A note on the ratio of the extreme to the root of the sum of squares in sequences of absolute values of Gaussian variables

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ABSTRACT: Let $Z_1, \dots, Z_n$ be i.i.d. standard normal variables, M_n the extreme among their absolute values, and Q_n their rooted sum of squares. Here we study the ratio $v_n = M_n/Q_n$. We show that $\mu(v_n) = \mu(M_n)/\mu(Q_n) \sim b_n/\sqrt{n} \sim \sqrt{2\log(n)/n}$, where $c_n \sim d_n$ denotes that $c_n/d_n \rightarrow 1$ for $n \rightarrow \infty$. Moreover, convergence happens in a stable manner, i.e. $v_n \sim \mu(v_n)$ in probability. For low dimensions we also derive explicit expressions for $\mu(v_n)$ and variance $\sigma^2(v_n)$. These expressions are calculated by using an appealing geometrical interpretation of v_n .

1 INTRODUCTION

Let $\{Z_n\}$ be i.i.d. standard normal variables and define the entities

$$M_n = \max_{i=1, \dots, n} (|Z_i|), \quad S_n = \sum_{i=1, \dots, n} |Z_i|, \quad Q_n = \sqrt{\sum_{i=1, \dots, n} Z_i^2}$$

$$\text{and } v_n = \frac{M_n}{Q_n}.$$

For a random variable X , denote with $\mu(X)$ its mean and with $\sigma^2(X)$ its variance. If $\{p_n\}$ and $\{q_n\}$ are sequences, then $q_n = o(p_n)$ means that $q_n/p_n \rightarrow 0$ ($n \rightarrow \infty$). For notational convenience, when it is clear from the context, we don't mention explicitly that $n \rightarrow \infty$ when we discuss asymptotic behavior. Also, we write $p_n \sim q_n$ as a shorthand for $p_n/q_n \rightarrow 1$ or, equivalently, $p_n = q_n(1 + o(1))$. Note that the tilde-operator defines an equivalence relation. Note that, if $p_n \sim q_n$ and $c_n \sim d_n$, then $p_n/c_n \sim q_n/d_n$.

As $Q_n \sim \sqrt{n}$ in probability (note that for all $\varepsilon > 0$ Chebychev's inequality implies

$$\begin{aligned} \Pr(|Q_n/\sqrt{n} - 1| < \varepsilon) \\ &= \Pr(|Q_n^2/n - 1|/(Q_n/\sqrt{n} + 1)) < \varepsilon \\ &\geq \Pr(|Q_n^2/n - 1| < \varepsilon) \\ &\geq \sigma(|Z_1|^2)/\varepsilon\sqrt{n} \rightarrow 0 \end{aligned}$$

and M_n is attracted by a stable law (meaning that there are sequences $\{a_n\}$ and $\{b_n\}$ such that $(M_n - b_n)/a_n$ converges in distribution to one of

the extreme value distributions), it is tempting to believe that v_n behaves asymptotically like $b_n/\sqrt{n}$. In this note we show that this intuition is indeed right and that in fact $\mu(v_n) = \mu(M_n)/\mu(Q_n) \sim b_n/\sqrt{n} \sim \sqrt{2\log(n)/n}$. Moreover, convergence of happens in a stable manner, i.e. $v_n \sim \mu(v_n)$ in probability.

A noteworthy result, that we use explicitly in this note, is that by Downey & Wright (2007). They show that for non-negative i.i.d. random variables X_i with finite second moment

$$\mu\left(\frac{\max(X_i)}{\sum X_i}\right) \sim \frac{\mu(\max(X_i))}{\mu(\sum X_i)}.$$

In particular this implies that $\mu(M_n/S_n) \sim \mu(M_n)/\mu(S_n)$, which mimicks $\mu(v_n) = \mu(M_n)/\mu(Q_n)$, though in a weaker manner.

Central element to the above is the observation that v_n has the same probability distribution as

$$\eta_n := \max\{|X_1|, \dots, |X_n|\} / X = (X_1, \dots, X_n) \sim U(B_n)$$

where $\sim U(B_n)$ denotes sampling from the uniform distribution U over B_n , being the surface of the n -dimensional unit hypersphere. This observation follows from the fact that points can be sampled randomly from B_n by sampling from $(Z_1, \dots, Z_n)$ and normalize the samples with Q_n (see Muller 1959). In Section 2, this observation allows us to calculate $\mu(v_n)$ and $\sigma^2(v_n)$ explicitly for low values of n , which is another aim of this note.

Also, by insensitivity of norms to scaling and translation, it follows that v_n can be interpreted as the max-distance relative to the Euclidian-distance of two arbitrary points in n -space, which is an appealing geometrical interpretation of v_n . Put differently, the ratio v_n can be interpreted as the longest edge of a hyperrectangle (also called orthotope), normalised by the space diagonal (which is Q_n).

In perspective, a vast amount of literature exists that deals with the properties of the ratio M_n/S_n . Application of this ratio can be found in the analysis of process speedup and the performance of scheduling (Downey & Wright 2007), on-line bin-batching and wireless link quality (Zhang 2001; Nguyen 2011) or in the modelling of high dimensional databases (Brabants 2005). Like with $v_n = M_n/Q_n$, the ratio M_n/S_n can be interpreted as the longest edge of a hyperrectangle, normalised by the sum of the lengths of its edges (which is S_n , taking one edge per dimension).

2 EXACT RESULT

In this section we calculate exact values for $\mu(v_n)$ and $\sigma^2(v_n)$ for low values of n . To do this, like mentioned in the introduction, we use the observation that v_n and $\eta_n := \max\{|X_1|, \dots, |X_n|\} | X = (X_1, \dots, X_n) \sim U(B_n)$ have the same distribution as a consequence of the fact that points can be sampled randomly from unit hypersphere B_n by sampling from $Z_1, \dots, Z_n$ and normalize the samples with Q_n .

Let $B_n^* := \{x \in B_n \mid x_1 \geq x_2 \geq \dots \geq x_n \geq 0\}$, i.e. B_n^* is the part of B_n of which all points are non-negative and define a non-increasing mapping. It is not difficult to see that by virtue of symmetry properties $\mu(\eta_n^k) = \mu(\eta_n^k | B_n^*)$, where $\mu(\eta_n^k | \cdot)$ denotes the conditional k -th moment of η_n . Note that $\text{vol}(B_n^*) = \text{vol}(B_n)/2^n n!$.

When integrating over B_n^* we switch to polar coordinates. That is, if $x \in B_n^*$, then $\varphi = (\varphi_1, \dots, \varphi_{n-1}) \in [0, \pi/2]^{n-1}$ and

$$x_1 = \cos(\varphi_1)\cos(\varphi_2) \cdots \cos(\varphi_{n-1})$$

$$x_2 = \sin(\varphi_1)\cos(\varphi_2) \cdots \cos(\varphi_{n-1})$$

$$x_3 = \sin(\varphi_2)\cos(\varphi_3) \cdots \cos(\varphi_{n-1})$$

....

$$x_{n-1} = \sin(\varphi_{n-2})\cos(\varphi_{n-1})$$

$$x_n = \sin(\varphi_{n-1})$$

the Jacobian of which is well known and satisfies

$$J_n(\varphi) = J_n(\varphi_1, \dots, \varphi_{n-1}) = \cos^{n-2}(\varphi_{n-1}) \cdot J_{n-1}(\varphi_1, \dots, \varphi_{n-2})$$

with $J_2(\varphi) = 1$. In terms of polar coordinates we have

$$B_n^* = \{x(\varphi) \in B_n \mid \tan(\varphi_1) \leq 1, \tan(\varphi_2) \leq \sin(\varphi_1), \\ \tan(\varphi_3) \leq \sin(\varphi_2), \dots, \tan(\varphi_{n-1}) \leq \sin(\varphi_{n-2})\}.$$

Clearly $\max\{x_1, \dots, x_n\} = x_1 = \cos(\varphi_1) \cdots \cos(\varphi_{n-1})$ for $x \in B_n^*$. Hence, the k -th moment of v_n is calculated as

$$\mu(v_n^k) = \mu(\eta_n^k) = \frac{2^n n!}{\text{vol}(B_n)} \int_{\tan(\varphi_j) \leq 1} (\cos(\varphi_1) \cdots \cos(\varphi_{n-1}))^k J_n(\varphi) d\varphi_{n-1} \cdots d\varphi_1. \\ \dots \int_{\tan(\varphi_{n-1}) \leq \sin(\varphi_{n-2})}$$

Evaluating this integral is quite cumbersome for large values of n , even for $k=1$ or $k=2$ (the latter being required to calculate $\sigma^2(v_n)$). In Appendix A we derive for lower dimensions

n	$\mu(v_n)$	approx. $\sigma^2(v_n)$	approx.
1	1	1	0
2	$\frac{2\sqrt{2}}{\pi}$	0.900	$\frac{1}{2} + \frac{1}{\pi} - \frac{8}{\pi^2}$
3	$\frac{3\sqrt{2}}{\pi} \times \arctan\left(\frac{1}{2}\sqrt{2}\right)$	0.831	$\frac{1}{3} + \frac{2}{\pi\sqrt{3}} - \frac{18}{\pi^2} \times \arctan^2\left(\frac{1}{2}\sqrt{2}\right)$
4	$\frac{8\sqrt{2}}{\pi} \times \arctan\left(\frac{1}{2}\sqrt{2}\right)$	0.779	

For $n=2$ the probability distribution has support $[1/\sqrt{2}, 1]$ and reads

$$F_2(z) = \Pr(v_2 \leq z) = \Pr(\cos(\varphi_1) \leq z \mid \varphi_1 \sim U[0, \pi/4]) \\ = 1 - \frac{4}{\pi} \arccos(z)$$

for z in the support.

3 MOMENT CONVERGENCE AND STABILITY OF v_n

Following the previous section, since $\text{vol}(B_n) = 2\pi^{n/2}/\Gamma(n/2)$, Q_n is chi-squared distributed with n degrees of freedom and density $X(t, n)$, and $M_n^k = \max\{|Z_1|^k, \dots, |Z_n|^k\}$ we have

$$\begin{aligned}
\mu(M_n^k) &= 2^n n! \int_{\tan(\varphi_1) \leq 1} \dots \int_{\tan(\varphi_{n-1}) \leq \sin(\varphi_{n-2})} \\
&\quad \int_0^\infty (r \cos(\varphi_1) \dots \cos(\varphi_{n-1}))^k \\
&\quad v^n(r) J_n(\varphi) r^{n-1} dr d\varphi_{n-1} \dots d\varphi_1 \\
&= \frac{2^n n!}{\text{vol}(B_n)} \int_{\tan(\varphi_1) \leq 1} \dots \\
&\quad \int_{\tan(\varphi_{n-1}) \leq \sin(\varphi_{n-2})} (\cos(\varphi_1) \dots \cos(\varphi_{n-1}))^k J_n(\varphi) d\varphi_{n-1} \dots d\varphi_1 \times \dots \\
&\quad \dots \times \text{vol}(B_n) \int_0^\infty \frac{1}{(2\pi)^{n/2}} e^{-r^2/2} r^{n-1+k} dr \\
&= \mu(v_n^k) \text{vol}(B_n) \int_0^\infty \frac{1}{(2\pi)^{n/2}} e^{-r^2/2} (r^2)^{(n-2+k)/2} d(r^2/2) \\
&= \mu(v_n^k) \frac{\text{vol}(B_n) \Gamma(n/2)}{2\pi^{n/2}} \int_0^\infty t^{k/2} \frac{1}{2^{n/2} \Gamma(n/2)} e^{-t/2} t^{n/2-1} dt \\
&= \mu(v_n^k) \int_0^\infty t^{k/2} X(t, n) dt \\
&= \mu(v_n^k) \mu(Q_n^k)
\end{aligned}$$

where Γ is the Gamma function, for which it is known that $\Gamma(k) = (k-1)!$ and $\Gamma(k+1/2) = \sqrt{\pi}(2k-1)!!/2^k$.

Furthermore, it is well-known in literature that $\mu(M_n) \sim b_n \sim \sqrt{2 \log(n)}$, where b_n is defined in the introduction and more precise in Appendix B. As to $\mu(Q_n)$, note that

$$\mu(Q_n) = \int_0^\infty \sqrt{x} \frac{x^{n/2-1} e^{-x/2}}{2^{n/2} \Gamma(n/2)} dx = \sqrt{2} \frac{\Gamma((n+1)/2)}{\Gamma(n/2)}$$

By observing that $\mu(Q_{n-1}) < \mu(Q_n) < \mu(Q_{n+1})$, we calculate straightforwardly that $n-1 = \mu(Q_n) \mu(Q_{n-1}) < \mu^2(Q_n) < \mu(Q_n) \mu(Q_{n+1}) = n$, so that $\sqrt{n-1} < \mu(Q_n) < \sqrt{n}$. Hence $\mu(Q_n) \sim \sqrt{n}$, implying

$$\mu(v_n) = \frac{\mu(M_n)}{\mu(Q_n)} \sim \frac{b_n}{\sqrt{n}} \sim \sqrt{\frac{2 \log(n)}{n}}.$$

Also, from Appendix it follows directly (note that $\mu(Q_n^2) = n$) that

$$\sigma(v_n^2) - \frac{\pi^2}{6nb_n^2} \rightarrow 0$$

so that $\sigma(v_n^2) \rightarrow 0$.

From Appendix B it also follows straightforwardly that

$$\begin{aligned}
\frac{\sigma^2(v_n)}{\mu^2(v_n)} &= \frac{\mu(v_n^2)}{\mu^2(v_n)} - 1 = \frac{\mu(M_n^2)/\mu(Q_n^2)}{\mu^2(M_n)/\mu^2(Q_n)} - 1 \\
&= \frac{(b_n^2/n)(1+o(1))}{(b_n/\sqrt{n})^2(1+o(1))} - 1 \rightarrow 0
\end{aligned}$$

and so (applying Chebychev's inequality), for all $\varepsilon > 0$

$$\begin{aligned}
&\Pr\left(\left|\frac{v_n - \mu(v_n)}{\mu(v_n)}\right| \leq \varepsilon\right) \\
&= \Pr\left(|v_n - \mu(v_n)| \leq \varepsilon \frac{\mu(v_n)}{\sigma(v_n)} \sigma(v_n)\right) \\
&\geq 1 - \frac{\sigma^2(v_n)}{\mu^2(v_n)} \frac{1}{\varepsilon^2} \rightarrow 1
\end{aligned}$$

Thus, we conclude that v_n is relatively stable, i.e. $v_n \sim \mu(v_n)$ in probability. Note that since $\mu(v_n) \rightarrow 0$ this is a stronger concept than absolute stability.

4 SIMULATION OF v_n

Following the previous elaboration, a Monte Carlo simulation is conducted to verify the asymptotic behaviour towards $\sqrt{2 \log(n)/n}$. The simulation code as included in Appendix C is used for this purpose. Figure 1 shows the result of 1000 runs. It is seen that the realisations are symmetrically distributed around the mean value and that the uncertainty vanishes large n .

Figure 1.

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APPENDIX A: DERIVATIONS OF EXACT RESULTS

For $n = 2$ we have

$$\mu(v_2) = \frac{2!2^2}{\text{vol}(B_2)} \int_0^{\pi/4} \cos(\varphi_1) d\varphi_1 = \frac{8}{2\pi} \left(\frac{1}{2} \sqrt{2} \right) = \frac{2\sqrt{2}}{\pi} \approx 0,900$$

Likewise, for $n = 3$ we calculate

$$\begin{aligned} \mu(v_3) &= \frac{3!2^3}{\text{vol}(B_3)} \int_0^{\pi/4} \int_0^{\arctan(\sin(\varphi_1))} \cos(\varphi_1) \cdot \cos^2(\varphi_2) d\varphi_2 d\varphi_1 \\ &= \frac{48}{4\pi} \int_0^{\pi/4} \cos(\varphi_1) \frac{1}{2} [\cos(\arctan(\sin(\varphi_1))) \\ &\quad + \dots + \arctan(\sin(\varphi_1))] d\varphi_1 \\ &= \frac{6}{\pi} \int_0^{\pi/4} \cos(\varphi_1) \left(\frac{1}{\sqrt{1+\sin^2(\varphi_1)}} \cdot \frac{\sin(\varphi_1)}{\sqrt{1+\sin^2(\varphi_1)}} \right. \\ &\quad \left. + \arctan(\sin(\varphi_1)) \right) d\varphi_1 \\ &= \frac{6}{\pi} \int_0^{1/\sqrt{2}} \frac{y}{1+y^2} + \arctan(y) dy \\ &= \frac{6}{\pi} [y \arctan(y)]_0^{1/\sqrt{2}} \\ &= \frac{3\sqrt{2}}{\pi} \arctan\left(\frac{1}{2}\sqrt{2}\right) \approx 0,831 \end{aligned}$$

and for $n = 4$

$$\begin{aligned} \mu(v_4) &= \frac{4!2^4}{\text{vol}(B_4)} \cdot \int_0^{\pi/4} \int_0^{\arctan(\sin(\varphi_1))} \int_0^{\arctan(\sin(\varphi_2))} \cos(\varphi_1) \cos^2(\varphi_2) \cos^3(\varphi_3) d\varphi_3 d\varphi_2 d\varphi_1 \\ &= \frac{384}{2\pi^2} \int_0^{\pi/4} \int_0^{\arctan(\sin(\varphi_1))} \cos(\varphi_1) \cos^2(\varphi_2) \frac{2}{3} \\ &\quad \sin(\arctan(\sin(\varphi_2))) d\varphi_2 d\varphi_1 + \dots \\ &\quad + \frac{384}{2\pi^2} \int_0^{\pi/4} \int_0^{\arctan(\sin(\varphi_1))} \cos(\varphi_1) \cos^2(\varphi_2) \\ &\quad \frac{1}{3} \sin(\arctan(\sin(\varphi_2))) \dots \end{aligned}$$

$$\dots \cos^2(\arctan(\sin(\varphi_2))) d\varphi_2 d\varphi_1$$

$$\begin{aligned} &= \frac{128}{\pi^2} \int_0^{\pi/4} \int_0^{\arctan(\sin(\varphi_1))} \cos(\varphi_1) \cos^2(\varphi_2) \frac{\sin(\varphi_2)}{\sqrt{1+\sin^2(\varphi_2)}} d\varphi_2 d\varphi_1 + \dots \\ &\quad \dots + \frac{64}{\pi^2} \int_0^{\pi/4} \int_0^{\arctan(\sin(\varphi_1))} \cos(\varphi_1) \cos^2(\varphi_2) \frac{\sin(\varphi_2)}{(1+\sin^2(\varphi_2))^{3/2}} d\varphi_2 d\varphi_1 \\ &= \frac{128}{\pi^2} \int_0^{\pi/4} \int_0^{\arctan(\sin(\varphi_1))} \cos(\varphi_1) \cos(\varphi_2) d\sqrt{1+\sin^2(\varphi_2)} d\varphi_1 + \dots \\ &\quad \dots + \frac{64}{\pi^2} \int_0^{\pi/4} \int_0^{\arctan(\sin(\varphi_1))} \cos(\varphi_1) \cos(\varphi_2) \sin(\varphi_2) d\frac{\sin(\varphi_2)}{\sqrt{1+\sin^2(\varphi_2)}} d\varphi_1 \\ &= \frac{128}{\pi^2} \left[\int_0^{\pi/4} \cos(\varphi_1) \frac{\sqrt{1+2\sin^2(\varphi_1)}}{1+\sin^2(\varphi_1)} d\varphi_1 - \frac{1}{2}\sqrt{2} \right] - \dots \\ &\quad \dots - \frac{128}{\pi^2} \left[\int_0^{\pi/4} \cos(\varphi_1) \int_0^{\arctan(\sin(\varphi_1))} \sqrt{2-\cos^2(\varphi_2)} d\cos(\varphi_2) d\varphi_1 \right] \\ &\quad + \dots \end{aligned}$$

$$\begin{aligned} &\dots + \frac{64}{\pi^2} \int_0^{\pi/4} \cos(\varphi_1) \left[\frac{\sin^2(\varphi_1)}{(1+\sin^2(\varphi_1))\sqrt{1+2\sin^2(\varphi_1)}} \right. \\ &\quad \left. - \int_0^{\frac{1}{\sqrt{1+\sin^2(\varphi_1)}}} \frac{2y^2-1}{\sqrt{2-y^2}} dy \right] d\varphi_1 \\ &= \frac{128}{\pi^2} A + \frac{64}{\pi^2} B. \end{aligned}$$

where

$$\begin{aligned} A &= \int_0^{1/\sqrt{2}} \frac{\sqrt{1+2y^2}}{1+y^2} dy - \frac{1}{2}\sqrt{2} \\ &\quad + \int_0^{\pi/4} \cos(\varphi_1) \int_0^{\frac{1}{\sqrt{1+\sin^2(\varphi_1)}}} \sqrt{2-y^2} dy d\varphi_1 \end{aligned}$$

and

$$\begin{aligned}
 B &= \int_0^{\pi/4} \cos(\varphi_1) \left[\frac{\sin^2(\varphi_1)}{(1+\sin^2(\varphi_1))\sqrt{1+2\sin^2(\varphi_1)}} - \int_{1/\sqrt{1+\sin^2(\varphi_1)}}^1 \frac{2y^2-1}{\sqrt{2-y^2}} dy \right] d\varphi_1 \\
 &= \int_0^{1/\sqrt{2}} \frac{y^2}{(1+y^2)\sqrt{1+2y^2}} dy \\
 &\quad - \int_0^{\pi/4} \cos(\varphi_1) \int_{1/\sqrt{1+\sin^2(\varphi_1)}}^1 \frac{2y^2-1}{\sqrt{2-y^2}} dy d\varphi_1 \\
 \text{Since} \\
 &\int_{1/\sqrt{1+\sin^2(\varphi_1)}}^1 \sqrt{2-y^2} dy \\
 &= \left[\frac{1}{2} y \sqrt{2-y^2} + \arcsin\left(\frac{1}{2} \sqrt{2} y\right) \right]_{1/\sqrt{1+\sin^2(\varphi_1)}}^1 \\
 &= \frac{1}{2} + \frac{1}{4} \pi - \frac{\sqrt{1+2\sin^2(\varphi_1)}}{2+2\sin^2(\varphi_1)} - \arcsin\left(\frac{1}{\sqrt{2+2\sin^2(\varphi_1)}}\right)
 \end{aligned}$$

we find

$$\begin{aligned}
 \int_0^{\pi/4} \cos(\varphi_1) \int_{1/\sqrt{1+\sin^2(\varphi_1)}}^1 \sqrt{2-y^2} dy d\varphi_1 &= \frac{1}{4} \sqrt{2} + \frac{1}{8} \pi \sqrt{2} \\
 - \int_0^{1/\sqrt{2}} \frac{\sqrt{1+2y^2}}{2+2y^2} dy - \int_0^{1/\sqrt{2}} \arcsin\left(\frac{1}{\sqrt{2+2y^2}}\right) dy.
 \end{aligned}$$

Also, we have

$$\begin{aligned}
 \int_0^{1/\sqrt{2}} \arcsin\left(\frac{1}{\sqrt{2+2y^2}}\right) dy &= \left[y \arcsin\left(\frac{1}{\sqrt{2+2y^2}}\right) \right]_0^{1/\sqrt{2}} \\
 &\quad + \int_0^{1/\sqrt{2}} \frac{2y^2}{(2+2y^2)\sqrt{1+2y^2}} dy \\
 &= \frac{1}{2} \sqrt{2} \arcsin\left(\frac{1}{3} \sqrt{3}\right) + \int_0^{1/\sqrt{2}} \frac{2y^2}{(2+2y^2)\sqrt{1+2y^2}} dy
 \end{aligned}$$

so that

$$\begin{aligned}
 A &= -\frac{1}{4} \sqrt{2} + \frac{1}{8} \pi \sqrt{2} - \frac{1}{2} \sqrt{2} \arcsin\left(\frac{1}{3} \sqrt{3}\right) \\
 &\quad + \int_0^{1/\sqrt{2}} \frac{1}{(2+2y^2)\sqrt{1+2y^2}} dy.
 \end{aligned}$$

Furthermore,

$$\begin{aligned}
 \int_{1/\sqrt{1+\sin^2(\varphi_1)}}^1 \frac{2y^2-1}{\sqrt{2-y^2}} dy &= \int_{1/\sqrt{1+\sin^2(\varphi_1)}}^1 \frac{2y^2-4}{\sqrt{2-y^2}} dy \\
 &\quad + \int_{1/\sqrt{1+\sin^2(\varphi_1)}}^1 \frac{3}{\sqrt{2-y^2}} dy \\
 &= -2 \int_{1/\sqrt{1+\sin^2(\varphi_1)}}^1 \sqrt{2-y^2} dy + 3 \int_{1/\sqrt{1+\sin^2(\varphi_1)}}^1 \frac{1}{\sqrt{2-y^2}} dy \\
 &= -2 \left[\frac{1}{2} + \frac{1}{4} \pi - \frac{\sqrt{1+2\sin^2(\varphi_1)}}{2+2\sin^2(\varphi_1)} \right. \\
 &\quad \left. - \arcsin\left(\frac{1}{\sqrt{2+2\sin^2(\varphi_1)}}\right) \right] + \dots \\
 &= -1 + \frac{1}{4} \pi + \frac{\sqrt{1+2\sin^2(\varphi_1)}}{1+\sin^2(\varphi_1)} \\
 &\quad - \arcsin\left(\frac{1}{\sqrt{2+2\sin^2(\varphi_1)}}\right)
 \end{aligned}$$

so that

$$\begin{aligned}
 B &= \int_0^{1/\sqrt{2}} \frac{(2y^2+1)-(1+y^2)}{(1+y^2)\sqrt{1+2y^2}} dy - \dots \\
 &\quad \dots - \int_0^{\pi/4} \left[-1 + \frac{1}{4} \pi + \frac{\sqrt{1+2\sin^2(\varphi_1)}}{1+\sin^2(\varphi_1)} \right. \\
 &\quad \left. - \arcsin\left(\frac{1}{\sqrt{2+2\sin^2(\varphi_1)}}\right) \right] d\sin(\varphi_1) \\
 &= \int_0^{1/\sqrt{2}} \frac{(2y^2+1)-(1+y^2)}{(1+y^2)\sqrt{1+2y^2}} dy \\
 &\quad - \int_0^{1/\sqrt{2}} \left[-1 + \frac{1}{4} \pi + \frac{\sqrt{1+2y^2}}{1+y^2} - \arcsin\left(\frac{1}{\sqrt{2+2y^2}}\right) \right] dy \\
 &= \int_0^{1/\sqrt{2}} \frac{-1}{\sqrt{1+2y^2}} dy \\
 &\quad + \int_0^{1/\sqrt{2}} \left[1 - \frac{1}{4} \pi + \arcsin\left(\frac{1}{\sqrt{2+2y^2}}\right) \right] dy \\
 &= \int_0^{1/\sqrt{2}} \frac{-1}{\sqrt{1+2y^2}} dy + \frac{1}{2} \sqrt{2} - \frac{1}{8} \pi \sqrt{2} + \frac{1}{2} \sqrt{2} \arcsin\left(\frac{1}{3} \sqrt{3}\right) \\
 &\quad + \int_0^{1/\sqrt{2}} \frac{2y^2}{(2+2y^2)\sqrt{1+2y^2}} dy
 \end{aligned}$$

$$= \frac{1}{2}\sqrt{2} - \frac{1}{8}\pi\sqrt{2} + \frac{1}{2}\sqrt{2} \arcsin\left(\frac{1}{3}\sqrt{3}\right) + \int_0^{1/\sqrt{2}} \frac{-2}{(2+2y^2)\sqrt{1+2y^2}} dy$$

Hence

$$\begin{aligned} \mu(v_4) &= \frac{128}{\pi^2} \left[-\frac{1}{4}\sqrt{2} + \frac{1}{8}\pi\sqrt{2} - \frac{1}{2}\sqrt{2} \arcsin\left(\frac{1}{3}\sqrt{3}\right) + \int_0^{1/\sqrt{2}} \frac{1}{(2+2y^2)\sqrt{1+2y^2}} dy \right] + \dots \\ &\quad \dots + \frac{64}{\pi^2} \left[\frac{1}{2}\sqrt{2} - \frac{1}{8}\pi\sqrt{2} + \frac{1}{2}\sqrt{2} \arcsin\left(\frac{1}{3}\sqrt{3}\right) + \int_0^{1/\sqrt{2}} \frac{-2}{(2+2y^2)\sqrt{1+2y^2}} dy \right] \\ &= \frac{8\sqrt{2}}{\pi} - \frac{32\sqrt{2}}{\pi^2} \arcsin\left(\frac{1}{3}\sqrt{3}\right) \\ &= \frac{8\sqrt{2}}{\pi} - \frac{32\sqrt{2}}{\pi^2} \arctan\left(\frac{1}{2}\sqrt{2}\right) \\ &\approx 0.779 \end{aligned}$$

As to the variance $\sigma^2(v_2)$ we calculate the second moment of $\mu_2(v_2)$ of v_2 as

$$\begin{aligned} \mu(v_2^2) &= \frac{2!2^2}{\text{vol}(B_2)} \int_0^{\pi/4} \cos^2(\varphi_1) d\varphi_1 \\ &= \frac{8}{2\pi} \left(\frac{1}{4} + \frac{\pi}{8} \right) = \frac{1}{\pi} + \frac{1}{2} \approx 0.818 \end{aligned}$$

so that

$$\sigma^2(v_2) = \frac{1}{\pi} + \frac{1}{2} - \frac{8}{\pi^2} \approx 0.00774.$$

Also

$$\begin{aligned} \mu(v_5^2) &= \frac{3!2^3}{\text{vol}(B_3)} \int_0^{\arctan(\sin(\varphi_1))} \int_0^{\pi/4} \cos^2(\varphi_1) \cdot \cos^3(\varphi_2) d\varphi_2 d\varphi_1 \\ &= \frac{48}{4\pi} \int_0^{\pi/4} \cos^2(\varphi_1) \left(-\frac{1}{3} \sin^3(\arctan(\sin(\varphi_1))) \right) d\varphi_1 \end{aligned}$$

$$= \frac{12}{\pi} \int_0^{\pi/4} \cos^2(\varphi_1) \left(\frac{\sin(\varphi_1)}{\sqrt{1+\sin^2(\varphi_1)}} - \frac{\sin^3(\varphi_1)}{3(1+\sin^2(\varphi_1))^{3/2}} \right) d\varphi_1$$

$$\begin{aligned} &= -\frac{12}{\pi} \int_0^{\pi/4} \frac{\cos^2(\varphi_1)}{\sqrt{2-\cos^2(\varphi_1)}} d\cos(\varphi_1) + \dots \\ &\quad \dots + \frac{12}{\pi} \int_0^{\pi/4} \frac{\cos^2(\varphi_1)(1-\cos^2(\varphi_1))}{3(2-\cos^2(\varphi_1))^{3/2}} d\cos(\varphi_1) \\ &= \frac{12}{\pi} \int_{1/\sqrt{2}}^1 \frac{y^2}{\sqrt{2-y^2}} dy - \frac{4}{\pi} \int_{1/\sqrt{2}}^1 \frac{y^2}{(2-y^2)^{3/2}} dy \\ &\quad + \frac{4}{\pi} \int_{1/\sqrt{2}}^1 \frac{y^4}{(2-y^2)^{3/2}} dy \\ &= \frac{12}{\pi} \left[\arcsin\left(\frac{y}{\sqrt{2}}\right) - \frac{1}{2}y\sqrt{2-y^2} \right]_{1/\sqrt{2}}^1 - \dots \\ &\quad \dots - \frac{4}{\pi} \left[\frac{y}{\sqrt{2-y^2}} - \arcsin\left(\frac{y}{\sqrt{2}}\right) \right]_{1/\sqrt{2}}^1 + \dots \\ &\quad \dots + \frac{4}{\pi} \left[\frac{1}{2}y\sqrt{2-y^2} + \frac{2y}{\sqrt{2-y^2}} - 3\arcsin\left(\frac{y}{\sqrt{2}}\right) \right]_{1/\sqrt{2}}^1 \\ &= \frac{4}{\pi} \left[\arcsin\left(\frac{y}{\sqrt{2}}\right) - y\sqrt{2-y^2} + \frac{y}{\sqrt{2-y^2}} \right]_{1/\sqrt{2}}^1 \\ &= \frac{1}{3} + \frac{2}{\pi\sqrt{3}}. \end{aligned}$$

so that

$$\sigma^2(v_3) = \frac{1}{3} + \frac{2}{\pi\sqrt{3}} - \frac{18}{\pi^2} \arctan^2\left(\frac{1}{2}\sqrt{2}\right) \approx 0.0100.$$

APPENDIX B: PROOF THAT,

$$\mu(M_n) \sim b_n, \mu(M_n^2) \sim b_n^2 \text{ AND } b_n \sim \sqrt{2\log(n)}$$

Let $\{Z_n\}$ and Z be i.i.d. standard normal variables (with distribution Φ and density ϑ). Let $\Phi_{abs}(x) = \Pr(|Z| \leq x)$ and denote its density with ϑ_{abs} . Also, we write Φ and Φ_{abs} as a shorthand for tail probabilities $1 - \Phi$ and $1 - \Phi_{abs}$ respectively. Obviously, for $x \geq 0$ we have $\vartheta_{abs}(x) = 2\vartheta(x)$ and $\Phi_{abs}(x) = 2\Phi(x) = \text{erfc}(x/\sqrt{2})$, where erfc is the complementary error function defined by and having first order Laurent expansion

$$\text{erfc}(x) := \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt = \frac{e^{-x^2}}{x\sqrt{\pi}} (1 + o(1)).$$

For the moments of $|Z|$ we have

$$\begin{aligned}\mu(|Z|^k) &= \sqrt{\frac{2}{\pi}} \int_0^\infty t^k e^{-t^2/2} dt = -\sqrt{\frac{2}{\pi}} \int_0^\infty t^{k-1} de^{-t^2/2} \\ &= (k-1) \sqrt{\frac{2}{\pi}} \int_0^\infty t^{k-2} e^{-t^2/2} dt = (k-1) \mu(|Z|^{k-2})\end{aligned}$$

so that $\mu(|Z|^{2i}) = (2i-1)!!$ and $\mu(|Z|^{2i+1}) = \sqrt{2/\pi} (2i)!!$. Notably, moments exist and are finite. Also, the inverse $\bar{\Phi}_{abs}^\leftarrow$ of $\bar{\Phi}_{abs}$ will be used below and follows from

$$\frac{1}{\bar{\Phi}_{abs}(x)} = \frac{1}{\text{erfc}^2(x/\sqrt{2})} = \frac{\pi}{2} x^2 e^{x^2} \frac{1}{(1+o(1))^2}$$

yielding

$$\bar{\Phi}_{abs}^\leftarrow(x) = \sqrt{W\left(\frac{2}{\pi x^2}(1+o(1))\right)}$$

where W is the Lambert W function, implicitly defined by $W(x)e^{W(x)} = x$.

With Λ we refer to the Gumbel distribution (also known as the doubly exponential distribution and being one of the three limiting cases of the GEV distribution, well known in extreme value theory) defined by $\Lambda(x) = \exp(-e^{-x})$, having mean γ and variance $\pi^2/6$, where $\gamma \approx 0.577$ is the Euler-Mascheroni constant. It is common knowledge that Φ satisfies the von Mises condition for being in the domain of attraction of Λ (von Mises 1936; Falk 1993), i.e.

$$\frac{\vartheta(x)}{\bar{\Phi}^2(x)} \int_x^\infty \bar{\Phi}(t) dt \rightarrow 1 \quad (x \rightarrow \infty).$$

It is easily seen that Φ_{abs} also satisfies this condition and is therefore also in the domain of attraction of Λ . Say that $\{a_n\}$ and $\{b_n\}$ are sequences for which $(M_n - b_n)/a_n \rightarrow \Lambda$ in distribution. Intuitively we would expect that $\mu(M_n) - (b_n + a_n\gamma) \rightarrow 0$ and that $\mu(M_n^2) - (a_n^2(\pi^2/6 + \gamma^2) + b_n^2 + 2a_nb_n\gamma) \rightarrow 0$. From Pickands (1968) (see also Nair 1981) we know that this is indeed the case. Furthermore, it is widely known that $\{a_n\}$ and $\{b_n\}$ are solutions to $b_n = \bar{\Phi}_{abs}^\leftarrow(1/n)$ and $a_n = r(b_n)$, where $r(x) = \bar{\Phi}_{abs}(x)/\vartheta_{abs}(x)$ is the reciprocal hazard function (Smith 1987). Hence

$$b_n = \sqrt{W\left(\frac{2n^2}{\pi}(1+o(1))\right)}$$

and

$$a_n = \frac{1/n}{\sqrt{2/\pi} e^{-b_n^2/2}} = \frac{1}{b_n} \sqrt{\frac{\pi}{2n^2}} b_n^2 e^{b_n^2/2} \sim \frac{1}{b_n}.$$

Hence (note that $\mu(M_n) = b_n + o(1)$ and $b_n \rightarrow \infty$), $\mu(M_n) \sim b_n$ and $\mu(M_n^2) \sim b_n^2$.

Finally, as $\log(x) - \log \log(x) \leq W(x) \leq \log(x) - \log \log(x)/2$ for $x \geq e$ (see Hoorfar 2008), we see that in terms of elementary functions $b_n \sim \sqrt{2 \log(n)}$. Consistently, Cramer (1946) and Deo (1972) approximate the location and scaling parameters $\{a_n\}$ and $\{b_n\}$ by $b_n = c_n - \log(4\pi \log(n))/2c_n + 2/c_n$ and $a_n = 1/c_n$, where $c_n = \sqrt{2 \log(n)}$.

APPENDIX C: MATLAB CODE FOR SIMULATION OF v_n

```
Clear
clg
n=1000;
hold on
for j=1:n,
z(1)=normrnd(0,1);
m(1)=abs(z(1));
s(1)=m(1);
q2(1)=z(1)^2;
for i=2:n,
z(i)=normrnd(0,1);
m(i)=max(m(i-1),abs(z(i)));
s(i)=s(i-1)+abs(z(i));
q2(i)=q2(i-1)+z(i)^2;
q(i)=sqrt(q2(i));
nu(i)=m(i)/q(i);
ex(i)=sqrt(2*log(i)/log(2.7181)/i);
end
loglog([1:n],nu,'b')
end
loglog([2:n],ex(2:n),'r')
xlabel('Sample size n')
ylabel('Nu')
grid
title('Results of 1000 simulations (blue lines) and
theoretical result (red line)')
hold off
```