1.0E+10 1.0</td>
<td>-7.27E+2</td>
<td>2.71E-1</td>
</tr>
<tr>
<td>1.0E+10 1.0 1.0E+10</td>
<td>-3.92E+3</td>
<td>1.46E+0</td>
</tr>
<tr>
<td>1.0      1.0E+10 1.0E+10</td>
<td>-1.07E+3</td>
<td>3.98E-1</td>
</tr>
<tr>
<td>1.0      1.0 1.0</td>
<td>-1.23E+3</td>
<td>4.58E-1</td>
</tr>
</tbody>
</table>

Table 3.7 Passive damping constants and damping ratios.

<table>
<thead>
<tr>
<th>normalized weighting factors</th>
<th>normalized performance values</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q_1$</td>
<td>$q_2$</td>
</tr>
<tr>
<td>1.0E+10</td>
<td>1.0</td>
</tr>
<tr>
<td>1.0</td>
<td>1.0E+10</td>
</tr>
<tr>
<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td>1.0E+10</td>
<td>1.0E+10</td>
</tr>
<tr>
<td>1.0E+10</td>
<td>1.0</td>
</tr>
<tr>
<td>1.0</td>
<td>1.0E+10</td>
</tr>
<tr>
<td>1.0</td>
<td>1.0</td>
</tr>
</tbody>
</table>

Table 3.8 Normalized performance parameters, passive damping.

Interesting is the situation where all three weighting factors have an equal
value. Here an 'optimal' passive dimensionless damping coefficient $k = 0.458$
is found. In practice this is a realistic value.

From the analysis presented in section 3.2.1 some conclusions may be drawn
about the dominance of certain feedback gains resulting from particular
combinations of weighting factors. Sometimes the extrapolation of results led
to an unstable closed-loop system. It is therefore important to examine the
stability boundary of an active suspension system with a control structure
based on state feedback. With the aid of this analysis it is possible to draw conclusions about the magnitude of particular gains or combinations of gains.

3.3 CLOSED-LOOP STABILITY ANALYSIS

In this section the stability of the actively controlled quarter-car model will be determined. With the aid of the stability analysis we are able to calculate the stability boundary for the state feedback gains of the actively controlled suspension system. In section 3.2.1 some remarks were made concerning the (in)stability of the closed-loop system. The concept of stability is extremely important, because almost every workable system is designed to be stable. The Routh-Hurwitz criterion will be used to check whether or not the roots of the characteristic polynomial have negative real parts. In this context it should be stated that the optimal control solution as described before will always lead to a stable closed-loop system.

The starting-point of the stability analysis are the linear differential equations of the quarter-car model which are given by

\[ m_a \ddot{z}_a - u - c(z_b - z_a) + c_i(z_a - z_r) = 0 \]
\[ m_b \ddot{z}_b + u + c_i(z_b - z_a) = 0 \]  
(1.1)

and the first-order road description given by

\[ \dot{z}_r + a v z_r = w \]  
(1.4)

Now, assume the following state feedback

\[ u = -K_c x = -k_{c1} z_r - k_{c2} z_a - k_{c3} \dot{z}_a - k_{c4} \dot{z}_r - k_{c4} \dot{z}_a - k_{c5} \dot{z}_b \]  
(3.13)

The closed-loop system reads

\[ \dot{x} = (A - BK_c)x + GW \]  
(3.14)

In order to determine the stability of the closed-loop system the following substitution can be made

\[ z_r = Z_r e^{\lambda t} \quad \dot{z}_r = \lambda \cdot Z_r e^{\lambda t} \]
\[ z_a = Z_a e^{\lambda t} \quad \dot{z}_a = \lambda \cdot Z_a e^{\lambda t} \quad \ddot{z}_a = \lambda^2 \cdot Z_a e^{\lambda t} \]  
(3.41)
\[ z_b = Z_b e^{\lambda t} \quad \dot{z}_b = \lambda \cdot Z_b e^{\lambda t} \quad \ddot{z}_b = \lambda^2 \cdot Z_b e^{\lambda t} \]

where \( Z_r, Z_a \) and \( Z_b \) are complex amplitudes. For the closed-loop system (3.14) this results in \((w = 0, \) homogeneous equation)
The characteristic equation can be found by calculating the determinant of the matrix in equation (3.42) and has the following structure

\[(\lambda + av) \cdot (\lambda^3 + a_3 \lambda^2 + a_2 \lambda + a_1 \lambda + a_0) = 0 \tag{3.43}\]

where

\[
a_0 = \frac{(c_s - k_{c3})c_i}{m_an_b} \quad a_1 = \frac{-k_{c6}c_i}{m_am_b} \\
a_2 = \frac{c_s - k_{c3} + c_i + k_{c2}}{m_a} \quad a_3 = \frac{-k_{c6} + k_{c5}}{m_b}
\]

As can be seen, the stability of the closed-loop system does not depend on the feedback gains \(k_{c1}\) and \(k_{c4}\). This is in agreement with the non-controllability as discussed in the beginning of this chapter.

A system is called stable if and only if the roots of the characteristic polynomial have negative real parts. Stability can readily be determined once all of its roots are computed. However, if the degree of the polynomial is three or higher, the analytical assessment of the roots is not a simple task. Fortunately, the knowledge of the exact location of the roots is not needed in determining the stability boundary. Here, the Routh-Hurwitz criterion will be used. The Routh-Hurwitz criterion is an algebraic method that indicates whether all roots of the characteristic equation have negative real parts without actually finding the roots.

According to the Routh-Hurwitz criterion a fourth-order system is stable if and only if:

1. all four coefficient \(a_0, a_1, a_2, a_3\) are positive and
2. \(a_1 a_2 a_3 - a_0 a_2^2 - a_1^2 > 0\).

The first claim will lead to the following constraints on the state feedback gains

\[
k_{c3} < c_i \tag{3.44a}\\
k_{c6} < 0 \tag{3.44b}\\
k_{c5} > \left(\frac{m_a}{m_b}\right)k_{c6} \tag{3.44c}
\]
Chapter 3 Active Suspension Control using Linear Quadratic Gaussian Control

\[ k_{c2} > \left( \frac{m_a}{m_b} \right) (c_s - c_s') - c_s - c_t \]  \hspace{1cm} (3.44d)

If the coefficient \( a_0 \) becomes negative, the system becomes monotonously unstable. Working out the second constraint equation results in

\[ (c_s + c_s + k_{c2}) \left( \frac{k_{c2}^2}{m_a m_b} - \frac{k_{c5} k_{c6}}{m_a^2} \right) + (c_s - k_{c3}) \left( \frac{k_{c5} k_{c6}}{m_a m_b} - \frac{k_{c5}^2}{m_a} \right) - \frac{k_{c6} k_{c5}}{m_a m_b} > 0 \]  \hspace{1cm} (3.45)

If the left-half member of (3.45) becomes negative then the quarter-car system becomes oscillatory unstable.

Now assume that only an unsprung mass velocity feedback exists \((k_{c2} = k_{c3} = k_{c5} = 0)\), then from

\[ -c_s \left( \frac{k_{c6}^2}{m_a^2} \right) > 0 \]  \hspace{1cm} (3.46)

it can be concluded that the closed-loop system is unstable. Thus, taking only the absolute velocity of the unsprung mass into account, will never stabilize the closed-loop system. This is in agreement with the observation made in section 3.2.1.

Figure 3.5 shows the stability boundary if \( k_{c6} \neq 0 \). The graph illustrates that at a given value of \( k_{c6} \), there exist a maximum value for gain \( k_{c5} \) for the system to remain stable.

![Figure 3.5 Active damping and stability.](image-url)
3.4 FULL-STATE VERSUS LIMITED STATE FEEDBACK CONTROL

This section deals with the comparison of results obtained with the different control structures (passive damping, full-state feedback, partial state feedback) assuming equal weighting of all three factors ($\bar{q}_1 = \bar{q}_2 = \bar{q}_3 = 1$). These results are summarized in the upper part of table 3.9. The numbers printed in boldface denote the increase of the preceding performance parameter compared with the passively damped system of the first row. Generally, ride comfort and suspension working space are improved and the variations of the dynamic tire load increase. Nevertheless the gains are very marginal considering the trouble that has to be taken. The differences between full-state feedback and limited state feedback are minimal. The partial state feedback is preferable because fewer states have to be determined. The overall performance of the full-state feedback system is better as indicated by the smaller performance index (P.I.) $J$.

<table>
<thead>
<tr>
<th>control structure</th>
<th>normalized performance values</th>
<th>P.I.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$P_{dyne}$</td>
<td>$z_a$</td>
</tr>
<tr>
<td>passive, $\kappa = 0.4579$</td>
<td>1.0752</td>
<td>0.0%</td>
</tr>
<tr>
<td>active, full state fb.</td>
<td>1.1784</td>
<td>9.6%</td>
</tr>
<tr>
<td>active, lim. state fb.</td>
<td>1.1827</td>
<td>10.0%</td>
</tr>
<tr>
<td>passive, $\kappa = 0.1494$</td>
<td>1.6581</td>
<td>54.2%</td>
</tr>
<tr>
<td>active, full state fb.</td>
<td>1.6581</td>
<td>54.2%</td>
</tr>
<tr>
<td>active, lim. state fb.</td>
<td>1.6581</td>
<td>54.2%</td>
</tr>
</tbody>
</table>

Table 3.9 Comparison of performance (concrete road, $v=30$ m/s).

The lower part of table 3.9 shows again all performance parameters. In this case a special weighting combination is used. The weighting factors $\bar{q}_1$ and $\bar{q}_3$ are chosen in such a way that the admissible maximum performance values for dynamic tire load variations and suspension working space meet the statements made in section 2.1. Under these circumstances a tremendous gain of about 45% on ride comfort can be achieved without too much increase of dynamic tire load variations or excessive suspension travel. Table 3.10 shows the feedback gains for both full and partial state feedback. Although suspension working space is not weighted in the cost function using partial state feedback ($\bar{q}_3 = 0$), its RMS value does not increase significantly. A very interesting result indicated in table 3.9 is that the passive suspension system with a fixed damping coefficient of 0.1494 (chosen to equal active $P_{dyne}$ performance) provides competitive performance compared with the two active
counterparts. When involving cost price and hardware requirements this would give the impression that an adaptive passive system is far more favorable. However, such a low damping causes a huge resonant peak at the sprung mass natural frequency that can not be accepted. Furthermore, softly damped suspension systems provide a very bad transient response. It will take several seconds before the motion of the sprung mass is damped out after a step input.

<table>
<thead>
<tr>
<th>normalized weighting factors</th>
<th>gain $k_1$ ($z_r$)</th>
<th>gain $k_2$ ($z_a$)</th>
<th>gain $k_3$ ($z_a$)</th>
<th>gain $k_4$ ($\dot{z}_r$)</th>
<th>gain $k_5$ ($\dot{z}_a$)</th>
<th>gain $k_6$ ($\dot{z}_a$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{q}_1$</td>
<td>3.278E-1</td>
<td>1.0</td>
<td>8.426E-5</td>
<td>-1024</td>
<td>-5409</td>
<td>8654</td>
</tr>
<tr>
<td>$\bar{q}_2$</td>
<td></td>
<td></td>
<td></td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\bar{q}_3$</td>
<td>3.336E-1</td>
<td>1.0</td>
<td>0.0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\bar{q}_4$</td>
<td></td>
<td></td>
<td></td>
<td>0</td>
<td>429</td>
<td>-375</td>
</tr>
<tr>
<td>$\bar{q}_5$</td>
<td></td>
<td></td>
<td></td>
<td>0</td>
<td>417</td>
<td>-2981</td>
</tr>
</tbody>
</table>

Table 3.10 State feedback gains.

Figure 3.6 presents a comparison between frequency response functions of three suspension systems. The settings correspond with the lower part of table 3.9. As already expected the gain in ride comfort meets with an increase of tire load variations and working space. Almost the entire contribution of this increase is situated near the wheel-hop natural frequency. However, it appears to be possible to improve both road holding and ride comfort in the lower frequency range (1 up to 10 Hz). The fully active system is characterized by its steady-state error (graph d). Due to this error the actuator force is also much larger in the low frequency range compared with the partial state feedback system. Only the course of the modulus of frequency response function (d) of the limited state feedback configuration is satisfying; it does not magnify the modulus near the sprung mass natural frequency (the amplitude ratio is always smaller than 1.0) and it doesn’t exhibit a steady-state offset. However, difficulties may be expected at a ramp input. When the vehicle goes up- or downhill with a constant speed, a constant vertical component of the speed of travel exists that causes the sprung and unsprung mass to diverge or approach depending on the feedback gains and the speed of travel.

The permissible values of dynamic tire load variations and suspension working space are not to be exceeded. But, including these admissible values in the determination of the control law would require an adaptive control strategy because the actual values depend on the road condition and the vehicle speed. Thus for optimal vibration isolation under a wide variety of road disturbance inputs, loadings and vehicle speeds it is necessary to design a family of controllers (set of gains). Each member of this family is appropriate
for a specific operating point within the operational envelope of the vehicle. Applying adaptive control it becomes necessary to identify the operating point of the vehicle in terms of road disturbance level, loading and vehicle speed. Some supervisor algorithm may then continuously monitor the average values of the conditions under which the vehicle is operating and then changes the control law setting.

![Figure 3.6 Frequency response functions of three suspension systems.](image)

(a) tire deflection  
(b) sprung mass acceleration  
(c) suspension working space  
(d) sprung mass displacement  
(e) actuator force
Rapid adaptation of control law parameters can, in principle, be realized but the system behavior cannot be predicted under those conditions from the theory so far developed. For the steady-state vibrational properties to change in a controlled manner, the variations of the coefficients must be accomplished slowly in comparison with the dynamics of the system. Adaptation over several seconds fits this specification.

From the former part it can be concluded that it is advisable to apply an adaptive control strategy emphasizing ride comfort for straight smooth/moderately uneven roads and accentuating road holding for handling, rough roads and discrete severe events like potholes. When using adaptive control it is necessary to monitor e.g. the dynamic tire load. Chapter 5 deals with the design of such an adaptive controller.

<table>
<thead>
<tr>
<th>control structure</th>
<th>rise time ( T_r ) (s)</th>
<th>delay time ( T_d ) (s)</th>
<th>settling time ( T_s ) (s)</th>
<th>% maximum overshoot</th>
<th>maximum ( z ) (( \text{m} / \text{s}^2 ))</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>unsprung mass</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>passive ( k = 0.4579 )</td>
<td>0.017</td>
<td>0.0%</td>
<td>0.021</td>
<td>0.0%</td>
<td>0.164</td>
</tr>
<tr>
<td>active, full state fb.</td>
<td>0.016</td>
<td>-5.3%</td>
<td>0.020</td>
<td>-1.9%</td>
<td>0.237</td>
</tr>
<tr>
<td>active, lim. state fb.</td>
<td>0.016</td>
<td>-4.1%</td>
<td>0.020</td>
<td>-1.5%</td>
<td>0.241</td>
</tr>
<tr>
<td><strong>sprung mass</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>passive ( k = 0.4579 )</td>
<td>0.124</td>
<td>0.0%</td>
<td>0.083</td>
<td>0.0%</td>
<td>1.051</td>
</tr>
<tr>
<td>active, full state fb.</td>
<td>0.103</td>
<td>-16.0%</td>
<td>0.071</td>
<td>-14.5%</td>
<td>0.355</td>
</tr>
<tr>
<td>active, lim. state fb.</td>
<td>0.249</td>
<td>101.4%</td>
<td>0.104</td>
<td>24.9%</td>
<td>0.336</td>
</tr>
<tr>
<td><strong>unsprung mass</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>passive ( k = 0.1494 )</td>
<td>0.015</td>
<td>0.0%</td>
<td>0.030</td>
<td>0.0%</td>
<td>0.507</td>
</tr>
<tr>
<td>active, full state fb.</td>
<td>0.015</td>
<td>-0.7%</td>
<td>0.020</td>
<td>-32.0%</td>
<td>0.463</td>
</tr>
<tr>
<td>active, lim. state fb.</td>
<td>0.015</td>
<td>-0.7%</td>
<td>0.020</td>
<td>-33.7%</td>
<td>0.464</td>
</tr>
<tr>
<td><strong>sprung mass</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>passive ( k = 0.1494 )</td>
<td>0.129</td>
<td>0.0%</td>
<td>0.123</td>
<td>0.0%</td>
<td>2.187</td>
</tr>
<tr>
<td>active, full state fb.</td>
<td>1.563</td>
<td>1115%</td>
<td>0.561</td>
<td>597.7%</td>
<td>1.969</td>
</tr>
<tr>
<td>active, lim. state fb.</td>
<td>0.491</td>
<td>280.7%</td>
<td>0.173</td>
<td>40.0%</td>
<td>0.702</td>
</tr>
</tbody>
</table>

Table 3.11 Comparison of performance, step input (height = 5 cm, \( v = 10 \text{m/s} \)).

As stated in section 2.2 we should also pay attention to the transient response of the controlled system. The transient behavior can be assessed using for example the step response. To do so, an obstacle (0.05 m high kerb) will be crossed with a forward velocity of 10 m/s. Figure 3.7 shows four important responses in case all weighting factors are equal to 1.0 (first part of table 3.9) and figure 3.8 shows the same responses in case of the special weighting
Chapter 3  Active Suspension Control using Linear Quadratic Gaussian Control

combination (second part of table 3.9). Table 3.11 gives the characteristic numerical values coupled with the step response (according to figure 2.8).

The first figure shows the displacement of the sprung mass. The ideal response is a fast response without the appearance of an overshoot. Only the active system with limited state feedback meets this requirement. The passive suspension shows an overshoot and the active counterpart based on full-state feedback appears to have a steady-state error.

The same steady-state error can be noticed in the graph presenting the suspension working space. In the real vehicle the axle will probably hit one of the bump stops causing a progressive non-linear stiffening of the suspension. From the tire load graph it can be seen that the tire looses contact with the ground (tire load = 0) during approximately 0.5 m (note the different x-scale!). Finally, the first peak of the sprung mass acceleration is in both active cases somewhat lower than in the passive situation (table 3.11).

Applying the special weighting combination in order to gain as much as possible on ride comfort without too much increase in dynamic tire load variations (stochastic case only) results in a much worse response. Again the sprung mass motion is only damped satisfactorily in combination with limited state feedback at the cost of rise time. Due to the low damping the passive system is vibrating quite a long time as indicated by the large value of the settling time. Also the overshoot of the unsprung mass is unacceptable. The full-state feedback controlled active systems shows even a larger steady-state error than in the former case. From the course of the tire load we can conclude that the tire is bouncing vehemently causing three times a loss of contact with the ground. In practice this cannot be accepted. In contrast to this, sprung mass accelerations are reduced significantly. The value of the first peak is halved compared with that one of figure 3.7.

The presence of a steady-state error in case of the active full-state feedback system is caused by the different feedback gains \( k_{c2} \) and \( k_{c3} \) with relation to states \( z_a \) and \( z_b \) and gain \( k_{c1} \). After having crossed the kerb the actuator will generate a permanent force (opposed to the suspension spring force) because these gains do not meet a special combination. If the quarter-car is at rest (accelerations and velocities are equal to zero) then the static equilibrium can be derived from equation (1.1) and (3.13)

\[-c_t(z_a - z_r) + c_b(z_b - z_a) - k_{c1}z_r - k_{c2}z_a - k_{c3}z_b = 0\]  

(3.47)
Because the axle and the road are only connected by means of a tire spring, 
\( z_r = z_a \). Therefore, relation (3.47) can be rewritten as

\[
\frac{z_b}{z_r} = \frac{c_s + k_{c1} + k_{c2}}{c_s - k_{c3}} \tag{3.48}
\]

The steady-state error of the sprung mass motion can only be eliminated if and only if

\[ k_{c3} = -k_{c1} - k_{c2} \tag{3.49} \]

Another possibility to eliminate the steady-state error is to include an integrator in the feedback loop according to

\[
u = -k_{c4} z_r - k_{c5} z_a - k_{c6} \dot{z}_r - k_{c7} \ddot{z}_a - k_{c8} \ddot{z}_b - k_{c7} \int (z_r - z_b) \, dt \tag{3.50} \]

The integrator term in the control structure is responsible for the elimination of the error because the presence of such an offset \( (z_r - z_b \neq 0) \) will be suppressed by means of feedback gain \( k_{c7} \) in combination with the feedback (PI-control instead of P-control). However, problems can still be expected for a ramp input. In case of a ramp, accelerations are still zero. Nevertheless the sprung and unsprung mass vertical velocities \( (\dot{z}_a \) and \( \dot{z}_b \)) are constant but not equal to zero (depending on the slope of the ramp). Thus in this case an offset is also present for the limited state feedback case.

In this chapter it has been shown that linear optimal control can be used quite easily for the design of a controller for an active suspension system. The main part of the analysis has been devoted to the selection of the appropriate weighting factors. The study has shown that it is not possible to improve both ride comfort and road holding at the same time. However, emphasizing either ride comfort or road holding leads to a substantial increase in performance compared with a passive suspension. The trade-off between ride comfort and road holding implies that the basic LQG feedback controller must be extended with adaptive properties. Both full-state and limited state feedback controllers have been calculated. The latter category exhibits similar performance as the full-state feedback system. The system with a feedback based on the absolute vertical sprung and unsprung mass velocity has the advantage that less states have to be determined when the controller is implemented in a real vehicle. Furthermore, no steady-state error will appear with a step input.

In the next chapter, the influence of a hardware reduction on the performance will be investigated. The active force generator will be replaced by an adjustable shock absorber.
Figure 3.7 Step responses (equal weighting).

--- passive, $k = 0.4579$

--- active, limited state feedback

--- active, full-state feedback
Figure 3.8 Step responses.

---
passive, $\kappa = 0.4579$

- - - active, limited state feedback

---
active, full-state feedback
SEMIB-ACTIVE SUSPENSION CONTROL

In this chapter the design and evaluation of control strategies will be discussed in combination with semi-active damper (SAD) systems. The distinction between active and semi-active lies in the field of force generation. Adjustable damper systems can only dissipate energy whereas active systems can also generate energy. However, this restriction does not necessarily imply that the performance of adjustable damper systems is inferior to active suspension systems.

From the active suspension analysis we know that it is not possible to improve ride comfort and road holding at the same time. Therefore, a distinction will be made between ride comfort control (section 4.1) and road holding control (section 4.2). The performance of a semi-active damper system strongly
depends on the range of the adjustable damper. This topic will be entered into
detail in section 4.3.

Semi-active suspension control is based on similar control algorithms as
applied to active suspension systems. However, the most important difference
between active and semi-active systems is that a semi-actively damped
suspension is only capable of dissipating energy. Since the adjustable damper
cannot supply power to the system, the best it can do is to generate no
damping force at all when the active solution requires a force that cannot be
generated by the damper. In practice this means that the damper is switched
to the lowest setting possible with the smallest damping constant. Drawing a
force versus relative velocity diagram (figure 4.1) turns up the important
differences between active and semi-active damper systems. The active system
can generate forces in all four quadrants whereas the semi-active system
reaches only two quadrants. It can also be seen that the continuously variable
damper possesses more resemblance to the actuator than the two-state
damper. In practice damping cannot be eliminated totally and it is also not
possible to manufacture a very firm damper. Therefore, it is not possible to
reach the entire area of both quadrants. The effective area is bounded by the
lowest and highest possible damping constants \( k_{low} \) and \( k_{high} \).

![Diagram](image)

**Figure 4.1** From active to semi-active.

The oncoming section will deal with the design of controllers for adjustable
damper systems. It is not the intention to design a control algorithm to adapt
the suspension damper to changing circumstances like road conditions,
loading or vehicle speed. These kinds of controllers are characterized by a slow
change of the control parameter (damping constant) and these suspension
systems are often denoted by the term adaptive suspension. The semi-active
suspension system is characterized by a rapidly adjustable damper that is
continuously tracking a desired force prescribed by a control law based on an active suspension system.

4.1 **RIDE COMFORT CONTROL**

The improvement of ride comfort can be obtained by softening the suspension. On the one side the suspension spring can be softened (relative control), on the other hand the suspension damper can be altered in such a way that the sprung mass accelerations are reduced (skyhook control). Both ways will be examined.

**Skyhook Control**

The starting-point of semi-active suspension control is active damping based on an ideal actuator, situated between sprung and unsprung mass with actuator force to be specified. Using optimal control in combination with limited state feedback the following control structure can be determined

\[
F_{\text{act}} = k_1 \dot{z}_a + k_2 \dot{z}_b \quad (4.1)
\]

where \( k_1 \) and \( k_2 \) are two feedback gains related to the unsprung and sprung mass absolute velocity respectively. The gains can be determined according to the procedure as discussed in section 3.2. Table 4.1 shows the weighting combinations and feedback gains considered in this study.

<table>
<thead>
<tr>
<th>weighting factor</th>
<th>( F_{\text{act}} = k_1 \dot{z}_a + k_2 \dot{z}_b )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \bar{q}_1 )</td>
<td>( \bar{q}_2 )</td>
</tr>
<tr>
<td>1</td>
<td>0.001</td>
</tr>
<tr>
<td>1</td>
<td>0.01</td>
</tr>
<tr>
<td>1</td>
<td>0.1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>10</td>
</tr>
<tr>
<td>1</td>
<td>100</td>
</tr>
<tr>
<td>1</td>
<td>1000</td>
</tr>
</tbody>
</table>

**Table 4.1** Feedback gains related to active damping.
First, a few comments on the normalized weighting factors and feedback gains. The three criteria of interest are road holding, ride comfort and suspension working space. Using optimal control they can be weighted in a quadratic criterion (equation 3.5) with weighting factor $\bar{q}_1$, $\bar{q}_2$ and $\bar{q}_3$ respectively. In table 4.1 weighting factor $\bar{q}_3$ (suspension working space) equals zero. Calculations of the RMS suspension working space value have shown that it is not necessary to weight this term because no excessive wheel travel will occur with the considered feedback combinations according to equation (4.1) (unless $\bar{q}_2$ gets extremely large, or a rough road is encountered). This enables us to calculate several feedback combinations by varying only one (normalized) weighting factor $\bar{q}_2$ (ride comfort). By making $\bar{q}_2$ very small, dynamic tire load variations are attached importance to and a large value of $\bar{q}_2$ accentuates ride comfort. The course of the gains $k_1$ and $k_2$ at varying $\bar{q}_2$ is very striking. For a small value of $\bar{q}_2$ (accentuating road holding) both gains become almost constant at decreasing $\bar{q}_2$. It seems that the reduction of dynamic tire load variations is limited by the structure of the feedback by means of the two velocity related states. Furthermore, $k_2$ becomes very large with increasing $\bar{q}_2$. Together with a rapidly decreasing $k_1$ this means that the actuator force is almost proportional to the sprung mass velocity $\dot{z}_s$ for large values of $\bar{q}_2$.

Figures 4.2 and 4.3 show the performance of the quarter-car equipped with an active suspension system in terms of normalized performance parameters. The RMS values of this active suspension system serve as a yardstick for the semi-active damper systems. Furthermore, the active suspension has been compared with a passive system with various damping rates ranging from 'soft' to 'firm'. Besides the analytically determined results (solid lines), simulation results have been included (dashed lines). Due to the non-linear character of adjustable damper systems, time domain simulations are necessary if the performance has to be evaluated. The curves obtained by simulation serve in the linear active suspension case merely as a verification of the simulation method.

The RMS values obtained by simulation are valid for a finite time. If this finite time is chosen long enough (328 seconds in this case), the RMS values will not differ much from those values obtained by analytical calculations and thus for an infinite time. The simulated model includes also a non-linear tire stiffness. The tire is only able to transmit compression forces. Especially for poorly damped (passive) suspension systems a difference exists between the analytically determined results and those obtained by simulation due to the
non-linear tire stiffness. All results presented in this chapter are valid for a concrete road with a speed of travel equal to 30 m/s.

From the performance related figures it can be seen that any improvement in ride comfort (lower level of the sprung mass acceleration) is accompanied by a deterioration of road holding as indicated by the increase in dynamic tire load variations. On the other hand, the possible improvement of the road holding ability is only marginal compared with the potential to improve the ride comfort parameter. This ride improvement is (theoretically) not limited. As \( q_2 \) gets larger, \( k_2 \) gets larger and ride comfort is improved correspondingly. A single point of contact exists between both (active and passive) curves at \( \bar{q}_2 = 0.32 \). This means that optimal control with \( \bar{q}_2 = 0.32 \) results in a passive feedback structure (\( k_2 = -k_1 = 1122 \) Ns/m). The active curve never intersects the passive one. Compared with the passive suspension system, for \( \bar{q}_2 > 0.32 \) any improvement of ride comfort is accompanied by a deterioration of the road holding and for \( \bar{q}_2 < 0.32 \) any improvement of the road holding ability is accompanied by an increase in sprung mass accelerations. A similar trade-off exists between suspension working space and ride comfort. Any ride comfort improvement goes with an increase of suspension working space, even in the active suspension case.

Figures 4.2 and 4.3 contain also a so-called 'optimal' passive configuration. This passive setting has been determined using optimal control in combination with limited state feedback. The feedback gain (= damping constant) related to \( \dot{z}_h - \dot{z}_u \) has been calculated for \( \bar{q}_1 = \bar{q}_2 = 1.0 \) and \( \bar{q}_3 = 0.0 \). Thus, ride comfort and road holding are equally weighted and similar to the active damping analysis (table 4.1) no weight has been put on suspension working space. The damping constant obtained with this method equals 727 Ns/m. This is equivalent to a realistic dimensionless damping coefficient \( \kappa \) equal to 0.27. Although this setting depends completely on the weighting combination chosen, it is labelled 'optimal' in the figures. This 'optimal' passive suspension configuration allows a fair comparison with other (semi-)active systems.
**Figure 4.2** RMS values, ride comfort versus road holding (cf. table 4.1).  
(—— analytical, - - - simulation)

**Figure 4.3** RMS values, working space versus ride comfort (cf. table 4.1).  
(—— analytical, - - - simulation)
Using a semi-active damper, whether discretely or continuously adjustable, it may now be possible to generate forces which attempt to approximate a desired actuator force $F_{act}$ according to equation (4.1) by adjusting the damping constant $k_s$ as expressed by

$$F_{sad} = k_s(\dot{z}_a - \dot{z}_o) \quad k_{low} \leq k_s \leq k_{high}$$

(4.2)

Since the semi-active damper cannot supply power to the system, the best it can do is to generate no damping force at all when $F_{act}$ and $F_{sad}$ differ from sign. However, in practice damping cannot be eliminated totally. In that case the damper is switched to lowest setting possible. With the considerations made above the following control strategy for the continuously variable damper can be constructed: if $F_{act}$ and $F_{sad}$ have the same sign (forces act in the same direction) then

$$k_s = \frac{k_1 \dot{z}_a + k_2 \dot{z}_o}{(\dot{z}_o - \dot{z}_a)} \quad k_{low} \leq k_s \leq k_{high} \quad F_{act} \cdot F_{sad} > 0$$

(4.3)

If both forces have different signs then

$$k_s = k_{low}, \quad F_{act} \cdot F_{sad} < 0$$

(4.4)

Taking only the signs of the actuator force and the damper force into consideration will lead to the following switching pattern for the 2-state damper

<table>
<thead>
<tr>
<th>$F_{act} \cdot F_{sad}$</th>
<th>HIGH</th>
<th>LOW</th>
</tr>
</thead>
<tbody>
<tr>
<td>$&gt; 0$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$&lt; 0$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 4.2 Skyhook control, 2-state damper.

Unlike controlling the damper by comparing the sign of the desired actuator force with the available damper force, one could also involve the magnitude of the desired force in the control algorithm of the 2-state damper. The damper will remain in its low setting when the desired force and the damper force are of opposite sign. When both forces possess the same sign, the selected damper setting will not always be 'high' like in table 4.2, but will also depend on the magnitude of the desired force and the two possible damper forces. The damper force that matches in an absolute way the best with the desired active force is chosen as the optimal setting. This control policy is shown in table 4.3 and will be labelled 'extended skyhook control'.

- 73 -
Table 4.3 Extended skyhook control, 2-state damper.

According to table 4.1 and figures 4.2 and 4.3, the ultimate way to reduce the sprung mass accelerations is to apply an actuator force $F_{act}$ which is merely proportional to the sprung mass absolute velocity $\dot{z}_b$ according to

$$F_{act} = k_{sky} \dot{z}_b$$  \hspace{1cm} (4.5)

where $k_{sky}$ represents the so-called skyhook damping constant. This configuration, together with a real skyhook system and a semi-active damper system with skyhook control are shown in figure 4.4.

![Skyhook control, from active to semi-active.](image)

Although there is a substantial difference between a real skyhook system and the active counterpart (the actively generated force acts on both sprung and unsprung mass; the real skyhook damper force acts only on the sprung mass), the frequency response functions of both systems show great correspondences. The five response functions of interest are plotted in figure 4.5.

The modulus of the tire deflection transfer function (graph a) of the active system with skyhook control differs from the curve valid for a real skyhook system. The reason is the point of action of the damper/actuator force. Not totally unexpected is the presence of a huge resonance peak nearby 12.6 Hz (unsprung mass natural frequency). Since the motion of the unsprung mass is
not damped properly in case of the skyhook damper or active system with skyhook control, a resonance peak is the result.

Figure 4.5 Frequency response functions of three suspension systems. (true skyhook, active with skyhook control and passive with $\kappa = 0.46$)

(a) tire deflection  
(b) sprung mass acceleration  
(c) suspension working space  
(d) sprung mass displacement  
(e) actuator/damper force

A small but not insignificant difference is the height of the resonance peak. In case of the true skyhook system the wheel-hop mode is not damped at all, resulting in an infinite modulus at that specific frequency. The active system with skyhook control however has a bounded finite amplitude. Although
small, the sprung mass velocity is not equal to zero near the wheel-hop
natural frequency. This, combined with the fact that the actuator force
depends solely on the sprung mass velocity, and that this force acts on both
sprung and unsprung masses, results in a finite modulus. In both skyhook
cases the damping selected is equal to the critical one ($\kappa = 1.0 \rightarrow k_{sky} = 2683$
Ns/m).

With exception of the area near the unsprung mass natural frequency, both
skyhook versions (true one and active approximation) offer a large
improvement in vibration isolation (graph b) over the entire frequency range
compared with the passive system with a fixed suspension damping.
Anticipating the simulation results of the semi-active system with skyhook
control, an improvement in ride comfort is to be expected at the cost of a
degradation of tire load variations and suspension working space.

Now that the performance of the active suspension with partial state feedback
has been evaluated, it is interesting to examine the potential of the semi-
active suspension system. Two kinds of adjustable dampers are considered.
The first one is continuously variable (CSAD) between $k_{low}$ and $k_{high}$. The
second damper is discretely controllable (DSAD) and has only two states: $k_{low}$ or
$k_{high}$. The damper rates chosen are:

- $\kappa_{low} = 0.05 \rightarrow k_{low} = 134$ Ns/m
- $\kappa_{high} = 1.0 \rightarrow k_{high} = 2683$ Ns/m

The adjustable damper has equal damping constants for the bump and
rebound motion. The low and high boundaries are not claimed to be an
optimum ratio. They are based on what is feasible in practice. Other ranges
are considered in section 4.3. No dynamics concerning the damper adjustment
are included in the simulation model. The bandwidth of the adjustment is
therefore limited by the simulation time step which is fixed and equal to 2 ms.

Figures 4.6 and 4.7 present the performance of the semi-active systems
together with the curves valid for the fully active system (dashed line) and a
range of passive systems with fixed damping rates (dash-dot line). Three semi-
active curves are plotted: (i) CSAD, continuously variable damper, (ii) DSAD,
discretely variable damper with 2-states, (iii) DESAD, 2-state damper with
extended control. Unlike on-off sign control of DSAD, the extended version
(DESAD) involves the magnitude of the desired force in the switching criterion
(table 4.3). Again weighting factor $\bar{q}_2$ varies along the curves. The optimum $k_1$
and $k_2$ are calculated and $F_{sad}$ is adjusted as explained before.
Figure 4.6 shows that especially the continuously variable damper performs almost as well as a true active force generator. For higher values of $\bar{q}_2$ (or larger values for gain $k_2$) the adjustable damper is not able to approximate the desired force prescribed by equation (4.1). Probably, the required force is too often in the wrong quadrant. Since the lower damping boundary is not equal to zero, both ride comfort and tire load variations are bounded. As soon as gain $k_2$ increases, and thus the desired system starts to resemble more and more a true skyhook system, working space as well as tire load variations increase. If $k_2$ approaches infinity, the curve will end in point 'M'. The two other curves valid for the 2-state systems will also end in point 'M'. The desired force for very large values of gain $k_2$ (and $k_1 = 0$) becomes so large that all three systems behave like an on-off controller. Obviously, the high-state damping rate $k_{\text{high}}$ multiplied with the relative velocity $\dot{z}_h - \dot{z}_a$ is too small to track the desired force $k_2 \dot{z}_h$. The 2-state damper system (DSAD) doesn't perform that well. The curve starts rather high for small values of $\bar{q}_2$ and ends in point 'M' where true skyhook behavior is desired. The large difference between the low and high state damping might be blamed for this behavior. At the same level of discomfort the continuously variable damper is preferable because the level of tire load variations is substantially lower. The 2-state damper with extended skyhook control performs better. However, the extra gain in ride comfort is attended with an extra loss in tire load variations. Nevertheless, it is surprising that the CSAD as well as the DESAD system are able to reduce the sprung mass acceleration to $\frac{3}{4}$ of the minimally achievable value obtain with a fixed passive damper setting ($\kappa = 0.07$).

Graph 4.7 illustrates again that the CSAD system resembles a true active system very much. For small values of $\bar{q}_2$ (emphasizing road holding) suspension working space can be reduced compared with the 'optimal' passive system. However, any gain in ride comfort is attended with an increase in suspension working space. Since the normalized RMS suspension deflection values are all smaller than 4.66 (see also section 2.1), there is no need for concern. The suspension will not hit the bump or rebound stops.

From the results as discussed above the conclusion can be drawn that the ride comfort improvement is almost completely owing to the skyhook damping component. The desired actuator force according to equation 4.1 can therefore be simplified to merely a feedback of the absolute vertical sprung mass velocity according to equation (4.5). The performances of the continuously adjustable as well as the 2-state damper system are now recalculated for this special situation. Figures 4.8 and 4.9 present the results.
Figure 4.6 Ride comfort versus road holding ($F_{act} = k_1 \ddot{z}_a + k_2 \ddot{z}_b$).

Figure 4.7 Suspension working space versus ride comfort ($F_{act} = k_1 \ddot{z}_a + k_2 \ddot{z}_b$).
Figure 4.8 Ride comfort versus road holding ($F_{act} = k_{sky} \dot{z}_b$).

Figure 4.9 Suspension working space versus ride comfort ($F_{act} = k_{sky} \dot{z}_b$).
Again the three lines for the semi-active systems are plotted: (i) continuously variable damper, (ii) 2-state damper with on-off control, (iii) 2-state damper with extended on-off control. The performance of the 2-state damper with plane skyhook control doesn't depend on the magnitude of the skyhook damping constant $k_{\text{sky}}$ because the feedback structure is fixed and thus only determined by a single gain. Therefore, this damper system regards only the sign of the desired actuator force. Since this sign doesn't depend on $k_{\text{sky}}$, a single point (M) can be found for the DSAD system.

Including the magnitude of the desired skyhook force (DESAD) improves the performance. The curve goes from the point 'low' for very small values of $k_{\text{sky}}$ (damper remains in its low state) to point 'M' for $k_{\text{sky}} \to \infty$. If the skyhook damping constant becomes very large, all three systems behave as on-off damper systems because the desired force $(k_{\text{sky}} \dot{z}_b)$ is either in the wrong quadrant or is too large (larger than $k_{\text{high}}(\dot{z}_b - \dot{z}_a)$). In that case the range between $k_{\text{low}}$ and $k_{\text{high}}$ is hardly used in the continuously variable damper case.

Looking at the ride comfort parameter, there seems to exist one optimal value for $k_{\text{sky}}$. Giving $k_{\text{sky}}$ the value of say 7500 Ns/m, it is possible to decrease the sprung mass acceleration to 3/4 of the minimum achievable value obtained with a passive suspension with a fixed damping coefficient $\kappa = 0.07$. This is in agreement with the observations done before.

The suspension working space is only affected for very small values of $k_{\text{sky}}$. This is not surprisingly since the adjustable damper will pretty much always remain in the low setting. The RMS value of the suspension deflection decreases rapidly for larger values of $k_{\text{sky}}$. The values remain all within the limitations of the admissible suspension deflection. Table 4.4 gives a summary of RMS results as a function of the skyhook damping constant and type of adjustable damper. The percentages are gains or losses with respect to the 'optimal' passive suspension with a fixed damping constant of 727 Ns/m.

With the help of extended skyhook control in combination with a discretely controllable semi-active damper it is possible to compete for the performance of the continuously variable damper. The explanation can be found when figure 4.10 is entered into detail. The dotted line is valid for the passive suspension system with a fixed damping coefficient of 0.27. The solid line is the CSAD system and the dashed one corresponds to the DESAD system. From the top figure it can be seen that the switching pattern of the continuously
controllable semi-active damper (CSAD) very much agrees with that of the
discretely adjustable damper (DSAD). Only during relatively short periods,
damper settings between the two boundary values ($k_{low}$ and $k_{high}$) are used.
The plotted results are valid for a concrete road and a speed of travel equal to
30 m/s. The three graphs beneath the switching patterns show the sprung
mass acceleration, the tire load and the suspension working space. From these
figures it can again be concluded that sprung mass accelerations can be
reduced at the cost of dynamic tire load variations and suspension working space.

<table>
<thead>
<tr>
<th>control structure</th>
<th>normalized performance values</th>
<th>$P_{den}$</th>
<th>$\ddot{z}_h$</th>
<th>$\ddot{z}_{h,iso}$</th>
<th>$z_a - z_b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>passive, $\kappa = 0.05$</td>
<td></td>
<td>2.6640</td>
<td>-108.8%</td>
<td>1.0911</td>
<td>15.5%</td>
</tr>
<tr>
<td>passive, $\kappa = 0.27$</td>
<td></td>
<td>1.2758</td>
<td>0.0%</td>
<td>1.2905</td>
<td>0.0%</td>
</tr>
<tr>
<td>passive, $\kappa = 1.00$</td>
<td></td>
<td>1.0140</td>
<td>20.5%</td>
<td>2.3589</td>
<td>-82.8%</td>
</tr>
</tbody>
</table>

**CSAD**

$k_{sky}$ = 500

2.6590 -108.4% 0.9625 25.4% 0.9388 35.4% 3.3589 -91.0%

$k_{sky}$ = 1000

2.6367 -106.7% 0.8592 33.4% 0.8623 40.6% 2.8250 -60.6%

$k_{sky}$ = 5,000

2.4920 -95.3% 0.7728 40.1% 0.7712 46.9% 2.1501 -19.8%

$k_{sky}$ = 10,000

2.4162 -89.4% 0.8228 36.2% 0.7735 46.7% 2.1003 -17.9%

$k_{sky}$ = 50,000

2.2957 -79.9% 1.2923 -0.1% 0.9615 33.8% 2.0746 -17.9%

$k_{sky}$ = 100,000

2.2808 -78.8% 1.4850 -15.1% 1.0224 29.6% 2.0652 -17.4%

$k_{sky}$ $\rightarrow \infty$

2.0820 -63.2% 1.7702 -37.2% 1.2686 12.7% 1.9696 -12.0%

**DESAD**

$k_{sky}$ = 500

2.6671 -109.1% 1.0645 17.5% 1.0080 30.6% 3.8119 -116.7%

$k_{sky}$ = 1000

2.6693 -109.2% 1.0071 22.0% 0.9542 34.3% 3.4780 -97.7%

$k_{sky}$ = 5,000

2.6448 -107.3% 0.9099 29.5% 0.8070 44.4% 2.4233 -37.8%

$k_{sky}$ = 10,000

2.6049 -104.2% 0.9752 24.4% 0.7801 46.3% 2.2386 -27.3%

$k_{sky}$ = 50,000

2.4278 -90.3% 1.3508 -4.7% 0.8612 40.7% 2.1365 -21.5%

$k_{sky}$ = 100,000

2.2992 -80.2% 1.5461 -19.8% 1.0997 30.5% 2.0725 -17.8%

$k_{sky}$ $\rightarrow \infty$

2.0820 -63.2% 1.7702 -37.2% 1.2686 12.7% 1.9696 -12.0%

**DSAD**

2.0820 -63.2% 1.7702 -37.2% 1.2686 12.7% 1.9696 -12.0%

Table 4.4 Comparison of RMS performance values.

(skyhook control, concrete road, 30 m/s)
Figure 4.10 Time domain example of semi-active suspension control.

--- CSAD, skyhook control, $k_{sky} = 7500$ Ns/m

--- DESAD, skyhook control, $k_{sky} = 7500$ Ns/m

--- passive suspension ($\kappa = 0.27$)
Besides the discussed RMS performance values, frequency response functions (FRFs) are of particular interest. However, judging FRFs of non-linear systems, such as adjustable damper systems, must be done with special care. In fact no transfer function exists because the amplitude ratio and phase angle depend completely on the road input signal. Therefore, power spectral densities (PSD) are calculated. The PSDs are calculated by fast fourier transform (FFT) of the time domain signals [5]. Figure 4.11 presents the results valid for a continuously variable damper system (CSAD) controlled by the skyhook algorithm with $k_{sky} = 7500$ Ns/m. Furthermore, the figures contain the curves for a true skyhook suspension also with $k_{sky} = 7500$ Ns/m and a passive suspension.

![Power spectral densities](image)

**Figure 4.11** Power spectral densities (CSAD, skyhook control).
(a) tire deflection  (b) sprung mass acceleration
(c) suspension working space  (d) damper force

Most striking is the tremendous decrease of the sprung mass acceleration level (graph b). Up to 20 Hz the reduction is noticeable. Above 20 Hz the rather smooth course of the spectral density of graph (b) changes into quite a chaos. An increased level of high frequency accelerations and higher harmonics can be noticed. A smooth spectral density could not be achieved although the calculated results were obtained by averaging over 40 frequency windows. The main reason for the increase of high frequency accelerations is
the non-linear character of the semi-active damper which is caused by the fast switching cycles. Although the damper is of a continuously variable type, there are still moments that it is necessary to switch the damper almost instantaneously from the high state to the low state or visa versa. Since the relative velocity is not necessarily equal to zero, the transition meets with a discontinuity (step-shape) in the damper force. These discontinuities initiate high frequency vibrations.

By the application of rubber bushes in the damper fastening or by the inclusion of damper dynamics limiting the control bandwidth, a reduction of the level of accelerations in the high frequency range can be expected. However, it is likely that semi-active damping is always accompanied by an increase in high frequency vibrations (noise and harshness). Even continuously variable dampers suffer from this problem. Similar to the active suspension system with limited state feedback, the improvement of ride comfort meets with the increase of tire load variations (graph a) and suspension working space (graph c), especially near 12.6 Hz (wheel-hop natural frequency).

From the results obtained with skyhook control the conclusion can be drawn that the gain in ride comfort is accompanied by a substantial increase in tire load variations. While driving straight ahead the increase of tire load variations is acceptable. However, during cornering, braking and accelerating the tire load variations may cause difficulties regarding the handling of the vehicle. Adaptation of the control algorithm according to the external circumstances is necessary.

Relative Control

Through the study of optimal control in combination with full-state feedback, it can be seen that the actuator cancels the suspension spring when ride comfort is emphasized (section 3.1). Trying to eliminate spring forces by controlling a damper is completely different from doing this by means of an actuator. But, for a good vibration isolation it is possible to minimize the suspension forces acting on the vehicle body by the application of a controllable damper. During oscillations there are times that the spring force and the damper force act in opposite directions. This situation occurs when the relative displacement \( z_a - z_b \) opposes the relative velocity \( \dot{z}_a - \dot{z}_b \). In this case the damper can be altered in such a way that the damper force and the spring force tend to cancel each other. If the relative displacement and the
relative velocity act in the same direction the damper is switched to the lowest setting possible \( k_{\text{low}} \) in order to minimize the overall suspension force. This control algorithm will be labelled 'relative' control [20].

The semi-active damper force \( F_{\text{sad}} \) and the suspension spring force \( F_s \) are given by

\[
F_{\text{sad}} = k_s (\dot{z}_b - \dot{z}_a) \quad k_{\text{low}} \leq k_s \leq k_{\text{high}}
\]

\[
F_s = c_s (z_b - z_a)
\]

If both forces possess a different sign, then it is possible to eliminate the spring force acting on the sprung mass by means of a continuously variable damper with a damping constant equal to

\[
k_s = \frac{-c_s (z_b - z_a)}{\dot{z}_b - \dot{z}_a} \quad k_{\text{low}} \leq k_s \leq k_{\text{high}} \quad F_s \cdot F_{\text{sad}} < 0
\]

If the relative displacement and the relative velocity have the same sign the damper is switched to the lowest setting possible

\[
k_s = k_{\text{low}} \quad F_s \cdot F_{\text{sad}} > 0
\]

The 2-state damper offers less control possibilities. The most basic switching algorithm consists of monitoring the signs of the damper and the spring force. If the relative displacement and the relative velocity possess the same sign then the damper is switched to the soft setting. In all other situations the firm setting will be selected. This switching pattern is summarized in table 4.5

<table>
<thead>
<tr>
<th>( F_s \cdot F_{\text{sad}} )</th>
<th>LOW</th>
<th>HIGH</th>
</tr>
</thead>
<tbody>
<tr>
<td>( F_s \cdot F_{\text{sad}} &gt; 0 )</td>
<td><img src="LOW" alt="" /></td>
<td></td>
</tr>
<tr>
<td>( F_s \cdot F_{\text{sad}} &lt; 0 )</td>
<td></td>
<td><img src="HIGH" alt="" /></td>
</tr>
</tbody>
</table>

Table 4.5 Relative control, 2-state damper.

Unlike controlling the damper by comparing the sign of the spring force with the passive damper force, one could also involve the magnitude of the spring force in the control algorithm. The damper will remain in its low setting when the spring force and the damper force are of the same sign. When both forces posses an opposite sign, the selected setting will depend on the magnitude of the spring force and the two possible damper forces. The damper force that matches in an absolute way the best with the current spring force is chosen as the optimal setting. This control policy is shown in table 4.6 and will be labelled 'extended relative control'.

- 85 -
Table 4.6 Extended relative control, 2-state damper.

Table 4.7 shows that the RMS ride comfort parameter can only be improved with the help of the continuously variable damper (CSAD). However, the improvement is coupled with a drastic increase in tire load variations and suspension working space. The overall performance is not so good compared with the systems based on skyhook control. Both 2-state damper systems act quite strangely. They both improve the road holding at the cost of ride comfort and an enormous amount of suspension working space. Probably, the low and high state damping constants in combination with the suspension deflection rate do no match very well with the spring force.

<table>
<thead>
<tr>
<th>control structure</th>
<th>$P_{\text{dyn}}$</th>
<th>$\ddot{z}_b$</th>
<th>$\ddot{z}_{b,\text{iso}}$</th>
<th>$z_a - z_b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>passive, $\kappa = 0.05$</td>
<td>2.6640</td>
<td>1.0911</td>
<td>15.5%</td>
<td>1.0388</td>
</tr>
</tbody>
</table>
| passive, $\kappa = 0.27$ | 1.2758 | 1.2905 | 0.0% | 1.4524 | 0.0% | 1.7590 | 0.0%
| passive, $\kappa = 1.00$ | 1.0140 | 2.3589 | -82.8% | 3.0643 | -111.0% | 1.0000 | 43.1%
| CSAD               | 2.4787 | 0.8580 | 33.5% | 0.7987 | 45.0% | 3.0816 | -75.2%
| DSAD               | 1.1350 | 1.8812 | -45.8% | 1.8952 | -30.5% | 5.8320 | -231.6%
| DESAD              | 1.1350 | 1.8812 | -45.8% | 1.8952 | -30.5% | 5.8320 | -231.6%

Table 4.7 Comparison of performance, relative control (concrete road, 30 m/s).

Figure 4.12 presents a comparison between power spectral densities of a semi-active damper system based on relative control and the 'optimal' passive system. In the mid-range (2-12 Hz) ride comfort can be improved as indicated by the lower response of graph (b). However, near the sprung mass natural frequency (1 Hz) the semi-active system in combination with relative control is not damped very well. These phenomena can also be noticed in all other three graphs. The gain in ride comfort goes with a loss of dynamic tire load variations nearby the wheel-hop natural frequency (largest contribution) and sprung mass natural frequency.

With the relative control strategy a large resonance peak showed up near the sprung mass natural frequency. This peak is more than what can be considered as acceptable in the conventional passive case. The axle motion on
the other hand is better damped than with skyhook control. Comparing these results with those valid for a fixed low damper setting (\( \kappa = 0.05 \)), it can be concluded that the damper will pretty much always remain at the lowest setting possible near the sprung mass natural frequency. The main reason for this resonance phenomenon is that both sprung mass and unsprung mass move in phase for low frequencies up to about 1 Hz. This means that the spring force (relative displacement) as well as the damper force (relative velocity) act in the same direction. Consequently, according to equation (4.8) the low damper setting will be selected. Since a poorly damped motion of the sprung mass provides also considerable motions of the vehicle body, this is the cause for the large value of the suspension working space.

Figure 4.12 Power spectral densities (CSAD, relative control).
(a) tire deflection  (b) sprung mass acceleration
(c) suspension working space  (d) damper force
Chapter 4 Semi-Active Suspension Control

Step Responses

Similar to the active suspension analysis it is possible to give some reflections on the step responses of semi-actively damped systems. Figure 4.13 shows these responses. The kerb height is 0.05 m and the forward velocity amounts 10 m/s. The semi-active system contains a continuously variable damper and is controller according to the skyhook algorithm with \( k_{\text{sky}} \) equal to 7500 Ns/m

From the sprung mass motion it is seen that the semi-active suspension system provides a slower response without overshoot. The settling time remains the same. From the course of the tire load the conclusion can be drawn that the wheel is bouncing vehemently causing several times a loss of contact with the ground. In practice this cannot be accepted. In contrast to this, sprung mass accelerations are reduced significantly. The value of the first peak is only \( \frac{1}{2} \) of the passive one. As indicated by the first peak of the suspension working space, the axle will probably hit one of the bump stops causing a progressive non-linear stiffening of the suspension when semi-active suspension control is applied in a real vehicle.

![Graph](image)

**Figure 4.13a** Step responses of a semi-active and passive suspension system.

- --- passive suspension (\( \kappa = 0.27 \))
- --- CSAD, skyhook control, \( k_{\text{sky}} = 7500 \) Ns/m

- 88 -
Figure 4.13b Step responses of a semi-active and passive suspension system.

- passive suspension ($\kappa = 0.27$)
- - - CSAD, skyhook control, $h_{sky} = 7500 \text{ Ns/m}$

4.2 ROAD HOLDING CONTROL

This section deals with the analysis of control algorithms that have the aim of improving road holding of the quarter car by means of reducing the dynamic tire load variations. From the results obtained from the limited state feedback analysis of section 3.2 (considering only states $z_a$ and $z_b$) it can be seen that, comparing the performance with a firm ($\kappa = 0.80$) fixed passive damper, the dynamic tire load variations could hardly be improved using active damping (2.9% reduction). The optimal minimizing active solution was (table 3.3, first row)
\[ F_{act} = -2319 \cdot \dot{z}_a + 826 \cdot \dot{z}_b \]  \hfill (4.9)

With the help of a semi-active damper, whether discretely or continuously controllable, it might be possible to generate forces, which attempt to approximate this desired actuator force \( F_{act} \). However, it is likely to expect only a slight decrease of dynamic tire load variations when this desired actuator force is the basis for a semi-active control strategy. The analysis of semi-active performance based on equation (4.9) will be left out of consideration because the reduction of tire load variations by means of active suspension control is marginally better than the optimal road holding setting with damping \( \kappa = 0.80 \) and because the decrease is even small compared with the 'optimal' passive configuration with damping \( \kappa = 0.27 \) (21.6\% reduction).

Roadhook Control

Table 3.5 shows that dynamic tire load variations can significantly be suppressed when in addition to \( \dot{z}_a \) and \( \dot{z}_b \) also state \( \dot{z}_r \) is considered. In practice, however, it is very difficult to measure or estimate the absolute elevation speed of the road irregularities \( \dot{z}_r \). In theory also problems are likely to be encountered because according to equation (1.4) there is a straight relationship between \( \dot{z}_r \) and white noise \( w \). Thus \( \dot{z}_r \) cannot be regarded as a 'smooth' signal. It is therefore plausible that the inclusion of state \( \dot{z}_r \) in whatever control strategy will result in a very chaotic control signal. Nevertheless it is worth trying it. The actuator force which is responsible for the minimization of the dynamic tire load variations is given by

\[ F_{act} = k_{road}(\dot{z}_r - \dot{z}_a) \]  \hfill (4.10)

where \( k_{road} \) is the 'roadhook' damping constant. This configuration is shown in figure 4.14.

**Figure 4.14** Roadhook control, from active to semi-active.
In order to suppress dynamic tire load variations a firm damping parallel to the tire would be favorable. Using active suspension it is possible to generate a force on the wheel axle that is equivalent to a tire damping force. In practice, however, the actuator force does not represent true damping alongside a tire since the actuator is located between sprung and unsprung mass and not between road and unsprung mass. By means of active suspension control the proposed feedback law is not stable (see section 3.3). From table 3.5 it can be seen that a stable feedback can only be established when state $\dot{z}_b$ is included in the control law. Because instability of the semi-active suspension systems can never occur (only damping is added), state $\dot{z}_b$ can be neglected.

Figure 4.15 presents five frequency response functions of three different suspension systems: (i) a passive system with fixed suspension damping, (ii) a passive system with equal values for the fixed suspension damping and the fixed tire damping ($k_t = k_s = 1000$ Ns/m) and (iii) an active system with fixed suspension damping ($k_s$) and roadhook control ($k_{road} = k_t$). In case of the active system the additional feedback from $\dot{z}_b$ or in this case $\dot{z}_b - \dot{z}_a$ is necessary because otherwise closed-loop stability is not ensured. Most striking are the changes of the response functions due to the presence of actual tire damping in combination with passive suspension damping. The amplitude of the tire deflection response function as well as the amplitude of the sprung mass acceleration transfer function is substantially smaller near the wheel-hop natural frequency (12.6 Hz) than in case no tire damping is present. This means that both ride comfort and road holding can be improved significantly by adding damping to the tire. Only a small increase of tire load variations can be noticed in the low frequency range up to 10 Hz. However, this pales into insignificance compared with the gain nearby the wheel-hop natural frequency (logarithmic scale!). In case of the active system with combined passive damping and roadhook control the same reduction of tire load variations can be achieved. This, however, is accompanied by a significant deterioration of the sprung mass acceleration. This is due to the location of the actuator: the passive tire damper force acts only between road surface and unsprung mass, the artificial tire damping by means of active roadhook control acts between sprung and unsprung mass and therefore deteriorates ride comfort significantly.
Figure 4.15 Frequency response functions of three suspension systems (passive system with tire damping, active with roadhook control and passive).

(a) tire deflection  
(b) sprung mass acceleration  
(c) suspension working space  
(d) sprung mass displacement  
(e) actuator/damper force

In case of the semi-active damper system a switching pattern is necessary. The desired actuator force $F_{ac}$ yields from equation (4.10) and can be approximated using an adjustable damper similar to the previous derived switching patterns for semi-active damper system.

While simulating the semi-active suspension system in the time domain some difficulties are likely to be encountered using 'roadhook control'. As already
Chapter 4 Semi-Active Suspension Control

mentioned above, state $\dot{z}_r$ might be difficult to be determined due to the stochastic nature of the road profile. This problem can be avoided in the analysis by using the second-order road model as introduced in chapter 3 (equation 3.2). To prevent integrator instability $\omega_q$ is reduced to $200\omega_1$ rad/s.

From the simulation results as summarized in table 4.8 the conclusion can be drawn that reducing tire load variations does not fit with semi-active suspension control. One possible solution is to retain the damper for a certain time at the damper setting $\kappa = 0.80$ if excessive wheel-hop occurs.

<table>
<thead>
<tr>
<th>control structure</th>
<th>normalized performance values</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_{\text{dyn}}$</td>
<td>$\dot{z}_b$</td>
</tr>
<tr>
<td>passive, $\kappa = 0.05$</td>
<td>2.6640</td>
</tr>
<tr>
<td>passive, $\kappa = 0.27$</td>
<td>1.2758</td>
</tr>
<tr>
<td>passive, $\kappa = 1.00$</td>
<td>1.0140</td>
</tr>
<tr>
<td><strong>CSAD</strong></td>
<td></td>
</tr>
<tr>
<td>$k_{\text{road}} = 100$</td>
<td>2.8081</td>
</tr>
<tr>
<td>$k_{\text{road}} = 1000$</td>
<td>1.3244</td>
</tr>
<tr>
<td>$k_{\text{road}} = 10,000$</td>
<td>1.2057</td>
</tr>
<tr>
<td>$k_{\text{road}} = 100,000$</td>
<td>1.2013</td>
</tr>
</tbody>
</table>

Table 4.8 Comparison of performance, roadhook control.

**Skyhook Control**

From the results achieved with skyhook control (absolute damping of the sprung mass) one could imagine that a 'skyhook' damper mounted between unsprung mass and (non-moving) ground (or sky) would give similar outcomes. Such a damper, if firm enough, would pull up the axle preventing it from any motion. In this case the RMS value of the dynamic tire load variation is given by

$$\text{RMS}(P_{\text{dyn}}) = c_t \sigma_r = 1120 \text{ N}$$

(4.11)

where $\sigma_r$ is the standard deviation of road displacement $z_r$ according to table 1.2 and $c_t$ the tire stiffness. From the value shown we can conclude that this is not the right solution to suppress dynamic tire load variations.
4.3 THE INFLUENCE OF THE DAMPER RANGE ON THE PERFORMANCE

This section deals with the evaluation of the performance of a semi-active damper system as a function of the damper range. All calculations carried out before were valid for a damper with a lower damping coefficient $k_{\text{low}} = 0.05$ and a higher damping coefficient $k_{\text{high}} = 1.0$. These damper ranges were chosen because they were feasible in practice. However, they were not claimed to be an optimal ratio. Therefore, the influence of the damper range on the performance will be studied in this section.

Figure 4.16 shows 6 different configurations considered in this study. System A corresponds to a fully active system with properties as described above. All four quadrants of the force versus relative velocity diagram can be reached. Ideal damper B can only reach two quadrants. Damper C up to F differ in the range of possible damper settings. Damper E corresponds to the damper used in all calculations as carried out before. It is assumed that the adjustable dampers are continuously variable within the predefined range. No dynamics concerning damper adjustment are included in the simulation model.

![Diagram of damper characteristics](image)

**Figure 4.16** Different damper characteristics.

- $B \iff k_{\text{low}} = 0 \text{ Ns/m} \quad k_{\text{high}} = \infty \text{ Ns/m}$
- $C \iff k_{\text{low}} = 0 \text{ Ns/m} \quad k_{\text{high}} = 2683 \text{ Ns/m}$
- $D \iff k_{\text{low}} = 134 \text{ Ns/m} \quad k_{\text{high}} = \infty \text{ Ns/m}$
- $E \iff k_{\text{low}} = 134 \text{ Ns/m} \quad k_{\text{high}} = 2683 \text{ Ns/m}$
- $F \iff k_{\text{low}} = 268 \text{ Ns/m} \quad k_{\text{high}} = 1342 \text{ Ns/m}$
The calculations carried out in the beginning of this chapter are now repeated for five other configurations. The desired actuator force is prescribed by equation 4.1. Table 4.9 provides several numerical simulation results and figures 4.17 and 4.18 give a graphical representation in terms of normalized performance values. First, some comments on the (normalized) performance parameters will be given. Quite unexpectedly, the performance of semi-active system B comes very close to that of active system A. Only for very large values of \( k_2 \) (and \( \tilde{q}_2 \)) a difference exists. Obviously, the lack of two entire quadrants (number 2 and 4) does not affect the performance much. Next closest is system C. The limited high damping rate reduces the ride comfort improvement. Considering system D it can be stated that the damper limitation on the firm side (\( k_{high} \)) reduces the ride comfort improvement for large values of \( \tilde{q}_2 \) (and thus \( k_2 \)). Due to the large feedback gain, the desired actuator force is sometimes too large to be generated with the adjustable damper. While studying figure 4.17 it becomes clear that the limitation of the low state damping rate, by means of \( k_{low} \), reduces the possibility to further increase the tire load variations. Obviously, the lower limit guarantees a minimal amount of damping to reduce undesired wheel-hop motions.

Table 4.9 provides a selection of simulation results in a numerical format. Besides the normalized RMS values, a gain with respect to the 'optimal' passive system (\( \kappa = 0.27 \)) is given. The last two columns need some explanation. Semi-active damper systems are not always effective. Sometimes the desired actuator force is in the wrong quadrant, sometimes it is beyond the reach of the damper. The last two columns present a number that represents a percentage describing whether the demanded force is within the reach of the damper settings. For 10 real time simulation minutes these moments have been memorized. During the 'dissipative %' the demanded force can be generated by the damper, during the 'active %' the force is beyond the reach of the damper. As expected there is a strong relation between performance and these percentages. Clearly, the performance is restricted by the damper range. The more feedback structure (4.1) differs from a passive feedback, the more often the desired force does not fit with the available damper force and the more likely performance degrades. As improving ride comfort is our goal (large \( \tilde{q}_2 \)), damper F does not fulfill the requirements well due to its narrow range. According to the table and figures, the largest value of \( \tilde{q}_2 \) does not guarantee the best ride comfort. Depending on the damper range an optimum value of \( \tilde{q}_2 \) can be determined. The more narrow the damper range is the smaller this optimum becomes.
Figure 4.17 RMS values, ride comfort versus road holding.

Figure 4.18 RMS values, suspension working space versus ride comfort.
Figures 4.19 and 4.20 give an impression of the force levels required for semi-active damping. The leftmost column presents the desired actuator force according to equation 4.1, the rightmost column shows the realized damper force. From the top downwards, weighting factor $q_2$ is increased. The larger $q_2$ becomes, the more the desired force is located in the 2nd and 4th quadrant. Figure 4.19 is valid for ideal damper B and figure 4.20 for damper E. As the semi-active system is a closed-loop system, the desired force changes with the applied damper at given weighting factor $q_2$. The application of a narrow range damper causes the desired force to differ because the applied damper force doesn't match always with the desired one. Finally graph 4.21 gives a three dimensional impression of the earlier explained 'active %'. For $q_2 = 0.32$ all semi-active systems act as a passive system with fixed damping rate. As this rate ($\kappa = 0.32$) is within the reach of all damper types, the active percentage equals zero (and thus '% dissipative' = 100). The more ride comfort is accentuated, the lower the passive percentage and thus the more often the desired force is not within the reach of the possible damper settings. The same is valid for road holding improvement.

![Diagram showing % active force levels and weighting factor q2](image-url)

**Figure 4.21** Damper deficiency graph.
| \( \bar{q}_1 \) | \( \bar{q}_2 \) | \( \bar{q}_3 \) | \( p_{\text{dyn}} \) | \( \% \) | \( \ddot{z}_2 \) | \( \% \) | \( \dot{z}_{\text{ISO}} \) | \( \% \) | \( \ddot{z}_2 - \dot{z}_1 \) | \( \% \) | \( \text{Passive} \) | \( \text{Active} \) |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.01 | 0 | A | 0.971 | 23.9 | 2.209 | -36.3 | \text{-}2.726 | -67.7 | 2.075 | -17.9 |
| B | 0.974 | 23.7 | 2.190 | -35.1 | 2.711 | -66.6 | 1.841 | -4.6 | 90.1 | 9.9 |
| C | 0.977 | 23.4 | 2.118 | -30.6 | 2.625 | -80.7 | 1.556 | 11.6 | 67.2 | 32.8 |
| D | 0.974 | 23.7 | 2.189 | -35.0 | 2.711 | -66.6 | 1.830 | -4.0 | 89.5 | 10.5 |
| E | 0.977 | 23.4 | 2.117 | -30.6 | 2.625 | -80.7 | 1.549 | 12.0 | 66.6 | 33.4 |
| F | 1.049 | 17.8 | 1.693 | -4.4 | 2.026 | -39.5 | 1.390 | 21.0 | 7.9 | 92.8 |
| 1 | 0.1 | 0 | A | 1.002 | 21.5 | 1.882 | -16.0 | 2.276 | -56.7 | 1.661 | 5.6 |
| B | 1.003 | 21.4 | 1.878 | -15.5 | 2.273 | -56.5 | 1.625 | 7.6 | 95.2 | 4.8 |
| C | 1.005 | 21.3 | 1.859 | -14.6 | 2.247 | -54.7 | 1.587 | 9.8 | 88.5 | 11.5 |
| D | 1.003 | 21.4 | 1.878 | -15.5 | 2.273 | -56.5 | 1.622 | 7.8 | 94.8 | 5.2 |
| E | 1.005 | 21.2 | 1.858 | -14.6 | 2.247 | -54.7 | 1.584 | 10.0 | 88.0 | 12.0 |
| F | 1.049 | 17.8 | 1.691 | -4.3 | 2.024 | -39.4 | 1.397 | 20.6 | 15.4 | 84.6 |
| 1 | 1 | 0 | A | 1.314 | -3.0 | 1.222 | 24.6 | 1.391 | 4.2 | 1.466 | 16.7 |
| B | 1.313 | -2.9 | 1.224 | 24.5 | 1.393 | 4 | 1.469 | 16.5 | 94.9 | 5.1 |
| C | 1.306 | -2.5 | 1.247 | 23.1 | 1.414 | 2.6 | 1.466 | 16.7 | 92.9 | 7.1 |
| D | 1.312 | -2.8 | 1.225 | 24.5 | 1.393 | 4.1 | 1.470 | 16.4 | 93.8 | 6.2 |
| E | 1.308 | -2.5 | 1.247 | 23.1 | 1.415 | 2.6 | 1.467 | 16.6 | 91.7 | 8.3 |
| F | 1.305 | -2.3 | 1.246 | 23.2 | 1.416 | 2.5 | 1.474 | 16.2 | 85.7 | 14.3 |
| 1 | 10 | 0 | A | 2.181 | -70.9 | 0.697 | 57.0 | 0.753 | 48.2 | 1.998 | -13.6 |
| B | 2.110 | -65.4 | 0.740 | 54.4 | 0.788 | 45.8 | 1.947 | -10.7 | 81.5 | 18.5 |
| C | 2.075 | -62.6 | 0.838 | 48.3 | 0.852 | 41.3 | 1.927 | -9.5 | 77.9 | 22.1 |
| D | 2.067 | -62.0 | 0.768 | 52.6 | 0.814 | 44.0 | 1.917 | -9.0 | 69.1 | 30.9 |
| E | 2.002 | -56.9 | 0.872 | 46.2 | 0.889 | 39.8 | 1.876 | -6.7 | 64.9 | 35.1 |
| F | 1.776 | -39.2 | 0.947 | 41.6 | 1.007 | 30.7 | 1.731 | 1.6 | 39.6 | 60.4 |
| 1 | 100 | 0 | A | 3.405 | -166.9 | 0.362 | 77.6 | 0.403 | 72.2 | 2.920 | -66.0 |
| B | 2.949 | -131.2 | 0.631 | 61.1 | 0.557 | 61.7 | 2.521 | 43.3 | 66.4 | 33.6 |
| C | 2.910 | -128.1 | 0.882 | 45.6 | 0.682 | 53.0 | 2.488 | 41.5 | 60.0 | 40.0 |
| D | 2.572 | -101.6 | 0.816 | 49.7 | 0.685 | 52.8 | 2.253 | 28.1 | 43.9 | 56.1 |
| E | 2.265 | -77.5 | 1.056 | 34.9 | 0.888 | 38.8 | 2.052 | 16.7 | 36.0 | 64.0 |
| F | 1.728 | -35.5 | 1.094 | 32.6 | 1.103 | 24.1 | 1.739 | 1.2 | 19.3 | 80.7 |
| 1 | 1000 | 0 | A | 5.220 | -300.9 | 0.180 | 88.9 | 0.234 | 83.9 | 5.591 | -217.9 |
| B | 3.577 | -180.4 | 1.057 | 34.8 | 0.559 | 61.5 | 3.108 | -76.7 | 55.6 | 44.4 |
| C | 3.333 | -161.2 | 1.287 | 20.6 | 0.690 | 52.5 | 2.831 | 60.9 | 44.7 | 55.3 |
| D | 2.849 | -123.3 | 1.436 | 11.5 | 0.815 | 43.9 | 2.433 | 38.3 | 32.6 | 67.4 |
| E | 2.284 | -79.0 | 1.443 | 11.0 | 1.013 | 30.3 | 2.068 | -17.6 | 17.6 | 82.4 |
| F | 1.634 | -28.1 | 1.257 | 22.5 | 1.292 | 15.2 | 1.703 | 3.2 | 5.0 | 95.0 |

Table 4.9 Semi-active suspension performance as a function of damper type.
\[ F_{act} = -2206 \cdot \ddot{z}_1 + 835 \cdot \dot{z}_2 \quad (\overline{q}_1 = 1.0, \overline{q}_2 = 0.01, \overline{q}_3 = 0.0, \kappa_{low} = 0.0, \kappa_{high} = \infty) \]

\[ F_{act} = -697 \cdot \ddot{z}_1 + 1709 \cdot \dot{z}_2 \quad (\overline{q}_1 = 1.0, \overline{q}_2 = 1.0, \overline{q}_3 = 0.0, \kappa_{low} = 0.0, \kappa_{high} = \infty) \]

\[ F_{act} = -230 \cdot \ddot{z}_1 + 6832 \cdot \dot{z}_2 \quad (\overline{q}_1 = 1.0, \overline{q}_2 = 10.0, \overline{q}_3 = 0.0, \kappa_{low} = 0.0, \kappa_{high} = \infty) \]

\[ F_{act} = -73 \cdot \ddot{z}_1 + 23624 \cdot \dot{z}_2 \quad (\overline{q}_1 = 1.0, \overline{q}_2 = 100.0, \overline{q}_3 = 0.0, \kappa_{low} = 0.0, \kappa_{high} = \infty) \]

**Figure 4.19** Active versus dissipative force generation (damper B).
Figure 4.20 Active versus dissipative force generation (damper E).
In this chapter the performance of semi-active damper systems has been evaluated. The semi-active suspension system is characterized by a rapidly adjustable damper. The control structures which use LQG control in combination with an active system, can also be applied to the semi-active system. In those cases where the sign of the desired active force does not match with the available damper force, the damper is switched off. Separate control strategies have been designed to improve either ride comfort or road holding. Simulation results have shown that it is possible to improve ride comfort significantly using the so-called skyhook control algorithm. This control structure is based on limited state feedback and considers only the absolute vertical velocity of the sprung mass. However, the gain in ride comfort is accompanied by a drastic increase of dynamic tire load variations. On the other hand it appeared not to be possible to decrease dynamic tire load variations significantly using an adjustable damper system.

Despite the control potentialities of (semi-)active suspensions, the conflict between ride comfort and road holding remains. This means that the suspension may be adjusted during the ride in order to improve either the ride comfort or road holding. Since the safety of the vehicle should not be endangered during cornering and braking or accelerating, an additional adaptive control loop must be designed which decides whether the suspension has to be setup for ride comfort or road holding. This is the research topic which will be examined in the next chapter.
SUSPENSION CONTROL WITH ADAPTIVE PROPERTIES

In this chapter suspension controllers as discussed in chapter 3 and 4 will be extended with adaptive properties. The aim of adaptive suspension control is to reduce the sprung mass accelerations as much as possible under given circumstances (road condition, speed, etc.). However, the desire for ride comfort may never interfere with the disability to handle the vehicle during cornering and braking or accelerating. Therefore, the adaptive suspension controller should be able to distinguish the different ride conditions and decide whether to concentrate on ride comfort or road holding. The potentialities of adaptive suspension control will be demonstrated on the basis of a semi-active damper system [39].
To apply the control strategies discussed in the previous chapters to a road vehicle is a substantial effort. The problem is that in spite of all effort done in the past, the conflict between road holding and ride comfort remains. However, it is not always necessary to maintain a good contact between tire and road. On a smooth road for example, dynamic tire load variations will not cause difficulties concerning the ability to handle the vehicle. For these kinds of roads passive suspension systems with a fixed damping rate (e.g. $\kappa = 0.45$) would restrict tire load variations too much even though these tire load variations were already very small due to the road condition. In these situations much on ride comfort could be gained if the dynamic tire load variations were not unnecessarily restricted so much.

Of course, more aspects play a part in the choice between ride comfort and road holding. If we were driving straight ahead on a rather smooth road, then the presence of dynamic tire load variations doesn't mind us. However, during cornering, braking, acceleration and side wind gusts the tire load must be kept as constant as possible depending on the heaviness of these events. Variations of the stochastic properties of the road surface condition should also be taken into consideration, especially when driving straight ahead. And last but not least events like potholes, speed humps and other deterministic obstacles must be regarded. Thus, from the foregoing the conclusion can be drawn that it is very important to have on-line stochastic and deterministic information of the tire load. As will be shown in chapter 6 it is possible to estimate the dynamic tire load using a state estimator. In this chapter it is assumed that a perfect knowledge of the tire load is available.

Considering the values of the dynamic tire load variations from section 4.1 the conclusion can be drawn that merely skyhook control is not an acceptable solution in practice. A reduction of the tire load variation can only be obtained if and only if the motion of the unsprung mass is taken into consideration. From section 4.2 we can conclude that there is no appropriate control strategy to improve road holding in combination with a semi-active damper system. A firm and fixed damping ($\kappa = 0.8$) provides the best results. If such a firm damper setting is not available then the highest setting possible is preferred as long as it is smaller than $\kappa = 0.8$. With these outcomes it is now possible to construct a new desired actuator force which is built up of each optimal solutions (in case of semi-active damping) but which form together also a conflicting solution according to

$$F_{act} = (1 - q_{adp}) \cdot k_{sky} \dot{z}_b + q_{adp} \cdot k_{rh} (\dot{z}_b - \dot{z}_a)$$  \hspace{1cm} (5.1)
where $q_{\text{adp}}$ is a weighting factor with a value between 0 and 1. Variable $k_{rh}$ denotes the optimal damping rate for minimum RMS tire load variations and is equal to $2147 \text{ Ns/m}$ ($\kappa_{rh} = 0.8$). Taking $q_{\text{adp}} = 0$ provides the best ride comfort ever possible (for a given value of $k_{sky}$) and $q_{\text{adp}} = 1$ takes care of the road holding capabilities.

From all simulations done in the past the optimal value of $k_{sky}$ for the best ride comfort amounts to about 7500 Ns/m. This value is valid for the continuously variable damper with $\kappa_{low} = 0.05$ and $\kappa_{high} = 1.0$. With this information it is now possible to examine the effect of the value of weighting factor $q_{\text{adp}}$ on the system performance. Figure 5.1 presents the normalized RMS performance values as a function of weighting factor $q_{\text{adp}}$. The continuously variable damper (CSAD) as well as the 2-state damper (DESAD) are considered. The latter one is controlled by extended on-off control according to Table 4.3.

The first thing that strikes is that the continuously variable damper performs better than the 2-state damper. The transition of the normalized RMS values at increasing $q_{\text{adp}}$ is much smoother for the CSAD system. This is advantageous because at every condition an appropriate setting can be found. The range for $q_{\text{adp}}$ from 0.0 to 0.5 is ineffective with the DESAD system. This is probably due to the two available damper settings with a rather large gap in between.

The problem is now to find out what is the right choice of weighting coefficient $q_{\text{adp}}$ under given external circumstances (road type, forward velocity, etc.). This is the task of the adaptive control algorithm. An important datum is the course of the dynamic tire load variations during the ride. First, the stochastic information such as the variance of the dynamic component must be derived.

The mean square (MS) values (equal to variance $\sigma^2$ when zero mean is assumed) of the dynamic tire load $P_{\text{dyn}}$ is defined by

$$MS\left(P_{\text{dyn}}\right) = \sigma^2_{P_{\text{dyn}}} = \lim_{T \to \infty} \frac{1}{T} \int_0^T (P_{\text{tire}} - P_{\text{stat}})^2 dt \quad (5.2)$$

where $P_{\text{tire}}$ is the total tire force and $P_{\text{stat}}$ is the static tire load. The variance can be obtained by letting $T$ approach infinity. Expression (5.2) can be rewritten to

$$\frac{\sigma^2_{P_{\text{dyn}}}(s)}{P^2_{\text{dyn}}(s)} = \frac{1}{Ts} \quad (5.3)$$

where $P_{\text{dyn}} = P_{\text{tire}} - P_{\text{stat}}$ and 's' is the Laplace variable.
Figure 5.1 Normalized performance values as a function of $q_{adp}$ (CSAD).

Figure 5.2 Normalized performance values as a function of $q_{adp}$ (DESAD).
Now assume the following first-order filter with time constant $\tau_s$,

$$\frac{\sigma_s^2(s)}{P_{dyn}(s)} = \frac{1}{\tau_s s + 1} \quad (5.4)$$

By taking $\tau_s$ very large it can be seen that $\sigma_s^2$ approaches the mean square value of $P_{dyn}$. In this case we have obtained a first-order system with a very slow response and therefore small bandwidth that is capable to approximate the variance of the dynamic tire load variations over a certain time depending on the value of $\tau_s$. By the application of this first-order filter we have gathered information about the variations of the dynamic tire load over a predefined period of time depending on the value of the time constant $\tau_s$ of the first-order system. In other words we have established a memory with a limited space of time by controlling the 'leakage' of early information from the integration process by adjusting the time constant. This long-period information is very useful for a slow adaptation of weighting coefficient $q_{adp}$. Using this information it is possible to react on relatively slow changes of road condition and forward vehicle speed.

From the point of view of overall safety we have to assume a maximum permissible value of the dynamic tire load. Even on a relatively smooth road surface and while driving straight ahead it is advisable to allow no more variations of the tire load than this prescribed maximum. The potential to change direction or to brake must remain. Therefore, in section 2.1 a maximum RMS value of the dynamic tire load variations depending on the static tire load $P_{stat}$ was proposed.

$$\text{RMS}(P_{dyn})_{max} = \frac{P_{stat}}{3} = \frac{(m_a + m_b)g}{3} = 762 \text{ N} \quad (5.5)$$

Thus, the maximum variance is given by

$$\text{MS}(P_{dyn})_{max} = \frac{P_{stat}^2}{9} = \sigma_{P_{dyn},\max}^2 = 5.801e + 6 \text{ N}^2 \quad (5.6)$$

While driving on a straight road without severe discrete events like potholes and without exercising any steering or braking maneuvers, the adaptive control algorithm should take care of the change of $q_{adp}$ in such a way that the 'current variance' $\sigma_s^2$ approaches the maximum allowable variance according to equation (5.6).

Now, define error $e_s$ between maximum allowable variance and 'current variance' as
\[ e_s = \sigma_{\text{dyn, max}}^2 - \sigma_s^2 \]  
(5.7)

The change of weighting factor \( q_s \) per unit of time is now given by

\[ \dot{q}_s = g_s \cdot e_s \]  
(5.8)

where \( g_s \) is a gain. Expression (5.8) represents nothing less than that if there exists a deviation between the desired variance of the dynamic tire load variations and the 'current variance', \( q_s \) will increase or decrease with a velocity that is proportional to error \( e_s \). In order to limit the value of \( q_s \) a saturation block must be added where the limits correspond to \( q_s = 0.0 \) and \( q_s = 1.0 \). The lower part of figure 5.3 shows the train of thought derived above in a block diagram. Instead of using \( P_{\text{dyn}} \) as an input for the adaptive controller, the ratio \( P_{\text{dyn}}/P_{\text{stat}} \) has been used.

This slow adaptive regulator with integral action is an input-output unstable system. Its unstable mode can give rise to difficulties under certain circumstances. Reset windup or integrator saturation can occur if the output saturates and the adaptive controller continues to integrate the error signal \( e_s \). The output of the integrator can then assume very large values, and it can take a long time to get it back to a normal value again. One way to avoid this problem is to stop updating the integrator when the output is limited. In this case the integrator will be reset every time saturation takes place.

The same train of thought can be pursued for the fast adaptive regulator. When driving over discrete events like potholes or kerbs it is necessary to monitor the tire load and to intervene into the controller if the variation of the tire load gets out of hand. An example of this phenomenon is shown in figure 4.13. From the course of the tire load it can be seen that the wheel is bouncing vehemently causing several times a loss of contact with the ground. These kinds of situations must be avoided. Braking or steering is hardly possible with such a variation of the tire load.

Due to the large time constant \( \tau_s \), the slow section of the adaptive controller is not able to react in time. Therefore, a faster adaptation is desirable. The problem is now that it is desirable to cross road irregularities such as potholes or speed humps as comfortably as possible. Adjusting the control algorithm for the ride comfort normally causes difficulties concerning the tire load variations. However, an intervention by means of a fast adaptive loop with the purpose to restrict these tire load variations would make the ride comfort a lot worse than desired. Therefore, the fast adaptation should not be too fast. For
example, the magnitude of the maximum acceleration of the sprung mass (discomfort) caused by a step disturbance is fixed by the state of the damper (figure 4.13). The lower the damping the more comfortable the transition. However, tire load variations get unacceptable large, especially a short time after the confrontation with the step. If the fast adaptive controller would react instantaneously on the inadmissible course of the tire load, the damper would definitely be switched to the firm setting which results in a very uncomfortable transition. Therefore, the fast adaptive loop should not be so fast that it makes such a transition uncomfortable, but should be fast enough to restrain the tire load variations a short moment after the confrontation with an obstacle.

![Diagram](image-url)

**Figure 5.3** Control structure of the adaptive system.

In order to be able to restrict the dynamic component of the tire load the following error function is proposed

\[ e_f = \left( \frac{P_{dy}}{P_{st}} \right)^4 \]  

(5.9)
As can be seen from figure 5.4 the fast adaptive error \( e_f \) will increase very much if the dynamic tire load \( P_{dyn} \) gets extremely large (either positive or negative). In order to avoid a conflict between the slow adaptive loop (which wants to prescribe a RMS value to the dynamic tire load variations) and the fast adaptive loop (which wants to restrict the dynamic tire load variations), error \( e_f \) must remain very small in the area between \(-P_{stat}/3\) and \(+P_{stat}/3\). This can very well be achieved using the fourth-degree weighting function.

![Weighting function of the dynamic tire load.](image)

Figure 5.4 Weighting function of the dynamic tire load.

As error \( e_f \) remains always positive it is not possible to use plain integral action in the fast adaptive loop. In that case the output of the integrator grows more and more until numerical saturation occurs. It is therefore better to use a first-order controller which is given by

\[
\frac{q_f(s)}{e_f(s)} = \frac{1}{\tau_f s + 1}
\]  

(5.10)

In this case \( \tau_f \) is a lot smaller than \( \tau_s \). When \( \tau_f \) becomes very small \( q_f \) approaches \( e_f \) and a proportional feedback has been obtained. By giving \( \tau_f \) a small value, a short duration memory can be constructed resulting in a PI feedback structure.

Overall weighting factor \( q_{adp} \) is now given by

\[
q_{adp} = q_s + q_f
\]  

(5.11)

Thus, if the variance of the dynamic tire load variations exceeds the maximum defined variance according to equation (5.6) both fast and slow adaptive
control loops will accumulate (with a different speed) weighting factor $q_{adp}$ in such a way that the tire load variations are bounded.

The following values for the time constants $\tau_s$ and $\tau_f$ are chosen

$$\tau_s = \frac{1}{f_s^0} = 2\pi \sqrt{\frac{m_b}{c_s}} = 0.937 \text{ s}$$

(5.12a)

$$\tau_f = \frac{1}{f_a^0} = 2\pi \sqrt{\frac{m_a}{c_s + c_f}} = 0.079 \text{ s}$$

(5.12b)

As can be seen, the values of the time constants correspond to the cycle time of the two natural motions of the 2 DOF quarter-car model. After some simulations, a value of $g_s = 1.0 \text{e-2}$ gave a good control behavior. Giving this gain a larger value will result in a kind of 'bang-bang' control of weighting factor $q_{adp}$.

The operation of the adaptive suspension controller will now be demonstrated on the basis of semi-active damping in combination with the continuously adjustable damper (type E from figure 4.16). Both stochastic and deterministic responses are calculated. Several simulations were carried out for different road conditions (table 1.2) and different forward velocities. Figure 5.5 presents the results. Besides the semi-active system with adaptive skyhook control, also the results of a similar system with plain skyhook control ($q_{adp} = 0$) have been plotted. As already expected, dynamic tire load variations increase in those cases where it is possible and allowed. On a very smooth asphalt road the adaptive controller is hardly active. The results coincide with those situations where $q_{adp} = 0.0$. However, on a concrete road it can be seen that the slow adaptive loop works very well. The RMS value of the dynamic tire load variations does not exceed the prescribed value of 762 N. Finally, for the rough road condition there is not much work to do for the adaptive controller. Even in the passive case the dynamic tire load variations will exceed the prescribed limit. The damper will pretty much remain in the firm setting that corresponds to $k_{rh}$.

In order to test the fast adaptation a step input was applied to the quarter-car model. Figure 5.6 presents the responses. Two curves are plotted: (i) passive suspension system with fixed damping ratio and (ii) a semi-active system with adaptive skyhook control ($k_{sky} = 7500 \text{ Ns/m}$). The responses of the latter system should be compared with the responses of the non-adaptive version according to figure 4.13.
Figure 5.5 Performance of the adaptively controlled system (CSAD).

The figures show that the fast adaptive loop reacts quite fast, but not too fast. The transition is still very comfortable: the height of the first acceleration peak is only $\frac{1}{2}$ of the peak value of the passive system and no overshoot of the sprung mass displacement will occur. The adaptation takes special care of the variation of the tire load after the confrontation with the step. The variations are well damped and a second loss of ground contact such as occurred with the non-adapting configuration (figure 4.13) can be avoided. One drawback of the adaptation is a higher second peak value of the sprung mass acceleration due to the firm damper setting necessary to restrict the tire load variations. The course of the damping coefficient shows that the damper is only adjusted shortly after the confrontation with the step. After that it will remain in its high state. The three weighting factors that take care of the adaptation are also plotted. The fast adaptation ($q_\text{f}$) reacts almost immediately after the confrontation with the step due to the excessive tire load changes. Although the slow adaptation accomplished by $q_\text{s}$ is not designed to cope with fast changes in driving conditions, it still contributes to the total weighting factor $q_{\text{adp}}$. Its contribution can be controlled by gain $g_\text{s}$. 

- 112 -
Figure 5.6a Step responses, CSAD with adaptive skyhook control.

(— passive, - - - semi-active)
Figure 5.6a Step responses, CSAD with adaptive skyhook control.
(— passive, - - - semi-active)

This chapter has dealt with the development of an adaptive extension of semi-active suspension control. An additional adaptive control loop has been designed which decides whether the suspension has to be setup for ride comfort or road holding. As long as the dynamic tire load component does not exceed a predetermined limit, the adaptive controller selects a setup that accentuates ride comfort. However, during cornering and braking or accelerating the tire load may not vary too much since that may endanger the safety. Under these circumstances the adaptive suspension controller permits a worsening of the ride comfort in favor of tire load control. The main input to the adaptive controller is the ratio of the dynamic tire load to the static tire load.

Most of the state variables must be know for the control structure as discussed above. It is often unrealistic to assume that all states of a system can be measured. Therefore, system states, like the absolute sprung mass velocity or the dynamic tire load, which cannot be measured can be estimated using a state estimator or Kalman filter. The next chapter will examine the application of state estimators in the quarter-car system.
IMPLEMENTATION OF SUSPENSION CONTROL USING A KALMAN FILTER

In this chapter the application of state estimators in controlled wheel suspension systems will be examined. It is often unrealistic to assume that all states of a system and the disturbances acting on the system can be measured. System states that cannot be measured can be estimated by a state estimator that is driven by measurements of the system. Such an estimator could be realized with the aid of an on-board computer in the vehicle. The difference between measurements and the estimated output quantities of the estimator will be fed back to minimize the estimation error.

In section 5 it has been illustrated that the semi-active suspension with skyhook control requires the feedback of the absolute vertical velocity of the
sprung mass. Determination of this velocity is quite difficult in a realistic environment where the vehicle possess many degrees of freedom (heave, pitch, roll, yaw, etc.). Furthermore, noise resulting from e.g. engine vibrations can corrupt the measurement signals substantially. In the quarter-car model application it is possible to determine the absolute velocity by integrating acceleration signals coming from an accelerometer. The major problem however is that the integrator, whether analog or digital, will spend most of its time at its upper or lower saturation limits, due to drift caused by very low frequency inputs (nearly D.C. signals). In order to prevent integrator drift one could consider the use of a low pass filter such as a first-order filter rather than a pure integrator.

This section will discuss the application of state estimators in active and semi-active suspension systems, avoiding the drawback of integrating acceleration signals as mentioned above. In this context it is important to be aware of the consequences of the so-called separation theorem. This theorem states that the control problem can be solved separately from the state estimator problem. First the theory of state estimators will be derived assuming that the suspension system is equipped with a linear actuator. After that some modifications are made such that the state estimator is also able to cope with non-linear semi-active suspension systems.

![The principle of a state estimator](image)

**Figure 6.1** The principle of a state estimator.
The idea of a state estimator is to implement a model of the real system in an on-board digital computer in parallel to the system itself. A fundamental problem is now how to describe a continuous-time system connected to a computer by A-D (analog-to-digital) and D-A (digital-to-analog) converters as depicted in figure 6.1.

6.1 AN INTRODUCTION TO DISCRETE-TIME SYSTEMS

To find the discrete-time equivalent of a continuous-time system is called sampling of a continuous-time system. To obtain the desired descriptions, it is necessary to describe the converters and the system. Assume that the continuous-time system is given by the following state-space formulation with order \( m \):

\[
\dot{x} = Ax + Bu + Gu \\
y = Cx + Du + v
\]  \hspace{1cm} (6.1a)

\( \dot{x} \) is the state equation with state vector \( x \), control signal \( u \) and process noise \( w \). The second equation is the measurement equation with output vector \( y \) and measurement noise \( v \). The process noise and measurement noise is represent by Gaussian white noise and have zero mean

\[
\text{mean}(w) = E\{w\} = 0 \quad \text{mean}(v) = E\{v\} = 0 \]  \hspace{1cm} (6.2a)

and covariances

\[
\text{cov}(w) = E\{ww^T\} = Q_f \quad \text{cov}(v) = E\{vv^T\} = R_f \]  \hspace{1cm} (6.2b)

Matrices \( Q_f \) and \( R_f \) can be regarded as the intensity of the process noise and measurement noise respectively. In chapter 1 the state-space equations of the quarter-car model have been derived including the stochastic road description.

A common situation in computer control is that the D-A converter is so constructed that it holds the analog signal constant until a new conversion is commanded. This is called zero-order-hold sampling of a system. Sampling often occurs with a periodic sample interval \( \delta t \). The system equations of the sampled system are now given by [4]

\[
x_{n+1} = \Phi x_n + \Gamma u_n + \Psi w_n \\
y_n = Cx_n + Du_n + v_n
\]  \hspace{1cm} (6.3a)

where

- 117 -
Chapter 6 Implementation of Suspension Control using a Kalman Filter

\[ \Phi = e^{A\delta t} \quad \Gamma = \int_{0}^{\delta t} e^{A\tau} \, d\tau \cdot B \quad \Psi = \int_{0}^{\delta t} e^{A\tau} \, d\tau \cdot G \]  

(6.4)

Matrices \( C \) and \( D \) are unchanged. \( \Phi, \Gamma \) and \( \Psi \) can be calculated using commercially available software like PC-Matlab or MatrixX.

In the simpler case where sample period \( \delta t \) is small compared with the time constants and cycle periods corresponding to the eigenvalues of state matrix \( A \), the following approximations may be used

\[ \Phi = I + A\delta t + \frac{A^2\delta t^2}{2} + \frac{A^3\delta t^3}{6} + \cdots = I + A\delta t \]  

(6.5a)

\[ \Gamma = \left( I\delta t + \frac{A\delta t^2}{2} + \frac{A^2\delta t^3}{6} + \cdots \right) B = B\delta t \]  

(6.5b)

\[ \Psi = \left( I\delta t + \frac{A\delta t^2}{2} + \frac{A^2\delta t^3}{6} + \cdots \right) G = G\delta t \]  

(6.5c)

A necessary condition for estimating the state variables \( x \) is that the system is observable. This means that all modes affect output \( y \). By constructing the observability matrix \( W_o \) it can be shown that observability is ensured if and only if the rank of this matrix equals \( m \)

\[ W_o^T = \begin{bmatrix} C & C\Phi & \cdots & C\Phi^{m-1} \end{bmatrix} \]  

(6.6)

where \( m \) is the order of the system.

6.2 KALMAN FILTER DESIGN FOR AN ACTIVE SUSPENSION SYSTEM

A Kalman filter is a state estimator designed to estimate the states of a stochastic system [37,38]. The linear time invariant state-space equation of such a predictor-type state estimator is given by

\[ \hat{x}_{n+1} = \Phi \hat{x}_n + \Gamma u_n + K_f d \left( y_n - C \hat{x}_n - D u_n \right) \]  

(6.7)

where \( \hat{x}_{n+1} \) represents the estimated state vector at \( t_{n+1} \). This state estimator is a one-step-ahead predictor. The dynamic behavior of the estimation error \( y_n - C \hat{x}_n - D u_n \) is prescribed by selecting the discrete feedback gain matrix \( K_f d \) of the Kalman filter.

The unknown actuator force \( u_n \) can be derived as a feedback of the estimated state vector \( \hat{x}_n \) by means of the discrete control feedback matrix \( K_{cd} \). This can either be a full-state feedback or a partial state feedback and reads
\[ u_n = -K_{cd} \hat{x}_n \] (6.8)

The discrete representation of \( K_c \) can be determined by using for example a discrete-time linear quadratic regulator algorithm instead of a continuous-time version.

The closed-loop Kalman filter can be found by substituting equation (6.8) in equation (6.7) and is given by

\[ \hat{x}_{n+1} = (\Phi - K_{fd} C - (\Gamma - K_{fd} D) K_{cd}) \hat{x}_n + K_{fd} y_n \] (6.9)

By combining equation (6.3) and (6.9) the observation error equation can be obtained by

\[ e_{n+1} = (\Phi - K_{fd} C) e_n + K_{fd} u_n - \Psi w_n \] (6.10)

where estimator error \( e_n \) is defined by

\[ e_n = \hat{x}_n - x_n \] (6.11)

It can be seen that \( K_{fd} \) may be designed to ensure that the eigenvalues of \( \Phi - K_{fd} C \) correspond to a rapid well-behaved decay of any estimator error. The state estimator becomes faster, but also more sensitive to measurement noise \( u_n \). Thus the Kalman filter provides a compromise between the speed of state reconstruction and the immunity to measurement noise. The balance between these two properties is determined by the magnitude of matrices \( Q_f \) and \( R_f \) in the design stage. An increase in reconstruction speed by making \( R_f \) small compared to \( Q_f \) is accompanied by a shift of the Kalman filter poles further into the left-half of the complex plane (in case of a continuous-time system).

A measure of the estimator error is the error covariance matrix \( S \) determined by

\[ S = \text{cov}(e_n) = E\{e_n e_n^T\} \] (6.12)

The Kalman filter design is optimal in that it selects the value of \( K_{fd} \) which minimizes the trace of \( S \) for \( n = 0, 1, ..., \infty \). In other words the Kalman filter algorithm selects a matrix \( K_{fd} \) that positions the eigenvalues of \( \Phi - K_{fd} C \) in such a way that the sum of the variances of the estimator error is minimized.

A method of computing the Kalman filter feedback gain matrix \( K_{fd} \) is required. The method chosen is that one due to Kalman and Bucy [21] in which it is assumed that the system to be observed is driven by white noise \( w \) and that
the observed signals are corrupted by white noise \( \nu \). Feedback matrix \( K_{fd} \) can be found by solving the discrete-time algebraic matrix Riccati equation given by

\[
S - \Phi S \Phi^T + \Phi S C^T (R_f + CSC^T)^{-1} CS \Phi^T - \Psi Q_f \Psi^T = 0
\]  

(6.13a)

\[
K_{fd} = \Phi S C^T (R_f + CSC^T)^{-1}
\]  

(6.13b)

where \( S \) is the solution of equation (6.13a) and represents the covariance matrix of the state estimation error. The filter gain \( K_{fd} \) is affected by \( Q_f \) and \( R_f \) which are basically measures of the amplitudes of noise affecting the system. The algorithm fails if \( Q_f = 0 \) since there is then insufficient data from which to develop an optimal filter feedback.

\[\text{Figure 6.2 State-space scheme of an active suspension system.}\]
The closed-loop system with discrete control feedback by $K_{cd}$ can also be expressed as
\[
\begin{bmatrix}
x_{n+1} \\
e_{n+1}
\end{bmatrix} = \begin{bmatrix}
\Phi - \Gamma K_{cd} & -\Gamma K_{cd} \\
0 & \Phi - K_{fd} C
\end{bmatrix}
\begin{bmatrix}
x_n \\
e_n
\end{bmatrix} + \begin{bmatrix}
0 \\
K_{fd}
\end{bmatrix} \Psi u_n + \begin{bmatrix}
0 \\
-\Psi
\end{bmatrix} w_n
\]  
(6.14)

The dynamics of the closed-loop system are determined by $(\Phi - \Gamma K_{cd})$ and $(\Phi - K_{fd} C)$ i.e. the dynamics of the corresponding control problem and the dynamics of the state estimator.

A block diagram of the complete system is shown in figure 6.2. Three areas can be distinguished: quarter-car, state estimator (Kalman filter) and controller.

### 6.3 KALMAN FILTER DESIGN FOR A SEMI-ACTIVE SUSPENSION SYSTEM

The previously derived theory concerns a linear suspension system with an actuator. In this section some modifications will be made such that the Kalman filter theory can also be applied to semi-active non-linear suspension systems [37,38].

The most important difference with an actuator system is that control signal $u_n$ (damper force) is not directly related to the estimated state variables $\hat{x}_n$ by equation (6.8). If we want to avoid measuring the damper force, then it is necessary to distinguish a control signal $\tilde{u}_n$ and an estimated control signal $\hat{u}_n$

\[
\begin{align*}
    u_n &= -K_{cd} x_n \\
    \hat{u}_n &= -K_{cd} \hat{x}_n
\end{align*}
\]  
(6.15a)

The major advantage of this approach is that non-linearities of the applied adjustable damper in the real car situation (damper characteristics, time delay, Coulomb friction, etc.) can be taken into account while estimating damper force $\tilde{u}_n$. Thus, $\tilde{u}_n$ does not need to depend linearly on $\hat{x}_n$ by $K_{cd}$. Other (time varying) relations may be used.

If $K_c$ is known in the continuous-time domain then the discrete control feedback matrix can be calculated using the pole (eigenvalue) placement gain selection. This algorithm calculates the discrete control feedback matrix $K_{cd}$ by varying $K_{cd}$ such that

\[
\text{eig}(\Phi - \Gamma K_{cd}) = \text{eig}(\Phi^c)
\]  
(6.16)
where $\Phi_c$ is the discrete-time closed-loop system matrix according to

$$\Phi^c = e^{(A-BK_c)\delta t} \tag{6.17}$$

Only one problem arises when equation (6.16) is used to calculated the
discrete equivalent representation of $K_c$: vector $u_n$ hasn't the same meaning
any more as $u(t)$, which is in our case a scalar representing the damper force.
Assume for example the following continuous-time control feedback matrix $K_c$
which is equivalent to a feedback by means of a passive damper with a
damping constant of 1000 Ns/m

$$K_c = \begin{bmatrix} 0 & 0 & 1000 & -1000 \end{bmatrix}$$

When using the pole placement algorithm the equivalent discrete control
feedback matrix $K_{cd}$ is given by

$$K_{cd} = \begin{bmatrix} 92736 & -6231 & 310 & 963 & -965 \end{bmatrix}$$

As can be seen this is not a limited state feedback anymore. Terms that were
zero in the continuous-time case are attached with a non-zero value in the
discrete-time case. Therefore, $u_n$ can not be interpreted any more as a damper
force. Note, the first gain of $K_{cd}$ can be chosen arbitrarily since state $z_r$ is not
controllable and can therefore not influence the eigenvalues of the closed-loop
system. Consequently, the continuous-time control feedback matrix must be
used to estimate the damper force instead of the discrete-time equivalent. The
new expression reads

$$\dot{u}_n = -K_c \dot{x}_n \tag{6.18}$$

Another difference with a fully active structure is that the discretely
controllable semi-active damper system (DSAD) has two feedback matrices
$K_c^{low}$ and $K_c^{high}$ each for one damper setting. In case of the continuously
adjustable damper system (CSAD) there is an infinite number of feedback
gains. The discrete representation of $K_c$ can also be determined by using for
example a discrete-time linear quadratic regulator algorithm instead of a
continuous-time version. However, the discrete-time feedback structure does
not offer a clear insight into the meaning of the feedback gains.

With the considerations made above, scheme 6.2 can be redesigned to figure
6.3. From this scheme it can be seen that the Kalman filter for the active
suspension is identical to that of the semi-active configuration.
Figure 6.3 State-space scheme of a 2-state semi-active suspension (DSAD).

Since we do not use the knowledge of the controller in the state estimator design stage, we might use the same procedures as described above for determining the discrete-time Kalman filter feedback matrix $K_{fd}$. For simulation use matrix equation (6.14) can be transformed to the following expression
\[
\begin{bmatrix}
\dot{x}_{n+1} \\
\dot{e}_{n+1}
\end{bmatrix} = 
\begin{bmatrix}
\Phi - \Gamma K_{cd} & 0 \\
0 & \Phi - \Gamma K_{cd} - K_{fd}(C - DK_{cd})
\end{bmatrix}
\begin{bmatrix}
x_n \\
e_n
\end{bmatrix} + 
\begin{bmatrix}
0 & \Psi \\
K_{fd} & -\Psi
\end{bmatrix}
\begin{bmatrix}
u_n \\
w_n
\end{bmatrix}
\] (6.19)

where in case of a 2-state damper \(K_{cd}\) equals

\[
K_{cd} = K_{cd}^{low} \quad \text{or} \quad K_{cd}^{high}
\]

If we look at equation (6.19) in detail then it strikes that the so-called separation principle is not valid here! The eigenvalues of the Kalman filter error equation (lower part of (6.19)) depend on the control feedback matrix \(K_{cd}\) that has two values in the DSAD case (soft and firm damping). Thus, the state estimator problem cannot be solved separately from the controller problem. This contradicts the separation theorem. Using the earlier derived Riccati equation for solving the state estimation feedback \(K_{fd}\) will not always result in a stable Kalman filter. Since the Riccati equation does not use the control feedback matrix \(K_{cd}\) the eigenvalues of the error equation given by

\[
\text{eig}(\Phi - \Gamma K_{cd} - K_{fd}(C - DK_{cd}))
\] (6.20)

will not always be stable.

However, it is still possible to solve Riccati equation (6.13) neglecting the control feedback \(K_{cd}\). Alas, Kalman filter stability is not ensured at all time. The stability depends on the shape of weighting matrices \(Q_f\) and \(R_f\) and the control feedback matrix \(K_{cd}\). Finding appropriate values for these weighting matrices turned out to be a time consuming business. A multi-model approach will carry us through the difficulty of instability.

### 6.4 Kalman Filter Design Using Multi-Model Techniques

As described in the last section neglecting the control feedback got us into trouble. In this section an optimal multi-model approach will be used to determine the discrete-time state estimator feedback matrix \(K_{fd}\). Normally, multi-model techniques are used in combination with the robustness synthesises of a closed-loop system preventing for example instabilities due to large variations of model parameters [33]. However, this approach will also be applicable to our problem.

Suppose we have the linear time-invariant model represented by

\[
\begin{alignat}{2}
x_{n+1} &= \Phi x_n + \Gamma u_n + \Psi w_n \\
y_n &= C x_n + D u_n + v_n
\end{alignat}
\] (6.3a, 6.3b)
with $k$ linear matching control feedback matrices

$$u_n = -K_{cd,i} \hat{x}_n \quad \text{for} \ i = 1, 2, \ldots, k$$  \hspace{1cm} (6.21)

Every control feedback matrix can be regarded as a different damper state in the semi-active suspension situation.

The $k$ closed-loop systems are given by

$$\hat{x}_{n+1} = (\Phi - \Gamma K_{cd,i})\hat{x}_n + \Psi \hat{w}_n + \Phi^c \hat{x}_n + \Psi \hat{w}_n \quad (6.22a)$$

$$\hat{y}_n = (C - DK_{cd,i})\hat{x}_n + \hat{u}_n = C^c \hat{x}_n + \hat{u}_n \quad (6.22b)$$

Where $\Phi^c$ and $C^c_i$ are the closed-loop system matrix and the closed-loop output matrix of the $i$th system respectively. For each system a Kalman filter can be considered according to

$$\hat{x}_{n+1} = \Phi^c \hat{x}_n + K_{fd} \left( y_n - C^c \hat{x}_n \right) \quad (6.23)$$

The multi-model Kalman filter feedback problem can now be formulated as finding one Kalman filter feedback matrix $K_{fd}$ such that by the application of a switching control strategy all $k$ Kalman filters (6.23) are stabilized.

The multi-model performance criterion is defined as

$$J_{tot} = \sum_{i=1}^{k} p_i \cdot J_i \quad (6.24)$$

with $p_i$ denoting a weighting scalar, and $J_i$ the performance index for each model given by the trace of covariance matrix $S$. The relative importance of the operating conditions can be expressed with the weighting scalars $p_i$.

A solution for the multi-model minimization problem can be found with the same numerical algorithm as used for the standard discrete-time linear quadratic output feedback (DLQOFB) problem. The numerical algorithm is time-consuming. The multi-model extension gives rise to a factor $k$ increase in calculation time, compared to the standard single model DLQOFB method. Because output feedback is normally used in combination with a controller problem, a substitution must be done in case DLQOFB is used to calculate the Kalman filter feedback matrix. The transformation is given in table 6.1.
Table 6.1 Correspondence between controller and Kalman filter design.

Kalman filter feedback matrix $K_{fd}$ is now equal to the transposed solution of the regulator feedback matrix $K_{od}$ which is the solution of the DLQOFB procedure

$$K_{fd} = K^T_{od} \quad (6.25)$$

and the estimator error covariance matrix $S_f$ is related to the regulator state covariance matrix $S_c$ by

$$S_f = \Phi^{-1} \left( S_c - \Psi Q_{fd} \Psi^T \right) \Phi^T \quad (6.26)$$

Since several closed-loop state feedback matrices $\Phi$ exist in the multi-model approach, a single estimator error covariance matrix $S_f$ is out of the question here. The next section deals with tuning aspects of a Kalman filter.

6.5 Tuning the Kalman Filter

In this section a Kalman filter will be designed to estimate all 5 states of the quarter-car model. As a design example the quarter-car is equipped with a 2-state damper. According to equation (6.3) one or more measurement signals $y$ have to be chosen such that observability is ensured. It has been chosen to use two accelerometers, mounted on the sprung and unsprung masses. The output equation becomes

$$y_n = C x_n + D u_n + v_n \quad (6.3b)$$

where

$$C = \begin{bmatrix} c_t & -c_s - c_t & c_s & 0 & 0 \\ \frac{m_a}{m} & c_s & \frac{m_a}{m} & 0 & 0 \\ 0 & -c_s & \frac{m_a}{m} & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1 \\ \frac{m_a}{m} \\ -1 \\ \frac{m_b}{m} \end{bmatrix}$$

Vector $v_n$ represents the measurement noise.

The determination of the discrete-time Kalman filter feedback matrix $K_{fd}$ in the multi-model sense can be summarized in general by:
Chapter 6 Implementation of Suspension Control using a Kalman Filter

- Determine control feedback matrices $K_{c,i}$
- Calculate closed-loop matrices $\Phi^c_i$ and $C^c_i$: use equation (6.22)
- Choose covariance matrix $Q_f$: use equation (1.8) $\rightarrow Q_f = 2\sigma^2_{av}$
- Choose intensity matrix $R_f$: $2 \times 2$ matrix
- Calculate feedback matrix $K_{fd}$: solve the multi-model problem

In the multi-model case the construction of all closed-loop multi-model matrices according to equation (6.22) must precede the calculation of the Kalman filter feedback matrix. Despite the presence of more than one model, one constant $Q_f$ and one constant $R_f$ matrix will be assumed which is valid for all models. This means that the intensity of the measurement noise sources and road excitation white noise source will not depend on the amount of damping in the model and will therefore be equal for all models.

The values of the elements in weighting matrices $\Psi \cdot Q_f \cdot \Psi^T$ and $R_f$ are not prescribed by a theory. Of course, it is possible to calculate expression $\Psi \cdot Q_f \cdot \Psi^T$, but it is not always guaranteed that this matrix is positive semidefinite. Choosing the right weighting matrices is an iterative process. After having calculated the Kalman filter feedback matrix a time-domain simulation must be carried out in order to be able to draw a conclusion about the closed-loop performance and the quality of estimation. To be able to judge the sensitivity of the Kalman filter to measurement noise it is necessary to include sensor noise in the simulation stage. By changing for example the value of the elements in matrix $R_f$ it is possible to control the estimation error due to this kind of noise. However, an almost absolute immunity to measurement noise goes with a sluggish estimation of the states. Thus a balance between these two properties must be found.

The determination of the Kalman filter feedback matrix will now be carried out for the concrete road condition and for a speed of travel equal to 30 m/s. Weighting matrix $\Psi \cdot Q_f \cdot \Psi^T$ looks like

$$
\Psi \cdot Q_f \cdot \Psi^T = \begin{bmatrix}
1.5e-9 & 6.0e-12 & 5.4e-17 & 9.0e-9 & 1.4e-13 \\
6.0e-12 & 2.4e-14 & 2.2e-19 & 3.6e-11 & 5.5e-16 \\
5.4e-17 & 2.2e-19 & 2.0e-24 & 3.3e-16 & 4.9e-21 \\
9.0e-9 & 3.6e-11 & 3.3e-16 & 5.5e-8 & 8.2e-13 \\
1.4e-13 & 5.5e-16 & 4.9e-21 & 8.2e-13 & 1.2e-17
\end{bmatrix}
$$

(6.27)

Matrix $R_f$ can be found by using the following rule of thumb

- 127 -
\[ R_f = n_f \cdot \sqrt{\text{diag}(C \cdot C^T)} = n_f \cdot \begin{bmatrix} 8.77e3 & 0 \\ 0 & 6.36e1 \end{bmatrix} \] (6.28)

where \( n_f \) is a 'noise factor' that can be changed such that a balance is found between reconstruction speed and sensitivity to measurement noise. From the diagonal form of equation (6.28) it can be seen that both noise sources are uncorrelated. Consequently, there is no interaction between the noise generated from each accelerometer.

In case of the 2-state semi-active damper system there are two closed-loop models. The relative importance of each model is expressed by a weighting factor. Here, both models are equally weighted \((p_1 = p_2 = 1)\). In practice this means that it is assumed that the soft damper stage is selected as often as the firm damper stage. As some parameters of the road description are included in the system matrix \( A \) \((a \text{ and } \nu)\) and the variance of the road irregularities appears in the covariance matrix \( Q_0 \), one set of parameters has been chosen as a starting-point in the design stage of the Kalman filter being \( a = 0.2 \text{ rad/m}, \sigma_a = 0.0056 \text{ m} \) and \( \nu = 30 \text{ m/s} \).

To illustrate the effect of the use of a state estimator on the closed-loop performance of a semi-active suspension system, a simulation will be carried out in combination with the adaptive control algorithm. To make this simulation of the total system (quarter-car + Kalman filter + controller) more realistic measurement noise must be included. It is assumed that the measurement noise is of a 'white' type and that the noise level is proportional to the intensity of white noise source \( w \). This means that driving over a smooth road results in less measurement noise than in case a rough section is crossed. For weighting matrices \( Q_f \) and \( R_f \) this means that their ratio is assumed to remain constant.

On a concrete road \((a = 0.2 \text{ rad/m}, \sigma_a = 0.0056 \text{ m})\) and at a forward velocity of \( 30 \text{ m/s} \) the following noise values are assumed:

- RMS noise level of the axle accelerometer = 1.0 m/s²
- RMS noise level of the body accelerometer = 0.1 m/s²

For this given speed and road condition, these values are approximately 10% of the RMS acceleration values of the passive suspension system with a fixed damping coefficient \( \kappa = 0.45 \) \((\text{RMS } \ddot{z}_a = 13.65 \text{ m/s}^2, \text{RMS } \ddot{z}_b = 1.07 \text{ m/s}^2)\). Figure 6.4 shows two time domain examples of the measured acceleration signals including the simulated white measurement noise. The additional noise signals are also plotted separately in the same figure.
Figure 6.4 Measured signals and measurement noise.

Measurement noise $v$ is assumed to be 'white' and the intensity matrix $C_{vw}$ is defined by

$$C_{vw} = \beta \begin{bmatrix} 1.0^2 & 0 \\ 0 & 0.1^2 \end{bmatrix} \cdot C_{ww}$$  \hspace{1cm} (6.29)

For a concrete road with parameters given above $\beta$ equals to

$$\beta = \frac{\delta t}{C_{ww}} = \frac{\delta t}{2 \sigma_{vw}^2} = \frac{0.002}{2 \cdot 0.0056^2 \cdot 0.2 \cdot 30} = 5.3146$$  \hspace{1cm} (6.30)

As can be seen from expression (6.29) the ratio of the variances of both noise sources (100:1) corresponds quite well with the ratio of the diagonal element in matrix $R_f$ (138:1). Care has to be taken when random number generators are used in the simulation phase for both road unevenness and measurement noise. One has to avoid that two successive samples are correlated since the Kalman filter theory derived here assumes uncorrelated random sequences.

The quality of estimation will be examined on the basis of the time domain history and RMS values. Firstly, the estimated variables should be able to follow the real variables in the time domain without being too noisy. The RMS values are useful to draw conclusions on the degradation of the performance.
due to the application of a state estimator. Figure 6.5 shows a time domain example of a simulation for a value of \( n_f = 1.0e-4 \). According to the theory as derived in chapter 6.3 the damper force must also be estimated. Since the dynamics of the damper are unknown and therefore not modelled, the estimated damper force is just a multiplication of the damping ratio with the estimated relative velocity.

The variables of main interest are the relative velocity between sprung and unsprung masses, the unsprung mass absolute velocity, the dynamic tire load and the damper force. From figure 6.5 it can be seen that especially the sprung mass velocity is difficult to estimate (the estimated states are plotted with dashed lines). A low frequency drift can be noticed. Although the estimated sprung mass velocity will not tend to infinity, the drift is unacceptable. Generally speaking the estimation is too slow for all variables of interest. Probably this is owing to the ratio of matrices \( Q_f \) and \( R_f \) or more specific the noise factor \( n_f \). Figure 6.6 gives again a comparison between estimation and real value but this time for \( n_f = 1.0e-6 \). This smaller value of \( n_f \) makes estimation of states faster but also more sensitive to measurement noise. Nevertheless, the estimated states in figure 6.6 reveal not to be too sensitive to measurement noise. The speed of reconstruction is all right. Probably \( n_f = 1.0e-8 \) provides a good compromise between noise sensitivity and reconstruction speed.

Now the operation of the estimation only depends on the choice of noise factor \( n_f \). In order to judge the quality of estimation and the sensitivity of the Kalman filter to measurement noise several values of \( n_f \) are considered. Table 6.2 gives the RMS estimation error for the relative velocity, the sprung mass absolute vertical velocity and the dynamic tire load.

<table>
<thead>
<tr>
<th>( \text{control structure} )</th>
<th>( P_{dyn} ) (N)</th>
<th>( \dot{z}_b ) (m/s)</th>
<th>( \dot{z}_a - \dot{z}_b ) (m/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n_f = 1.0e-4 )</td>
<td>264.8</td>
<td>0.0273</td>
<td>0.0758</td>
</tr>
<tr>
<td>( n_f = 1.0e-5 )</td>
<td>199.9</td>
<td>0.0151</td>
<td>0.0336</td>
</tr>
<tr>
<td>( n_f = 1.0e-6 )</td>
<td>179.9</td>
<td>0.0105</td>
<td>0.0343</td>
</tr>
<tr>
<td>( n_f = 1.0e-7 )</td>
<td>178.2</td>
<td>0.0081</td>
<td>0.0336</td>
</tr>
<tr>
<td>( n_f = 1.0e-8 )</td>
<td>178.1</td>
<td>0.0074</td>
<td>0.0346</td>
</tr>
<tr>
<td>( n_f = 1.0e-9 )</td>
<td>185.3</td>
<td>0.0125</td>
<td>0.0584</td>
</tr>
<tr>
<td>( n_f = 1.0e-10 )</td>
<td>236.3</td>
<td>0.0102</td>
<td>0.1173</td>
</tr>
</tbody>
</table>

Table 6.2 DSAD, adaptive extended skyhook control including Kalman filter.
**Figure 6.5** Time domain example of a controlled suspension system including a Kalman filter (DSAD, adaptive extended skyhook control, $n_f = 1.0e-4$).

(— true variable, - - - estimated variable)
Figure 6.6 Time domain example of a controlled suspension system including a Kalman filter (DSAD, adaptive extended skyhook control, $n_f = 1.0 \times 10^{-8}$).

(--- true variable, - - estimated variable)
The percentages in Table 6.2 stand for the ratio of the RMS estimation error to the true RMS value of the concerning parameters. As can be seen from the table only one value of \( n_f \) (1.0e-8) gives satisfactory results. Especially the sprung mass absolute velocity and dynamic tire load seem to be difficult to estimate. The large values of the sprung mass velocity estimation error are mainly caused by low frequency estimation errors as can be observed from figures 6.5 and 6.6. The dynamic tire load is more corrupted by high frequency estimation errors.

As can be seen from equation 1.13, state matrix \( A \) contains information of the road description. In practice the speed of travel \( v \) and road roughness parameter \( a \) vary. In theory this would imply that the quarter-car model is not time invariant any more. By adapting the time varying parameters, the Kalman filter theory as derived above becomes very complex. However, one important property of a state estimator is also that it is robust to parameter variations within a certain range due to the internal feedback. As long as measured output \( y_n \) and estimated output \( \hat{y}_n \) differ, the state estimator will adjust the estimated variables such that measurement and estimation coincide. Simulations showed that a variation of parameters \( a \) and \( v \) doesn’t influence the estimation of important states and derived variables very much. It is therefore not necessary to modify state matrix \( A \), derived matrices \( \Phi \) and \( \Gamma \) and Kalman filter feedback \( K_{fil} \) during the ride.

Figure 6.7 presents several simulation results for different velocities in combination with a concrete road. The Kalman filter has been designed for a speed of travel equal to 30 m/s. The state estimator matrices and feedback gain remain unchanged if the vehicle operates at another speed. It can be seen that the overall performance of the controlled suspension system is worsening somewhat when a Kalman filter is included in the system. This is a usual property of control systems including state estimators. The tire load variations and suspension working space are slightly larger if a Kalman filter is used. Since the deviations from the ideal (not realistic) system with optimal state information is very small, the conclusion can be drawn that the Kalman filter works well, even when the measurement signals are corrupted by noise and when time invariant matrices are assumed.
Figure 6.7 Performance of DSAD, adaptive extended skyhook control, including a Kalman filter.

This chapter has dealt with the application of a state estimator or Kalman filter in the quarter-car system. For a linear system such as the active suspension system it quite easy to construct a Kalman filter. However, due to the non-linearities of the semi-active damper system such a solution is not evident any more. Sometimes estimator instability can not be avoided. This problem can be solved using multi-model techniques. The multi-model analysis considers each damper setting as a different linear suspension system. By taking all these system descriptions it is now possible to determine a single stable Kalman filter using the multi-model optimization.

Considering the application of a Kalman filter in practice, more attention should be paid to the robustness of the estimation. Besides the unknown dynamics of the damper also variations of the vehicle parameters such as the mass of the vehicle body (changes considerable with the load) or non-linear stiffnesses must be considered. Although very time consuming, the multi-model technique to determine a stable Kalman filter internal feedback is applicable for this analysis.
THREE DIMENSIONAL RIDE
AND HANDLING MODEL
MODELING A FULL-VEHICLE SYSTEM

The analysis of suspension control will now be continued with the full-vehicle application. The modeling, the definition of design criteria and the control system design as carried out in the quarter-car part will be repeated for the full-vehicle application. The full-vehicle is not just a connection of four quarter-cars. Besides the roll and pitch degree of freedom, the full-vehicle offers the possibility to perform steering maneuvers. The road holding aspect of the quarter-car model can now be enhanced with handling related aspects. Especially the influence of the vertical tire dynamics on the lateral vehicle dynamics is of particular interest. Furthermore, the full-vehicle contains four suspensions and is excited by four road inputs. Suspension control and performance might be enhanced by these extra control potentialities and inputs compared with the quarter-car application.
Chapter 7 Modeling a Full-Vehicle System

This chapter will deal with the modeling of a full-vehicle model. Besides the derivation of the equations of motion, attention will be paid to road roughness modeling in combination with multi-axle vehicles. The model developed represents a 3-dimensional vehicle with 13 degrees of freedom (DOF). These DOFs are selected to describe both the ride and handling aspects of the vehicle motions. The model should be able to describe the major rigid body modes up to 20 Hz. Although the first bending and torsion mode of the vehicle body are situated near 25 Hz, these flexible body modes are neglected. Their contribution to the total dynamic behavior is assumed to be small. The model should also contain some lateral dynamics because it should be able to describe handling aspects. Especially the interaction between vertical (ride) and lateral (road holding) dynamics is of considerable interest. The handling aspects require a (non-linear) tire model. The first part of this chapter describes the modeling aspects and at the end of this chapter the model is subjected to some preliminary tests to verify its correctness and to gain insight into its dynamic behavior.

7.1 THE VEHICLE MODEL

The 13 DOF model is shown in figure 7.1. The vehicle model is based on the 400 series of Volvo. Nevertheless, the model has been kept universal to such an extent that it can easily be adjusted such that it is applicable for a broad range of passenger cars. The DOFs can be split up into:

- Vehicle body, 6 DOFs \((x_b, y_b, z_b, \phi_b, \theta_b, \psi_b)\),
- Two independent front axles each with 1 DOF \((\phi_{a1L}, \phi_{a1R})\),
- Rigid rear axle with 2 DOFs \((z_{a2}, \phi_{a2})\),
- Powertrain with 3 DOFs \((z_e, \phi_e, \theta_e)\).

For ride analysis, at least three DOFs are necessary for the vehicle body (heave \(z_e\), roll \(\phi_e\) and pitch \(\theta_e\)). The other three (longitudinal \(x_e\), lateral \(y_e\) and yaw \(\psi_e\)) are necessary if handling forms a part of the study. The vertical DOF of both front axles \((\phi_{a1L}, \phi_{a1R})\) speak for themselves. The rigid rear axle needs at least two DOFs \((z_{a2}, \phi_{a2})\) to be able to cope with asymmetric road unevennesses. The 400 series of Volvo are all front wheel driven. They possess a transverse engine with two fastenings at the front side and one fastening at the rear side. Because of compliance in the engine/transmission mounts, the system will vibrate in six directions. Therefore, the powertrain (engine, gearbox, clutch) forms a part of the model also because several natural frequencies of the powertrain are situated in the 10-20 Hz zone. Of all the directions of motion, the most important degrees of freedom for ride analysis are the bounce \(z_e\), roll \(\phi_e\) and pitch \(\theta_e\). The contribution of the powertrain longitudinal/lateral jerk

- 138 -
(for-and-aft/side shake) and yaw is neglected because none of these modes are excited directly by vertical road unevennesses.

\[ \Phi_b \]
\[ \Theta_b \]
\[ \Psi_b \]

**Figure 7.1** 13 DOF vehicle model.

Before describing the derivation of the equations of motion, some vehicle parts, such as the wheel suspensions, powertrain and tires are discussed in more detail.
7.1.1 The Suspension

This section deals with the construction of the front and rear suspension. Since suspension kinematics play an important part in both ride and handling, more attention will be paid to the suspension layout [17].

Front Suspension

The VOLVO 400 series front suspension is based on the well-known Mc Pherson type struts. The stub axle is connected to the vehicle body by means of a strut with an integrated spring/damper element on the one side and a wishbone on the other side. In the real car the lower arms are not directly connected to the vehicle body. Instead of that, they are attached to a subframe which carries the complete powertrain. For reasons of noise isolation, the subframe is fixed to the vehicle body by means of rubber bushes. The wishbones are also fastened to the subframe by means of rubber bushes with predefined elasto-kinematic properties. In the vehicle model the subframe has been neglected. Its rigid body natural frequencies are far beyond 30 Hz. Also the rubber bushes of the lower wishbone arm have been neglected. The complete front suspension is shown in figure 7.2

![Figure 7.2 Front suspension.](image)

The geometry of the Mc Pherson strut front suspension has been approximated by substitution of a swing axle suspension. The inclination point of the swing axle is situated in the virtual reaction point of the Mc Pherson strut [12]. This virtual point is obtained as shown in figure 7.3.
Figure 7.3 Reaction point and roll center of a Mc Pherson strut.

The reaction point (= pole of the wheel kinematics) lies at the intersection of the axis of the lower control arm and the line perpendicular to the strut. The roll center is located on the centerline of the vehicle at the intersection with the line from the center of the tire contact to the virtual reaction point. For small deviations from the equilibrium, it is allowed to replace the Mc Pherson strut with a swing axle suspension with point of application in the virtual reaction point. For small rotations, this configuration contains the same kinematics as the real Mc Pherson strut. The relatively small caster angle, present in the real vehicle, has been neglected.

Rear Suspension

The rear suspension of the 400 series is based on a light weight five link rigid rear axle containing two Watt-linkages and a Panhard rod. This rod takes care of the transmission of lateral tire forces to the vehicle body during cornering. This configuration offers a rather constant wheel camber and a constant track width. Because this construction is a one degree over-engineered kinematic system, bushes are necessary in at least one joint. The complete rear suspension is shown in figure 7.4.
The vehicle model contains the same kinematics as the real rear axle. It is assumed that the rigid axle moves only in a vertical plane, perpendicular to the longitudinal vehicle body axis. The Panhard rod is also present in the vehicle model. Due to its initial oblique angle, an (undesired) coupling between vertical and lateral dynamics has been obtained. The larger the initial angle, the stronger the coupling. In the real vehicle the shock absorbers are situated on the lower arms and the coil springs on top of the axle beam. In the vehicle model both spring and damper elements are positioned on top of the axle beam. This implies that the damping constants must be transformed with the square of the position ratio.

7.1.2 The Powertrain

As already explained before, the powertrain contains six degrees of freedom. Three of them are only used in the intended ride analysis (bounce, roll and pitch). Modeling the powertrain as a rigid body on spring/damper elements is quite awkward. Firstly, the direction of the principle axes of the engine/transmission body does not coincide at all with those of the vehicle body (which have an orientation identically to the Cartesian coordinate system of the origin). This implies that several modes of vibration are coupled. The pitch motion for example is coupled with a roll motion (warp mode). As it is very difficult to derive the equations of motion of such a system with rotated principle axes, the introduction of an oblique frame has been rejected. Instead, the 3 DOFs of the powertrain part has been tuned in such a way that the major (decoupled) bounce, roll and pitch modes are present at the right frequencies.
A second difficulty is the mounting of the engine/transmission. The mounts are not only designed to absorb the rigid body modes but far more important are the internal excitations resulting from the combustion process and inherent imbalances in the reciprocating/rotating masses. Therefore, engine mounts are given very specific characteristics manifested as frequency dependent damping and stiffness properties. To overcome these complications, the (frequency independent) stiffnesses and damping constants are fixed to their values attained at or nearby the rigid body natural frequencies of the engine/transmission system. This guarantees at least that the behavior near the resonant frequencies is described correctly. It is of course possible to model frequency dependent stiffness/damping properties by means of series and parallel connections of spring and damper elements, but this is beyond the scope of this work.

With a proper design of the mounting system the mass of the engine/transmission can be utilized as a vibration absorber attenuating other vibrations, such as the vertical vibrations arising from the front wheel road excitations. For this purpose the mounts are designed to provide a vertical resonance frequency close to that of the front wheel-hop natural frequency (typically 12-14 Hz).

7.1.3 The Tires

The tire plays a crucial role in vehicle dynamics. The tire accomplishes essentially three basic functions:

- Support the vehicle weight, cushioning road irregularities,
- Develop lateral forces forcornering,
- Develop longitudinal forces for acceleration and braking.

The vertical supporting function is the easiest one to model. It can be represented by a (non-linear) spring. The non-linearity originates from the fact that the tire is only able to transmit a compression force. If the wheel loses contact with the road the vertical tire stiffness and force become zero. For reasons of rolling resistance and heat conductivity, vertical tire damping is (unfortunately) very small. Furthermore, it appears to be frequency and speed dependent. Therefore, tire damping has been disregarded. Likewise, the enveloping effect of road irregularities resulting from the finite length of the tire contact patch has been neglected.

The lateral tire forces are responsible for the cornering capabilities of the vehicle. When a rolling tire is subjected to side force, the tire will drift to the
side. An angle \( \alpha \), known as slip angle, will be created between the wheel center plane and the direction of travel (figure 2.3). The relation between the cornering force and slip angle is rather complex.

It is not always necessary that slip angles are initiated by a steering wheel input. On a rough road, for example, slip angles might be generated due to a small lateral motion of the tire contact point caused by wheel load variations. The extent of the lateral deviation depends on the suspension geometry. This mechanism may be responsible for generating a (small) slip angle and consequently a lateral force. At all four wheels such a mechanism may be present. Since it is not necessary that all four side forces cancel each other, this mechanism might influence directional stability (steering wheel corrections of the driver are necessary, preventing the car from deviating from driving straight on). As illustrated in table 7.1, four possible models may be employed to describe the lateral tire forces.

<table>
<thead>
<tr>
<th>steady-state</th>
<th>linear</th>
<th>non-linear</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>constant cornering stiffness</td>
<td>Magic Formula</td>
</tr>
<tr>
<td>transient</td>
<td>constant cornering stiffness</td>
<td>Magic Formula</td>
</tr>
<tr>
<td></td>
<td>+</td>
<td>load dependent</td>
</tr>
<tr>
<td></td>
<td>constant relaxation length</td>
<td>relaxation length</td>
</tr>
</tbody>
</table>

Table 7.1 Tire models.

At small slip angles the side force \( F_y \) (N) - slip angle \( \alpha \) (rad) relationship may be represented by a linear function containing a coefficient \( C_{F\alpha} \), known as the cornering stiffness (N/rad)

\[
F_y = C_{F\alpha} \cdot \alpha
\]  (7.1)

The cornering stiffness is dependent on many variables, of which the vertical tire load is the most important one. Figure 7.5 illustrates this influence.

In linear analyses the load influence is commonly disregarded. This tire model is used here only for analytical calculations. In computer simulations it is better to use a load dependent cornering stiffness or for example the Magic Formula. The latter has the advantage that it is also valid for large slip angles.
Figure 7.5 Cornering stiffness $C_{Rt}$ versus vertical tire load $F_z$.

The Magic Formula [2] is an empirical tire model that is partly based on physical insight into tire force generating properties. For the simple case of pure side slip the side force $F_y$ versus slip angle $\alpha$ relation according to the Magic Formula reads

$$F_y = D \sin \left( C \left( B \{ \alpha + S_h \} - E \{ B \{ \alpha + S_h \} - \arctan \{ B \{ \alpha + S_h \} \} \} \right) \right) + S_v$$  \hspace{1cm} (7.2)$$

with $B$, $C$, $D$ and $E$ being the four characterizing coefficients according to

- **shape factor** $\hspace{1cm} C = a_0$
- **peak factor** $\hspace{1cm} D = a_1 F_z^2 + a_2 F_z$
- **cornering stiffness** $\hspace{1cm} BCD = a_3 \cdot \sin \left( 2 \cdot \arctan \left( \frac{F_z}{a_4} \right) \right) \cdot (1 - a_5 |\gamma|)$
- **stiffness factor** $\hspace{1cm} B = \frac{BCD}{CD}$
- **curvature factor** $\hspace{1cm} E = a_8 F_z + a_9$
- **horizontal shift** $\hspace{1cm} S_h = a_6 \gamma + a_9 F_z + a_{10}$
- **vertical shift** $\hspace{1cm} S_v = a_{11} F_z \gamma + a_{12} F_z + a_{13}$

where $a_0$..$a_{13}$ are the 14 parameters that characterize the influence of the vertical load $F_z$ and camber angle $\gamma$ on the side force generation. Because of ply-steer, conicity and camber the characteristics of the side force will have an off-set. This is described by the shifts $S_h$ and $S_v$. 

---

- 145 -
Chapter 7 Modeling a Full-Vehicle System

![Graph showing side force F_y as a function of slip angle at different vertical tire loads F_z.]

**Figure 7.6** Side force $F_y$ as a function of slip angle at different vertical tire loads $F_z$.

The side force versus slip angle characteristic for several vertical tire loads is illustrated in Figure 7.6. The horizontal and vertical shifts have been omitted and the camber angle is assumed to be zero. As the vehicle model does not contain rotating wheels, the longitudinal force characteristics have not been included. This simplification makes implementation of the Magic Formula tire model much easier since the formulae for combined cornering and braking are very awkward to deal with.

The mechanism of generating lateral slip forces is not an instantaneous process, but lags with respect to the actual development of slip. This time lag is caused by the deflection of the tire sidewalls, carcass and rubber tread elements in the lateral direction. The lag is closely related to the distance travelled, typically taking one-half revolution of the tire. This distance is often referred to as the relaxation length. The time lag in developing the lateral force depends on the speed of travel of the vehicle. In vehicle dynamics analysis the relaxation length is often used to describe the side force lag resulting from a steering wheel input. Depending on the speed of travel, the turning behavior becomes more or less sluggish. However, the relaxation effect plays also an important role in the loss of cornering force when a tire operates on a rough road surface. At varying vertical tire loads, the sidewalls are continuously being straightened out and compressed. During cornering, the tire must then roll through its relaxation length to again build up a lateral force. As a consequence, the tire is observed to have lower lateral force capabilities on rough roads. Therefore, to achieve the best road-holding
performance, the suspension should be designed to minimize vertical tire load variations.

The relaxation effect can be described by a simple pragmatic tire model that is especially suitable to be used in vehicle simulation studies. This tire model was compared with a more complex adapted bare string tire model by Takahashi et al. [34]. Both models showed good agreement; moreover the models agreed with experimental results in the range of wavelengths $L$ not less than about two meters (i.e. about ten times the tire contact length).

The relation between wavelength $L$ (m) and frequency $f$ (Hz) is given by

$$f = \frac{v}{L} \quad (7.3)$$

where $v$ represents the forward vehicle speed in (m/s). Now assume a speed of travel of 30 m/s (108 km/h), then the simple transient tire model is able to describe the dynamic tire properties well up to 15 Hz ($L \geq 2$ m). Because this frequency extends beyond the wheel-hop natural frequency (13 Hz) this simple tire model can be helpful in the design and tuning of (semi-)active suspension systems.

The simple transient tire model extension consists of a first-order approximation of the side force lag and is given for small slip angles (no sliding in the contact zone), constant vertical load and disregarding the turn slip by

$$\sigma \frac{dF_y}{ds} + F_y = F_y'' \quad (7.4)$$

with $F_y$ the dynamic side force in (N), $s$ the distance rolled in (m), $\sigma$ the relaxation length in (m) and $F_y''$ the steady-state side force. It is well known that the side force properties show essential non-linearities in slip angle $\alpha$ and vertical load $F_z$. The following non-linearities can be introduced for the relaxation length

$$\sigma = \sigma(F_z) = b_1 F_z + b_2 F_z^2 + b_3 F_z^3 \quad (7.5)$$

The coefficients of the third-order polynomial functions for the expression of the load dependency of $\sigma$ were found by regression of measured data. Table 7.2 gives the coefficients $b_1, b_3$ and figure 7.7 shows the relaxation length as a function of the vertical load. Measurements carried out by Takahashi et al. showed that the speed dependency of the relaxation length is negligible. The
effect of the non-linear dependency of $F_y$ on the slip angle $\alpha$ [28] has not been used here.

\[
\sigma = \sigma(F_z) = b_1 F_z + b_2 F_z^2 + b_3 F_z^3
\]

<table>
<thead>
<tr>
<th>$b_1$</th>
<th>$b_2$</th>
<th>$b_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.886e-04</td>
<td>-3.650e-08</td>
<td>1.269e-12</td>
</tr>
</tbody>
</table>

195/60R14H, pressure: 2.0 bar, speed: 90 km/h

**Table 7.2** Polynomial coefficients of $\sigma(F_z)$.

The distance derivative of $F_y$ can be written in terms of a time derivative using $s = v \cdot t$ and assuming a constant forward speed $v$

\[
\frac{dF_y}{ds} = \frac{dF_y}{v \cdot dt} \quad (7.6)
\]

![Figure 7.7](image)

**Figure 7.7** Relaxation length $\sigma$ as a function of vertical tire load $F_z$.

For simulation purpose, equation (7.4) may be rewritten as

\[
\dot{F}_y = A_\sigma \cdot F_y + B_\sigma \cdot \dot{F}_y \quad (7.7)
\]

with

\[
A_\sigma = \frac{-v}{\sigma(F_z)} = \left[ \frac{-v}{b_1 F_z + b_2 F_z^2 + b_3 F_z^3} \right] \quad B_\sigma = -A_\sigma
\]
As can be seen from the coefficients \( A_\sigma \) and \( B_\sigma \) of equation (7.7), the side force relation is highly non-linear. At each simulation time-step matrices \( A_\sigma \) and \( B_\sigma \) have to be recalculated.

Care must be taken with the implementation of equation (7.7) in a simulation program. To prevent integrator instability it is necessary to bound the coefficient of \( A_\sigma \) according to

\[
\frac{v}{\sigma(F_y)} = \frac{1}{\tau} = \omega_b \leq \frac{\pi}{\delta t}
\]

(7.8)

where \( \tau \) is the time constant and \( \omega_b \) is the break-frequency of the first-order filter. \( \delta t \) denotes the time step of the integration procedure. If for example \( \delta t \) equals 0.002 s, \( \omega_b \) should be smaller than 500 \( \pi \) rad/s.

Figure 7.8 shows an example of wave forms of the side force \( F_y \) at various wavelengths \( L \) and fixed speed (30 m/s). The slip angle \( \alpha \) has been kept constant (1.0 deg) during a sinusoidal variation of the normal load. The static load is 2286 N and the amplitude of the dynamic variation equals 0.9 times the static load.

![Figure 7.8 Side force under time-varying vertical load (\( \alpha = 1 \) deg).](image-url)
The loss of side force is composed of the 'static' loss and 'dynamic' loss. The first one is caused by the non-linear relation between the cornering stiffness and the vertical load. The latter one is due to the transient effect represented by the varying relaxation length. The dynamic loss is the difference between the total loss and the static loss. Figure 7.9 illustrates the influence of the relaxation length on the side force generation. The full-vehicle model is subjected to a lane-change on an uneven road. Using only the steady-state tire description (Magic Formula), the side force variations are fully dependent on the vertical tire load variations. Both forces contain the same frequency contents. Only the amplitudes of the side force are related to the magnitude of the tire load by means of non-linear relation (7.2). The transient tire model extension combined with the Magic Formula shows a much smoother course of the side force. The transient tire model with load dependent relaxation length acts as a low-pass filter.

![steady-state tire model](image1)

![transient tire model](image2)

**Figure 7.9** Side force example of a vehicle subjected to a lane-change.

Using relation (7.7) in linear analysis is not possible due to its non-linear character. Therefore, a linear transient tire model with a constant relaxation length will be used in combination with the linear steady-state tire model with a constant cornering stiffness. In all handling simulations the non-linear Magic Formula tire model will be used with the non-linear transient tire model with a load dependent relaxation length.
The equations of motion of the vehicle model are derived by hand using the method of Lagrange. This method requires the calculation of the kinetic and potential energy. Analytical analyses of vehicle vibrations including control algorithm optimization require a model with a minimal number of degrees of freedom (DOF). Of course, it is possible to use standard multi-body software the generate the equations of motion of the vehicle. However, many of the contemporary software packages are not able the generate a minimal set of (linear) equations of motion in a symbolic way.

The full-vehicle model contains 5 bodies (main body, 2 front axles, 1 rear axle and engine/transmission), each with 6 degrees of freedom. This implies that the vehicle model numbers 30 degrees of freedom. However, it contains several kinematic constraints such as two hinges (front suspension) and two joints (rear axle). Thus, the number of degrees of freedom is reduced to 13 through of 17 (non-linear) constraint equations.

It is assumed that the rotations of the bodies are small. This means that rotations about the three axes attached to the body can be superimposed. The Lagrangian method used here requires an inertial coordinate system. Since the allowed rotations are typically small, the vehicle model developed is not suitable for handling situations with large yaw angles, such as steady-state cornering and J-turns. It is however possible to describe lane-change and slalom maneuvers very well. For large angles it is better to take the Newton-Euler equations or the modified method of Lagrange as a starting point for the derivations of the equations of motion. These methods are based on a moving coordinate system, rather than an inertial one used with the standard Lagrangean method. The force and moment expressions are quite complicated due to coupling terms. Besides that, the elimination of the constraint forces is very time-consuming with the Newton-Euler method.

The equation of Lagrange in his standard form is given by

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_i} \right) - \left( \frac{\partial T}{\partial q_i} \right) + \left( \frac{\partial V}{\partial q_i} \right) + \left( \frac{\partial D}{\partial q_i} \right) = Q_i$$

(7.9)

with $T$ as the kinetic energy, $V$ the potential energy, $D$ the Rayleigh's dissipation function, $q_i$ the generalized coordinates and $Q_i$ the generalized forces. The complete derivation of the equations of motion can be found in appendix A. The final result of the exercise explained above is a set of thirteen 2nd-order differential equations in matrix form given by
\[ M \ddot{\mathbf{z}} + K \dot{\mathbf{z}} + C \mathbf{z} = E_i^y F_i^y + E_i^z F_i^z \]  

(7.10)

with \( M \) being the mass matrix, \( C \) the stiffness matrix and \( K \) the damping matrix. \( E_i, E_i^y \) and \( E_i^z \) are input matrices for the suspension force, lateral tire force and vertical tire force respectively. Vector \( \mathbf{z} \) (cf. figure 7.1) with generalized coordinated reads

\[
\mathbf{z} = [x_b \quad y_b \quad z_b \quad \phi_b \quad \theta_b \quad \psi_b \quad \phi_a^{1L} \quad \phi_a^{1R} \quad \phi_{a2} \quad z_{a2} \quad \phi_{e} \quad \theta_{e}]^T
\]

The three force vectors representing the suspension forces, tire side forces and vertical tire forces are

\[
F_s = \begin{bmatrix} F_{s1L} & F_{s1R} & F_{s2L} & F_{s2R} \end{bmatrix}^T,
\]

\[
F_i^y = \begin{bmatrix} F_{i1L}^y & F_{i1R}^y & F_{i2L}^y & F_{i2R}^y \end{bmatrix}^T,
\]

\[
F_i^z = \begin{bmatrix} F_{i1L}^z & F_{i1R}^z & F_{i2L}^z & F_{i2R}^z \end{bmatrix}^T,
\]

where '1' stands for front axle, '2' for rear axle, 'L' for left and 'R' for right. There is no need for a separate input for the steering angle, since the steering wheel input corresponds to an extra side force on the front wheels (valid for small steering and slip angle only). The model offers two possible lateral tire force characteristics: a linear model with a constant cornering stiffness \( C_{Fa} \) and a non-linear tire model based on the Magic Formula.

The thirteen 2nd-order differential equations of the vehicle can be converted to twenty-six 1st-order differential equations according to

\[
\dot{x}_o = A_o x_o + F_i u_i + S_i \delta + T_i^y F_i^y + T_i^z F_i^z
\]

(7.11)

with

\[
A_o = \begin{bmatrix} \textbf{0} & \textbf{I} \\ -M^{-1}C & -M^{-1}K \end{bmatrix}, \quad x_o = \begin{bmatrix} \mathbf{z} \\ \dot{\mathbf{z}} \end{bmatrix}
\]

and

\[
F_i = \begin{bmatrix} \textbf{0} \\ M^{-1}E_i \end{bmatrix}, \quad T_i^y = \begin{bmatrix} \textbf{0} \\ M^{-1}E_i^y \end{bmatrix}, \quad T_i^z = \begin{bmatrix} \textbf{0} \\ M^{-1}E_i^z \end{bmatrix}, \quad S_i = \begin{bmatrix} \textbf{0} \\ M^{-1}E_i^y \end{bmatrix}
\]

\( A_o \) is the system matrix, \( F_i \) represents the input matrix for the active suspension force \( u_i \), \( T_i^y \) and \( T_i^z \) stand for the input matrices for the lateral and vertical tire force \( F_i^y \) and \( F_i^z \) respectively and \( S_i \) represents the input matrix for the front wheel steering angle \( \delta \) (equal for left and right).
7.2 ROAD MODEL

In this section a four-wheel road elevation model will be developed. The model contains the frequency dependent correlation between left and right track and the correlation arising from the fact that the input at the rear wheels is approximately a delayed version of that at the front wheels. Due to the off-tracking of the vehicle rear axle, the model is only valid for straight-ahead running and driving though wide curves.

7.2.1 Two Track Road Model

Following Rill [31] one may derive a road model as shown in figure 7.10. The unevennesses of the left and right track are made up by the combination of two uncorrelated random processes $w_z$ (in-phase, heave) and $w_o$ (out-of-phase, roll).

![Figure 7.10 Road model.](image)

The state-space equation of the road unevenness shaping filter is given by

\[ \dot{x}_r = A_r x_r + G_r w_r \]  \hspace{1cm} (7.12a)

\[ y_r = C_r x_r \]  \hspace{1cm} (7.12b)

with matrices $A_r$, $G_r$, $C_r$ equal

\[ A_r = \begin{bmatrix} -v \beta_1 & 0 \\ 0 & -v \beta_2 \end{bmatrix} \quad G_r = \begin{bmatrix} v g_1 & 0 \\ 0 & v g_2 \end{bmatrix} \quad C_r = \begin{bmatrix} 1 & -T_w/2 \\ 1 & +T_w/2 \end{bmatrix} \]

where $v$ denotes the forward vehicle speed in (m/s) and $T_w$ the track width in (m).
State vector $\mathbf{x}_r$, white noise vector $\mathbf{w}_r$, and track displacement $\mathbf{y}_r$ read

$$
\mathbf{x}_r = \begin{bmatrix} z_m \\ \phi_m \end{bmatrix}, \quad \mathbf{w}_r = \begin{bmatrix} w_z \\ w_0 \end{bmatrix}, \quad \mathbf{y}_r = \begin{bmatrix} z_{rL} \\ z_{rR} \end{bmatrix}
$$

The multi-variable transfer function matrix $G(s)$ between white noise input $\mathbf{w}_r$ and road elevation $z_r$ is given by

$$
G(s) = \frac{Y_r(s)}{W_r(s)} = C_r(sI - A_r)^{-1}G_r
$$

with 's' being the Laplace variable. Matrix $G(s)$ reads

$$
G(s) = \begin{bmatrix}
\frac{ug_1}{s-v\beta_1} & -\frac{T_mug_2}{2(s-v\beta_2)} \\
\frac{ug_1}{s-v\beta_1} & \frac{T_mug_2}{2(s-v\beta_2)} \\
\frac{ug_1}{s-v\beta_1} & \frac{H_{zz}(s)}{H_{qq}(s)} \\
\frac{ug_1}{s-v\beta_1} & \frac{H_{zz}(s)}{H_{qq}(s)}
\end{bmatrix}
$$

(7.14)

The power spectral density of the left ($S_{LL}$) or right ($S_{RR}$) track can be derived by substituting $s = j\omega$

$$
S_{LL}(\omega) = S_{RR}(\omega) = H_{zz}(j\omega)H_{zz}(j\omega) \cdot S_{zz}(\omega) + H_{zz}(j\omega)H_{zz}(j\omega) \cdot S_{zz}(\omega)
$$

$$+ H_{qq}(j\omega)H_{qq}(j\omega) \cdot S_{qq}(\omega) + H_{qq}(j\omega)H_{qq}(j\omega) \cdot S_{qq}(\omega)
$$

(7.15)

with $\overline{H}$ being the complex conjugate of $H$. Since the product of a complex number and its complex conjugate is equal to the magnitude of the number squared, the following relations hold

$$
S_{LL}(\omega) = S_{RR}(\omega) = |H_{zz}(\omega)|^2 S_{zz}(\omega) + |H_{zz}(\omega)|^2 S_{zz}(\omega)
$$

$$+ |H_{qq}(\omega)|^2 S_{qq}(\omega) + |H_{qq}(\omega)|^2 S_{qq}(\omega)
$$

(7.16a)

$$
S_{LL}(\omega) = S_{RR}(\omega) = \frac{v^2g_1^2}{(\omega^2 + v^2\beta_1^2)} S_{zz}(\omega) + \frac{T_m^2v^2g_2^2}{4(\omega^2 + v^2\beta_2^2)} S_{qq}(\omega)
$$

(7.16b)

with $S_{zz}$ and $S_{qq}$ representing the power spectral density of the heave- and roll excitation respectively. As the cross-spectral density terms $S_{zz} = S_{qq}$ are all equal to zero (uncorrelated random sequences), no cross terms will appear in equation (7.16b).

The coherence $\Gamma(\omega)$ between the left and right track can be derived from

$$
\Gamma(\omega) = \frac{|S_{LR}(\omega)|}{\sqrt{S_{LL}(\omega) \cdot S_{RR}(\omega)}}
$$

(7.17)
with \( S_{LR} (= S_{RL}) \) being the cross-spectral density according to

\[
S_{LR}(\omega) = H_{zz}(j\omega) \cdot \overline{H_{sy}}(j\omega) \cdot S_{sy}(\omega) + H_{sy}(j\omega) \cdot \overline{H_{sy}}(j\omega) \cdot S_{yz}(\omega) + H_{yz}(j\omega) \cdot \overline{H_{zy}}(j\omega) \cdot S_{zy}(\omega) + H_{zy}(j\omega) \cdot \overline{H_{zy}}(j\omega) \cdot S_{yx}(\omega)
\]

\[
S_{LR}(\omega) = S_{RL}(\omega) = \frac{v^2 g_1^2}{(\omega^2 + \nu^2 \beta_1^2)} S_{zz}(\omega) - \frac{T_w^2 v^2 g_2^2}{4(\omega^2 + \nu^2 \beta_2^2)} S_{yy}(\omega)
\]

The coherence function can be simplified to

\[
\Gamma(\omega) = \frac{(1 - \alpha) \omega^2 + (\beta_2^2 - \alpha \beta_1^2)v^2}{(1 + \alpha) \omega^2 + (\beta_2^2 + \alpha \beta_1^2)v^2}
\]

with

\[
\alpha = \left( \frac{T_w g_2}{2 g_1} \right)^2
\]

For \( \alpha = 0.0 \), \( \beta_1 = \beta_2 \) and \( g_1 = 1 \) the coherence between left and right track is always equal to 1. For \( \alpha = 1.0 \) in combination with \( \beta_1 = \beta_2 \) and \( g_1 = 1/2\sqrt{2} \) the left and right tracks are fully uncorrelated. Figure 7.11 gives a three-dimensional impression of a realization of a road profile according to the theory described above. Three values of \( \alpha \) are considered (0.0, 0.5 and 1.0). Now assume that \( S_{zz} = S_{sy} = S_{ww} \) (the white noise amplitudes have been scaled through \( g_1 \) and \( g_2 \) appearing in \( G_r \)), then

\[
S_{LL}(\omega) = S_{RR}(\omega) = \frac{\omega^2 (1 + \alpha) v^2 g_1^2 + (\beta_2^2 + \alpha \beta_1^2)v^4 g_1^2}{(\omega^2 + \nu^2 \beta_1^2)(\omega^2 + \nu^2 \beta_2^2)} S_{ww}(\omega)
\]

The power spectral density \( S_{ww}(\omega) \) of white noise \( w \) is defined as

\[
S_{ww}(\omega) = \frac{\sigma^2}{4\pi v}
\]

with \( \sigma/v \) being the intensity of white noise \( w \). The intensity function reads

\[
C_{ww}(\tau) = \begin{bmatrix} q & 0 \\ v & 0 \\ 0 & q/v \end{bmatrix} \delta(\tau)
\]

The speed independent spectral densities can be obtained by substituting

\[
\omega = \Omega \cdot v = \frac{2\pi \cdot v}{L} \quad S(\Omega) = S(\omega) \cdot v = S(f) \cdot v
\]

where \( \Omega \) represents the wave number or road frequency in (1/m).
Figure 7.11 Road elevation as a function of coherence factor $\alpha$.

$S_{LL}(\Omega)$ or $S_{RR}(\Omega)$ take a form as

$$S_{LL}(\Omega) = S_{RR}(\Omega) = \frac{\Omega^2(1 + \alpha) + \left(\beta_2^2 + \alpha \beta_1^2\right)}{\left(\Omega^2 + \beta_1^2\right)\left(\Omega^2 + \beta_2^2\right)} g_1^2 \cdot S_{ww}(\Omega)$$

(7.25)
The speed independent coherency function is given by

$$\Gamma(\Omega) = \frac{(1-\alpha)\Omega^2 + \beta_2^2 - \alpha\beta_1^2}{(1+\alpha)\Omega^2 + \beta_2^2 + \alpha\beta_1^2}$$  \hspace{1cm} (7.26)$$

The spectral density function as well as the coherence function are plotted in figure 7.12. The essential parameter values for the road model as described above are given in table 7.3.

<table>
<thead>
<tr>
<th>symbol</th>
<th>unit</th>
<th>value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td></td>
<td>0.750</td>
</tr>
<tr>
<td>$\beta_1$</td>
<td></td>
<td>0.114</td>
</tr>
<tr>
<td>$\beta_2$</td>
<td></td>
<td>1.200</td>
</tr>
<tr>
<td>$\delta_1$</td>
<td></td>
<td>1.000</td>
</tr>
<tr>
<td>$\delta_2$</td>
<td></td>
<td>1.228</td>
</tr>
<tr>
<td>$q$</td>
<td></td>
<td>5.35e-6</td>
</tr>
<tr>
<td>$T_w$</td>
<td>m</td>
<td>1.410</td>
</tr>
</tbody>
</table>

Table 7.3 Road model parameters.

Figure 7.12 Power spectral density and coherence function of left and right track.
Chapter 7 Modeling a Full-Vehicle System

The parameters shown in table 7.3 are valid for a concrete road. The spectral density of the left or right track corresponds to the PSD of the first-order road model as discussed with the quarter car model analysis (section 1.2) with \( a = 0.15 \) and \( \sigma = 0.0056 \). The coherence between the left and right track depends of course on the track width. It is quite clear that the closer the left track approaches the right track, the higher the coherence gets. In accordance with a real road, the low frequency road excitations are more correlated than the high frequency roughnesses at a given track width.

7.2.2 Pure Time Delay Approximation

The road input at the rear wheels are identical to the disturbances at the front wheels, except for a pure time delay. This time delay \( \tau \) between the front and rear wheel excitation equals

\[
\tau = \frac{w_b}{v} \tag{7.27}
\]

with \( w_b \) representing the wheelbase (m) and \( v \) the speed of travel (m/s). In computer simulations and calculations it is often not possible to formulate the pure time delay. For example, the combination of a pure time delay and a variable time-step integrator may be very cumbersome to program. In this section three methods to approximate a pure time delay are discussed:

- First-order approximation
- Padé approximation
- Discrete time shift

First-order approximation

It is possible to approximate a pure time delay by placing \( n \) identical first-order systems with a time constant equal to \( \tau/n \) in series. The sum of all time constants always equals the wheelbase time delay \( \tau \). The transfer function of such a set of first-order systems is given by

\[
H_f(s) = \left( \frac{\tau}{n s + 1} \right)^n \tag{7.28}
\]

For an infinite number of first-order systems this expression reads as

\[
\lim_{n \to \infty} H_f(s) = e^{-\tau s} \tag{7.29}
\]

It can be seen that this is the Laplace transform of a pure time delay. For a finite number of systems this method only approximates the pure time delay over a finite bandwidth. Figure 7.13 shows 5 systems (1st-order up to 9th-
Besides the amplitude ratio and phase angle, also the relative phase error between the true time delay and approximation has been included. The figures show clearly that the more systems are connected in series, the better the approximation will be. However, the major drawback of the first-order approximation is the number of systems needed for the wheelbase time delay application. Besides the phase deviation, also an amplitude error appears.

**Figure 7.13** Pure time delay approximation by means of a series connection of first-order systems.
The series connection acts as a low-pass filter. This implies that the rear wheel road unevennesses are not only delayed, but also filtered. A proper choice of $n$ depends on the wheelbase, vehicle speed and frequency range of excitation. Suppose, it is desirable to describe a frequency range between 0 and 20 Hz, furthermore $v = 30 \text{ m/s}$ and $w_b = 2.5 \text{ m}$, then it is necessary to have more than 100 first-order systems in series to meet with the amplitude criterion. As this number is far too large, this method of approximation is not suitable for the wheel-base time delay.

**Padé Approximation**

The transfer function of a $n$th order Padé approximation of a pure time delay is defined as

$$H_p(s) = \frac{2 - ts + \frac{(ts)^2}{2!} - \frac{(ts)^3}{3!} + \ldots + (-1)^n \frac{(ts)^n}{n!}}{2 + ts + \frac{(ts)^2}{2!} + \frac{(ts)^3}{3!} + \ldots + \frac{(ts)^n}{n!}}$$

(7.30)

Both numerator and denominator contain the same coefficients. Only the sign of the uneven numerator coefficients differs. The amplitude ratio of this Padé filter always equals 1. The phase angle approximation of a pure time delay changes with the order of the filter. Figure 7.14 shows five examples (1st-order up to 9th-order).

Unlike the first-order approximation, the Padé filter does not function as a low-pass filter. This is very convenient in our wheel-base time delay application. Furthermore, the Padé filter approximates the phase lag of a pure time delay much better than the series connected first-order filters.

Table 7.4 gives an impression of the relative phase error as a function of vehicle speed and filter order $(n)$ at 20 Hz. The table shows that the approximation for low speeds is more troublesome than for higher speeds. At 50 m/s a 4th-order Padé filter is adequate, at 10 m/s even a 10th-order filter is not sufficient. It is therefore recommendable to adjust the order with the speed of travel.
Figure 7.14 Pure time delay approximation by means of a Padé approximation.

For convenience, the transfer function (7.30) of the Padé filter can be converted to a state-space model (with matrices $A_p$, $B_p$, $C_p$, $D_p$ and state vector $\dot{x}_p$). It can now be combined with the road unevenness description according to equation (7.12). The complete road model reads

\[
\begin{bmatrix}
\dot{x}_r \\
\dot{x}_p
\end{bmatrix} =
\begin{bmatrix}
A_r & 0 \\
B_p C_r & A_p
\end{bmatrix}
\begin{bmatrix}
x_r \\
\dot{x}_p
\end{bmatrix} +
\begin{bmatrix}
G_r \\
0
\end{bmatrix} w_r
\]  

(7.31a)
with 2 Padé filters (one for each track) according to

$$A_p = \begin{bmatrix} A_{pL} & 0 \\ 0 & A_{pR} \end{bmatrix}, \quad B_p = \begin{bmatrix} B_{pL} & 0 \\ 0 & B_{pR} \end{bmatrix}, \quad C_p = \begin{bmatrix} C_{pL} & 0 \\ 0 & C_{pR} \end{bmatrix}, \quad D_p = \begin{bmatrix} D_{pL} & 0 \\ 0 & D_{pR} \end{bmatrix}$$

The order of the total road model equals \(2 + 2n_p\), with \(n_p\) being the order of the time delay approximation by means of a Padé filter.

<table>
<thead>
<tr>
<th>Padé order</th>
<th>Speed of travel (m/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>10</td>
</tr>
<tr>
<td>1</td>
<td>90.4</td>
</tr>
<tr>
<td>2</td>
<td>81.2</td>
</tr>
<tr>
<td>3</td>
<td>72.4</td>
</tr>
<tr>
<td>4</td>
<td>64.1</td>
</tr>
<tr>
<td>5</td>
<td>56.1</td>
</tr>
<tr>
<td>6</td>
<td>48.6</td>
</tr>
<tr>
<td>7</td>
<td>41.5</td>
</tr>
<tr>
<td>8</td>
<td>34.9</td>
</tr>
<tr>
<td>9</td>
<td>28.8</td>
</tr>
<tr>
<td>10</td>
<td>23.2</td>
</tr>
</tbody>
</table>

**Table 7.4** Relative phase error (%) at 20 Hz as a function of vehicle speed and order of Padé filter \((w_b = 2.5\, \text{m})\).

**Discrete time shift**

As the name already suggests, the discrete time shift is based on an array with numbers containing the road elevation over a predefined period of time \(\tau\). Suppose we simulate our vehicle model with a fixed time step \(\delta t\), the size of the array then equals

$$n = \text{round} \left( \frac{\tau}{\delta t} \right) + 1$$  \hspace{1cm} (7.32)
As \( n \) has to be an integer number, the delay time does not always correspond to the wheel-base time delay. It is then appropriate to adjust the speed of travel a bit such that \( n \) is a whole number. Normally, the array operations are based on shifts. Every new time-step the entire contents of the array is moved up one place (LIFO, last in first out). For low speeds \( n \) might become very large (e.g. \( n = 251 \) at \( v = 10 \) m/s, \( w_b = 2.5 \) m, \( \delta t = 1 \) ms). In that case the shift operation becomes very time consuming. More elaborate methods do exist to overcome this problem. Even more problematic becomes the method when an integration routine with variable time-step \( \delta t \) is used.

It is possible to describe a pure time delay by using discrete time state-space equations. Suppose we want to delay the signal \( w \) with 2 time steps then

\[
\begin{align*}
x_1 &= w \\
x_2 &= x_1 \\
x_3 &= x_2
\end{align*}
\]

In matrix notation this is equivalent with

\[
x_{n+1} = \Phi_d \cdot x_n + \Gamma \cdot w
\]  \hspace{1cm} (7.33)

where

\[
\Phi_d = \begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}, \quad \Gamma_d = \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}, \quad x_n = \begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
\]

It is possible to transform the continuous time differential equations of the vehicle model to the discrete time domain by means of the z-transform. By having these discrete time equations it is now easy to combine the vehicle equations with the delay equations as clarified above. However, this is not recommendable because the number of delay steps might be large (likely more than 100). The sizes of the matrices get unacceptably large to deal with despite the many zeros. Including the discrete time shift into the system description has the advantage that the total system does not contain any time delays. In that case, optimizations, simulations and linear analysis are more easy to carry out.
7.3 DRIVER MODEL

As handling forms an aspect of the suspension design study, it is not sufficient to use merely open-loop steering tests. Depending on the kind of suspension modification, more or less steering effort is needed to maintain the vehicle on the desired track. The steering effort can be a measure of handling performance. For this reason, handling studies of the driver-vehicle closed-loop model system are considered to be important for judging the effect of suspension modifications on handling. The driver-vehicle closed-loop system was previously studied by McRuer [25].

![Figure 7.15 Driver-vehicle model.](image)

The original model of McRuer et al. has been employed for corrections of the lateral deviation of the vehicle due to disturbances like side-wind gusts. However, with a proper choice of gains, the model is also suitable for modest handling tests, with small lateral deviations, such as lane changes or slalom tests. The driver holds the steering wheel and he looks at a preview point $P$ (figure 7.15). The driver should correct for the difference $\varepsilon$ between the intended path $r$ and the lateral viewpoint location $y_p$ at a distance $l_r$ in front of the vehicle by means of steering angle $\delta$. The path deviation $\varepsilon$ reads for small yaw angles $\psi_b$

$$\varepsilon = r - y_p = r - (y_b + l_r \psi_b)$$

(7.34)

According the McRuer the behavior of the human controller can be modelled according to the following transfer function

$$H_d(s) = \frac{\delta(s)}{\varepsilon(s)} = k_d \cdot \frac{\tau_z s + 1}{\tau_z s + 1} \cdot \frac{1}{\tau_s s + 1} \cdot e^{-\tau_d s}$$

(7.35)
with \( k_d \) as a gain, \( \tau_1 \) and \( \tau_2 \) the phase lead and lag time constant respectively, \( \tau_n \) the neuromuscular time constant (characterizing the slowness of the human muscular system) and \( \tau_d \) the time delay of the driver. For modeling convenience the pure time delay can be approximated by a Padé filter. Using a first-order Padé approximation, transfer function (7.35) can be rewritten to

\[
H_d(s) = K_d \frac{\tau_1 s + 1}{\tau_2 s + 1} \frac{1}{\tau_n s + 1} \frac{2 - \tau_d s}{2 + \tau_d s}
\]  

(7.36)

According to McRuer, the human controller combined with an arbitrary system will behave like the so-called cross-over model

\[
H_d \cdot H_v = \frac{\omega_c}{s} \cdot e^{-\tau_d s}
\]  

(7.37)

with \( \omega_c \) representing the cross-over frequency and \( H_v \) the transfer function of the vehicle between \( y_p \) and \( \delta \). Depending on the system to be controlled, \( \tau_1 \) and \( \tau_2 \) can be chosen such that the series connection of driver and vehicle corresponds to equation (7.37). This means that the human controller adapts himself to the system to be controlled. From equation (7.37) it can be seen that the closed-loop man-machine system will behave like an ordinary servo system (with a time delay). Figure 7.16 illustrates the closed-loop driver-vehicle system with \( H_d \) as the driver model frequency response function (FRF) and \( H_v \) as the vehicle model FRF.

\[
\begin{align*}
\text{Figure 7.16 Closed-loop driver-vehicle model.}
\end{align*}
\]

Having the linear vehicle model, it is quite easy to determine the open-loop frequency response \( H_v \) of a steering input \( \delta \) to the viewpoint output \( y_p \). Figure 7.17 shows this open-loop response for a speed of 80 km/h and a look-ahead distance of 10 m. It can be seen that the lateral control of a vehicle corresponds well to a scaled double integrator (scaling factor = 182). This information is very useful when values for the time constants \( \tau_1 \) and \( \tau_2 \) and gain \( K_d \) have to be chosen.
A double integrator in series with the driver model according to equation (7.36) can be converted to the cross-over model (7.37) when \(\tau_0 = 0\) and \(\tau_1\) is large. The cross-over frequency \(\omega_c\) chosen is equal to 4.0 [25]. This gain is not arbitrary, it is strongly dependent on time delay \(\tau_d\). A large value will make the closed-loop cross-over model unstable; with a small value the bandwidth of the servo system is limited. The other values chosen are: \(\tau_1 = 10\), \(\tau_2 = 0\), \(\tau_d = 0.2\), \(\tau_e = 0.2\), \(K_d = 0.0016\). These values are all valid for a speed of travel equal to 80 km/h and a preview distance of 10 m. Other speeds or look-ahead distances result in other gains \(K_d\) and \(\tau_1\).

![Magnitude and phase plots](image)

**Figure 7.17** Open loop frequency response of viewpoint output to steering input (dashed = scaled double integrator).

The linear driver model according to equation (7.36) can be converted to a state-space model. This model takes a form as

\[
\dot{x}_d = A_d x_d + B_d (r - y_p) \tag{7.38a}
\]

\[
y_d = C_d x_d + D_d (r - y_p) = \delta \tag{7.38b}
\]
Chapter 7 Modeling a Full-Vehicle System

7.4 Natural Frequencies and Mode Shapes

To gain more insight into the dynamic behavior of the vehicle model it is important to have a closer look at the eigenvalues of the linear vehicle model. For linear analysis it is possible to combine the following four sub-systems:

- vehicle model with 13 DOF (26 states), equation (7.11),
- driver model (2 states), equation (7.38),
- transient tire model (4 states), equation (7.7),
- road unevenness model + Padé filter (2 + 2n_p states), equation (7.31).

The state-space equations of the complete model can be written as

\[ \dot{x} = Ax + Bu \quad (7.39) \]

with

\[
A = \begin{bmatrix}
A_d & -S_{d}D_{d}P & SC_d & T_{t}C_r & 0 \\
-B_{d}P & A_d & 0 & 0 & 0 \\
0 & 0 & A_r & 0 & 0 \\
0 & 0 & 0 & A_{d} & 0 \\
B_{d}C_r & A_{d} & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
F_{t} & S_{d}D_{d} & 0 & T_{t}C_r & 0 \\
0 & B_{d} & 0 & 0 & 0 \\
0 & 0 & B_{r} & 0 & 0 \\
0 & 0 & 0 & 0 & B_{r} \\
\end{bmatrix}
\]

\[
x^T = [x_v, x_d, x_t, x_r, x_p]
\]

\[
u^T = [u_a, r, F_{t}, F_{t}^2, u]
\]

with \( P \) representing the path deviation of the look-ahead point according to equation (7.31)

\[ P = [0, 1, 0, 0, 0, l_r, 0, 0, 0, 0, 0] \]

This model still contains some undefined inputs such as the vertical and lateral tires forces. In case of a linear model the steady-state lateral tire force depends on the vehicle states according to

\[ F_{t}^y = G_{t}^y \cdot x_u = \text{diag}(C_{Fa1}, C_{Fa2}, C_{Fa2}) \cdot E_{t}^y \cdot x_u \quad (7.40) \]

where \( C_{Fa} \) stands for the cornering stiffness. In case of a linear vertical tire stiffness, the following relation is valid
\[ F_i^e = G_i^v \cdot \tilde{x}_v + R_i \begin{bmatrix} \gamma_r \\ \gamma_p \end{bmatrix} = \text{diag}(c_i, c_i, c_i, c_i) \begin{bmatrix} E_i^v \cdot \tilde{x}_v + \begin{bmatrix} C_r & 0 \\ D_p C_r & C_p \end{bmatrix} \begin{bmatrix} \tilde{x}_r \\ \tilde{x}_p \end{bmatrix} \end{bmatrix} \] (7.41)

where \( c_i \) represents the vertical tire stiffness, \( R_i \) the road elevation input matrix and \( \tilde{x}_r \) the vertical road input according to

\[
\begin{bmatrix} \gamma_r \\ \gamma_p \end{bmatrix}^T = [z_{r,1L} \quad z_{r,1R} \quad z_{r,2L} \quad z_{r,2R}]^T
\]

Table 7.5 shows the eigenvalues, natural frequencies, damping and damped natural frequencies of the 13 DOF vehicle model. A distinction has been made between a vehicle model without lateral tire properties (virtually standing on ice) and one with lateral tire properties. According to the side-slip theory, the tires act as lateral dampers, with a damping constant depending on the speed of travel. Only at zero speed the lateral tire properties can be modelled as a pure spring. Through the suspension geometry it is imaginable that the tires influence the vertical dynamics for instance through the change of the lateral displacement of the tire contact point and camber angle resulting from vertical suspension displacements.

Besides the vehicle model with equations of motion derived by hand, an equivalent model has been developed using the multi-body program BAMMS. BAMMS stands for Bond graph based Algorithm for Modeling Multi-body Systems [41]. BAMMS was developed at the Delft University of Technology and is now managed and further developed at TNO Road Vehicles Research Institute in Delft. Modeling a 13 DOF vehicle model in BAMMS is quite easy. It is just a matter of connecting bodies with spring/damper elements and defining the constraints between bodies. Since in general the model in BAMMS is not linearized, it is especially suitable for time domain simulations with large (roll, pitch and yaw) angles. BAMMS can derive the linearized set of equations of motion around the point of operation and eigenvalue analysis may be conducted. However, BAMMS is not able to generate a linearized model with a minimum number of degrees of freedom. For reasons of speed (in the formalism phase), the degrees of freedom are fixed by the number of bodies. The constraints are not eliminated in BAMMS. Instead Langrangean multipliers are used. The 13 DOF vehicle in BAMMS (actually it is 17 DOF model because it contains rolling wheels) will contain 102 states! The minimum number of states necessary to describe the behavior of the vehicle is 17 \* 2 = 34. The large number of states puts BAMMS in an unfavorable position for control analysis. Therefore, it has only been used for checking the eigenvalues.
Before going into the vibration modes, a few terms will be explained first. The 'undamped' natural frequency is the frequency of free vibration in the absence of damping according to
\[ f_{0,i} = \frac{1}{2\pi} \text{abs}(\lambda_i) \] (7.42)
where \( \lambda_i \) (i = 1 system order) are the eigenvalues of the linearized system. For complex eigenvalues the damping ratio is defined as follows
\[ \beta = \frac{-\text{real}(\lambda_i)}{\text{abs}(\lambda_i)} \] (7.43)
A real eigenvalue corresponds to a damping ratio \( |\beta| \geq 1 \) (monotonous motion). For a complex eigenvalue with \( 0 \leq \beta \leq 1 \), the damped natural frequency reads
\[ f_{n,i} = \frac{1}{2\pi} \text{imag}(\lambda_i) \] (7.44)
The actual (damped) natural frequency is always smaller than the 'undamped' natural frequency.

The vehicle model contains several modes of vibration. In general, the following modes can be distinguished:

- Bounce, the translational component of ride vibrations of a body in the direction of the vertical axis (z-axis).
- Roll, the rotational component of ride vibrations of a body about the longitudinal axis (x-axis).
- Pitch, the rotational component of ride vibrations of a body about the lateral axis (y-axis).
- Yaw, the rotational motion of a body about the vertical axis (z-axis).
- Wheel-hop, the vertical oscillatory motion of a wheel.
- Tramp is a form of wheel-hop in which a pair of wheels hop in opposite direction.

The mode numbers in table 7.5 correspond to the following modes of vibration:

- sprung mass bounce : mode 10
- sprung mass roll : mode 8
- sprung mass pitch : mode 9
- front axle wheel-hop : mode 3
- front axle tramp : mode 4
- rear axle wheel-hop : mode 5
- rear axle tramp : mode 2
- engine bounce : mode 7
Chapter 7 Modeling a Full-Vehicle System

- engine roll : mode 1
- engine pitch : mode 6

In general, the bounce and pitch motions are coupled. A special situation exists where pitch and bounce are decoupled. The requirements for this case are [12]:

1. Both suspension springs (front and rear) must have the same static deflection,
2. The pitch moment of inertia $I_y$ must be equal to $I_y = -m_b \cdot l_{a1} \cdot l_{a2}$ where $l_{a1}$ and $l_{a2}$ stand for the distance between c.g. of the vehicle body and position of the front and rear axle respectively.

In the decoupled situation, one mode of vibration is a uniform bounce, where both springs act in the same phase, the other a pitch about the center of gravity when both springs act in opposite phase. The decoupled situation is undesired since it produces the maximum angular amplitude. Coupled pitch can be distinguished from bounce by means of the center of rotation. If the center is outside the wheelbase, the motion is predominantly bounce, if the center is within the wheelbase, the motion will be predominantly pitch. The location of the centers of rotation are dependent on the stiffness distribution, the center of gravity, the mass and the pitch moment of inertia. More about this later on.

Table 7.5 shows the eigenvalues, natural frequencies and damping ratios of the hand-derived vehicle model compared with those of the semi-automatically derived model using BAMMS. All eigenvalues correspond very well with exception of mode 1 (roll of engine) The reason for this difference might be the additional constraints necessary in BAMMS to 'freeze' 3 DOFs of the 6 DOF powertrain body with respect to the vehicle body.
<table>
<thead>
<tr>
<th>Mode</th>
<th>Eigenvalues</th>
<th>Natural Frequency</th>
<th>Damping Ratio</th>
<th>DAMPED Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-28.57 +/- 90.94 i</td>
<td>15.17</td>
<td>0.300</td>
<td>14.47</td>
</tr>
<tr>
<td>2</td>
<td>-10.82 +/- 84.50 i</td>
<td>13.56</td>
<td>0.127</td>
<td>13.45</td>
</tr>
<tr>
<td>3</td>
<td>-27.99 +/- 76.03 i</td>
<td>12.89</td>
<td>0.346</td>
<td>12.10</td>
</tr>
<tr>
<td>4</td>
<td>-23.98 +/- 81.31 i</td>
<td>13.49</td>
<td>0.283</td>
<td>12.94</td>
</tr>
<tr>
<td>5</td>
<td>-20.66 +/- 80.64 i</td>
<td>13.25</td>
<td>0.248</td>
<td>12.83</td>
</tr>
<tr>
<td>6</td>
<td>-22.94 +/- 79.65 i</td>
<td>13.19</td>
<td>0.277</td>
<td>12.68</td>
</tr>
<tr>
<td>7</td>
<td>-16.84 +/- 77.08 i</td>
<td>12.56</td>
<td>0.214</td>
<td>12.27</td>
</tr>
<tr>
<td>8</td>
<td>-2.67 +/- 14.82 i</td>
<td>2.40</td>
<td>0.177</td>
<td>2.36</td>
</tr>
<tr>
<td>9</td>
<td>-3.03 +/- 8.92 i</td>
<td>1.50</td>
<td>0.321</td>
<td>1.41</td>
</tr>
<tr>
<td>10</td>
<td>-2.49 +/- 7.44 i</td>
<td>1.25</td>
<td>0.317</td>
<td>1.18</td>
</tr>
<tr>
<td>11</td>
<td>0.00 +/- 0.00 i</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>12</td>
<td>0.00 +/- 0.00 i</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>13</td>
<td>0.00 +/- 0.00 i</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
</tbody>
</table>

**MATLAB (linear)**

**BAMMS (non-linear)**

<table>
<thead>
<tr>
<th>Mode</th>
<th>Eigenvalues</th>
<th>Natural Frequency</th>
<th>Damping Ratio</th>
<th>DAMPED Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-24.00 +/- 83.81 i</td>
<td>13.87</td>
<td>0.275</td>
<td>13.34</td>
</tr>
<tr>
<td>2</td>
<td>-10.83 +/- 84.68 i</td>
<td>13.59</td>
<td>0.127</td>
<td>13.48</td>
</tr>
<tr>
<td>3</td>
<td>-28.40 +/- 75.95 i</td>
<td>12.91</td>
<td>0.350</td>
<td>12.09</td>
</tr>
<tr>
<td>4</td>
<td>-23.94 +/- 80.99 i</td>
<td>13.44</td>
<td>0.283</td>
<td>12.89</td>
</tr>
<tr>
<td>5</td>
<td>-20.27 +/- 80.02 i</td>
<td>13.14</td>
<td>0.246</td>
<td>12.74</td>
</tr>
<tr>
<td>6</td>
<td>-21.53 +/- 79.28 i</td>
<td>13.07</td>
<td>0.262</td>
<td>12.62</td>
</tr>
<tr>
<td>7</td>
<td>-16.29 +/- 74.50 i</td>
<td>12.14</td>
<td>0.214</td>
<td>11.86</td>
</tr>
<tr>
<td>8</td>
<td>-2.69 +/- 14.41 i</td>
<td>2.33</td>
<td>0.184</td>
<td>2.29</td>
</tr>
<tr>
<td>9</td>
<td>-3.03 +/- 8.84 i</td>
<td>1.49</td>
<td>0.325</td>
<td>1.41</td>
</tr>
<tr>
<td>10</td>
<td>-2.49 +/- 7.46 i</td>
<td>1.25</td>
<td>0.316</td>
<td>1.19</td>
</tr>
<tr>
<td>11</td>
<td>0.00 +/- 0.00 i</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>12</td>
<td>0.00 +/- 0.00 i</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>13</td>
<td>0.00 +/- 0.00 i</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
</tbody>
</table>

**Table 7.5: Natural Frequencies and damping.**

**with lateral tire properties (100 km/h)**

<table>
<thead>
<tr>
<th>Mode</th>
<th>Eigenvalues</th>
<th>Natural Frequency</th>
<th>Damping Ratio</th>
<th>DAMPED Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-28.59 +/- 90.93 i</td>
<td>15.17</td>
<td>0.300</td>
<td>14.47</td>
</tr>
<tr>
<td>2</td>
<td>-13.96 +/- 83.90 i</td>
<td>13.54</td>
<td>0.164</td>
<td>13.35</td>
</tr>
<tr>
<td>3</td>
<td>-27.99 +/- 76.03 i</td>
<td>12.89</td>
<td>0.346</td>
<td>12.10</td>
</tr>
<tr>
<td>4</td>
<td>-23.95 +/- 81.32 i</td>
<td>13.49</td>
<td>0.283</td>
<td>12.94</td>
</tr>
<tr>
<td>5</td>
<td>-20.81 +/- 80.60 i</td>
<td>13.25</td>
<td>0.250</td>
<td>12.83</td>
</tr>
<tr>
<td>6</td>
<td>-23.00 +/- 79.61 i</td>
<td>13.19</td>
<td>0.278</td>
<td>12.67</td>
</tr>
<tr>
<td>7</td>
<td>-16.84 +/- 77.08 i</td>
<td>12.56</td>
<td>0.214</td>
<td>12.27</td>
</tr>
<tr>
<td>8</td>
<td>-3.87 +/- 13.91 i</td>
<td>2.36</td>
<td>0.268</td>
<td>2.21</td>
</tr>
<tr>
<td>9</td>
<td>-3.05 +/- 8.89 i</td>
<td>1.49</td>
<td>0.324</td>
<td>1.41</td>
</tr>
<tr>
<td>10</td>
<td>-2.49 +/- 7.44 i</td>
<td>1.25</td>
<td>0.317</td>
<td>1.18</td>
</tr>
<tr>
<td>11</td>
<td>-7.00 +/- 2.20 i</td>
<td>1.17</td>
<td>0.954</td>
<td>0.35</td>
</tr>
<tr>
<td>12</td>
<td>0.00 +/- 0.00 i</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>13</td>
<td>0.00 +/- 0.00 i</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
</tbody>
</table>

**with lateral tire properties (100 km/h)**

<table>
<thead>
<tr>
<th>Mode</th>
<th>Eigenvalues</th>
<th>Natural Frequency</th>
<th>Damping Ratio</th>
<th>DAMPED Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-24.05 +/- 84.10 i</td>
<td>13.92</td>
<td>0.275</td>
<td>13.39</td>
</tr>
<tr>
<td>2</td>
<td>-13.09 +/- 84.55 i</td>
<td>13.62</td>
<td>0.153</td>
<td>13.46</td>
</tr>
<tr>
<td>3</td>
<td>-28.42 +/- 75.92 i</td>
<td>12.90</td>
<td>0.351</td>
<td>12.08</td>
</tr>
<tr>
<td>4</td>
<td>-23.93 +/- 80.90 i</td>
<td>13.43</td>
<td>0.284</td>
<td>12.87</td>
</tr>
<tr>
<td>5</td>
<td>-20.24 +/- 79.80 i</td>
<td>13.10</td>
<td>0.246</td>
<td>12.70</td>
</tr>
<tr>
<td>6</td>
<td>-21.76 +/- 79.48 i</td>
<td>13.11</td>
<td>0.264</td>
<td>12.65</td>
</tr>
<tr>
<td>7</td>
<td>-16.26 +/- 74.47 i</td>
<td>12.13</td>
<td>0.213</td>
<td>11.85</td>
</tr>
<tr>
<td>8</td>
<td>-4.51 +/- 15.62 i</td>
<td>2.59</td>
<td>0.277</td>
<td>2.49</td>
</tr>
<tr>
<td>9</td>
<td>-2.99 +/- 8.81 i</td>
<td>1.48</td>
<td>0.322</td>
<td>1.40</td>
</tr>
<tr>
<td>10</td>
<td>-2.49 +/- 7.45 i</td>
<td>1.25</td>
<td>0.317</td>
<td>1.19</td>
</tr>
<tr>
<td>11</td>
<td>-5.92 +/- 3.69 i</td>
<td>1.11</td>
<td>0.849</td>
<td>0.59</td>
</tr>
<tr>
<td>12</td>
<td>0.00 +/- 0.00 i</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>13</td>
<td>0.00 +/- 0.00 i</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
</tbody>
</table>
The difference between the model with and without lateral tire properties can be seen from mode 2 (rear axle tramp) and mode 8 (vehicle body roll). With lateral tire properties, whether represented by a linear tire model (cornering stiffness) or a more elaborate non-linear tire model (Magic Formula), the real parts of modes 2 and 8 are larger than in case no lateral tire properties are considered. This means that the tires add some damping to both modes. This is not surprising, because according to the side-slip theory the tires act like lateral dampers with a damping constant depending on the speed of travel.

All mode shapes have been plotted using BAMMS. Figure 7.18 shows these mode shapes. Without a mode shape plotting facility, it is very difficult to distinguish the different modes. Although the eigenvectors contain all information about the shape of each mode, it is hard to assess and visualize the modes of vibration because the state variables do not have the same units (rotations and translations are mixed up). The low frequency modes can be distinguished quite easily. Pure pitch and pure bounce are not present in the vehicle model. Mode 9 is predominantly pitch (the pole is located near the front suspension) and mode 10 is predominantly bounce (the pole is located somewhat behind the car). Also mode 8 does not give any problems. This mode corresponds to the roll motion of the vehicle body. Mode 11 represents the yaw motion of the car. The shape of this mode depends on the distribution of the cornering stiffness between the front and rear axle and on the speed of travel. This mode is typically speed dependent. For low speeds the yaw motion is characterized by two (negative) real eigenvalues and for higher speeds the yaw motion becomes oscillatory (negative real part ± imaginary part). The range between 12 and 15 Hz is more complicated; 7 modes are located in this narrow zone. The easiest to trace are mode 2 (rear axle wheel-hop), mode 5 (rear axle tramp) and mode 1 (engine roll). Mode 3, 4 and 6 are more difficult to distinguish. Carefully looking at the mode shape reveals that mode 4 corresponds to the front wheel tramp (left and right wheel hop in opposite phase) and mode 3 is the front wheel-hop (in-phase). Finally, mode 3 and 7 correspond to two modes of vibration of the engine. Both are coupled pitch and bounce. Mode 7 is predominantly pitch and mode 3 is predominantly bounce.

This chapter has been dealing with an introduction into the modeling aspects of a full-vehicle system. The vehicle model contains 13 degrees of freedom (DOFs). The equations of motion have been derived using the method of Lagrange. Complex suspension kinematics have been simplified. Since a full-vehicle contains four wheels, a new stochastic road model has been developed. The model used comprises the correlation between left and right track and the
time delay between front and rear wheel excitations. The modeling of the tire characteristics need special attention. Both steady-state and transient properties have been included in the model. Finally, a driver model has been developed and discussed. Since the full-vehicle has more degrees of freedom than a quarter-car, additional performance criteria are necessary to describe the performance in an objective way. Chapter 8 will deal with the definition of performance criteria based on the full-vehicle model. Special attention will be paid to the assessment of ride comfort and handling.
Figure 7.18a Mode shapes of the 13 DOF vehicle model at 100 km/h (BAMMS).
Figure 7.18b Mode shapes of the 13 DOF vehicle model at 100 km/h (BAMMS).
ADDITIONAL CRITERIA FOR
FULL-VEHICLE SUSPENSION DESIGN

This chapter deals with the definition of additional criteria to design or tune wheel suspension systems of a complete vehicle. The larger number of degrees of freedom (DOFs) of a full-vehicle requires an extension of the design criteria as discussed in chapter 2. Two items are entered into detail: (i) the perception of ride and (ii) the assessment of handling. Unlike the single DOF of the quarter-car model sprung mass, the main body of the full-vehicle contains six DOFs. These additional DOFs in combination with the multi-wheel road input require some (re)definitions of the assessment of ride comfort. Furthermore, it is now possible to study the handling aspects of the full-vehicle. With the quarter-car model the handling
aspects were restricted to road-holding (dynamic tire load variations). The full-vehicle model with its lateral and longitudinal DOF offers the possibility to perform handling maneuvers. The interaction between vertical and lateral dynamics is of major importance in the suspension design phase. The extended ride and handling criteria are both applied to the standard vehicle with fixed spring and damper rates. The outcomes serve as a yardstick to judge other suspension systems.

8.1 Perception of Ride

Ride is a subjective perception, normally associated with the level of discomfort experienced when travelling in a vehicle. The tactile vibrations transmitted to the passenger’s body through the seat, and at the hands and feet, are the factors most commonly associated with ride. Additionally, the general comfort level can be influenced by several other factors, such as, temperature, interior space, noise, smell, sight, ventilation, and many other elements. Some of the above factors, such as vibrations can be measured objectively.

The judgment of ride comfort in a vehicle is still an area of controversy in the automotive community. The judgment of ride vibrations depends on several variables such as:

- seating position
- comfort scaling
- multi-direction input

Seating Position

The seating position has a large influence on ride. Unlike the quarter-car situation, the human body is not only exposed to vertical vibrations. Due to the correlated road input (left-right and front-rear) the full-vehicle model becomes a multi-input system and therefore the driver and passengers will sense longitudinal, lateral and rotational accelerations as well. The ratio of these accelerations depends strongly on the seating position and the speed of travel. The wheelbase filtering mechanism is the phenomenon that is responsible for the speed influence.

As the vehicle moves on the road, the roughness input at the front wheels acts subsequently on the rear wheels, delayed in time by the interval equal to the wheelbase divided by speed. With pure sinusoidal road excitations, pure bounce motion inputs will occur if the wavelength \((L)\) equals the wheelbase
(\(w_b\)) of the vehicle. The same is true for wavelengths which have an integer multiple equal to the wheelbase. In a similar fashion, a pure pitch motion input will be seen at a wavelength that is twice the wheelbase, or at shorter wavelengths that have an odd integer multiple equal to twice the wheelbase. Mathematically, the following expressions are valid

\[
\begin{align*}
\text{bounce:} & \quad L = \frac{w_b}{n} \quad \text{for } n = 1, 2, 3... \\
\text{pitch:} & \quad L = \frac{2w_b}{2n - 1} \quad \text{for } n = 1, 2, 3...
\end{align*}
\]

(8.1a) (8.1b)

The situation as described above is valid for a vehicle with decoupled pure pitch and bounce modes. Nevertheless, the wheelbase filtering mechanism will also show up with a vehicle with coupled modes. Instead of pure pitch or pure bounce, the motion will show predominantly pitch or predominantly bounce. At high speeds a passenger car tends to experience vertical bounce vibrations more or less as predicted with the quarter-car model. At low speeds the time delay between the front and rear wheel excitations approaches the period of the pitch mode, readily exciting the pitch mode.

**Comfort Scaling**

Several studies have been devoted to find out the tolerance related to discomfort of the human body in a seated position at various frequencies of excitation. Pure sinusoidal inputs are often used in the attempt to establish quantified levels of discomfort, or equal levels of sensation, as a function of frequency. The International Organization for Standardization (ISO) set of curves is one of the many results gathered from these studies [18].

These ISO curves describe the relation between level of acceleration and frequency of excitation at different durations of the vibration exposure. These kinds of curves have been developed for vertical (chapter 2.1) as well as longitudinal and lateral vibrations. The ISO curves illustrate that the human body shows a minimum tolerance to vertical vibrations in the frequency range between 4 and 8 Hz. This sensitivity is well recognized as the result from the vertical resonance of the abdominal cavity. At frequencies above or below this range the tolerance increases in proportion to the frequency (logarithmic scales).

The human sensitivity for longitudinal (fore/aft) vibrations is somewhat different from that of the vertical. The region of maximum sensitivity occurs
in the 1 to 2 Hz range. This sensitivity is generally recognized to result from the fore/aft resonance of the upper torso.

The tolerance curves are generally derived from pure sinusoidal inputs, whereas the ride environment in a car contains frequencies over a broad spectrum. A practical solution is that the acceleration signals are filtered in inverse proportion to the amplitude of the tolerance curves. The filter characteristics for vertical and longitudinal accelerations according to ISO 2631 are shown in figure 8.1

Although the original ISO curves are valid in the range from 1 to 80 Hz, the weighting function has been extrapolated to the range in between 0.5 and 1 Hz. Accelerations outside the 0.5-80 Hz range will not be considered.

![Figure 8.1 Frequency dependent weighting functions for horizontal and vertical vibration perception derived from ISO 2631.](image)

The area beneath a spectral density of accelerations is equal to the mean-square acceleration (MS). In case a weighting function is applied, the spectral density should be multiplied with the square of the weighting function before the mean-square or root-mean-square (RMS) value is calculated. Yet it has been found by many engineers that calculated seat vibrations, even weighted according to tolerance curves, bear little correlation to the subjective rating that will be obtained in road tests. Therefore, weighted as well as non-weighted RMS acceleration values will be used side by side.

In science of dynamics, it is a common practice to present frequency domain information in log-log format. The log-log format offers a clear insight for
understanding the dynamics and the asymptotical behavior of complex systems. For ride purpose, however, this distorts the relative importance of vibrations at various frequencies. Presentation of the ride spectra in lin-lin format is more meaningful because the area under the plot is indicative of the mean-square accelerations. The log-log format creates the impression that vibrations are generally equally important across the entire spectrum. Nevertheless, both formats will be used side by side in this thesis.

Multi-Direction Input

As already mentioned above, a 3-dimensional vehicle driving on an uneven road can be regarded as a multi-input system. Besides vertical accelerations, the driver and passengers will also sense longitudinal, lateral and rotational accelerations. Therefore, (weighted) root-mean-square (RMS) accelerations in each direction \((x, y \text{ and } z)\) are combined by means of a vector sum according to the following formula to obtain an overall RMS value.

\[
\text{RMS}_{\text{res}} = \sqrt{\text{MS}_x + \text{MS}_y + \text{MS}_z}
\]  \hspace{1cm} (8.2)

8.1.1 Evaluation of the Passive Vehicle Ride

In order to gain more inside into the ride performance of a standard passive vehicle, some tests have been carried out. The tests are related to RMS performance values of several variables, such as the sprung mass acceleration, tire load variations and suspension working space. The passive car is assumed to drive straight on over a random road with a speed of travel of 120 km/h. The lateral (transient) tire properties and the driver model are also included in the performance analysis besides all vertical dynamics. The driver model guarantees that the car will remain close to a straight line (zero reference signal). The root mean square (RMS) acceleration levels are evaluated for various positions in the car. The calculation of the RMS values is based on the covariance technique using a modified version of the Lyapunov equation. This method, which has already been discussed in the quarter-car part (chapter 3), must be adjusted for the time delayed inputs. The steady-state covariance equation for a system subjected to two pairs of time delayed inputs (left and right track) reads [14]
\[ A_t \cdot C_{\text{struk}}^T + C_{\text{struk}} \cdot A_t^T + G_t \cdot C_{\text{struk}} \cdot G_t^T + C_{\text{wkr}}^T(1,1) \cdot \left( G_t(\cdot,3) \cdot G_t(\cdot,1)^T \right)^T + e \cdot A_t^T \cdot G_t(\cdot,1) \cdot G_t(\cdot,3)^T \]
\[ + C_{\text{wkr}}^T(2,2) \cdot \left( G_t(\cdot,4) \cdot G_t(\cdot,2)^T \right)^T + e \cdot A_t^T \cdot G_t(\cdot,2) \cdot G_t(\cdot,4)^T \] = 0

with
\[ \dot{x}_t = A_t x_t + G_t u \quad \text{(8.4a)} \]
\[ y_t = C_t x_t \quad \text{(8.4b)} \]

where \( A_t \) stands for the complete state matrix of the vehicle model + road model + transient tire model + driver model, and \( C_{\text{wkr}} \) is the intensity matrix of the white noise input \( u \) for the shaping filter according to equation (7.23). Finally, \( \tau \) denotes the time delay between the front and rear wheel excitations according to equation (7.27).

The calculation of the state covariance matrix of a system with a time delay requires a special format of the road description. The rear wheel road excitations must be modeled as an uncorrelated random process. Thus, in contradiction with the reality, the front and rear wheel road inputs are assumed to be two fully uncorrelated random processes. The total system description (vehicle + road model) reads as follows
\[ A_t = \begin{bmatrix} A_r & R_c C_{r4} \\ 0 & A_{r4} \end{bmatrix} \quad G_t = \begin{bmatrix} 0 \\ B_{r4} \end{bmatrix} \quad C_t = \begin{bmatrix} C_r & R_c C_{r4} \\ 0 & C_{r4} \end{bmatrix} \]

where \( A_{r4} (4\times4) \), \( B_{r4} (4\times4) \) and \( C_{r4} (4\times4) \) are the state-space matrices of the road description
\[ A_{r4} = \begin{bmatrix} A_r & 0 \\ 0 & A_r \end{bmatrix} \quad B_{r4} = \begin{bmatrix} B_r & 0 \\ 0 & B_r \end{bmatrix} \quad C_{r4} = \begin{bmatrix} C_r & 0 \\ 0 & C_r \end{bmatrix} \]

with \( A_r, B_r, C_r \) as the state-space model of the two track road description according to equation (7.12).

The mean square (MS) values of all outputs in vector \( y_t \) can easily be computed according to
\[ \text{MS}(y_t) = \text{diag}(C_t \cdot C_{\text{struk}} \cdot C_t^T) \quad \text{(8.5)} \]
Chapter 8 Additional Criteria for Full-Vehicle Suspension Design

The influence of the seating position on the accelerations of the vehicle body will now be entered into detail. Figures 8.2, 8.3 and 8.4 illustrate the longitudinal, lateral and vertical accelerations of the vehicle body as a function of the longitudinal, lateral and vertical distance from the center of gravity (c.g.). All cross-sections have been made through the center of gravity of the main body.

The longitudinal (fore/aft) accelerations are mainly caused by the pitch input of the time delayed rear wheel road excitation. The minimum acceleration level is located in the c.g. of the vehicle body. The acceleration level increases almost proportionally with the vertical distance from the c.g. The fore/aft accelerations are the largest near the roof. The distribution of accelerations is almost equal for the left and right side of the car. The small difference is caused by the Panhard rod of the rear suspension.

On first sight, the course of the lateral accelerations is rather peculiar. Two phenomena play a role. First of all, the lateral accelerations are caused by the roll motion of the car. The vehicle is directly excited in its roll mode, due to the preprogrammed cross-correlation of the left and right road input. The level of lateral accelerations depends on the location of the roll axis (The line which joins the rear and front suspension roll centers). The roll center of the front suspension is located just above the road surface (figure 7.3) and the roll center of the rear suspension is positioned in the middle of the Panhard rod. Therefore, the lateral accelerations resulting from the roll motion are the largest near the roof. The second aspect that is responsible for the presence of lateral accelerations is the coupling of the vertical dynamics and lateral dynamics through the Panhard rod in the rear suspension. The yaw DOF is directly excited by the vertical (symmetric) motion of the rear axle and main body. This phenomenon is only present when the tires are able to generate side forces. Although a constant (tire load independent) cornering stiffness is used in this linear analysis, the phenomenon will still remain when a more elaborate tire model is used. The relaxation effect of the side force generation softens the negative influence of the coupling a bit. The lateral acceleration level is the highest at the rear-end of the car. From the bottom plot it can be seen that the yaw centroid (lowest level of lateral accelerations) is located at the front wheel.

The distribution of the vertical accelerations is illustrated in figure 8.4. The vertical accelerations are the most important variables to assess ride comfort. The figures show that the most comfortable zone is located somewhat in front
of the c.g. of the vehicle body. The vertical accelerations increase in the lateral direction due to the roll input of the road irregularities. This level increases also in the direction of the front- and rear-end of the car. The wheelbase filtering mechanism as explained above plays an important role in the resulting ride comfort. The most important variable is the speed of travel that fixes the time delay between front and rear axle input. From the accelerations at the seat positions, the conclusion can be drawn that the accelerations at the front seat position are somewhat lower than those at the rear seat positions.

The influence of the speed of travel on several performance parameters has also been examined and the results are plotted in figure 8.5. The two uppermost figures show the speed influence on the six components of the vehicle body. The longitudinal, lateral and yaw (angular) accelerations remain almost constant as the speed increases. The other three accelerations increase with the speed of travel. The two figures in the middle illustrate the longitudinal, lateral and vertical accelerations at the position of the front and rear seats. Also the vector sum of these three components has been plotted. It can be concluded that the rear seats are somewhat more uncomfortable over the entire range of speeds. The two figures at the bottom show the RMS values of the tire load variations and the suspension deflection. The tire load variations at the rear axle are larger than at the front axle, although the front and rear suspension parameters (mass, stiffness, etc.) are almost identical. This difference is in direct relation to the type of suspension. The rigid rear axle is not only excited in the vertical direction (wheel-hop), but also a roll excitation (tramp) is present due to the cross-correlated left and right road track. This means that the tire load variations are built up of both contributions. The larger the moment of inertia of the rear axle about the longitudinal axis, the larger the roll contribution. A heavy rigid rear axle with a large moment of inertia is therefore not favorable. The small difference in tire load variations at the left and right rear side is caused by the asymmetric mounting of the Panhard rod to the rigid axle. The right side is somewhat larger and therefore 'heavier' than the left side. The centroid of the tramp mode is located left of the line of symmetry of the rear axle.
Figure 8.2 Longitudinal vehicle body accelerations.
Figure 8.3 Lateral vehicle body accelerations.
Figure 8.4 Vertical vehicle body accelerations.
Figure 8.5 RMS performance values as a function of speed of travel.

8.2 ASSESSMENT OF HANDLING

The cornering behavior of a car is an important performance mode often equated with handling. Handling is a commonly used term to denote the responsiveness of a vehicle to a driver input, or the ease of control. The driver-vehicle combination is a closed-loop system. The driver is the controller, the vehicle the system to be controlled and the environment is an information and disturbance source. A car handles well if it is possible to maintain the vehicle on the desired course without too much exertion of the driver. The most important aspect of assessing the handling of a car is its capacity to avoid
traffic accidents. Good handling implies that the risk for the driver and vehicle to get involved in an accident is reduced to a minimum.

Unlike the quarter-car model, the full-vehicle model offers the possibility to perform steering maneuvers. The road-holding aspect of the quarter-car model (assessed by the dynamic tire load variations) can now be extended with some more realistic aspects related to handling. Good road-holding is one of the requirements for good handling. If the dynamic tire load variations due to road unevennesses are small with respect to the static tire load (good road-holding), side slip force variations are small too. A small variation of the tire slip forces during a maneuver may indicate that the handling of the car is good in this particular situation. The car will at least not deviate too much from the desired path due to tire slip force variations. However, many more aspects such as the motion of the vehicle (roll, pitch and yaw angle) and the steering effort play a role in assessing handling.

Tests have to be developed in order to be able to analyze and judge the behavior of the driver-vehicle system \cite{45}. Nowadays, a vast range of tests exist to analyze the lateral and longitudinal behavior of a car. The methods of judging the handling of a car may be subdivided into:

- **Subjective rating.** The handling of the car is judged by test drivers. No absolute standard is applied. The judgment is carried out by making comparisons with other vehicles.

- **Objective rating.** The handling qualities are judged with the aid of absolute quantities obtained through measurements. Data of several vehicle configurations or competitor cars are compared with each other.

The subjective rating is a powerful tool for fine-tuning the handling of a vehicle. The final vehicle setup is achieved through the process of subjective judgment using several test drivers each with different qualifications. For this reason, this method is still of major importance in the contemporary design of a vehicle.

The handling tests may be divided into:

- **Closed-loop test.** With this method the driver-vehicle closed-loop system is considered. The driver has to follow a predefined path and has to apply steering and/or braking maneuvers at particular moments. The ability of the driver to steer and brake at the right moments is of major influence to the results obtained. The drawback of
this method is a large spread in results due to the different qualities of the drivers.

- **Open-loop test.** Only the vehicle is characterized. The influence of the driver is eliminated as much as possible. Sometimes, a programmed steering and/or braking machine is used. The loop is open because the driver is not allowed to make corrections. The advantage is good reproducibility of the test, even with very complex maneuvers.

Due to the non-availability of a test vehicle to evaluate the influence of different suspension settings on the handling properties, the method of subjective rating is not discussed in this thesis. Objective rating is the remaining possibility the judge the handling properties. Important features of a quantitative analysis is the selection of test procedure and measurement signals.

**Test Procedures**

Due to the large number of specific test procedures and external conditions, it is not possible to record the handling of a specific car in one single test. Only certain aspects of handling such as braking or cornering can be analyzed and compared with different vehicle configurations. Roughly speaking, the following aspects can be analyzed:

- **Driving straight on.** Possible tests are: stability at high speed, braking behavior, side wind sensitivity and aquaplaning.

- **Transient motions.** Different kinds of steering wheel inputs may be used: one period of a sine, continuous sine, pulse, stochastic signal.

- **Cornering.** The following test can be carried out: steady-state cornering, non-steady-state cornering, braking in curves, response to throttle and aquaplaning.

Most of these tests can be carried out in an open-loop or closed-loop configuration. Sometimes standard test procedures are used according to e.g. the International Organization for Standardization (ISO). Some examples are: ISO 4138 'Steady-State Cornering', ISO 7975 'Braking in a Turn', ISO 7401 'Lateral Transient Response Test Procedure', ISO/TR 3888 'Test Procedure for Severe Lane-Change Maneuver, ISO/TR 8349, 'Measurement of Road Surface Friction' and ISO/TR 8350 'High-friction Test Track Surface - Specifications'.

The vehicle model as discussed in the previous chapter is not suitable for large yaw angles. Therefore, steady-state cornering cannot be judged here. Also, longitudinal tire slip characteristics were disregarded. Consequently, the
straight line braking tests cannot be carried out. What remains are the transient lateral response tests. The response of the vehicle to a steering wheel input is an essential criterion to judge the handling behavior. The transient test can be performed either in open or in closed-loop. The vehicle behavior can be described in both time and frequency domain. One of the oldest test procedures is the double lane-change. The aim of this test is to study the transient behavior of the closed-loop driver-vehicle system. The test shows a realistic correspondence with the daily routine of a driver on the road. A sudden pull out is combined with a fast steering maneuver necessary to return on the original track. It can be imagined that this test corresponds to a situation where the driver must avoid a collision with an obstacle on the road.

The double lane-change test is described in ISO/TR 3888 [19] and the dimensions of the test track are shown in figure 8.6. The velocity of the car should be kept as constant as possible and the entering speed should at least be 80 km/h.

![Figure 8.6 Dimensions in (m) of the double lane-change test track according to ISO/TR 3888.](image)

The variables $w_1$, $w_2$ and $w_3$ represent the width of the track (m) in the three phases

- $w_1 = 1.1 \cdot w_v + 0.25$
- $w_2 = 1.2 \cdot w_v + 0.25$
- $w_3 = 1.3 \cdot w_v + 0.25$

where $w_v$ stands for the width of the vehicle in (m). According to ISO/TR 3888 the offset is equal to 3.5 m. Actually, ISO/TR 3888 is not a norm but it can be regarded as a recommendation. Tests carried out by several car manufactures and institutes have shown that it is not possible to define objective criteria because the influence of the driver has a commanding influence on the
measurement results. The transient behavior of the driver-vehicle system can be judged from:

- Steering wheel angle
- Lateral acceleration
- Yaw rate
- Body slip angle
- Roll angle

The double lane-change test can be performed very well with the vehicle-driver system as developed in chapter 7. It is a typical example of a useful time domain simulation that takes up a limited amount of simulation time. All non-linearities concerning the tire (lateral) force generation process must be used since the vertical tire load and side slip angles vary over a broad range during the double lane-change maneuver.

Another way of studying the non-steady-state steering behavior of a vehicle is looking at frequency response functions. This corresponds typically to an open-loop test. The vehicle can be regarded as a black box with a steering wheel angle as an input, and several important variables such as lateral acceleration or yaw rate as an output. The steering wheel is rotated according to a sinusoidal function of time. The amplitude and frequency may vary. If the system is completely linear, the output variables will also show a sine shape, however, with different amplitude and phase. By repeating this test for various frequencies of steering wheel excitation, it is possible to construct a frequency response function (magnitude and phase as a function of frequency). If the vehicle model is completely linear, the frequency response functions can easily be calculated using the linear set of differential equations. In such a case there is no real need for time-domain simulations.

Besides the double lane-change simulation, the frequency response functions (FRFs) will be used to assess handling. The three most important variables for the FRFs are:

- Lateral acceleration \( \frac{\ddot{y}_s(j\omega)}{\delta(j\omega)} \)
- Yaw rate \( \frac{\psi_h(j\omega)}{\delta(j\omega)} \)
- Roll angle \( \frac{\phi_h(j\omega)}{\delta(j\omega)} \)

Since the tire side slip responses are speed depended, the FRFs depend also on the speed of travel. The meaning of these FRFs with respect to handling will be
illustrated in section 8.2.1, where the FRFs for the standard car with passive suspension elements will be discussed in detail.

Although the frequency response tests are strongly related to handling, their relation to road-holding is very weak. While calculating the FRFs, the road is assumed to be flat. Therefore, the tire load will only vary due to the load transfer and the roll/pitch dynamics. Since the model is linear, the cornering stiffnesses and relaxation lengths of the tires remain constant.

8.2.1 Evaluation of the Passive Vehicle Handling

The handling of the standard passive vehicle will be illustrated using (1) a time domain simulation of a double lane-change, and (2) the frequency response functions calculated for various speeds.

Double Lane-Change

A time-domain simulation of a double lane-change has been carried out for the standard passive vehicle. This closed-loop test must be performed with a driver model. The model as discussed in chapter 7.3 requires the definition of a reference path. This path \( r \) has been built up from sine functions using the longitudinal coordinate \( x_b \) as the input variable (cf. figure 8.6).

\[
0 < x_b < 15 \Rightarrow r = 0
\]
\[
15 < x_b < 45 \Rightarrow r = \left( \text{offset} + \frac{w_x - w_1}{2} \right) \cdot \left( \frac{x - 15}{30} - \frac{1}{2\pi} \cdot \sin \left( \frac{2\pi(x - 15)}{30} \right) \right)
\]
\[
45 < x_b < 70 \Rightarrow r = \left( \text{offset} + \frac{w_x - w_1}{2} \right)
\]
\[
70 < x_b < 95 \Rightarrow r = \left( \text{offset} + \frac{w_x - w_1}{2} \right) \cdot \left( 1 - \frac{x - 70}{25} \cdot \frac{1}{2\pi} \cdot \sin \left( \frac{2\pi(x - 70)}{25} \right) \right)
\]
\[
x_b > 96 \Rightarrow r = 0
\]

The reference path determines more or less how well the vehicle should corner in between the cones and must therefore be selected with care. Also, the selection of the driver model gains and the look-ahead distance are very important tuning parameters. Since no longitudinal dynamics (braking and/or accelerating) are modeled, the speed of travel remains constant during the maneuver and it equals 80 km/h. The following variables have been plotted:

- path described by the c.g. of the total vehicle
- steering angle of the front wheels
- lateral acceleration at the c.g. of the total vehicle
- roll angle of the vehicle body
- yaw rate of the vehicle
- body slip angle at the c.g. of the total vehicle
- vertical tire load at all four wheels
- lateral tire slip force at all four wheels
- vertical sprung mass acceleration at both front and rear seat positions
- damper forces in all four suspensions
- deflection of all four wheel suspensions

The lateral acceleration \( \ddot{y}_v \) in the center of gravity of the total vehicle is given by

\[
\ddot{y}_v = \dot{y}_v + \delta x \cdot \dot{\psi}_b - \delta z \cdot \dot{\phi}_b
\]  
(8.6)

where \( \delta x \) and \( \delta z \) (\( \delta y = 0 \)) represent the shift of the c.g. with respect to the c.g. of the vehicle body itself.

\[
\delta x = \frac{2m_a l_{a1} + m_a l_{a2} + m_e l_e}{m_b + 2m_{a1} + m_{a2} + m_e} \tag{8.7a}
\]

\[
\delta z = \frac{2m_a h_{a1} + m_a h_{a2} + m_e h_e}{m_b + 2m_{a1} + m_{a2} + m_e} \tag{8.7b}
\]

The body slip angle \( \beta \) has also been determined in the c.g. of the total vehicle. Its tangent is equal to the ratio of the local lateral and local longitudinal velocity of the reference point. Angle \( \beta \) reads

\[
\beta = \arctan \left( \frac{\dot{y}_b + \delta x \cdot \dot{\psi}_b - \delta z \cdot \dot{\phi}_b \cos \psi_b - (\dot{x}_b + \delta z \cdot \dot{\phi}_b) \sin \psi_b}{\dot{x}_b + \delta x \cdot \dot{\phi}_b \cos \psi_b + (\dot{y}_b + \delta x \cdot \dot{\psi}_b - \delta z \cdot \dot{\phi}_b) \sin \psi_b} \right) \tag{8.8}
\]

All above mentioned signals are of considerable interest when handling and ride comfort are studied. Figure 8.7a and b illustrate the results for the standard car. Unlike the ISO prescription, the road surface is not plane. In order to judge the relation between road-holding and handling, stochastic road unevennesses have been included in the simulation. From the course of the vehicle it can be seen that the driver model is not able to guide the vehicle entirely through the lane-change without hitting one or more cones. However, it is quite easy to adjust either the reference path or the driver model parameters in such a way that the driver-vehicle combination manages the double lane-change without any problems. The adjustments on the driver model would for example imply a reduction of the neuromuscular time constant and/or delay time of the driver. The phase lead and lag time constants may now be adjusted in such a way that the driver acts much faster. However, a robot-like driver does not reflect the reality. Therefore, the
shortcoming of the driver model in combination with the chosen reference path will be taken for granted.

The second plot shows the steering angle necessary to perform the double lane-change. The steering angle generated by the driver model looks quite smooth and the values are not too large (note: small angles are used, $\cos(\delta) = 1$, $\sin(\delta) = \delta$). The next plot shows the lateral acceleration. The maneuver is quite severe as indicated by the peak values (0.8g). There is a small ripple on the acceleration signal caused by the road unevennesses. The ripple is the largest near the peak values. This is due to the tire side force variations becoming larger in these regions. The roll angle never exceeds a value of 3 degrees.

The most interesting plots are those of the vertical and lateral tire forces. The influence of road unevennesses and load transfer can be spotted very nicely with the vertical tire load. Quite unexpected, the side force variations remain rather small. Only when the vertical tire load decreases (mainly due to the load transfer), the lateral force variations become larger. This filtering phenomenon can be explained easily considering the first-order transient tire model. The relaxation length of a tire at large vertical load is larger than at small vertical load. The time constant of the non-linear first-order system is proportional to the relaxation length. This means that the bandwidth of the first-order relaxation system of the tire with the larger vertical load is lower than the bandwidth of the tire with the lower load. The tire with a larger vertical load is therefore relatively insensitive to road irregularities (small bandwidth, much of the load variations is filtered out). The opposite is true for the tire with a small load. The smaller the vertical load, the larger the bandwidth, the less the filtering effect. In case the tire load approaches zero, the relaxation length equals zero and therefore a one to one relationship between vertical and lateral tire load variations arises. This situation must be avoided in the simulation because the bandwidth of the first-order relaxation systems becomes infinite. In order to guarantee integrator stability the relaxation length has therefore been limited. The larger ripple nearby the peak value of the lateral acceleration corresponds to the larger variation of the lateral tire forces.

From the tire load plots the conclusion can be drawn that vertical tire load variations are quite harmless with respect to the degradation of the handling capabilities. The road-holding to handling relation is not so strong. The largest side force variations can be found at those tires were the vertical load
is the lowest. However, these load variations are rather harmless because the average side force is quite small.

Tests have shown that body slip angle $\beta$ is an important quantity. A good subjective rating of the driver corresponds to a small body slip angle. Sometimes four-wheel steering algorithms are designed in such a way that $\beta = 0$. The body slip angle depends on the location of the reference point on the vehicle body. In this case the slip angle has been calculated at the center of gravity of the total vehicle. The magnitude of this angle can only be judged in comparison with other vehicles or vehicle configurations. The same can be said about the other variables: yaw rate, vertical seat accelerations, damper force and suspension working space (figure 8.7b).

**Frequency Response Functions**

The frequency response functions (FRFs) are calculated using the state-space equations of the linear 13 DOF vehicle model given by

\[
\dot{x}_v = A_v x_v + S_i \delta \\
\chi_v = C_v x_v + S_o \delta
\]

with

\[
\chi_v^T = [\dot{\chi}_v \ \psi_b \ \phi_b]
\]

and $\delta$ the steering angle of the front wheels (equal for left and right wheel). Equation (8.9a) contains also the four linear first-order systems for the tire relaxation properties. The total order of system matrix $A_v$ equals 30. The complex FRF can be calculated according to

\[
\frac{\chi_v(j\omega)}{\delta(j\omega)} = C_v \cdot (j\omega I - A_v)^{-1} S_i + S_o
\]

where $j^2 = -1$ and $\omega$ is the frequency in rad/s. The FRFs are calculated for a linear model. This implies tires with constant cornering stiffnesses and relaxation lengths.
Figure 8.7a Double lane-change, standard vehicle (80 km/h).
Figure 8.7b Double lane-change, standard vehicle (80 km/h).
Lateral acceleration FRF

Figure 8.8a gives an example of the frequency response function of the lateral acceleration with respect to the steering input at the front wheels. The vehicle configuration corresponds to the standard car with fixed spring and damper rates. Situations at various speeds have been plotted. From the figures it can be seen that the gain starts to drop after approximately 0.5 Hz. In the range from 2 up to 3 Hz the gain is minimal. The more understeered the vehicle, the higher the bandwidth. The phase angle between lateral acceleration and steering angle becomes larger with increasing frequency but decreases again after 2 Hz. For good handling, the gain should not decrease in the range from 0 to 1 Hz. In this range live saving steering maneuvers might be necessary in critical situations. Little and slow reaction of the car (low gain and large phase lag) might take the driver by surprise and even overburden him.

Yaw rate FRF

The magnitude plot (figure 8.8b) shows a small resonance peak at 0.5 Hz. The higher the speed of travel, the larger the resonance peak will be and, the lighter the yaw damping. The phase angle between steering input and yaw rate becomes gradually larger with increasing frequency. According to Weir and Di Marco [42] the gain at 45° phase lag is an important quantity. In this frequency range the driver is very sensitive. Weir and Di Marco approximate the FRF of the yaw rate with a first-order system. From a first-order system it is known that the bandwidth is defined at that particular frequency where the gain becomes smaller than 3 dB attenuation. This corresponds to a phase angle of -45°. The time constant or equivalent delay time is given by

$$T_{eq} = \frac{1}{2\pi f_{\phi=-45^\circ}}$$

This time constant is speed dependent and ranges from about 0.12 (180 km/h) up to 0.21 seconds (60 km/h). Subjective tests have shown that the best cornering behavior corresponds to a large yaw rate gain and a good yaw damping [3].

Roll angle FRF

The roll angle is not so often considered in handling analyses. Body roll is undesired and it is an inherent property to any kind of passive wheel suspension designed to attenuate road irregularities. Ordinary vehicles for
everyday use have a center of gravity that is located above the rolling axis of the suspension. This implies that the vehicle body will lean over during cornering. This roll motion is undesired because the driver and the passengers are visually bothered by the inclination of the horizon. Furthermore, the geometry of the suspension (wheel camber) might be affected noticeably due the tilted position of the vehicle body. And last but not least, the roll dynamics might affect handling considerably because they can cause severe undesired tire load and slip angle variations. Body roll can be reduced with the aid of roll stabilizers. However, ride comfort gets worse, especially with asymmetric road excitations (e.g. crossing a pothole with one wheel). Figure 8.9 illustrates the FRF of the body roll with respect to the steering angle. The gain should be as small as possible thereby limiting roll angles. The steady-state gain gets larger with increasing speed. This is in agreement with the increase in the steady-state gain for the lateral acceleration at higher speeds. At 2.3 Hz a resonance peak occurs. This peak corresponds to the roll natural frequency. At this frequency the damping of the roll mode decreases with increasing speed. This is not favorable for either ride comfort or handling.

This chapter has dealt with the definition of additional performance criteria that are necessary to describe the performance of the full-vehicle model in an objective way. Unlike with the quarter-car model, the ride comfort parameter can be extended with lateral and longitudinal accelerations. Furthermore objective criteria have been designed to assess handling. It has been chosen to use a double lane-change test on an uneven road as an experiment to investigate the relation between road holding and handling. In addition frequency response functions have been used. The ride and handling performance of a full-vehicle with a standard passive suspension system has been evaluated in order to have a standard for similar vehicles with controlled suspensions.

The next chapter will deal with the design of the suspension controller based on the full-vehicle model. Similar to the quarter-car analysis the design of an intelligent suspension system has been based on LQG control.
Figure 8.8a & b Lateral acceleration and yaw rate FRF.
Figure 8.8c Roll angle frequency response function.
ACTIVE SUSPENSION DESIGN
USING LINEAR OPTIMAL CONTROL

The design of an active suspension system for a full-vehicle model application as will be discussed in this chapter is based on the well-known theory of linear optimal control [7]. The aim of the control design is to improve the ride comfort of the passengers and to guarantee (or improve) the tire to road contact (road holding). The condition of limited suspension travel plays a role as well. This chapter has been split up in several sections for reasons of readability and because of the many different types of controllers to be reviewed.

A full-vehicle system is a system with multi-inputs and multi-outputs (MIMO). During the ride, the vehicle will continuously be excited by four road inputs at
each wheel. According to section 7.2 we know that the road excitations are not fully independent. A cross-correlation between the left and right road track exists and the rear wheels are excited with the same road irregularities as the front wheels, except for a pure time delay. In the control system design it is possible to include the a priori knowledge of the road disturbances acting on the vehicle. This chapter has been built up according to complexity of the road excitation. The following cases are examined:

- Identical left and right road track,
  - Identical front and rear wheel excitation,
  - Uncorrelated front and rear wheel excitation,
  - Delayed rear wheel excitation,
- Cross-correlated left and right road track,
  - Uncorrelated front and rear wheel excitation,
  - Delayed rear wheel excitation,
- Uncorrelated left and right road track,
  - Uncorrelated front and rear wheel excitation,
  - Delayed rear wheel excitation.

We will start with the special and unrealistic case that all four wheels are simultaneously excited by the same road irregularities. After that, the case of uncorrelated and delayed rear wheel inputs with equal left/right inputs will be looked at. Both cases correspond with the design of a controller for a quarter car and half car situation respectively. The third dimension starts to play a role when a cross-correlation between the left and right road track is assumed. A controller will be designed for both delayed and uncorrelated rear axle excitation. Finally, the assumption of correlated left and right road tracks is ignored. Again the delayed and non-delayed versions are considered. The complexity of the controller and the control design effort strongly depends on the assumptions made regarding the disturbances acting on the vehicle. It is a question whether knowledge of the disturbance can improve the performance of the actively suspended vehicle. Therefore, all cases will be compared to each other.

One of the conditions for a sensible use of optimal control is that the vehicle model is as simple as possible; this means a model with as little states as possible. The starting-point for the full-vehicle active suspension design is not the 13 DOF model as described in the previous chapter but a reduced model with 7 DOFs. This model contains all the basic elements of the 13 DOF model, however the following degrees of freedom have been omitted:
• the longitudinal vehicle DOF: \(x_b\),
• the lateral vehicle DOF: \(y_b\),
• the yaw DOF: \(\psi_b\),
• all powertrain related DOFs: \(z_e, \phi_e, \theta_e\).

The location of the center of gravity, the mass and the moments of inertia of the main body have been adjusted for the elimination of the engine body in such a way that the preload of the suspension and tire springs remains unchanged compared to the 13 DOF model. The shifts of the center of gravity read

\[
\delta x = \frac{m_e l_e}{m_b + m_e} \quad \delta y = \frac{m_e w_e}{m_b + m_e} \quad \delta z = \frac{m_e h_e}{m_b + m_e}
\]  

with \(m_e\) being the mass of the engine, \(m_b\) the mass of the vehicle body, and \((l_e, w_e, h_e)\) the distance in \(x, y\) and \(z\) direction of the c.g. of the vehicle body to the c.g. of the engine. The mass of the vehicle body and the moments of inertia change according to

\[
m_b^* = m_b + m_e
\]  

\[
I_{xb}^* = I_{xb} + I_{xe} + m_e \left(l_e^2 + h_e^2\right) - m_b \left(\delta_y^2 + \delta_z^2\right)
\]  

\[
I_{yb}^* = I_{yb} + I_{ye} + m_e \left(l_e^2 + w_e^2\right) - m_b \left(\delta_y^2 + \delta_x^2\right)
\]  

\[
I_{zb}^* = I_{zb} + I_{ze} + m_e \left(w_e^2 + h_e^2\right) - m_b \left(\delta_x^2 + \delta_z^2\right)
\]

Furthermore all dimensions in the longitudinal, lateral and vertical direction change to

\[
l' = l - \delta x \quad w' = w - \delta y \quad h' = h - \delta z
\]

It is of no use to design a controller based on linear optimal control (with full-state feedback) for a large vehicle model, since state feedback control requires that all states of the vehicle must be known and measurable (the use of state observers is left out of consideration in this stage). Besides that, it is likely that the longitudinal, lateral and yaw DOF of the vehicle are of minor influence to the vertical dynamics. The 7 DOF model contains all basic degrees of freedom to reasonably evaluate ride comfort and tire load variations. The DOFS are: heave, pitch and roll of the main body, left and right front axle heave and rigid rear axle heave and tramp. A slight modification has been made in the rear suspension. The Panhard rod of the 13 DOF model has been substituted by a parallel slider in the middle of the rear axle for the 7 DOF model. This modification turned out to be necessary because during the first attempts to
design a controller, a strong asymmetric control law was found (different feedback gains for the left and right suspension actuators), resulting from the asymmetry of the Panhard rod based rear suspension.

The resemblance between 13 DOF and 7 DOF model can be verified by comparing the eigenvalues of both models. The natural frequencies and damping values are shown in table 9.1.

<table>
<thead>
<tr>
<th>mode</th>
<th>eigenvalues</th>
<th>natural frequency</th>
<th>damping ratio</th>
<th>damped frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>13 DOF</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>-10.82 +/- 84.50 i</td>
<td>13.56</td>
<td>0.127</td>
<td>13.45</td>
</tr>
<tr>
<td>3</td>
<td>-27.99 +/- 76.03 i</td>
<td>12.89</td>
<td>0.346</td>
<td>12.10</td>
</tr>
<tr>
<td>4</td>
<td>-23.98 +/- 81.31 i</td>
<td>13.49</td>
<td>0.283</td>
<td>12.94</td>
</tr>
<tr>
<td>5</td>
<td>-20.66 +/- 80.64 i</td>
<td>13.25</td>
<td>0.248</td>
<td>12.83</td>
</tr>
<tr>
<td>8</td>
<td>-2.67 +/- 14.82 i</td>
<td>2.40</td>
<td>0.177</td>
<td>2.36</td>
</tr>
<tr>
<td>9</td>
<td>-3.03 +/- 8.92 i</td>
<td>1.50</td>
<td>0.321</td>
<td>1.41</td>
</tr>
<tr>
<td>10</td>
<td>-2.49 +/- 7.44 i</td>
<td>1.25</td>
<td>0.317</td>
<td>1.18</td>
</tr>
</tbody>
</table>

| 7 DOF |                |                   |               |                 |
| 2    | -10.87 +/- 84.55 i | 13.57             | 0.128         | 13.46           |
| 3    | -23.83 +/- 72.89 i | 12.21             | 0.311         | 11.60           |
| 4    | -23.99 +/- 81.21 i | 13.48             | 0.283         | 12.93           |
| 5    | -21.42 +/- 80.52 i | 13.26             | 0.257         | 12.82           |
| 8    | -2.67 +/- 14.82 i  | 2.40              | 0.177         | 2.36            |
| 9    | -3.02 +/- 8.83 i   | 1.49              | 0.324         | 1.41            |
| 10   | -2.52 +/- 7.42 i   | 1.25              | 0.321         | 1.18            |

Table 9.1 Natural frequencies and damping.

For clearness' sake, the meaning of the mode numbers will be repeated once more:

- rear axle tramp : mode 2
- front axle wheel-hop : mode 3
- front axle tramp : mode 4
- rear axle wheel-hop : mode 5
- sprung mass roll : mode 8
- sprung mass pitch : mode 9
- sprung mass bounce : mode 10

Table 9.1 proves that all modes with exception of mode 3 correspond very well. The small difference between both models with relation to mode 3 (front wheel-hop) must be due to the effect of the engine body. Table 7.5 shows that two eigenvalues related to drivetrain rigid body modes (bounce and pitch) are
located very close to the eigenvalues corresponding to the front wheel-hop mode. The possibility of a coupling between both modes can't be ruled out. It is therefore likely that the absence of the engine in the 7 DOF model is responsible for the small difference in the eigenvalues of mode 3.

After having designed a controller for the simplified 7 DOF model, the performance is evaluated on the complex 13 DOF vehicle model, because this model is more realistic with respect to the real car. Important point of consideration is that the performance is evaluated for a car in a realistic environment. This means that besides lateral tire properties, a realistic road is assumed with a correlated left and right track and a delayed rear wheel road excitation. This implies that some of the controllers that are based on simplified disturbance assumptions, such as the non-delayed rear disturbance, are somewhat misjudged. This choice has been made because it is of no use to evaluate the performance of a car in a situation which will never occur. Another important difference between the 7 and 13 DOF model is the location of the center of gravity. The results (such as feedback gain matrices) obtained with the 7 DOF model analysis must be transformed to the 13 DOF model coordinate system. As only the z-coordinate forms a part of both models, the following transformation must be made

\[ z_{b_{13}}^z = z_{b_{7}}^z + \delta x \cdot \theta_b \]  

(9.4)

with \( \delta x \) representing the shift of the center of gravity according to equation (9.1).

### 9.1 State Feedback Control

Linear optimal control is based on the minimization of a quadratic cost function. Similar to the quarter-car model analysis, it is possible to define a quadratic performance index \( J \) according to

\[
J = \lim_{T \to \infty} \frac{1}{2} \int_0^T \begin{bmatrix}
q_1 \left[ a_{p1L}^2 + a_{p1R}^2 + a_{p2L}^2 + a_{p2R}^2 \right] + \\
q_2 \left[ (F_{11L} - F_{11R})^2 + (F_{12L} - F_{12R})^2 + (F_{22L} - F_{22R})^2 \right] + \\
q_3 \left[ d_{a1L}^2 + d_{a1R}^2 + d_{a2L}^2 + d_{a2R}^2 \right] + \\
q_4 \left[ F_{a1L}^2 + F_{a1R}^2 + F_{a2L}^2 + F_{a2R}^2 \right]
\end{bmatrix} dt
\]

(9.5)

where \( a_{p1L}, a_{p1R}, a_{p2L}, a_{p2R} \) represent the multi-directional vector sum of vehicle body accelerations at the c.g. of the left/right front and left/right rear passengers respectively according to
Chapter 9 Active Suspension Design using Linear Optimal Control

\[
\begin{align*}
\alpha_{p1L}^2 &= \ddot{x}_{p1L}^2 + \dot{y}_{p1L}^2 + \dot{z}_{p1L}^2 \quad \alpha_{p1R}^2 &= \ddot{x}_{p1R}^2 + \dot{y}_{p1R}^2 + \dot{z}_{p1R}^2 \\
\alpha_{p2L}^2 &= \ddot{x}_{p2L}^2 + \dot{y}_{p2L}^2 + \dot{z}_{p2L}^2 \quad \alpha_{p2R}^2 &= \ddot{x}_{p2R}^2 + \dot{y}_{p2R}^2 + \dot{z}_{p2R}^2
\end{align*}
\]  

(9.6)

Besides the heave accelerations, the roll and pitch angular accelerations are weighted automatically since they are included in equation (9.6). The longitudinal, lateral and vertical accelerations of, e.g., the right front passenger are given by

\[
\begin{align*}
\dot{x}_{p1R} &= h_{\text{seat}1} \dot{\theta}_b \\
\dot{y}_{p1R} &= -h_{\text{seat}1} \dot{\phi}_b \\
\ddot{z}_{p1R} &= \ddot{z}_b + w_{\text{seat}1} \ddot{\phi}_b - l_{\text{seat}1} \dot{\theta}_b
\end{align*}
\]  

(9.7)

where \( l_{\text{seat}1} \), \( w_{\text{seat}1} \) and \( h_{\text{seat}1} \) fix the longitudinal, lateral and vertical position of the c.g. of the passenger with respect to the c.g. of the vehicle body. The deflections of the four suspension elements are denoted by \( d_{s1L}, d_{s1R}, d_{s2L}, d_{s2R} \) and the dynamic vertical tire forces are described by the difference between the tire load \( F_t \) and the static tire load \( F_t^0 \). Finally, it is possible to weight the actuator effort \( F_a \) separately. The four contributions to the performance criterion can be weighted with factors \( q_1, q_2, q_3 \) and \( q_4 \). It is of no use to weight each suspension variable (deflection, dynamic tire load and actuator force) separately since the more weighting factors are present, the more combinations are possible. If weighting factors are desired which account for the difference in wheel loads, it is better to consider the following weighting structure

\[
J = \lim_{T \to \infty} \frac{1}{T} \int_0^T \left\{ q_1 \left( \frac{\alpha_{p1L}^2 + \alpha_{p1R}^2 + \alpha_{p2L}^2 + \alpha_{p2R}^2}{F_{t1}^0} \right)^2 + q_2 \left( \frac{F_{11L} - F_{t1}^0}{F_{t1}^0} \right)^2 + \left( \frac{F_{11R} - F_{t1}^0}{F_{t1}^0} \right)^2 + \left( \frac{F_{12L} - F_{t2}^0}{F_{t2}^0} \right)^2 + \left( \frac{F_{12R} - F_{t2}^0}{F_{t2}^0} \right)^2 \right\} + dt \\
q_3 \left( d_{s1L}^2 + d_{s1R}^2 + d_{s2L}^2 + d_{s2R}^2 \right) + q_4 \left( F_{a1L}^2 + F_{a1R}^2 + F_{a2L}^2 + F_{a2R}^2 \right) 
\]  

(9.8)

The modified weighting criterion shows that the ratio of the dynamic tire load to the static tire load is used instead of the plain dynamic tire load. For road holding it is not so important to have knowledge about the absolute value of the tire load variations. Moreover, the relation with respect to the static tire load is of importance, because this gives a better representation when the limit of reduced safety is reached (tire loses contact with the ground). Since the static rear tire load is about 60% of the static front tire load, it is preferable to use the ratio instead of the absolute value. As the maximum
Chapter 9 Active Suspension Design using Linear Optimal Control

Suspension travel of the front as well as the rear suspension is assumed to be equal, there is no reason to weight the front and rear suspension deflection differently.

Having the quadratic performance index it is now possible to search for a state feedback matrix $K_c$ that provides a minimum $J$ at given weighting factors. Before going into the design of the controller, the state-space equations of the 7 DOF will be discussed briefly. These state-space equations read as follows

$$\begin{align*}
\dot{x}_v &= A_v x_v + F_v^0 u_v + R_v^0 z_r, \\
y_v &= C_v x_v + F_v^0 u_v + R_v^0 z_r,
\end{align*}$$

(9.9a)

(9.9b)

with

$$\begin{align*}
x_v^T &= \begin{bmatrix} z_b & \dot{\phi}_b & \phi_{a1L} & \phi_{a1R} & z_{a2} & \dot{\phi}_{a2} & \phi_{a2} & \dot{\phi}_{a2} & \dot{\phi}_{a2} & \dot{\phi}_{a2} \end{bmatrix}^T, \\
u_v^T &= \begin{bmatrix} u_{\phi_L} & u_{\phi_R} & u_{\omega_L} & u_{\omega_R} \end{bmatrix}, \\
z_r &= \begin{bmatrix} z_{r1L} & z_{r1R} & z_{r2L} & z_{r2R} \end{bmatrix}^T.
\end{align*}$$

$x_v$ represents the state vector, the actuator forces are present in vector $u_v$, and the four road inputs are taken together in vector $z_r$. In order to be able to use standard LQG software it is necessary to convert quadratic cost function (9.8) to the three weighting matrices $Q_c$, $N_c$, and $R_c$. The most convenient way to do this, is to write down all weighted variables as an output of the system according to equation (9.9b). The state weighting matrix $Q_c$, cross-product weighting matrix $N_c$ and control effort weighting matrix $R_c$ can easily be calculated from

$$\begin{align*}
Q_c &= C_v^T \cdot q \cdot C_v, \\
N_c &= C_v^T \cdot q \cdot F_v^0, \\
R_c &= F_v^0 \cdot q \cdot F_v^0 + \text{diag}(q_4, q_4, q_4, q_4)
\end{align*}$$

(9.10a)

(9.10b)

(9.10c)

where $q$ is a diagonal matrix according to

$$q = \begin{bmatrix} \text{diag}(q_1, \ldots, q_1) & 0 & 0 \\
0 & \text{diag}(q_2, q_2, q_2, q_2) & 0 \\
0 & 0 & \text{diag}(q_3, q_3, q_3, q_3) \end{bmatrix}.$$

- 209 -
Output equation (9.9b) contains 20 outputs:
- $y(01) .. y(12)$: accelerations at each seat (3 directions, 4 seats),
- $y(13) .. y(16)$: dynamic tire load / static tire load at each wheel,
- $y(17) .. y(20)$: deflection of each wheel suspension.

Before discussing the results in section (9.1.4), some theory with respect to road surface modeling will be treated briefly in the next three sections (9.1.1-9.1.3).

### 9.1.1 Identical Left and Right Road Track

The fully correlated left and right track is the first simplification made. This means that the vehicle is purely excited by heave and pitch inputs. The front/rear road description decides whether the vehicle is additionally excited by a pitch input. The active suspension design of this symmetric road excitation assumption corresponds completely with active control of a half-car model. Actually the 7 DOF vehicle could be reduced to a 4 DOF model, since the roll DOF is never excited (only valid for a symmetric car) because the left wheels sense the same road irregularities as the right wheels. With the exception of calculation effort, the half vehicle model offers no advantages over the full-vehicle model in this case of symmetric excitation. The fully correlated left/right case has been split up in three situations:

1. Identical front and rear wheel excitation,
2. Uncorrelated front and rear wheel excitation,
3. Delayed rear wheel excitation.

The model for case 1 will be prepared in section 9.1.1.1. It looks very much like the model for the design of a suspension controller for a quarter-car. Situation 2 will be treated in section 9.1.1.2 and assumes two uncorrelated random processes, one for each axle. Situation 3 will be dealt with in section 9.1.1.3 and assumes a single random process at the front axle only, and a delayed version of the same process at the rear axle.

The road model as introduced in section 7.2 can be simplified to the following shaping filter

\[ \ddot{x}_r = A_r x_r + B_r w_r \]  \hspace{1cm} (9.11a)

\[ y_r = C_r x_r \]  \hspace{1cm} (9.11b)

with

\[ A_r = -\nu\beta_1 \quad B_r = \nu g_1 \quad C_r = 1 \]
Actually, road model (7.12) could also be used with \( \alpha \) equal to zero (equation (7.20)) and \( \beta_2 = \beta_1 \). However, such a model would contain one superfluous state (roll excitation). It is therefore better to eliminate this state from the beginning. The values of \( \beta_1 \) and \( g_1 \) correspond to those of table 7.3.

The vehicle model is almost identical to that of equation (9.9). The one and only simplification concerns the road input. Since there is no need for a separate left and right track road input the road vector has been modified to

\[
z_r = \begin{bmatrix} z_{r1} \\ z_{r2} \end{bmatrix}
\]

which implies that the first and second column, and third and fourth column of \( R_t \) and \( R_v \) can be combined.

### 9.1.1.1 Identical Front and Rear Wheel Input

The assumption that all four wheels of the car are excited (in-phase) with the same road input is quite unrealistic. However, at high speeds, the wheelbase time delay between the front and rear wheel excitation is so short that its influence on the vertical dynamics can be neglected. The road input and output vector \( R_t \) and \( R_v \) of the vehicle model can be simplified once more by adding together the front and rear road input. Road model (9.11) can now be augmented to vehicle model (9.9) according to

\[
\dot{x}_r = A_r x_r + F_t^i u_v + R_t^i w_1 \\
y_r = C_r x_r + F_v^o u_v
\]

(9.12a)  
(9.12b)

with

\[
A_r = \begin{bmatrix} A_v & R_t^i C_r \\ 0 & A_r \end{bmatrix}, \quad F_t^i = \begin{bmatrix} F_v^o \\ 0 \end{bmatrix}, \quad R_t^i = \begin{bmatrix} 0 \\ B_r \end{bmatrix}, \quad C_r = \begin{bmatrix} C_v & R_t^i C_r \\ 0 & C_r \end{bmatrix}, \quad F_v^o = \begin{bmatrix} F_v^o \\ 0 \end{bmatrix}
\]

\[
x_r = \begin{bmatrix} x_v \\ x_r \end{bmatrix}, \quad y_r = \begin{bmatrix} y_v \\ y_r \end{bmatrix}
\]

The sizes of the matrices are:

\[
A_r = 15 \times 15, \quad F_t^i = 15 \times 4, \quad R_t^i = 15 \times 1
\]

\[
C_r = 20 \times 15, \quad F_v^o = 20 \times 4
\]
9.1.1.2 Uncorrelated Front and Rear Wheel Input

The uncorrelated front/rear case assumes that the vehicle is excited by two completely uncorrelated random processes, each for one axle, with identical statistical properties. Although this situation is not in agreement with the real car, it is interesting to study the influence of the non-delayed road disturbance.

The state-space equations of the road model take a form as

\[
\begin{align*}
\dot{x}_{r2} &= A_{r2} x_{r2} + B_{r2} w_{r2} & (9.13a) \\
y_{r2} &= C_{r2} x_{r2} & (9.13b)
\end{align*}
\]

with

\[
A_{r2} = \begin{bmatrix} -v \beta_1 & 0 \\ 0 & -v \beta_1 \end{bmatrix}, \quad B_{r2} = \begin{bmatrix} v g_1 & 0 \\ 0 & v g_1 \end{bmatrix}, \quad C_{r2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]

\[
x_{r2} = \begin{bmatrix} z_{r1} \\ z_{r2} \end{bmatrix}, \quad w_{r2} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}
\]

where \( w_1 \) and \( w_2 \) are two uncorrelated random processes and \( z_{r1} \) and \( z_{r2} \) represent the road elevation at the front and rear axle respectively. This road model can be combined with the 7 DOF vehicle model to

\[
\begin{align*}
\dot{x}_i &= A_i x_i + F_i^u u_i + R_i w_{r2} & (9.14a) \\
y_i &= C_i x_i + F_i^o u_i & (9.14b)
\end{align*}
\]

with

\[
A_i = \begin{bmatrix} A_{u} & R_{i}^o C_{r2} \\ 0 & A_{r2} \end{bmatrix}, \quad F_i^u = \begin{bmatrix} F_{u} \\ 0 \end{bmatrix}, \quad R_i = \begin{bmatrix} 0 \\ B_{r2} \end{bmatrix}, \quad C_i = \begin{bmatrix} C_{u} & R_{i}^o C_{r2} \\ 0 & C_{r2} \end{bmatrix}, \quad F_i^o = \begin{bmatrix} F_{u}^o \\ 0 \end{bmatrix}
\]

\[
x_i = \begin{bmatrix} x_u \\ x_{r2} \end{bmatrix}, \quad y_i = \begin{bmatrix} y_u \\ y_{r2} \end{bmatrix}
\]

The sizes of the matrices are:

\[
A_i = 16 \times 16, \quad F_i^u = 16 \times 4, \quad R_i = 16 \times 2 \\
C_i = 20 \times 16, \quad F_i^o = 20 \times 4
\]
Chapter 9 Active Suspension Design using Linear Optimal Control

9.1.1.3 Delayed Rear Wheel Input

The time delay between the front and rear wheels has been approximated using a Padé filter (section 7.2.1) with order \( n_p \) and state-space matrices \( A_p \), \( B_p, C_p \) and \( D_p \). Since the left and right tracks are identical, only one Padé filter is necessary. Together with the road description according to equation (9.11), it is possible to combine road model and Padé filter to one total road model according to

\[
\begin{align*}
\dot{x}_{r2} &= A_{r2}x_{r2} + B_{r2}w_{r2} \\
y_{r2} &= C_{r2}x_{r2}
\end{align*}
\]  

(9.15a)\hspace{1cm} (9.15b)

with

\[
A_{r2} = \begin{bmatrix} A_r & 0 \\ B_p C_r & A_p \end{bmatrix}, \quad B_{r2} = \begin{bmatrix} B_r \\ 0 \end{bmatrix}, \quad C_{r2} = \begin{bmatrix} C_r & 0 \\ D_p C_r & C_p \end{bmatrix}
\]

\[
x_{r2} = \begin{bmatrix} z_{r1} \\ x_p \end{bmatrix}, \quad y_{r2} = \begin{bmatrix} z_{r1} \\ y_p \end{bmatrix}, \quad w_{r2} = w_1
\]

Finally, the total road model can be combined with the 7 DOF vehicle similar to equation (9.12). The sizes of the matrices are:

\[
A_t = (15 + n_{pode}) \times (15 + n_{pode}), \quad F_t = (15 + n_{pode}) \times 4, \quad R_t = (15 + n_{pode}) \times 1
\]

\[
C_t = 20 \times (15 + n_{pode}), \quad F_t^o = 20 \times 4
\]

9.1.2 Correlated Left and Right Road Track

The cross-correlated left and right road track case has been split up in two situations

1. Uncorrelated front and rear wheel excitation,
2. Delayed rear wheel excitation.

The correlated left/right track case is based on the road model as discussed in section 7.2. The state-space equations of this road model read

\[
\begin{align*}
\dot{x}_{r2} &= A_{r2}x_{r2} + B_{r2}w_{r2} \\
y_{r2} &= C_{r2}x_{r2}
\end{align*}
\]  

(9.16a)\hspace{1cm} (9.16b)

with matrices \( A_{r2}, B_{r2}, C_{r2} \) equal to
\[
A_{r2} = \begin{bmatrix}
-\frac{\nu(\beta_1 + \beta_2)}{2} & -\frac{\nu(\beta_1 - \beta_2)}{2} \\
-\nu(\beta_1 - \beta_2) & -\nu(\beta_1 + \beta_2)
\end{bmatrix}
\quad
B_{r2} = \begin{bmatrix}
\nu g_1 & -\frac{\nu g_2 T_w}{2} \\
\nu g_1 & \frac{\nu g_2 T_w}{2}
\end{bmatrix}
\quad
C_{r2} = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\]

and

\[
x_{r2} = \gamma_{r2} = \begin{bmatrix}
z_{r1L} \\
z_{r1R}
\end{bmatrix}
\quad
w_{r2} = \begin{bmatrix}
w_1 \\
w_2
\end{bmatrix}
\]

At first sight this model looks different from equation (7.12). However, the state-space equations (7.12) have only be transformed to the above set of equations using a similarity transformation. The transformation matrix is equal to matrix \(C_r\) from equation (7.12b). This transformation has been done in order to obtain the road input at each front wheel \(z_{r1L}\) and \(z_{r1R}\) as variables in state vector \(x_{r2}\), rather than the original heave and roll components. In section 7.2 it has been demonstrated that this model is able to describe the frequency dependent cross-correlation between the left and right track. For long wavelengths the left track is almost identical to the right track (high correlation) and for short wavelengths both excitations are almost independent (low correlation).

### 9.1.2.1 Uncorrelated Front and Rear Wheel Input

The uncorrelated front/rear case assumes that the vehicle is excited by two completely uncorrelated random processes, each for one axle, with identical statistical properties. The state-space equations of the total road model are composed of two identical road models according to equation (9.16). It reads

\[
\dot{x}_{r4} = A_{r4}x_{r4} + B_{r4}w_{r4}
\quad
(9.17a)
\]

\[
\gamma_{r4} = C_{r4}x_{r4}
\quad
(9.17b)
\]

with

\[
A_{r4} = \begin{bmatrix}
A_{r2} & 0 \\
0 & A_{r2}
\end{bmatrix}
\quad
B_{r4} = \begin{bmatrix}
B_{r2} & 0 \\
0 & B_{r2}
\end{bmatrix}
\quad
C_{r4} = \begin{bmatrix}
C_{r2} & 0 \\
0 & C_{r2}
\end{bmatrix}
\]

and

\[
x_{r4}^T = \gamma_{r4}^T = \begin{bmatrix}
z_{r1L} & z_{r1R} & z_{r2L} & z_{r2R}
\end{bmatrix}
\quad
w_{r4}^T = \begin{bmatrix}
w_1 & w_2 & w_3 & w_4
\end{bmatrix}
\]

The total road model can be combined with the 7 DOF vehicle similar to equation (9.14). The sizes of the matrices are:

- 214 -
\[ A_i = 18 \times 18, \quad F_i^o = 18 \times 4, \quad R_i = 18 \times 4 \]
\[ C_i = 20 \times 18, \quad F_i^o = 20 \times 4 \]

### 9.1.2.2 Delayed Rear Wheel Input

Similar to the uncorrelated left/right case, the time delay between the front and rear wheels has been approximated using two Padé filters, each with order \( n_p \) and state-space matrices \( A_p, B_p, C_p \), and \( D_p \). Since the left and right track are not identical, two Padé filters are necessary according to

\[ \dot{x}_{p2} = A_p x_{p2} + B_p y_{r2} \]
\[ y_{p2} = C_p x_{p2} + D_p y_{r2} \]  

with

\[ A_p = \begin{bmatrix} A_p & 0 \\ 0 & A_p \end{bmatrix}, \quad B_p = \begin{bmatrix} B_p & 0 \\ 0 & B_p \end{bmatrix}, \quad C_p = \begin{bmatrix} C_p & 0 \\ 0 & C_p \end{bmatrix}, \quad D_p = \begin{bmatrix} D_p & 0 \\ 0 & D_p \end{bmatrix} \]

Together with the road description according to equation (9.16), it is possible to combine road model and Padé filter to one total road model according to

\[ \dot{x}_{r4} = A_{r4} x_{r4} + B_{r4} u_{r4} \]
\[ y_{r4} = C_{r4} x_{r4} \]

with

\[ A_{r4} = \begin{bmatrix} A_{r2} & 0 \\ B_{r2} C_{r2} & A_{r2} \end{bmatrix}, \quad B_{r4} = \begin{bmatrix} B_{r2} \\ 0 \end{bmatrix}, \quad C_{r4} = \begin{bmatrix} C_{r2} & 0 \\ D_{r2} C_{r2} & C_{r2} \end{bmatrix} \]

and

\[ x_{r4}^T = [z_{r1L} \quad z_{r1R} \quad x_{pL} \quad x_{pR}] \]
\[ y_{r4}^T = [z_{r1L} \quad z_{r1R} \quad z_{r2L} \quad z_{r2R}] \]
\[ u_{r4}^T = [u_1 \quad u_2] \]

Finally, the total road model can be combined with the 7 DOF vehicle similar to equation (9.12). The sizes of the matrices are:

\[ A_i = (16 + 2n_{pade}) \times (16 + 2n_{pade}), \quad F_i^o = (16 + 2n_{pade}) \times 4, \quad R_i = (16 + 2n_{pade}) \times 2 \]
\[ C_i = 20 \times (16 + 2n_{pade}), \quad F_i^o = 20 \times 4 \]
9.1.3 Uncorrelated Left and Right Road Track

The uncorrelated road track situation assumes that the left track irregularities are completely independent of the right track irregularities. The validity of this assumption depends on the road description in the long wavelength range. If the road has been modeled as integrated white noise rather than (low-pass) filtered white noise, the low frequency inputs are not bounded. This implies that the possibility exists that the left track raises one meter and that the right drops one meter at the same time. Therefore, the uncorrelated track input makes only sense with low frequency bounded road inputs (e.g. road model 7.12). The theory of the control design is completely identical to the situation of cross-correlated left and right tracks (section 9.2). The only difference is connected with the value of the parameters of the road model. According to equation (7.19) and (7.20), both tracks are fully uncorrelated if and only if $\alpha$ equals 1.0 and $\beta_2 = \beta_1$. Parameter $g_1$ must be corrected in order to obtain the same spectral density for the road input as with the fully correlated case. Its value changes from 1.0 to $0.5\sqrt{2}$. From equation (7.20) it is now easy to calculate $g_2$. Its value changes from 1.228 to 0.992. All matrix equations of section 9.2 remain unchanged.

9.1.4 Evaluation of the Active Vehicle Ride and Handling

This section compares all seven proposed control structures as discussed in the previous seven sections. The comparison is based on the RMS values of the selected outputs, spectral densities and a handling test. The starting-point of the control system design is the selection of the weighting factors. The following values have been chosen for the performance index according to equation (9.8)

- ride comfort: $q_1$ is free to choose

- road holding: $q_2 = \frac{1}{\sigma_{2,\text{max}}^2} = \frac{1}{(1/3)^2} = 9$  \hspace{1cm} (9.20a)

- suspension deflection: $q_3 = \frac{1}{\sigma_{3,\text{max}}^2} = \frac{1}{(0.05/3)^2} = 3600$  \hspace{1cm} (9.20b)

- actuator force: $q_4 = \frac{1}{\sigma_{4,\text{max}}^2} = \frac{1}{(2000/3)^2} = 2.25e-6$  \hspace{1cm} (9.20c)

The selection of the values for the weighting factors is based on the maximum admissible variance of the outputs. The maximum suspension deflection equals ± 0.05 m, the maximum actuator force is assumed to be ± 2000 N and the maximum of the ratio of the dynamic tire load to the static tire load is ±1.
The vehicle will operate 99.97% of the time within the assumed ranges if the standard deviation \( \sigma \) is equal to one third of the maximum value. Weighting factor \( q_1 \) is free of choice. A large value of \( q_1 \) will emphasize ride comfort, a small value accentuates the two other performance parameters (tire and suspension deflection).

The design conditions for the active suspension are:
- The controller will be designed for a 7 DOF vehicle model with suspension springs with stiffnesses equal to those of the passive car and without suspension dampers,
- The performance is evaluated on a 13 DOF vehicle model on a realistic road (cross-correlated track, time delayed rear wheel input),
- The speed of travel is equals to 120 km/h,
- The actuators are ideal (infinite bandwidth and force potential),
- In those cases where needed, a 4th-order Padé filter will be used for the approximation of the wheelbase time delay.

RMS Values

The performance of the 13 DOF vehicle driving on a random road surface has been computed using the covariance method. This method, which has already been discussed in chapter 8, must be modified slightly for the active suspension system. Instead of the open-loop total state-space matrix \( A_t \) and output matrix \( C_t \), the closed-loop matrices \( A_t^c \) and \( C_t^c \) must be used. These matrices are given by

\[
A_t^c = A_t - F_t^c \cdot K_c \\
C_t^c = C_t - F_t^c \cdot K_c
\]

(9.21a)
(9.21b)

where \( K_c \) represents the matrix with state feedback gains.

After having composed the total state-space representation of the vehicle model augmented with the road model (section 9.1.1 up to 9.1.3), and after having selected the weighting factors and composed the weighting matrices, it is possible to calculate the full-state feedback gains using standard LQG software. To be able to calculate the performance of the 13 DOF vehicle model, state feedback matrix \( K_c \) must be converted from the 7 DOF form to the 13 DOF form. This implies a shift of the feedback gains to the appropriate states and adding zeros at the engine related states as well as the longitudinal, lateral and yaw degree of freedom. Furthermore a state transformation must be...
applied in order to compensate for the shift of the center of gravity (equation 9.4). In the case that the left and right tracks were assumed identical in the design phase, the feedback gains related to the road elevation at the front and rear axle have been spread over the left and right wheels. In case all four wheels are excited by the same road disturbance, the feedback gain related to the road elevation has been divided by 4 and distributed over all four road inputs.

Table 9.2 up to 9.4 show the results for three values of weighting factor $q_1$: 0.1, 1 and 10. A large value of $q_1$ means that ride comfort is emphasized over the other three weighted variables. For $q_1 = 10$ (table 9.4) it can be seen that irrespective of the road description, all acceleration levels are substantially lower than those of the passive car with exception of the lateral and yaw angular acceleration. The gain in the total accelerations at the front and rear seat position is almost 50%. Not fully unexpected, the gain in ride comfort is accompanied by an increase in tire load variations and suspension working space. The amount of space used by the active suspension is still a lot smaller than the available 0.1 m. A rule of thumb is that 6 times the RMS value must be smaller than 0.1 m. This guarantees at least that 99.7% of the time the suspension deflection is within the range of $\pm 0.05$ m (equilibrium is assumed to be in the middle). The increase of tire load variations is the largest at the front wheels. Whether an increase of 100% over the standard passive car is acceptable depends on the situation. During cornering it is likely that such an increase will affect the handling of the car substantially, but while driving straight on, there is no need to be concerned. For small values of $q_1$ both tire load variations and suspension deflections can be reduced. The degradation of the ride comfort related variables is surprisingly small.

The difference in performance between the uncorrelated and the delayed rear wheel inputs is very striking. The benefits of the wheelbase time delay are considerable. The tire load variations at the rear wheels and the rear suspension deflection can be reduced by 15 to 25% with regard to the uncorrelated case depending on the value of $q_1$. The improvement in the level of pitch accelerations is also tremendous: 40 - 50% compared with the non-delayed control version. The performance at the front end of the vehicle is not affected by the time-delay assumption in the control design phase. Figure 9.1 illustrates the effect of the wheelbase time delay on the performance for different values of $q_1$ (ride comfort weighting factor). The left and right tracks are assumed to be correlated. The dashed lines correspond to delayed rear wheel inputs, the solid ones to uncorrelated front/rear wheel inputs and the
dotted one refers to the standard passive car. The improvement of the rear tire load variations, the rear suspension wheel travel and the accelerations at the rear seats is accompanied by a small increase in front seat accelerations and body roll accelerations. Instead of the plain tire load variations, the ratio between the dynamic tire load and the static tire load has been plotted. The effect of the wheelbase time delay can be interpreted as the situation that the rear suspension has 'preview' information available from state measurements at the front. Therefore, the actuators in the rear suspension are able to react on disturbances which still have not reached the rear wheels. If a bump approaches the rear axle, the actuators are able to anticipate on the oncoming bump by means of retracting the rear suspension. The amount of preview time depends on the vehicle speed (75 ms at 120 km/h), and so it may be anticipated that as the speed increases (and the preview time decreases) the improvement in performance is reduced. More about this later on.

Figure 9.2 compares all rear wheel delayed versions based on different cross-correlation assumptions. The results shown in the tables and figure indicate clearly that over a range of conditions, the effect of including the cross-correlation information in the control design phase is of minor benefit. For large values of $q_1$ the extra gain resulting from the delayed input in combination with (un)correlated tracks is somewhat larger than in the case of identical left and right tracks. These results justify the design of a suspension controller based on a half vehicle model. This simplifies not only the control design stage, but reduces also the number of required Padé filters from two to one. These results agree with the findings of Abdel Hady et al. [1].
<table>
<thead>
<tr>
<th>front/rear road description</th>
<th>identical tracks</th>
<th>correlated tracks</th>
<th>uncorrelated tracks</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>variable</td>
<td>place unit</td>
<td>RMS values</td>
</tr>
<tr>
<td></td>
<td>longitudinal acceleration</td>
<td>c.g. m/s^2</td>
<td>0.023 -15% 0.020 -3% 0.011 47%</td>
</tr>
<tr>
<td></td>
<td>lateral acceleration</td>
<td>c.g. m/s^2</td>
<td>0.113 11% 0.114 11% 0.109 14%</td>
</tr>
<tr>
<td></td>
<td>vertical acceleration</td>
<td>c.g. m/s^2</td>
<td>0.578 1% 0.569 3% 0.582 0%</td>
</tr>
<tr>
<td></td>
<td>roll acceleration</td>
<td>c.g. 1/s^2</td>
<td>0.860 -11% 0.860 -11% 0.860 -11%</td>
</tr>
<tr>
<td></td>
<td>pitch acceleration</td>
<td>c.g. 1/s^2</td>
<td>0.507 -15% 0.453 -3% 0.234 47%</td>
</tr>
<tr>
<td></td>
<td>yaw acceleration</td>
<td>c.g. 1/s^2</td>
<td>0.058 28% 0.059 24% 0.054 31%</td>
</tr>
<tr>
<td></td>
<td>dynamic tire load</td>
<td>1L N</td>
<td>410 -3% 407 -2% 408 -3%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1R N</td>
<td>410 -3% 407 -3% 408 -3%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2L N</td>
<td>383 12% 385 11% 326 25%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2R N</td>
<td>387 12% 390 12% 329 26%</td>
</tr>
<tr>
<td></td>
<td>suspension deflection</td>
<td>1L mm</td>
<td>3.34 21% 3.67 13% 3.71 12%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1R mm</td>
<td>3.32 22% 3.66 14% 3.70 13%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2L mm</td>
<td>4.24 2% 3.53 18% 2.98 31%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2R mm</td>
<td>4.24 2% 3.53 19% 2.98 31%</td>
</tr>
<tr>
<td></td>
<td>longitudinal acceleration</td>
<td>f-seat m/s^2</td>
<td>0.068 -12% 0.063 -5% 0.036 41%</td>
</tr>
<tr>
<td></td>
<td>lateral acceleration</td>
<td>f-seat m/s^2</td>
<td>0.165 1% 0.165 1% 0.162 3%</td>
</tr>
<tr>
<td></td>
<td>vertical acceleration</td>
<td>f-seat m/s^2</td>
<td>0.649 -1% 0.640 0% 0.652 4%</td>
</tr>
<tr>
<td></td>
<td>vector sum</td>
<td>f-seat m/s^2</td>
<td>0.673 -1% 0.664 0% 0.673 1%</td>
</tr>
<tr>
<td></td>
<td>longitudinal acceleration</td>
<td>r-seat m/s^2</td>
<td>0.097 -15% 0.089 -4% 0.048 44%</td>
</tr>
<tr>
<td></td>
<td>lateral acceleration</td>
<td>r-seat m/s^2</td>
<td>0.235 4% 0.236 4% 0.231 6%</td>
</tr>
<tr>
<td></td>
<td>vertical acceleration</td>
<td>r-seat m/s^2</td>
<td>0.818 -5% 0.775 0% 0.661 15%</td>
</tr>
<tr>
<td></td>
<td>vector sum</td>
<td>r-seat m/s^2</td>
<td>0.856 -5% 0.818 0% 0.702 14%</td>
</tr>
</tbody>
</table>

Table 9.2 RMS performance values for $q_1 = 0.1$ ($v = 120$ km/h).
<table>
<thead>
<tr>
<th>left/right track description</th>
<th>identical tracks</th>
<th>correlated tracks</th>
<th>uncorrelated tracks</th>
</tr>
</thead>
<tbody>
<tr>
<td>front/rear road description</td>
<td>identical</td>
<td>uncorrelated</td>
<td>delayed</td>
</tr>
<tr>
<td>variable</td>
<td>place</td>
<td>unit</td>
<td>RMS values</td>
</tr>
<tr>
<td>longitudinal acceleration</td>
<td>c.g.</td>
<td>m/s²</td>
<td>0.018 ± 9%</td>
</tr>
<tr>
<td>lateral acceleration</td>
<td>c.g.</td>
<td>m/s²</td>
<td>0.132 ± 3%</td>
</tr>
<tr>
<td>vertical acceleration</td>
<td>c.g.</td>
<td>m/s²</td>
<td>0.417 ± 30%</td>
</tr>
<tr>
<td>roll acceleration</td>
<td>c.g.</td>
<td>1/s²</td>
<td>0.655 ± 16%</td>
</tr>
<tr>
<td>pitch acceleration</td>
<td>c.g.</td>
<td>1/s²</td>
<td>0.401 ± 15%</td>
</tr>
<tr>
<td>yaw acceleration</td>
<td>c.g.</td>
<td>1/s²</td>
<td>0.074 ± 5%</td>
</tr>
<tr>
<td>dynamic tire load</td>
<td>1L</td>
<td>N</td>
<td>488 ± 23%</td>
</tr>
<tr>
<td>dynamic tire load</td>
<td>1R</td>
<td>N</td>
<td>492 ± 24%</td>
</tr>
<tr>
<td>dynamic tire load</td>
<td>2L</td>
<td>N</td>
<td>512 ± 18%</td>
</tr>
<tr>
<td>dynamic tire load</td>
<td>2R</td>
<td>N</td>
<td>527 ± 19%</td>
</tr>
<tr>
<td>suspension deflection</td>
<td>1L</td>
<td>mm</td>
<td>3.70 ± 12%</td>
</tr>
<tr>
<td>suspension deflection</td>
<td>1R</td>
<td>mm</td>
<td>3.70 ± 13%</td>
</tr>
<tr>
<td>suspension deflection</td>
<td>2L</td>
<td>mm</td>
<td>5.17 ± 20%</td>
</tr>
<tr>
<td>suspension deflection</td>
<td>2R</td>
<td>mm</td>
<td>5.21 ± 20%</td>
</tr>
<tr>
<td>longitudinal acceleration</td>
<td>f-seat</td>
<td>m/s²</td>
<td>0.056 ± 8%</td>
</tr>
<tr>
<td>lateral acceleration</td>
<td>f-seat</td>
<td>m/s²</td>
<td>0.166 ± 1%</td>
</tr>
<tr>
<td>vertical acceleration</td>
<td>f-seat</td>
<td>m/s²</td>
<td>0.469 ± 27%</td>
</tr>
<tr>
<td>vector sum</td>
<td>f-seat</td>
<td>m/s²</td>
<td>0.501 ± 25%</td>
</tr>
<tr>
<td>longitudinal acceleration</td>
<td>r-seat</td>
<td>m/s²</td>
<td>0.078 ± 8%</td>
</tr>
<tr>
<td>lateral acceleration</td>
<td>r-seat</td>
<td>m/s²</td>
<td>0.239 ± 2%</td>
</tr>
<tr>
<td>vertical acceleration</td>
<td>r-seat</td>
<td>m/s²</td>
<td>0.547 ± 29%</td>
</tr>
<tr>
<td>vector sum</td>
<td>r-seat</td>
<td>m/s²</td>
<td>0.602 ± 26%</td>
</tr>
</tbody>
</table>

Table 9.3 RMS performance values for $q_1 = 1$ ($v = 120$ km/h).
<table>
<thead>
<tr>
<th>variable</th>
<th>place</th>
<th>unit</th>
<th>identical</th>
<th>RMS values</th>
<th>identical</th>
<th>RMS values</th>
<th>identical</th>
<th>RMS values</th>
<th>identical</th>
<th>RMS values</th>
<th>identical</th>
<th>RMS values</th>
</tr>
</thead>
<tbody>
<tr>
<td>longitudinal acceleration</td>
<td>c.g.</td>
<td>m/s²</td>
<td>0.013</td>
<td>0.013</td>
<td>0.013</td>
<td>0.008</td>
<td>0.008</td>
<td>0.013</td>
<td>0.008</td>
<td>0.008</td>
<td>0.008</td>
<td>0.008</td>
</tr>
<tr>
<td>lateral acceleration</td>
<td>c.g.</td>
<td>m/s²</td>
<td>0.192</td>
<td>-0.50%</td>
<td>0.192</td>
<td>-0.50%</td>
<td>0.181</td>
<td>-0.42%</td>
<td>0.196</td>
<td>-0.58%</td>
<td>0.162</td>
<td>-0.27%</td>
</tr>
<tr>
<td>vertical acceleration</td>
<td>c.g.</td>
<td>m/s²</td>
<td>0.240</td>
<td>0.237</td>
<td>0.247</td>
<td>0.247</td>
<td>0.237</td>
<td>0.247</td>
<td>0.237</td>
<td>0.247</td>
<td>0.247</td>
<td>0.247</td>
</tr>
<tr>
<td>roll acceleration</td>
<td>c.g.</td>
<td>1/s²</td>
<td>0.362</td>
<td>0.362</td>
<td>0.360</td>
<td>0.360</td>
<td>0.343</td>
<td>0.394</td>
<td>0.335</td>
<td>0.394</td>
<td>0.394</td>
<td>0.394</td>
</tr>
<tr>
<td>pitch acceleration</td>
<td>c.g.</td>
<td>1/s²</td>
<td>0.290</td>
<td>0.284</td>
<td>0.175</td>
<td>0.175</td>
<td>0.284</td>
<td>0.175</td>
<td>0.284</td>
<td>0.175</td>
<td>0.284</td>
<td>0.175</td>
</tr>
<tr>
<td>yaw acceleration</td>
<td>c.g.</td>
<td>1/s²</td>
<td>0.123</td>
<td>-0.57%</td>
<td>0.123</td>
<td>-0.57%</td>
<td>0.125</td>
<td>-0.61%</td>
<td>0.130</td>
<td>-0.67%</td>
<td>0.106</td>
<td>-0.36%</td>
</tr>
<tr>
<td>dynamic tire load</td>
<td>1L</td>
<td>N</td>
<td>812</td>
<td>-0.10%</td>
<td>814</td>
<td>-0.10%</td>
<td>854</td>
<td>-115%</td>
<td>815</td>
<td>-108%</td>
<td>854</td>
<td>-115%</td>
</tr>
<tr>
<td>dynamic tire load</td>
<td>1R</td>
<td>N</td>
<td>843</td>
<td>-0.11%</td>
<td>844</td>
<td>-0.11%</td>
<td>874</td>
<td>-120%</td>
<td>845</td>
<td>-113%</td>
<td>875</td>
<td>-121%</td>
</tr>
<tr>
<td>dynamic tire load</td>
<td>2L</td>
<td>N</td>
<td>828</td>
<td>-0.91%</td>
<td>828</td>
<td>-0.91%</td>
<td>721</td>
<td>-66%</td>
<td>834</td>
<td>-92%</td>
<td>692</td>
<td>-60%</td>
</tr>
<tr>
<td>dynamic tire load</td>
<td>2R</td>
<td>N</td>
<td>879</td>
<td>-0.99%</td>
<td>879</td>
<td>-0.99%</td>
<td>765</td>
<td>-73%</td>
<td>886</td>
<td>-101%</td>
<td>734</td>
<td>-66%</td>
</tr>
<tr>
<td>suspension deflection</td>
<td>1L</td>
<td>mm</td>
<td>5.68</td>
<td>-0.34%</td>
<td>5.95</td>
<td>-0.41%</td>
<td>6.25</td>
<td>-48%</td>
<td>5.94</td>
<td>-40%</td>
<td>6.23</td>
<td>-47%</td>
</tr>
<tr>
<td>suspension deflection</td>
<td>1R</td>
<td>mm</td>
<td>5.80</td>
<td>-0.37%</td>
<td>6.08</td>
<td>-0.43%</td>
<td>6.34</td>
<td>-49%</td>
<td>6.07</td>
<td>-43%</td>
<td>6.32</td>
<td>-49%</td>
</tr>
<tr>
<td>suspension deflection</td>
<td>2L</td>
<td>mm</td>
<td>7.62</td>
<td>-0.76%</td>
<td>7.58</td>
<td>-0.75%</td>
<td>6.45</td>
<td>-49%</td>
<td>7.59</td>
<td>-76%</td>
<td>6.42</td>
<td>-48%</td>
</tr>
<tr>
<td>suspension deflection</td>
<td>2R</td>
<td>mm</td>
<td>7.80</td>
<td>-0.80%</td>
<td>7.77</td>
<td>-0.79%</td>
<td>6.60</td>
<td>-52%</td>
<td>7.77</td>
<td>-79%</td>
<td>6.57</td>
<td>-61%</td>
</tr>
<tr>
<td>longitudinal acceleration</td>
<td>f-seat</td>
<td>m/s²</td>
<td>0.047</td>
<td>0.046</td>
<td>0.046</td>
<td>0.034</td>
<td>0.047</td>
<td>0.034</td>
<td>0.047</td>
<td>0.034</td>
<td>0.034</td>
<td>0.034</td>
</tr>
<tr>
<td>lateral acceleration</td>
<td>f-seat</td>
<td>m/s²</td>
<td>0.203</td>
<td>0.203</td>
<td>0.203</td>
<td>0.194</td>
<td>0.207</td>
<td>0.24</td>
<td>0.207</td>
<td>0.24</td>
<td>0.207</td>
<td>0.24</td>
</tr>
<tr>
<td>vertical acceleration</td>
<td>f-seat</td>
<td>m/s²</td>
<td>0.270</td>
<td>0.267</td>
<td>0.278</td>
<td>0.278</td>
<td>0.264</td>
<td>0.283</td>
<td>0.264</td>
<td>0.283</td>
<td>0.264</td>
<td>0.283</td>
</tr>
<tr>
<td>vector sum</td>
<td>f-seat</td>
<td>m/s²</td>
<td>0.341</td>
<td>0.338</td>
<td>0.341</td>
<td>0.341</td>
<td>0.339</td>
<td>0.334</td>
<td>0.339</td>
<td>0.334</td>
<td>0.339</td>
<td>0.334</td>
</tr>
<tr>
<td>longitudinal acceleration</td>
<td>r-seat</td>
<td>m/s²</td>
<td>0.057</td>
<td>0.056</td>
<td>0.056</td>
<td>0.036</td>
<td>0.056</td>
<td>0.034</td>
<td>0.057</td>
<td>0.033</td>
<td>0.034</td>
<td>0.034</td>
</tr>
<tr>
<td>lateral acceleration</td>
<td>r-seat</td>
<td>m/s²</td>
<td>0.295</td>
<td>0.295</td>
<td>0.280</td>
<td>-14%</td>
<td>0.302</td>
<td>0.255</td>
<td>0.302</td>
<td>0.255</td>
<td>0.302</td>
<td>0.255</td>
</tr>
<tr>
<td>vertical acceleration</td>
<td>r-seat</td>
<td>m/s²</td>
<td>0.297</td>
<td>0.297</td>
<td>0.278</td>
<td>0.278</td>
<td>0.295</td>
<td>0.282</td>
<td>0.295</td>
<td>0.282</td>
<td>0.295</td>
<td>0.282</td>
</tr>
<tr>
<td>vector sum</td>
<td>r-seat</td>
<td>m/s²</td>
<td>0.423</td>
<td>0.423</td>
<td>0.396</td>
<td>0.52%</td>
<td>0.426</td>
<td>0.382</td>
<td>0.426</td>
<td>0.382</td>
<td>0.426</td>
<td>0.382</td>
</tr>
</tbody>
</table>

Table 9.4 RMS performance values for $q_1 = 10$ ($v = 120$ km/h).
Figure 9.1 RMS performance values as a function of weighting factor $q_1$.

- = active system, uncorrelated rear wheel input
--- = active system, delayed rear wheel input
----- = passive reference system
Figure 9.2 RMS Performance values as a function of weighting factor $q_1$.

- = active system, identical tracks
- - = active system, correlated tracks
- - - = active system, uncorrelated tracks
- - - - = passive reference system
Spectral Densities

One of the objections of RMS values is that the calculated numbers represent an average value. An RMS value doesn't contain any information in what frequency range the performance is improved, and in which range the performance degrades. An alternative way of representing the performance is to look at power spectral densities (PSD) or transfer functions (magnitude and phase relation between one output and one input). The PSDs are preferable to the transfer functions because the full-vehicle is a system with multi-inputs. Since the area beneath a power spectral density is equal to the mean square (MS) value, there exists a one to one relationship between RMS values and PSDs.

Figure 9.3a and 9.3b show the spectral densities of the variables of interest. The plots contain three lines: the solid one corresponds to an active suspension based on a cross-correlated road and uncorrelated rear axle excitation, the dashed line refers to the same active suspension, however in this case with a delayed rear wheel excitation assumption, and finally the dotted line corresponds to the standard passive car. The car is driving straight on with a speed of 120 km/h for all three cases. The situation chosen is the case with $q_1 = 10$. All plots illustrate that the major gain in ride comfort is obtained in the low frequency range from 0 up to 10 Hz. The tremendous gain in pitch accelerations owing to the built in wheelbase preview can be spotted very nicely. The degradation of the tire load variations as occurs with both active suspension systems is mainly concentrated around the wheel-hop natural frequency (12-13 Hz). The gain in tire load variations resulting from the built in wheelbase preview as could be seen from the RMS values is very difficult to distinguish owing to the logarithmic scales. Nevertheless, the peak value of the dynamic tire load is lower in the wheelbase preview case.
Figure 9.3a Power spectral densities \((v = 120 \text{ km/h}, q_1 =10)\).

- = active system without wheelbase preview
- - = active system with wheelbase preview
----- = passive system
Figure 9.3b Power spectral densities \(v = 120 \text{ km/h}, q_1 = 10\).

- = active system without wheelbase preview
- - = active system with wheelbase preview
- --- = passive system

Eigenvalues

The eigenvalues and natural frequencies of the 13 DOF vehicle model with an active suspension based on full-state feedback have been calculated for \(q_1 = 10\) and \(v = 120 \text{ km/h}\). The lateral and longitudinal dynamics have been disabled (\(c_{Fu} = 0\)). Table 9.5 presents the results. Three times an eigenvalue equal to zero occurs. Each of them corresponds to the longitudinal, lateral and yaw degree of freedom. The other modes are far more difficult to distinguish. Due to the full-state feedback, several modes of vibration are coupled. The three drive-train related eigenvalues (mode 1, 6 and 7) can be found quite easily. The sprung mass roll natural frequency (mode 8) is reduced from 2.36 to 0.88 Hz. This change is quite drastic and might affect the handling of the car (excessive roll during cornering). The natural frequency of the other two sprung mass modes (9, 10) are reduced in a similar fashion. This indicates that the suspension is very soft (low stiffness) but still well damped (only the vehicle body related modes). On the other hand, is the damping of the wheel-
hop modes (2, 3, 4, 5) rather poor. This is not so favorable since damping is necessary to reduce the tire load variations.

<table>
<thead>
<tr>
<th>mode</th>
<th>eigenvalues</th>
<th>natural frequency</th>
<th>damping ratio</th>
<th>damped frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>without lateral tire properties</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>-28.57 +/- 90.94 i</td>
<td>15.17</td>
<td>0.300</td>
<td>14.47</td>
</tr>
<tr>
<td>2</td>
<td>-10.82 +/- 84.50 i</td>
<td>13.56</td>
<td>0.127</td>
<td>13.45</td>
</tr>
<tr>
<td>3</td>
<td>-27.99 +/- 76.03 i</td>
<td>12.89</td>
<td>0.346</td>
<td>12.10</td>
</tr>
<tr>
<td>4</td>
<td>-23.98 +/- 81.31 i</td>
<td>13.49</td>
<td>0.283</td>
<td>12.94</td>
</tr>
<tr>
<td>5</td>
<td>-20.66 +/- 80.64 i</td>
<td>13.25</td>
<td>0.248</td>
<td>12.83</td>
</tr>
<tr>
<td>6</td>
<td>-22.94 +/- 79.65 i</td>
<td>13.19</td>
<td>0.277</td>
<td>12.68</td>
</tr>
<tr>
<td>7</td>
<td>-16.84 +/- 77.08 i</td>
<td>12.56</td>
<td>0.214</td>
<td>12.27</td>
</tr>
<tr>
<td>8</td>
<td>-2.67 +/- 14.82 i</td>
<td>2.40</td>
<td>0.177</td>
<td>2.36</td>
</tr>
<tr>
<td>9</td>
<td>-3.03 +/- 8.92 i</td>
<td>1.50</td>
<td>0.321</td>
<td>1.41</td>
</tr>
<tr>
<td>10</td>
<td>-2.49 +/- 7.44 i</td>
<td>1.25</td>
<td>0.317</td>
<td>1.18</td>
</tr>
<tr>
<td>11</td>
<td>0.00 +/- 0.00 i</td>
<td>0.00</td>
<td>0.000</td>
<td>0.00</td>
</tr>
<tr>
<td>12</td>
<td>0.00 +/- 0.00 i</td>
<td>0.00</td>
<td>0.000</td>
<td>0.00</td>
</tr>
<tr>
<td>13</td>
<td>0.00 +/- 0.00 i</td>
<td>0.00</td>
<td>0.000</td>
<td>0.00</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>mode</th>
<th>eigenvalues</th>
<th>natural frequency</th>
<th>damping ratio</th>
<th>damped frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>without lateral tire properties</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>-28.56 +/- 91.00 i</td>
<td>15.18</td>
<td>0.299</td>
<td>14.48</td>
</tr>
<tr>
<td>2</td>
<td>-7.34 +/- 76.41 i</td>
<td>12.22</td>
<td>0.096</td>
<td>12.16</td>
</tr>
<tr>
<td>3</td>
<td>-21.88 +/- 88.41 i</td>
<td>14.50</td>
<td>0.240</td>
<td>14.07</td>
</tr>
<tr>
<td>4</td>
<td>-3.87 +/- 80.93 i</td>
<td>12.90</td>
<td>0.048</td>
<td>12.88</td>
</tr>
<tr>
<td>5</td>
<td>-4.09 +/- 80.16 i</td>
<td>12.77</td>
<td>0.051</td>
<td>12.76</td>
</tr>
<tr>
<td>6</td>
<td>-27.11 +/- 82.28 i</td>
<td>13.79</td>
<td>0.313</td>
<td>13.10</td>
</tr>
<tr>
<td>7</td>
<td>-18.41 +/- 77.08 i</td>
<td>12.61</td>
<td>0.232</td>
<td>12.27</td>
</tr>
<tr>
<td>8</td>
<td>-4.69 +/- 5.54 i</td>
<td>1.16</td>
<td>0.646</td>
<td>0.88</td>
</tr>
<tr>
<td>9</td>
<td>-4.58 +/- 5.80 i</td>
<td>1.18</td>
<td>0.620</td>
<td>0.92</td>
</tr>
<tr>
<td>10</td>
<td>-3.31 +/- 2.37 i</td>
<td>0.65</td>
<td>0.814</td>
<td>0.38</td>
</tr>
<tr>
<td>11</td>
<td>0.00 +/- 0.00 i</td>
<td>0.00</td>
<td>0.000</td>
<td>0.00</td>
</tr>
<tr>
<td>12</td>
<td>0.00 +/- 0.00 i</td>
<td>0.00</td>
<td>0.000</td>
<td>0.00</td>
</tr>
<tr>
<td>13</td>
<td>0.00 +/- 0.00 i</td>
<td>0.00</td>
<td>0.000</td>
<td>0.00</td>
</tr>
</tbody>
</table>

**Table 9.5** Natural frequencies of the active suspension system based on full-state feedback ($v = 120$ km/h).
Speed Influence

The influence of the speed of travel starts to play a role when a delayed rear wheel input is assumed in the design phase. Since the rear wheel inputs are simply a delayed version of the front wheel inputs, it is likely that the active suspension performance depends on the amount of wheelbase preview information. For the case of the cross-correlated road tracks, several controllers have been calculated at varying speeds (10 - 200 km/h). These calculations have been carried out for $q_1 = 10$. This means that ride comfort is emphasized. As already explained before, the wheelbase time delay has been approximated using a Padé filter. Unfortunately, the order $(n_p)$ of the Padé approximation couldn’t be adjusted for the speed of travel. According to table 7.4 it is necessary to adjust the order of the filter at a fixed admissible relative phase error. However, the LQG software used didn’t allow for a better approximation than a fourth-order. For larger values of $n_p$, no stabilizing solution could be found. This implies that high frequency excitations are not delayed properly at low speeds. The influence of the speed of travel on the ride performance is illustrated in figure 9.4.

The figures present a gain in performance with respect to the uncorrelated rear wheel input situation. A positive value indicates that the variable concerned is improved. The variables of main interest are of course those situated at the rear side of the car: rear tire load variations and rear suspension working space. Both variables take advantage of the wheelbase preview information. The improvement of the longitudinal as well as the pitch accelerations is considerable for speeds larger than about 90 km/h. The maximum gain in pitch accelerations is close to 40% for a speed equal to 120 km/h. At this particular speed the wheelbase time delay amounts to 75 ms. This corresponds to a frequency of 13.3 Hz and is very close to the natural frequencies of the wheel-hop modes. It is imaginable that the interaction between all four actuators in combination with the coinciding frequencies is responsible for the large gain in pitch angular accelerations at this particular speed. On the other hand the vertical and roll accelerations of the sprung mass deteriorate slightly in the same range of speeds. The level of vibrations at the rear seat can be improved by 10% if the delayed rear wheel input is included in the control design phase. This gain is mainly caused by the improvement in pitch angular accelerations. The largest gains can be found for the rear tire load variations and rear suspension working space. From 70 km/h the wheelbase preview effect reduces the tire load variations up to about 20%.
Figure 9.4 Extra gain in RMS performance values through wheelbase time-delay as a function of the forward speed ($q_1 = 10$).
Chapter 9 Active Suspension Design using Linear Optimal Control

The low speed behavior is quite strange, especially around 65 km/h. At this speed all variables located at the rear side of the car deteriorate a little. Remarkably, a slight improvement in tire load variations and suspension working space at the front side of the car can be observed. However, note that the wheelbase time delay approximation couldn't be described well for low speeds. For reasons of stability, the order of the Padé filter was restricted to 4. However, a higher order is preferable for low velocities. At higher speeds the benefit of the wheelbase preview diminishes somewhat. At 200 km/h the tire load variations at the rear suspension benefit the most from the wheelbase preview.

Handling Test

Similar to the handling analysis of the standard passive car, both double lane-change test and frequency response functions will be calculated in order to gain more insight into handling of the actively suspended vehicle.

Double lane-change

The results of the lane-change time domain simulation are plotted in figure 9.5a and 9.5b. All conditions with exception of the active suspension control correspond to the standard passive car. The results are valid for $q_1 = 10$ (emphasizing ride comfort). As the feedback gains are typically speed dependent, the control feedback structure has been recalculated for $v = 80$ km/h. In the control design phase as well as in the time-domain simulation, cross-correlated road tracks and a delayed rear wheel road input have been assumed.

From the position of the vehicle it can be seen that the maneuverability is not so good compared to the standard passive car. A serious part of the track has been cut off. These findings agree with the course of the steering angle as a function of time. For the sake of clearness, also the steering angle of the standard car has been plotted in the same figure (dashed line). The rather poor response can also be seen from the lateral acceleration. Again the curve of the standard passive vehicle has been plotted in the same figure (dashed line). The ripples nearby the peak values are quite serious. The course of the roll angle gives an explanation of the above findings. This angle is almost 3 times larger than the roll angle of the standard car (dashed line). The reason for this undesired behavior must be the active suspension. Since the active suspension control algorithm is based on a full-state feedback structure,
position related states are fed back too. The weighting factors were chosen to improve ride comfort. A good ride comfort is generally obtained from a soft suspension (low stiffness and damping rates). Therefore, the large roll angle can only be explained from a reduction of the overall roll stiffness by the LQG algorithm. This is in agreement with the findings of the natural frequency analysis (table 9.5). The reduction of the roll stiffness is very unfavorable for handling.

The tire force plots are also very interesting. From the RMS results we know that the gain in ride comfort is accompanied by an increase in tire load variations. The increase in the RMS values of the vertical tire load variations is 40\% for the front axle and 45\% for the rear axle compared with the standard car. If we compare figure 8.7a and 9.5a, clearly the increase in vertical tire load variations can be seen. Furthermore, it strikes that the load transfer is not equally distributed between the front and rear axle. The contribution of the rear suspension is minimal, indicating a very low roll stiffness of the rear suspension. The large variation of the front tire load is not favorable for handling. Since the average cornering stiffness decreases considerably, this vehicle can be typified as very understeered. The sum of the tire side forces at both front wheels is smaller in the active suspension case. Consequently, the cornering power is reduced. Sometimes the vertical tire load (and thus also the side force) becomes zero. Despite the increase in vertical tire load variations due to the road irregularities, the side force variations are only slightly worse compared to the standard vehicle. The variations are only larger in those cases were the vertical tire load decreases drastically due to the load transfer. The tires with the largest load are particularly insensitive to load variations caused by road unevennesses. The yaw rate and body slip angle differ only marginally from the standard car (dashed lines). Only at t = 4 s the values are large due to the loss of side force. Up to now, only negative aspects of the active suspension configuration have been discussed. The real improvement can be observed from the vertical accelerations of the sprung mass at the front and rear seat positions. The reduction in RMS values (62\% for the front seat and 58\% for the rear seat) is impressive. The portion from the roll acceleration is obvious. The ride comfort gain is attended with worse handling and an unacceptably low roll stiffness. The reduction of the handling capabilities is only partially owing to the increase of vertical tire load variations. The reduced roll stiffness and unbalanced load transfer are to be blamed too.
Chapter 9 Active Suspension Design using Linear Optimal Control

The required actuator forces to obtain the gain in ride comfort is immense. Peaks of more than 5000 N occur. Actually, these large values are not necessary for the ride improvement while driving straight on. They arise from the reduced roll stiffness. As the vehicle corners, the body is additionally excited by lateral acceleration forces. In the LQG design phase we didn't reckon with this kind of excitation, because the number of degrees of freedom had been reduced from 13 to 7. During cornering the reduction of the roll stiffness by means of actively generating counteracting forces requires much power. Not only the suspension springs, but also the roll stabilizers (largest contribution to the roll stiffness) must be conquered. Finally, the suspension working space is far too large. In reality the bump-stops (non-linear spring characteristic) would have restricted the suspension deflection considerably.

From the lane-change result the conclusion can be drawn that LQG in combination with full-state feedback must be handled with care. A correct solution based on an in-plane vehicle model might become unusable if it is applied in a real vehicle due to the omission of the out-of-plane degrees of freedom. One possible solution is to weight for example the roll angle in the quadratic cost function. This implies that it is possible to attach importance to the roll stiffness depending on the corresponding weighting factor. The distribution of the roll stiffness between front and rear suspension must be reckoned with. The same problems might appear with the pitch angle during braking and accelerating. All in all the LQG design becomes rather complicated to deal with.

Another possibility is to design separate anti-roll/pitch controllers that prevent the vehicle from rolling during cornering and braking [40]. These kinds of controllers may not react on road unevennesses because this would stiffen the suspension with respect to ride comfort. The third possibility is to use the standard LQG algorithm for the 'ride' model in combination with limited state feedback. The feedback structure may not contain position related states. This guarantees at least a fixed steady-state behavior of the car. Section 9.2 will deal with this solution.
Figure 9.5a Double lane change, active suspension, full-state feedback (80 km/h).
Figure 9.5b Double lane change, active suspension, full-state feedback (80 km/h).
Frequency Response Functions

The frequency response functions (FRFs) were calculated in a similar fashion as with the standard passive vehicle using the linear state-space model description. Figures 9.6a up to 9.6c present the results. They are valid for \( q_1 = 10 \). Since the feedback gains are typically speed dependent, for every speed of travel the LQG problem has been solved before the transfer functions were calculated. Besides the actively suspended vehicle (solid lines) also the FRFs of the standard car have been plotted in the figures (dashed lines). The lateral acceleration FRF as well as the yaw rate FRF show a considerable decrease in the gain response. Although the steady-state gains are identical to the standard car, they drop sooner. This means that the vehicle with active suspension based on full-state feedback is not so responsive. It is likely that the reduced roll stiffness is the main reason for this behavior. The yaw rate response is damped very well, even at higher velocities. From the roll angle FRF it is can be seen that the low frequency gains have increased tremendously. These results correspond with the findings obtained from the double lane-change test and the eigenvalue analysis.

![lateral acceleration / steering angle](image)

**Figure 9.6a** Lateral acceleration frequency response function.
Figure 9.6b & c Roll angle and yaw rate frequency response function.
9.2 OUTPUT FEEDBACK CONTROL

The full-state feedback control as discussed in the previous sections has one major disadvantage: all states of the vehicle system must be measurable or reconstructable. Many states of the 7 DOF model are absolute (and therefore almost impossible to measure) states. Furthermore, the augmented road model (which is necessary to be able to weight the dynamic tire load) involves measuring the road irregularities. Depending on the extensiveness of the road model, it might be necessary to measure the road elevation at each wheel. If a Padé filter is included within the road model, the same filter must be implemented in the real car since the controller contains feedback gains that correspond to the states of the Padé filter. All in all the full-state feedback controller is far from ready to be implemented in a car. One possibility to overcome the difficulties of measuring some state variables is the application of a state estimator. With the help of more easily accessible measurement signals it is possible to estimate the states of the system using an internal model of the car. Although the application of a state estimator is a step towards practice, a reduction of the number of feedback gains is preferable. From the quarter-car analysis it could be seen that the performance of an active suspension system with limited state feedback control is only slightly worse compared with full-state feedback when a sensible choice for the states with feedback gains has been made. Therefore, this chapter will deal with optimal control using limited state feedback [40].

The limited state feedback as discussed in this chapter is based on output feedback. As the name already suggests, output feedback uses the outputs of a system rather than all the states as with state feedback control. The control effort is given by

\[ u = K_c y = K_c C x \]  \hspace{1cm} (9.22)

where \( K_c \) represents a matrix with feedback gains and vector \( y \) contains the outputs. Those outputs are a selection of the states or a combination of these. This selection can be made by output matrix \( C \) of the vehicle. The control system designer is free to use any combination of states. The closed-loop system becomes

\[ \dot{x} = (A + BK_c C)x + Gw \]  \hspace{1cm} (9.23)

The theory of finding the output feedback gains has already been discussed in the section dealing with the quarter-car model and will not be repeated here.
This section has been split up in four sections. Each section deals with a specific structure of the output feedback. The following four structures will be examined:

- Active damping: feedback of all velocity related states.
- Partially coupled active damping: partially coupled feedback of all velocity related states.
- Decoupled active damping: decoupled feedback of all velocity related states.
- Passive damping: decoupled feedback of all four suspension deflection velocities.

The choice of using only velocity related states is based on the fact that damping, whether active or passive, is the keyword in suspension tuning. Looking at spectral density plots for example, it can be roughly stated that stiffness variations influence the location of the natural frequencies (horizontal position on frequency axis) and damping fixes the height of the spectral density (vertical position). Minimizing the variance of a variable (e.g. tire load variations) means minimizing the area beneath the spectral density. In practical situations this may imply optimizing the damping. The distinction between active and passive damping is quite straightforward. Passive damping is attended with dissipation of energy only, active damping involves also energy supply. The term decoupled needs some more explanation. Decoupled is related to the absence of interaction among the four suspension actuators. In case of passive damping, the term decoupled denotes that each actuator is able to generate a force proportional to the relative velocity of the corresponding wheel suspension. For example, the actuator in the left rear suspension generates a force that is proportional to the relative velocity of the left rear suspension only (actuator = damper). In case of active damping it is imaginable that the left rear actuator is not only able to generate a force with respect to the relative velocity of the left rear actuator, but also the deflection velocity of the left front suspension might be considered as an input to the controller. This is of course one of the many possible combinations. In case the term decoupled is used, one can imagine to cut the car in four sections (four quarter-cars). The actuator in each part does not know anything about the other three parts. Of course, a car cannot be considered fully as a decoupled system. Some interaction due to roll and pitch cannot be omitted. More details about the feedback structures can be found in the oncoming sections.

Similar to the full-state feedback controller, the limited state feedback controller will be designed for the 7 DOF vehicle model and evaluated on the 13
DOF vehicle model. Actually, the full-state feedback as discussed in the previous sections is in fact also a kind of limited state feedback. Several LQG controllers have been designed for a 7 DOF vehicle model and implemented in a 13 DOF model. The 14 states of the 7 DOF vehicle model can be regarded as a selection of the 26 states of the 13 DOF model. It would have been possible to use output feedback in combination with the 13 DOF model from the beginning, and just eliminate the undesired 12 states (engine, etc.) in the output equation. This might have worked well, but from the theory of output feedback we know that the feedback gains are found using an iterative algorithm. Except that output feedback requires much longer calculation time, good initial stabilizing feedback gains are also necessary. Furthermore, more minimizing solutions may exist in the limited state feedback case. Therefore, the state feedback analysis as well as the output feedback analysis are conducted on the simple 7 DOF model.

The road model description used in this section is based on the cross-correlated track model as discussed in chapter 7. Both uncorrelated and delayed rear wheel excitation will be discussed. Similar to the full-state feedback analysis, the wheelbase time delay has been approximated by a Padé filter of 4th-order. The quadratic criterion as well as the weighting factors are identical to the full-state feedback analysis. The design speed is 120 km/h.

9.2.1 Active Damping

The active damping case is based on feedback of velocities. The 7 DOF vehicle model contains 14 states; 7 of them are velocities. They are

\[
\begin{bmatrix}
\dot{z}_b & \phi_b & \dot{\theta}_b & \phi_{a1L} & \phi_{a1R} & \dot{z}_{a2} & \phi_{a2}
\end{bmatrix}
\]

It is quite straightforward to compose an output matrix \( C \) which together with state vector \( X \) makes these 7 outputs. For reasons of understanding, a modified version is preferred to the above solution. The following outputs are selected

\[
y^T = [\dot{d}_{s1L} \quad \dot{d}_{s1R} \quad \dot{d}_{s2L} \quad \dot{d}_{s2R} \quad v_{b1L} \quad v_{b1R} \quad v_{b2L} \quad v_{b2R}]
\]

where \( \dot{d}_s \) denotes the deflection velocity of the appropriate wheel suspension and \( v_b \) the absolute velocity of the sprung mass at the location of the upper damper fastening. Since in the model the main body has only three degrees of freedom (heave, roll and pitch), one output \( v_{b1} \) is superfluous. The number of outputs is however not restricted by any rule. Therefore, all four velocity
outputs will be used. According to appendix A the relative and absolute velocities take the following form
\[
\dot{s}_{1L} = \left( \frac{x_0}{l_0} \left( h_{p1} - h_{s1} \right) - \frac{z_0}{l_0} \left( w_{p1} - w_{s1} \right) \right) \dot{\theta}_b + \left( \frac{x_0}{l_0} l_{s_a} - \frac{z_0}{l_0} l_{c_a} \right) \dot{\phi}_{a1L} \tag{9.24a}
\]
\[
\dot{s}_{1R} = -\left( \frac{x_0}{l_0} \left( h_{p1} - h_{s1} \right) - \frac{z_0}{l_0} \left( w_{p1} - w_{s1} \right) \right) \dot{\theta}_b - \left( \frac{x_0}{l_0} l_{s_a} - \frac{z_0}{l_0} l_{c_a} \right) \dot{\phi}_{a1R} \tag{9.24b}
\]
\[
\dot{s}_{2L} = \dot{z}_{a2} - w_{s2} \dot{\theta}_{a2} - \dot{z}_b + l_{a2} \dot{\theta}_b + w_{s2} \dot{\phi}_b \tag{9.24c}
\]
\[
\dot{s}_{2R} = \dot{z}_{a2} + w_{s2} \dot{\theta}_{a2} - \dot{z}_b + l_{a2} \dot{\theta}_b - w_{s2} \dot{\phi}_b \tag{9.24d}
\]
\[
v_{b1L} = \dot{z}_b - l_{a1} \dot{\theta}_b - w_{s1} \dot{\phi}_b \tag{9.24e}
\]
\[
v_{b1R} = \dot{z}_b - l_{a1} \dot{\theta}_b + w_{s1} \dot{\phi}_b \tag{9.24f}
\]
\[
v_{b2L} = \dot{z}_b - l_{a2} \dot{\theta}_b - w_{s2} \dot{\phi}_b \tag{9.24g}
\]
\[
v_{b2R} = \dot{z}_b - l_{a2} \dot{\theta}_b + w_{s2} \dot{\phi}_b \tag{9.24h}
\]

From equations (9.24) it is easy to compose the output matrix \( C \) according to equation (9.22) where \( K_\omega \) is a matrix with \( 4 \times 8 \) gains. The output feedback gains have been calculated for a variety of weighting factors. The values of \( q_3 \) (tire load variations), \( q_3 \) (suspension working space) and \( q_4 \) (actuator force) remain constant, while \( q_1 \) has been varied from 0.01 to 100. The calculations were performed for both uncorrelated and delayed rear wheel excitations. Figure 9.7 shows the results. The solid lines correspond to the situation of uncorrelated rear wheel input, the dashed lines refer to the delayed rear wheel input and the dotted lines stand for the standard passive car.

With increasing \( q_1 \) ride comfort can be improved at the cost of tire load variations and a little more suspension working space. Most surprising is the effect of the wheelbase time delay. Although there is neither a feedback on the road elevation related variables nor a feedback on Padé filter related states, the effect of wheelbase preview is still present. This might look very strange, but in fact it is not. The actuators in the rear suspension are fed back not only with state variables of the rear side of the car, but also the suspension deflection velocity and the absolute sprung mass velocity at the front side of the car play a role. If the front side of the car crosses a bump, the front suspension tries to isolate the vehicle body from the disturbance. If a motion in the front suspension is initiated, the two actuators in the rear suspension know already from the four outputs at the front side of the car that a bump is approaching. This mechanism enables these two actuators to retract the rear suspension at the moment the bump crosses the rear suspension.
Figure 9.7 RMS performance values as a function of weighting factor $q_1$.

--- = active damping, uncorrelated rear wheel input
- - - = active damping, delayed rear wheel input
       = passive reference system
Chapter 9 Active Suspension Design using Linear Optimal Control

9.2.2 Partially Coupled Active Damping

The partially coupled version of the active damping looks very similar to the structure as discussed in section 9.2.1. The same outputs are used

\[ \dot{y}^T = \begin{bmatrix} \dot{d}_{s1L} & \dot{d}_{s1R} & \dot{d}_{s2L} & \dot{d}_{s2R} & v_{b1L} & v_{b1R} & v_{b2L} & v_{b2R} \end{bmatrix} \]

However, the feedback structure has been modified to

\[
\mathbf{u} = \mathbf{K}_c \mathbf{y} = \begin{bmatrix} u_{1L} \\ u_{1R} \\ u_{2L} \\ u_{2R} \end{bmatrix} = \begin{bmatrix} -k_{s1} & 0 & 0 & -k_{a1} & 0 & 0 & 0 \\ 0 & -k_{s1} & 0 & 0 & 0 & -k_{a1} & 0 \\ -k_{s2} & 0 & -k_{s2} & 0 & -k_{a2} & 0 \\ 0 & -k_{s2} & 0 & -k_{s2} & 0 & -k_{a2} & 0 \end{bmatrix} \begin{bmatrix} \dot{d}_{s1L} \\ \dot{d}_{s1R} \\ \dot{d}_{s2L} \\ \dot{d}_{s2R} \\ v_{b1L} \\ v_{b1R} \\ v_{b2L} \\ v_{b2R} \end{bmatrix}
\]

(9.25)

As can be seen, feedback matrix \( \mathbf{K}_c \) has a partially coupled structure. The software used to solve the output feedback problem contains the possibility to prescribe some terms of the feedback matrix with a value equal to zero. The actuators in the front suspension are controlled completely without interaction. The rear suspension actuators are not only controlled with the rear suspension related variables, but also the relative and absolute velocities at the front suspension are integrated within the feedback structure. This structure looks very similar to the active damping of a quarter-car. In both cases the relative velocity as well as the absolute sprung mass velocity are used. Gains \( k_{s1} \) and \( k_{s2} \) correspond to passive damper constants for the front and rear suspension respectively and gains \( k_{a1} \) and \( k_{a2} \) are active 'skyhook' damping constants. The coupling for the rear suspension actuators with the front suspension related variables has been included in the feedback structure in order gain profit of the wheelbase preview effect as found with the full-state feedback and active damping case. The two coupling gains are \( k_{cs2} \) and \( k_{ca2} \).

The performances of the above system has been calculated for severe values of weighting factor \( q_1 \) (ride comfort). Figure 9.8 illustrates the results. The effect of the feedback coupling of the rear actuators can be spotted very nicely. The gain in tire load variations and suspension deflection in the rear suspension is considerable. Up to \( q_1 \) equal to 2, ride comfort can be improved substantially.
without any loss in tire load variations and suspension deflection at the rear axle, and with only a minor increase if both variables at the front suspension. Also the improvement in pitch angular accelerations is impressive in the delayed rear wheel case. For small values of $q_1$ (0.01) an improvement of 25% in tire load variations and 45% in suspension deflection at the rear side can be obtained with the combination of coupling and delayed rear wheel input. These gains are with respect to the standard passive car. The same gains valid for the uncorrelated rear wheel case are: 7% reduction of rear tire load variations and 24% reduction of rear suspension working space. The influence of the coupling is also at low values of $q_1$ substantial. Compared with the fully coupled version of active damping (section 9.2.1) the conclusion can be drawn that both systems perform equally well. However, the partially coupled version contains only 6 feedback gains against 16 for the fully coupled version. These six feedback gains as a function of weighting factor $q_1$ are shown in figure 9.8b.

With exception of the two coupling gains ($k_{cu2}$, $k_{ca2}$) there is not much difference in the uncorrelated and delayed rear wheel excitation assumption in the design phase. For small values of $q_1$ (emphasizing road holding) it can be seen that from the extra information of the two outputs at the front suspension it is mainly the relative velocity that is used. Coupling gain $k_{ca2}$ is almost as large as the rear suspension passive damping constant $k_{s2}$ and the absolute velocity of the sprung mass at the front end of the car does not play any role (gain $k_{ca2} = 0$). However, for large values of $q_1$ (emphasizing ride comfort) the most important coupling information is the absolute velocity at the front side of the car.
Figure 9.8a RMS performance values as a function of weighting factor $q_1$.

--- partially coupled active damping, uncorr. rear wheel input
- - - partially coupled active damping, delayed rear wheel input
----- passive reference system
Figure 9.8b Feedback gains as a function of weighting factor $q_1$.
- - = partially coupled active damping, uncorr. rear wheel input
--- = partially coupled active damping, delayed rear wheel input
----- = passive reference system

9.2.3 Decoupled Active Damping

The structure for the decoupled version of the active damping looks very similar to the one discussed in section 9.2.2. The same outputs are used. However, the feedback structure has been modified to
\[
\mathbf{u} = \mathbf{K}_c \mathbf{y} = \begin{bmatrix} u_{1L} \\ u_{1R} \\ u_{2L} \\ u_{2R} \end{bmatrix} = \begin{bmatrix} -k_{s1} & 0 & 0 & 0 & -k_{a1} & 0 & 0 & 0 \\ 0 & -k_{s1} & 0 & 0 & 0 & -k_{a1} & 0 & 0 \\ 0 & 0 & -k_{s2} & 0 & 0 & 0 & -k_{a2} & 0 \\ 0 & 0 & 0 & -k_{s2} & 0 & 0 & 0 & -k_{a2} \end{bmatrix} \begin{bmatrix} \dot{d}_{s1L} \\ \dot{d}_{s1R} \\ \dot{d}_{s2L} \\ \dot{d}_{s2R} \\ v_{b1L} \\ v_{b1R} \\ v_{b2L} \\ v_{b2R} \end{bmatrix}.
\]

(9.26)

As can be seen, feedback matrix \( \mathbf{K}_c \) has a decoupled structure. This structure is similar to the active damping of a quarter-car. In both cases the relative velocity as well as the absolute sprung mass velocity are used. Gains \( k_{s1} \) and \( k_{s2} \) correspond to passive damper constants of the front and rear suspensions respectively and gains \( k_{a1} \) and \( k_{a2} \) are active damping constants. They are better known under the name 'skyhook' damping constants.

The decoupled version has been studied for two reasons:

1. What is the effect of the omission of the coupling and thus the introduction of a delayed input?,
2. Can a quarter-car based solution compete with a solution based on a full-vehicle model?

It is expected that the introduction of a delayed input is of no influence on the decoupled active suspension system. Figure 9.9a confirms this. Both systems based on uncorrelated and delayed rear wheel input respectively show an almost equal performance. These results do not only illustrate that decoupling eliminates the wheelbase preview effect, but they also show that in this case the full-vehicle can be approximated as four quarter-cars i.e. the interaction among all four wheel suspensions is small. The best ride comfort at both seat positions can be found for \( q_1 = 10 \). As \( q_1 \) grows, the tire load variations increase and therefore the sprung mass accelerations in the 10-14 Hz zone increase as well. Together with the tire load variations, the suspension travel increase as \( q_1 \) gets larger. The rate of increase is significantly larger that with the coupled version of active damping. For \( q_1 \) equal to 100 the tire load variations at the rear wheels are unacceptably large.

Figure 9.9b shows the gains of the feedback structure. Not completely unexpected, the skyhook principle makes its appearance again. With
increasing $q_1$, the passive damping constants $k_{a1}$ and $k_{a2}$ decrease and the gains related to the absolute velocity of the vehicle body, $k_{a1}$ and $k_{a2}$ (active damping constants or skyhook damping constants) increase. The increase of tire load variations and suspension travel can be explained by the decreasing passive damping. For a large value of $q_1$, the actuator forces are almost proportional to the sprung mass velocity at each wheel suspension. In this case, all motions of the unsprung masses are damped very poorly since the controller does not respond to unsprung mass related state variables.

While comparing the results obtained with the coupled and decoupled active control algorithms (cf. figure 9.8a), it strikes that the difference in performance is quite small. For a given value of weighting factor $q_1$ the gain in ride comfort is larger with the decoupled version. Consequently, the degradation of tire load variations and suspension deflection are larger too. However, the same results can be obtained for the coupled version with a larger value of $q_1$.

![Graphs showing RMS performance values as a function of weighting factor $q_1$.](image)

**Figure 9.9a** RMS performance values as a function of weighting factor $q_1$.

--- decoupled active damping, uncorrelated rear wheel input

- - decoupled active damping, delayed rear wheel input

----- passive reference system
Figure 9.9b RMS values and feedback gains as a function of factor $q_1$.

- = decoupled active damping, uncorrelated rear wheel input
- - = decoupled active damping, delayed rear wheel input
- - - = passive reference system
9.2.4 Passive Damping

The output feedback structure is also suitable for optimizing passive suspension systems. With an appropriate selection of the output variables it is, for example, quite easy to optimize the suspension damping. Only four output variables are necessary. Each of them represents the deflection velocity \( \ddot{d}_s \) of the corresponding wheel suspension

\[
\dot{y}^T = [\ddot{d}_{s1L}, \ddot{d}_{s1R}, \ddot{d}_{s2L}, \ddot{d}_{s2R}]
\]

The actuator (damper) forces \( u \) are related to the outputs \( y \) by

\[
u = K_{c,y} \begin{bmatrix} u_{1L} \\ u_{1R} \\ u_{2L} \\ u_{2R} \end{bmatrix} = \begin{bmatrix} -k_{s1} & 0 & 0 & 0 \\ 0 & -k_{s1} & 0 & 0 \\ 0 & 0 & -k_{s2} & 0 \\ 0 & 0 & 0 & -k_{s2} \end{bmatrix} \begin{bmatrix} \ddot{d}_{s1L} \\ \ddot{d}_{s1R} \\ \ddot{d}_{s2L} \\ \ddot{d}_{s2R} \end{bmatrix}
\]

As can be seen, feedback matrix \( K_c \) has a decoupled structure. The gains \( k_{s1} \) and \( k_{s2} \) represent the front and rear suspension damping constants. Since the 7 DOF vehicle model is fully symmetrical, the left and right suspension damping constants are identical.

Figure 9.10 shows the calculated results. Most striking is the rather disappointing effect of the variation of the passive damping constant according to figure 9.10b. Sprung mass accelerations, tire load variations and suspension working space do not vary a lot as the damping is increased or decreased. Very unusual is the course of the damping constant as a function of weighting factor \( q_1 \) for the delayed rear wheel assumption in the control design phase. Against all expectations the damping constant of the front suspension increases as ride comfort is emphasized. This contradicts the uncorrelated rear wheel input case completely. Most surprisingly, the ride comfort at both seat positions improves slightly with this rather strange configuration. The two upper plots show that the gain is merely a result from the improved vertical accelerations at the c.g. Both roll and pitch accelerations deteriorate significantly at large values of \( q_1 \). At the design speed (120 km/h) the optimal damper setting for ride comfort is approximately half of that of the standard passive car. For minimal tire load variations, the front suspension of the passive reference car is damped correctly. The rear dampers could be firmer. However, the resulting decrease of tire load variations is only very small. According to figures 9.8 and 9.9 the actively damped car still shows a considerable improvement with respect to the optimally damped passive vehicle of figure 9.10.
Figure 9.10a RMS Performance values as a function of weighting factor $q_1$.

- = passive system, uncorrelated rear wheel input

- - = passive system, delayed rear wheel input

- - - = passive reference system
Figure 9.10b Passive damping constant as a function of weighting factor $q_1$.
- = passive damping, uncorrelated rear wheel input
-- = passive damping, delayed rear wheel input
.... = passive reference system

9.2.5 Evaluation of the Active Vehicle Ride and Handling

This section deals with the comparison of all active suspension systems based on linear optimal control with limited state feedback. Furthermore, the performance of the full-state feedback system is compared with that of the output feedback case.

RMS values

Table 9.6 shows a comparison in terms of RMS values for three systems based on limited state feedback for $q_1$ equal to 10. The optimum passive damping has not been included in the table. The extra improvement obtained from the time delay inclusion in combination with a coupled structure in the design phase is obvious. However, compared with fully coupled versions there is no real advantage to include all interactions. The performance of the partially coupled system can compete with the fully coupled system. Furthermore, the calculation effort for solving the output feedback problem for the coupled version is far more difficult due to the number of gains (16 for the coupled version against 6 for the partially coupled version).

Comparing the results from the full-state feedback analysis with the limited state feedback analysis tells us that there is a small difference in performance. Figure 9.11 shows three systems: the first system is based on full-state feedback with a cross-correlated track and a delayed rear wheel assumption (dashed lines), the second system concerns the limited state feedback active suspension without any interaction (solid lines) and the third system is an active suspension based on partially coupled active damping; all
with delayed rear wheel road input. At the same level of discomfort at both seat positions, the system based on full-state feedback performs better with respect to tire load variations and suspension working space. The possible ride comfort improvement at the rear seat position for the full-state feedback is unequalled. The main reason for this improvement is the possibility to soften the passive springs with the application of full-state feedback. Since the control law includes displacement related variables, the passive springs can be softened or eliminated completely by means of an actuator that acts as a spring with a negative stiffness. The softening, however, is not always desired. It might reduce the stiffness of the suspension so much that the handling of the car gets into danger during cornering and braking (excessive roll and pitch).

The advantage of the built in wheelbase preview can be observed very nicely in the pitch angular accelerations of both systems with full/partial interaction. Up to \( q_1 \approx 2 \), the tire load variations as well as the suspension working space are not deteriorated for both preview systems compared with the passive reference system. The optimal value for ride comfort (at the seat positions) is located somewhere between \( q_1 = 10 \) and 20 for all three systems.

From the point of view of calculation effort and ease of implementation the systems based on partial state feedback are preferable. The difference in performance is rather small. Since coupling of the actuators and in particular the coupling between rear and front actuators can improve the performance at the rear suspension considerably, it is preferable to include the wheelbase time delay in the design phase and couple the feedback law. Therefore, the partially coupled active damping (full-vehicle model solution) is preferred to the decoupled active damping (quarter-car model solution).
<table>
<thead>
<tr>
<th>suspension type</th>
<th>coupled active damping</th>
<th>partial coupl. act. damping</th>
<th>decoupled active damping</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>uncorrelated</td>
<td>delayed</td>
<td>uncorrelated</td>
</tr>
<tr>
<td></td>
<td>RMS values</td>
<td></td>
<td>RMS values</td>
</tr>
<tr>
<td></td>
<td>c.g.</td>
<td>m/s²</td>
<td>0.012 ±40%</td>
</tr>
<tr>
<td>front/rear road description</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>unit</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>place</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>RMS values</td>
<td></td>
</tr>
<tr>
<td>front/rear road description</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>variable</td>
<td>RMS values</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>RMS values</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Figure 9.11 Comparison of three active suspension systems.

--- = active system, decoupled active damping (9.2.2)
- - - = active system, full-state feedback (9.1.2.2)
- - = active system, partially coupled active damping (9.2.3)
----- = passive reference system
Power Spectral Densities

Similar to the analysis dealing with full-state feedback control design, power spectral densities (PSD) have been calculated for the cases of partial state feedback. Figure 9.12 presents the results. Only delayed rear wheel inputs are considered. The solid lines refer to the active damping configuration without interaction, the dashed lines correspond to the partly coupled actively damped system and the dotted lines represent the passive reference system. All plotted results are valid for a speed of travel equal to 120 km/h and $q_1 = 10$.

Similar to the full-state feedback system, the major part of the improvement of ride comfort is located in the frequency range from 0 to 12 Hz. The influence of the (partial) coupling can only be indicated at the PSD of pitch angular accelerations. Due to the logarithmic scale the decrease of tire load variations at the rear suspension in the 12-14 Hz range cannot be clearly seen.

Figure 9.12a Power spectral densities ($v = 120 \text{ km/h}$, $q_1 = 10$).
- = active decoupled damping
- - = active coupled damping
- --- = passive reference system
Figure 9.12b Power spectral densities ($v = 120$ km/h, $q_1 = 10$).
- = active decoupled damping
-- = active coupled damping
----- = passive reference system
Chapter 9 Active Suspension Design using Linear Optimal Control

Eigenvalues

The eigenvalues and natural frequencies of the 13 DOF vehicle model with active suspension based on partially coupled limited state feedback are shown in table 9.7. The results are valid for $q_1 = 10, v = 120$ km/h and include the wheelbase time delay in the control systems design phase. The powertrain rigid body related modes (1, 6 and 7) are quite easy to recognize because the location of the eigenvalues in the complex plane does not change a lot. Since the suspension stiffnesses remain unchanged compared to the standard car with passive suspension elements, all wheel-hop/tramp related modes (2, 3, 4 and 5) can be distinguished easily (imaginary part does not change). The damping of these modes is rather poor. It is likely that the road holding degrades. The sprung mass related modes are all pure real modes. It is not possible to distinguish the different mode shapes because the eigenvalues are all real. This means that the motions are damped in such a way that the vehicle body never responds in an oscillatory way after a perturbation. Mode numbers 11, 12 and 13 correspond to the longitudinal, lateral and yaw degree of freedom respectively.

Speed Influence

Similar to the full-state feedback case, it is expected that the speed of travel has an influence on the suspension performance, in particular with the (partially) coupled case. Several controllers have been calculated in the range from 5 up to 200 km/h for the partially coupled active damping system. Both uncorrelated and delayed rear wheel inputs have been considered. The time delay has been approximated with a 4th-order Padé filter and all calculations have been carried out for $q_1 = 10$. The RMS performance values of the parameters of interest are all plotted in figure 9.13a. The six feedback gains as a function of the speed of travel are shown in figure 9.13b. The solid lines hold for the system with uncorrelated rear wheel inputs in the control system design phase. The dashed curves belong to the system with wheelbase time delay and the dotted lines refer to the standard car with passive suspension.

The rather rough curves refering to the controller based on the wheelbase time delay assumption is remarkable. The calculations of the feedback structures turned out to be very time-consuming (several days on a 486 PC). Sometimes different starting values were needed because the algorithm didn't converge to a stable solution. Therefore, the results including the time delay
approximation based on a Padé filter must be dealt with with some caution. The uncorrelated case didn’t cause any problem.

<table>
<thead>
<tr>
<th>mode</th>
<th>eigenvalues</th>
<th>natural frequency</th>
<th>damping ratio</th>
<th>damped frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>without lateral tire properties</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>-28.57 +/- 90.94 i</td>
<td>15.17</td>
<td>0.300</td>
<td>14.47</td>
</tr>
<tr>
<td>2</td>
<td>-10.82 +/- 84.50 i</td>
<td>13.56</td>
<td>0.127</td>
<td>13.45</td>
</tr>
<tr>
<td>3</td>
<td>-27.99 +/- 76.03 i</td>
<td>12.89</td>
<td>0.346</td>
<td>12.10</td>
</tr>
<tr>
<td>4</td>
<td>-23.98 +/- 81.31 i</td>
<td>13.49</td>
<td>0.283</td>
<td>12.94</td>
</tr>
<tr>
<td>5</td>
<td>-20.66 +/- 80.64 i</td>
<td>13.25</td>
<td>0.248</td>
<td>12.83</td>
</tr>
<tr>
<td>6</td>
<td>-22.94 +/- 79.65 i</td>
<td>13.19</td>
<td>0.277</td>
<td>12.68</td>
</tr>
<tr>
<td>7</td>
<td>-16.84 +/- 77.08 i</td>
<td>12.56</td>
<td>0.214</td>
<td>12.27</td>
</tr>
<tr>
<td>8</td>
<td>-2.67 +/- 14.82 i</td>
<td>2.40</td>
<td>0.177</td>
<td>2.36</td>
</tr>
<tr>
<td>9</td>
<td>-3.03 +/- 8.92 i</td>
<td>1.50</td>
<td>0.321</td>
<td>1.41</td>
</tr>
<tr>
<td>10</td>
<td>-2.49 +/- 7.44 i</td>
<td>1.25</td>
<td>0.317</td>
<td>1.18</td>
</tr>
<tr>
<td>11</td>
<td>0.00 +/- 0.00 i</td>
<td>0.00</td>
<td>0.000</td>
<td>0.00</td>
</tr>
<tr>
<td>12</td>
<td>0.00 +/- 0.00 i</td>
<td>0.00</td>
<td>0.000</td>
<td>0.00</td>
</tr>
<tr>
<td>13</td>
<td>0.00 +/- 0.00 i</td>
<td>0.00</td>
<td>0.000</td>
<td>0.00</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>mode</th>
<th>eigenvalues</th>
<th>natural frequency</th>
<th>damping ratio</th>
<th>damped frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>without lateral tire properties</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>-28.86 +/- 90.68 i</td>
<td>15.15</td>
<td>0.303</td>
<td>14.43</td>
</tr>
<tr>
<td>2</td>
<td>-2.20 +/- 85.39 i</td>
<td>13.59</td>
<td>0.026</td>
<td>13.59</td>
</tr>
<tr>
<td>3</td>
<td>-3.56 +/- 76.69 i</td>
<td>12.22</td>
<td>0.047</td>
<td>12.21</td>
</tr>
<tr>
<td>4</td>
<td>-5.25 +/- 82.82 i</td>
<td>13.21</td>
<td>0.063</td>
<td>13.18</td>
</tr>
<tr>
<td>5</td>
<td>-2.79 +/- 80.43 i</td>
<td>12.81</td>
<td>0.035</td>
<td>12.80</td>
</tr>
<tr>
<td>6</td>
<td>-30.54 +/- 76.23 i</td>
<td>13.07</td>
<td>0.372</td>
<td>12.13</td>
</tr>
<tr>
<td>7</td>
<td>-18.90 +/- 77.31 i</td>
<td>12.67</td>
<td>0.238</td>
<td>12.30</td>
</tr>
<tr>
<td></td>
<td>-127.50 +/- 0.00 i</td>
<td>20.29</td>
<td>1.000</td>
<td>0.00</td>
</tr>
<tr>
<td></td>
<td>-37.14 +/- 0.00 i</td>
<td>5.91</td>
<td>1.000</td>
<td>0.00</td>
</tr>
<tr>
<td></td>
<td>-24.11 +/- 0.00 i</td>
<td>3.84</td>
<td>1.000</td>
<td>0.00</td>
</tr>
<tr>
<td></td>
<td>-6.36 +/- 0.00 i</td>
<td>1.01</td>
<td>1.000</td>
<td>0.00</td>
</tr>
<tr>
<td></td>
<td>-2.65 +/- 0.00 i</td>
<td>0.42</td>
<td>1.000</td>
<td>0.00</td>
</tr>
<tr>
<td></td>
<td>-0.78 +/- 0.00 i</td>
<td>0.12</td>
<td>1.000</td>
<td>0.00</td>
</tr>
<tr>
<td>11</td>
<td>0.00 +/- 0.00 i</td>
<td>0.00</td>
<td>0.000</td>
<td>0.00</td>
</tr>
<tr>
<td>12</td>
<td>0.00 +/- 0.00 i</td>
<td>0.00</td>
<td>0.000</td>
<td>0.00</td>
</tr>
<tr>
<td>13</td>
<td>0.00 +/- 0.00 i</td>
<td>0.00</td>
<td>0.000</td>
<td>0.00</td>
</tr>
</tbody>
</table>

Table 9.7 Natural frequencies of the active suspension system based on limited state feedback ($v = 120$ km/h).
From the point of ride comfort, the improvement in the RMS level of vertical accelerations is astonishing. The vertical seat accelerations remain almost constant for speeds higher than 100 km/h. Of course this gain is not for free. Both front and rear tire load variations increase with the speed of travel. The magnitude at the front tire is slightly larger in excess of 100 km/h in case the wheelbase time delay has been included in the control system design phase. The tire load ratio of the rear tires is considerably larger than at the front wheels. However, the delayed control version manages to reduce the loss above 90 km/h. This is one of the major advantages of including the time delay in the control design phase in combination with a coupled structure. The same phenomenon can be spotted with the suspension working space.

The 6 feedback gains of the case of the uncorrelated rear wheel input (solid lines) are almost speed independent. This makes the implementation of the suspension controller a lot easier, since it is sufficient to use only six gain values. The gains of the delayed version (dashed lines) are rather chaotic. Some of them vary an awful lot with the forward speed.

**Handling Test**

**Double Lane-Change**

A simulation has been carried out with the actively suspended vehicle, in a similar fashion as with the standard car. The feedback gains for the partially coupled feedback structure have been calculated for $q_1 = 10$ and a forward speed equal to 80 km/h. Figure 9.14 illustrates the results.

The plots tell us that the vehicle with active suspension handles well. The driver manages to pass the double lane-change quite well. It is even a bit better than with the standard car. During the motion necessary to return to the original lane (after 80 m), the vehicle is even a bit oversteered. There is a small lateral overshoot at approximately 100 m. This can also be noticed from the steering angle and lateral acceleration. Both curves differ only slightly from the standard car (dashed lines). Unlike the full-state feedback, the roll angle of the partial state feedback is not exceptionally large. The roll motion is even a bit better damped in the active case compared to the standard car.
Figure 9.13a RMS performance values as a function of speed ($q_1 = 10$).

- = active coupled damping, uncorrelated rear wheel input
- - = active coupled damping, delayed rear wheel input
--- = passive reference system
Figure 9.13b Feedback gains as a function of speed ($q_1 = 10$).

--- = active decoupled damping, uncorrelated rear wheel input

-- = active decoupled damping, delayed rear wheel input

The aim of the active suspension is to improve ride comfort. From the RMS values we know that this is accompanied by an increase in vertical tire load variations due to road unevennesses. Both front and rear tire load variations increase (19% and 38% for the front and rear axle respectively). At 3.5 s the increase of the dynamic load variations coincides with a decrease of the average right rear tire load due to the load transfer. At this moment the rear tires lose too much cornering power. The average side force decreases to such a degree that the car oversteers. This can also be nicely seen from the yaw rate and body side slip angle. Similar to all other double lane-change
simulations, the lateral tire force variations due to vertical tire load variations get only out of hand when the average vertical tire load approaches zero.

The gain in ride comfort is considerable (48% and 46% for the front and rear seat respectively according to the RMS calculations). It is even better than the full-state feedback active suspension because the contribution of the roll angular acceleration to the seat vertical accelerations is substantially less. Also the required actuator forces are quite small. The rear actuators require somewhat more power. The peaks coincide with the maxima of the roll velocity of the sprung mass. Finally, the suspension deflection is not larger than the working space used in the standard car. The deflection due to the road unevennesses is far less than the space required to counteract the couple due to the lateral g-forces through the action of a roll angle.

From the simulation results the conclusion can be drawn that despite the increased level of dynamic tire load variation, the handling has not been affected seriously. The only situations to be avoided are those cases where the load transfer and the dynamic tire load variations coincide in such a way that the average cornering power will be reduced too much. If this situation occurs at the rear axle the vehicle becomes oversteered, if it happens at the front axle the vehicle becomes understeered.
Figure 9.14a Double lane-change, active suspension, limited state feedback (80 km/h)
Figure 9.14b Double lane-change, active suspension, limited state feedback (80 km/h)
Frequency Response Functions

The three frequency response functions (FRFs) of interest are plotted in figure 9.15a up to c. The lateral acceleration and yaw rate FRFs of the actively suspended vehicle (solid lines) do not show much difference with the standard car (dashed lines). The damping of the lateral acceleration FRF in the range from 2 up to 3 Hz is better due to a better damped roll mode and the phase lag between steering input and lateral acceleration is slightly smaller. A noticeable difference can be spotted in figure 9.15c (roll angle FRF). The vehicle with active suspension based on limited state feedback (partially coupled) shows a much better roll damping. The gain between 2 and 3 Hz is substantially lower. Furthermore, the gain starts to drop sooner. These findings imply not only that the handling is improved, but also the ride. The better roll damping arises from the feedback of the absolute vertical velocity of the sprung mass at all four wheel suspensions. If the vehicle body rolls (during cornering) the resulting roll rate (difference between absolute vertical velocity of the left and right side of the vehicle divided by the distance between the suspension top mountings) is fed back into the controller and the actuators generate a force proportional to the roll rate. The distribution among the front and rear actuators depends on the gains of the feedback structure. From the double lane-change maneuver (figure 9.14b) it can be seen that both actuators in the rear suspension are almost entirely responsible for the extra roll damping.

This chapter has dealt with the design and evaluation of an intelligent suspension system for a full-vehicle model based on LQG control. The full-vehicle application offers some more degrees of freedom in the control design phase than the quarter-car model. In addition to the full- and partial state feedback, the effect of the road description on the performance has been investigated. Especially the effect of the track correlation and wheelbase time delay on the performance parameters has been studied. The results showed that the extra information arising from the cross-correlated road tracks does not result in greater control potentials. However, including the wheelbase time delay in the control design phase has a large influence on the performance. Especially the tire load variations and suspension travel of the two rear suspensions can benefit from the extra information of the wheelbase time delay.
Chapter 9 Active Suspension Design using Linear Optimal Control

Figure 9.15a & b Lateral acceleration and yaw rate FRF.

--- active suspension, partially coupled damping
--- standard car, passive suspension
Figure 9.15c Roll angle frequency response function.

--- = active suspension, partially coupled damping
--- = standard car, passive suspension

The LQG control analysis has also shown that the performance of partial state feedback systems can compete with suspensions based on full-state feedback. The partial state feedback considers only the velocities of the sprung and unsprung mass at the suspension mounts and can therefore be conceived as active damping. The exchange of state information among the four suspensions may enhance the performance noticeably. The main conclusions applicable to the full-vehicle model are generally similar to those of the quarter-car model. In general, the ride comfort of the passengers can be improved substantially albeit at the cost of a degradation of the road holding ability.

After the determination of the ride performance the vehicles with active suspension have been exposed to the double-lane change maneuver on an uneven road surface. The results have shown that moderate tire load variations do not affect the handling. The tire side force variations are only slightly worse compared with the standard vehicle. The variations are only larger in those cases were the vertical tire load decreases drastically due to
the load transfer. The vehicle with active suspension based on full-state feedback did not perform very well in the double lane-change test. Besides the quite harmless increase in tire load variations, the unacceptable low roll stiffness and the unbalanced load transfer contribute to the degradation of the handling. The latter two phenomena are caused by the full-state feedback structure of the active suspension. The feedback of position related states must be avoided. The vehicle with a suspension based on partial state feedback (velocities only) performed well compared with the standard configuration. The only situations to be avoided are those cases where the load transfer and the dynamic tire load variations coincide in such a way that the average cornering force will be reduced too much. An adaptation of the suspension controller is necessary to avoid these situations. This adaptation will be discussed in chapter 10.
SUSPENSION CONTROL WITH ADAPTIVE PROPERTIES

Similar to the quarter-car analysis, suspension control with adaptive properties is necessary if the controller has to be implemented in a real vehicle. The aim of suspension control is to improve ride comfort as much as possible. From the evaluation of performance of active suspension systems carried out in the preceding chapter it is known that the ride comfort improvement is accompanied with an increase in tire load variations when the vehicle operates on an uneven road. These tire load variations may affect the handling of a vehicle during cornering and/or braking or accelerating. Therefore, an adaptation is required to overcome problems related to road holding. Unlike the quarter-car model, the full-vehicle model offers the possibility to study the relations between road holding and handling.
The ride comfort improvement obtained with the LQG controllers as discussed in chapter 9, is accompanied with an increase in tire load variations. This means that road holding deteriorates on an uneven road. As shown with the double lane-change simulation results, this deterioration may not necessarily lead to a serious degradation of the handling of a vehicle. Firstly, moderate variations of the tire load do not affect the cornering potential of the tires. Secondly, the most common situation of straight ahead driving on moderately uneven roads does not demand generation of large lateral tire forces. In this case the handling aspect may be ignored and the suspension can be tuned for optimal ride comfort. On the other hand situations may arise during cornering and/or braking where it is advantageous to reduce the load variations due to road unevennesses of a particular tire that has the lowest load at any given moment. In that case a large handling potential is preferred at the cost of ride comfort. The conflict between ride comfort and handling calls for an adaptive control algorithm that considers not only distinction of different road profiles but also whether large tire slip forces are needed in situations involving braking and/or cornering.

Figure 10.1 shows an outline of the proposed adaptive controller that can be regarded as an outer feedback loop to the earlier discussed LQG controller [40]. The adaptation is based on gain scheduling of output feedback matrix \( K \), and shows much correspondence to the structure used in the quarter-car analysis as discussed in chapter 5. Gain scheduling means that the elements of the feedback gain matrix are adjusted during the ride depending on the external situation. The vertical tire loads are the main input for the adaptive controller. They can be estimated very well with a Kalman filter and several accelerometers attached to the vehicle body and axles. In this section it is assumed that the tire loads are known. Chapter 11 will deal with the design of a Kalman filter that can be used to estimate the dynamic tire load. The most important variable that relates road-holding to handling is the ratio of the dynamic tire load \( P_{dyn} \) to the static tire load \( P_{stat} \). However, one modification has been made.

When calculating the tire load ratio, it is recommendable to use the average vertical tire load (denoted by \( \overline{P}_{tire} \)) rather than the (constant) static tire load \( P_{stat} \), because the average tire load may vary considerably during cornering and braking due to the load transfer. The average value (which is actually not a constant) can be regarded as the vertical tire load while driving over a smooth road surface. This so-called moving average can be calculated by filtering the
tire load with a 1st-order filter with a sufficiently low cut-off frequency. The transfer function of such a filter is given by

$$\frac{\bar{P}_{\text{tire}}(s)}{\bar{P}_{\text{tire}}(s)} = \frac{1}{\tau_m s + 1} \quad (10.1)$$

A bandwidth of 3 Hz ($\tau_m = 1/(2\pi \cdot 3)$) filters all the contributions of the road unevennesses but doesn’t affect the load transfer information because steering maneuvers take place typically below 1 Hz. Since the moving average value lags the real average, higher order filters are not recommendable. The introduction of the tire load ratio based on the moving average value reflects the tire side force sensitivity to the variation with respect to the average value. The tire with a large average vertical load is therefore relatively insensitive to road irregularities compared to a tire at a low load running over the same irregularities.

Figure 10.1 Outline of the adaptive controller.

In addition, the ratio has been weighted according to the 4th-degree function of figure 5.4. This because moderate tire load variations do not substantially
affect the cornering potential of the tires. Unlike a 2nd-degree function, the 4th-degree weighting function attaches almost no importance to small and moderate variations. The weighting ratio $r_w$ is always positive. The resulting number is a weighted ratio of dynamic to the average tire load and is given by

$$r_w = \left( \frac{\dot{p}_{\text{dye}}}{\bar{p}_{\text{tire}}} \right)^4$$

(10.2)

It is the most important variable to attach value to. If this ratio equals zero or is very close to zero then the tire load variations do not affect handling. If it gets close to one, the handling of the vehicle in this particular situation gets into the danger zone.

One of the easiest ways to reduce tire load variations due to road unevennesses of a particular suspension is simply to add more passive damping. Since the adaptation is based on gain scheduling, output feedback matrix $K_n$ can be adjusted by introducing more passive damping. The actuator force of a particular suspension can be expressed as

$$F_{act} = F_{rc} \cdot (1 - q_{adp}) + F_{rh} \cdot q_{adp}$$

(10.3)

where $q_{adp}$ is a weighting factor with a value between 0 and 1. Each suspension will be adapted separately. Thus four weighting factors $q_{adp}$ exist for the full-vehicle model. The force responsible for the ride comfort improvement $F_{rc}$ can be the result of a partial state feedback (active damping) as discussed in section 9.2.

The road holding force $F_{rh}$ is composed of passive damping. The corresponding suspension damping constants have been calculated using the LQG limited state feedback controller as discussed in section 9.2.4. The normalized weighting factors chosen are: $\tilde{q}_1 = 0$, $\tilde{q}_2 = 1000$, $\tilde{q}_3 = 0$ and $\tilde{q}_4 = 1/2000^2$. This means that very much importance has been attached to the reduction of tire load variations. The resulting feedback is equivalent to passive damping which guarantees a minimum amount of tire load variations due to road unevennesses. The damping ratios obtained are 2439 Ns/m and 2639 Ns/m for the front and rear suspension respectively. These values are calculated for a speed of travel equal to 120 km/h and for a concrete road. Although the wheelbase time delay has been included in the LQG calculations, the optimal damping ratios do hardly depend on the speed. This is not so strange since the contribution of the sprung mass motion to the tire load variations is small compared to the contribution of the axle motions. The wheel-hop mode is
dominantly present in the tire load variations. Therefore finding the damping ratio that corresponds to optimal road holding can be treated as if the vehicle was excited by four independent road excitations (on the condition that the suspension is of an independent type).

The gain scheduling as discussed in this section has been achieved in such a way that the actuator force corresponds to that of active damping (partially coupled) if the weighted tire load ratio equals zero and passive (firm) damping if the weighted ratio equals one. The link between weighted ratio $r_w$ and gain scheduler input $q_{adp}$ has been accomplished by proportional control added with integral action (PI-controller). The proportional gain equals one and the integral action retains the selected feedback for a certain amount of time. The adaptation is determined by

$$\tau_a \dot{q}_{adp} + q_{adp} = r_w \quad 0 \leq q_{adp} \leq 1$$

where $\tau_a$ is a time constant that controls the integral action. Its value is chosen equal to $\tau_a = 1/(2\pi \cdot 3)$. The integral action has been included because it will take some time to settle the tire load variations whenever a serious oscillation takes place. Weighting factor $q_{adp}$ has been limited between 0 and 1.

Unlike figure 10.1 suggests, all four wheel suspensions contain an individual adaptive loop. This has the advantage that only those suspensions that really contribute to the deterioration of handling are adjusted while the other wheel suspensions still contribute to the improvement of ride comfort. The adaptive loop is virtually inactive while driving straight on unless large tire load variations occur due to very rough road conditions or discrete events like potholes.

Figure 10.2 presents simulation results of the adaptive controller in combination with limited state feedback based on partially coupled active damping. Besides the vertical tire load (including moving average) and lateral tire forces, the weighted tire load ratio $r_w$ and the input for the gain scheduler ($q_{adp}$) have been plotted. The results should be compared with the plots of figure 9.14 which are valid for an identical vehicle but with a non-adaptive suspension controller. It can be clearly seen that the adaptive loop only intervenes in those situations where desired. Especially the side force variations at the rear axle are reduced considerably preventing the vehicle from becoming oversteered compared with the non-adaptive version. Although the average side force of the unloaded tire is still low compared to the total
side force of the entire axle, it can influence the handling noticeably. The deterioration of ride comfort (not plotted) during the intervention of the adaptive loop is only for a short period of time and is not serious.

This chapter has dealt with the development of an adaptive controller. The adaptation has been based on gain scheduling. This means that the gains of the state feedback matrix are adjusted during the ride depending on the operating point of the vehicle. The adaptation has been based on the ratio of the dynamic tire load to the average tire load. If this ratio becomes large than the dynamic tire load variations may affect the handling. In this case, the adaptive suspension controller will automatically adjust the feedback gains in order to restrict the tire load variations of a particular wheel. The demand for safety goes with a brief worsening of the ride comfort during this intervention.

The implementation of adaptive suspension control as described above requires the knowledge of several states of the vehicle. States like the absolute vertical velocity of the vehicle body or the dynamic tire load cannot be measured directly. It is however possible to estimate the required signals with the aid of an internal model of the vehicle. The unknown states can be estimated by a Kalman. This topic will be examined in chapter 11.
Figure 10.2 Double lane-change, active suspension based on limited state feedback (partially coupled active damping) with adaptive properties.
KALMAN FILTER DESIGN FOR AN ACTIVE SUSPENSION SYSTEM

In this chapter a Kalman filter will be designed to estimate unknown or unaccessible states of the full-vehicle. In the preceding chapters it has been illustrated that the most important variables of interest are the absolute sprung mass vertical velocity at each wheel suspension, the deflection rate of each suspension and the tire load ratio (dynamic load over static load). Apart from the relative velocity, the other eight variables are difficult (tire load) or impossible (absolute velocity) to measure. However, with the aid of an (internal) model of the vehicle and other more easily accessible measurement signals it is possible to estimate the unknown states.
The first section of this chapter deals with the selection of the internal (vehicle) model. It is the basis of the Kalman filter. Besides the knowledge of the system to be observed, measurement signals are required. The selection of the sensors will be discussed in the second section. The third section deals with some theoretical aspects of discrete-time Kalman filters specifically for the full-vehicle application. The basic theory of Kalman filtering has already been explained in the quarter-car part and will therefore not be repeated. Once the internal model and sensors are chosen the Kalman filter is ready to be tuned. This is a very important phase because a balance has to be found between speed of state reconstruction and sensor noise sensitivity. The quality of estimation will be studied using both RMS values and time domain simulations. The vehicle will also be subjected to the double lane-change handling maneuver in order to provoke certain problems related to signal offsets.

11.1 Selection of the Internal Model

The theory of state estimation is based on the knowledge of the plant in the form of an internal model. This internal model should be a good representation of the real vehicle. This means that it should be able to describe all dynamics of interest. Nevertheless, it is not necessary to include all dynamics of the real car in the internal model. Since we are not interested in flexible body dynamics, the number of degrees of freedom (DOF) is finite. Furthermore, for the ease of design and calculation effort the internal model should not contain too many DOFs. It is obvious to choose the 13 DOF full-vehicle model as described in chapter 7 and derived in appendix A. However, this model meets with two objections: (i) the rigid body powertrain dynamics are not modelled accurately enough, (ii) the handling related DOFs cause serious problems due to linearization and non-linear tire characteristics.

As already pointed out in chapter 7, the rigid body powertrain dynamics are difficult to describe with linear elements. Especially the engine mounts cause serious problems because their stiffness and damping characteristics are typically frequency dependent. Furthermore, the tilted orientation of the principle axes of the powertrain/transmission has been neglected in the modeling phase. It is expected that the influence of the transmission on the dynamic behavior of the vehicle (ride and handling) is rather small. Therefore, it has been chosen to integrate the powertrain into the main vehicle body. The number of degrees of freedom is now reduced from 13 to 10.
The second problem concerns the handling degrees of freedom. Three of the 10 DOFs describe the out-of-plane vehicle dynamics: \( x_b, y_b \) and \( \psi_b \). The most important drawback of the 10 DOF model is that the out-of-plane dynamics have been linearized (small yaw angles). This implies that it is for example not allowed to drive though corners because in this situation the internal model is not valid anymore. Unlike the quarter-car model, the handling dynamics are coupled with the ride dynamics. During cornering for example, the centrifugal forces cause the vehicle body to roll. This implies that the roll dynamics are not merely excited by road unevennesses. The same occurs with the pitch DOF during braking or accelerating. And last but not least, the tire load very much depends on the load transfer during cornering and/or braking/accelerating. It is possible to couple the ride dynamics with the handling dynamics by means of linear tire slip characteristics such as constant cornering stiffnesses. However, this would imply that the internal model is only valid for straight ahead driving and moderate handling maneuvers. Even in this case the cornering stiffness cannot be made load dependent, since the internal model has to be linear. With linear tire parameters, estimation errors are expected because the internal model will show up a larger roll angle for example than the real vehicle during (severe) cornering. The load transfer could also be larger because the linear tire characteristics do not describe the side force saturation with larger slip angles. Estimation problems are to occur if the condition of the road surface differs from the initial one (other \( \mu \)-values). One possibility is the use of non-linear state estimators. However, the description of the tire characteristics is still very complex because slip angles, vertical tire loads and road surface condition must be known a priori. Since we are not interested in estimating the handling related state variables \( x_b, y_b \) and \( \psi_b \) (they do not contribute to any kind of active suspension control algorithm), another method to estimate the tire slip forces has been chosen. This method is based on measuring the longitudinal, lateral and yaw acceleration, and will be discussed after the description of the internal vehicle model.

The 10 DOF model can be simplified ones more by deleting states \( x_b, y_b \) and \( \psi_b \). The resulting model corresponds completely to the 7 DOF vehicle model as used with the active suspension control design (chapter 9). As the four wheel suspensions are connected to the main vehicle body by means of constraints, coupling terms between longitudinal/lateral tire slip forces and ride dynamics exist. These forces have been omitted in the active suspension control algorithm design phase. The seven 2nd-order equations of motion of the vehicle model are given by
\[ M \ddot{\tilde{z}} + K \dot{\tilde{z}} + C \tilde{z} = E_s u_a + E_{ix} E_{ix}^r + E_{iy} E_{iy}^r + E_{iz} E_{iz}^r \]  

(11.1)

with

\[
\tilde{z}^T = [z_b \ \phi_b \ \theta_b \ \phi_{a1L} \ \phi_{a1R} \ \phi_{a2} \ \phi_{a2}]
\]

\( M, K \) and \( C \) are the mass, damping and stiffness matrix respectively, \( E_s \) represents the input matrix for the active suspension force \( u_a \) and \( E_{ix}, E_{iy} \) and \( E_{iz} \) stand for the input matrices for the longitudinal, lateral and vertical tire forces \( F_{ix}, F_{iy} \) and \( F_{iz} \) respectively. The state-space equation of this 7 DOF vehicle model is as follows

\[
\dot{x}_u = A_u x_u + F_i u_a + R_i z_r + S_i \delta + T_{ix} E_{ix}^r + T_{iy} E_{iy}^r
\]  

(11.2)

with

\[
A_u = \begin{bmatrix} 0 & I \\ -M^{-1}C & -M^{-1}K \end{bmatrix} \quad x_u = \begin{bmatrix} \tilde{z} \\ \dot{\tilde{z}} \end{bmatrix}
\]

and

\[
F_i = \begin{bmatrix} 0 \\ M^{-1}E_s \end{bmatrix} \quad T_{ix} = \begin{bmatrix} 0 \\ M^{-1}E_{ix} \end{bmatrix} \quad T_{iy} = \begin{bmatrix} 0 \\ M^{-1}E_{iy} \end{bmatrix} \quad S_i = \begin{bmatrix} 0 \\ M^{-1}E_{iz} \end{bmatrix}
\]

\( A_u \) is the system matrix, \( F_i \) represents the input matrix for the active suspension force \( u_a \), \( T_{ix} \) and \( T_{iy} \) stand for the input matrices for the longitudinal and lateral tire forces \( F_{ix} \) and \( F_{iy} \) respectively, \( S_i \) is an input matrix for the front wheel steering angle \( \delta \) (equal for left and right wheels) and \( R_i \) the road input matrix with \( z_r \) as a vector with four road elevations. The outputs of the vehicle system can be selected as desired. Mostly the outputs are corrupted by measurement noise \( v \). The output equation can be considered as

\[
\gamma_u = C_u x_u + F_o u_a + R_o z_r + S_o \delta + T_{ox} E_{ix} + T_{oy} E_{iy} + S_o E_{iz} + v
\]  

(11.3)

This model with 14 states serves as a starting point for the Kalman filter design. However, it does not contain the four road elevations at each wheel suspension in the state vector. As stated in the introduction of this chapter, one of the reasons for the use of a state estimator, is the possibility to estimate the dynamic tire load. In order to be able to estimate this load, it is necessary to augment the vehicle model with a (simple) road unevenness model. Furthermore, the Kalman filter design algorithm requires that the internal model is excited by white noise.
The background theory of the two track road model with cross-correlated tracks has already been discussed in chapter 7. The delayed road excitation between front and rear wheels cause some problems. Although the wheelbase time delay can be approximated with a Padé filter, it is not desired to include the time delay in the system description. Firstly, the number of augmented states is quite large (more than 10 for low speeds), secondly, the parameters of the Padé approximation depend on the vehicle speed. This implies a linear system description with time-variant parameters. From the quarter-car model Kalman filter design we know that the quality of estimation depends very little on the accuracy of the road description. Therefore, it has been chosen to omit the wheelbase time delay. This situation corresponds to fully uncorrelated front and rear wheel road inputs. Furthermore, one speed of travel and one road condition is assumed. The state-space equations of the above described road model can be expressed as

\[
\dot{x}_{r4} = A_{r4} x_{r4} + B_{r4} w_{r4} \\
y_{r4} = C_{r4} x_{r4}
\]

with

\[
A_{r4} = \begin{bmatrix}
-v \beta_1 & 0 & 0 & 0 \\
0 & -v \beta_2 & 0 & 0 \\
0 & 0 & -v \beta_1 & 0 \\
0 & 0 & 0 & -v \beta_2 \\
\end{bmatrix}
\]

\[
B_{r4} = \begin{bmatrix}
u g_1 & 0 & 0 & 0 \\
0 & v g_2 & 0 & 0 \\
0 & 0 & v g_1 & 0 \\
0 & 0 & 0 & v g_2 \\
\end{bmatrix}
\]

\[
C_{r4} = \begin{bmatrix}
1 & -T_w/2 & 0 & 0 \\
1 & +T_w/2 & 0 & 0 \\
0 & 0 & 1 & -T_w/2 \\
0 & 0 & 1 & +T_w/2 \\
\end{bmatrix}
\]

and

\[
x_{r4}^T = [z_{m1} \quad \phi_{m1} \quad z_{m2} \quad \phi_{m2}] \\
y_{r4}^T = [z_{r1L} \quad z_{r1R} \quad z_{r2L} \quad z_{r2R}] \\
w_{r4}^T = [w_1 \quad w_2 \quad w_3 \quad w_4]
\]

Vector \( \chi_{r4} \) contains the four road elevations at each wheel and vector \( w_{r4} \) contains four uncorrelated random white noise processes. One of the conditions for the design of a Kalman filter is a system excited by white noise. The above road model meets this requirement.
Both vehicle model (14 states) and road model (4 states) can be combined to one total internal model given by

\[
\dot{x}_t = A_t x_t + F_{it} u_a + R_{it} w_{r4} + T_{it}^{x} F_t^x + T_{it}^{y} F_t^y \\
y_t = C_t x_t + F_{ot} u_a + R_{ot} w_{r4} + T_{ot}^{x} F_t^x + T_{ot}^{y} F_t^y + v
\]

(11.4a)\hspace{1cm}(11.4b)

with

\[
A_t = \begin{bmatrix} A_o & R_t C_{r4} \\ 0 & A_{r4} \end{bmatrix} \quad F_{it} = \begin{bmatrix} F_i \\ 0 \end{bmatrix} \quad R_{it} = \begin{bmatrix} 0 \\ B_{r4} \end{bmatrix} \quad T_{it}^{x} = \begin{bmatrix} T_i^{x} \\ 0 \end{bmatrix} \quad T_{it}^{y} = \begin{bmatrix} T_i^{y} \\ 0 \end{bmatrix}
\]

\[
C_t = \begin{bmatrix} C_o & R_t C_{r4} \end{bmatrix} \quad F_{ot} = F_o \quad R_{ot} = R_o \quad T_{ot}^{x} = T_o^{x} \quad T_{ot}^{y} = T_o^{y}
\]

and

\[
x_t = \begin{bmatrix} x_o \\ x_{r4} \end{bmatrix} \quad y_t = y_o
\]

The total internal model has three inputs: (i) white noise \( w_{r4} \), (ii) four longitudinal tire forces \( F_i^x \) and (iii) four lateral tire forces \( F_i^y \). The separate input for the steering angle \( \delta \) has been omitted because a steering wheel input can be converted to an extra side force at the front wheels. The white noise road inputs are unknown but necessary for the Kalman filter design. The eight tire forces can be regarded as known inputs. Although they can’t be measured directly without using very expensive equipment, it is at least possible to estimate them by using for example accelerometers. The actuator forces \( u_a \) are known from the control system design. In the oncoming section it will be explained how the tire slip forces can be estimated.

**Lateral Tire Force Estimation**

The tire side slip forces \( F_i^y \) can be estimated using the three unused equations of motion of the 10 DOF vehicle model. The following differential equation is valid for the lateral \( (y_b) \) degree of freedom

\[
F_i^y + F_{i2}^y = (m_b + 2m_{a1} + m_{a2}) \ddot{y}_b + m_{a2} t_p \ddot{z}_b - m_{a2} t_p \dot{\phi}_b
- \left(2m_{a1} h_{p1} + m_{a2} (h_{p2} - w_{p2} t_p) \right) \ddot{\phi}_b + (2m_{a1} v_{a1} + m_{a2} v_{a2}) \ddot{\psi}_b
- m_{a1} a_{1d} (\dot{\phi}_{a1L} + \dot{\phi}_{a1R}) - m_{a2} a_{2d} \ddot{\phi}_{a2} + m_{a2} (w_{p2} t_p + h_r) \ddot{\phi}_{a2}
\]

(11.5)

where \( F_{i1}^y \) and \( F_{i2}^y \) represent the total side force at the front and rear axle respectively. Appendix A provides an explanation of the symbols used in equation (11.5) and other oncoming equations. Figure A.1 and A.2 also
illustrates the meaning of the symbols. For the yaw ($\psi_b$) degree of freedom the differential equation takes a form as

$$F_{r1}^{\gamma} \cdot l_{a1} + F_{r2}^{\gamma} \cdot l_{a2} = (2m_a l_{a1} + m_a l_{a2}) \ddot{y}_b + m_a l_{a2} t_p \ddot{z}_b - m_a l_{a2} t_p \ddot{\phi}_b$$

$$- \left( \frac{2m_a l_{a1} h_p}{l_{a1} - l_{a2}} + m_a l_{a2} \left( h_p - w_{p2} t_p \right) \right) \ddot{\phi}_b$$

$$+ \frac{2m_a l_{a1} g}{l_{a1} - l_{a2}} + m_a l_{a2} g + 2F_{r1}^{\gamma} l_{a1} + 2F_{r2}^{\gamma} l_{a2}$$

$$+ \left( \frac{I_{zb} + 2I_{za1} + I_{za2}}{l_{a1} - l_{a2}} + \frac{2m_a \left( (w_{p1} + l_a c_a)^2 + I_{a1}^2 \right)}{l_{a1} - l_{a2}} + m_a l_{a2}^2 \ddot{\psi}_b \right)$$

$$- \frac{m_a l_{a1} l_s a \left( \ddot{\phi}_{a1} \frac{1}{l_{a1}} + \ddot{\phi}_{a2} \frac{1}{l_{a2}} \right) - m_a l_{a2} t_p \ddot{z}_a + m_a l_{a2} t_p + h_r \ddot{\phi}_{a2}}{l_{a1} - l_{a2}}$$

(11.6)

During cornering, the tire side slip forces are mainly determined by the lateral ($y$) and yaw ($\psi$) degree of freedom. All other degrees of freedom describe the influence of suspension kinematics on the side force generation. If, for example, the vehicle body moves up and down, both front and rear axles describe a motion prescribed by the kinematics of the suspension. Beside a vertical motion of the axles, also a lateral component appears. These motions are attended by very small accelerations compared with the lateral acceleration of the entire vehicle during cornering. Therefore these terms can be neglected. This situation corresponds to a horizontal swing axle front suspension ($\phi_{a1}^0 = 0$ and $c_a = 1, s_a = 0$) and a horizontal location of the Panhard rod ($\phi_p^0 = 0$ and $t_p = 0$). The degrees of freedom that now remain are $\gamma_b, \phi_b, \psi_b$ and $\phi_{a2}$. The influence of the rear axle tramp ($\phi_{a2}$) on the dynamic tire side force is negligible since the distance between the Panhard rod mounting and the c.g. of the rear axle ($h_r$) is very small. From the three remaining variables the total side force at the front and rear axle can now be calculated according to

$$F_{r1}^{\gamma} = \frac{2m_a l_{a1} (l_{a1} - l_{a2}) - m_a l_{a2} \ddot{y}_b}{l_{a1} - l_{a2}}$$

$$- \frac{2m_a h_p (l_{a1} - l_{a2}) + 2m_a l_{a1} g + m_a l_{a2} g + 2F_{r1}^{\gamma} l_{a1} + 2F_{r2}^{\gamma} l_{a2}}{l_{a1} - l_{a2}} \ddot{\phi}_b$$

$$+ \frac{I_{zb} + 2I_{za1} + I_{za2} + 2m_a \left( (w_{p1} + l_a c_a)^2 + I_{a1}^2 \frac{1}{l_{a1} l_{a2}} \right)}{l_{a1} - l_{a2}} \ddot{\psi}_b$$

(11.7a)
\[
F_{t1}^y = \frac{m_{a2}(l_{a1} - l_{a2}) + m_{b1}l_{a1}}{l_{a1} - l_{a2}} \ddot{y}_b + \frac{-m_{a2}h_{p1}(l_{a1} - l_{a2}) + 2m_{a2}l_{a1}g + m_{a2}l_{a2}g + 2F_{r1}^{0yi} + 2F_{r2}^{0yi}}{l_{a1} - l_{a2}} \ddot{\phi}_b \\
+ \frac{I_{zb} + 2I_{za1} + I_{za2} + 2m_{a1}(w_{p1} + l_{ca})^2 + m_{a2}(l_{a2} - l_{a1})^2}{l_{a1} - l_{a2}} \ddot{y}_b
\]

(11.7b)

From the above equations it can be seen that the total side force at an axle can be calculated if we know the lateral, roll and yaw acceleration of the vehicle body. All three accelerations can be measured quite easily by attaching accelerometers on various places on the vehicle body. The lateral acceleration can be measured by a single accelerometer in the c.g. of the vehicle body. The yaw angular acceleration can be calculated if an additional accelerometer is attached to the vehicle body also pointing in the lateral direction. The best place is located at the front side of the car on the same height as the other lateral accelerometer. Finally, the roll acceleration can be calculated from two vertical accelerometer signals, each mounted for example at the front suspension strut top mounting. The contribution of the roll acceleration to the total tire side force is expected to be small.

The internal model described by equation (11.4) requires a tire side slip force for each wheel. This is only necessary for the front suspension because the rear suspension is of a rigid axle type. This can also be seen from matrix \( E_t^y \) (appendix A). Column 3 and 4 are identical indicating that both side slip forces of the rear wheels may be added together. For the front suspension it is assumed that the side slip forces can be distributed according to the current vertical tire load

\[
F_{t1L}^y = \frac{F_{t1L}^z}{F_{t1L}^z + F_{t1R}^z} \cdot F_{t1}^y \quad F_{t1R}^y = \frac{F_{t1R}^z}{F_{t1L}^z + F_{t1R}^z} \cdot F_{t1}^y
\]

(11.8)

It is assumed that the tire to road friction coefficient (\( \mu \)-value) is equal for both wheels. The vertical tire loads can be estimated with the aid of the Kalman filter. Values from the previous time step are used to distribute the side force.

**Longitudinal Tire Force Estimation**

The traction/braking forces \( (F_t^x) \) in the tire contact surface are fairly difficult to estimate. The following relation can be derived from the omitted differential equation of the 10 DOF model in the longitudinal direction
\[ F_{t1L}^x + F_{t1R}^x + F_{t2L}^x + F_{t2R}^x = (m_b + 2m_{a1} + m_{a2}) \ddot{x}_b \]
\[ + \left( 2m_{a1}(h_{p1} + l_a s_a) + m_{a2}(h_{p2} + l_p s_p - h_r) \right) \dot{\theta}_b \]  

(11.9)

The longitudinal acceleration can be measured quite easily by mounting an accelerometer in the c.g. of the vehicle body. The pitch angular acceleration can be calculated from two vertical accelerometer signals. One accelerometer should be mounted for example on the vehicle body at the front suspension strut top mounting and an other one at the rear suspension strut top mounting. For the yaw (\(\psi_b\)) degree of freedom the differential equation reads

\[ \left( F_{t1L}^y - F_{t1R}^y \right) (w_{p1} + l_a c_a) + \left( F_{t2L}^y - F_{t2R}^y \right) \frac{T_p}{2} + F_{t1L1a1} + F_{t2L2a2} \]
\[ = \left( 2m_{a1} \dot{b}_1 + m_{a2} \dot{b}_2 \right) \ddot{b}_b + m_{a2} \dot{b}_2 \dot{z}_b - m_{a2} \dot{b}_2 \dot{p} \dot{\theta}_b \]
\[ - \left( 2m_{a1} \dot{a}_1 h_{p1} + m_{a2} \dot{a}_2 h_{p2} - w_{p2} \dot{a}_2 \right) \dot{\phi}_b \]
\[ + \left( L_{2a} + 2L_{x1a1} + L_{x2a2} + 2m_{a1} \left( w_{p1} + l_a c_a \right)^2 + l_{a1}^2 + m_{a2} \right) \ddot{\psi}_b \]
\[ - m_{a1} l_{x1a1} (\dot{\phi}_{a1L} + \dot{\phi}_{a1R}) - m_{a2} l_{x2a2} \dot{z}_{a2} + m_{a2} l_{x2a2} \left( w_{p2} \dot{p} + h_r \right) \dot{\phi}_{a2} \]  

(11.10)

In order to simplify the expressions, it is assumed that no combination of lateral and longitudinal forces (braking/accelerating in corners) will occur.

The problem is now that with these two differential equations it is impossible to estimate the longitudinal tire forces at each wheel. There are four unknown tire forces with only two equations. Therefore a simplification must be made.

The simplification implies that it is assumed that the braking and driving forces at the left and right wheels are identical. This assumption is allowed if the coefficient of friction between tire and road is equals for both tracks and if the vertical tire load is equal for the left and right wheels. Furthermore, the distribution of the traction and braking forces must be equal for the left and right wheels.

The contribution of the longitudinal tire forces to the yaw DOF can now be neglected. The equation that remains can be described by

\[ F_{t1}^x + F_{t2}^x = (m_b + 2m_{a1} + m_{a2}) \ddot{x}_b \]
\[ + \left( 2m_{a1}(h_{p1} + l_a s_a) + m_{a2}(h_{p2} + l_p s_p - h_r) \right) \dot{\theta}_b \]  

(11.11)

The total traction or braking force can easily be calculated from the longitudinal and pitch angular acceleration. However, there are still two unknown forces and one equation. The next problem is related to the
distribution of driving and braking forces between front and rear axle. In case of traction \( \ddot{x}_b \neq 0 \) it is assumed that all the driving forces are generated at the front wheels. For a front-wheel driven car the following relation is valid

\[
\ddot{x}_b < 0 \quad F_{t1}^x = (m_b + 2m_{a1} + m_{a2}) \ddot{x}_b + (2m_{a1}(h_{p1} + l_p s_a) + m_{a2}(h_{p2} + l_p s_p - h_r)) \ddot{\omega}_b
\]  

(11.12)

In case of braking \( \ddot{x}_b = 0 \) it is assumed that the distribution of the braking forces between front and rear axle is proportional to the current vertical tire load

\[
\ddot{x}_b < 0 \quad F_{t1}^x = \frac{F_{t1}^z}{F_{t1}^z + F_{t2}^z} \left( (m_b + 2m_{a1} + m_{a2}) \ddot{x}_b + (2m_{a1}(h_{p1} + l_p s_a) + m_{a2}(h_{p2} + l_p s_p - h_r)) \ddot{\omega}_b \right)
\]  

(11.13a)

\[
\ddot{x}_b < 0 \quad F_{t2}^x = \frac{F_{t2}^z}{F_{t1}^z + F_{t2}^z} \left( (m_b + 2m_{a1} + m_{a2}) \ddot{x}_b + (2m_{a1}(h_{p1} + l_p s_a) + m_{a2}(h_{p2} + l_p s_p - h_r)) \ddot{\omega}_b \right)
\]  

(11.13b)

The simplifications made above imply that the estimation of the longitudinal tire forces is incorrect in situations with different road track friction coefficients \( \mu \)-split or during driving or braking in curves (different vertical tire loads left and right due to the load transfer).

**11.2 Selection of the Measurement Signals**

The design of a Kalman filter requires the selection of the measurement variables. A necessary condition for estimating the state variables \( x_t \) is that the system is observable. This means that all modes affect output \( y_t \). The observability can be checked by constructing the observability matrix \( W_o \)

\[
W_o = \begin{bmatrix} C_t & C_t A_t & \cdots & C_t A_t^{m-1} \end{bmatrix}^T
\]  

(11.14)

Similar to the quarter-car model, accelerometer signals are chosen in the full-vehicle application. These sensors are very small and can be attached almost anywhere on the vehicle. Other possible measurement devices are displacement sensors attached between axle and vehicle body.

Seven accelerometers are used in this phase: three mounted on the vehicle main body and one sensor for each wheel axle. All sensors are attached in such a way that they are sensitive to vertical accelerations only. As the vehicle body of the 7 DOF model contains only three degrees of freedom (heave, roll and pitch), three accelerometers are sufficient for the main body. These
sensors are located at the suspension strut top mountings of the left and right front suspension and the left rear suspension (figure 11.1). The corresponding accelerations are as follows

\begin{align}
    a_{b1L} &= \ddot{z}_b - w_{x1}\dot{\phi}_b - l_{a1}\ddot{\phi}_b - g \\
    a_{b1R} &= \ddot{z}_b + w_{x1}\dot{\phi}_b - l_{a1}\ddot{\phi}_b - g \\
    a_{b2L} &= \ddot{z}_b - w_{x2}\dot{\phi}_b - l_{a2}\ddot{\phi}_b - g
\end{align}

(11.15a) (11.15b) (11.15c)

The other four accelerometers are assumed to be positioned on the axles at the suspension strut bottom mounting (figure 11.1). These accelerations are given by

\begin{align}
    a_{a1L} &= \ddot{z}_b - w_{y1}\dot{\phi}_b - l_{a1}\ddot{\phi}_b - l_{x}\ddot{\phi}_{a1L} - g \\
    a_{a1R} &= \ddot{z}_b + w_{y1}\dot{\phi}_b - l_{a1}\ddot{\phi}_b + l_{x}\ddot{\phi}_{a1R} - g \\
    a_{a2L} &= \ddot{\phi}_{a2} - w_{y2}\ddot{\phi}_{a2} - g \\
    a_{a2R} &= \ddot{\phi}_{a2} + w_{y2}\ddot{\phi}_{a2} - g
\end{align}

(11.15d) (11.15e) (11.15f) (11.15g)

Figure 11.1 Location of the accelerometers.
It might look quite strange to subtract the acceleration due to gravity (\(g\)) as occurs with all equations (11.15) because the direction of the acceleration due to gravity corresponds to the positive vertical z-direction (downwards). Figure 11.2 gives an explanation. An accelerometer is a device that does not measure accelerations. The sensor can be considered as a single degree of freedom system consisting of a mass with a spring and damper. Providing that the system is well tuned, the deflection of the spring is a measure of the acceleration level. Any motion of the sensor in its sensing direction results in a motion of the mass. Thus, a positive downwards acceleration (\(\ddot{z}\)) is coupled with an upwards motion of the mass due to inertia effects. Now, let us say that the expansion of the spring corresponds to a positive output signal of the accelerometer. The acceleration of gravity causes the mass to go downwards resulting in a compression of the spring and therefore a negative output signal will be measured. Although both accelerations point downward, they result in opposite output signals! Dropping the sensor results in zero output.

![Outline of an accelerometer.](image)

**Figure 11.2** Outline of an accelerometer.

The acceleration due to gravity (\(g\)) must be added to all acceleration signals in order to create a zero output in a steady-state situation. This normally happens in the control computer. Checking out the observability of this seven sensor configuration tells us that the rank of \(W_0\) equals 7. Thus the requirement of observability has been fulfilled. After having defined the linear set of equations (11.15a) up to (11.15g) it is possible to construct the output equation of the set of state-space equations.

The estimation of the longitudinal and lateral tire forces requires once again three accelerometers: two in the lateral direction (lateral and yaw acceleration) and one in the longitudinal direction. Figure 11.1 gives an indication of the location of these three sensors. The required roll and pitch angular accelerations can be calculated from the three vertical accelerometers mounted on the vehicle body. The total number of required sensor equals now 10.
In the real vehicle the acceleration signals as indicated above act somewhat differently. The above set of outputs considers that the sensors do always measure the accelerations in the direction of the positive z-axis of the fixed coordinate system. This means that even in a tilted position of the vehicle body, the sensors point perpendicular to the road plane. In reality the accelerometers rotate according to the possibly changed orientation of the vehicle body. As the vehicle rolls during cornering for example, the accelerometers measure a component (sine of roll angle) of the lateral acceleration and a reduced component (cosine of roll angle) of the vertical acceleration. The same occurs with the pitch influence during braking and accelerating. The longitudinal and lateral acceleration must also be transformed to the local moving reference frame. Therefore, the following expressions for the acceleration signals must be used in all time-domain simulations

\[
a_{b1L} = \left( \dot{x}_b + \dot{h}_b \ddot{\theta}_b + w_{s1} \ddot{\psi}_b \right) \cdot \cos \psi_b + \left( \dot{y}_b - h_s \ddot{\phi}_b + l_{s1} \ddot{\psi}_b \right) \cdot \sin \psi_b \cdot \sin \theta_b
- \left( \dot{\theta}_b - h_s \ddot{\phi}_b + l_{s1} \ddot{\psi}_b \right) \cdot \cos \psi_b - \left( \dot{x}_b + h_s \ddot{\phi}_b + w_{s1} \ddot{\psi}_b \right) \cdot \sin \psi_b \cdot \sin \phi_b
+ \left( \ddot{z}_b - w_{s1} \ddot{\psi}_b - l_{s1} \ddot{\theta}_b - g \right) \cdot \cos \phi_b \cdot \cos \theta_b
\]

(11.16a)

\[
a_{b1R} = \left( \dot{x}_b + h_s \ddot{\phi}_b - w_{s1} \ddot{\psi}_b \right) \cdot \cos \psi_b + \left( \dot{y}_b - h_s \ddot{\phi}_b + l_{s1} \ddot{\psi}_b \right) \cdot \sin \psi_b \cdot \sin \phi_b
- \left( \dot{\theta}_b - h_s \ddot{\phi}_b + l_{s1} \ddot{\psi}_b \right) \cdot \cos \psi_b - \left( \dot{x}_b + h_s \ddot{\phi}_b - w_{s1} \ddot{\psi}_b \right) \cdot \sin \psi_b \cdot \sin \phi_b
+ \left( \ddot{z}_b + w_{s1} \ddot{\psi}_b - l_{s1} \ddot{\theta}_b - g \right) \cdot \cos \phi_b \cdot \cos \theta_b
\]

(11.16b)

\[
a_{b2L} = \left( \dot{x}_b + h_s \ddot{\phi}_b + w_{s2} \ddot{\psi}_b \right) \cdot \cos \psi_b + \left( \dot{y}_b - h_s \ddot{\phi}_b + l_{s2} \ddot{\psi}_b \right) \cdot \sin \psi_b \cdot \sin \phi_b
- \left( \dot{\theta}_b - h_s \ddot{\phi}_b + l_{s2} \ddot{\psi}_b \right) \cdot \cos \psi_b - \left( \dot{x}_b + h_s \ddot{\phi}_b + w_{s2} \ddot{\psi}_b \right) \cdot \sin \psi_b \cdot \sin \phi_b
+ \left( \ddot{z}_b - w_{s2} \ddot{\psi}_b - l_{s2} \ddot{\theta}_b - g \right) \cdot \cos \phi_b \cdot \cos \theta_b
\]

(11.16c)

\[
a_{a1L} = \left( \dot{x}_b + (h_{p1} + l_{s2}) \ddot{\theta}_b + (w_{p1} + l_{s2}) \ddot{\psi}_b \right) \cdot \cos \psi_b
\left( \dot{y}_b - h_{p2} \ddot{\phi}_b + l_{s1} \ddot{\psi}_b - l_{s2} \ddot{\phi}_{a1L} \right) \cdot \sin \psi_b \cdot \sin \theta_b
- \left( \dot{\theta}_b - h_{p2} \ddot{\phi}_b + l_{s1} \ddot{\psi}_b - l_{s2} \ddot{\phi}_{a1L} \right) \cdot \cos \psi_b
- \left( \dot{x}_b + (h_{p1} + l_{s2}) \ddot{\phi}_b + (w_{p1} + l_{s2}) \ddot{\psi}_b \right) \cdot \sin \psi_b \cdot \sin \phi_b
+ \left( \ddot{z}_b - w_{p2} \ddot{\psi}_b - l_{s2} \ddot{\theta}_b - l_{s2} \ddot{\phi}_{a1L} - g \right) \cdot \cos \phi_b \cdot \cos \theta_b
\]

(11.16d)
\[ a_{1R} = \left( \ddot{x}_b + (h_{p1} + l_s c_a) \ddot{\theta}_b - (w_{p1} + l_s c_a) \dot{\psi}_b \right) \cos \psi_b \cdot \sin \theta_b + \left( \dot{y}_b - h_{p1} \dot{\theta}_b + l_{a1} \dot{\psi}_b - l_s \ddot{\theta}_{a1R} \right) \cos \psi_b \cdot \sin \phi_{a1R} - \left( \ddot{x}_b + (h_{p1} + l_s c_a) \ddot{\theta}_b - (w_{p1} + l_s c_a) \dot{\psi}_b \right) \cos \psi_b \cdot \sin \theta_b + \left( \ddot{z}_b + w_{p1} \dot{\phi}_b - l_{a1} \ddot{\theta}_b + l_s c_a \dddot{\theta}_{a1R} - g \right) \cos \phi_{a1R} \cdot \cos \theta_b \] (11.16e)

\[ a_{2L} = \left( \ddot{x}_b + (h_{p2} + l_s c_p - h_r) \ddot{\theta}_b + w_{a2} \dot{\psi}_b \right) \cos \psi_b \cdot \sin \theta_b + \left( \dot{y}_b + t_p \ddot{z}_b - (h_{p2} - w_{p2} t_p) \ddot{\theta}_b - l_{a2} \dot{\phi}_b \right) \cos \psi_b \cdot \sin \phi_{a2} - \left( \ddot{x}_b + (h_{p2} + l_s c_p - h_r) \ddot{\theta}_b + w_{a2} \dot{\psi}_b \right) \cos \psi_b \cdot \sin \theta_b + \left( \ddot{z}_b + w_{a2} \dddot{\theta}_{a2} - g \right) \cos \phi_{a2} \cdot \cos \theta_b \] (11.15f)

\[ a_{2R} = \left( \ddot{x}_b + (h_{p2} + l_s c_p - h_r) \ddot{\theta}_b + w_{a2} \dot{\psi}_b \right) \cos \psi_b \cdot \sin \theta_b + \left( \dot{y}_b + t_p \ddot{z}_b - (h_{p2} - w_{p2} t_p) \ddot{\theta}_b - l_{a2} \dot{\phi}_b \right) \cos \psi_b \cdot \sin \phi_{a2} - \left( \ddot{x}_b + (h_{p2} + l_s c_p - h_r) \ddot{\theta}_b + w_{a2} \dot{\psi}_b \right) \cos \psi_b \cdot \sin \theta_b + \left( \ddot{z}_a + w_{a2} \dddot{\theta}_{a2} - g \right) \cos \phi_{a2} \cdot \cos \theta_b \] (11.16g)

In order to estimate the tire slip forces, one longitudinal and two lateral accelerometers are required. The first lateral accelerometer and the sensor measuring the acceleration in the longitudinal direction are assumed to be situated in the center of gravity of the vehicle body. The second lateral accelerometer is situated at a distance \( l_{a12} \) in front of the other lateral \( g \)-sensor and a distance \( l_{a22} \) below the same sensor. The accelerations become

\[ a_{lon} = (\ddot{x}_b \cdot \cos \psi_b + \ddot{y}_b \cdot \sin \psi_b) \cdot \cos \theta_b - (\dddot{z}_b - g) \cdot \sin \theta_b \] (11.16h)
\[ a_{lat1} = (\ddot{y}_b \cdot \cos \phi_b - \dot{x}_b \cdot \sin \psi_b) \cdot \cos \phi_b + (\ddot{z}_b - g) \cdot \sin \phi_b \]  
(11.16i)

\[ a_{lat2} = \left( \left( \ddot{y}_b - h_{lat2} \ddot{\theta}_b + l_{lat2} \ddot{\psi}_b \right) \cdot \cos \psi_b - \left( \dot{x}_b + h_{lat2} \ddot{\theta}_b \right) \cdot \sin \psi_b \right) \cdot \cos \phi_b + (\ddot{z}_b - l_{lat2} \ddot{\theta}_b - g) \cdot \sin \phi_b \]  
(11.16j)

The contribution of the acceleration due to gravity to the longitudinal, lateral and vertical acceleration is clarified in figure 11.3.

**Figure 11.3** Contribution of the acceleration of gravity \((g)\) to the longitudinal, lateral and vertical accelerometer signals.
11.3 Discrete-Time Kalman Filter

The continuous time state-space equation of a Kalman filter is defined by

\[
\dot{x}_t = A_t \dot{x}_t + F_t u_x + T^x_{it} \dot{F}^x_t + T^y_{it} \dot{F}^y_t + K_t (y_t - C_t \dot{x}_t - F_{ot} u_a - T^x_{ot} \dot{F}^x_t - T^y_{ot} \dot{F}^y_t) \tag{11.17}
\]

With exception of two differences, this 18th-order system agrees almost with equation (11.4). Firstly, the Kalman filter does not contain the white noise road input because this noise is unknown. Secondly, equation (11.17) contains a feedback of the measured output \(y_t\) and the estimated output \(\hat{y}_t\) given by

\[
\dot{\hat{y}}_t = C_t \dot{x}_t + F_{ot} u_a + T^x_{ot} \dot{F}^x_t + T^y_{ot} \dot{F}^y_t \tag{11.18}
\]

This feedback guarantees that the estimated output \(\hat{y}_t\) follows the true output \(y_t\) if a proper selection of gain matrix \(K_t\) is made. If this is the case, it is likely that the estimated states \(\dot{x}_t\) follow the true state \(x_t\). The selection of feedback gain \(K_t\) must be made in such a way that the closed-loop Kalman filter \((A_t - K_t C_t)\) is stable and is fast enough to follow the measurement signals. However, the Kalman filter should also be robust to e.g. measurement errors such as noise \(u\). The speed of reconstruction can be assessed by means of the eigenvalues of the closed-loop Kalman filter. A fast reconstruction goes with fast dynamics and thus eigenvalues that are situated far away in the left half of the complex plane.

Since a Kalman filter is a typical system to be implemented in a digital signal processor (DSP), the continuous-time state-space description (11.4) must be converted to an equivalent discrete-time system given by

\[
x_{n+1} = \Phi_t x_n + \Gamma_t u_n + \Psi_t w_n + \Theta_t^x \dot{F}^x_t + \Theta_t^y \dot{F}^y_t \tag{11.19}
\]

where

\[
\Phi_t = e^{A_t \delta t} \quad \Gamma_t = \int_0^{\delta t} e^{A_t \delta \tau} d\tau \cdot F_{it} \\
\Psi_t = \int_0^{\delta t} e^{A_t \delta \tau} d\tau \cdot R_t \quad \Theta_t^x = \int_0^{\delta t} e^{A_t \delta \tau} d\tau \cdot T^x_{it} \\
\Theta_t^y = \int_0^{\delta t} e^{A_t \delta \tau} d\tau \cdot T^y_{it}
\]

The measurements are assumed to be sampled with a periodic interval \(\delta t\) and the D-A (digital to analog) converter holds the analog signal constant until a new conversion is commanded. The discrete-time closed-loop Kalman filter is given by
\[
\hat{x}_{n+1} = (\Phi_t - K_{fd} C_t) \hat{x}_n + (\Gamma_t - K_{fd} P_{ot}) u_n \\
+ (\Theta_t^x - K_{fd} T_{ot}^x) \hat{F}_{t,n}^x + (\Theta_t^y - K_{fd} T_{ot}^y) \hat{F}_{t,n}^y + K_{fd} y_n
\]  

(11.20)

Figure 11.4 shows a block diagram of the Kalman filter as described above.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure11_4}
\caption{Block diagram of a discrete-time Kalman filter.}
\end{figure}

A method of computing the observer feedback gain matrix \( K_{fd} \) is required. The method chosen is that one based on the theory developed by Kalman and Bucy [21] in which it is assumed that the system to be observed is driven by white noise \( w_n \) and the observed signals are corrupted by white noise \( v_n \). Feedback matrix \( K_{fd} \) can be found by solving the discrete-time algebraic Riccati equation.

\begin{align*}
S - \Phi_t S \Phi_t^T + \Phi_t S C_t^T (R_{fd} + C_t S C_t^T)^{-1} C_t S \Phi_t^T - \Psi_t Q_{fd} \Psi_t^T &= 0 \\
K_{fd} &= \Phi_t S C_t^T (R_{fd} + C_t S C_t^T)^{-1}
\end{align*}

(11.21a)

(11.21b)

where \( S \) is the covariance matrix of the state estimation error. Besides the discrete time state matrix \( \Phi_t \) and output matrix \( C_t \), the covariance matrices \( Q_{fd} \) and \( R_{fd} \) of white noise input \( w_n \) and measurement noise \( v_n \) respectively are required.
The continuous-time covariance matrix $Q_f (4 \times 4)$ of the white noise related to the road input is given by

$$Q_f = \frac{q}{v} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$  \hspace{1cm} (11.22)

with $q/v$ being the intensity of the white noise $w_n$. The discrete-time equivalent can be found by the following procedure [46]. First, let us define matrix $H$ given by

$$H = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} = \begin{bmatrix} -A_t & R_{it} \cdot Q_f \cdot R_{it}^T \\ 0 & A_t^T \end{bmatrix}$$  \hspace{1cm} (11.23)

The discrete-time equivalent of $H$ can be determined by

$$H^d = \varepsilon H^x = \begin{bmatrix} H_{11}^d & H_{12}^d \\ H_{21}^d & H_{22}^d \end{bmatrix}$$  \hspace{1cm} (11.24)

The discrete-time equivalent of continuous-time road noise covariance matrix $Q_f$ is found through

$$Q_{fd} = \frac{H_{22}^d \cdot H_{12}^d + H_{12}^d \cdot H_{22}^d}{2}$$  \hspace{1cm} (11.25)

The discrete-time equivalent of the continuous-time measurement noise intensity matrix $R_f (7 \times 7)$ can easily be obtained from

$$R_{fd} = \frac{R_f}{\delta t}$$  \hspace{1cm} (11.26)

Similar to the Kalman filter design for the quarter-car model, the following rule of thumb is used to determine the magnitude of covariance matrix $R_f$

$$R_f = n_f \sqrt{\text{diag}(C_t \cdot C_t^T)}$$  \hspace{1cm} (11.27)

where $n_f$ is a noise factor which can be chosen freely. All matrices necessary to calculate the Kalman filter feedback matrix $K_{fd}$ are now known. The Kalman filter is ready to be tuned in order to fulfil the requirements of speed of reconstruction and noise sensitivity.

### 11.4 Tuning the Kalman Filter

The main variable to tune the estimation properties of the Kalman filter as described above is noise factor $n_f$. A small value of $n_f$ makes the estimation of
the states to become faster but also more sensitive to measurement noise. Therefore, a compromise between the speed of reconstruction and noise sensitivity must be found. The design of the Kalman filter feedback matrix is based on the following assumptions:

- Sampling time $\delta t = 0.001$ s.
- Concrete road $v = 120$ km/h.

The operation of the closed-loop Kalman filter will be checked using RMS estimation error values and time-domain simulations. The main variables of interest are:

- The relative velocity of each wheel suspension $(\dot{d}_{s1L}, \dot{d}_{s1R}, \dot{d}_{s2L}, \dot{d}_{s2R})$ according to equation (9.24a-d).
- The absolute vertical velocity of the sprung mass at each wheel suspension top strut mounting $(v_{b1L}, v_{b1R}, v_{b2L}, v_{b2R})$ according to equation (9.24e-h).
- The vertical tire load of each wheel $(F_{t1L}, F_{t1R}, F_{t2L}, F_{t2R})$ according to equation (9.24a-d).

The eight estimated variables of interest for the output feedback (relative and absolute velocities) can be written as

$$\hat{\mathbf{y}}_e = C_e \hat{x}_t$$

where $\hat{\mathbf{y}}_e$ is a vector containing the four relative and absolute velocities, and $C_e$ is an $8 \times 18$ output matrix. The RMS estimation error can now easily be calculated from

$$RMS(\mathbf{y}_e - \hat{\mathbf{y}}_e) = \sqrt{\text{diag}(C_e \cdot S \cdot C_e^T)}$$

(11.29)

In case a Kalman filter is used, the unknown actuator forces $u_a$ can be calculated from

$$u_a = K_o \hat{\mathbf{y}}_e = K_o C_e \hat{x}_t$$

(11.30)

A more elaborate way to examine the performance of the active suspension system as well as the Kalman filter is combining the state-space description of the 13 DOF continuous-time vehicle model with the discrete-time Kalman filter. However, it is not possible to mix continuous-time systems with discrete-time systems in one state-space description. Therefore, either the vehicle model must be transformed to a discrete-time representation or the Kalman filter must be transformed to a continuous-time representation. The latter possibility is preferred because the performance calculations in the
continuous-time domain are more easily to carry out. Using the modified Lyapunov equation (8.3) there is no need to approximate the delayed rear wheel road input. The continuos-time representation of the discrete-time closed-loop Kalman filter (11.18) can be expressed as

$$\dot{\hat{x}}_{kf} = A_{kf} \hat{x}_{kf} + B_{kf} \dot{u}_a + K_f \gamma_t$$  \hspace{1cm} (11.31)

where $A_{kf}$ is the continuous-time closed-loop system matrix of the Kalman filter, $B_{kf}$ the continuous-time closed-loop input matrix for the actuator force and $K_f$ the continuous-time Kalman filter feedback matrix. The known inputs (longitudinal and lateral tire slip forces) are omitted because the performance calculations are carried out while driving straight on with a constant speed of travel. The continuous-time equivalents of the discrete-time state-space description of the closed-loop Kalman filter can be calculated as follows

$$P = \ln \begin{bmatrix} (\Phi_t - K_f C) - K_f F_{ot} \\ 0 \end{bmatrix} \frac{1}{\delta t} = \begin{bmatrix} A_{kf} & B_{kf} \\ P_{21} & P_{22} \end{bmatrix}$$  \hspace{1cm} (11.32a)

$$P = \ln \begin{bmatrix} (\Phi_t - K_f C) - K_f F_{ot} \\ 0 \end{bmatrix} \frac{1}{\delta t} = \begin{bmatrix} A_{kf} & K_f \\ P_{21} & P_{22} \end{bmatrix}$$  \hspace{1cm} (11.32b)

The state-space equations of the vehicle model and Kalman filter can be combined according to

$$\begin{bmatrix} \dot{x}_t \\ \dot{\hat{x}}_{kf} \end{bmatrix} = \begin{bmatrix} A_t & F_t K_s C_s \\ K_s C_s & A_{kf} + (B_{kf} + K_f F_{ot}) K_s C_s \end{bmatrix} \begin{bmatrix} x_t \\ \hat{x}_{kf} \end{bmatrix} + \begin{bmatrix} R_t & 0 \\ 0 & K_f \end{bmatrix} \begin{bmatrix} \omega_t \\ \nu \end{bmatrix}$$  \hspace{1cm} (11.33)

In the performance calculations the 13 DOF (26 states) vehicle model has been augmented with a 2 track road model (4 states), a transient tire model (4 states) and a driver model (2 states). Along with the 36 states related to the total driver-vehicle-environment system, 18 states from the Kalman filter must be added. The total state-space model (11.33) contains now 54 states and the performance can easily be calculated from equation (8.3) using the modified Lyapunov equation that accounts for the delayed rear wheel input.

The Kalman filter design is independent of the active suspension feedback law since the internal vehicle model used in the Kalman filter is an open-loop model. In order to examine the influence of the feedback by means of estimated variables on the active suspension performance, the following output feedback has been considered.
\[
\begin{bmatrix}
  u_{1L} \\
  u_{1R} \\
  u_{2L} \\
  u_{2R}
\end{bmatrix} =
\begin{bmatrix}
  -308 & 0 & 0 & 0 & 7648 & 0 & 0 & 0 \\
  0 & -308 & 0 & 0 & 0 & 7648 & 0 & 0 \\
  -255 & 0 & -254 & 0 & -6220 & 0 & 21631 & 0 \\
  0 & -255 & 0 & -254 & 0 & -6220 & 0 & 21631
\end{bmatrix}
\begin{bmatrix}
  \hat{d}_{s1L} \\
  \hat{d}_{s1R} \\
  \hat{d}_{s2L} \\
  \hat{d}_{s2R} \\
  \hat{b}_{b1L} \\
  \hat{b}_{b1R} \\
  \hat{b}_{b2L} \\
  \hat{b}_{b2R}
\end{bmatrix}
\]  
(11.34)

This feedback law corresponds to the results of the partially coupled active damping analysis carried out in section 9.2.2. The gains shown above are valid for \( q_1 = 10 \) and a forward speed equal to 120 km/h.

Firstly, the RMS estimation errors are calculated for several values of \( n_f \). The measurement noise intensity matrix is assumed to be of a diagonal form and takes a form as

\[
R_f = n_f \sqrt{\text{diag}(C_i \cdot C_i^T)} = n_f \cdot 
\begin{bmatrix}
  778 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 779 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 717 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 55786 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 55786 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 9953 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 9900 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 9900
\end{bmatrix}
\]  
(11.35)

From the diagonal form it can be seen that all noise sources are uncorrelated. Thus, there is no interaction between the noise generated from each accelerometer. After having determined covariance matrices \( Q_f \) (equation 11.22) and \( R_f \), it is easy to calculate the discrete time equivalents \( Q_{fd} \) and \( R_{fd} \) based on equation (11.25) and (11.26). It is now just a matter of transforming the continuous-time state-space equations of the seven DOF vehicle model augmented with the road-space model and solving the discrete-time algebraic Riccati equation (11.21) for several values of noise factor \( n_f \). The best Kalman filter is an estimator with a good balance between the speed of reconstruction and sensitivity to measurement noise.

Figure 11.5 presents the relative estimation error as a function of noise factor \( n_f \). Twelve variables are considered: the deflection rate of each wheel
suspension \( v_{\text{rel}} \) or \( \dot{d}_s \)), the absolute vertical velocity of the sprung mass at the strut top mounting at the four corners of the car \( v_{\text{abs}} \) or \( v_b \) and the dynamic tire load of each wheel \( P_{\text{dyn}} \) or \( P_{\text{t}} \). The numbers presented are relative errors. This means that first the RMS values of each variable of interest must be calculated for the closed-loop system. After that the RMS estimation errors are calculated and finally the RMS relative error is the ratio of both numbers. The relative RMS estimation errors are more useful than the plain RMS estimation errors since the closed-loop performance of the actively suspended vehicle using a Kalman filter depends on the quality of estimation.

![Graphs showing RMS relative estimation errors as a function of noise factor \( n_f \).]

\[ \text{Figure 11.5 RMS relative estimation errors as a function of noise factor } n_f. \]
\[ \text{(--- = front suspension, - - - = rear suspension)} \]

From figure 11.5 it can be seen that, with exception of the relative velocity of the front suspension, the quality of estimation improves as \( n_f \) gets smaller. This is not surprising because the speed of reconstruction and thus the quality of estimation improves as the amount of assumed measurement noise decreases. The model reduction from 13 to 7 DOFs must be the reason why the estimation of the front suspension deflection velocity deteriorates when \( n_f \) decreases. The omission of powertrain body might cause estimation errors because the seven DOF internal model does not contain a powertrain with its rigid body natural frequencies very close to the front axle wheel-hop and
tramp natural frequencies. Apparently, the demand for estimation accuracy conflicts with the desired robustness. Another reason might be the ratio of the numbers in the measurement intensity matrix \( R_f \) (equation (11.35)). The values that correspond to the accelerometers mounted on the front axle are considerably larger than those corresponding to the rear axle sensors. This indicates that the level of assumed measurement noise is larger for the front axle sensors. This might result in a deterioration of the quality of estimation.

From the 12 relative estimation errors an average value can be determined and plotted. The fourth plot of figure 11.5 shows that the 'optimal value' for factor \( n_f \) equals 3.0e-9. The sensitivity to measurement noise cannot be assessed by means of these RMS values unless the stochastic (white noise) properties are included in the covariance calculations. Time-domain simulations including sensor noise is one of the possibilities to examine the deterioration of the estimations caused by sensor noise. Table 11.1 presents the RMS values for \( n_f = 3.0e-9 \). Whether the magnitude of the relative errors is acceptable can only be gathered from the closed-loop performance and time-domain simulations.

<table>
<thead>
<tr>
<th>variable</th>
<th>place</th>
<th>unit</th>
<th>RMS value</th>
<th>RMS error</th>
<th>Relative error</th>
</tr>
</thead>
<tbody>
<tr>
<td>relative velocity</td>
<td>1L</td>
<td>m/s</td>
<td>0.2761</td>
<td>0.0654</td>
<td>24%</td>
</tr>
<tr>
<td>relative velocity</td>
<td>1R</td>
<td>m/s</td>
<td>0.2756</td>
<td>0.0687</td>
<td>25%</td>
</tr>
<tr>
<td>relative velocity</td>
<td>2L</td>
<td>m/s</td>
<td>0.2236</td>
<td>0.0149</td>
<td>7%</td>
</tr>
<tr>
<td>relative velocity</td>
<td>2R</td>
<td>m/s</td>
<td>0.2405</td>
<td>0.0152</td>
<td>6%</td>
</tr>
<tr>
<td>absolute velocity</td>
<td>1L</td>
<td>m/s</td>
<td>0.0153</td>
<td>0.0016</td>
<td>11%</td>
</tr>
<tr>
<td>absolute velocity</td>
<td>1R</td>
<td>m/s</td>
<td>0.0152</td>
<td>0.0016</td>
<td>11%</td>
</tr>
<tr>
<td>absolute velocity</td>
<td>2L</td>
<td>m/s</td>
<td>0.0114</td>
<td>0.0018</td>
<td>16%</td>
</tr>
<tr>
<td>absolute velocity</td>
<td>2R</td>
<td>m/s</td>
<td>0.0114</td>
<td>0.0021</td>
<td>18%</td>
</tr>
<tr>
<td>dynamic tire load</td>
<td>1L</td>
<td>N</td>
<td>744</td>
<td>52</td>
<td>7%</td>
</tr>
<tr>
<td>dynamic tire load</td>
<td>1R</td>
<td>N</td>
<td>742</td>
<td>53</td>
<td>7%</td>
</tr>
<tr>
<td>dynamic tire load</td>
<td>2L</td>
<td>N</td>
<td>626</td>
<td>65</td>
<td>10%</td>
</tr>
<tr>
<td>dynamic tire load</td>
<td>2R</td>
<td>N</td>
<td>680</td>
<td>62</td>
<td>9%</td>
</tr>
</tbody>
</table>

Table 11.1 RMS estimation errors \((n_f = 3.0e-9)\).

Table 11.2 presents the closed-loop RMS performance values for three systems. The first two systems are active suspensions with output feedback based on the partially coupled active damping structure (equation (11.34)). The third system corresponds to the standard passive vehicle and serves as a reference. The two active systems differ with respect to the output feedback. The first column corresponds to an ideal system where all states (or outputs) are measurable. The numbers in the second column are valid for an identical active system, however, the feedback has been achieved by means of
estimated variables. The first system is purely academic and can never be realized in practice, the second system shows more resemblance with a practical vehicle. The RMS values do not account for the performance degradation due to sensor noise $v_n$.

The closed-loop performance of the active system including a reduced order Kalman does not degrade much despite the moderate relative estimation errors (Table 11.1). The dynamic tire load variations at both front wheels are only slightly less compared with the ideal system. Also the rear suspension working space is insignificantly larger. All other parameters are almost identical. The percentages as plotted in both first columns are related to the gain (or loss) in performance with respect to the standard passive vehicle (third column). Table 11.2 clearly illustrates that the degradation of performance due to the inclusion of a Kalman filter is only marginal. However, the numbers do not account for the loss in performance caused by sensor noise.

<table>
<thead>
<tr>
<th>SUSPENSION TYPE</th>
<th>Active Damping</th>
<th>Passive</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>direct</td>
<td>Kalman filter</td>
</tr>
<tr>
<td>variable</td>
<td>place</td>
<td>unit</td>
</tr>
<tr>
<td>longitudinal acceleration</td>
<td>c.g.</td>
<td>m/s²</td>
</tr>
<tr>
<td>lateral acceleration</td>
<td>c.g.</td>
<td>m/s²</td>
</tr>
<tr>
<td>vertical acceleration</td>
<td>c.g.</td>
<td>m/s²</td>
</tr>
<tr>
<td>roll acceleration</td>
<td>c.g.</td>
<td>1/s²</td>
</tr>
<tr>
<td>pitch acceleration</td>
<td>c.g.</td>
<td>1/s²</td>
</tr>
<tr>
<td>yaw acceleration</td>
<td>c.g.</td>
<td>1/s²</td>
</tr>
<tr>
<td>dynamic tire load</td>
<td>1L</td>
<td>N</td>
</tr>
<tr>
<td>dynamic tire load</td>
<td>1R</td>
<td>N</td>
</tr>
<tr>
<td>dynamic tire load</td>
<td>2L</td>
<td>N</td>
</tr>
<tr>
<td>dynamic tire load</td>
<td>2R</td>
<td>N</td>
</tr>
<tr>
<td>suspension deflection</td>
<td>1L</td>
<td>mm</td>
</tr>
<tr>
<td>suspension deflection</td>
<td>1R</td>
<td>mm</td>
</tr>
<tr>
<td>suspension deflection</td>
<td>2L</td>
<td>mm</td>
</tr>
<tr>
<td>suspension deflection</td>
<td>2R</td>
<td>mm</td>
</tr>
<tr>
<td>longitudinal acceleration</td>
<td>f-seat</td>
<td>m/s²</td>
</tr>
<tr>
<td>lateral acceleration</td>
<td>f-seat</td>
<td>m/s²</td>
</tr>
<tr>
<td>vertical acceleration</td>
<td>f-seat</td>
<td>m/s²</td>
</tr>
<tr>
<td>vector sum</td>
<td>f-seat</td>
<td>m/s²</td>
</tr>
<tr>
<td>longitudinal acceleration</td>
<td>r-seat</td>
<td>m/s²</td>
</tr>
<tr>
<td>lateral acceleration</td>
<td>r-seat</td>
<td>m/s²</td>
</tr>
<tr>
<td>vertical acceleration</td>
<td>r-seat</td>
<td>m/s²</td>
</tr>
<tr>
<td>vector sum</td>
<td>r-seat</td>
<td>m/s²</td>
</tr>
</tbody>
</table>

Table 11.2 RMS closed-loop performance values (partially coupled active damping, $q_1 = 10$, $v = 120$ km/h, without sensor noise).

- 302 -
Figure 11.6 Time domain example of the quality of estimation (120 km/h).
(----- = real variable, ------ = estimated variable)
A time-domain simulation has been carried out in order to give an impression of the estimation quality. Figure 11.6 shows the 12 variables of interest. The speed of travel is 120 km/h and the vehicle is driving straight on. In the simulation phase it is necessary to include some measurement noise. The sensor noise is assumed to be of a white discrete type and the RMS values are chosen according to:

- RMS sensor noise of body mounted accelerometer = 0.04 m/s²
- RMS sensor noise of axle mounted accelerometer = 2.0 m/s²

These values correspond approximately to 10% of the nominal level of accelerations at the position where the accelerometer is mounted. The figures show that the deflection rate of the front suspension is the variable that is the most difficult to estimate. The estimated signals are drawn with a dashed line, the true values are plotted using a solid line type. The neglect of the transmission in the internal model is probably the reason for the small deviations. The other variables can be estimated very well.

The time domain-simulation including handling maneuvers is also of considerable interest because the results should point out whether the estimation of the known inputs (tire slip forces) is correct. Furthermore, the sensitivity of the estimation to the offset of the accelerometer signals caused by the small components of the acceleration due to gravity and the lateral and longitudinal acceleration during cornering and braking is of considerable interest. Therefore the actively suspended vehicle is subjected to a double lane-change maneuver. The speed of travel is 80 km/h. The internal model remains unchanged. This means that the model is not adapted for the change in speed of travel (assumed: 120 km/h, reality: 80 km/h). The speed plays only a role in the road model. Figure 11.7 presents the results.

The tire side forces are estimated using two lateral accelerometers (lateral and yaw acceleration) and the two vertical accelerometers (roll acceleration). Since the front suspension is of an independent type it is necessary to calculate the tire side forces of the left and right wheel separately. For the rear axle only the total side force (left + right) is required because the rear suspension is of a rigid axle type. Figure 11.7 shows that the tire side forces are estimated well. The estimated forces are somewhat corrupted by noise. Both road unevennesses and measurement noise (assumed to be of a white type) are responsible for the corruption. The dynamic tire load can also be estimated accurately enough. Even the load transfer is estimated correctly. However, serious problems arise with the absolute vertical sprung mass velocities. These signals seem to drift away.
Figure 11.7 Quality of estimation during a lane-change maneuver (80 km/h).

(— = real variable, --- = estimated variable)
The main reason is the small contribution of the lateral and gravitational acceleration to the vertical sprung mass acceleration caused by a roll angle of the vehicle body. Both accelerations are interpreted as a small (almost constant) vertical acceleration. The Kalman filter is not able to distinguish both small components from the true vertical acceleration. This results in a drift of the estimated absolute velocities. However, the signals do not approach infinity. It takes about 10 seconds for the Kalman filter to return the estimated absolute velocities to the real ones. The estimation of the relative velocity seems to be quite insensitive to the offset of the acceleration signals.

One possibility to overcome the drift problem is to correct the measured accelerations for the lateral, longitudinal and gravitational acceleration components. Since we know the roll and pitch angle from the internal model it is quite easy to correct the accelerometer signals. First of all the roll, pitch and yaw angular accelerations must be calculated. They are given by

\[
a_{\text{roll}} = \frac{a_{b1R} - a_{b1L}}{2u_1} \tag{11.36a}
\]

\[
a_{\text{pitch}} = \frac{a_{b2L} - a_{b1L} + (w_{z2} - w_{z1}) \cdot a_{\text{roll}}}{l_{a1} - l_{a2}} \tag{11.36b}
\]

\[
a_{\text{yaw}} = \frac{a_{\text{lat1}} - a_{\text{lat2}} + h_{\text{lat2}} \cdot a_{\text{roll}}}{l_{\text{lat1}}} \tag{11.36c}
\]

All ten accelerometer signals can now be compensated for the lateral, longitudinal and gravitational acceleration. Samples of the estimated roll and pitch angles of a previous time step are used for the compensation. The ten corrected accelerations (denoted with the superscript c) read as follows

\[
a_{\text{roll}}^c = \frac{a_{b1L}}{\cos \delta_b \cdot \cos \theta_b} + g \left( a_{\text{lon}} + h_{s1} a_{\text{pitch}} + w_{z1} a_{\text{yaw}} \right) \cdot \tan \delta_b
\]

\[
+ \left( a_{\text{lat}} - h_{s1} a_{\text{roll}} + l_{a1} a_{\text{yaw}} \right) \cdot \tan \delta_b \tag{11.37a}
\]

\[
a_{\text{pitch}}^c = \frac{a_{b1R}}{\cos \phi_b \cdot \cos \theta_b} + g \left( a_{\text{lon}} + h_{s1} a_{\text{pitch}} - w_{z1} a_{\text{yaw}} \right) \cdot \tan \delta_b
\]

\[
+ \left( a_{\text{lat}} - h_{s1} a_{\text{roll}} + l_{a1} a_{\text{yaw}} \right) \cdot \tan \delta_b \tag{11.37b}
\]

\[
a_{\text{yaw}}^c = \frac{a_{b2L}}{\cos \phi_b \cdot \cos \theta_b} + g \left( a_{\text{lon}} + h_{s2} a_{\text{pitch}} + w_{z2} a_{\text{yaw}} \right) \cdot \tan \delta_b
\]

\[
+ \left( a_{\text{lat}} - h_{s2} a_{\text{roll}} + l_{a2} a_{\text{yaw}} \right) \cdot \tan \delta_b \tag{11.37c}
\]
\[ a_{\text{on}}^c = \frac{a_{\text{on}}}{\cos \hat{\theta}_b} \cdot \tan \hat{\theta}_b + g \left( a_{\text{lat}} - h_p a_{\text{roll}} + l_a a_{\text{yaw}} \right) \cdot \tan \hat{\theta}_b \]  

\[ a_{\text{on}}^R = \frac{a_{\text{on}}^R}{\cos \hat{\theta}_b} + g \left( a_{\text{lat}} - h_p a_{\text{roll}} + l_a a_{\text{yaw}} \right) \cdot \tan \hat{\theta}_b \]  

\[ a_{\text{on}}^L = \frac{a_{\text{on}}^L}{\cos \hat{\theta}_b} + g \left( a_{\text{lat}} + h_p a_{\text{roll}} + l_a a_{\text{yaw}} \right) \cdot \tan \hat{\theta}_b \]  

\[ a_{\text{lat}} = \frac{a_{\text{lat}} + g \sin \hat{\theta}_b}{\cos \hat{\theta}_b} \]  

\[ a_{\text{lat1}} = \frac{a_{\text{lat1}} + g \sin \hat{\theta}_b}{\cos \hat{\theta}_b} \]  

\[ a_{\text{lat2}} = \frac{a_{\text{lat2}} + g \sin \hat{\theta}_b}{\cos \hat{\theta}_b} \]  

The double lane-change test must now be repeated in order to judge the influence of the corrections. Figure 11.8 presents the results.

The drift phenomenon has been reduced considerably compared with figure 11.7. However, it can not be eliminated totally. This is probably owing to the difference between estimated and true roll angle. The compensation as proposed above works only satisfactory when the estimation of the absolute roll and pitch angle is correct. Since the correction is based on estimated angles it cannot be guaranteed that the compensation is effective in all situations. While driving on a leaning road surface or during the ascent or descent of a hill a constant component of the lateral or longitudinal acceleration is present which cannot be corrected using the estimated roll and pitch angle. The almost constant offset of the acceleration requires other measures such as for example high-pass filtering of the vertical acceleration signals.
Figure 11.8 Quality of estimation during a lane-change maneuver using corrected accelerometer signals (80 km/h).

(— = real variable, - - - = estimated variable)
From the foregoing the conclusion can be drawn that any offset in the vertical acceleration signals causes serious estimation problems. The offset of the acceleration signal can be divided into

- Very low frequency drift (0-0.2 Hz) caused by leaning road surfaces, ascending or descending a hill or steady-state cornering and/or braking.
- Low frequency drift (0.2-1 Hz) caused by transient handling maneuvers.

The low frequency offset caused by roll and pitch dynamics can be compensated very well using the corrected acceleration signals according to equations (11.36) and (11.37) provided that the estimation of the roll and pitch angle works satisfactory. The very low frequency components can only be eliminated by using high-pass filters or by the application of a stabilized platform. Since the latter solution is very expensive, the application of high-pass filters is one of the remaining possibilities to eliminate the offset. These filters should be selected and tuned in such a way that the very low frequency acceleration components (0-0.2 Hz) are completely removed and that the low frequency vertical accelerations (0.2-2 Hz) are not distorted too much. The filters should only be applied to the vertical accelerometer signals mounted on the vehicle body and axle. The two lateral and longitudinal accelerometer signals may not be high-pass filtered because they should also sense steady-state accelerations caused by (steady-state) cornering and/or braking or accelerating.

The effectiveness of high-pass filters depends on the type of filter and cut-off frequency selected. The higher the order and the higher the cut-off frequency, the sooner an offset will be removed. However, high-pass filtering can distort the low frequency acceleration signals considerably. Especially the accelerometers mounted on the vehicle body require a sensible tuning. If too much of the low frequency signal contents is removed, much of the low frequent body information will be thrown away. Furthermore, the phase distortion is very important. The higher the order of the high-pass filter the more the phase distortion. Therefore the order should be selected as low as possible preferably first-order. The transfer function of a first-order high-pass filter is given by

\[
H_f(s) = \frac{\tau s}{\tau s + 1}
\]

(11.38)

where \(\tau\) denotes the time constant which is equal to
\[ \tau = \frac{1}{2\pi f_0} \]  

(11.39)

with \( f_0 \) the cut-off frequency in Hz. Figure 11.9 shows several frequency response functions of first-order high-pass filters with different cut-off frequencies and figure 11.10 presents the corresponding step responses.

**Figure 11.9** Frequency response functions of first-order high-pass filters with different cut-off frequencies.

**Figure 11.10** Step responses of first-order high-pass filters with different cut-off frequencies.
A high cut-off frequency is favorable with respect to the elimination of an offset. However, the phase distortion equals already +45° at 1 Hz for a high-pass filter with a cut-off frequency of 1 Hz. Since the sprung mass natural frequencies are quite close to 1 Hz, it is questionable whether this phase lead is acceptable. The step responses tell us that it is desired to have a high cut-off frequency because the offset (step) is eliminated very fast. The elimination time is approximately inversely proportional to the cut-off frequency. (10 sec. for the 0.1 Hz filter, 1 sec. for the 1 Hz filter). In practice this time should not be too long. Ten seconds is all right while ascending or descending a hill, but is probably too long for e.g. large bends on highway exits. A cut-off frequency of 0.5 Hz seems to be a good compromise between speed of elimination (= 2 sec.) and phase distortion. In order to examine the influence of high-pass filtering of the seven vertical accelerometer signals on the estimation of the state variables calculations and simulations must be carried out.

Similar to the first tuning stage, the performance of the 13 DOF vehicle model augmented with seven sensor filters and a Kalman filter must be calculated. Assume that the state-space description of the first-order high-pass filters is given by

\[
\begin{align*}
\dot{x}_f &= A_f x_f + B_f y_f, \\
y_f &= C_f x_f + D_f y_f
\end{align*}
\]

(11.40a) (11.40b)

with \(A_f(7 \times 7), B_f(7 \times 7), C_f(7 \times 7)\) and \(D_f(7 \times 7)\) given by

\[
A_f = \begin{bmatrix}
-2\pi f_0 & 0 & 0 \\
0 & \ddots & 0 \\
0 & 0 & -2\pi f_0
\end{bmatrix}, \quad B_f = \begin{bmatrix}
1 & 0 & 0 \\
0 & \ddots & 0 \\
0 & 0 & 1
\end{bmatrix}, \quad C_f = A_f, \quad D_f = B_f
\]

Both vehicle model and high-pass filters can be combined to one total model given by

\[
\begin{align*}
\dot{x}_t &= A_t x_t + F_{it} u_{ia} + R_{it} z_e + T_{it}^x F_{t}^x + T_{it}^y F_{t}^y, \\
y_{it} &= C_t x_t + F_{ot} u_{oa} + R_{ot} z_e + T_{ot}^x F_{t}^x + T_{ot}^y F_{t}^y + v
\end{align*}
\]

(11.41a) (11.41b)

with

\[
A_t = \begin{bmatrix}
A_f & 0 \\
B_f C_f & A_f
\end{bmatrix}, \quad F_{it} = \begin{bmatrix}
F_i \\
B_f F_o
\end{bmatrix}, \quad R_{it} = \begin{bmatrix}
R_i \\
B_f R_o
\end{bmatrix}, \quad T_{it}^x = \begin{bmatrix}
T_{i}^x \\
B_f T_{o}^x
\end{bmatrix}, \quad T_{it}^y = \begin{bmatrix}
T_{i}^y \\
B_f T_{o}^y
\end{bmatrix}
\]

\[
C_t = \begin{bmatrix}
D_f C_f & C_f \\
D_f F_o & C_f
\end{bmatrix}, \quad F_{ot} = D_f F_o, \quad R_{ot} = D_f R_o, \quad T_{ot}^x = D_f T_{o}^x, \quad T_{ot}^y = D_f T_{o}^y
\]
and

\[ x_i = \begin{bmatrix} x_v \\ x_f \end{bmatrix} \]

The RMS estimation error as a function of noise factor \( n_f \) has been plotted in figure 11.11. Most striking is the enormous increase in the estimation error of the sprung mass absolute vertical velocity. The quality of estimation of the relative velocity as well as the dynamic tire load is quite insensitive to the high-pass filters. It is likely that the cut-off frequency selected is too close to the sprung mass natural frequencies. The phase distortion of the high-pass filters corrupt the vertical sprung mass accelerations in such a way that the Kalman filter is not able to reconstruct the low frequency information of the sprung mass absolute vertical velocities.

![Graphs showing RMS estimation errors as a function of noise factor](image)

**Figure 11.11** RMS relative estimation errors as a function of noise factor \( n_f \). The accelerometer signals are high-pass filtered.

(--- = front suspension, - - - = rear suspension)

A time-domain simulation illustrates the effect of the signal distortion of the applied high-pass filter. Figure 11.12 presents the results for driving straight on with a speed of 120 km/h.
Figure 11.12 Time domain example of the quality of estimation (120 km/h). The accelerometer signals are high-pass filtered.

(----- = real variable, - - - = estimated variable)
Despite the absence of an accelerometer signal offset a low frequent drift can be observed with the sprung mass absolute vertical velocity. This is caused by the phase lead of the filtered accelerometer signals. The absolute velocities are distorted in such a way that the principle of skyhook damping is probably partially ineffective. The quality of estimation can only be improved by lowering the cut-off frequency of the high-pass filters resulting in less phase distortion in the 1 Hz range. However, the elimination of acceleration offset will take too much time and it is therefore not recommendable to lower the cut-off frequency. The one and only remaining possibility to get rid of the phase distortion is to include the high-pass filters also in the internal model. The seven DOF vehicle model (14 states) including the two track road model (4 states) must be augmented with seven first-order high-pass filters (7 states). The total internal model contains 25 states. The design of the Kalman filter must now be based on this extended model. Since both real vehicle and internal model contain the same dynamics (with exception of the drivetrain), it is expected that the extended version performs almost as well as the system without high-pass filters.

Figure 11.13 RMS relative estimation errors as a function of noise factor $n_f$. The accelerometer signals are high-pass filtered and the internal model contains also high-pass filters.

(— = front suspension, - - - = rear suspension)
Figure 11.13 presents the relative RMS estimation errors as a function of noise factor \( n_f \). Compared with figure 11.11 the relative estimation error of the absolute sprung mass velocity has been reduced by a factor 2 by including the high-pass filters in the internal model. Nevertheless, the errors are still very large compared with the initial design (figure 11.7).

The quality of estimation can be improved by altering the diagonal element of matrix \( R_f \) (equation 11.33). A reduction of the first three diagonal elements (assuming less sensor noise of the three body mounted accelerometers) result in a further deterioration of the relative velocity of the front suspension. Again the need for more accuracy meets with the problem of lacking sufficient robustness.

Figure 11.14 shows an example of the quality of estimation while driving straight on. The difference is obvious when the results are compared with those of figure 11.12. However, the absolute body velocity still cannot be estimated satisfactory accurately.

The desired quality of estimation is still not obtained despite all effort done in the previous sections. Obviously, the estimation quality is restricted by the seven DOF vehicle model. Any alteration in the noise intensity matrix \( R_f \) to improve the quality of estimation of the sprung mass absolute vertical velocity results in a further degradation of the estimation of the front suspension deflection rate. The omission of the powertrain should be blamed. It is therefore interesting to examine whether a ten DOF vehicle model as an internal model performs better. In addition to the seven DOF model the ten DOF model contains the three powertrain related DOFs. Similar to all other calculations and simulations carried out in the previous sections RMS estimation errors and time domain plots must be calculated in order to judge the new design. Figure 11.15 presents the RMS relative estimation errors as a function of noise factor \( n_f \).
Figure 11.14 Time domain example of the quality of estimation (120 km/h). The accelerometer signals are high-pass filtered and the internal model contains also high-pass filters, \( n_f = 1e-10 \).

(— = real variable, - - - = estimated variable)
The relative error of the front suspension deflection rate has been improved substantially compared with figure 11.13 (from 53 % in case of the seven DOF model to 6 % for the ten DOF model when \( n_f \) is equal to 1.0e-10). However, the estimation errors of the vehicle body absolute velocities have not been improved at all. This is not a shortcoming of the Kalman filter based on the ten DOF vehicle model but just owing to an incorrect choice of measurement intensity matrix \( R_f \). Reducing the first three diagonal elements by a factor 100 gives much better results without affecting the other variables of interest. In this case matrix \( R_f \) looks like

\[
R_f = n_f \begin{bmatrix}
8 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 8 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 7 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 55786 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 55786 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 9953 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 9900 \\
\end{bmatrix}
\]  

(11.42)

**Figure 11.15** RMS relative estimation errors as a function of noise factor \( n_f \). The accelerometer signals are high-pass filtered and the internal model (10 DOF) contains also high-pass filters. 

(- - - = front suspension, - - - = rear suspension)
Table 11.3 presents the RMS estimation errors for $n_f = 1\text{e-10}$. Compared with the initial design based on the simplified seven DOF vehicle model (table 11.1) the final Kalman filter based on a ten DOF vehicle model with high-pass filters applied in both real vehicle and internal model performs even better. The number of calculations to be carried out each sampled time step has increased accordingly. The Kalman filter state-space equation grew from 18th-order to 31st-order.

<table>
<thead>
<tr>
<th>variable</th>
<th>place</th>
<th>unit</th>
<th>$\text{RMS value}$</th>
<th>$\text{RMS error}$</th>
<th>Relative error</th>
</tr>
</thead>
<tbody>
<tr>
<td>relative velocity</td>
<td>1L</td>
<td>m/s</td>
<td>0.2908</td>
<td>0.0205</td>
<td>7%</td>
</tr>
<tr>
<td>relative velocity</td>
<td>1R</td>
<td>m/s</td>
<td>0.2900</td>
<td>0.0190</td>
<td>7%</td>
</tr>
<tr>
<td>relative velocity</td>
<td>2L</td>
<td>m/s</td>
<td>0.2323</td>
<td>0.0143</td>
<td>6%</td>
</tr>
<tr>
<td>relative velocity</td>
<td>2R</td>
<td>m/s</td>
<td>0.2554</td>
<td>0.0237</td>
<td>9%</td>
</tr>
<tr>
<td>absolute velocity</td>
<td>1L</td>
<td>m/s</td>
<td>0.0156</td>
<td>0.0013</td>
<td>8%</td>
</tr>
<tr>
<td>absolute velocity</td>
<td>1R</td>
<td>m/s</td>
<td>0.0155</td>
<td>0.0013</td>
<td>8%</td>
</tr>
<tr>
<td>absolute velocity</td>
<td>2L</td>
<td>m/s</td>
<td>0.0108</td>
<td>0.0010</td>
<td>9%</td>
</tr>
<tr>
<td>absolute velocity</td>
<td>2R</td>
<td>m/s</td>
<td>0.0107</td>
<td>0.0011</td>
<td>10%</td>
</tr>
<tr>
<td>dynamic tire load</td>
<td>1L</td>
<td>N</td>
<td>775</td>
<td>65</td>
<td>8%</td>
</tr>
<tr>
<td>dynamic tire load</td>
<td>1R</td>
<td>N</td>
<td>773</td>
<td>64</td>
<td>8%</td>
</tr>
<tr>
<td>dynamic tire load</td>
<td>2L</td>
<td>N</td>
<td>640</td>
<td>65</td>
<td>10%</td>
</tr>
<tr>
<td>dynamic tire load</td>
<td>2R</td>
<td>N</td>
<td>711</td>
<td>65</td>
<td>9%</td>
</tr>
</tbody>
</table>

Table 11.3 RMS estimation errors (10 DOF internal model, $n_f = 1.0\text{e-10}$)

The final test concern a time-domain simulation of the final Kalman filter during a double lane-change. Figure 11.16 presents the results. Unfortunately, the elimination of the low frequency estimation errors of the sprung mass absolute vertical velocity is not effective. This is probably owing to the high-pass filters within the internal model. The application of these filters is the reason that the gain of the closed-loop Kalman filter at zero frequency (DC-gain) is not equal to one because the high-pass filters have a zero DC-gain. Therefore the Kalman filter is not able to eliminate the after-effects of the acceleration signal offset. From the results obtained with high-pass filters the conclusion can be drawn that high-pass filtering of the acceleration signals cannot be used in a full-vehicle application. A reasonable fast suppression of the signal drift cannot be combined with the desired quality of estimation.
Figure 11.16 Quality of estimation during a lane-change maneuver (80 km/h). The accelerometer signals are high-pass filtered and the internal model (10 DOF) contains also high-pass filters, $n_f = 1e-10$. (--- = real variable, - - - = estimated variable)
Since high-pass filtering of the acceleration signals does not yield to satisfactory results, another method to eliminate the estimation drift must be looked after. One way to deal with low-frequency corrupted measurement signals is to model the (nearly) constant disturbances. If, instead of the white measurement noise, integrated white measurement noise is assumed, it might be possible to eliminate the estimation drift. Assume now that the measurement noise can be modeled according to

\[ \dot{z} = \xi \]  

(11.43)

where \( \xi \) represents white noise. The increase in fluctuations of \( \psi \) reflects the likely variations in the slowly varying disturbances. Both vehicle model and sensor noise model can be combined to one total model given by

\[
\begin{align*}
\dot{x}_t &= A_t x_t + F_{it} u_{it} + R_{it} z_r + T_{it}^x F_{i}^x + T_{it}^y F_{i}^y + N_{it} \xi \\
y_{it} &= C_i x_t + F_{ot} u_{ot} + R_{ot} z_r + T_{ot}^x F_{i}^x + T_{ot}^y F_{i}^y
\end{align*}
\]  

(11.44a)

(11.44b)

with

\[
A_t = \begin{bmatrix} A_i & 0 \\ 0 & 0 \end{bmatrix}, \quad F_{it} = \begin{bmatrix} F_i \\ 0 \end{bmatrix}, \quad R_{it} = \begin{bmatrix} R_i \\ 0 \end{bmatrix}, \quad T_{it}^x = \begin{bmatrix} T_i^x \\ 0 \end{bmatrix}, \quad T_{it}^y = \begin{bmatrix} T_i^y \\ 0 \end{bmatrix}, \quad N_{it} = \begin{bmatrix} 0 \\ I_7 \end{bmatrix}
\]

\[
C_i = \begin{bmatrix} C_v & I_7 \end{bmatrix}, \quad F_{ot} = F_i, \quad R_{ot} = R_i, \quad T_{ot}^x = T_i^x, \quad T_{ot}^y = T_i^y
\]

and

\[
x_t = \begin{bmatrix} x_v \\ u \end{bmatrix}
\]

The determination of the Kalman filter feedback gain \( K_f \) can now be carried out according to the procedures as described before. The Kalman filter that results from combining the internal model with the sensor noise model has the property that the constant (non-zero) measurement disturbances, such as for instance the acceleration offset caused by leaning road surfaces, are always compensated so that zero steady-state estimation error result. As expected, this is achieved by the "integrating action" of the Kalman filter.

The closed-loop transfer function between estimated output \( \hat{y}_{\hat{u}} \) and measurement signals \( y_u \) of the continuous-time Kalman filter is given by

\[
\frac{\hat{y}_{\hat{u}}(s)}{y_u(s)} = C_i \left(sI_{25} - A_t + K_f C_t\right)^{-1} K_f
\]  

(11.45)

working out equation (11.45) with the aid of equation (11.44) results in
\[
\frac{\hat{y}_r(s)}{y_r(s)} = [C_v \quad I_\tau] \left[ \begin{array}{cc}
sI_{18} - A_v + K_f^p C_v & K_f^p \\
K_f^p & sI_\tau + K_f^i
\end{array} \right]^{-1} \left[ \begin{array}{c} K_f^p \\ K_f^i \end{array} \right]
\] (11.46)

where

\[
C_v = [C_v \quad I_\tau] \\
(K_f^p)^{-1}(sI_{25} - A_v + K_f^p C_v)^{-1} = \left[ \begin{array}{cc}
sI_{18} - A_v + K_f^p C_v & K_f^p \\
K_f^p & sI_\tau + K_f^i
\end{array} \right]^{-1} \\
K_f = \left[ \begin{array}{c} K_f^p \\ K_f^i \end{array} \right]
\]

The DC-gain ($\omega = 0$) can be found by substituting $s = j \cdot \omega = 0$

\[
\frac{\hat{y}_r(0)}{y_r(0)} = [C_v \quad I_\tau] \left[ \begin{array}{cc}
-A_v + K_f^p C_v & K_f^p \\
K_f^p & K_f^i
\end{array} \right]^{-1} \left[ \begin{array}{c} K_f^p \\ K_f^i \end{array} \right] \\
= [C_v \quad I_\tau] \left[ \begin{array}{cc}
-A_v & K_f^p \\
0 & K_f^i
\end{array} \right] \left[ \begin{array}{cc}
I_{18} & 0 \\
C_v & I_\tau
\end{array} \right]^{-1} \left[ \begin{array}{c} K_f^p \\ K_f^i \end{array} \right] \\
= [C_v \quad I_\tau] \left[ \begin{array}{cc}
I_{18} & 0 \\
-C_v & I_\tau
\end{array} \right]^{-1} \left[ \begin{array}{cc}
-A_v^{-1} & K_f^p K_f^i^{-1} \\
0 & K_f^i
\end{array} \right] \left[ \begin{array}{c} K_f^p \\ K_f^i \end{array} \right] \\
= [C_v \quad I_\tau] \left[ \begin{array}{cc}
-A_v^{-1} & A_v^{-1} K_f^p K_f^i^{-1} \\
C_v A_v^{-1} & -C_v A_v^{-1} K_f^p K_f^i^{-1} + K_f^i
\end{array} \right] \left[ \begin{array}{c} K_f^p \\ K_f^i \end{array} \right] = I_\tau
\] (10.47)

Since the DC-gain matrix equals the identity matrix, the estimated outputs coincide with the measured acceleration signals even if the mean value is not equal to zero.

The Kalman filter design will now be repeated for several values of the noise intensity factor $n_r$. The internal vehicle model is assumed to contain 7 DOFs. The total number of states of the Kalman filter equals 25 (14 related to the vehicle model, 4 related to the road model and 7 related to the sensor noise model). Figure 11.17 presents the RMS relative estimation errors as a function of $n_r$. The figures show that the estimation of the sprung mass absolute velocity degrades slightly compared with the initial design (figure 11.5). Although the smallest average error can be found for very small values of

- 321 -
noise factor $n_f$, this is not the best solution. Time domain simulations showed that the Kalman filter based on for instance $n_f = 1e^{-10}$ became unstable due to measurement noise. No problems occurred for larger values of $n_f$. Figure 11.18 shows the simulation results of the double lane-change maneuver in combination with the Kalman filter with integral action. The results clearly show that the estimation of the sprung mass absolute velocity is no longer attended with a drift phenomenon.

Figure 11.17 RMS relative estimation errors as a function of noise factor $n_f$. The internal model (7 DOF) contains also the sensor noise model. (— = front suspension, - - - = rear suspension)
Figure 11.18 Quality of estimation during a lane-change maneuver (80 km/h). The internal model (7 DOF) contains also the sensor noise model, $n_f = 1e-9$.

(—- = real variable, - - - = estimated variable)
CONCLUSIONS and RECOMMENDATIONS

This thesis deals with the control system design of intelligent suspension systems for automotive application. Unlike contemporary suspensions with fixed characteristics, the adjustment of particular suspension attributes may enhance performance noticeably. The main point of examination is concentrated on the design of the controller that adjusts the suspension for varying external conditions such as speed, road condition and steering maneuver. The potential of suspension control has been illustrated on the basis of active and semi-active force generators. The analysis has been based on the theory of linear optimal control. Linear optimal control offers the possibility to emphasize quantifiable issues like ride comfort or road holding very easily by altering the weighting factor of a quadratic criterion.
Conclusions and Recommendations

It has been determined that the quarter-car analysis is very useful to gain insight into the problematics related to suspension control. However, the handling aspects of road holding are difficult to describe by one variable and is therefore a rather complex phenomenon. The effect of the variation of the tire load (due to road irregularities) on the cornering behavior of a car can not be described by steady-state tire relations in combination with the quarter-car model.

In order to improve the overall suspension performance, various control solutions have been discussed. In case of an active suspension system it is possible to apply the theory of optimal control. The optimal control theory can be used for full state feedback as well as for limited state feedback. Several weighting combinations have been examined. The selection of the weighting factors can be made easier by the application of normalized performance variables. The analysis has shown that despite the presence of an active force generator, and thus a freedom to generate an arbitrary force, the conflict between ride comfort and road holding remains. This is mainly because the actuator force acts on both axle and vehicle body in combination with a two degree of freedom model with only a single actuator. The gain in ride comfort and road holding is only marginal compared with a well-tuned passive suspension. The main advantage of an actively controlled suspension system is that it is possible to change one or more parameters of the system such that in certain cases ride comfort can be improved at the cost of road holding or vice versa. The performance of the limited state feedback configuration (feedback of sprung and unsprung mass absolute velocity only) can compete with those of a full-state feedback system (feedback of displacements as well as velocities). Limited state-feedback is preferable because fewer states have to be determined. Full-state feedback systems need integral action in order to eliminate the steady-state offset when the vehicle is subjected to a step input. Although the name 'linear optimal control' assumes that the final controller is optimal, the results depend completely on the weighting factor chosen. When a wrong set of weighting factors is chosen, these results may be far from optimal.

The contradiction between ride comfort and road holding suggests that there may exist some limitations of the quarter-car model. The limitations are mainly caused by the presence of the so-called invariant points in the transfer functions. Probably a more specific area in the frequency domain could be marked where ride comfort can be improved without making road holding (dynamic tire load variations) worse. Furthermore areas might exist where it
is absolutely 'forbidden' to improve ride comfort because any improvement is coupled with a deterioration of the tire load variations and thus road holding capabilities. Optimal control in combination with frequency dependent weighting might provide the best prospects for a more successful non-adaptive solution of the suspension control problem.

As active suspension systems possess many drawbacks, such as high energy consumption and cost price, semi-active damping has been discussed as well. Separate control strategies to improve ride comfort or road holding have been proposed. Simulation results showed that it is possible to improve ride comfort significantly using the so-called skyhook control algorithm (feedback of absolute sprung mass velocity). However, this is accompanied by a drastic increase in dynamic tire load variations. On the other hand it appeared not to be possible to decrease dynamic tire load variations compared with a standard passive suspension system. The performance of semi-active damper systems strongly depends on the range of the damper. The larger the range between soft and firm, the larger is the possible gain in ride comfort and the more the road holding will deteriorate. Semi-active damper systems controlled by an LQG solution will always suffer from harshness, even if a continuously adjustable damper is used.

The additional adaptive control loop as proposed in chapter 5 takes care of the dynamic tire load variations in situations were the safety is endangered. As long as the dynamic tire load component does not exceed a predetermined limit, ride comfort can be improved significantly. However, during cornering and braking or accelerating the adaptive suspension controller permits a worsening of the ride comfort in favor of tire load control. The main input to the adaptive controller is the ratio of the dynamic tire load to the static tire load. Control law adaptation provides a substantial boost in performance compared with fixed parameter LQG based controllers.

In order to be able to control a suspension system several state variables must be known. These states can be estimated with the help of a state estimator or Kalman filter. For a linear system such as the active suspension system it quite easy to construct a Kalman Filter. However, due to the non-linearities of the semi-active damper system such a solution is not evident anymore. Sometimes estimator instability can not be avoided. This problem can be solved using multi-model techniques. A special study should be devoted to the robustness of the Kalman filter, particularly to parameter variations.
Although very time consuming, the multi-model technique to determine a stable Kalman filter internal feedback is applicable for this analysis.

The second part of the thesis deals with a full-vehicle model. The full-vehicle is not just a connection of four quarter-cars. Besides the roll and pitch degree of freedom, the full-vehicle offers the possibility to perform steering maneuvers. The road holding aspect of the quarter-car model can now be enhanced with handling related aspects. Especially the influence of vertical dynamics such as tire load variations of lateral dynamics is of particular interest.

Similar to the quarter-car analysis the design of an intelligent suspension system has been based on LQG control. The results showed that the extra information arising form the cross-correlated road tracks does not result in extra control potentialities. However, including the wheelbase time delay in the control design phase has a large influence on the performance. Especially the tire load variations and suspension travel of the two rear suspensions can benefit from the extra information of the wheelbase time delay. Furthermore, the LQG control analysis has shown that the performance of partial state feedback systems can compete with suspensions based on full-state feedback. The partial state feedback considers only the velocities of the sprung and unsprung mass at the suspension mounts and can therefore be denoted by active damping. The exchange of state information among the four suspensions does enhance the performance noticeably. The controller based on partial state feedback can also profit from the wheelbase time delay. However, the extra gain appears only with a coupled feedback structure. The full-vehicle model results correspond very well to the quarter-car model findings. In general the ride comfort of the passengers can be improved substantially at the cost of a degradation of the road holding ability.

After the determination of the ride performance, the vehicles with an active suspension have been exposed to a double-lane change maneuver on an uneven road surface. The results show that moderate tire load variations do not affect the handling. The tire side force variations are only slightly worse compared with the standard vehicle with a passive suspension. The side force variations are only larger in those cases were the vertical tire load decreases drastically due to the load transfer. However, these variations are rather harmless because the average side force is already very low. The vehicle with active suspension based on full-state feedback did not perform well in the double lane-change test. Besides the quite harmless increase in tire load
variations, the unacceptably low roll stiffness and the unbalanced load transfer contribute to the degradation of the handling. The latter two phenomena are caused by the full-state feedback structure of the active suspension. Therefore the feedback of position related states must be avoided. The vehicle with a suspension based on partial state feedback performed well compared with the standard configuration. The only situations to be avoided are those cases where the load transfer and the dynamic tire load variations due to road unevennesses coincide in such a way that the average cornering force will be reduced too much. An adaptation of the suspension controller is necessary to avoid these situations. The double lane-change test has shown that the relation between road holding and handling is not so strong. Including handling related criteria (such as tire side force variations) in the quadratic cost function rather than road holding related criteria (vertical tire load variations) would be an interesting extension to the controller design based on LQG control. However, a problem may be expected with the non-linear relation between the vertical tire load and the tire slip forces.

The adoption as proposed in chapter 10 has been based on gain scheduling. This means that the gains of the state feedback matrix are adjusted during the ride depending on the state of operation of the vehicle. The adaptation has been based on the ratio of the dynamic tire load to the momentary average tire load. If this ratio becomes large than the dynamic tire load variations affect the handling and therefore the adaptive suspension controller will automatically adjust the feedback gains in order to restrict the tire load variations of a particular wheel. The demand for safety goes with a brief worsening of the ride comfort during this intervention.

The implementation of adaptive suspension control requires the knowledge of several states of the vehicle. States like the absolute vertical velocity of the vehicle body or the dynamic tire load cannot be measured directly. The unknown states have been estimated by a Kalman filter. The measured 'vertical' accelerations of the vehicle body and axles serve as an input to the Kalman filter. A major problem is the elimination of the contribution of the gravitational, lateral and longitudinal accelerations to the sensor signals. During cornering and braking or accelerating the offset of the acceleration signals due to the tilted position of the vehicle body causes the estimated signals to drift. This drift phenomenon can be suppressed by the application of a Kalman filter with an internal proportional-integral feedback.
Appendix A Equations of Motion of a Full-Vehicle Model

EQUATIONS OF MOTION OF A FULL-VEHICLE MODEL

In this appendix the equations of motion of a full-vehicle model will be derived using the Lagrangean method. The vehicle model has 13 DOFs: 3 translations and 3 rotations for the vehicle body, 2 rotations for the front suspension, 1 translation and 1 rotation for the rear axle and finally 1 translation and 2 rotations for the engine/transmission. The simplified, 13 degrees of freedom (DOF) model is shown in figure 7.1. The geometry of the McPherson strut front suspension has been approximated by substitution of a swing axle suspension (more about this in chapter 7). The rear axle is characterized by a rigid beam connected to the vehicle body by means of a spring and Panhard rod. The equations of motion are derived using the
method of Lagrange. This method requires the calculation of the kinetic and potential energy.

The equation of Lagrange is given by

$$\frac{d}{dt}\left(\frac{\partial T}{\partial q_i}\right) - \frac{\partial T}{\partial \dot{q}_i} + \frac{\partial V}{\partial q_i} + \frac{\partial D}{\partial q_i} = Q_i$$  \hspace{1cm} (A.1)$$

with \( T \) the kinetic energy, \( V \) the potential energy, \( D \) the Rayleigh's dissipation function, \( q_i \) the generalized coordinates and \( Q_i \) the generalized forces.

A full-vehicle model with 5 bodies contains 30 degrees of freedom, 6 DOFs for each body. The bodies are: main body (index 'b'), 2 front axles (indices 'a1L' and 'a1R'), 1 rear axle (index 'a2') and powertrain (index 'e'). The total kinetic energy of the vehicle is given by

$$T = T_b + T_{a1L} + T_{a1R} + T_{a2} + T_e$$ \hspace{1cm} (A.2)$$

with

$$T_b = \frac{1}{2} m_b (\dot{x}_b^2 + \dot{y}_b^2 + \dot{z}_b^2) + \frac{1}{2} I_{xb} \dot{\theta}_b^2 + \frac{1}{2} I_{yb} \dot{\psi}_b^2 + \frac{1}{2} I_{zb} \dot{\psi}_b^2$$

$$T_{a1L} = \frac{1}{2} m_{a1} (\dot{x}_{a1L}^2 + \dot{y}_{a1L}^2 + \dot{z}_{a1L}^2) + \frac{1}{2} I_{xa1} \dot{\theta}_{a1L}^2 + \frac{1}{2} I_{ya1} \dot{\theta}_{a1L}^2 + \frac{1}{2} I_{za1} \dot{\psi}_{a1L}^2$$

$$T_{a1R} = \frac{1}{2} m_{a1} (\dot{x}_{a1R}^2 + \dot{y}_{a1R}^2 + \dot{z}_{a1R}^2) + \frac{1}{2} I_{xa1} \dot{\theta}_{a1R}^2 + \frac{1}{2} I_{ya1} \dot{\theta}_{a1R}^2 + \frac{1}{2} I_{za1} \dot{\psi}_{a1R}^2$$

$$T_{a2} = \frac{1}{2} m_{a2} (\dot{x}_{a2}^2 + \dot{y}_{a2}^2 + \dot{z}_{a2}^2) + \frac{1}{2} I_{xa2} \dot{\theta}_{a2}^2 + \frac{1}{2} I_{ya2} \dot{\theta}_{a2}^2 + \frac{1}{2} I_{za2} \dot{\psi}_{a2}^2$$

$$T_e = \frac{1}{2} m_{e} (\dot{x}_e^2 + \dot{y}_e^2 + \dot{z}_e^2) + \frac{1}{2} I_{xe} \dot{\theta}_e^2 + \frac{1}{2} I_{ye} \dot{\varphi}_e^2 + \frac{1}{2} I_{ze} \dot{\varphi}_e^2$$

Because of the number of coordinates, equation A.2 suggests that the vehicle model has 30 degrees of freedom. However, it contains several kinematic constraints such as hinges and ball joints. The number of degrees of freedom is reduced to 13 by means of 17 (non-linear) constraint equations. It is assumed that the rotations of the bodies are small. This means rotations about the three different axes attached to the body can be superimposed. The position of a body in the 3-dimensional space is determined by the position in the global reference frame and the orientation of the body fixed coordinated system. The parameters chosen to describe the orientation are the Cardan angles. The chosen sequence of rotation is:

- rotation about the x-axis of inertial reference frame (angle \( \phi \)),
- rotation about the new y-axis (angle \( \theta \)),
- rotation about the new z-axis (angle \( \psi \)).
For large angles, the order is not unique; a different sequence implies a different transformation matrix.

The transformation matrix from inertial reference frame to body fixed coordinate system is given by the multiplication of matrices for the three successive rotations according to

\[
S = \begin{bmatrix}
\cos \psi & -\sin \psi & 0 \\
\sin \psi & \cos \psi & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\cos \theta & 0 & \sin \theta \\
0 & 1 & 0 \\
-\sin \theta & 0 & \cos \theta
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \phi & -\sin \phi \\
0 & \sin \phi & \cos \phi
\end{bmatrix}
\] (A.3a)

It is assumed that the inertial reference frame is parallel to the body fixed coordinate system at \( t = 0 \). Expression (A.3a) has been expanded up to the second-order. The following transformation matrix, valid for small angles only, is obtained after expansion

\[
S = \begin{bmatrix}
1 - \frac{\theta^2}{2} - \frac{\psi^2}{2} & -\psi & \theta \\
\phi \theta + \psi & 1 - \frac{\phi^2}{2} - \frac{\psi^2}{2} & -\phi \\
-\theta + \phi \psi & \phi + \theta \psi & 1 - \frac{\phi^2}{2} - \frac{\theta^2}{2}
\end{bmatrix}
\] (A.3b)

It is necessary to include the second-order of magnitude terms because, during the derivation of the equations of motion, first-order terms might reappear when calculating the partial derivatives of the kinetic and potential energy.

For the longitudinal direction the following constraint equations are valid

\[
x_{a1L} = x_b + l_{a1} \left( 1 - \frac{\theta_a^2}{2} - \frac{\psi_a^2}{2} \right) + \left( h_{p1} + l_{a2} s_{a1} \right) \theta_b + \left( w_{p1} + l_{a2} c_{a1} \right) \psi_b
\] (A.4a)

\[
x_{a1R} = x_b + l_{a1} \left( 1 - \frac{\theta_a^2}{2} - \frac{\psi_a^2}{2} \right) + \left( h_{p1} + l_{a2} s_{a1} \right) \theta_b - \left( w_{p1} + l_{a2} c_{a1} \right) \psi_b
\] (A.4b)

\[
x_{a2} = x_b + l_{a2} \left( 1 - \frac{\theta_a^2}{2} - \frac{\psi_a^2}{2} \right) + \left( h_{p2} + l_p s_p - h_r \right) \theta_b
\] (A.4c)

\[
x_b = x_b + l_a \left( 1 - \frac{\theta_a^2}{2} - \frac{\psi_a^2}{2} \right) + h_a \theta_a
\] (A.4d)

with \( s_a = \sin(\phi_{a1}^0), c_a = \cos(\phi_{a1}^0), s_p = \sin(\phi_p^0) \) and \( c_p = \cos(\phi_p^0) \) representing the sine and cosine of the initial rotation of the swing axle and Panhard rod respectively. For the lateral direction the equations read
\[ y_{a1L} = y_b - w_{p1} \left( 1 - \frac{\phi_b^2}{2} - \frac{\psi_b^2}{2} \right) - l_a c_a \left( 1 - \frac{\phi_{a1L}^2}{2} - \frac{\psi_{a1L}^2}{2} \right) - l_a s_a \phi_{a1L} \]  
\[ -h_{p1} \phi_b + l_{a1} (\phi_b \theta_b + \psi_b) \]  
\[ y_{a1R} = y_b + w_{p1} \left( 1 - \frac{\phi_b^2}{2} - \frac{\psi_b^2}{2} \right) + l_a c_a \left( 1 - \frac{\phi_{a1R}^2}{2} - \frac{\psi_{a1R}^2}{2} \right) - l_a s_a \phi_{a1R} \]  
\[ -h_{p1} \phi_b + l_{a1} (\phi_b \theta_b + \psi_b) \]  
\[ y_{a2} = y_b + l_a c_a (\phi_b \theta_b + \psi_b) - w_{p2} \left( \frac{\phi_b^2}{2} + \frac{\phi_{a2}^2}{2} \right) - l_p c_p \left( \frac{\phi_p^2}{2} \right) \]  
\[ -l_p s_p \phi_p - h_{p2} \phi_b + h_r \phi_{a2} \]  
\[ y_c = y_b - h_c \phi_b + l_c (\phi_b \theta_b + \psi_b) \]  

For the vertical direction the constraint equations take the form

\[ z_{a1L} = z_b + l_a c_a (\phi_b \psi_b - \theta_b) - w_{p1} (\theta_b \psi_b + \phi_b) - l_a c_a (\theta_b \psi_b + \phi_{a1L}) \]  
\[ +h_{p1} \left( 1 - \frac{\phi_b^2}{2} - \frac{\theta_b^2}{2} \right) + l_a s_a \left( 1 - \frac{\phi_{a1L}^2}{2} - \frac{\theta_{a1L}^2}{2} \right) \]  
\[ z_{a1R} = z_b + l_a c_a (\phi_b \psi_b - \theta_b) + w_{p1} (\theta_b \psi_b + \phi_b) + l_a c_a (\theta_b \psi_b + \phi_{a1R}) \]  
\[ +h_{p1} \left( 1 - \frac{\phi_b^2}{2} - \frac{\theta_b^2}{2} \right) + l_a s_a \left( 1 - \frac{\phi_{a1R}^2}{2} - \frac{\theta_{a1R}^2}{2} \right) \]  
\[ z_{a2} = z_b + l_p c_p (\phi_b \psi_b - \theta_b) \]  
\[ +w_{p2} (\phi_b + \phi_{a2}) - h_r \left( 1 - \frac{\phi_{a2}^2}{2} - \frac{\theta_{a2}^2}{2} \right) + l_a c_a (\phi_b \psi_b - \theta_b) \]  

For the rotation about the y-axis the following equations are valid

\[ \theta_{a1L} = \theta_b \]  
\[ \theta_{a1R} = \theta_b \]  
\[ \theta_{a2} = \theta_b \]  

Finally, the equations for the yaw rotation become

\[ \psi_{a1L} = \psi_b \]  
\[ \psi_{a1R} = \psi_b \]  
\[ \psi_{a2} = \psi_b \]
Appendix A Equations of Motion of a Full-Vehicle Model

\[ \Psi_c = \Psi_b \quad (A.4.r) \]

For convenience, the rotation of the Panhard rod \( \phi_p \) can be written in terms of the vertical displacement of the rear axle \( z_{a2} \). This can be accomplished through equation (A.4.k). Herewith, \( \phi_p \) becomes

\[ \phi_p = \frac{z_{a2} - z_b - w_{p2} \dot{z}_b + l_{a2} \theta_b - w_{p2} \phi_{a2} - (h_{p2} + l_p s_p - h_r)}{l_p c_p} \quad (A.5) \]

and \( y_{a2} \) becomes

\[ y_{a2} = y_b + t_p (h_{p2} + l_p s_p - h_r) + t_p z_b + (h_{p2} - w_{p2} t_p) \phi_b \]

\[ -l_{a2} \dot{t}_p \theta_b + l_{a2} \dot{z}_b - t_p z_{a2} + (w_{p2} t_p + h_r) \phi_{a2} \quad (A.6) \]

with \( t_p = \tan(\phi_p^0) \).

After having determined the velocities of the bodies with the aid of the constraint equations it is possible the calculate the kinetic energy. It is necessary to include 2nd-order terms in the velocity expressions because 1st-order terms may appear after time differentiation of the partial derivatives of the kinetic energy. The (linearized) time derivatives of the partial derivatives are given by

\[ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{x}_b} \right) - \left( \frac{\partial T}{\partial x_b} \right) = (m_b + 2m_{a1} + m_{a2} + m_c) \ddot{x}_b \]

\[ + \left( 2m_{a1} (h_{p1} + l_{a} s_a) + m_{a2} (h_{p2} + l_p s_p - h_r) + m_c h_r \right) \ddot{\theta}_b \quad (A.7) \]

\[ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{y}_b} \right) - \left( \frac{\partial T}{\partial y_b} \right) = (m_b + 2m_{a1} + m_{a2} + m_c) \ddot{y}_b + m_{a2} t_p (\ddot{z}_b - \ddot{z}_{a2}) + m_{a2} (w_{p2} t_p + h_r) \ddot{\phi}_{a2} \]

\[ - \left( 2m_a t_{a1} + m_{a2} (h_{p2} - w_{p2} t_p) + m_c h_r \right) \ddot{\phi}_b - m_{a2} l_{a2} t_p \ddot{\phi}_b \]

\[ + \left( 2m_a t_{a1} + m_{a2} l_{a2} + m_c l_{a2} \right) \ddot{\psi}_b - m_{a2} l_{a2} s_{a2} \ddot{\phi}_{a2} - m_a l_{a2} s_{a2} \ddot{\phi}_{a1R} \]

\[ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{z}_b} \right) - \left( \frac{\partial T}{\partial z_b} \right) = m_{a2} t_{p1} \ddot{\phi}_b + (m_b + 2m_{a1} + m_{a2} t_{p1}^2) \ddot{z}_b - m_{a2} t_{p1} \ddot{z}_{a2} \]

\[ - m_{a2} t_{p1} (h_{p2} - w_{p2} t_p) \ddot{\phi}_b - \left( 2m_{a1} t_{a1} + m_{a2} t_{a2}^2 \right) \ddot{\theta}_b \]

\[ + m_{a2} l_{a2} t_{p1} \ddot{\psi}_b + m_{a2} t_{p1} (w_{p2} t_p + h_r) \ddot{\phi}_{a2} - m_{a2} l_{a2} c_{a2} \ddot{\phi}_{a2} - m_a l_{a2} c_{a2} \ddot{\phi}_{a1R} \]
Appendix A Equations of Motion of a Full-Vehicle Model

\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \phi_b} \right) - \left( \frac{\partial T}{\partial \psi_b} \right) = -2m_{a1} h_{p1} + m_{a2} \left( h_{p2} - w_{p2} t_p \right) + m_{k} \left( h_{p2} - w_{p2} t_p \right) \dot{z}_b \\
\quad + \left( I_{a1} + 2m_{a1} \left( w_{p1}^2 + h_{p1}^2 \right) + m_{a2} \left( h_{p2} - w_{p2} t_p \right)^2 + m_k h_{c}^2 \right) \dot{\phi}_b \\
- \left( 2m_{a1} \left( h_{p1} \right) + m_{a2} \left( h_{p2} - w_{p2} t_p \right) + m_k h_{c} \right) \ddot{\psi}_b + m_{a2} \left( 2w_{p2} t_p \right) \ddot{\phi}_b \\
+ m_{a2} \left( h_{p2} - w_{p2} t_p \right) \ddot{z}_b - m_{a2} \left( w_{p2} t_p + h_{r} \right) \left( h_{p2} - w_{p2} t_p \right) \ddot{\phi}_a \\
+ m_{a1} \left( c_w w_{p1} + s_a h_{p1} \right) \ddot{\phi}_{a1L} + m_{a1} \left( c_w w_{p1} + s_a h_{p1} \right) \ddot{\phi}_{a1R} \\
\frac{d}{dt} \left( \frac{\partial T}{\partial \psi_b} \right) - \frac{d}{dt} \left( \frac{\partial T}{\partial \psi_b} \right) = -2m_{a1} \left( h_{p1} \right) + m_{a2} \left( h_{p2} + t_p - h_{r} \right) + m_{k} h_{c} \ddot{z}_b + m_{a2} \left( 2w_{p2} t_p \right) \ddot{\phi}_b \\
- \left( 2m_{a1} \left( h_{p1} \right) + m_{a2} \left( h_{p2} - w_{p2} t_p \right) + m_k h_{c} \right) \ddot{\psi}_b + m_{a2} \left( 2w_{p2} t_p \right) \ddot{\phi}_b \\
+ m_{a2} \left( h_{p2} - w_{p2} t_p \right) \ddot{z}_b - m_{a2} \left( w_{p2} t_p + h_{r} \right) \left( h_{p2} - w_{p2} t_p \right) \ddot{\phi}_a \\
+ \left( I_{a1} + 2I_{a1} + I_{a2} + I_{x2} + 2m_{a1} \left( \left( w_{p1} + l_c \right)^2 \right) + m_{a2} \left( \left( w_{p2} + l_c \right)^2 \right) \ddot{\phi}_b \\
- m_{a1} \left( h_{p2} + l_c s_p - h_{r} \right)^2 + \left( \left( l_c \right)^2 \right) + m_k h_{c}^2 \right) \ddot{\phi}_a \\
\frac{d}{dt} \left( \frac{\partial T}{\partial \phi_{a1L}} \right) - \frac{d}{dt} \left( \frac{\partial T}{\partial \phi_{a1L}} \right) = -m_{a1} \left( h_{p1} \right) + m_{a2} \left( h_{p2} + t_p - h_{r} \right) \ddot{z}_b + m_{a2} \left( 2w_{p2} t_p \right) \ddot{\phi}_b \\
- \left( 2m_{a1} \left( h_{p1} \right) + m_{a2} \left( h_{p2} - w_{p2} t_p \right) + m_k h_{c} \right) \ddot{\psi}_b + m_{a2} \left( 2w_{p2} t_p \right) \ddot{\phi}_b \\
+ m_{a2} \left( h_{p2} - w_{p2} t_p \right) \ddot{z}_b - m_{a2} \left( w_{p2} t_p + h_{r} \right) \left( h_{p2} - w_{p2} t_p \right) \ddot{\phi}_a \\
+ \left( I_{a1} + m_{a1} \right) \ddot{\phi}_{a1L} - m_{a1} \left( h_{p2} + l_c s_p - h_{r} \right)^2 + \left( \left( l_c \right)^2 \right) + m_k h_{c}^2 \right) \ddot{\phi}_{a1R} \\
\frac{d}{dt} \left( \frac{\partial T}{\partial \phi_{a1R}} \right) - \frac{d}{dt} \left( \frac{\partial T}{\partial \phi_{a1R}} \right) = -m_{a1} \left( h_{p1} \right) + m_{a2} \left( h_{p2} + t_p - h_{r} \right) \ddot{z}_b + m_{a2} \left( 2w_{p2} t_p \right) \ddot{\phi}_b \\
- \left( 2m_{a1} \left( h_{p1} \right) + m_{a2} \left( h_{p2} - w_{p2} t_p \right) + m_k h_{c} \right) \ddot{\psi}_b + m_{a2} \left( 2w_{p2} t_p \right) \ddot{\phi}_b \\
+ m_{a2} \left( h_{p2} - w_{p2} t_p \right) \ddot{z}_b - m_{a2} \left( w_{p2} t_p + h_{r} \right) \left( h_{p2} - w_{p2} t_p \right) \ddot{\phi}_a \\
+ \left( I_{a1} + m_{a1} \right) \ddot{\phi}_{a1R} - m_{a1} \left( h_{p2} + l_c s_p - h_{r} \right)^2 + \left( \left( l_c \right)^2 \right) + m_k h_{c}^2 \right) \ddot{\phi}_{a1R} \\
\frac{d}{dt} \left( \frac{\partial T}{\partial \phi_{a2}} \right) - \frac{d}{dt} \left( \frac{\partial T}{\partial \phi_{a2}} \right) = m_{a2} \left( 2w_{p2} t_p \right) \ddot{z}_b - m_{a2} \left( \ddot{\phi}_b - m_{a2} \left( 2w_{p2} t_p \right) \ddot{\phi}_b \\
+ m_{a2} \left( h_{p2} - w_{p2} t_p \right) \ddot{z}_b - m_{a2} \left( w_{p2} t_p + h_{r} \right) \ddot{\phi}_a \\
+ m_{a2} \left( h_{p2} - w_{p2} t_p \right) \ddot{z}_b - m_{a2} \left( w_{p2} t_p + h_{r} \right) \ddot{\phi}_a \right) \ddot{\phi}_{a2}
\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \theta_{a2}} \right) - \left( \frac{\partial T}{\partial \theta_{a2}} \right) = m_{a2}(w_{p2}t_p + h_r)(\ddot{y}_b + m_{a2}z_p^* + h_r)(\ddot{z}_b - \ddot{z}_{a2})
\]
\[
- m_{a2}(w_{p2}t_p + h_r)(h_p + w_{p2}t_p)\ddot{\phi}_b - m_{a2}L_{a2}t_p(w_{p2}t_p + h_r)\ddot{\phi}_b
\]
\[
+ m_{a2}L_{a2}(w_{p2}t_p + h_r)\ddot{\psi}_b + \left( I_{a2} + m_{a2}(w_{p2}t_p + h_r)^2 \right)\ddot{\phi}_{a2}
\]
\[
\frac{d}{dt} \left( \frac{\partial T}{\partial z_e} \right) - \left( \frac{\partial T}{\partial z_e} \right) = m_{a2}\ddot{z}_e
\]
\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \phi_e} \right) - \left( \frac{\partial T}{\partial \phi_e} \right) = I_{x e}\phi_e
\]
\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \theta_e} \right) - \left( \frac{\partial T}{\partial \theta_e} \right) = I_{y e}\theta_e
\]

The potential energy \( V_{\text{susp}} \) can be expressed as
\[
V_{\text{susp}} = \frac{1}{2} c_{s1} \left( d_{s1L}^2 + d_{s1R}^2 \right) + \frac{1}{2} c_{s2} \left( d_{s2L}^2 + d_{s2R}^2 \right)
\]
\[
+ \frac{1}{2} c_{t} \left( d_{1L}^2 + d_{1R}^2 + d_{2L}^2 + d_{2R}^2 \right)
\]
(A.8)

with \( d_{s1} \) the deformation of the front suspension (\( d_{s1L}, d_{s1R} \)) and rear suspension (\( d_{s2L}, d_{s2R} \)) in (m), \( c_{s1} \) the suspension stiffness in (N/m), \( d_{t} \) the deformation of the front and rear tires and \( c_{t} \) the tire stiffness. The Rayleigh's dissipation function \( D \) is defined by
\[
D_{\text{susp}} = \frac{1}{2} k_{a1} \left( d_{s1L}^2 + d_{s1R}^2 \right) + \frac{1}{2} k_{a2} \left( d_{s2L}^2 + d_{s2R}^2 \right)
\]
(A.9)

With \( k_{a} \) the damping constants in (Ns/m) and \( \dot{d}_{i} \) the deformation velocities in (m/s). As the front suspension is situated at an oblique angle, the method of calculating the potential energy is not recommendable. Besides that, we would like the suspension and tire deflection to be available for possible non-linear or active force inputs. Therefore, the method of virtual work of (generalized) forces is used.

**Front Suspension**

The geometry of the Mc Pherson strut front suspension has been approximated by substitution of a swing axle suspension. The inclination point of the swing axle is situated in the virtual reaction point of the Mc Pherson strut. This virtual point is obtained as shown in figure 7.3. The reaction point lies at the intersection of the axis of the lower control arm and the line perpendicular to the strut with an origin at the strut top mounting.
The roll center is located on the centerline of the vehicle at the intersection with the line from the center of the tire contact to the virtual reaction point. For small deviations from the static position, it is allowed to replace the McPherson strut with a swing axle suspension with length $l_o$ and hinge point in the virtual reaction point. For small rotations, this configuration shows the same kinematics as the real McPherson strut. The simplified front suspension is shown in figure A.1 (for the sake of clearness the swing axles are drawn shorter than in reality).

**Figure A.1** Front suspension.

The virtual work $\delta W_{s1}$ of an arbitrary force $F_{s1}$ in the McPherson strut front suspension is given by

\[
\delta W_{s1L} = -F_{s1L} \cdot \delta \phi_{s1L} - F_{s1L}^y \cdot \delta y_{s1L} - F_{s1L}^z \cdot \delta z_{s1L}
\]

(A.10a)

\[
\delta W_{s1R} = -F_{s1R} \cdot \delta \phi_{s1R} - F_{s1R}^y \cdot \delta y_{s1R} - F_{s1R}^z \cdot \delta z_{s1R}
\]

(A.10b)

with

\[
\delta y_{s1L} = (h_{p1} - h_{s1}) \delta \phi_b + l_s a \delta \phi_{a1L} - (w_{p1} - w_{s1})(\psi_b \delta \phi_b + \psi_\delta \psi_b)
\]

\[-l_s c_a (\delta \phi_{a1L} + \psi_\delta \psi_b)
\]

\[
\delta y_{s1R} = -(h_{p1} - h_{s1}) \delta \phi_b - l_s a \delta \phi_{a1R} - (w_{p1} - w_{s1})(\psi_b \delta \phi_b + \psi_\delta \psi_b)
\]

\[-l_s c_a (\delta \phi_{a1R} + \psi_\delta \psi_b)
\]

\[
\delta z_{s1L} = -(w_{p1} - w_{s1}) \delta \phi_b - l_s a \delta \phi_{a1L} - (h_{p1} - h_{s1})(\psi_b \delta \phi_b + \psi_\delta \psi_b)
\]

\[-l_s a (\delta \phi_{a1L} + \psi_\delta \psi_b - (w_{p1} + l_s c_a - w_{s1})(\theta_\delta \delta \psi_b + \psi_\delta \delta \phi_b)
\]

\[\]
\[ \delta z_{s1R} = (w_{p1} - w_{s1}) \delta \phi_b + l_c a \delta \phi_{a1R} - (h_{p1} - h_{s1}) (\phi_b \delta \phi_b + \psi_b \delta \psi_b) \]
\[ -l_s a (\phi_{a1R} \delta \phi_{a1R} + \psi_b \delta \psi_b) + (w_{p1} + l_c a - w_{s1}) (\theta_b \delta \psi_b + \psi_b \delta \theta_b) \]

and
\[ F_{s1L} = \frac{y_0}{l_0} F_{s1L} \quad F_{s1L} = \frac{z_0}{l_0} F_{s1L} \quad F_{s1R} = \frac{y_0}{l_0} F_{s1R} \quad F_{s1R} = \frac{z_0}{l_0} F_{s1R} \]

where \( l_0 \) represents the initial length of the strut given by
\[ l_0 = \sqrt{y_0^2 + z_0^2} \quad y_0 = w_{p1} - w_{s1} + l_c a \quad z_0 = h_{p1} - h_{s1} + l_s a \]

Note: no work is done in the x-direction since the initial position of the suspension is in the y-z plane. The second-order terms in the virtual displacements are necessary because of presence of the constant pretension forces. The virtual work expression can now be rewritten to
\[ \delta W_{s1L} = -F_{s1L} \left[ \left( \frac{-y_0}{l_0} (h_{p1} - h_{s1}) - \frac{z_0}{l_0} (w_{p1} - w_{s1}) \right) \phi_b \delta \phi_b + \left( \frac{y_0}{l_0} l_s a - \frac{z_0}{l_0} l_c a \right) \delta \phi_{a1L} \right] \]
\[ -F_{s1}^0 \left( \phi_b \delta \phi_b - \left( \frac{y_0}{l_0} l_s a + \frac{z_0}{l_0} l_c a \right) \delta \phi_{a1L} \right) \]
\[ -\left( w_{p1} + l_c a - w_{s1} \right) (\theta_b \delta \psi_b + \psi_b \delta \theta_b) \]
\[ (A.11a) \]

\[ \delta W_{s1R} = -F_{s1R} \left[ \left( \frac{-y_0}{l_0} (h_{p1} - h_{s1}) + \frac{z_0}{l_0} (w_{p1} - w_{s1}) \right) \phi_b \delta \phi_b - \left( \frac{y_0}{l_0} l_s a - \frac{z_0}{l_0} l_c a \right) \delta \phi_{a1R} \right] \]
\[ -F_{s1}^0 \left( \phi_b \delta \phi_b - \left( \frac{y_0}{l_0} l_s a + \frac{z_0}{l_0} l_c a \right) \delta \phi_{a1R} \right) \]
\[ -\left( w_{p1} + l_c a - w_{s1} \right) (\theta_b \delta \psi_b + \psi_b \delta \theta_b) \]
\[ (A.11b) \]

where \( F_{s1}^0 \) represents the pretension force of the front suspension spring according to
\[ F_{s1}^0 = \frac{l_0}{l_s} m_{a3} g \left( l_a - l_w \right) - \frac{-m_{g} g l_{a2} + m_{g} g \left( l_a - l_{a2} \right)}{2 \left( l_{a1} - l_{a2} \right)} \frac{c_{sw}}{z_0 c_a - y_0 s_a} \]
\[ (A.12) \]

The front suspension contains a spring with stiffness \( c_{s1} \) in (N/m), a damper with damping constant \( k_{s1} \) in (Ns/m) and a force actuator capable of generating force \( u_{s1} \). The suspension forces read
Appendix A  Equations of Motion of a Full-Vehicle Model

\[ F_{s1L} = c_{s1} \left( \left( \frac{\gamma}{l_c} \left( h_{p1} - h_{s1} \right) - \frac{\gamma}{l_c} \left( w_{p1} - w_{s1} \right) \right) \phi_h + \left( \frac{\gamma}{l_c} L_s a - \frac{\gamma}{l_c} L_c a \right) \phi_{a1L} \right) + F^0_{s1} \]
\[ + k_{s1} \left( \left( \frac{\gamma}{l_c} \left( h_{p1} - h_{s1} \right) - \frac{\gamma}{l_c} \left( w_{p1} - w_{s1} \right) \right) \phi_h + \left( \frac{\gamma}{l_c} L_s a - \frac{\gamma}{l_c} L_c a \right) \phi_{a1L} \right) + u_{s1L} \]
(A.13a)

\[ F_{s1R} = c_{s1} \left( \left( -\frac{\gamma}{l_c} \left( h_{p1} - h_{s1} \right) + \frac{\gamma}{l_c} \left( w_{p1} - w_{s1} \right) \right) \phi_h - \left( \frac{\gamma}{l_c} L_s a - \frac{\gamma}{l_c} L_c a \right) \phi_{a1R} \right) + F^0_{s1} \]
\[ + k_{s1} \left( \left( -\frac{\gamma}{l_c} \left( h_{p1} - h_{s1} \right) + \frac{\gamma}{l_c} \left( w_{p1} - w_{s1} \right) \right) \phi_h - \left( \frac{\gamma}{l_c} L_s a - \frac{\gamma}{l_c} L_c a \right) \phi_{a1R} \right) + u_{s1R} \]

Rear Suspension

The rear suspension consists of a rigid rear axle connected to the vehicle body by means of a Panhard rod. This rod has an initial rotation of \( \phi^0_p \) and is situated a distance \( h \), beneath the center of gravity of the rear axle. The center of gravity is located above the centerline of the beam connecting the left and right wheel. The complete rear suspension is shown in figure A.2

![Figure A.2 Rear Suspension.](image)

The virtual work \( \delta W_{s2} \) of an arbitrary force \( F_{s2} \) in the rear suspension is given by

\[ \delta W_{s2} = -F_{s2L} \cdot \delta d_{s2L} = -F_{s2L} \cdot \left( \delta z_{a2} - w_{p2} \delta \phi_{a2} - \delta z_h + w_{p2} \delta \phi_h + l_{r2} \delta \theta_h \right) \]
\[ - F^0_{s2} \cdot \left( \left( h_{s2} - h_{p2} - l_{sp} \right) \delta \theta_h + \left( h_{s2} - h_{p2} \right) \delta \phi_h \right) \]
(A.14a)
\[
\delta W_{s2R} = -F_{s2R} \delta d_{s2R} = -F_{s2R} \left( \delta z_{a2} + w_{z2} \delta \phi_{a2} - \delta x_b - w_{z2} \delta \phi_b + l_{a2} \delta \theta_b \right) \\
- F_{s2}^0 \left( (h_{z2} - h_{p2}) \delta \phi_b + (h_{s2} - h_{p2}) \delta \phi_b \right) + h_{x2} \delta \phi_{a2} - l_{p2} \delta \phi_p \delta \phi_p
\]

(A.14b)

where \( F_{s2}^0 \) represents the pretension of the front suspension spring according to

\[
F_{s2}^0 = -\frac{m_b g l_{a1} + m_c g (l_{a1} - l_c)}{2(l_{a1} - l_{a2})}
\]

Similar to the front suspension, the rear suspension contains a spring with stiffness \( c_{s2} \) in (N/m), a damper with damping constant \( k_{s2} \) in (Ns/m) and a force actuator capable of generating force \( u_{s2} \). The suspension forces read

\[
F_{s2L} = c_{s2} (z_{a2} - w_{s2} \phi_{a2} - z_b + l_{a2} \delta \theta_b + w_{s2} \phi_b) + F_{s2}^0 \\
+ k_{s2} (\dot{z}_{a2} - w_{s2} \dot{\phi}_{a2} - \dot{z}_b + l_{a2} \dot{\theta}_b + w_{s2} \dot{\phi}_b) + u_{s2L}
\]

(A.15a)

\[
F_{s2R} = c_{s2} (z_{a2} + w_{s2} \phi_{a2} - z_b + l_{a2} \phi_b - w_{s2} \phi_b) + F_{s2}^0 \\
+ k_{s2} (\dot{z}_{a2} + w_{s2} \dot{\phi}_{a2} - \dot{z}_b + l_{a2} \dot{\theta}_b - w_{s2} \dot{\phi}_b) + u_{s2R}
\]

(A.15b)

**Roll Stabilizers**

The vehicle contains two roll stabilizers. They prevent the vehicle body from excessive rolling during cornering. The contribution of the roll stabilizers to the dynamics of the vehicle model can be described easily using the potential energy. As the stabilizers can be regarded as springs, the potential energy reads

\[
V_{roll} = \frac{1}{2} c_{r1} (\phi_b - \phi_{a1})^2 + \frac{1}{2} c_{r2} (\phi_b - \phi_{a2})^2
\]

(A.16)

with \( c_{r1} \) and \( c_{r2} \) the roll stiffness in (Nm/rad) of the front and rear stabilizers respectively and \( \phi_{a1} \) given by

\[
\phi_{a1} = \frac{z_{a1L} - z_{a1L}}{2w_{p1} + 2L_c a} = \frac{2w_{p1} + L_c a (\phi_{a1L} + \phi_{a1R})}{2w_{p1} + 2L_c a}
\]

(A.17)

The partial derivatives of \( V_{roll} \) to the generalized coordinates are

\[
\frac{\partial V_{roll}}{\partial \phi_b} = c_{r1} \left( \frac{L_c a}{w_{p1} + L_c a} \right)^2 \phi_b + c_{r2} (\phi_b - \phi_{a2}) - c_{r1} \left( \frac{L_c a}{w_{p1} + L_c a} \right)^2 (\phi_{a1L} + \phi_{a1R})
\]
Appendix A Equations of Motion of a Full-Vehicle Model

\[
\frac{\partial V_{roll}}{\partial \theta_{a1L}} = \frac{c_1}{4} \left( \frac{l_a c_a}{w_{p1} + l_a c_a} \right)^2 (\phi_{a1L} + \phi_{a1R}) - \frac{c_1}{2} \left( \frac{l_a c_a}{w_{p1} + l_a c_a} \right)^2 \phi_b
\]

\[
\frac{\partial V_{roll}}{\partial \theta_{a1R}} = \frac{c_1}{4} \left( \frac{l_a c_a}{w_{p1} + l_a c_a} \right)^2 (\phi_{a1L} + \phi_{a1R}) - \frac{c_1}{2} \left( \frac{l_a c_a}{w_{p1} + l_a c_a} \right)^2 \phi_b
\]

\[
\frac{\partial V_{roll}}{\partial \phi_{a2}} = c_2 (\phi_{a2} - \phi_b)
\]

Front and Rear Tires

Similar to the suspension forces, the vertical tire deformations belonging to the virtual work \( \delta W_z^v \) can be derived according to

\[
\delta W^v_{t1L} = -F^v_{t1L} \cdot \left( \begin{array}{c}
\delta s_b + w_{p1} \delta \theta_b + l_{a1} \delta \psi_b + l_{w1} \delta \phi_{a1L} \\
(h_{p1} \phi_b \delta \phi_b + \theta_b \delta \theta_b) + (l_{w1} \delta \psi_b + h_{t1}) (\phi_{a1L} \delta \phi_{a1L} + \theta_b \delta \theta_b) \\
(\psi_b \delta \psi_b + \psi_b \delta \phi_b) - l_{a1} (\phi_b \delta \psi_b + \psi_b \delta \phi_b)
\end{array} \right)
\]

(A.18a)

\[
\delta W^v_{t1R} = -F^v_{t1R} \cdot \left( \begin{array}{c}
\delta s_b - w_{p1} \delta \theta_b - l_{a1} \delta \psi_b - l_{w1} \delta \phi_{a1R} \\
(h_{p1} \phi_b \delta \phi_b + \theta_b \delta \theta_b) + (l_{w1} \delta \psi_b + h_{t1}) (\phi_{a1R} \delta \phi_{a1R} + \theta_b \delta \theta_b) \\
(\psi_b \delta \psi_b + \psi_b \delta \phi_b) - l_{a1} (\phi_b \delta \psi_b + \psi_b \delta \phi_b)
\end{array} \right)
\]

(A.18b)

\[
\delta W^v_{t2L} = -F^v_{t2L} \cdot \left( \begin{array}{c}
\delta s_{a2} + \frac{T_x}{2} \delta \phi_{a2} \\
(h_{p2} + l_{p2} s_p - h_r + h_{t2}) \theta_b \delta \theta_b + h_{p2} \phi_b \delta \phi_b \\
(h_{t2} - h_r) \phi_{a2} \delta \phi_{a2} + l_{p2} \delta \psi_b - l_{a2} (\phi_b \delta \psi_b + \psi_b \delta \phi_b) + \frac{T_x}{2} (\theta_b \delta \psi_b + \psi_b \delta \theta_b)
\end{array} \right)
\]

(A.18c)

\[
\delta W^v_{t2R} = -F^v_{t2R} \cdot \left( \begin{array}{c}
\delta s_{a2} - \frac{T_x}{2} \delta \phi_{a2} \\
(h_{p2} + l_{p2} s_p - h_r + h_{t2}) \theta_b \delta \theta_b + h_{p2} \phi_b \delta \phi_b \\
(h_{t2} - h_r) \phi_{a2} \delta \phi_{a2} + l_{p2} \delta \psi_b - l_{a2} (\phi_b \delta \psi_b + \psi_b \delta \phi_b) - \frac{T_x}{2} (\theta_b \delta \psi_b + \psi_b \delta \theta_b)
\end{array} \right)
\]

(A.18d)

where \( F^v \) and \( F^v \) represent the pretension of the front and rear tires respectively according to
Appendix A Equations of Motion of a Full-Vehicle Model

\[
F_{t1}^0 = \frac{-m_b g l_{a2} + m_c g (l_e - l_{a2})}{2(l_{a1} - l_{a2})} - m_a g \\
F_{t2}^0 = \frac{-m_b g l_{a1} + m_c g (l_a - l_e)}{2(l_{a1} - l_{a2})} - \frac{m_a g}{2}
\]

(A.19)

The vertical tire forces \( F_i^z \) read

\[
F_{11L}^z = c_i \left( z_{11L} - z_b + \omega_p t_p \phi_b + l_{a1} \theta_b + l_{a1} c_a \phi_{a1L} \right) + F_{11L}^0
\]

(A.20a)

\[
F_{11R}^z = c_i \left( z_{11R} - z_b - \omega_p t_p \phi_b + l_{a1} \theta_b - l_{a1} c_a \phi_{a1R} \right) + F_{11L}^0
\]

(A.20b)

\[
F_{22L}^z = c_i \left( z_{22L} - z_a + \frac{T_a}{2} \phi_a \right) + F_{22L}^0
\]

(A.20c)

\[
F_{22R}^z = c_i \left( z_{22R} - z_a - \frac{T_a}{2} \phi_a \right) + F_{22L}^0
\]

(A.20d)

with \( z_i \) the road input. The tires are also able to generate side slip forces \( F_i^y \). The virtual work of these forces become

\[
\delta W_{11L}^y = F_{11L}^y \cdot \left( \delta y_b - h_{p1} \delta \phi_b + l_{a1} \delta \psi_b - (l_{a1} s_a + h_{t1}) \delta \phi_{a1L} \right)
\]

(A.21a)

\[
\delta W_{11R}^y = F_{11R}^y \cdot \left( \delta y_b - h_{p1} \delta \phi_b + l_{a1} \delta \psi_b - (l_{a1} s_a + h_{t1}) \delta \phi_{a1R} \right)
\]

(A.21b)

\[
\delta W_{12L}^y = F_{12L}^y \cdot \left( \delta y_b + t_p \delta z_b - \left( h_{p2} - w_p t_p \right) \delta \phi_b - l_{a2} t_p \delta \theta_b \right) + l_{a2} \delta \psi_b - t_p \delta z_{a2} - \left( h_{t2} - w_p t_p - h_e \right) \delta \phi_{a2}
\]

(A.21c)

\[
\delta W_{12R}^y = F_{12R}^y \cdot \left( \delta y_b + t_p \delta z_b - \left( h_{p2} - w_p t_p \right) \delta \phi_b - l_{a2} t_p \delta \theta_b \right) + l_{a2} \delta \psi_b - t_p \delta z_{a2} - \left( h_{t2} - w_p t_p - h_e \right) \delta \phi_{a2}
\]

(A.21d)

In case of a linear tire model, the cornering forces are linearly related with the slip angle through the load dependent cornering stiffness \( C_{Fa} \) at each wheel.

\[
F_{11L}^y = C_{Fa1} \cdot \alpha_{1L} = C_{Fa1} \cdot \left( -\frac{v_{1L}}{u} + \delta \right) = C_{Fa1} \cdot \left( -\frac{\dot{y}_{1L}}{u} + \psi_b + \delta \right)
\]

(A.22a)

\[
F_{11R}^y = C_{Fa1} \cdot \alpha_{1R} = C_{Fa1} \cdot \left( -\frac{v_{1R}}{u} + \delta \right) = C_{Fa1} \cdot \left( -\frac{\dot{y}_{1R}}{u} + \psi_b + \delta \right)
\]

(A.22b)

\[
F_{22L}^y = C_{Fa2} \cdot \alpha_{2L} = C_{Fa2} \cdot \left( -\frac{v_{2L}}{u} \right) = C_{Fa2} \cdot \left( -\frac{\dot{y}_{2L}}{u} + \psi_b \right)
\]

(A.22c)

\[
F_{22R}^y = C_{Fa2} \cdot \alpha_{2R} = C_{Fa2} \cdot \left( -\frac{v_{2R}}{u} \right) = C_{Fa2} \cdot \left( -\frac{\dot{y}_{2R}}{u} + \psi_b \right)
\]

(A.22d)

with \( \alpha_i \) being the side slip angle in (rad), \( u \) the local (moving reference frame) lateral side slip velocity in (m/s), \( v \) the speed of travel in (m/s), \( \delta \) the steering angle at the front wheels in (rad) (assumed to be equal for the left and right
wheel) and $\dot{y}_i$, the global (fixed reference frame) lateral velocity at each wheel contact point according to

$$
\dot{y}_{1L} = \dot{y}_b - h_p\dot{\phi}_b + l_{a1}\dot{\psi}_b - (l_w s_a + h_{t1})\dot{\phi}_{a1L} \tag{A.23a}
$$

$$
\dot{y}_{1R} = \dot{y}_b - h_p\dot{\phi}_b + l_{a1}\dot{\psi}_b - (l_w s_a + h_{t1})\dot{\phi}_{a1R} \tag{A.23b}
$$

$$
\dot{y}_{2L} = \dot{y}_b + t_p\dot{z}_b - (h_{p2} - w_p\dot{\phi}_p)\dot{\phi}_b - l_{a2}\dot{\phi}_{a2} 
+ t_p\dot{\phi}_p - (h_{t2} - w_p\dot{\phi}_p - h_r)\dot{\phi}_{a2} \tag{A.23c}
$$

$$
\dot{y}_{2R} = \dot{y}_b + t_p\dot{z}_b - (h_{p2} - w_p\dot{\phi}_p)\dot{\phi}_b - l_{a2}\dot{\phi}_{a2} 
+ t_p\dot{\phi}_p - (h_{t2} - w_p\dot{\phi}_p - h_r)\dot{\phi}_{a2} \tag{A.23d}
$$

**Powertrain**

The modeling aspects of the engine/transmission system are discussed in detail in chapter 7. Figure A.3 shows the engine including the dimensions.

![Figure A.3 The engine/transmission.](image)

The potential energy of the deformation of the engine/transmission mounts reads

$$
V_{\text{eng}} = \frac{1}{2} c_{e1} (d_{b1L}^2 + d_{b1R}^2) + \frac{1}{2} c_{e2} d_{b2}^2 \tag{A.24}
$$

with $c_{e1}$ and $c_{e2}$ the stiffnesses of the engine bushes and $d_i$ the deformation of the two front and rear fastenings according to
\[ a_{c1L}^2 = h_{b1}^2(\phi_e - \phi_b)^2 + h_{b1}^2(\theta_e - \theta_b)^2 + (z_e - z_b - w_{b1}(\phi_e - \phi_b) - l_{01}\theta_e + l_{01}\theta_b)^2 \]
\[ a_{c1R}^2 = h_{b1}^2(\phi_e - \phi_b)^2 + h_{b1}^2(\theta_e - \theta_b)^2 + (z_e - z_b + w_{b1}(\phi_e - \phi_b) - l_{01}\theta_e + l_{01}\theta_b)^2 \]
\[ a_{b2}^2 = h_{b2}^2(\phi_e - \phi_b)^2 + h_{b2}^2(\theta_e - \theta_b)^2 + (z_e - z_b - l_{02}\theta_e + l_{02}\theta_b)^2 \]

The partial derivatives of \( V_{eng} \) to the generalized coordinates are given by

\[ \frac{\partial V_{eng}}{\partial z_b} = (2c_{e1} + c_{e2})(z_b - z_e) - (2c_{e1}l_{e1} + c_{e2}l_{e2})\theta_b + (2c_{e1}l_{b1} + c_{e2}l_{b2})\theta_e + 2F_{e1}^0 + F_{e2}^0 \]

\[ \frac{\partial V_{eng}}{\partial \phi_b} = (2c_{e1}(w_{b1}^2 + h_{b1}^2) + c_{e2}h_{b2}^2)(\phi_b - \phi_e) 
+ 2F_{e1}^0(l_{c1}\psi_b - (h_e + h_{b1})\phi_b) + F_{e2}^0(l_{c2}\psi_b - (h_e + h_{b2})\phi_b) \]

\[ \frac{\partial V_{eng}}{\partial \theta_b} = (2c_{e1}(l_{b1}^2 + h_{b1}^2) + c_{e2}(l_{b2}^2 + h_{b2}^2))\theta_b + (2c_{e1}l_{b1} + c_{e2}l_{b2})(z_e - z_b) 
- (2c_{e1}(l_{c1}l_{b1} + h_{b1}^2) + c_{e2}(l_{c2}l_{b2} + h_{b2}^2))\theta_e 
- 2F_{e1}^0(l_{c1} + (h_e + h_{b1})\theta_b) - F_{e2}^0(l_{c2} + (h_e + h_{b2})\theta_b) \]

\[ \frac{\partial V_{eng}}{\partial \psi_b} = 2F_{e1}^0(-l_{b1}\phi_e + l_{c1}\phi_b) + F_{e2}^0(-l_{b2}\phi_e + l_{c2}\phi_b) \]

\[ \frac{\partial V_{eng}}{\partial z_e} = (2c_{e1} + c_{e2})(z_e - z_b) + (2c_{e1}l_{e1} + c_{e2}l_{e2})\theta_b - (2c_{e1}l_{b1} + c_{e2}l_{b2})\theta_e - 2F_{e1}^0 - F_{e2}^0 \]

\[ \frac{\partial V_{eng}}{\partial \phi_e} = (2c_{e1}(w_{b1}^2 + h_{b1}^2) + c_{e2}h_{b2}^2)(\phi_e - \phi_b) 
+ 2F_{e1}^0(-l_{b1}\psi_b + h_{b1}\phi_e) + F_{e2}^0(-l_{b2}\psi_b + h_{b2}\phi_e) \]

\[ \frac{\partial V_{eng}}{\partial \theta_e} = (2c_{e1}(l_{b1}^2 + h_{b1}^2) + c_{e2}(l_{b2}^2 + h_{b2}^2))\theta_e + (2c_{e1}l_{b1} + c_{e2}l_{b2})(z_b - z_e) 
- (2c_{e1}(l_{c1}l_{b1} + h_{b1}^2) + c_{e2}(l_{c2}l_{b2} + h_{b2}^2))\theta_b 
+ 2F_{e1}^0(l_{b1} + h_{b1}\theta_e) + F_{e2}^0(l_{b2} + h_{b2}\theta_e) \]

where \( F_{e1}^0 \) and \( F_{e2}^0 \) represent the pretension of the front and rear engine bush respectively according to

\[ F_{e1}^0 = \frac{-m_{e1}g}{2(l_{b1} - l_{b2})} \quad F_{e2}^0 = \frac{-m_{e2}g}{(l_{b1} - l_{b2})} \] (A.25)
From the partial derivatives, the stiffness matrix $C$ can be composed. In the same way damping matrix $K$ can be calculated using the Rayleigh's dissipation function. These calculations are omitted because it is just a matter of replacing the stiffnesses $c_{e1}$ and $c_{e2}$ with damping constants $k_{e1}$ and $k_{e2}$.

**Gravity**

The potential energy due to gravity reads

$$V_{\text{grav}} = -g(m_b z_b + m_{a1} z_{a1L} + m_{a1} z_{a1R} + m_{a2} z_{a2} + m_c z_c) \quad (A.26)$$

in which $g$ denotes the gravitational acceleration. The partial derivatives of $V_{\text{grav}}$ to the generalized coordinates are given by

$$\frac{\partial V_{\text{grav}}}{\partial z_b} = -g(m_b + 2m_{a1})$$

$$\frac{\partial V_{\text{grav}}}{\partial \phi_b} = -2m_{a1}g(l_{a1} \psi_b - h_{p1} \phi_b) - m_{a2}g(l_{a2} \psi_b - h_{p2} \phi_b)$$

$$\frac{\partial V_{\text{grav}}}{\partial \theta_b} = 2m_{a1}g(l_{a1} + (h_{p1} + l_a s_a) \theta_b) + m_{a2}g(h_{p2} + l_p s_p - h_r) \theta_b$$

$$\frac{\partial V_{\text{grav}}}{\partial \psi_b} = -2m_{a1}gl_{a1} \phi_b - m_{a2}gl_{a2} \phi_b$$

$$\frac{\partial V_{\text{grav}}}{\partial \phi_{a1L}} = m_{a1} gl_a (s_a \phi_{a1L} + c_a)$$

$$\frac{\partial V_{\text{grav}}}{\partial \phi_{a1R}} = m_{a1} gl_a (s_a \phi_{a1R} - c_a)$$

$$\frac{\partial V_{\text{grav}}}{\partial z_{a2}} = -m_{a2}g$$

$$\frac{\partial V_{\text{grav}}}{\partial \phi_{a2}} = -m_{a2}gh_a \phi_a$$

$$\frac{\partial V_{\text{grav}}}{\partial \phi_p} = m_{a2}gl_p s_p \phi_p$$
\[
\frac{\partial V_{\text{grav}}}{\partial z_e} = -m_e g
\]

**Equations of Motion**

The final result of the exercise explained above is a set of 2nd-order differential equations in matrix form given by

\[
M \ddot{\mathbf{z}} + K \dot{\mathbf{z}} + C \mathbf{z} = \mathbf{E}_a F_s + \mathbf{E}_l^x F_l^x + \mathbf{E}_t^z F_t^z
\]  \hspace{1cm} \text{(A.27)}

With \( M \) being the mass matrix, \( C \) the stiffness matrix and \( K \) the damping matrix. \( \mathbf{E}_a, \mathbf{E}_l^x \) and \( \mathbf{E}_t^z \) are input matrices for the suspension force, lateral tire force and vertical tire force respectively. Vector \( \mathbf{z} \) with generalized coordinates reads

\[
\mathbf{z}^T = [x_b \ y_b \ z_b \ \phi_b \ \theta_b \ \psi_b \ \phi_{a1L} \ \phi_{a1R} \ z_{a2} \ \phi_{a2} \ z_e \ \phi_e \ \theta_e]
\]

The three force vectors representing the suspension forces, tire side forces and vertical tire forces are given by

\[
\mathbf{F}_s = \begin{bmatrix} F_{s1L} & F_{s1R} & F_{s2L} & F_{s2R} \end{bmatrix},
\]

\[
\mathbf{F}_l^x = \begin{bmatrix} F_{l1L}^x & F_{l1R}^x & F_{l2L}^x & F_{l2R}^x \end{bmatrix},
\]

\[
\mathbf{F}_t^z = \begin{bmatrix} F_{t1L}^z & F_{t1R}^z & F_{t2L}^z & F_{t2R}^z \end{bmatrix}
\]

The upper triangle of the (symmetric) mass matrix \( M \) is given by

\[
M(1,1) = m_b + 2m_{a1} + m_{a2} + m_e
\]

\[
M(1,5) = 2m_{a1}(h_{p1} + l_{sa}) + m_{a2}(h_{p2} + l_{p2} - h_r) + m_e h_e
\]

\[
M(2,2) = m_b + 2m_{a1} + m_{a2} + m_e
\]

\[
M(2,3) = m_{a2} l_p
\]

\[
M(2,4) = -2m_{a1} h_{p1} - m_{a2}(h_{p2} - w_{p2} t_p) - m_e h_e
\]

\[
M(2,5) = -m_{a2} l_{a2} t_p
\]

\[
M(2,6) = 2m_{a1} l_{a1} + m_{a2} l_e
\]

\[
M(2,7) = -m_{a1} l_{sa}
\]

\[
M(2,8) = -m_{a1} l_{sa}
\]

\[
M(2,9) = -m_{a2} t_p
\]

\[
M(2,10) = m_{a2}(w_{p2} t_p + h_r)
\]
Appendix A Equations of Motion of a Full-Vehicle Model

\[ M(3, 3) = m_s + 2m_a + m_a t_p^2 \]
\[ M(3, 4) = -m_a t_p (h_{p2} - w_{p2} t_p) \]
\[ M(3, 5) = -2m_{a1} l_{a1} - m_a l_{a2} \]
\[ M(3, 6) = m_a l_{a2} t_p \]
\[ M(3, 7) = -m_a l_{a} c_a \]
\[ M(3, 8) = m_a l_{a} c_a \]
\[ M(3, 9) = -m_a t_p^2 \]
\[ M(3, 10) = m_a t_p (w_{p2} t_p + h_r) \]
\[ M(4, 4) = I_{x_a} + 2m_a (l_{w1}^2 + h_{p1}^2) + m_a (h_{p2} - w_{p2} t_p)^2 + m_e h_e^2 \]
\[ M(4, 5) = m_a t_a (h_{p2} - w_{p2} t_p) \]
\[ M(4, 6) = -2m_{a1} l_{a1} h_{p1} - m_a l_{a2} (h_{p2} - w_{p2} t_p) - m_e h_e \]
\[ M(4, 7) = m_{a1} l_{a1} (c_a w_{p1} + s_a h_{p1}) \]
\[ M(4, 8) = m_{a1} l_{a1} (c_a w_{p1} + s_a h_{p1}) \]
\[ M(4, 9) = m_a t_p (h_{p2} - w_{p2} t_p) \]
\[ M(4, 10) = -m_a t_p (w_{p2} t_p + h_r) (h_{p2} - w_{p2} t_p) \]
\[ M(5, 5) = I_{y_b} + 2I_{y_{a1}} + I_{y_{a2}} + 2m_a (l_{a1}^2 + (h_{p1} + l_{s_a})^2) \]
\[ + m_a (l_{a2} t_p^2 + (h_{p2} + l_{p} s_{p} - h_r)^2) + m_e h_e^2 \]
\[ M(5, 6) = -m_a l_{a2} t_p \]
\[ M(5, 7) = m_{a1} l_{a1} l_{a} c_a \]
\[ M(5, 8) = -m_{a1} l_{a1} l_{a} c_a \]
\[ M(5, 9) = m_a l_{a2} t_p^2 \]
\[ M(5, 10) = -m_a l_{a2} t_p (w_{p2} t_p + h_r) \]
\[ M(6, 6) = I_{x_b} + 2I_{x_{a1}} + I_{x_{a2}} + I_{z_e} + 2m_a (l_{a1}^2 + (w_{p1} + l_{c_a})^2) + m_a l_{a2}^2 + m_e h_e^2 \]
\[ M(6, 7) = -m_{a1} l_{a1} l_{a} s_a \]
\[ M(6, 8) = -m_{a1} l_{a1} l_{a} s_a \]
Appendix A Equations of Motion of a Full-Vehicle Model

\[ M(6,9) = -m_{a2}l_{a2}t_p \]

\[ M(6,10) = m_{a2}l_{a2} \left( w_{p2}t_p + h_r \right) \]

\[ M(7,7) = I_{x1} + m_{a1}l_a^2 \]

\[ M(8,8) = I_{x1} + m_{a1}l_a^2 \]

\[ M(9,9) = m_{a2} \left( 1 + t_p^2 \right) \]

\[ M(9,10) = -m_{a2}l_{a2} \left( w_{p2}t_p + h_r \right) \]

\[ M(10,10) = I_{x2} + m_{a2} \left( w_{p2}t_p + h_r \right)^2 \]

\[ M(11,11) = m_e \]

\[ M(12,12) = I_{xe} \]

\[ M(13,13) = I_{ye} \]

The upper triangle of stiffness matrix \( C \) takes a form as

\[ C(3,3) = 2c_{e1} + c_{e2} \]

\[ C(3,5) = -2c_{e1}l_{e1} - c_{e2}l_{e2} \]

\[ C(3,11) = -2c_{e1} - c_{e2} \]

\[ C(3,13) = 2c_{e1}l_{h1} + c_{e2}l_{h2} \]

\[ C(4,4) = c_{r1} \left( \frac{l_a c_a}{w_p + l_a c_a} \right)^2 + c_{r2} + 2c_{r1} \left( w_{b1}^2 + h_{b1}^2 \right) + c_{e2}h_{b2}^2 + 2m_{a1}gh_{p1} + m_{a2}gh_{p2} \]

\[ \quad + 2F_{x1}^0 \left( \frac{w_{p1}}{l_a} \left( h_{p1} - h_{s1} \right) - \frac{w_{s1}}{l_a} \left( w_{p1} - w_{s1} \right) \right) + 2F_{x2}^0 \left( h_{s2} - h_{p2} \right) \]

\[ \quad + 2F_{x1}^0 h_{p1} + 2F_{x2}^0 h_{p2} - 2F_{x1}^0 (w_{x1} + h_{b1}) - F_{x2}^0 (w_{x2} + h_{b2}) \]

\[ C(4,6) = -2m_{a1}g_{l1}a1 - m_{a2}g_{l2}a2 - 2F_{l1}^0 g_{l1}a1 - 2F_{l2}^0 g_{l2}a2 + 2F_{c1}^0 l_{c1} + F_{c2}^0 l_{c2} \]

\[ C(4,7) = -c_{r1} \left( \frac{l_a c_a}{w_p + l_a c_a} \right)^2 \]

\[ C(4,8) = -c_{r1} \left( \frac{l_a c_a}{w_p + l_a c_a} \right)^2 \]

\[ C(4,10) = -c_{r2} \]

\[ C(4,12) = -2c_{r1} \left( w_{b1}^2 + h_{b1}^2 \right) - c_{e2}h_{b2}^2 \]
Appendix A Equations of Motion of a Full-Vehicle Model

\[ C(5, 5) = 2c_{r_1}(l_{r_1}^2 + h_{r_1}^2) + c_{r_2}(l_{r_2}^2 + h_{r_2}^2) + 2m_a g (h_{p_1} + l_{a_1}) + m_{a_2} g (h_{p_2} + l_{a_2} - h_r) + F_{r_1}^0(h_{r_2} - h_{p_2} - l_{p_2} - h_r) - F_{r_2}^0(h_r + h_{b_2}) - 2F_{c_1}(h_r + h_{b_1}) + 2F_{c_2}(h_{p_2} + l_{p_2} - h_r - h_{b_2}) \]

\[ C(5, 11) = 2c_{r_1}l_{r_1} + c_{r_2}l_{r_2} \]

\[ C(5, 13) = -2c_{r_1}(l_{b_1}l_{r_1} + h_{b_1}) - c_{r_2}(l_{b_2}l_{r_2} + h_{b_2}) \]

\[ C(6, 6) = 2F_{c_1}^0(\frac{y_0}{l_0}(w_{p_1} + l_{a_1} - w_{s_1}) - \frac{y_0}{l_0}(h_{p_1} + l_{a_1} - h_{s_1})) \]

\[ C(6, 12) = -2F_{c_1}^0l_{b_1} - F_{c_2}^0l_{b_2} \]

\[ C(7, 7) = \frac{c_{r_1}}{4} \left( \frac{l_{a_1}c_{a_1}}{w_{p_1} + l_{a_1}} \right)^2 + m_{a_1} g l_{a_1} s_{a_1} - F_{c_1}^0 \left( \frac{y_0}{l_0} l_{s_1} c_{a_1} + \frac{y_0}{l_0} l_{c_1} \right) + F_{c_1}^0(l_{s_1} s_{a_1} + h_{b_1}) \]

\[ C(7, 8) = \frac{c_{r_1}}{4} \left( \frac{l_{a_1}c_{a_1}}{w_{p_1} + l_{a_1}} \right)^2 + m_{a_1} g l_{a_1} s_{a_1} - F_{c_1}^0 \left( \frac{y_0}{l_0} l_{s_1} c_{a_1} + \frac{y_0}{l_0} l_{c_1} \right) + F_{c_1}^0(l_{s_1} s_{a_1} + h_{b_1}) \]

\[ C(8, 8) = \frac{c_{r_1}}{4} \left( \frac{l_{a_1}c_{a_1}}{w_{p_1} + l_{a_1}} \right)^2 + m_{a_1} g l_{a_1} s_{a_1} - F_{c_1}^0 \left( \frac{y_0}{l_0} l_{s_1} c_{a_1} + \frac{y_0}{l_0} l_{c_1} \right) + F_{c_1}^0(l_{s_1} s_{a_1} + h_{b_1}) \]

\[ C(10, 10) = c_{r_2} + 2F_{c_2}^0(h_{r_2} - h_r) - 2F_{c_2}^0h_r - m_{a_2} g h_r \]

\[ C(11, 11) = 2c_{r_1} + c_{r_2} \]

\[ C(11, 13) = -2c_{r_1}l_{b_1} - c_{r_2}l_{b_2} \]

\[ C(12, 12) = 2c_{r_1}(w_{b_1}^2 + h_{b_1}^2) + c_{r_2}(h_{b_2}^2 + 2F_{c_2}^0h_{b_1} + F_{c_2}^0l_{b_2}) \]

\[ C(13, 13) = 2c_{r_1}(l_{b_1}^2 + h_{b_1}^2) + c_{r_2}(l_{b_2}^2 + h_{b_2}^2) + 2F_{c_1}^0l_{b_1} + F_{c_2}^0l_{b_2} \]

Equation (A.14), (A.18) and (A.25) show that there are some second-order terms with the Panhard rod rotation \( \phi_p \) as the generalized coordinate. Since this degree of freedom is replaced with the vertical displacement of the rear axle \( z_{a_2} \), a conversion must take place. The contribution of the gravity and pretension of the tire and suspension spring to stiffness matrix \( C \) due to Panhard rod DOF can easily be described by the following relation

\[ C = C + \left( 2F_{c_2}^0 - 2F_{c_2}^0 + m_{a_2} g \right) \cdot l_{p_2} s_{p_2} \cdot T_{p}^T \cdot T_{p} \quad (A.27) \]

with \( T_{p} \) representing the relation between Panhard rod orientation \( \phi_p \) and vertical displacement of the rear axle \( z_{a_2} \) according to equation (A.5)

\[ T_{p} = \frac{1}{l_p c_p} \begin{bmatrix} 0 & 0 & -1 & -w_{p_2} & l_{a_2} & 0 & 0 & 0 & 1 & -w_{p_2} & 0 & 0 \end{bmatrix} \]
Appendix A  Equations of Motion of a Full-Vehicle Model

It can easily be seen that the contribution of equation (A.27) to the stiffness matrix is zero since the pretension and gravity terms cancel each other.

The upper triangle of damping matrix $K$ becomes

$K(3,3) = 2k_{c1} + k_{c2}$

$K(3,5) = -2k_{c1}l_{c1} - k_{c2}l_{c2}$

$K(3,11) = -2k_{c1} - k_{c2}$

$K(3,13) = 2k_{c1}l_{b1} + k_{c2}l_{b2}$

$K(4,4) = 2k_{c1}(u_{b1}^2 + h_{b1}^2) + k_{c2}h_{b2}^2$

$K(4,12) = -2k_{c1}(u_{b1}^2 + h_{b1}^2) - k_{c2}h_{b2}^2$

$K(5,5) = 2k_{c1}(l_{c1}^2 + h_{b1}^2) + k_{c2}(l_{c2}^2 + h_{b2}^2)$

$K(5,11) = 2k_{c1}l_{c1} + k_{c2}l_{c2}$

$K(5,13) = -2k_{c1}(l_{c1}l_{c1} + h_{b1}^2) - k_{c2}(l_{c2}l_{c2} + h_{b2}^2)$

$K(11,11) = 2k_{c1} + k_{c2}$

$K(11,13) = -2k_{c1}l_{b1} - k_{c2}l_{b2}$

$K(12,12) = 2k_{c1}(u_{b1}^2 + h_{b1}^2) + k_{c2}h_{b2}^2$

$K(13,13) = 2k_{c1}(l_{b1}^2 + h_{b1}^2) + k_{c2}(l_{b2}^2 + h_{b2}^2)$
The suspension force input matrix $E_s$ can be derived from the virtual work $\delta W_s$

$$
E_s = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & -1 & -1 \\
\frac{20}{l_0} (h_{p1} - h_{s1}) & -\frac{20}{l_0} (w_{p1} - w_{s1}) & 0 & -l_{a2} \\
\frac{20}{l_0} (h_{p1} - h_{s1}) + \frac{20}{l_0} (w_{p1} - w_{s1}) & w_{s2} & -w_{s2} & l_{a2}
\end{bmatrix}
$$

The side slip force input matrix $E_i^y$ equals

$$
E_i^y = \begin{bmatrix}
0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 \\
0 & 0 & t_p & t_p \\
-h_{p1} & -h_{p1} & -h_{p2} + w_{p2} t_p & -h_{p2} + w_{p2} t_p \\
0 & 0 & -l_{a2} t_p & -l_{a2} t_p \\
l_{a1} & l_{a1} & l_{a2} & l_{a2} \\
-l_{w} s_a - h_{t1} & 0 & 0 & 0 \\
0 & -l_{w} s_a - h_{t1} & 0 & 0 \\
0 & 0 & -t_p & -t_p \\
0 & 0 & -h_{t2} + w_{p2} t_p + h_r & -h_{t2} + w_{p2} t_p + h_r \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
$$
And finally the vertical tire force input matrix $E^v_i$ is determined by

$$E^v_i = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-1 & -1 & 0 & 0 \\
w_{p1} & -w_{p1} & 0 & 0 \\
l_{a1} & l_{a1} & 0 & 0 \\
0 & 0 & 0 & 0 \\
l_w c_a & 0 & 0 & 0 \\
0 & -l_w c_a & 0 & 0 \\
0 & 0 & -1 & -1 \\
0 & 0 & \frac{t_w}{2} & -\frac{t_w}{2} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}$$

**List of Symbols**

- $c_a$: cosine of initial rotation of front swing axle
- $c_p$: cosine of initial rotation of Panhard rod
- $c_{e1}$: stiffness engine mount, vertical direction, front side
- $c_{e2}$: stiffness engine mount, vertical direction, rear side
- $c_{fu}$: cornering stiffness
- $c_{r1}$: stiffness roll stabilizer, front suspension
- $c_{r2}$: stiffness roll stabilizer, rear suspension
- $c_{s1}$: stiffness front suspension
- $c_{s2}$: stiffness rear suspension
- $c_i$: stiffness tire, vertical direction
- $C$: stiffness matrix
- $d_i$: deformation of spring $i$
- $h_{a1}$: $z$-distance c.g. vehicle body to c.g. front axle
- $h_{a2}$: $z$-distance c.g. vehicle body to c.g. rear axle
- $h_{b1}$: $z$-distance c.g. engine to front engine mounting
- $h_{b2}$: $z$-distance c.g. engine to rear engine mounting
- $h_e$: $z$-distance c.g. vehicle body to c.g. engine
- $h_{p1}$: $z$-distance c.g. vehicle body to reaction point front axle
Appendix A Equations of Motion of a Full-Vehicle Model

\[ h_{p2} \] z-distance c.g. vehicle body to Panhard rod fastening  
\[ h_r \] z-distance c.g. rear axle to Panhard rod fastening  
\[ h_{s1} \] z-distance c.g. vehicle body to Mc Pherson strut top mounting  
\[ h_{s2} \] z-distance c.g. vehicle body to spring seat rear suspension  
\[ h_{r1} \] z-distance c.g. front axle to tire track contact point  
\[ h_{r2} \] z-distance c.g. rear axle to tire track contact point

\[ I_{xa1} \] moment of inertia front axle about x-axis  
\[ I_{ya1} \] moment of inertia front axle about y-axis  
\[ I_{za1} \] moment of inertia front axle about z-axis  
\[ I_{xa2} \] moment of inertia rear axle about x-axis  
\[ I_{ya2} \] moment of inertia rear axle about y-axis  
\[ I_{za2} \] moment of inertia rear axle about z-axis  
\[ I_{xb} \] moment of inertia vehicle body about x-axis  
\[ I_{yb} \] moment of inertia vehicle body about y-axis  
\[ I_{zb} \] moment of inertia vehicle body about z-axis  
\[ I_{xe} \] moment of inertia engine about x-axis  
\[ I_{ye} \] moment of inertia engine about y-axis  
\[ I_{ze} \] moment of inertia engine about z-axis  

\[ k_{e1} \] damping constant engine mount, vertical direction, front side  
\[ k_{e2} \] damping constant engine mount, vertical direction, rear side  
\[ k_s1 \] damping constant front suspension  
\[ k_s2 \] damping constant rear suspension  
\[ K \] damping matrix

\[ l_a \] distance reaction point to c.g. front axle  
\[ l_{a1} \] x-distance c.g. vehicle body to c.g. front axle  
\[ l_{a2} \] x-distance c.g. vehicle body to c.g. rear axle  
\[ l_{b1} \] x-distance c.g. engine to front engine mounting  
\[ l_{b2} \] x-distance c.g. engine to rear engine mounting  
\[ l_e \] x-distance c.g. vehicle body to c.g. engine  
\[ l_{e1} \] x-distance c.g. vehicle body to front engine mounting  
\[ l_{e2} \] x-distance c.g. vehicle body to rear engine mounting  
\[ l_{p} \] length of Panhard rod  
\[ l_{s} \] distance reaction point to Mc Pherson strut bottom mounting  
\[ l_w \] distance reaction point to front wheel center

\[ m_{a1} \] mass of front axle  
\[ m_{a2} \] mass of rear axle
Appendix A  Equations of Motion of a Full-Vehicle Model

\[ \begin{align*} 
    m_b & \quad \text{mass of vehicle body} \\
    m_e & \quad \text{mass of engine/transmission} \\
    M & \quad \text{mass matrix} \\
    s_a & \quad \text{sinus of initial rotation of front swing axle} \\
    s_p & \quad \text{sinus of initial rotation of Panhard rod} \\
    T & \quad \text{kinetic energy} \\
    v & \quad \text{speed of travel} \\
    V & \quad \text{potential energy} \\
    w_{a1} & \quad y\text{-distance c.g. vehicle body to c.g. right front axle} \\
    w_{b1} & \quad y\text{-distance c.g. engine to right front engine mounting} \\
    w_{p1} & \quad y\text{-distance c.g. vehicle body to reaction point of right front axle} \\
    w_{p2} & \quad y\text{-distance c.g. vehicle body to Panhard rod fastening} \\
    w_{x1} & \quad y\text{-distance c.g. vehicle body to Mc Pherson strut top mounting, right} \\
    w_{x2} & \quad y\text{-distance c.g. vehicle body to spring seat right rear suspension} \\
    W & \quad \text{work} \\
    x & \quad \text{longitudinal displacement} \\
    y & \quad \text{lateral displacement} \\
    z & \quad \text{vertical displacement} \\
    \alpha & \quad \text{slip angle} \\
    \delta & \quad \text{steering angle at the front wheels} \\
    \phi & \quad \text{roll angle} \\
    \theta & \quad \text{pitch angle} \\
    \psi & \quad \text{yaw angle} 
\end{align*} \]
BIBLIOGRAPHY


This thesis deals with the control system design of intelligent suspension systems for automotive application. Unlike contemporary suspensions with fixed characteristics, the adjustment of particular suspension properties may enhance performance noticeably. The main point of examination is concentrated on the design of the controller that adjusts the suspension for varying external conditions such as speed, road condition and steering maneuver. The potential of suspension control has been illustrated on the basis of active and semi-active force generators. The distinction between active and semi-active lies in the field of energy consumption. A practical example of a semi-active device is an adjustable damper that is only able to dissipate energy. The active suspension is characterized by an effort source such as a hydraulic cylinder or electric motor. The thesis has been split in two
parts. In the first part only a single wheel station of a car (quarter-car) will be
treated whereas the second part deals with a full-vehicle. The quarter-car
analyses are very useful to gain insight into the problems related to
suspension control. It makes the analysis and synthesis of the full-vehicle
much easier.

The quarter-car model has only two degrees of freedom that can be
represented by two second-order differential equations. The roughnesses of
the road profile in a stochastic sense can be approximated by a first-order
differential equation. By having now five state variables it is possible to define
performance criteria. These criteria are necessary once suspension systems
have to be optimized objectively. The performance of the quarter-car
suspension system can roughly be assessed quantitatively in terms of ride
comfort and road holding. It is well known that the improvement of ride
comfort (a low level of accelerations of the vehicle body) of suspension systems
with a fixed setup conflicts with the variations of the tire load which are a
measure of the road holding ability. The controlled wheel suspension offers
the possibility to adjust the setup and may therefore relieve the conflict
between ride comfort and road holding. Other performance considerations that
may play a role in suspension design are the relative displacement between
axle and vehicle body (suspension working space) and, particularly in cases of
deterministic disturbances like kerbs or speed humps, the motion of the
sprung mass.

In order to improve the overall suspension performance, various control
solutions have been discussed. In case of an active suspension system it is
possible to apply the theory of linear quadratic Gaussian (LQG) control. The
LQG algorithm requires that the system is disturbed by white noise and is
based on the minimization of a quadratic performance criterion. This cost
function may contain the quantitative performance criteria as mentioned
above. The relative importance of the different performance parameters can be
expressed by weighting factors to match. After having chosen the weighting
factor, the optimal control algorithm will select a state feedback gain matrix
such that the cost function is minimized. The optimal control theory has been
used for full-state feedback as well as for limited state feedback.

The full-state feedback considers all the states of the system and uses this
information to compose an active suspension force applied between axle and
vehicle body. On the other hand, the limited state feedback looks only at
particular states. The study on suspension control has been concentrated on
the effect of the choice of the weighting factors on the performance of the closed-loop active suspension system. The analysis has shown that despite the presence of an active force generator, and thus a freedom to generate an arbitrary force, the conflict between ride comfort and road holding remains. This is mainly because the actuator force acts on both axle and vehicle body in combination with a two degree of freedom model with only a single actuator. Therefore, the main advantage of a controlled suspension system is the possibility to change one or more parameters of the system such that in certain cases ride comfort can be improved at the cost of road holding or vice versa. The control analysis has been repeated for limited state feedback. The evaluation of performance has shown that there is only a very little difference between full-state and limited state feedback controlled systems. The limited state feedback is more favorable because fewer states have to be measured when the controller is implemented in a real vehicle.

As active suspension systems possess many drawbacks, such as high energy consumption and cost price, the performance of semi-active damper systems has been compared with fully active suspensions. The semi-active suspension system is characterized by a rapidly adjustable damper. This can either be a continuously variable damper or a discretely switchable damper. When damping is made variable, a stratagem for controlling is needed. The control structures which use LQG control in combination with an active system, can also be applied to the semi-active system but now without guarantee of optimality. In those cases where the sign of the desired active force does not match with the sign of the available damper force, the damper is switched off. Separate control strategies were designed to improve either ride comfort or road holding. Because adjustable damper systems are typically non-linear, time domain simulations are necessary to evaluate the performance. Simulation results have shown that it is possible to improve ride comfort significantly using the so-called skyhook control algorithm. This control structure is based on limited state feedback and considers only the absolute vertical velocity of the sprung mass. However, the gain in ride comfort is accompanied by a drastic increase of dynamic tire load variations. On the other hand it appeared not to be possible to decrease dynamic tire load variations significantly using an adjustable damper system.

Despite the control potentialities of (semi-)active suspensions, the conflict between ride comfort and road holding remains. This means that the suspension may be adjusted during the ride in order to improve either the ride comfort or road holding. Since the safety of the vehicle should not be
Summary

endangered during cornering and braking or accelerating, an additional adaptive control loop has been designed which decides whether the suspension has to be setup for ride comfort or road holding. As long as the dynamic tire load component does not exceed a predetermined limit, the adaptive controller selects a setup that accents ride comfort. However, during cornering and braking or accelerating the tire load may not vary too much since that may endanger the safety. Under these circumstances the adaptive suspension controller permits a worsening of the ride comfort in favor of tire load control. The main input to the adaptive controller is the ratio of the dynamic tire load to the static tire load.

Most of the state variables must be known for the control structure as discussed above. It is often unrealistic to assume that all states of a system can be measured. Therefore, system states that cannot be measured have been estimated using a state estimator or Kalman filter. For a linear system such as the active suspension system it quite easy to construct a state estimator. However, due to the non-linearities of the semi-active damper system such a solution is not evident any more. Sometimes estimator instability can not be avoided. This problem can be solved using multi-model techniques. The multi-model analysis considers each damper setting as a different linear suspension system. By taking all these system descriptions it is now possible to determine a single stable Kalman filter using the multi-model optimization.

The second part of the thesis deals with a full-vehicle model. The full-vehicle is not just a connection of four quarter-cars. Besides the roll and pitch degree of freedom, the full-vehicle offers the possibility to perform steering maneuvers. The road holding aspect of the quarter-car model can now be enhanced with handling related aspects. Especially the influence of vertical dynamics such as tire load variations on lateral dynamics is of particular interest.

The full-vehicle part starts with an introduction into the modeling aspects. The vehicle model contains 13 degrees of freedom (DOFs): six for the main vehicle body, one for each axle and three for the powertrain. The engine related DOFs have been included because several natural frequencies of the powertrain as a rigid body on elastic mounts are in the important frequency range between 10 and 15 Hz. The equations of motion have been derived using the method of Lagrange. Complex suspension kinematics have been simplified. Since a full-vehicle contains four wheels, a new stochastic road model must be developed. The model used comprises the correlation between
left and right track and the time delay between front and rear wheel excitations. The modeling of the tire characteristics need special attention. Both steady-state and transient properties have been included in the model.

Since the full-vehicle model has more degrees of freedom than a quarter-car, additional performance criteria are necessary to describe the performance in an objective way. In addition to the criteria used in the quarter-car model analysis, the ride comfort parameter can be extended to include lateral and longitudinal accelerations. Furthermore objective criteria are desired to assess handling. To develop such criteria it was decided to conduct a double lane-change test on an uneven road as an experiment to investigate the relation between road holding and handling. In addition frequency response functions are used. The ride and handling performance of a full-vehicle with a standard passive suspension system has been evaluated in order to serve as a standard for comparing similar vehicles with controlled suspensions.

Similar to the quarter-car analysis the design of an intelligent suspension system has been based on LQG control. However, the full-vehicle application offers some more degrees of freedom in the control design phase. In addition to the full- and partial state feedback, the effect of the road description on the performance has been investigated. Especially the effect on the track correlation and wheelbase time delay on the performance parameters has been studied. The results showed that the extra information arising from the cross-correlated road tracks does not result in greater control potentials. However, including the wheelbase time delay in the control design phase has a large influence on the performance. Especially the tire load variations and suspension travel of the two rear suspensions can benefit from the extra information of the wheelbase time delay. Furthermore, the LQG control analysis has shown that the performance of partial state feedback systems can compete with suspensions based on full-state feedback. The partial state feedback considers only the velocities of the sprung and unsprung mass at the suspension mounts and can therefore be conceived as active damping. The exchange of state information among the four suspensions may enhance the performance noticeably. The main conclusion applicable to the full-vehicle model are generally similar to those of the quarter-car model. In general, the ride comfort of the passengers can be improved substantially albeit at the cost of a degradation of the road holding ability.

After the determination of the ride performance the vehicles with active suspension were exposed to the double-lane change maneuver on an uneven
road surface. These simulations should point out to what extent the increase in tire load variations affects the handling. The results show that moderate tire load variations do not affect the handling. The tire side force variations are only slightly worse compared with the standard vehicle. The variations are only larger in those cases were the vertical tire load decreases drastically due to the load transfer. The vehicle with active suspension based on full-state feedback did not perform very well in the double lane-change test. Besides the quite harmless increase in tire load variations, the unacceptable low roll stiffness and the unbalanced load transfer contribute to the degradation of the handling. The latter two phenomena are caused by the full-state feedback structure of the active suspension. The feedback of position related states must be avoided. The vehicle with a suspension based on partial state feedback (velocities only) performed well compared with the standard configuration. The only situations to be avoided are those cases where the load transfer and the dynamic tire load variations coincide in such a way that the average cornering force will be reduced too much. An adaptation of the suspension controller is necessary to avoid these situations.

The adaptation has been based on gain scheduling. This means that the gains of the state feedback matrix are adjusted during the ride depending on the operating point of the vehicle. The adaptation has been based on the ratio of the dynamic tire load to the average tire load. If this ratio becomes large than the dynamic tire load variations may affect the handling. Then, the adaptive suspension controller will automatically adjust the feedback gains in order to restrict the tire load variations of a particular wheel. The demand for safety goes with a brief worsening of the ride comfort during this intervention.

The implementation of adaptive suspension control requires the knowledge of several states of the vehicle. States like the absolute vertical velocity of the vehicle body or the dynamic tire load cannot be measured directly. It is however possible to estimate the required signals with the aid of an internal model of the vehicle. The unknown states have been estimated by a Kalman filter. The vertical accelerations of the vehicle body and axles serve as an input to the Kalman filter. A major problem is to eliminate the contribution of the gravitational, lateral and longitudinal accelerations to the sensor signals. During cornering and braking or accelerating the offset of the acceleration signals due to the tilted position of the vehicle body cause the estimated signals to drift. The drift phenomena can be suppressed by means of a Kalman filter with proportional-integral action.
SAMENVATTING

Dit proefschrift beschrijft het ontwerp van regelingen voor een intelligent wielophangingssysteem toegepast in personenwagens. De huidige systemen bevatten meestal passieve elementen zoals veren en dempers welke vaak een vaste instelling hebben. De optimale veer-demper afstelling wordt door de fabrikant eenmalig gekozen zodanig dat het betreffende voertuig gedurende de gehele levenscycles en op een grote verscheidenheid van wegdekken aan de vooraf opgelegde specificaties met betrekking tot comfort en wegligging zal voldoen. Vaak is de gekozen instelling een compromis tussen beide criteria. Afhankelijk van het type auto en de filosofie van de fabrikant zal de nadruk komen te liggen op comfort of wegligging. Door nu karakteristieke parameters van een wielophanging, zoals veersteifheid en/of demping variabel te maken, of door extra krachtsactuatoren aan een wielophanging toe te voegen, kan
men de prestaties van een auto in termen van comfort en wegliggings
aanzienlijk verbeteren doordat men optimaal in kan spelen op de heersende
rijomstandigheden. Het variabel maken van wielophangingsparameters
vraagt om een regelaar welke een optimale afstelling moet garanderen op
basis van gemeten grootheden, zoals rijsnelheid of stuurwieluitslag, die het
werkpunt van de auto karakteriseren. In dit proefschrift wordt er vanuit
gegaan dat de wielophanging een actuator (actieve vering) of een geregelde
demper (semi-actieve vering) bevat. Actieve en semi-actieve systeem
onderscheiden zich voornamelijk op het gebied van energieverbruik. Geregelde
dempersystemen kunnen alleen energie dissiperen. Het verstellen van de
demper kost nagenoeg geen energie. Actieve veersystemen worden
gekarakteriseerd door een krachtbron (b.v. hydraulische actuator) welke in
theorie elke gewenste kracht in de wielophanging kunnen introduceren. Dit
gaat meestal gepaard met een aanzienlijk energieverbruik.

Het proefschrift is opgedeeld in twee delen. In het eerste deel zal er aandacht
besteed worden aan een vierde deel van een auto (kwart-voertuig); het tweede
deel behandelde een volledige auto. De analyse op basis van het kwart-voer-
tuigmmodel dient hoofdzakelijk om inzicht te verkrijgen in de problematiek van
geregelde wielophangingen. Dit inzicht en de resultaten verkregen uit deze
analyse vergemakkelijken de analyse van de regeling op basis van een
volledig voertuig.

Het kwart-voertuigmmodel bestaat uit twee lichamen: de wielas en carrosserie.
Deze twee lichamen zijn onderling verbonden met een veer en demper. In het
geval van actieve vering zal er ook een krachtsactuator tussen as en
carrosserie bevinden. De autoband wordt gekarakteriseerd door een niet-
lineaire veer welke alleen drukkrachten kan genereren. De dynamica van dit
tweevoudig massa-veersysteem kan beschreven worden door twee tweede-orde
differentiaal vergelijkingen. De stochastische wegdekoneffenheden kunnen
benaderd worden door een vormend eerste-orde filter welke witte ruis als
ingang heeft.

Alvorens over te gaan tot het analyseren en optimaliseren van het geregelde
systeemgedrag, is het noodzakelijk eerst de objectieve ontwerp-criteria te
definieren. Het doel van de regeling is zowel het comfort als het weggedrag
van het voertuig te verbeteren. Het comfort van een voertuig is onder andere
gerelateerd aan de trillingshinder die de inzittenden ondervinden tijdens het
rijden over een oneven wegdek. Een maat voor het comfort is het (verticale)
versnellingsniveau van de carrosserie. Hoe lager het versnellingsniveau, hoe
comfortabeler de auto. Het weggedrag van een voertuig is een stuk moeilijker te karakteriseren. Als een auto over een oneffen wegdek rijdt, zal de contactkracht tussen band en wegdek variëren. Tijdens het rechtuit rijden zijn deze bandkrachtvariaties nauwelijks van invloed op de wegligging. Bevindt het voertuig zich echter in een bocht, of wordt er geremd, dan zullen de verticale bandkrachtvariaties koersafwijkingen kunnen veroorzaken doordat de slipkrachten in de langs- en dwarsrichting van de band variëren ten gevolgen van verticale bandkrachtvariaties. Hieruit volgt dat het voertuig een beter wegligging zal hebben als de dynamische bandkrachtvariaties laag zijn. Naast comfort en wegligging spelen met name de werkruimte (dit is de ruimte tussen as en carrosserie), geluidssproductie, kostprijs en het energieverbruik een belangrijke rol bij het optimaliseren van een wielophanging.

Nu het systeem en de ontwerp-criteria bekend zijn, kan er gestart worden met het ontwerp van de regeling. In het geval van actieve vering is er gebruik gemaakt van de lineaire optimale regeltheorie. De optimale regeltheorie veronderstelde dat het systeem (= het voertuig) verstoord wordt door witte ruis en is gebaseerd op een minimalisatie van een kwadratisch criterium. Dit criterium bevat de kwantificeerbare grootheden zoals versnellingsniveau van de carrosserie, bandkrachtvariaties, werkruimte en actuatorkracht. De bijdrage van ieder ontwerp-criterium aan de totale kostenfunctie kan tot uitdrukking gebracht worden met weegfactoren. Nadat een keuze van weegfactoren is gemaakt, kan er met behulp van de optimale regeltheorie een toestandsterugkoppeling berekend worden die de kwadratische kostenfunctie minimaliseert. Naast volledige toestandsterugkoppeling kan er ook een gedeeltelijke toestandsterugkoppeling berekend worden. De studie met betrekking tot de regeling gebaseerd op de optimale regeltheorie is met name toegespitst op de keuze van de weegfactoren. De studie heeft aangetoond dat, ondanks de aanwezigheid van een actieve krachtsgenerator tussen as en carrosserie, het conflict tussen comfort en wegligging blijft voortbestaan. Dit komt hoofdzakelijk voort uit het feit dat de actuatorkracht zowel op de wielas als op de carrosserie werkt. Maar, het actieve veersysteem kan, afhankelijk van de gekozen waarden voor de weegfactoren, het comfort ten koste van de wegligging of de wegligging ten koste van het comfort verbeteren, afhankelijk van de rijomstandigheden. Het prestatieverschil tussen volledige en gedeeltelijke toestandsterugkoppeling is van geringe betekenis. Een gedeeltelijke toestandsterugkoppeling geniet de voorkeur omdat er minder sensoren noodzakelijk zijn als de regeling gerealiseerd moet worden.
Samenvatting

Omdat actieve veersystemen een groot aantal nadelen hebben, zoals het energieverbruik en de kostprijs, is de analyse en evaluatie van de regeling van een kwart voertuig herhaald voor het geval van een semi-actief veersysteem (geregelde demper). Het uitgangspunt van de regeling is wederom de lineaire optimale regeltheorie. Een regelbare demper kan niet elke gewenste kracht leveren. In situaties waar de regeling een kracht voorschrijft die niet door een geregelde demper gegenereerd kan worden, zal de demper uitgeschakeld worden. Daar geregelde dempersystemen niet-lineair zijn, moet het prestatieniveau gebaseerd worden op simulatiereultaten. Deze resultaten hebben aangetoond dat het comfort aanzienlijk verbeterd kan worden als de zogenaamde skyhook regeling wordt gebruikt. Deze regeling is gebaseerd op de terugkoppeling van de absolute verticale carrosseriesnelheid. Echter, de comfortverbetering gaat gepaard met een toename van de dynamische bandkrachtvariaties hetgeen een verslechtering van de wegligging tot gevolg heeft. Het is niet mogelijk gebleken de wegligging met behulp van een geregelde demper te verbeteren. Een stugge, vaste instelling van de demper biedt de minste bandkrachtvariaties.

Ondanks het regelpotentieel van (semi-)actieve veersystemen kan men uit het voorgaande concluderen dat het conflict tussen comfort en wegligging nog steeds bestaat. Maar bepaalde parameters van de wielophanging kunnen worden veranderd zodanig dat hetzij comfort hetzij wegligging verbeterd kunnen worden afhankelijk van de rijomstandigheden. Tijdens het rechtuitrijden kan de instelling volledig op comfort gericht worden omdat er slechts geringe slipkrachten aan de banden noodzakelijk zijn. Tijdens het rijden door bochten, remmen of optrekken kan de nadruk gelegd worden op wegligging omdat in die gevallen de band optimaal contact moet houden met de weg. Het is dus noodzakelijk de regeling uit te breiden met een adaptieve lus. De adaptiviteit zoals die in dit proefschrift ontworpen is, is gebaseerd op de dynamische bandkracht. Zolang de bandkrachtvariaties door wegdekoneffenheden niet een vastgestelde limiet overschrijden, zal de regeling het comfort verbeteren. Worden de bandkrachtvariaties te groot tijdens het nemen van een bocht, of tijdens het remmen of accelereren, dan zal de regeling afhankelijk van de ernst van de manoeuvre de wegligging trachten te verbeteren ten koste van het comfort.

Een aantal toestandsgrootheden van het voertuig moeten bekend zijn als bovengenoemde regeling in een voertuig geïmplementeerd moet worden. Het is vaak idealistisch te veronderstellen dat alle toestanden van het voertuig meetbaar zijn. Zowel de absolute verticale snelheid van de carrosserie als de
Samenvatting


Het tweede deel van het proefschrift beschrijft het ontwerp van de regeling op basis van een volledig-voertuigmodell. Een volledig voertuig is niet zomaar een samenstelling van vier kwart-voertuigmodellen. Naast het rollen in bochten en duiken tijdens remmen en optrekken is het ook mogelijk om manoeuvres geïntialiseerd door een stuurwieluitslag uit te voeren. De weggingsaspecten zoals die naar voren kwamen bij het kwart-voertuigmodel kunnen nu aangevuld worden met aspecten gerelateerd aan het weggedrag. Vooral de invloed van de dynamische bandkrachtvariaties op het rijgedrag van het voertuig zijn van belang.

Samenvatting

Omdat het volledige voertuigmodel meer vrijheidsgraden bezit dan een kwartvoertuigmodel, is het noodzakelijk de ontwerp-criteria uit te breiden. Het comfort wordt nu niet alleen meer bepaald door verticale versnellingen, maar ook de laterale, longitudinale en rotatieversnellingen van de carrosserie. Daarnaast is het noodzakelijk om objectieve criteria met betrekking tot het rijgedrag te definiëren. Er is gekozen om een dubbele rijbaanwisseling op een oneffen wegdek te gebruiken als experiment op simulatieneiveau om de relatie tussen wegligging en weggedrag te bestuderen. Daarnaast zijn er ook frequentieresponsesies gebruikt. Het comfort en het rijgedrag van een standaard voertuig met passieve wielophanging dient als vergelijkingsmaatstaf voor identieke voertuigen uitgerust met actieve vering.

In overeenstemming met de kwart-voertuiganalyse is er bij het ontwerp van de regeling gebruikt gemaakt van de optimale regeltheorie. Het volledige voertuig biedt echter een grotere ontwerp-flexibiliteit. Naast volledige en gedeeltelijke toestandsterugkoppeling is er ook gekeken naar het effect van de wegdekbeschrijving op de prestaties van de regeling. De berekeningen hebben aangetoond dat de extra informatie die voortkomt uit de correlatie tussen het linker en rechter wegdekspoor geen noemenswaardige prestatieverbetering oplevert. Maar de tijdsvertraging tussen voor- en achterwiel kan wel uitgenut worden om de prestaties te verbeteren. Vooral de dynamische bandkrachtvariaties en werkruimte van de achterwielophanging kunnen aanzienlijk verbeterd worden. Bovendien is er aangetoond dat een regeling op basis van gedeeltelijke toestandsterugkoppeling de prestaties benaderd van een regeling op basis van volledige toestandsterugkoppeling. De gekozen structuur van de terugkoppeling omvat alleen verticale snelheden van de carrosserie en assen. Echter, ook bij het volledige voertuig gaat een verbetering van comfort gepaard met een verslechtering van de wegligging (meer dynamische bandkrachtvariaties)

Na het regelaarontwerp zijn de verschillende voertuigconfiguraties onderworpen aan de dubbele rijbaanwisseling op een oneffen wegdek. Deze simulatie moet aantonen in hoeverre de toename van dynamische bandkrachtvariaties een gevolg heeft voor het weggedrag. De resultaten hebben aangetoond dat matige bandkrachtvariaties geen noemenswaardige verslechtering van het weggedrag tot gevolg hebben. De band die ontlast wordt door de gewichtsverdrijving is het meest gevoelig voor wegedekonoffenfenden en zal dus de meeste variaties in slipkrachten vertonen. Het voertuig met actieve vering gebaseerd op volledige toestandsterugkoppeling vertoont een aanzienlijke verslechtering van het weggedrag. Naast de vrij
onschadelijke toename van de dynamische bandkrachtvariaties, bezit dit voertuig een onacceptabel lage rolstijfheid en een slechte rolstijfheidsverdeling tussen voor- en achteras. De rolproblemen zijn veroorzaakt door de volledige toestandsterugkoppeling omdat hier ook positie gerelateerde toestanden teruggekoppeld worden. Dit soort terugkoppelingen moet dus vermeden worden. Ondanks de toename van de dynamische bandkrachtvariaties blijkt het voertuig met een regeling gebaseerd op gedeeltelijke toestandsterugkoppeling geen slechter weggedrag te bezitten dan de standaard auto met passieve wielophanging. Het comfort is uiteraard wel sterk verbeterd. Situaties die vermeden moet worden zijn situaties waar de gewichtsoverdracht samenvalt met de bandkrachtvariaties door wegdekoneffenheden zodanig dat de gemiddelde spoorkracht te veel gereduceerd wordt. Een adaptatie van de regeling is dus noodzakelijk.

Deze adaptatie is gebaseerd op het aanpassen van de terugkoppelingsterkeringsfactoren. De verhouding tussen de dynamische bandkracht en de gemiddelde bandkracht bepaald of de nadruk gelegd moet worden op comfort of op wegligging. Als deze verhouding groot is dan kunnen de bandkrachtvariaties de wegligging nadelig beïnvloeden zodanig dat het voertuig niet meer optimaal beheersbaar is. In dit geval zal de adaptieve regelaar trachten de bandkrachtvariaties van het betreffende wiel te beperken. Dit gaat gepaard met een kortstondige comfortverslechtering.

De implementatie van deze adaptieve regeling verlangt dat meerdere toestanden van het voertuig bekend zijn. Toestanden zoals de absolute verticale snelheid van de carrosserie en de verticale bandkracht zijn niet of moeilijk meetbaar. In overeenstemming met de analyse van het kwartvoertuigmodel kunnen deze toestanden worden afgeschat met een Kalman filter. De verticale versnellingen van de vier assen en op drie punten op de carrosserie dienen als ingang voor het Kalman filter. Het is niet eenvoudig gebleken om de bijdrage van de zwaartekrachtsversnelling en de dwars- en langsversnelling van het voertuig aan de meetsignalen te elimineren als de carrosserie niet volledig horizontaal ligt. Zonder compensatie zullen de geschatte toestanden een driftverschijnsel vertonen. De drift is opgevangen door een extra integrerende terugkoppeling in het Kalman filter.
Paul Venhovens was born in Roosendaal, the Netherlands, on January 4, 1967. After graduating in May 1985 from the pre-university education (Atheneum) at the Norbertuscollege in Roosendaal, he studied Mechanical Engineering at the Delft University of Technology, from which he graduated in December 1989 on the evaluation of the performance of a two DOF vehicle model using optimal control in combination with state estimators. In January 1990 he became a doctorate student at the Vehicle Research Laboratory where he has been working towards his PhD thesis. From December 1993 he will be engaged as a Postdoctoral Researcher at the Transportation Research Institute of the University of Michigan (UMTRI) in Ann Arbor, Michigan, U.S.A.
The book is finished,
the work is done.

Address of the author:
The University of Michigan
Transportation Research Institute
2901 Baxter Road, Ann Arbor, Michigan 48109-2150
U.S.A.