RANDOMIZED UMD BANACH SPACES AND DECOUPLING INEQUALITIES FOR STOCHASTIC INTEGRALS

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Abstract. In this paper we prove the equivalence of decoupling inequalities for stochastic integrals and one-sided randomized versions of the UMD property of a Banach space as introduced by Garling.

1. Introduction

In recent years, decoupling inequalities have been used to construct theories of stochastic integration in UMD Banach spaces [4, 13, 15]. The basic idea underlying this approach is to use abstract decoupling inequalities to estimate stochastic integrals

\[ \int_0^T \phi(t) \, dW(t), \]

where \( \phi \) is a process with values in a UMD space \( E \) and \( W \) is a standard Brownian motion, with its decoupled analogue

\[ \int_0^T \phi(t) \, d\tilde{W}(t), \]

where \( \tilde{W} \) is a standard Brownian motion independent of \( \phi \) and \( W \). This decoupled integral is easier to handle, as it is defined in a pathwise sense. Indeed, using a general two-sided decoupling inequality for \( E \)-valued tangent sequences, McConnell [13] was able to show that a strongly measurable \( E \)-valued process is stochastically integrable with respect to \( W \) if and only if its trajectories, viewed as \( E \)-valued functions, are stochastically integrable with respect to \( \tilde{W} \). His techniques depend heavily on the equivalence of the UMD property and geometric notions related to \( \zeta \)-convexity. Decoupling inequalities for tangent sequences may be found in [7, 9, 13, 14, 17].

Earlier, Garling [4] had derived a two-sided decoupling inequality for stochastic integrals of elementary \( E \)-valued processes directly from the definition of the UMD property. More precisely, he proved that a Banach space \( E \) is a UMD space if and only if for some (for all) \( 1 < p < \infty \) there exist constants \( 0 < c \leq C < \infty \) such that for all elementary \( E \)-valued processes \( \phi \), we have

\[ cE \left\| \int_0^T \phi \, d\tilde{W}(t) \right\|^p \leq E \left\| \int_0^T \phi \, dW(t) \right\|^p \leq C E \left\| \int_0^T \phi \, d\tilde{W}(t) \right\|^p. \]

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These inequalities, combined with the operator-theoretic approach to stochastic integration of Banach space-valued functions developed in [16], was used in [15] to construct a systematic theory of stochastic integration for $E$-valued processes. In particular, necessary and sufficient conditions for $L^p$-stochastic integrability were obtained, analogues of the Itô isometry and the Burkholder-Davis-Gundy inequalities were proved, and McConnell’s result was recovered as a corollary via standard stopping time arguments.

Various applications of the decoupling inequalities in (1.1) require only one of the two a priori estimates. An analysis of the proof of (1.1) in [4] shows moreover that one-sided decoupling inequalities can be derived from one-sided versions of the UMD property which were introduced subsequently by Garling in [5]. These properties are called UMD$^-$ and UMD$^+$ below. These properties can be used as in [15] to obtain generalized theories of stochastic integration in which the necessary and sufficient conditions and two-sided estimates for stochastic integrals are replaced by necessary conditions or sufficient conditions, respectively, with one-sided estimates.

The stochastic integration theory in [15] has many consequences and applications. For instance, many results in the theory of stochastic evolution equations in Hilbert spaces (cf. [3] and the references therein), have analogues in UMD$^-\ PW$ Banach spaces. Therefore, we believe it is important to know the largest class of spaces for which one can construct a stochastic integration theory as in [15]. The aim of the present paper is to show that this is the class of UMD$^-\ PW$ Banach spaces. It is shown that the validity of the second one-sided a priori estimate in (1.1) for all elementary processes implies the UMD$^-\ PW$ property. With the same ideas one can prove that $E$ has property UMD$^+_\ PW$ if for some $1 < p < \infty$ the left estimate in (1.1) holds for all elementary $E$-valued processes, so we include this too. The proofs are based on Skorohod embedding techniques from [4], the Maurey-Pisier characterizations of finite cotype and estimates for randomized sums in spaces of finite cotype.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n \geq 1}, P)$ be a filtered probability space, and let $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ be a probability space. Both probability spaces are assumed to be rich enough for constructions as below. We shall consider random variables and processes on $(\Omega \times \tilde{\Omega}, \mathcal{F} \times \tilde{\mathcal{F}}, P \times \tilde{P})$. On this probability space we use the filtration $(\mathcal{F}_n \otimes \tilde{\mathcal{F}})_{n \geq 1}$. In most cases our random variables and processes are extensions to $\Omega \times \tilde{\Omega}$ of variables and processes on $\Omega$ or $\tilde{\Omega}$. Integration over $\Omega$ and $\tilde{\Omega}$ will be denoted by $\mathbb{E}$ and $\mathbb{E}^\Omega$.

Let $(r_n)_{n \geq 1}$ be a Rademacher sequence on $(\Omega, \mathcal{F}, P)$ and let $\mathcal{G}_0 = \{\emptyset, \Omega\}$ and $\mathcal{G}_n = \sigma(r_k, k = 1, \ldots, n)$. Recall that a martingale difference sequence $(d_n)_{n=1}^N$ is a Paley-Walsh martingale difference sequence if it is a martingale difference sequence with respect to the filtration $(\mathcal{G}_n)_{n=0}^N$.

Recall that a Banach space $E$ is a UMD$(p)$ space for $p \in (1, \infty)$ if there exists a constant $C_p > 0$ such that for every $N \geq 1$, every martingale difference sequence $(d_n)_{n=1}^N$ in $L^p(\Omega, E)$ and every $\{\pm 1\}$-valued sequence $(\varepsilon_n)_{n=1}^N$, we have
\[
\left( \mathbb{E} \left\| \sum_{n=1}^N \varepsilon_n d_n \right\|^p \right)^{\frac{1}{p}} \leq C_p \left( \mathbb{E} \left\| \sum_{n=1}^N d_n \right\|^p \right)^{\frac{1}{p}}.
\]

Similarly, we say $E$ is a UMD$^\*$$(p)$ space if one only considers Paley-Walsh martingales in the definition of UMD$(p)$. In [11], Maurey has shown that UMD$^\*$$(p)$ already implies UMD$(p)$. It was shown by Burkholder in [1] that if $E$ is UMD$(p)$
space for some \( p \in (1, \infty) \), then \( E \) is a UMD\((p)\) space for all \( p \in (1, \infty) \). Spaces with this property will be referred to as \textit{UMD spaces}. For the theory of UMD spaces we refer the reader to [1, 2] and references given therein.

Let \( (\tilde{r}_n)_{n \geq 1} \) be a Rademacher sequence on \( \Omega \).

\textbf{Definition 1.1.} Let \( E \) be a Banach space and let \( p \in (1, \infty) \).

1. The space \( E \) is a UMD\(_\text{PW}^-(p)\) space if there exists a constant \( C_p^- > 0 \) such that for every \( N \geq 1 \), every Paley-Walsh martingale difference sequence \( (d_n)_{n=1}^N \in L^p(\Omega, E) \), we have

\[
(1.2) \quad \left( \mathbb{E} \left\| \sum_{n=1}^N d_n \right\|^p \right)^{\frac{1}{p}} \leq C_p^- \left( \mathbb{E} \mathbb{E} \left\| \sum_{n=1}^N \tilde{r}_n d_n \right\|^p \right)^{\frac{1}{p}}.
\]

2. The space \( E \) is a UMD\(_\text{PW}^+(p)\) space if there exists a constant \( C_p^+ > 0 \) such that for every \( N \geq 1 \), every Paley-Walsh martingale difference sequence \( (d_n)_{n=1}^N \in L^p(\Omega, E) \), we have

\[
(1.3) \quad \left( \mathbb{E} \mathbb{E} \left\| \sum_{n=1}^N \tilde{r}_n d_n \right\|^p \right)^{\frac{1}{p}} \leq C_p^+ \left( \mathbb{E} \left\| \sum_{n=1}^N d_n \right\|^p \right)^{\frac{1}{p}}.
\]

The corresponding notion of UMD\(_-\) and UMD\(_+\) spaces, where arbitrary martingale difference sequences are allowed, has been studied by Garling in [5]. It was shown there that if \( E \) is a UMD\(_\pm\)(p) space for some \( p \in (1, \infty) \), then \( E \) is a UMD\(_\pm\)(p) space for all \( p \in (1, \infty) \). Thus, both definitions are independent of \( p \in (1, \infty) \) and spaces with this property will be referred to as \textit{UMD\(_-\)} and \textit{UMD\(_+\)} spaces. In [5] these properties are called LERMT (Lower Estimates for Random Martingale Transforms) and UERMT (Upper Estimates for Random Martingale Transforms) respectively. We preferred the notation UMD\(_-\) and UMD\(_+\), since it emphasizes the relation with UMD. Here the superscript \(-\) stands for Lower and the superscript \(+\) stands for Upper. Similarly, one can show that UMD\(_\text{PW}^-\)(p) are \( p \)-independent and these will denoted by UMD\(_\text{PW}^L\). It seems to be an open problem if UMD\(_\text{PW}^-\) implies UMD\(_-\) and if UMD\(_\text{PW}^+\) implies UMD\(_+\).

We list some results on UMD\(_-\) and UMD\(_+\) spaces, the proofs of which can be found in [5]:

- If \( E \) is a UMD\(_+\) space, then its dual \( E^* \) is a UMD\(_-\) space. If \( E^* \) is a UMD\(_+\) space, then its predual \( E \) is a UMD\(_-\) space.
- Every UMD\(_-\) space has finite cotype. Every UMD\(_+\) space is super-reflexive.
- \( E \) is a UMD space if and only if it is both UMD\(_-\) and UMD\(_+\).

Similar results hold for UMD\(_\text{PW}^-\) and UMD\(_\text{PW}^+\) spaces.

It was shown in [5] that \( L^1 \) is a UMD\(_-\) space. It can be shown that if \( E \) is a UMD\(_-\) space and if \( (S, \Sigma, \mu) \) is a \( \sigma \)-finite measure space, then \( L^p(\Sigma; E) \) is a UMD\(_-\) space for all \( p \in [1, \infty) \). A similar result holds for UMD\(_+\) for \( p \in (1, \infty) \).

Apart from trivial cases, the space \( L^1(S, \mu) \) is an example of a UMD\(_-\) space that is not UMD. It appears to be unknown if there exist UMD\(_+\) or UMD\(_\text{PW}^+\) spaces that are not UMD (cf. [6, Problem 4.2]).
2. Main result

Let \( W \) be a Brownian motion on \((\Omega, \mathcal{F}, \mathbb{P})\) and let \((\mathcal{F}_t)_{t \geq 0}\) be the augmented filtration induced by \( W \). Similarly, let \( \tilde{W} \) be a Brownian motion on \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})\) and let \((\tilde{\mathcal{F}}_t)_{t \geq 0}\) be the augmented filtration induced by \( \tilde{W} \).

Let \( E \) be a real Banach space. A process \( \phi: [0, \infty) \times \Omega \to E \) will be called an elementary process if it is of the form

\[
\phi(t, \omega) = 1_{[0]}(t)\xi_0(\omega) + \sum_{n=1}^N 1_{(t_{n-1}, t_n]}(t)\xi_n(\omega),
\]

where \( 0 \leq t_0 < \cdots < t_N < \infty \), \( \xi_n \) is an elementary \( \mathcal{F}_{t_{n-1}} \)-measurable random variable, \( n = 1, \ldots, N \) and \( \xi_0 \) is \( \mathcal{F}_0 \)-measurable. The stochastic integral \( \int_0^\infty \phi(t) \, dW(t) \) is defined in the usual way and is an element of \( L^p(\Omega; E) \) for all \( p \in [1, \infty) \).

**Theorem 2.1** (Garling). For a UMD space \( E \) and \( p \in (1, \infty) \) the following statements hold:

1. There exists a constant \( c_p > 0 \) such that for all elementary processes \( \phi \),

   \[
   \mathbb{E} \left\| \int_0^\infty \phi(t) \, dW(t) \right\|^p \leq c_p \mathbb{E} \tilde{\mathbb{E}} \left\| \int_0^\infty \phi(t) \, d\tilde{W}(t) \right\|^p.
   \]

2. There exists a constant \( c_p > 0 \) such that for all elementary processes \( \phi \),

   \[
   \mathbb{E} \tilde{\mathbb{E}} \left\| \int_0^\infty \phi(t) \, d\tilde{W}(t) \right\|^p \leq c_p \mathbb{E} \left\| \int_0^\infty \phi(t) \, dW(t) \right\|^p.
   \]

Conversely, if (2.1) and (2.2) hold for all elementary processes \( \phi \), then \( E \) is a UMD space.

Inspection of the proof in [4, Theorem 2] shows that (2.1) only requires UMD\(^-\) and (2.2) only requires UMD\(^+\). The main result of this paper reads as follows.

**Theorem 2.2.** Let \( E \) be a Banach space \( E \) and let \( p \in (1, \infty) \). The following statements hold:

1. If there exists a constant \( c_p > 0 \) such that (2.1) holds for all elementary processes, then \( E \) is a UMD\(^{-} \) space.

2. If there exists a constant \( c_p > 0 \) such that (2.2) holds for all elementary processes, then \( E \) is a UMD\(^{+} \) space.

Although these results are in some sense not surprising, they appear to be new and nontrivial to prove.

For the proof we need some lemmas. The first lemma is well-known and follows from the strong Markov property.

**Lemma 2.3.** Let \( \tau_0 = 0 \) and define inductively

\[
\tau_n = \inf\{ t \geq \tau_{n-1} : |W_t - W_{\tau_{n-1}}| = 1 \}, \quad 1 \leq n \leq N.
\]

Then \( \{\tau_n\}_{n=1}^N \) is an increasing sequence of stopping times and \( (\Delta \tau_n, \Delta W_n)_{n=1}^N \) is an i.i.d. sequence of random vectors, where

\[
\Delta \tau_n = \tau_n - \tau_{n-1}, \quad \Delta W_n = W_{\tau_n} - W_{\tau_{n-1}}, \quad 1 \leq n \leq N.
\]

Moreover \( (\Delta W_n)_{n=1}^N \) is a Rademacher sequence adapted to \( (\mathcal{F}_{\tau_n})_{n=1}^N \).
The next lemma gives some important properties of the independent Brownian motion $\tilde{W}$ at random times. Such stopped Brownian motions $\tilde{W}$ are not Gaussian random variables in general, but in this case they inherit some important properties.

**Lemma 2.4.** For $1 \leq n \leq N$, let $\Delta \tilde{W}_n = \tilde{W}_{\tau_n} - \tilde{W}_{\tau_{n-1}}$. Then $(\Delta \tilde{W}_n)_{n=1}^N$ is an i.i.d. sequence of symmetric random variables, which is independent of $(\Delta W_n)_{n=1}^N$. Furthermore, each $\Delta W_n$ has finite moments of all orders.

**Proof.** For all $1 \leq n \leq N$, $\Delta \tilde{W}_n$ is symmetric, because $\Delta \tilde{W}_n(\omega, \cdot)$ is symmetric for each $\omega \in \Omega$. It follows from the strong Markov property of $(W, \tilde{W})$ that $(\Delta W_n, \Delta \tilde{W}_n)_{n=1}^N$ is an i.i.d. sequence. So in order to prove the independence of $(\Delta W_n)_{n=1}^N$ and $(\Delta \tilde{W}_n)_{n=1}^N$, it is enough to show that $\Delta W_1 = W_{\tau_1}$ and $\Delta \tilde{W}_1 = \tilde{W}_{\tau_1}$ are independent. The following argument is shown to us by Tuomas Hytönen. For every Brownian motion $B$ on $\Omega$ we introduce the following two stopping times:

$$\tau_{\pm}^B = \inf\{t \geq 0 : B_t = \pm 1\}.$$ 

Note that $\tau_1 = \tau_-^W \wedge \tau_+^W$ and for the Brownian motion $-W$, we have $\tau_-^W = \tau_-^W$ and $\tau_+^W = \tau_+^W$. Let $B \in \mathbb{R}$ be some Borel measurable set. Since $(W, \tilde{W})$ is identically distributed with $(-W, W)$ it follows that

$$\mathbb{P}(W_{\tau_1} = 1, \tilde{W}_{\tau_1} \in B) = \mathbb{P}(\tau_-^W < \tau_+^W, \tilde{W}_{\tau_1} \in B) = \mathbb{P}(\tau_-^W < \tau_+^W, \tilde{W}_{\tau_1} \in B) = \mathbb{P}(W_{\tau_1} = 1, \tilde{W}_{\tau_1} \in B).$$

Clearly,

$$\mathbb{P}(W_{\tau_1} = 1, \tilde{W}_{\tau_1} \in B) + \mathbb{P}(W_{\tau_1} = -1, \tilde{W}_{\tau_1} \in B) = \mathbb{P}(\tilde{W}_{\tau_1} \in B).$$

Hence

$$\mathbb{P}(W_{\tau_1} = 1, \tilde{W}_{\tau_1} \in B) = \frac{1}{2} \mathbb{P}(\tilde{W}_{\tau_1} \in B) = \mathbb{P}(W_{\tau_1} = 1) \mathbb{P}(\tilde{W}_{\tau_1} \in B).$$

The same holds for $-1$. This proves the independence.

For $0 < p < \infty$ we have

$$\mathbb{E} \mathbb{E}\left|\Delta \tilde{W}_n\right|^p = \mathbb{E} \mathbb{E}\left|\tilde{W}_{\tau_1}\right|^p = g_p\mathbb{E} \tau_1^{p/2},$$

where $g_p$ is the $p$-th moment of a standard Gaussian random variable and the statement follows from the elementary fact that $\tau_1$ has finite moments of all orders.

□

Below we will consider adapted and measurable processes $\phi : [0, \infty) \times \Omega \to E$ that take values in a finite-dimensional subspace of $E$. Since $n$-dimensional subspaces of $E$ are isomorphic to $\mathbb{R}^n$, one may construct the stochastic integral for such processes $\phi$ that satisfy $t \mapsto \phi(t, \omega) \in L^2(0, \infty, E)$ for almost all $\omega \in \Omega$. By the Burkholder-Davis-Gundy inequalities we have for all $p \in (1, \infty)$ and for $\phi$ as above, $\int_0^\infty \phi(t) \, d\tilde{W}(t) \in L^p(\Omega; E)$ if $\phi \in L^p(\Omega; L^2(0, \infty; E))$. In this case the decoupled stochastic integral $\int_0^\infty \phi(t) \, d\tilde{W}(t)$ is defined pathwise as an element of $L^p(\Omega; L^p(\tilde{W}; E))$. Moreover, if (2.1) or (2.2) holds for all elementary processes one may extend this to all processes as above. In fact, Garling proved (2.1) and (2.2) for this class of processes.

The next lemma is a variation of an example in [5]. We include a proof for convenience.
Lemma 2.5. Let $E = c_0$ and $p \in [1, \infty)$. There does not exist a constant $c_p > 0$ such that for all elementary processes $\phi$, (2.1) holds.

Proof. Assume there exists a constant $c_p > 0$ such that for all elementary processes $\phi$, (2.1) holds. Then we may extend (2.1) to all measurable and adapted processes $\phi \in L^p(\Omega; L^2(0, \infty; E))$ that take values in a finite-dimensional subspace of $E$. For each $N \geq 1$, we will construct a process $\phi$ as above and such that

$$
\left( \mathbb{E} \left\| \int_0^\infty \phi(t) \, dW(t) \right\|^p \right)^{1/p} = N \quad \text{and} \quad \left( \mathbb{E} \left\| \int_0^\infty \phi(t) \, dW(t) \right\|^p \right)^{1/p} \leq K_p \sqrt{N}.
$$

Here $K_p > 0$ is some universal constant. This gives a contradiction.

We modify an example in [5] in such a way that the martingale differences arise as stochastic integrals. We use the notation of Lemmas 2.3 and 2.4. Fix an integer $N \geq 1$. Let $D = \{-1, 1\}^N$, and for each $e = (e_n)_{n=1}^N \in D$ define the process $\phi_e : [0, \infty) \times \Omega \to \mathbb{R}$ by

$$
\phi_e(t) = \begin{cases} 
    e_n 1_{A_{e,n}} & \text{for } t \in (\tau_{n-1}, \tau_n], \ n = 1, \ldots, N, \\
    0 & \text{for } t = 0 \text{ or } t > \tau_N,
\end{cases}
$$

where $A_{e,1} = \Omega$ and for $2 \leq n \leq N$,

$$
A_{e,n} = \{\Delta W_1 = e_1, \ldots, \Delta W_{n-1} = e_{n-1}\}.
$$

Then each $\phi_e$ is stochastically integrable with

$$
\int_0^\infty \phi_e(t) \, dW(t) = \sum_{n=1}^N \Delta W_n e_n 1_{A_{e,n}}.
$$

Define $\phi : [0, \infty) \times \Omega \to l^\infty(D)$ by $\phi = (\phi_e)_{e \in D}$. Then $\phi$ is stochastically integrable and for almost all $\omega \in \Omega$ and $e \in D$ we have $\left| \left( \int_0^\infty \phi(t) \, dW(t) \right)(\omega)(e) \right| \leq N$. For almost all $\omega \in \Omega$ and $e = (\Delta W_n(\omega))_{n=1}^N$ we have $\left| \left( \int_0^\infty \phi(t) \, dW(t) \right)(\omega)(e) \right| = N$. This shows that

$$
\left( \mathbb{E} \left\| \int_0^\infty \phi(t) \, dW(t) \int_{l^\infty(D)}^p \right\rangle \right)^{1/p} = N, \quad \text{for all } p \in [1, \infty).
$$

On the other hand, we have

$$
\int_0^\infty \phi(t) \, d\tilde{W}(t) = \sum_{n=1}^N \Delta \tilde{W}_n v_n,
$$

where for $1 \leq n \leq N$, $v_n = (e_n 1_{A_{e,n}})_{e \in D}$.

For $\omega \in \Omega$ and $e \in D$ let $k(\omega, e)$ be 0 if $\Delta W_1(\omega) \neq e_1$ and let $k(\omega, e)$ be the maximum of all integers $n \leq N$ such that $\Delta W_i(\omega) = e_i$ for all $i \leq n$ if $\Delta W_1(\omega) = e_1$. For almost all $\omega \in \Omega$ and for all $e \in D$, $\left( \sum_{n=1}^N \Delta \tilde{W}_n v_n \right)(\omega)(e)$ is equal to

$$
-\Delta \tilde{W}_{k(\omega, e)+1}(\omega, \cdot) \Delta W_{k(\omega, e)+1}(\omega) + \sum_{n=1}^{k(\omega, e)} \Delta \tilde{W}_n(\omega, \cdot) \Delta W_n(\omega), \quad \text{if } k(\omega, e) < N,
$$

$$
\sum_{n=1}^N \Delta \tilde{W}_n(\omega, \cdot) \Delta W_n(\omega), \quad \text{if } k(\omega, e) = N.
$$

Of course we have for all $k \leq N$,

$$
-\tilde{W}_k \Delta W_k + \sum_{n=1}^{k-1} \Delta \tilde{W}_n \Delta W_n = 2 \sum_{n=1}^{k-1} \Delta \tilde{W}_n \Delta W_n - \sum_{n=1}^{k} \Delta \tilde{W}_n \Delta W_n.
$$
We obtain that for almost all $\omega \in \Omega$,
\[
\left\| \int_0^\infty \phi(t, \omega) \, d\tilde{W}(t) \right\|_{l^\infty(D)} \leq 3 \sup_{k \leq N} \left\| \sum_{n=1}^k \Delta \tilde{W}_n(\omega, \cdot) \Delta W_n(\omega) \right\|.
\]
Since for almost all $\omega \in \Omega$, $(\Delta \tilde{W}_n(\omega, \cdot))_{n=1}^N$ is a sequence of independent centered Gaussian random variables on $\tilde{\Omega}$, we have by the Lévy-Octaviani inequalities for independent symmetric random variables (see [9, Section 1.1]) for almost all $\omega \in \Omega$,
\[
\tilde{E} \sup_{k \leq N} \left\| \sum_{n=1}^k \Delta \tilde{W}_n(\omega, \cdot) \Delta W_n(\omega) \right\| \leq 2^p \tilde{E} \left| \sum_{n=1}^N \Delta \tilde{W}_n(\omega, \cdot) \Delta W_n(\omega) \right|^p
\]
\[
= 2^p \tilde{E} \sum_{n=1}^N \Delta \tilde{W}_n(\omega, \cdot) \Delta W_n(\omega) \left| \Delta \tilde{W}_n(\omega, \cdot) \right| \left| \Delta W_n(\omega) \right| = 2^p \tilde{E} \left| \Delta \tilde{W}_n(\omega, \cdot) \right|^p = 2^p g_p^p \tau N(\omega)^{p/2}.
\]
Here $g_p^p$ is the $p$-th moment of a standard Gaussian random variable. We may conclude that
\[
\left( \tilde{E} \left\| \int_0^\infty \phi(t) \, d\tilde{W}(t) \right\|_{l^\infty(D)} \right)^{1/p} \leq 6 g_p(\tau N^{p/2})^{1/p}.
\]
Recall that the sequence $(\tau_n - \tau_{n-1})_{n=1}^N$ is identically distributed. For $p = 2$ we obtain
\[
(\tilde{E} \tau_N^{p/2})^{1/p} = (\tilde{E} \tau_N)^{1/2} = \left( \tilde{E} \sum_{n=1}^N \tau_n - \tau_{n-1} \right)^{1/2}
\]
\[
= \left( \sum_{n=1}^N \tilde{E} (\tau_n - \tau_{n-1}) \right)^{1/2} = \left( \sum_{n=1}^N \tilde{E} \tau_1 \right)^{1/2} = \sqrt{N} \sqrt{\tilde{E} \tau_1}.
\]
For $1 \leq p < 2$ we have by Hölder’s inequality,
\[
(\tilde{E} \tau_N^{p/2})^{1/p} \leq (\tilde{E} \tau_N)^{1/2} = \sqrt{N} \sqrt{\tilde{E} \tau_1}.
\]
Finally for $p > 2$, by the triangle inequality in $L^{p/2}(\Omega)$,
\[
(\tilde{E} \tau_N^{p/2})^{1/p} = \left( \tilde{E} \left( \sum_{n=1}^N \tau_n - \tau_{n-1} \right)^{p/2} \right)^{1/p} \leq \left( \sum_{n=1}^N \left( \tilde{E} (\tau_n - \tau_{n-1})^{p/2} \right)^{2/p} \right)^{1/2}
\]
\[
= \left( \sum_{n=1}^N \tilde{E} \tau_1^{p/2} \right)^{1/2} = \sqrt{N} (\tilde{E} \tau_1^{p/2})^{1/p}.
\]
By Lemma 2.4 this proves that for all $p \in [1, \infty)$ and some universal constant $K_p$,
\[
\left( \tilde{E} \left\| \int_0^\infty \phi(t) \, d\tilde{W}(t) \right\| \right)^{1/p} \leq K_p \sqrt{N}.
\]
Since $l^\infty(D)$ can be identified isometrically with a finite-dimensional subspace of $c_0$, this completes the proof. \hfill \Box

**Corollary 2.6.** Let $E$ be a Banach space. If there exists a constant $c_p > 0$ such that for all elementary processes (2.1) holds, then $E$ has finite cotype.

**Proof.** It follows from the above example that $c_0$ is not finitely representable in $E$. Hence the Maurey-Pisier Theorem (see [12]) implies that $E$ has finite cotype. \hfill \Box
Proof of Theorem 2.2. We may assume that the martingale starts at zero \( \text{(see [1, Remark 1.1])} \). Let \( (r_n)_{n=1}^N \) be a Rademacher sequence on the probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) and let \( (d_n)_{n=1}^N \) be an \( E \)-valued martingale difference sequence with respect to the filtration \( (\sigma(r_1, r_2, \ldots, r_n))_{n=0}^N \). We may write \( d_n = r_n f_n(r_1, \ldots, r_{n-1}) \) for \( n = 1, \ldots, N \), for some \( f_n : \{-1, 1\}^{n-1} \to E \). Let \( (\tilde{r}_n)_{n=1}^N \) be a Rademacher sequence on the probability space \( (\Omega, \mathcal{F}, \tilde{\mathbb{P}}) \).

(1): We will show that there exists a constant \( C_p^* > 0 \) only depending on \( E \) such that

\[
\mathbb{E} \left\| \sum_{n=1}^N d_n \right\|^p \leq (C_p^*)^p \mathbb{E} \tilde{\mathbb{E}} \left\| \sum_{n=1}^N \tilde{r}_n d_n \right\|^p.
\]

We use the notation of Lemmas 2.3 and 2.4. Define a process \( \phi \) by

\[
\phi(t) = \begin{cases} 0 & \text{if } t = 0 \text{ or } t > \tau_N, \\
 f_n(\Delta W_1, \ldots, \Delta W_{n-1}) & \text{if } t \in (\tau_{n-1}, \tau_n), \ n = 1, \ldots, N \\
 \end{cases}
\]

The process \( \phi \) is stochastically integrable and we have

\[
\mathbb{E} \left\| \int_0^\infty \phi(t) \, dW(t) \right\|^p = \mathbb{E} \left\| \sum_{n=1}^N \Delta W_n f_n(\Delta W_1, \ldots, \Delta W_{n-1}) \right\|^p
\]

\[
= \mathbb{E} \left\| \sum_{n=1}^N \tilde{r}_n f_n(r_1, \ldots, r_{n-1}) \right\|^p = \mathbb{E} \left\| \sum_{n=1}^N d_n \right\|^p.
\]

Also, we have

\[
\mathbb{E} \tilde{\mathbb{E}} \left\| \int_0^\infty \phi(t) \, d\tilde{W}(t) \right\|^p = \mathbb{E} \tilde{\mathbb{E}} \left\| \sum_{n=1}^N \Delta \tilde{W}_n f_n(\Delta W_1, \ldots, \Delta W_{n-1}) \right\|^p.
\]

By Lemma 2.4, Corollary 2.6 and [10, Proposition 9.14], we have

\[
\mathbb{E} \tilde{\mathbb{E}} \left\| \sum_{n=1}^N \Delta \tilde{W}_n x_n \right\|^p \leq K_p \mathbb{E} \tilde{\mathbb{E}} \left\| \sum_{n=1}^N \tilde{r}_n x_n \right\|^p,
\]

where \( (x_n)_{n=1}^N \) is a sequence in \( E \) and \( K_p > 0 \) is some constant depending only on \( E \) and \( p \). By conditioning (cf. [8, Lemma 3.11]) this result extends to

\[
\mathbb{E} \tilde{\mathbb{E}} \left\| \sum_{n=1}^N \Delta \tilde{W}_n X_n \right\|^p \leq K_p \mathbb{E} \tilde{\mathbb{E}} \left\| \sum_{n=1}^N \tilde{r}_n X_n \right\|^p,
\]

where \( (X_n)_{n=1}^N \) is a sequence of \( E \)-valued random variables independent of \( (\Delta \tilde{W}_n)_{n=1}^N \) and independent of \( (\tilde{r}_n)_{n=1}^N \). By Lemmas 2.3 and 2.4, we may apply \( (2.4) \) to the random variables \( X_n = f_n(\Delta W_1, \ldots, \Delta W_{n-1}) \) for \( 1 \leq n \leq N \) to obtain:

\[
\mathbb{E} \tilde{\mathbb{E}} \left\| \sum_{n=1}^N \Delta \tilde{W}_n f_n(\Delta W_1, \ldots, \Delta W_{n-1}) \right\|^p \leq K_p \mathbb{E} \tilde{\mathbb{E}} \left\| \sum_{n=1}^N \tilde{r}_n f_n(\Delta W_1, \ldots, \Delta W_{n-1}) \right\|^p
\]

\[
= K_p \mathbb{E} \tilde{\mathbb{E}} \left\| \sum_{n=1}^N \tilde{r}_n f_n(r_1, \ldots, r_{n-1}) \right\|^p \overset{\text{(ii)}}{=} K_p \mathbb{E} \tilde{\mathbb{E}} \left\| \sum_{n=1}^N \tilde{r}_n r_n f_n(r_1, \ldots, r_{n-1}) \right\|^p
\]

\[
= K_p \mathbb{E} \tilde{\mathbb{E}} \left\| \sum_{n=1}^N \tilde{r}_n d_n \right\|^p.
\]
In (i), we used that \((r_1, \ldots, r_N, \tilde{r}_1, \ldots, \tilde{r}_N)\) and \((r_1, \ldots, r_N, \tilde{r}_1, \ldots, \tilde{r}_N)\) are identically distributed. By assumption we have
\[
\mathbb{E} \left\| \int_0^\infty \phi(t) \, dW(t) \right\|_p \leq c_p \mathbb{E} \left\| \int_0^\infty \phi(t) \, d\tilde{W}(t) \right\|_p.
\]
We may conclude that (2.3) holds with constant \(c_pK_p\).

(2): We will show that there exists a constant \(C'_p > 0\) only depending on \(E\) such that
\[
(2.5) \quad \mathbb{E} \left\| \sum_{n=1}^N \tilde{r}_n d_n \right\|_p \leq (C'_p)^p \mathbb{E} \left\| \sum_{n=1}^N d_n \right\|_p.
\]
Let \(\phi\) be as before. By Lemmas 2.3, 2.4 and [10, Lemma 4.5] and the same arguments as before we have
\[
\begin{align*}
\mathbb{E} \left\| \sum_{n=1}^N \tilde{r}_n d_n \right\|_p &= \mathbb{E} \left\| \sum_{n=1}^N \tilde{r}_n f_n(r_1, \ldots, r_{n-1}) \right\|_p \\
&= \mathbb{E} \left\| \sum_{n=1}^N \tilde{r}_n f_n(\Delta W_1, \ldots, \Delta W_{n-1}) \right\|_p \\
&\leq \frac{1}{(\mathbb{E} |W_1|)^p} \mathbb{E} \left\| \sum_{n=1}^N \Delta \tilde{W}_n f_n(\Delta W_1, \ldots, \Delta W_{n-1}) \right\|_p.
\end{align*}
\]
By assumption we have
\[
\mathbb{E} \left\| \int_0^\infty \phi(t) \, d\tilde{W}(t) \right\|_p \leq c_p \mathbb{E} \left\| \int_0^\infty \phi(t) \, dW(t) \right\|_p.
\]
We may conclude that (2.5) holds with constant \(\frac{c_p}{\mathbb{E} |W_1|}\). \(\square\)

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