CHAPTER 9

ON THE PROBABILITY DISTRIBUTION OF WAVE FORCE AND AN INTRODUCTION TO THE CORRELATION DRAG COEFFICIENT AND THE CORRELATION INERTIAL COEFFICIENT

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INTRODUCTION

In many design problems the sea state is specified as part of the design criteria. For offshore structures it is important to know the maximum probable wave height and the maximum probable wave force which might be experienced during the life expectancy of the structure, for example the 50 or 100 year wave height and period and the 50 or 100 year wave force. Based on previous analysis of wave and wave force measurements, practically no correlation was found between the measured "apparent wave height" and "apparent wave force." Several authors have proposed statistical distribution of drag and mass coefficients, with recommendations for use in design.

The statistical distribution of the calculated drag (or mass) coefficients has no relationship to the statistical distribution of the measured wave drag forces (or inertial forces). Borgman (1964) has considered the statistical distribution of wave forces.

This paper presents methods for predicting the probability distributions of peak wave drag and inertial forces, and a method is proposed which might be used to predict the most probable maximum wave force once the sea state is specified. It is found that there is a good correlation between the probability distributions of wave heights and the probability distributions of peak drag forces. For example, if the most probable maximum wave height is given from a statistical distribution of wave height, then it is possible to predict the most probable maximum drag (or inertial) force, and the most probable maximum force (combined drag and inertial force).

PROBABILITY DISTRIBUTION OF WAVE HEIGHTS AND THE MOST PROBABLE MAXIMUM WAVE HEIGHT

THE RAYLEIGH DISTRIBUTION

It was first shown by Longuet-Higgins (1952) that if the sea state is Gaussian and has a narrow spectrum, then the probability distribution
of wave heights follows the Rayleigh distribution, hence

\[ p(H) \, dH = 2 \frac{H}{H_r^2} e^{-\left(\frac{H}{H_r}\right)^2} \, dH \]  \hspace{1cm} (1)

where

\[ p(H) \, dH \] is probability density of \( H \)
\( H = H_i \) is individual wave height of the distribution
\( H_r \) is the root mean square wave height; i.e.

\[ H_r^2 = \frac{1}{N} \sum H_i^2 \]  \hspace{1cm} (2)

The arithmetic mean \( \bar{H} = \frac{1}{N} \sum H_i \) is related as follows:

\[ \bar{H} = \sqrt{\frac{\pi}{4} H_r} = 0.625 H_s \]  \hspace{1cm} (3)

where \( H_s \) is the significant wave height.

The integral of Eq. (1) leads to the cumulative probability distribution, whence

\[ P(H) = 1 - e^{-\left(\frac{H}{H_r}\right)^2} \]  \hspace{1cm} (4)

Rearranging the terms in Eq. (4) and taking the natural logarithm \( \ln \) of both sides one obtains

\[ \frac{H}{H_r} = \sqrt{\ln \frac{1}{1 - P(H)}} = \sqrt{\ln \frac{N}{N - N \, P(H)}} \]  \hspace{1cm} (5)

where \( \ln \) signifies natural logarithm, and \( N \) the total number of waves in a record.

Now the highest wave in a record of \( N \) waves, say at least 100 waves, should correspond to

\[ N - N \, p(H) = 1 \text{ or for } P(H) = \frac{N - 1}{N} \approx \frac{N}{N + 1} \]

Now a most probable maximum wave height \( H_{\text{max}} \) can be defined accordingly:

\[ H_{\text{max}} = H_r \sqrt{\ln N} \]  \hspace{1cm} (6)

The above equation also follows from the work by Longuet-Higgins (1952) for large values of \( N \).
The significant wave period $T_s$ is approximately equal to the mean wave period

$$T_s \approx \bar{T} = \frac{1}{N} \sum T_i$$  \hspace{1cm} (7)

The total number of waves $N$ can be estimated approximately from

$$N \approx \frac{t}{T_s}$$  \hspace{1cm} (8)

where $t$ is the length of record in seconds and $T_s$ is the significant wave period in seconds. In general $t$ is limited by the duration of the storm for steady state condition, when $H_s$ and $T_s$ remain relatively constant.

**THE WEIBULL DISTRIBUTION**

When the sea state deviates from Gaussian, then the Rayleigh distribution also deviates, but not so much as might be expected. However, if greater accuracy is required, one can obtain improvement by use of the Weibull distribution. Weibull (1951) presented a distribution function which has been used successfully for other engineering problems. The Weibull cumulative probability distribution, applied to wave height, for example, can be written as follows:

$$P(H) = 1 - e^{-A \left( \frac{H}{H_r} \right)^m}$$  \hspace{1cm} (9)

where $A$ and $m$ are constants. From Eq. (9) the probability density distribution becomes

$$p(H) dH = A m \left( \frac{H}{H_r} \right)^{m-1} e^{-A \left( \frac{H}{H_r} \right)^m} dH$$  \hspace{1cm} (10)

Although the Weibull distribution has no theoretical foundation, it has the advantage of being easily manipulated for use with empirical data.

If one takes the logarithm of Eq. (9) twice, rearranging terms one obtains

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* One could also make use of the Gamma type distributions, similar to the work on wave height and wave period distributions originally proposed by Putz (1952) before Longuet-Higgins developed the Rayleigh distribution for wave heights.
Figure 1. Cumulative Distribution of Wave Height and Drag Force (Weibull Plot)
\[ \ln \ln \frac{1}{1 - P(H)} = \ln A + m \ln \frac{H}{H_r} \]  

Eq. (11) is that of a straight line, \( \ln A \) being the intercept and \( m \) the slope. Thus, empirically one can obtain \( A \) and \( m \) to determine the best fit expression for Eq. (11). It might be noted for \( A = 1 \), \( \ln A = 0 \) and \( m = 2 \), one obtains the Rayleigh distributions given by Eqs. (1) or (4).

Special graph paper can be constructed for the Weibull distribution function so that one can plot directly \( H \) vs. \( P(H) \) and eliminate the need for taking the corresponding logarithms. Figure 1 is a graph for the Weibull probability distribution data. The data in Figure 1 are for wave heights and drag forces from a record by Wiegel (1954). If these data followed the Rayleigh distribution, then the slope of the line for wave height would be \( m = 2 \) and drag force would be \( m = 1 \). The reason that the slopes \( m \) deviate considerably from theory in Figure 1 is the fact that many of the smaller waves were not entered in the tabulation. However, the Weibull distribution appears to be a good approximation for a record which is trimmed at the low end.

The most probable maximum wave for the Weibull distribution becomes

\[ H_{\text{max}} = H_r \frac{m}{A} \ln N \]  

**PROBABILITY DISTRIBUTION OF PEAK DRAG FORCES AND PEAK INERTIAL FORCES**

**LINEAR THEORY**

According to Morison, et. al. (1950), the equation for wave force is given by

\[ f(t) = f_D(t) + f_I(t) \]  

where

\[ f_D(t) = \frac{1}{2} \rho C_D D u |u| \]  

is the drag force

\[ f_I(t) \]  

\[ \text{Eq. (13)* is general and applicable also for higher order wave theory, provided that there is no secondary influence caused by the pile being immersed in the wave, or vorticity as might be caused by viscosity or wind-air-water interface effects.} \]
According to linear theory, the horizontal particle velocity is given by
\[ u(t) = \frac{\pi H}{T} \frac{\cosh k (d + z)}{\sinh k d} \cos(kx - \sigma t) \] (16)
and the horizontal particle acceleration by
\[ \ddot{u}(t) = \frac{2 \pi^2 H}{T^2} \frac{\cosh k (d + z)}{\sinh k d} \sin(kx - \sigma t) \] (17)

where
- \( k = \frac{2 \pi}{L} \) = wave number
- \( \sigma = \frac{2 \pi}{T} \) = wave frequency
- \( d \) = water depth
- \( z \) = distance measured negative downward from the undisturbed mean water surface
- \( \theta = (kx - \sigma t) \)

The maximum drag force occurs at \( \theta = 0 \), whence \( \cos \theta = 1 \), \( \sin \theta = 0 \); the maximum inertial force occurs at \( \theta = \pi / 2 \), whence \( \cos \theta = 0 \), \( \sin \theta = 1 \).

It will be convenient to write Eqs. (14) and (15) respectively for maximum or peak values as follows:
\[ f_D = K_1 H^2 \] at \( \theta = 0 \) (18)
\[ f_i = K_2 H \] at \( \theta = \pi / 2 \) (19)
where, from Eqs. (15) and (16)

\[ K_1 = \frac{1}{2} \rho C_D D \left( \frac{\pi}{T} \frac{\cosh k (d + z)}{\sinh k d} \right)^2 \]  
and
\[ K_2 = \frac{\pi}{4} \rho C_m D^2 \frac{2 \pi^2}{T^2} \frac{\cosh k (d + z)}{\sinh k d} \]  

at \( \theta = 0 \) (20)

and
\[ K_2 = \frac{\pi}{4} \rho C_m D^2 \frac{2 \pi^2}{T^2} \frac{\cosh k (d + z)}{\sinh k d} \]  

at \( \theta = \pi/2 \) (21)

Knowing the probability distribution of wave heights and assuming for the time being that \( C_D, C_m, D, T \) and \( d \) are constant, one can obtain the probability distributions for \( f_D \) and \( f_i \) respectively as follows:

\[ p (f_D) \, df_D = p (H) \, dH \] (22)

and
\[ p (f_i) \, df_i = p (H) \, dH \] (23)

From Eqs. (18) and (19), respectively, one obtains by differentiation

\[ df_D = 2 K_1 H \, dH \] (24)

and
\[ df_i = K_2 \, dH \] (25)

It then follows from Eqs. (1), (18), (19), (22), (23), (24) and (25) that the probability distributions for peak drag forces and peak inertial forces respectively are as follows:

\[ p (f_D) \, df_D = \frac{f_D}{K_1 H_r^2} e^{-\frac{f_D}{K_1 H_r^2}} \, df_D \] (26)

and
\[ p (f_i) \, df_i = \frac{2 f_i}{K_2 H_r} e^{-\left(\frac{f_i}{K_2 H_r}\right)^2} \, df_i \] (27)

* Assuming \( T \) is constant infers that there is no correlation between \( H \) and \( T \), and \( T \) is the same at all levels of \( H \) in the joint probability distribution between \( H \) and \( T \). If \( C_D \) and \( C_m \) are not constant, then a better form of the probability distribution would be \( p (f_D/C_D) \) and \( p (f_i/C_m) \) where \( C_D \) and \( C_m \) are functions of the wave parameters and pile size.
Now let
\[ f_D = f_Da = K_1 H^2 = K_1 H_r^2 \]  
(28)

and
\[ f_{ir} = f_{ir} = K_2^2 H^2 = K_2^2 H_r^2 \]  
(29)

thus
\[
p (f_D) \, df_D = \frac{1}{f_Da} \, e^{-\frac{f_D}{f_Da}} \, df_D
\]  
(30)

and
\[
p (f_i) \, df_i = \frac{2 f_i}{f_{ir}} \, e^{-\left(\frac{f_i}{f_{ir}}\right)^2} \, df_i
\]  
(31)

The cumulative distributions for Eqs. (30) and (31) are given respectively by
\[
P (f_D) = 1 - e^{-\frac{f_D}{f_Da}}
\]  
(32)

and
\[
P (f_i) = 1 - e^{-\left(\frac{f_i}{f_{ir}}\right)^2}
\]  
(33)

If \( f_D \) is proportional to \( H^2 \) and \( f_i \) to \( H \), according to linear theory, then the distribution of \( (f_D/f_Da)^{1/2}, (f_i/f_{ir}) \) and \( (H/H_r) \) should coincide; i.e., all three relations follow the nondimensional form of the Rayleigh distribution.

From the above equation one obtains the following for the most probable maximum drag and maximum inertial forces, respectively:
\[ f_{D(max)} = f_Da \ln N \]  
(34)

and
\[ f_{i(max)} = f_{ir} \sqrt{\ln N} \]  
(35)
WEIBULL DISTRIBUTION

For the Weibull distribution one can write the following equations for drag force and inertial force respectively:

\[ P(f_D) = 1 - e^{-A_1 \left( \frac{f_D}{f_{Da}} \right)^{m_1}} \]  
\[ P(f_i) = 1 - e^{-A_2 \left( \frac{f_i}{f_{ir}} \right)^{m_2}} \]

where \( A_1, A_2, m_1 \) and \( m_2 \) are empirical constants.

The drag and inertial forces can still be obtained by use of Morison's equations, but \( u \) and \( \dot{u} \) should be obtained by use of higher order wave theory.

The most probable maximum drag and inertial forces for the Weibull distribution are obtained respectively as follows:

\[ f_D^{\text{max}} = f_{Da} \left( \frac{1}{A_1} \right)^{1/m_1} \text{f n N} \]  
\[ f_i^{\text{max}} = f_{ir} \left( \frac{1}{A_2} \right)^{1/m_2} \text{f n N} \]

PROBABILITY DISTRIBUTION OF PEAK FORCES (COMBINED DRAG AND INERTIAL FORCES)

LINEAR WAVE THEORY

Returning now to Eqs. (13) thru (17) and writing the wave force equation as

\[ f = K_1 H^2 \cos \theta \mid \cos \theta \mid + K_2 H \sin \theta \]  
for the phase position \( \frac{\pi}{2} \leq \theta < \frac{\pi}{2} \), one may obtain the maximum positive force by setting \( df/d\theta = 0 \) at \( \theta = \beta \), whence

\[ \beta = \arcsin \left( \frac{K_2}{2 K_1 H} \right) \]
Figure 2. Relationships for Peak Force in Terms of Peak Drag and Peak Inertial Forces
for
\[-\frac{\pi}{2} < \beta < \frac{\pi}{2}\]
but since the maximum force will always occur at or ahead of the crest, one can write
\[0 < \beta < \frac{\pi}{2}\]
A similar equation can be derived for the maximum negative force.

Using the identity \(\cos^2 \theta + \sin^2 \theta = 1\), one obtains the expression for maximum force as follows:
\[f_m = K_1 H^2 + \frac{1}{4} \frac{K_2^2}{K_1}\] (42)

for
\[0 < \beta < \frac{\pi}{2}\]

Using Eqs. (18) and (19), Eq. (42) becomes
\[f_m = f_D \left[ 1 + \frac{1}{4} \left( \frac{f_1}{f_D} \right)^2 \right] \text{ for } 0 < \frac{f_1}{f_D} < 2\]
otherwise
\[f_m = f_1 \text{ for } \frac{f_1}{f_D} \geq 2\] (43)

In Figure 2 the curve for Airy theory, reproduced from Reid and Bretschneider (1953), is based on
\[F_m = F_{Dm} \left[ 1 + \frac{1}{4} \left( \frac{F_{im}}{F_{Dm}} \right)^2 \right] \text{ for } 0 < \frac{F_{im}}{F_{Dm}} < 2\]
otherwise
\[F_m = F_{im} \text{ for } \frac{F_{im}}{F_{Dm}} \geq 2\] (44)
Once $f_D$ and $f_i$ have been determined for any particular level of probability, then $f_m$ can be determined by use of Eq. (43). For example,

$$f_{ma} = f_{Da} \left[ 1 + \frac{1}{4} \left( \frac{f_{ir}}{f_{Da}} \right)^2 \right] \text{ for } 0 \leq \frac{f_{ir}}{f_{Da}} \leq 2$$

otherwise

$$f_{ma} = f_{ir} \text{ for } \frac{f_{ir}}{f_{Da}} \geq 2 \quad (46)$$

The most probable maximum (peak) wave force becomes

$$f_{m(max)} = f_{D(max)} \left[ 1 + \frac{1}{4} \left( \frac{f_{i(max)}}{f_{D(max)}} \right)^2 \right] \text{ for } 0 \leq \frac{f_{i(max)}}{f_{D(max)}} \leq 2$$

otherwise

$$f_{m(max)} = f_{i(max)} \text{ for } \frac{f_{i(max)}}{f_{D(max)}} \geq 2 \quad (47)$$

Eq. (43) can also be written as follows:

$$\frac{f_m}{1/2 \rho D u^2} = C_D + \frac{C_m}{C_D} \left( \frac{\pi}{4} \right)^2 \left( \frac{D u}{u^2} \right)^2 \text{ for}$$

$$0 \leq \frac{\pi}{2} \frac{C_m}{C_D} \frac{D u}{u^2} \leq 2$$

otherwise

$$\frac{f_m}{1/2 \rho D u^2} = C_m \frac{\pi}{2} \frac{D u}{u^2} \text{ for}$$

$$\frac{\pi}{2} \frac{C_m}{C_D} \frac{D u}{u^2} \geq 2 \quad (48)$$
The use of Eq. (48) for data analysis was illustrated in an earlier report by Bretschneider (1957). The first form of Eq. (48) is that of a straight line $Y = b + mx$, where $b = C_D$, the intercept, and $C_m^2/C_D$ is the slope of the line. Thus, for a set of the proper data the first form of Eq. (48) can be used to solve simultaneously for $C_D$ and $C_m$.

The second form of Eq. (48) is also that of a straight line but passes through the origin. This does not mean that $C_D$ is zero, but that the drag force is zero and $C_D$ vanishes because all the force is inertial force. This would be observed on the wave record when the maximum force occurs at a phase position $\pi/2$ ahead of the wave crest. In using Eq. (48) one should always check to see which of the two forms is applicable. Except for large piles and relatively low wave heights, the first form of Eq. (48) will generally apply.

Also of interest in Eq. (48) is the parameter $\frac{D \ddot{u}}{u^2}$, which is a form of the Froude number, where the acceleration $g$ due to gravity is replaced with the particle acceleration $\ddot{u}$. Iverson and Balent (1951) first used the above parameter, and since then it has been called the Iverson number.

PROBABILITY DISTRIBUTION OF PEAK FORCES

The probability distribution of peak forces (combined drag and inertial) can be obtained in a simple manner as that used for drag and inertial forces; i.e.:

$$P (f_m) \, df_m = P (H) \, dH$$

(49)

where $f_m$ was previously given by Eq. (42). Using Eq. (42) one obtains the proper relationship for $df_m = 2 K_1 H \, dH$, and together with the Rayleigh distribution for wave height, one obtains the probability density for peak force. It then follows that the cumulative probability distribution becomes:

$$P (f_m) = 1 - e^{\left[ \frac{f_m}{f_{ir}} \right]^2 / \left( \frac{f_{ir}}{f_D} \right)^2}$$

for $0 < \frac{f_{ir}}{f_D} < 2$

otherwise

(50)
The most probable maximum wave force, based on Eq. (50), will be given by Eq. (47) previously derived.

WEIBULL DISTRIBUTION

The probability distribution of maximum forces for the Weibull distribution should be determined by use of empirical data. If there exists a change from pure inertial force to predominantly drag force for a particular record, then one should observe this fact by a change in the slope of the data on the Weibull plot.

INTERPRETATION OF CORRELATION DRAG AND CORRELATION INERTIAL COEFFICIENTS

In the previous development on probability distributions of wave forces, it is assumed that $C_D$ and $C_m$ were constants. Previous analysis by Wiegel, et. al. (1957) shows a vast scatter of the values of $C_D$ and $C_m$.

In order to correlate empirical data on the probability distribution of drag forces (and inertial forces) with the probability distribution of wave heights, it will be convenient to introduce the terms "correlation drag coefficient" $C_{Dr}$ and "correlation inertial coefficient" $C_{mr}$ which are determined respectively from the following:

\[
C_{Dr} = \frac{f_{Da}}{\frac{1}{2} \rho D u_r^2} = \frac{f_{Da}}{\frac{P}{1} H D^2}
\]  
\[\text{and}\]
\[
C_{mr} = \frac{4 f_{ir}}{\pi \rho D^2 u_r} = \frac{f_{ir}}{\frac{P}{2} D^2}
\]

where

$f_{Da}$ is average of maximum drag forces;
$u_r$ is root mean square average of particle velocity
(where $f_D$ is max, $\theta = 0$);
is root mean square average of maximum inertial forces; and

\( u_r \) is root mean square average of particle acceleration

(where \( f_i \) is max, \( 0 \leq \frac{\pi}{2} \), or the phase position where
the surface profile intersects mean water elevation).

\( P_1 \) and \( P_2 \) are given as functions of \( H/d \), \( d/T^2 \) and \( S/d \) in

tables by Skjelbreia, et. al. (1961) for the fifth order Stokes wave theory. Thus, \( C_{Dr} \) and \( C_{mr} \) can apply to conditions of high order wave theory.

In order to evaluate \( C_{Dr} \) and \( C_{mr} \) from measured data, it is

necessary to have simultaneous recordings of wave elevation and wave
force traces, say 15 to 20 minutes duration. From the wave elevation
trace one reads off the individual wave height, such as according to the
zero up-crossing method proposed by Pierson (1955), whence

\[ H_r^2 = \frac{1}{N} \sum H_i^2 \]  \hspace{1cm} (53)

where \( N \) is the total number of waves and \( H_i \) is the individual wave
height. From the wave force trace, at crest position, one reads the
individual peak drag forces and calculates

\[ f_{Da} = \frac{1}{N} \sum f_{Di} \]  \hspace{1cm} (54)

where \( f_{Di} \) is the individual peak drag force.

Also from the wave force trace, at phase position where the surface
profile and still water intersect, one reads the individual peak inertial
forces and calculates

\[ f_{ir}^2 = \frac{1}{N} \sum f_{ii}^2 \]  \hspace{1cm} (55)

where \( f_{ii} \) is the individual peak inertial force.

Now from the wave height (or wave force) trace one calculates the
mean wave period

\[ T = \frac{t}{N} \]  \hspace{1cm} (56)

where \( t \) is the length of record in seconds and \( T \) is the mean wave period
in seconds.

Using \( H_r \), \( T \), \( d \) (depth of water) and \( S \) (depth of submergence)
for wave force determinations, one can calculate \( u_r \) and \( u^{-r} \). This can
be done by use of the appropriate wave theory. It then follows that \( C_{Dr} \) and \( C_{mr} \) can be calculated by use of Eqs. (51) and (52) respectively.

A check on the adequacy of the "correlation drag coefficient" can be made by applying the calculated coefficient \( C_{Dr} \) to the distribution of wave heights to determine the corresponding distribution of drag forces and compare the results with the distribution of measured drag forces.

An alternate approach to the check can be taken by considering simultaneously the probability distributions of wave height and of drag force, and calculate an apparent drag coefficient based on corresponding drag force and wave height at each level of percent probability. The constancy of \( C_D \) at all levels of percent probability (or minimum deviations from \( C_{Dr} \)) is an index of the degree of correlation. A check on the adequacy of the "correlation inertial coefficient" \( C_{mr} \) can be made in a similar manner as described above.

If the "correlation drag coefficient" and the "correlation inertial coefficient" are found to be relatively constant (or else some function of the variables \( H, T, d, S \) and \( D \)), then one can calculate \( f_{Da} \) and \( f_{ir} \) as follows:

\[
f_{Da} = \frac{1}{2} \rho C_{Dr} D u_r^2
gives (57)
\]

and

\[
f_{ir} = \frac{4}{\pi} \rho C_{mr} D^2 u_r
\]
gives (58)

It then follows that the probability distributions of the peak forces (drag, inertial, and combined drag and inertial) can be calculated from the corresponding equations given earlier.

ANALYSIS OF SOME DATA PRESENTLY AVAILABLE

There are several sources of data which might be analyzed for demonstration purposes. References to these data are summarized in a report by Bretschneider, Collins and Pick (1965). Evidently there are two sources of data which would be quite useful, namely Wiegel, et. al. (1954 and 1957) and Skjelbreia, et. al. (1960), the latter of which is not available at present. Therefore, the following demonstration is limited to the data by Wiegel (1954 and 1957), but it would be of great advantage to apply the same demonstration, for the sake of the "state of the art," also to the data used by Skjelbreia, et. al. (1960).
Figure 3. Measured Wave Height vs. Measured Wave Drag Force (Joint Probability Distribution of Wave Height and Drag Force)
The main text of this paper mentions three types of probability distribution: Rayleigh, Weibull, and the Gamma type. The Rayleigh distribution is theoretical and the other two are empirical, which require a statistical analysis of empirical data. In some respects the empirical distributions tend to account for the nonlinear aspects, which are not allowed for by the Rayleigh distribution.

Figure 1 is a typical example of a wave height and drag force distribution, plotted on a Weibull distribution graph. It is seen from Figure 1 that the relationship for the Weibull distribution is a straight line with straight slope $m$. For the Rayleigh distribution the slope should be $m = 2$ for wave height and $m = 1$ for drag force, based on linear wave theory. Obviously the corresponding slopes were not verified, first because the data are incomplete for low wave heights, and second, the waves and forces are not described by linear wave theory. Nevertheless, there exists a good empirical relationship for calculating the correlation drag coefficients, provided high order wave theory is used to calculate the particle velocities.

For purposes of demonstrating the correlation drag coefficient and the correlation inertial coefficient, reference is made to the data by Wiegel, et. al. (1954 and 1957). The wave height data were grouped in one-foot increments. Tables 1, 2 and 3 correspond to the original data by Wiegel, et. al. given in his Tables I, II and III.

Figure 3 shows a scatter diagram or a joint distribution of $H$ and $f_D$ of the original data, point by point. As can be seen, there is tremendous scatter, but a trend of increasing $f_D$ with increasing $H$. The corresponding marginal distribution for $H$ and $f_D$ are obtained respectively by summation independent of each other. Figure 4 shows cumulative marginal distributions of drag force and wave height plotted on log normal graph paper. The fifth order gravity wave theory was used to calculate particle velocities.

The root mean square average, or the overall correlation drag coefficient for Figure 3, is approximately $C_{Dr} = 0.75$. The drag coefficient for various wave height and wave force probabilities ranges between 0.4 and 0.8 for large and small wave heights, respectively. The lower value of $C_{Dr}$ might be associated with a high Reynolds number effect. If $C_{Dr} = 0.75$ were applied to all wave heights, most of the distribution of forces would be verified, except that the maximum values would be too high by a factor of 1.5 to 2.0.
Figure 4. Cumulative Distribution of Wave Height and Drag Force
Figure 5. Ranked Wave Height vs. Ranked Drag Force
(Correlation between Marginal Distribution of Wave Height and Marginal Distribution of Drag Force)
Figure 6. Measured Wave Height vs. Measured Drag Force (Joint Probability Distribution of Wave Height and Drag Force)
Figure 7. Cumulative Distribution of Wave Height and Drag Force
If, for example, \( f_D \) and \( H \) were correlated on a plot for various values of \( p = 0.5, 1, 5, 10, 20, 50, \) etc., one would obtain a continuous smooth curve. Immediately it becomes obvious that the original data, when ranked, should also follow a similar relationship. That is, if the wave heights are ranked in one column and the wave forces ranked in another column, then the corresponding ranked data should follow a smooth curve with minimum deviation. The original data of Wiegel (1957) (Table I) have been ranked according to wave height and according to drag force and the corresponding values have been plotted in Figure 5. Figure 3, of course, represents the data as a joint probability distribution between wave force and wave height, but with practically no correlation. This is the same as plotting or correlating the cumulative marginal distribution of wave drag force versus the cumulative marginal distribution of wave height.

Figure 5 shows that there is a definite correlation between the marginal distribution of wave height and the marginal distribution of wave force. In Figure 5 the deviations from a straight line are in part associated with the deviation of \( C_D \) from \( C_{Dr} = 0.75 \) given in Figure 4.* Figure 5 shows a tremendous improvement in correlation from that given by the same data in Figure 4. This is as expected from any "gun shot" block of data.

Figure 6 shows a scatter diagram or a joint distribution between \( H \) and \( f_D \) obtained from Table 2 by Wiegel, et. al. There again appears to be absolutely no correlation between measured apparent \( f_D \) and the corresponding measured apparent \( H \). The joint distribution between \( H \) and \( f_D \) is practically zero.

Figure 7 shows the corresponding marginal cumulative distribution for \( f_D \) and \( H \). The correlation drag coefficient was calculated to be \( C_{Dr} = 0.57 \). Drag coefficients \( C_D \) were calculated using the fifth order gravity wave theory for all levels of probability and for this record was found to be very nearly a constant at \( C_D = 0.57 \). Thus there is perfect correlation between the marginal probability distribution of peak drag forces and the marginal probability distribution of wave heights.

* It should be noted that \( C_D \) in Figure 4 does not represent a statistical distribution of \( C_D \), but rather values of \( C_D \) for various levels of wave force probability. The same interpretation is given for \( C_D \) for the other figures of the report.
Figure 8. Ranked Wave Height vs. Ranked Drag Force  
(Correlation between Marginal Distribution of Wave Height and Marginal Distribution of Drag Force)
Figure 9. Cumulative Distribution of Wave Height and Drag Force

DATA FROM WIEGEL (1954)
RECORD I-V
N = 3 - 294
D = 24"
S = 33'
d = 46'
T = 13.6

PERCENT OF TIME WAVE HEIGHT (AND DRAG FORCE)
IS EQUAL TO OR GREATER THAN INDICATED
Figure 10. Ranked Wave Height and Ranked Drag Force
(Correlation between Marginal Distribution of Wave Height and Marginal Distribution of Drag Force)

DATA FROM WIEGEL (1954)
RECORD N
N = 136 - 334
D = 12 INCHES
Figure 8 gives a plot of ranked $H$ and ranked $f_D$ or a correlation between the corresponding marginal distributions for the same data of Figure 6. It shows a very good degree of correlation between two marginal probability distributions.

Figure 9 shows cumulative probability distributions for $f_D$ and $H$ based on data originally given in an earlier report by Wiegel (1954). Again there is an excellent correlation in the constancy of drag coefficient for all levels of probabilities, average value being about $C_{D^r} = 0.59$.

Figure 10 shows a plot of ranked $f_D$ versus ranked $H$ or a correlation between the corresponding marginal distribution presented on linear paper for another wave record. For low wave heights it appears that $f \sim H$, mostly inertial force; intermediate heights $f \sim H^2$, mostly drag forces; and for large wave heights $f \sim H^2$, reflecting a high Reynolds number effect.

Figure 11 is a summary of drag coefficients based on Figures 4, 7 and 9. Two curves are essentially the same for a pile diameter $D = 2.0$ feet. The other curve shows a possibility on the effect of pile diameter, but this is not conclusive. Additional data for $D = 1$ foot from the original report by Wiegel, et al. (1954) have been analyzed in a similar manner. The results of these analyses are shown in Figure 12. The average drag coefficient is $C_{D^r} = 0.7$, but is not conclusive. For the higher waves $C_D$ becomes 0.5 or less, and is in better agreement with the results for $D = 2.0$ feet. Considering all data, $C_{D^r}$ is about $0.65 \pm 20\%$ and $C_D$ at the one percent level is about $0.5 \pm 20\%$.

Data from Wiegel (1957) (Table 3) were used to determine cumulative distributions for wave height and inertial forces. These results are shown in Figure 13. The correlation inertial coefficient for this record was determined to be $C_{mr} = 3.1$, using linear wave theory to calculate particle acceleration. The variation of $C_m$ from $C_{mr}$ for various levels of probability is not satisfactory. Some of this variation, perhaps, is because the linear wave theory might not be satisfactory for calculating particle accelerations. The value of $C_{mr} = 3.1$ is higher than the theoretical value of 2.0. Wiegel calculated an average value of $C_m = 2.5$ with a standard deviation of 1.2, based on many additional data not given in Table 3, and also noted that this value of 2.5 was higher than given by theory. Using some of Wiegel's data, Borgman (1964) calculated a $C_m = 5.1$, and also noted that this value was considerably higher than given by theory, and that "high values of $C_m$ though seem to characterize the
Figure 11. Summary of Drag Coefficients vs. Percent Level, from Figures 4, 7 and 9
Figure 12. Summary of Drag Coefficients vs. Percent Level.
Figure 13. Cumulative Distribution of Wave Height and Inertial Force
Davenport data, particularly for the larger waves. One other possible explanation is that the measured values of inertial force include large second order contributions of drag force. The drag force is proportional to the square of the wave height, and this would help to explain the increase in calculated $C_m$ with increase in wave height. The data are not in suitable form to determine the effect of drag force, since the phase positions are not given.

CONCLUSIONS

1. Although there may be no correlation between individual apparent wave force and apparent wave height, there can be a very good degree of correlation between the corresponding marginal probability distributions.

2. Statistical distributions of calculated drag and inertial coefficients from measured "apparent" wave heights and the corresponding "apparent" wave forces as determined in the past are almost meaningless to the design engineer.

3. Average values of $C_D$ and $C_m$ as determined in the past (weighted toward many lower wave heights) when used for the design wave (maximum waves) should be expected to predict maximum probable forces on the conservative side of safety.

4. Statistical distributions of wave forces correlated with statistical distributions of wave height can have a lot of meaning and value to the design engineer. The procedures used in this report should be investigated in greater detail using new data.

5. The correlation coefficient of drag and the correlation coefficient of inertia should be investigated in greater detail using new and better data than is presently available. The results of such an analysis should permit the design engineer to select better values of $C_D$ and $C_m$ for the design wave.

ACKNOWLEDGEMENTS

The work presented here is part of a larger research program on ocean waves, supported by the U. S. Navy Bureau of Yards and Docks under contract NBy-45815.

REFERENCES


TABLE 1
WAVE HEIGHT AND DRAG FORCE DISTRIBUTIONS
(Data from Wiegel, et.al., 1957)

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\( d = 48 \text{ ft.} \)
\( S = 42.5 \text{ ft.} \)
\( D = 1.0 \text{ ft.} \)
\( H_r = 9.5 \text{ ft.} \)
\( f_{Da} = 18 \text{ lbs.} \)
\( T = 14.5 \text{ sec.} \)
\( C_{Dr} = 0.75 \)

NOTE: All calculations made by slide rule
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d = 47 ft.
S = 33 ft.
D = 2.0 ft.
H_r = 9.1 ft.
f_Da = 20.3 lbs.
T = 13.4 sec.
C_{Dr} = 0.57
### TABLE 3

**WAVE HEIGHT AND INERTIAL FORCE DISTRIBUTIONS**
(Data from Wiegel, et. al., 1957)

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$\text{d} = 47 \text{ ft.}$

$\text{S} = 33 \text{ ft.}$

$\text{D} = 24 \text{ in.}$

$H_r = 9.1 \text{ ft.}$

$f_{ir} = 33.0 \text{ lbs.}$

$T = 13.3 \text{ sec.}$

$C_{mr} = 3.1$