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SYSTEMS FOR AUTOMATIC COMPUTATION AND PLOTTING OF POSITION FIXING PATTERNS

BY

IR. H. PH. VAN DER SCHAAF †

Chief Engineer, Rijkswaterstaat

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Itr. H. PH. VAN DER SCHAAF †

Chief Engineer, Rijkswaterstaat

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Any correspondence should be addressed to

RIJKSWATERSTAAT
DIRECTIE WATERHUISHOUING EN WATERBEWEGING
THE HAGUE — NETHERLANDS

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The work begun by Ir. H. Ph. van der Schraaf will be continued by Ir. C. W. Corbet.
The views in this report are the author's own.
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Synopsis

In the past ten years there has been a growing tendency to use hyperbolic, circular and arc of circle patterns for position fixing on water-covered areas. Rijkswaterstaat Communications No. 2 (see bibl. 1) discusses the computations associated with the Decca pattern for the Netherlands Delta Project as one of the first examples of the use of electronic computers by the Survey Department of the ‘Rijkswaterstaat’. * The introduction of automatic plotting equipment has now made it possible to reduce manual computing and plotting work to an absolute minimum. Under the supervision of Prof. Dr. D. Eckhart, acting as our consultant on computerization problems, a complete software package has been developed for the computation and plotting of position fixing patterns. The systems on which this software is based are described in this issue of ‘Communications’.

In the simplest systems, the depiction of the earth’s surface by a system of map projection is assumed to provide an undistorted representation, thus enabling plane hyperbolic, circular and arc of circle patterns to be used. This mathematical model must, however, be refined if the inaccuracies inherent in the type of projection used are to be taken into account. The plane pattern (see bibl. 1), or alternatively the accompanying grid or graticule, may be modified for this purpose. Systems of this kind are described, not only with regard to the chart patterns but also with regard to the relationship between the chart and the transmitted patterns together with the calibration necessary to determine the pattern constants.

* Netherlands State Water Board
Introduction

Pattern charts are used to fix the position of ships graphically. Position lines are traced on them, i.e. loci corresponding to the readings which would be obtained with position fixing instruments at the same location. A position pattern consists theoretically of an infinite number of such position lines; in practice the presentation is confined to a specific pattern of selected readings or points marked at predetermined intervals.

Since the position is determined by the intersection of two position lines a pattern chart must contain at least two specific patterns. And since a position is normally expressed in rectangular or geographical coordinates, it is necessary to show either a 'grid' of coordinate intersections based on round values for $X$ and $Y$ coordinates in the local system of rectangular coordinates or on a 'graticule' of intersections between parallels and meridians based on round values for $\phi$ and $\lambda$.

Position patterns are therefore plotted against a predetermined background consisting either of a grid or of a graticule. Position fixing as such is effected by graphic interpolation between the plotted position lines, which are based on round values. The rectangular or geographical coordinates of the position can then be read off. Without pattern charts these position fixing systems would entail numerical derivation of the position readings in terms of $X$ and $Y$ or $\phi$ and $\lambda$.

Specific patterns can be plotted for the different position fixing systems on the basis of a simple geometrical relationship between the position lines. The following possibilities may be used:

A. **Hyperbolic patterns**, based on confocal hyperbola branches in the plane. These patterns are used, for example, in conjunction with the Decca or Hi-Fix position fixing systems, the readings being taken either on 'decometers' in the case of Decca or on counters in the case of Hi-Fix.

B. **Circular patterns**, based on concentric circles, used for decometer or counter readings in the two-range Decca or Hi-Fix system. This pattern is also used whenever distances are measured from fixed points; this is done electronically when radio-log, tellurometer, geodimeter or the Cubic and Hydrolites systems are used.

C. **Arc of circle patterns**, made up of a series of arcs of a circle for sextant readings at fixed sighting points.

These three types of position fixing patterns are particularly suitable for automatic plotting because of the mathematical relationship between the position lines in a single pattern. Automatic plotting will be considered in detail in this issue of Rijkswaterstaat Communications. It is, therefore, appropriate that a brief review of the
development of automatic processes for this particular application should be given in this introduction.

Before the appearance of automatic computing and plotting equipment, the drawing of position patterns on charts was an extremely laborious task. Straight or curved position lines had to be drawn by hand on the paper; the actual plotting work was preceded by equally elaborate manual computing. To draw a straight line the coordinates of at least two points had to be calculated. Compasses could be used to draw arcs of circles if the central point was in a convenient position, but the coordinates of this central point and the radius of the circle had to be calculated. In all other cases the coordinates of large numbers of points had to be calculated and the points plotted on the chart. These points then had to be joined manually using a ruler or spline. The first step towards automation was the use of a computer, which led to an enormous reduction in the time needed for calculations. Because of the mathematical relationship between the coordinates of the points on each individual position line and also between the position lines themselves, this is an application which produces a maximum saving of labour. Once programmes have been written for different types of pattern, each specific pattern requires very little input data.

The next stage in the development of automation was the appearance of the electronically controlled coordinatograph. Once calculated on the computer and reproduced on punched tape or cards, the coordinates of the points referred to above could be used as direct input for the automatic coordinatograph. The needle unit controlled by the input then marked out all these points in succession, but manual tracing was still necessary.

In the automatic plotting unit built in the next stage of development, the needle was replaced by a plotting stylus which automatically connected the points on the chart. The automatic plotting unit could, if desired, be directly controlled by the computer. Rijkswaterstaat Communications No. 2 (see bibl. 1), published in 1960, deals with the method used for preparing the Decca patterns for the Delta Project. It describes one of the first applications of the computer and electronic coordinatograph by the Survey Department of 'Rijkswaterstaat'. Electronic plotting machines were not available commercially at that time. Subsequent developments, however, have made the method described in the above publication obsolete, since, with the equipment available today, the charts could have been plotted even more efficiently.

Allowance was, nevertheless, made for the fact that the plane hyperbolic pattern (type A) does not give a completely accurate representation of the geometric position of points on the earth's surface when the area considered is large. In principle the geographical position lines forming a spherical, or more precisely a spheroidal, hyperbolic pattern on a sphere or ellipsoid have to be represented on the plane surface by means of a particular map projection. The mathematical model of plane, confocal hyperbolic branches then need refining, i.e. the plane hyperbolic branches must be modified. This correction is necessary if, for example, patterns are being plotted for the North Sea Continental Shelf. A general method of doing this is described later.

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In Chapter 6 of Communications No. 2 (see bibl. 1), reference is also made to the difference between transmitted patterns and chart patterns. If the speed of propagation of radio waves were constant, the chart pattern obtained in the manner described above would be a correct representation of the transmitted pattern. In practice, however, the behaviour of radio waves when passing over areas of varying electromagnetic conductivity leads to discrepancies between the two patterns.

Practical interpretation of these discrepancies has to be based on calibration measurements. For the measurements the Decometer readings resulting from the transmitted pattern and specific ship positions are compared with synchronous position fixes. The latter have either been made on land by means of direction or distance measurements from the shore marks known or are based on readings from a different pattern system the accuracy of which is sufficient to be used as a means of calibration.

The above refinements have been developed from experience with hyperbolic position patterns, but they also apply in principle mutatis mutandis, to circular and arc of circle patterns. Up to now, however, there has been no need to apply the refinements to the latter types of pattern because of the much smaller areas of operation in which they are used.

The following are examples of the different types of pattern charts used in the Netherlands. Hyperbolic patterns (type A) were used in the Delta Project (the RWS Delta Chain 0B), the enlargement of the outer harbour at Ymuiden (the Ymuiden Hi-Fix Chain), studies of the Waddenzee along the coast of Groningen (Groningen Hi-Fix Chain), the enlargement of the outer harbour at the Hook of Holland (Europoort Hi-Fix Chain) and of the outer harbour at Scheveningen (Scheveningen Hi-Fix Chain), for control of the northern part of the Continental Shelf (Decca Navigator Chain 9B, Frisian Islands Chain), for navigation through the approach and access channel to Europoort and also for control of the southern part of the Continental Shelf (Decca Navigator Chain 2E Europoort) and recently in the Waddenzee studies carried out between the islands and the Frisian coast and between the Ysselmeer dam and the coast of Noord-Holland (Wadden Sea-Fix Chain). Circle pattern charts (type B) have been used in laying a sewage outlet pipe in the sea at Scheveningen building harbour moles at the Hook of Holland and building the dam in the Brouwershavense Gat.

Arc of circle patterns (type C) for sextant readings have been used in laying underwater mains in the Western Scheldt, building the Benelux Tunnel, the construction of the Hellegat dam, laying mains under the Hollandsch Diep, building the Heinenoord Tunnel and for the dumping of gravel in the Nieuwe Waterweg.
Figure 1. Types of position patterns
1. Definition of the specific position fixing pattern and chart sheet

1.1. The plane hyperbolic pattern

On a plane surface a hyperbola is an infinite number of points having an identical difference in distance from two given points, the focal points. By varying the differences in distance it is possible to arrange the points on the plane surface in accordance with a set of an infinite number of hyperbolae. This set is therefore defined by the position of the focal points.

A specific hyperbola in the set can be indicated by the difference in distance mentioned above or alternatively by the value assumed by a given function of this difference in distance. In either case it is essential to distinguish between positive and negative differences in distance so that the values can be considered as hyperbolic coordinates relating to the correct branches of the hyperbolic system.

In the hyperbolic position fixing system, the focal points are formed by the master \( M \) and slave \( S_I \) stations; the hyperbolic coordinate is the lane number \( L \), which can be read off on a deecometer. The relationship between such a coordinate \( L_i \) of the point \( S_i \) (see fig. 1) and the difference in distance \( S_iM - SI S_i \) is expressed by the formula

\[
L_i = \frac{S_iM + MSI - SI S_i}{\lambda} + k
\]

If we define \( S_iM = a_i \), \( MSI = b \) (base), \( SI S_i = c_i \) and \( \lambda = l \) (where \( \lambda \) is the wavelength), then

\[
L_i = l(a_i + b - c_i) + k
\]

in which \( l \) is a multiplicative constant and \( k \) an additive constant (index correction).

When the earth is considered as a flat surface, the general definition of this plane hyperbolic pattern is given by the rectangular coordinates \( X_M, Y_M \) and \( X_{SI}, Y_{SI} \) in a Cartesian system of terrestrial coordinates and by the constants \( l \) and \( k \), while the specific definition is obtained by choosing a series of discrete values for \( L \), e.g. by taking equal values for an interval \( \Delta L \).

1.2. The circular pattern

A circle is the loci of points situated at an equal distance from a given point. The set of circles forming a circular pattern is defined by the common central point \( M \) with the coordinates \( X_M, Y_M \) (see figure 1).
Discrete values for the radii \( r \) of these concentric circles again produce a specific pattern. The parameter with which a particular circle in the pattern is identified is generally a function of the radius. Since the distance from the fixed point \( M \) is often determined electronically, the concept of lane number \( L \) is again used, in linear relationship to the radius \( r \); thus:

\[
L_i = 2lr_i + k
\]

where \( l = \frac{1}{\lambda} \) is a multiplicative constant and \( k \) an additive constant (index correction). If the reading for \( L \) corresponds to the radius \( r \) in metres, then \( l = 1/2 \) and \( k = 0 \).

1.3. The arc of circle pattern

The arc of a circle is a set of points characterized by an identical difference in direction from two given points.

When the arc of circle pattern (see figure 1) is used, the given points \( L \) and \( R \) form the directional points for sextant measurements. The measured angle \( LSR = \gamma \) has a specific arc of circle as position line. In geometrical terms a particular angle \( \gamma \) would give two arcs of circle but by introducing the convention that the angle \( \gamma \) is identical to the azimuth from \( S \) to \( R \), less the azimuth from \( S \) to \( L \), one of these arcs of circle is eliminated. For a point \( S_i \) it follows that \( \angle LSR = \gamma_i \) (see fig. 1). This sequence of points runs anticlockwise. If the locus excluded in this way is referred to, the order in which \( L \) and \( R \) are mentioned is reversed.

Since the actual differences in direction, and not a function of them, are read off on the circle graduation of the sextant, the general definition of the pattern is given by the terrestrial coordinates of the sighting points \( X_L, Y_L; X_R, Y_R \). The specific definition is obtained by choosing discrete values for \( \gamma \), e.g. by taking equal values for an interval \( \Delta \gamma \).

1.4. Definition of the chart sheet

The limits of a section of the specific fixing pattern to be represented on a chart sheet are shown in figure 1. The problem is simply to plot this section to a given scale on the chart. Its position must therefore also be defined.

It should be possible to give the \( X \) and \( Y \) coordinates of the angular points of this area, but if it forms a rectangle it will be sufficient to work with the length \( p \) and width \( q \) of the rectangle, together with the \( X \) and \( Y \) coordinates of one of the angular points and the azimuth of one of the sides.
In this way two limiting factors are arrived at, the first relating to the values of the parameters used to define the position lines that are to be drawn on the chart and the second to the section or sections of each position line to be represented.

1.5. Conclusion

The position pattern and chart sheet have now been defined, while the geometrical solution to the problem has been given. However, for the elaboration of the computational and plotting techniques an analytic solution of the problems involved is needed.
2. The consequences of using an automatic plotting unit

The design of the automatic plotting unit influences the computing and plotting system to be used. The explanation given below refers to an automatic unit in which a plotting stylus is moved by step-type motors over a drawing table. This movement takes place in steps in eight principal directions which can be distinguished with respect to the azimuth they make with the \( y \)-axis of the coordinate system of the instrument (automatic plotting units are also available with 16 and 24 operating directions). The \( y \)-axis motor generates steps in the \(+y\) or \(-y\) direction, the \( x \)-motor steps perpendicular to it, i.e. in the \(+x\) or \(-x\) direction. Simultaneous actuation of both motors causes the plotting styles to travel diagonally. Figure 2 illustrates these eight principal directions. The step size is fixed; a particular automatic plotting unit may, for example be based on a step of 0.1 mm.

![Figure 2. Principal directions in automatic plotting](image)

The drawing stylus is therefore driven by motors which in their turn are actuated by impulses generated in the computer. If the automatic plotting unit is coupled directly to the computer (on-line system), the resulting chart is to be considered as the computer output. If there is no direct connection (off-line system) a system must be designed to use the output results obtained from the computer as the input for the automatic plotting unit. In both cases the stylus must receive successive orders to move a specific number of steps in one of the main directions. The computer must select both the
direction and the number of steps. The computer also gives the order to lower the stylus onto the drawing table or raise it from the table.

Figure 3 is an enlarged illustration of a network of squares having a side length of 0.1 mm. Assuming that the stylus is located at position 1, 14 steps in the \( +x \) direction bring it from 1 to 2 and 14 steps in the \( +y \) axis from 1 to 3, while 14 combined \( +x \) and \( +y \) steps bring it from 1 to 4. The lines thus described by the stylus would be 1-2, 1-3 and 1-4.

An example of a movement which does not coincide with one of the main directions would be from 5 to 6, entailing a total \( x \)-movement of 14 steps and an \( y \)-movement of 2 steps. The line from 5 to 6 can, however, only be drawn by the stylus in basic steps in one of the main directions, in other words the stylus must travel across a polygon. The computer must be programmed to produce the series of elementary steps. Such a programme can be based on the following system.

At the beginning of each step the direction which gives the shortest distance from the finishing point of the step to the line connecting the starting and finishing points of the polygon is selected from the main directions which would bring the finishing point of the step closer to the finishing point of the polygon to be described.

The result of this selection is demonstrated for different directions in figure 3, which shows clearly that a choice always has to be made between not more than two pos-
sibilities. The result is that zig-zag connection lines are obtained whose zig-zag pattern varies with the azimuth of the line to be represented. If the steps are sufficiently small this raises no practical problems.

Similarly it would be possible to develop systems which approximate curves of the second or higher power and to work out appropriate programmes for them. In the present publication we shall only be concerned with the method discussed above, viz. approximating the straight line connecting two points that are given in $x$- and $y$-coordinates.

The system of operation described has the following implication for the design of computing and plotting systems for position patterns:

1. A position line must be built up from chords, although the arc is the geometrical form with which we are actually concerned. The distance between the chord and arc must therefore not exceed the margin of plotting accuracy.

2. The points on the arc serving as the ends of these chords must be calculated in the system of plotting coordinates, viz. $x$-$y$ system. As the first stage of the calculation it is therefore logical to convert the coordinates of points indicated in the terrestrial $X$-$Y$ system, into coordinates of points in the $x$-$y$ system.

![Figure 4. The terrestrial and chart coordinate system](image.png)
Taking into account the location of the chart concerned, e.g. by the $X$, $Y$ coordinates of one angular point and the azimuth of one of the sides, for which in figure 4 point $I$ and the side $I-4$ have been chosen, the conversion formulae are as follows:

\[
\begin{align*}
x &= (X - X_I) \cos \overline{I.4} - (Y - Y_I) \sin \overline{I.4} \\
y &= (Y - Y_I) \cos \overline{I.4} + (X - X_I) \sin \overline{I.4}
\end{align*}
\]

(2)

Using this coordinate conversion, the methods of calculation to be developed can also be based on the $x,y$ coordinate system of the automatic plotting unit, the origin of which is considered to coincide with the angular point $I$ of the chart.
3. Elements of system analysis for the specific plane hyperbolic chart pattern

3.1. Introduction of parameters for points on a position line

A position line in a specific pattern is defined by the value allocated to the parameter $L$. For successive points on line $L_i$, $x,y$ coordinates must be calculated to enable the plotting stylus to draw the connecting lines. We therefore need a parameter by means of which successive points $S_{i,j}$ can be determined unambiguously on line $L_i$. It is not sufficient to use the $x$ or $y$ coordinate because the possibility exists that line $x=constant$ and line $y=constant$ have two points of intersection with line $L_i$. As an appropriate parameter we therefore introduce one of the coordinates based on the oblique angled system of axes $\xi-\eta$, formed by the asymptotes of the hyperbola. The origin of this system is centre $O$ of base $M\,SI$, and the equation for the hyperbola is:

$$\xi\eta = \frac{b^2}{16}$$

(3)

For positive values of $\xi$ and $\eta$ the hyperbola section is indicated to which the lane number $L$ is added. The expression $\frac{b^2}{16}$ is a pattern constant illustrating that the shape of the pattern is solely dependent on $b$. All position lines have the same equation in $\xi$ and $\eta$. Either $\xi$ or $\eta$ can be used as a parameter to indicate a point on the position line. Each position line has a separate $\xi, \eta$ system. These systems vary with the angle $\alpha$ between the asymptotes and the base $MSI$. The relationship between the system $\xi_i, \eta_i$ defined by angle $\alpha_i$ and lane number $L_i$ is expressed by the formula:

$$\cos \alpha_i = \frac{b-(L_i-k)\lambda}{b}$$

(4)

where $\alpha_i < 200^\circ$. The conversion of a point $S_{i,j}$ given in $\alpha_i, \xi_{i,j}, \eta_{i,j}$ into $x_{i,j}, y_{i,j}$ is effected by means of the expressions:

$$x_{s_{i,j}} = x_o + \xi_{i,j} \sin (\bar{OM} + \alpha_i) + \eta_{i,j} \sin (\bar{OM} - \alpha_i)$$

(5a)

$$y_{s_{i,j}} = y_o + \xi_{i,j} \cos (\bar{OM} + \alpha_i) + \eta_{i,j} \cos (\bar{OM} - \alpha_i)$$

(5b)

It is necessary to introduce into these conversion formulae (5) either increasing values
for $\eta$, in which case the $\zeta_{i,j}$ values corresponding to $\eta_{i,j}$ are obtained from (3), or increasing values for $\xi$, the $\eta_{i,j}$ values corresponding to $\xi_{i,j}$ again being obtained from (3).

Figure 5 shows that for line $L_i$ with $\eta$ as the parameter, the plotting stylus must be brought onto the edge of the sheet at point $A$. The line from $A$ to $B$ must then be drawn and the stylus must be raised again at point $B$. If $\xi$ is used as the parameter, line $L_i$ would run in the opposite direction, i.e. from $B$ to $A$. 
3.2. Calculating the parameters for the points of intersection of a position line with the chart edge

The first problem is to determine the values of $\xi$ and $\eta$ which apply to the points of intersection between a position line $L_i$ and the chart edge. Figure 6 shows that there may be 0, 2, 4 or 6 of these points of intersection, depending on the lane number $L_i$ and on the position of the chart with respect to the pattern. Points of tangency are irrelevant for our purposes.

If, in accordance with figure 5, the terrestrial distance between angle points 1 and 2 or 3 and 4 is $p$, and the distance between angle points 1 and 4 or 2 and 3 is $q$, this implies analytical testing in 2 phases:

1. Does the hyperbola branch $L_i$ have real points of intersection with the 4 lines: $x=0$, $x=p$, $y=0$ and $y=q$?

2. If so, for the first 2 lines with $x=$ constant the condition must be met that $0 \leq y \leq q$ and for the last 2 lines with $y=$ constant that $0 \leq x \leq p$.

In the situation illustrated in figure 5, line $L_i$ accordingly has no real points of intersection with the line $x=0$, but two such intersections with each of the other lines; the first examination therefore gives 6 points of intersection, of which, however, only two comply with the second criterion of the test.

Figure 6. Possible variations in the number of intersections between lanes and the edges of the chart
The following system of formulae gives us the result in terms of $\xi$ and $\eta$. If in (3) we substitute from (5a):

$$\xi_i = \frac{(x-x_o) - \eta_i \sin (\Omega M - \alpha_i)}{\sin (\Omega M + \alpha_i)}$$

we obtain the quadratic equation:

$$\{\sin (\Omega M - \alpha_i)\} \eta_i^2 - (x-x_o)\eta_i + \frac{b^2}{16} \sin (\Omega M + \alpha_i) = 0$$  \hspace{1cm} (6a)

and from (5b) in (3):

$$\xi_i = \frac{(y-y_o) - \eta_i \cos (\Omega M - \alpha_i)}{\cos (\Omega M + \alpha_i)}$$

we obtain the quadratic equation:

$$\{\cos (\Omega M - \alpha_i)\} \eta_i^2 - (y-y_o)\eta_i + \frac{b^2}{16} \cos (\Omega M + \alpha_i) = 0$$  \hspace{1cm} (6b)

The first test is an examination of the discriminant of equation (6a):

$$D_x = (x-x_o)^2 - \frac{b^2}{4} \sin (\Omega M + \alpha_i) \sin (\Omega M - \alpha_i) > 0$$  \hspace{1cm} (7a)

with $x=0$ or $x=p$.

and for equation (6b):

$$D_y = (y-y_o)^2 - \frac{b^2}{4} \cos (\Omega M + \alpha_i) \cos (\Omega M - \alpha_i) > 0$$  \hspace{1cm} (7b)

with $y=0$ or $y=q$.

The second test is applied in two phases to the equations remaining from the first test.

The first criterion is that the $\eta$ roots must be positive, i.e.

$$\eta = \frac{(x-x_o) + \sqrt{D_x}}{2 \sin (\Omega M - \alpha_i)} > 0 \hspace{1cm} \text{for } x=0, \text{ or } x=p$$

$$\eta = \frac{(x-x_o) - \sqrt{D_x}}{2 \sin (\Omega M - \alpha_i)} > 0 \hspace{1cm} \text{for } x=0, \text{ or } x=p$$
The positive roots $\eta_i$ obtained in this way are supplemented by means of (3) with the appropriate $\xi_i$ values and substituted in the conversion formula (5b), for the lines $x=$ constant, or (5a), for the lines $y=$ constant.

The result of this latter phase is tested against the second criterion giving 0, 2, 4 or 6 values for $\xi_i$, $\eta_i$ which results in points on the edge of the sheet. They must now be sorted into pairs by order of magnitude for $\eta_i$ or $\xi_i$; the smallest value for each pair shows the beginning of the line to be drawn and the largest value the end of the line. It is therefore necessary to plot 0, 1, 2 or 3 arcs for which beginning and end, but not the intermediate, points have been ascertained.

### 3.3. Selecting the position lines on a chart sheet

We are now in a position to determine the method for selecting the position lines which must appear on a particular chart sheet. Formula (1) is used to calculate lane number $L_m$, e.g. for the centre of the chart sheet. As the initial value $L_S$ we take the highest multiple of $\Delta L$ included in $L_m$. After processing $L_s$ we go on to lane $L_S + \Delta L$; if $L_S$ was calculated with $\eta$ as the parameter, lane $L_S + \Delta L$ is calculated with $\xi$ as the parameter etc. We now add $\Delta L$ to the lane number again and again until either the maximum lane number of the pattern is exceeded or an examination of the points of intersection shows no more intersections.

We then turn to $L_S - \Delta L$; we subtract $\Delta L$ from the previous lane number until either the minimum lane number for the pattern is passed or an examination of the points of intersection shows no more intersections.

### 3.4. Calculating intermediate points on the position lines to be plotted

The problem that remains is to determine the values $\eta$ or $\xi$ for the points between $A$ and $B$ of the lane $L_i$ to be plotted. That in each case the distance between the chord and the corresponding arc may not exceed the plotting accuracy $t$, for example $t = 0.1$ mm, is taken as a criterion. This distance represents a distance of $s$. $t$ in the terrain if the chart scale is $1 : s$. 

\[
\eta = \frac{(y - y_o) + \sqrt{D_y}}{2 \cos (OM - \alpha_i)} > 0 \quad \text{for } y = 0, \text{ or } y = q
\]

\[
\eta = \frac{(y - y_o) - \sqrt{D_y}}{2 \cos (OM - \alpha_i)} > 0 \quad \text{for } y = 0, \text{ or } y = q
\]
Figure 7 shows an extreme example of the first chord from \( A \) to point \( I \). The requirement is to determine \( \eta_I \) from \( \eta_A \) in such a way that the above tolerance is not exceeded. At point \( R \) where the tangent to the arc runs parallel with the chord from \( A \) to point \( I \), the maximum distance between the chord and the arc is reached. Chord and tangent are extended to the asymptotes, which they reach at points \( P_k \) and \( Q_k \), and \( P_r \) and \( Q_r \), respectively. The area of the trapezoid \( P_kQ_kQ_rP_r \) is determined by the formula:

\[
A = \frac{b^2 \sin \alpha \cos \alpha}{16} \left( \frac{\eta_I}{\eta_A} + \frac{\eta_A}{\eta_I} - 2 \right)
\]  

(8)
If we assume this area to be equal to

\[ A = \frac{P_k Q_k + P_r Q_r}{2} \times s \times t \]

we obtain a complicated equation for \( \eta_1 \).
The minimum variable distance which could occur in the latter expression is \( b \sin \alpha \), i.e. for \( P',Q',r \), being the tangent perpendicular to the base \( MSL \), with the point of contact \( R' \) on the position line.
The area \( A \) in equation (8) is therefore certainly larger than \( stb \sin \alpha \). If we introduce this value in the left-hand term, \( \eta_1 \) will always be too small as an unknown value, i.e. it will always fall within the specified tolerance. We now take the expression \( \nu' = \frac{\eta_1}{\eta_A} \) as the unknown. Equation (8) now becomes:

\[ \frac{b^2 \sin \alpha \cos \alpha}{16} \left( \nu' + \frac{1}{\nu'} - 2 \right) = stb \sin \alpha \]

or:

\[ \nu + \frac{1}{\nu'} - 2 - \frac{16 st}{b \cos \alpha} = 0 \]

Hence

\[ \nu' = \left(1 + \frac{8 st}{b \cos \alpha}\right) \pm \sqrt{\left(1 + \frac{8 st}{b \cos \alpha}\right)^2 - 1} \]

The product of the two roots is 1 and in our case \( \nu' > 1 \), so that:

\[ \nu' = \left(1 + \frac{8 st}{b \cos \alpha}\right) + \sqrt{\left(1 + \frac{8 st}{b \cos \alpha}\right)^2 - 1} \] (9)

where \( \nu' \) is the factor by which \( \eta_A \) must be multiplied to give \( \eta_1 \).
This factor is a constant for lane \( L_i \) so that for the following intermediate point 2:

\[ \eta_2 = \nu' \times \eta_1 = (\nu')^2 \times \eta_A \]

For the final intermediate point \( n \) we obtain:

\[ \eta_n = \nu' \times \eta_{n-1} = (\nu')^n \times \eta_A \]

This final intermediate point must be linked with \( B \)
It is, however, more elegant to divide the whole arc from $A$ to $B$ with $n$ intermediate points in such a way that:

$$
\eta_B = v^{n+1} \eta_A
$$

(10)

We then obtain in the opposite direction:

$$
\xi_A = v^{n+1} \xi_B
$$

(10a)

so that the same number of intermediate points $n$ is reached. The multiplicative constant $v'$ must be decreased to $v$ in such a way that equation (10) is satisfied.

For this purpose $v'$, calculated in accordance with formula (9), is considered as an approximation of $v$.

If we now assume

$$
\eta_B = (v')^{n+1} \eta_A
$$

it follows for $n+1$ that:

$$
(n+1) = \frac{\log \eta_B - \log \eta_A}{\log v'}
$$

(11)

In general $(n+1)$ is an improper fraction; the entier of which (i.e. the greatest integer not exceeding $n+1$) must be introduced as $n$.

After determining $n$, $v$ follows from:

$$
\log v = \frac{\log \eta_B - \log \eta_A}{n+1}
$$

(12)

Finally, the point parameters $\eta$ from $A$ to $B$ or $\xi$ from $B$ to $A$ form a geometrical series with a ratio $v > 1$.

3.5. Summary of input and formulae to be programmed on the basis of the system analysis

3.5.1. Input constants

Pattern: $X_M, Y_M, X_{Sl}, Y_{Sl}, l, k$. 

25
Chart area: \( X_1, Y_1, \bar{4}, p, q \).
Plotting: \( s, \Delta L, t \).

3.5.2. Preliminary calculations

Transformation:

\[
\begin{align*}
    x &= (X - X_1) \cos \bar{4} - (Y - Y_1) \sin \bar{4} \\
    y &= (Y - Y_1) \cos \bar{4} + (X - X_1) \sin \bar{4}
\end{align*}
\]

for the points \( M \) and \( Sl \).
Middle of the base:

\[
\begin{align*}
    x_o &= \frac{x_M + x_{Sl}}{2} \\
    y_o &= \frac{y_M + y_{Sl}}{2}
\end{align*}
\]

Chart centre:

\[
\begin{align*}
    x_c &= \frac{p}{2} \\
    y_c &= \frac{q}{2}
\end{align*}
\]

Distances:

\[
\begin{align*}
    b &= MSl = +\sqrt{(x_M - x_{Sl})^2 + (y_M - y_{Sl})^2} \\
    MC &= +\sqrt{(x_M - x_c)^2 + (y_M - y_c)^2} \\
    SI/C &= +\sqrt{(x_{Sl} - x_c)^2 + (y_{Sl} - y_c)^2}
\end{align*}
\]

Initial lane number:

\[
L_S = \Delta L \text{ entier} \left( \frac{1(b + MC - SI/C) + k}{\Delta L} \right)
\]
Constant $c$:

$$c = \frac{b^2}{16}$$

Direction $\overline{OM}$:

$$\overline{OM} = \arctan \left( \frac{x_M - x_o}{y_M - y_o} \right)$$

3.5.3. Limitation of lane calculation $L_s$

Constant $\alpha$:

$$\cos \alpha = \frac{bl - (L_s - k)}{bl} \quad (\alpha < 200^\circ)$$

(4)

a. Points of intersection with 1,2 : $y=0$

Discriminant $D_y$:

$$D_y = (y - y_o)^2 - 4c \cos (\overline{OM} + \alpha) \cos (\overline{OM} - \alpha)$$

(7b)

General condition: $D_y > 0$

Roots:

$$\eta = \frac{(y - y_o) \pm \sqrt{D_y}}{2 \cos (\overline{OM} - \alpha)}$$

Condition: $\eta > 0$

Corresponding $\xi$:

$$\xi = \frac{c}{\eta}$$

Corresponding $x$:

$$x = x_o + \xi \sin (\overline{OM} + \alpha) + \eta \sin (\overline{OM} - \alpha)$$

(5a)

Condition:

$$0 < x < p$$
b. Points of intersection with 3,4: \( y = q \)
   as under a but with \( y = q \).
c. Points of intersection with 1,4: \( x = 0 \)
   Discriminant \( D_x \):
   \[
   D_x = (x-x_o)^2 - 4c \sin (\overline{OM}+x) \sin (\overline{OM}-x)
   \]  
   (7a)
   General condition:
   \[ D_x > 0 \]
   Roots:
   \[
   \eta = \frac{(x-x_o) + \sqrt{D_x}}{2 \sin (\overline{OM}-x)}
   \]
   Condition:
   \[ \eta > 0 \]
   Corresponding \( \xi \):
   \[
   \xi = \frac{c}{\eta}
   \]
   Corresponding \( y \):
   \[
   y = y_o + \xi \cos (\overline{OM}+x) + \eta \cos (\overline{OM}-x)
   \]  
   (5b)
   Condition:
   \[ 0 < y < q \]
d. Points of intersection with 2,3: \( x = p \)
   as under c but with \( x = p \)
e. Results of limitation:
   Sorting the permissible \( \eta \) roots in order of magnitude, in pairs \( \eta_A, \eta_B \).

3.5.4. Calculating intermediate points for plotting

Formulae:

\[
x = x_o + \xi \sin (\overline{OM}+x) + \eta \sin (\overline{OM}-x)
\]  
(5a)
\[
y = y_o + \xi \cos (\overline{OM}+x) + \eta \cos (\overline{OM}-x)
\]  
(5b)
Determination of \( v' \):

\[
v' = \left( 1 + \frac{8st}{b \cos \alpha} \right) + \sqrt{\left( 1 + \frac{8st}{b \cos \alpha} \right)^2 - 1}
\]  \hspace{1cm} (9)

Number of intermediate points:

\[
n = \text{entier} \left( \frac{\log \eta_B - \log \eta_A}{\log v'} \right)
\]  \hspace{1cm} (11)

Determination of \( v \):

\[
\log v = \frac{\log \eta_B - \log \eta_A}{n+1}
\]  \hspace{1cm} (12)

To be substituted in formulae 5a and 5b:

starting point: \( \eta_A \), \( \xi_A = \frac{c}{\eta_A} \)

1st intermediate point: \( \eta_1 = \eta \), \( \xi_1 = \frac{c}{v \eta} \)

2nd intermediate point: \( \eta_2 = \eta_1 \), \( \xi_2 = \frac{c}{v \eta_1} \)

nth intermediate point: \( \eta_n = \eta_{n-1} \), \( \xi_n = \frac{c}{v \eta_{n-1}} \)

end point: \( \eta_B \), \( \xi_B = \frac{c}{\eta_B} \)

3.5.5. Calculating 1st sequence of lanes

Start:

\[
L = L_s + \Delta L
\]

Calculation as in 3.5.3. but sorting mentioned in 3.5.3\(^*\) in accordance with order of magnitude of pairs \( \xi_A, \xi_B \), then as in 3.5.4.

General:

\[
L_{j+1} = L_j + \Delta L
\]
\[ L_k = L_{k-1} + \Delta L, \text{ if the results e) for } L_{k+1} = L_k + \Delta L \text{ gives no value for } \eta, \xi. \]

Sorting is carried out alternately in accordance with 3.5.3.e) and 3.5.5.e).

3.5.6. Calculating 2nd sequence of lanes

Start:

\[ L = L_s - \Delta L \]

Calculation: as in 3.5.5.

General:

\[ L_{j+1} = L_j - \Delta L \]

End:

\[ L_k = L_{k-1} - \Delta L, \text{ if the result e) for } L_{k+1} = L_k - \Delta L \text{ gives no values for } \eta \text{ and } \xi. \]
4. Elements of system analysis for plotting the specific circular and arc of circle patterns

The arc of circle pattern must be considered as a variant of the circular pattern. The principle of the circular pattern system is that concentric circles are plotted which are split up into arcs of circle at the points where the circles intersect the edges of the chart. In arc of circle patterns only arcs of circle are plotted, the centres of which vary. The edges of the chart often make it necessary to split these arcs up again. Here too it is possible to start by plotting one circular position line provided that it is situated within the area of the chart.

4.1. The parameter $\psi$ for points on a circle (on arc of circle)

It is obvious to introduce a system of polar coordinates, $r$, $\psi$, to calculate the coordinates of points on a circle with centre $M$, indicated by the coordinates $x_M$, $y_M$. The circle is then determined by the choice of the radius $r_i$, for which, in accordance with 1.2,

$$r_i = \frac{L_i - k}{2l}$$

(13)

A point $j$ on this circle is determined by the azimuth $\psi_j$ of the radius vector $r_i$. This azimuth $\psi$ therefore serves as a point parameter. We then obtain:

$$x_{i,j} = x_M + r_i \sin \psi_j$$

$$y_{i,j} = y_M + r_i \cos \psi_j$$

(14)

4.2. Calculating the point parameters for the points of intersection of a circle with the edge of the chart

The equation for a circle with radius $r$ in the $x,y$ chart coordinate system is as follows:

$$(x - x_M)^2 + (y - y_M)^2 = r^2$$
The coordinates for the points of intersection with the lines $x=0$, $x=p$, $y=0$ and $y=q$ are to be calculated. It is useful to observe a specific sequence for this purpose. This sequence, together with the conditions with which the results must comply, is given in the following formulae.

\begin{align*}
S_1: x_1 &= 0, \ y_1 = y_M + \sqrt{r^2 - x_M^2} \\
S_2: x_2 &= x_M - \sqrt{r^2 - (q - y_M)^2}, \ y_2 = 0 \\
S_3: x_3 &= x_M + \sqrt{r^2 - (q - y_M)^2}, \ y_3 = 0 \\
S_4: x_4 &= p, \ y_4 = y_M + \sqrt{r^2 - (p - x_M)^2} \\
S_5: x_5 &= p, \ y_5 = y_M - \sqrt{r^2 - (p - x_M)^2} \\
S_6: x_6 &= x_M + \sqrt{r^2 - y_M^2}, \ y_6 = 0 \\
S_7: x_7 &= x_M - \sqrt{r^2 - y_M^2}, \ y_7 = 0 \\
S_8: x_8 &= 0, \ y_8 = y_M + \sqrt{r^2 - x_M^2}
\end{align*}

<table>
<thead>
<tr>
<th>Sequence</th>
<th>1st test</th>
<th>2nd test</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_1$</td>
<td>$r^2 &gt; x_M^2$</td>
<td>$q \geq y_1 \geq 0$</td>
</tr>
<tr>
<td>$S_2$</td>
<td>$r^2 &gt; (q - y_M)^2$</td>
<td>$p \geq x_2 \geq 0$</td>
</tr>
<tr>
<td>$S_3$</td>
<td>$r^2 &gt; (q - y_M)^2$</td>
<td>$p \geq x_3 \geq 0$</td>
</tr>
<tr>
<td>$S_4$</td>
<td>$r^2 &gt; (p - x_M)^2$</td>
<td>$q \geq y_4 \geq 0$</td>
</tr>
<tr>
<td>$S_5$</td>
<td>$r^2 &gt; (p - x_M)^2$</td>
<td>$q \geq y_5 \geq 0$</td>
</tr>
<tr>
<td>$S_6$</td>
<td>$r^2 &gt; y_M^2$</td>
<td>$p \geq x_6 \geq 0$</td>
</tr>
<tr>
<td>$S_7$</td>
<td>$r^2 &gt; y_M^2$</td>
<td>$p \geq x_7 \geq 0$</td>
</tr>
<tr>
<td>$S_8$</td>
<td>$r^2 &gt; x_M^2$</td>
<td>$q \geq y_8 \geq 0$</td>
</tr>
</tbody>
</table>

Figure 8. Possible variations in the number of intersections between circles and the edges of the chart
Since the circle is a closed curve, 8, 6, 4, 2 or 0 roots, corresponding to an equivalent number of real points of intersection, satisfy the requirements. Figure 8 shows the various possibilities.

As regards the circular pattern, the arcs to be plotted are indicated by the roots which satisfy both tests, arranged in pairs according to the numerical order of the point $S$ concerned, and beginning with the lowest odd number.

If there are no real points of intersection the circle concerned may be situated inside or outside the chart. In the former case it will be plotted but in the latter case (outside the chart) this is unnecessary.

This means that if there are no valid roots, a further test must be carried out to determine whether the condition $x_M + r \leq p$ is fulfilled to enable plotting of this position line.

The valid points of intersection are expressed in point parameters $\psi$ according to the formula:

$$\psi_{MS} = \arctan \left( \frac{x_S - x_M}{y_S - y_M} \right)$$  \hspace{1cm} (15)

4.3. Calculating the point parameters for the starting and end points of the position lines to be plotted in the case of an arc of circle pattern

In the arc of circle pattern $x_M$ and $y_M$ are not given; these coordinates are calculated from:

$$r = \frac{LR}{2 \sin \gamma}$$  \hspace{1cm} (16)

$$\begin{align*}
x_M &= x_L + r \sin (\psi_{LR} + \frac{1}{2} \pi - \gamma) \\
y_M &= y_L + r \cos (\psi_{LR} + \frac{1}{2} \pi - \gamma)
\end{align*}$$  \hspace{1cm} (17)

The result for $\psi$ obtained by formula (15) in accordance with 4.2, serve as a basis for selecting the part of the circle to be plotted, running from $\psi_{MR}$ to $\psi_{ML}$. If $\psi_{MR} < \psi_{ML}$, the point parameters will increase from $\psi_{MR}$ to $\psi_{ML}$. If $\psi_{MR} > \psi_{ML}$, increasing point parameters are obtained by splitting up the arc from $\psi_{MR}$ to $2\pi$ and from 0 to $\psi_{ML}$.

The criterion for plotting is that the arcs or sections of arcs, found in accordance with 4.2, will only be accepted if they are located within the range indicated. This means that the point $L$ may serve as the end point and the point $R$ as the starting point of an arc which is to be plotted; this is illustrated in figure 9.
In this way each arc of circle to be plotted is therefore fixed by starting point $A$ having parameter $\psi_{MA}$ and by the end point $B$ having parameter $\psi_{MB}$.

If plotting is effected in a clockwise direction, a positive $\Delta \psi$ is introduced for the intermediate points. If plotting is carried out in an anti-clockwise direction, $\Delta \psi$ is negative and we start at point $B$ and finish at point $A$. An alternating plotting direction can be programmed.

4.4. Calculation of intermediate points $K$ on the arcs of circle to be plotted

Variations in the azimuth $\psi_{MK}$ follow an interval $\Delta \psi$. This produces a distance on the chart for which a tolerance $t$ can be introduced.

In figure 10 this distance is indicated by the geographical distance $s.t$. A conservative value for $\Delta \psi'$ applicable to $\Delta \psi$ is derived as follows.

From

$$\frac{PK}{s \cdot t} = \frac{PQ}{PK} = \frac{2r}{PK}$$

we obtain

$$PK = \sqrt{2r \cdot s \cdot t} \quad \text{while} \quad PK \approx \frac{r \cdot \Delta \psi'}{2}$$

so that:

$$\Delta \psi' = \sqrt{\frac{8st}{r}}$$

In order to divide the arc of circle $AB$ into equal parts for plotting purposes, the number of intermediate points $n$ can be taken as:
Figure 10. Approximation of plotting arcs

\[ n = \text{entier} \left\{ \frac{\psi_{MB} - \psi_{MA}}{\sqrt{\frac{8St}{r}}} \right\} \] (18)

so that the interval is determined by:

\[ \Delta \psi = \frac{\psi_{MB} - \psi_{MA}}{n + 1} \]

The coordinates are calculated in accordance with formulae 14.

4.5. Selecting the position lines on a chart sheet

We begin by calculating the line parameter for the centre of the chart, obtaining a
value $L_c$ for the circular pattern and a value $\gamma_c$ for the arc of circle pattern. Plotting begins either with the initial value $L_s$, which is the largest multiple of $\Delta L$ included in $L_c$, or at $\gamma_s$, which is the largest multiple of $\Delta \gamma$ included in $\gamma_c$. Further processing is effected as described in 3.3.

4.6. Summary of the input and formulae to be programmed on the basis of system analysis in the case of a circular pattern

4.6.1. Input constants

Pattern: $X_M$, $Y_M$, $l$, $k$
Chart area: $X_1$, $Y_1$, $1.4$, $p$, $q$
Plotting: $s$, $\Delta L$, $t$

4.6.2. Preliminary calculations

Conversion:

\[ x_M = (X_M - X_1) \cos 1.4 - (Y_M - Y_1) \sin 1.4 \]
\[ y_M = (Y_M - Y_1) \cos 1.4 + (X_M - X_1) \sin 1.4 \]

Chart centres:

\[ x_c = \frac{p}{2} \]
\[ y_c = \frac{q}{2} \]

Initial line parameter:

\[ L_s = \Delta L \ \text{entier} \left( \frac{2l \sqrt{(x_c - x_M)^2 + (y_c - y_M)^2 + k}}{\Delta L} \right) \]

Initial radius:

\[ r_s = \frac{L_s - k}{2l} \]
4.6.3. **Limitation of line calculation** \( L_S \)

Test: from points of intersection \( S_i \) to \( S_8 \) in accordance with 4.2. with \( r = r_S \) and with no points of intersection for \( x_M + r_S \leq p \)

Sorting of the permissible intersection points \( S \): in pairs, odd numbered points \( S = A \) with immediately following even numbered point \( S = B \).

Result of limitation: express coordinates for sorted points \( A \) and \( B \) in their point parameter \( \psi \) by substitution in the formula

\[
\psi = \arctan \left( \frac{x - x_M}{\gamma - y_M} \right)
\]

and in the case of a complete circle introduce:

\[
\psi_{MA} = 0, \psi_{MB} = 2\pi
\]

4.6.4. **Calculating intermediate points for plotting**

Formulae:

\[
x = x_M + r_S \sin \psi \]
\[
y = y_M + r_S \cos \psi
\]

Number of intermediate points:

\[
n = \text{entier} \left( \frac{\psi_{MB} - \psi_{MA}}{\frac{8\pi t}{r_s}} \right)
\]

Determination of \( \Delta \psi \):

\[
\Delta \psi = \frac{\psi_{MB} - \psi_{MA}}{n + 1}
\]

Substitute in formulae (14): starting point: \( \psi_{MA} \)
1st intermediate point: \( \psi_{MA} + \Delta \psi \)
2nd intermediate point: \( \psi_{MA} + 2\Delta \psi \)
\( n \)th intermediate point: \( \psi_{MA} + n\Delta \psi \)
end point: \( \psi_{MB} \)
4.6.5. Calculating the 1st sequence of position lines

Start: \( L = L_s + \Delta L \)
Calculation as for \( L_s \), but sorting in sequence \( BA \)
General: \( L_{j+1} = L_j + \Delta L \)
End: \( L_k = L_{k-1} + \Delta L \) if the results in accordance with 4.6.3. for: \( L_{k+1} = L_k + \Delta L \) yield no values for \( \psi \).
Sorting is effected alternately in sequence \( AB \) and \( BA \).

4.6.6. Calculating the 2nd sequence of position lines

Start: \( L = L_s - \Delta L \)
Calculation: as in 4.6.5.
General: \( L_{j+1} = L_j - \Delta L \)
End: as in 4.6.5.

4.7. Summary of input and formulae to be programmed on the basis of system analysis in the case of an arc of circle pattern

4.7.1. Input constants

Pattern: \( X_L, Y_L, X_R, Y_R \)
Chart area: \( X_1, Y_1, \bar{1.4}, p, q \)
Plotting: \( s, \Delta \gamma, t \)

4.7.2. Preliminary calculations

Conversion:

\[
x = (X - X_1) \cos \bar{1.4} - (Y - Y_1) \sin \bar{1.4}
\]
\[
y = (Y - Y_1) \cos \bar{1.4} + (X - X_1) \sin \bar{1.4}
\]
for points \( L \) and \( R \)
Distance \( LR \):

\[
LR = \sqrt{(X_R - X_L)^2 + (Y_R - Y_L)^2}
\]
Azimuth $LR$:

$$\psi_{LR} = \arctan \left( \frac{x_R - x_L}{y_R - y_L} \right)$$

Chart centres:

$$x_c = \frac{p}{2}$$

$$y_c = \frac{q}{2}$$

Initial line parameter:

$$\gamma_s = \Delta \gamma \text{ entier} \left( \frac{\arctan \left( \frac{x_L - x_c}{y_L - y_c} \right) - \arctan \left( \frac{x_R - x_c}{y_R - y_c} \right)}{\Delta \gamma} \right)$$

Initial radius:

$$r_s = \frac{LR}{2 \sin \gamma_s}$$

Coordinates of initial centre point:

$$x_{M_s} = x_L + r_s \sin \left( \psi_{LR} + \frac{1}{2} \pi - \gamma_s \right)$$

$$y_{M_s} = y_L + r_s \cos \left( \psi_{LR} + \frac{1}{2} \pi - \gamma_s \right)$$

4.7.3. Limitation of line calculation

Output data: results of limitation in accordance with 4.6.3.

Calculation of $\psi_{MR}$ and $\psi_{ML}$:

$$\psi_{MR} = \arctan \frac{x_R - x_M}{y_R - y_M}$$

$$\psi_{ML} = \arctan \frac{x_L - x_M}{y_L - y_M}$$
Determining the range of the position line:

1st $\psi_{MR} < \psi_{ML}$ range of point parameters $\psi_{MR} \rightarrow \psi_{ML}$

2nd $\psi_{MR} > \psi_{ML}$ range of point parameters $\psi_{MR} \rightarrow 2\pi \rightarrow 0 \rightarrow \psi_{ML}$

Selecting of the $S$ points within this range:
Compare $\psi_{MS}$ with this range and check whether a discarded $A$ point can be replaced by point $R$ or a discarded $B$ point by point $L$.
Results of limitation: indicate the starting point of an arc of circle to be plotted with $\psi_{MA}$ and the end point with $\psi_{MB}$.

4.7.4. Calculating intermediate points for plotting

See 4.6.4.

4.7.5. Calculating 1st and 2nd sequence of position lines

As in 4.6.5. and 4.6.6. with substitution of $\gamma$ for $L$. 
5. Influence on the hyperbolic chart pattern, if the earth is considered as a sphere or as an ellipsoid

5.1. General considerations

In the case of the plane, hyperbolic pattern the distances \( a_i, b \) and \( c_i \) are calculated from formula (1) according to the location of the points in the plane of the chart projection. If \( n \) is the number of lanes in the pattern, then \( l = \frac{n}{2b} \).

We now write lane number \( L_i \) according to this pattern as:

\[
L_i = \frac{n(a_i + b - c_i)}{2b} + k
\]  
(19)

The mathematical model for the pattern is refined by locating it on an earth considered as a sphere or ellipsoid. We then obtain a spherical or spheroidal pattern, for which the formula for the lane number \( \hat{L}_i \) is:

\[
\hat{L}_i = \frac{n(a_i + \hat{b} - \hat{c}_i)}{2\hat{b}} + k
\]  
(20)

in which the distances \( a_i, b \) and \( c_i \) are taken over the surface of the sphere or ellipsoid.

The general constants which define this pattern are the geographical coordinates of the Master and Slave stations: \( \lambda_M, \varphi_M, \lambda_S, \varphi_S \), the multiplicative constant \( T = \frac{n}{2b} \) and the additive constant \( k \).

Figure 11 shows the difference between \( \bar{L} \) and \( \hat{L} \) for points in the Ymuiden Hi-Fix pattern and figure 12 for points in the Frisian Islands Decca pattern, \( \bar{L} \) being calculated in accordance with formula (19) and \( \hat{L} \) in accordance with formula (20) for distances on the ellipsoid.
Figure 11. Ymuiden Hi-Fix pattern, stereographic projection
<table>
<thead>
<tr>
<th>Point</th>
<th>( L_N )</th>
<th>( \hat{L}_N )</th>
<th>( L_Z )</th>
<th>( \hat{L}_Z )</th>
<th>( \Delta L = \hat{L} - L )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>175.568</td>
<td>175.572</td>
<td>333.235</td>
<td>333.235</td>
<td>-0.004</td>
</tr>
<tr>
<td>2.</td>
<td>248.112</td>
<td>248.114</td>
<td>283.855</td>
<td>283.855</td>
<td>0.000</td>
</tr>
<tr>
<td>3.</td>
<td>312.825</td>
<td>312.826</td>
<td>226.803</td>
<td>226.804</td>
<td>-0.001</td>
</tr>
<tr>
<td>4.</td>
<td>125.631</td>
<td>125.633</td>
<td>341.910</td>
<td>341.910</td>
<td>0.000</td>
</tr>
<tr>
<td>5.</td>
<td>235.515</td>
<td>235.516</td>
<td>243.453</td>
<td>243.454</td>
<td>-0.001</td>
</tr>
<tr>
<td>6.</td>
<td>338.403</td>
<td>338.403</td>
<td>168.951</td>
<td>168.952</td>
<td>0.000</td>
</tr>
<tr>
<td>7.</td>
<td>34.989</td>
<td>34.990</td>
<td>222.071</td>
<td>222.072</td>
<td>-0.001</td>
</tr>
<tr>
<td>8.</td>
<td>158.908</td>
<td>158.909</td>
<td>88.907</td>
<td>88.908</td>
<td>-0.001</td>
</tr>
<tr>
<td>9.</td>
<td>346.319</td>
<td>346.320</td>
<td>79.209</td>
<td>79.210</td>
<td>-0.001</td>
</tr>
</tbody>
</table>
Figure 12. Decca patterns of the Frisian Islands Chain, U.T.M. projection
| Point | \( \lambda = 3^\circ \) | \( \varphi = 53^\circ \) | \( \lambda = 3^\circ \) | \( \varphi = 54^\circ \) | \( \lambda = 3^\circ \) | \( \varphi = 55^\circ \) | \( \lambda = 4^\circ \) | \( \varphi = 53^\circ \) | \( \lambda = 4^\circ \) | \( \varphi = 54^\circ \) | \( \lambda = 4^\circ \) | \( \varphi = 55^\circ \) | \( \lambda = 5^\circ \) | \( \varphi = 53^\circ \) | \( \lambda = 5^\circ \) | \( \varphi = 54^\circ \) | \( \lambda = 5^\circ \) | \( \varphi = 55^\circ \) | \( \lambda = 6^\circ \) | \( \varphi = 53^\circ \) | \( \lambda = 6^\circ \) | \( \varphi = 54^\circ \) | \( \lambda = 6^\circ \) | \( \varphi = 55^\circ \) |
|-------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| \( L_R \) | 77.135         | \( \widehat{L_R} \) | 77.079         | \( L_R \) | 145.479        | \( \widehat{L_R} \) | 145.474        | \( L_R \) | 226.088        | \( \widehat{L_R} \) | 226.154        | \( L_R \) | 63.336         | \( \widehat{L_R} \) | 63.281         | \( L_R \) | 144.730        | \( \widehat{L_R} \) | 144.727        | \( L_R \) | 242.360        | \( \widehat{L_R} \) | 242.434        | \( L_R \) | 95.305         | \( \widehat{L_R} \) | 95.335         | \( L_R \) | 206.61          | \( \widehat{L_R} \) | 206.149        | \( L_R \) | 90.367         | \( \widehat{L_R} \) | 90.288         |
| \( L_G \) | 277.987        | \( \widehat{L_G} \) | 277.984        | \( L_G \) | 227.008        | \( \widehat{L_G} \) | 226.980        | \( L_G \) | 186.080        | \( \widehat{L_G} \) | 186.024        | \( L_G \) | 144.782        | \( \widehat{L_G} \) | 144.727        | \( L_G \) | 159.663        | \( \widehat{L_G} \) | 159.596        | \( L_G \) | 159.663        | \( \widehat{L_G} \) | 159.596        | \( L_G \) | 126.630        | \( \widehat{L_G} \) | 126.553        | \( L_G \) | 131.860        | \( \widehat{L_G} \) | 131.832        | \( L_G \) | 94.513         | \( \widehat{L_G} \) | 94.466         | \( L_G \) | 90.367         | \( \widehat{L_G} \) | 90.288         |
| \( \Delta \widehat{L} = \widehat{L} - \widehat{L} \) | -0.056          |                         | -0.003          |                         | -0.028          |                         | +0.066          |                         | -0.056          |                         | -0.005          |                         | -0.036          |                         | +0.074          |                         | -0.067          |                         | -0.017          |                         | +0.003          |                         | -0.046          |                         | +0.082          |                         | -0.077          |                         | -0.032          |                         | +0.013          |                         | -0.047          |                         | +0.088          |                         | -0.079          |                         |
In the area of $40 \times 40$ km for which the Ymuiden Chain was designed, these differences, arranged according to the point location, amount to the following values in thousandths of a lane:

<table>
<thead>
<tr>
<th></th>
<th>North pattern</th>
<th>South pattern</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>-1 0 -1</td>
<td>-1 -1 -1</td>
</tr>
<tr>
<td>-2</td>
<td>-1 -1 -1</td>
<td>0 -1 -1</td>
</tr>
<tr>
<td>-4</td>
<td>-2 -1 -1</td>
<td>0 0 -1</td>
</tr>
</tbody>
</table>

These differences are of no real practical significance, because the scale divisions on the Decometers used represent 5 thousandths of a lane.

On the other hand in the $201 \times 228$ km area of the Continental Shelf in accordance with figure 12, we find the following difference:

<table>
<thead>
<tr>
<th></th>
<th>Red pattern</th>
<th>Green pattern</th>
</tr>
</thead>
<tbody>
<tr>
<td>-66</td>
<td>-74 -82</td>
<td>+56 +67 +77</td>
</tr>
<tr>
<td>+5</td>
<td>+3 -13</td>
<td>+28 +36 +46</td>
</tr>
<tr>
<td>+56</td>
<td>+55 +49</td>
<td>+3 +5 +17 +28</td>
</tr>
</tbody>
</table>

In the latter case the conclusion can be drawn that the spheroidal pattern gives a better mathematical model in practice than the plane pattern in the U.T.M. projection.

In the introduction we saw that the chart pattern is superimposed on a grid or graticule in order to make graphic position fixing possible. The chart patterns for the chain can now be used either in the form of plane, hyperbolic patterns and the graticule modified, or alternatively the plane, hyperbolic pattern can be modified and the grid or graticule used in the chart projection chosen is maintained.

The first method has the advantage that the programmes already discussed remain unchanged. The graticule can be easily adjusted as we shall see below. In fact we are now introducing an entirely new chart projection, the principle of which is that spherical or spheroidal hyperbolae are represented as plane hyperbolae.

The second method has the advantage that the chart projection chosen allows undistorted additions to be made to the fixing charts, using data from other charts of the same projection. The modification of plane hyperbolae to obtain position lines in accordance with the projection chosen does, however, require additional programming, which will be discussed below.

For both methods it is necessary to determine the length of the geodetic on the ellipsoid between two points of which the latitude and longitude $\lambda_1$, $\varphi_1$, and $\lambda_2$ and $\varphi_2$ are given (the second principal problem of geodesy). J. C. P. De Kruijff has found an elegant solution to this problem using an iterative programme. The basic principles of his method and the corresponding programme in Algol '60 were published in the Tijdschrift voor Kadaster en Landmeetkunde, June 1967 (see bibl. 2). Applying this method
it is simple to calculate by automatic computer lane number $\hat{L}_i$ using formula (20) for a point $i$, given in geographic coordinates $\lambda_i, \varphi_i$ in relation to a pattern defined in accordance with the general pattern constants $\lambda_M, \varphi_M, \lambda_{SI}, \varphi_{SI}, \bar{l}$ and $k$. Before elaborating the methods indicated above (to be discussed in sections 5.2 and 5.3), it must be pointed out that the plane pattern can be superimposed on an unmodified grid or graticule in the chart projection used, provided that a correction $\Delta L = \hat{L} - \hat{L}$, valid for that particular place, is applied to each reading $\hat{L}$. Because of the principles on which they are based, these corrections will show a regular pattern in the area under consideration. It will be necessary to analyze a sufficient number of points in the area covered (as in figure 12) to compile a general chart on which the density of these corrections is sufficient so that interpolation for practical purposes is justified.

5.2. Modification of the graticule

In the case of the plane hyperbolic pattern, the relative position of the transmitters to each other can be determined in accordance with a traditional chart projection. To plot the graticule, the intersections of the meridians and parallels which are to be shown must first be calculated in spheroidal-hyperbolic coordinates in accordance with formula (20) in lanes $\hat{L}_1$ and $\hat{L}_2$, after which the rectangular coordinates of the point of intersection of lanes $\hat{L}_1 = \hat{L}_1$ and $\hat{L}_2 = \hat{L}_2$ are calculated and plotted in the system of the plane pattern.

This problem of determining $X_S, Y_S$, of which lane number $L_1$ and $L_2$ for point $S$ are given, has already been described on pages 44, 45 and 46 of Rijkswaterstaat Communications No. 2 (see bibl. 1), where an iterative solution is arrived at. Verstelle published a direct solution in the Supplement to the International Hydrographic Review of December 1963 (see bibl. 3). His method has been programmed by the Survey Department of the ‘Rijkswaterstaat’. In actual fact it is Ballarin’s method for a plane surface.

Modification of the graticule by the method indicated above was used for the provisional pattern charts of the Continental Shelf on a scale of 1 : 50000. Since 1970, however, the method described in 5.3 has been used. For practical purposes it is perfectly feasible to use corresponding charts plotted by these two systems in conjunction with one another.

However, the method described in this section only results in a one-to-one correspondence if the relative positions of the transmitters to each other are determined in the following manner instead of by traditional chart projection.

In $M SI, SI_2$ (figure 13) the distances over the ellipsoid are given by $\hat{b}_1, \hat{b}_2$ and $\hat{s}$. For the point $SI_2$ we obtain:
Figure 13. The transmitter positions of Master and Slaves

\[
\hat{L}_1 = n_1 \frac{\hat{b}_1 + \hat{b}_2 - \hat{s}}{b_1} + k_1
\]

\[
\bar{L}_1 = n_1 \frac{\bar{b}_1 + \bar{b}_2 - \bar{s}}{\bar{b}_2} + k_1
\]

For a one-to-one correspondence projection in \( Sl_2 \) the following condition must therefore be met:

\[
\frac{\hat{b}_1 + \hat{b}_2 - \hat{s}}{\hat{b}_1} = \frac{\bar{b}_1 + \bar{b}_2 - \bar{s}}{\bar{b}_1}
\]

And for a one-to-one correspondence projection in \( Sl_1 \) :

\[
\frac{\hat{b}_1 + \hat{b}_2 - \hat{s}}{\hat{b}_2} = \frac{\bar{b}_1 + \bar{b}_2 - \bar{s}}{\bar{b}_2}
\]

This means that for the relative positions in the plane, the following condition must be met:

\[
\hat{b}_1 : \bar{b}_1 = \hat{b}_2 : \bar{b}_2 = \hat{s} : \bar{s}
\]

A further consequence of this is that discrepancies will occur when working out a system with three slaves. As long as these discrepancies remain less than five-thousandths of a lane, they can be disregarded for practical purposes.
5.3. Modification of the plane hyperbolic pattern

In the International Hydrographic Review, vol. XXVII-2-1950, Prof. BALLARIN published an article entitled ‘Geometric properties of position lines in hyperbolic navigation and their lay-out on the reference ellipsoid’. (see bibl. 4). In it the author described a solution to the problem of converting intersections of $L_1, L_2$ into the corresponding geographical position $\phi, \lambda$. If the results $\phi, \lambda$ were to be converted into $X, Y$ by means of the formulae for the chart projection used, this would in principle represent a solution to the problem of plotting points in the spheroidal chart pattern.

In the publication referred to earlier, VERSTELLE advanced certain practical objections to BALLARIN’s method. For the purpose of preparing charts for the Continental Shelf a new method has been worked out and programmed in accordance with ideas developed by the Mathematical Department of I.T.C. The theory of this method is described by KUBIK and others in Rijkswaterstaat Communications No. 12 (see bibl. 7). In the following a short outline of this method and its practical consequences are given.

If we base the method described in 5.2. and 5.3. on the same system of $X$ and $Y$ coordinates, then the transmitters acquire the same $X, Y$ coordinates in both systems of calculation. For a given point $S$ with readings $L_1, L_2$ and coordinates $X, Y$ a different pair of coordinates, $X', Y'$, applicable to point $S'$ in the system of method 5.3 is calculated in the plane pattern, in accordance with $L_1 = \hat{L}_1$ and $L_2 = \hat{L}_2$, using the intersection of lanes $\hat{L}_1$ and $\hat{L}_2$. Figure 14 is a diagram showing both points. This means that corrections $\Delta X'$ and $\Delta Y'$ must be made to the result as calculated in the programme under discussion, in order to correct the location of $S'$ based on

Figure 14. Discrepancy between plane chart pattern and spherical, or spheroidal, chart pattern
plane hyperbolae to give the corresponding position in a system of spheroidal hyperbolae.

For the area covered by the Frisian Islands Chain, shown in figure 12, the position of the given 12 points S in X and Y coordinates was obtained from U.T.M. tables, while the position of the corresponding points S' was determined according to the method described.

These points results in the following $\Delta X'$ and $\Delta Y'$ corrections in metres:

$$
\begin{align*}
X' & : -104 & -118 & -132 & -143 \\
   & - 95 & - 70 & - 55 & - 48 \\
   & -245 & -146 & - 57 & - 33 \\
\Delta Y' & : - 63 & - 47 & - 23 & + 11 \\
           & +  4 & +  2 & -  2 & -  4 \\
           & + 24 & + 36 & + 50 & + 32
\end{align*}
$$

The necessary corrections will show a regular pattern over the area concerned because of the fundamental law on which they are based. This law can be approximated by means of adaptation formulae for $\Delta X'$ and $\Delta Y'$.

We have found that formulae of the following type:

$$
\Delta = a + bx + cy + dx^2 + ey + fy^2 + g x^3 + h x^2 y + i x y^2 + j y^3 +
+ k x^3 y + l x^2 y^2 + m x y^3 + n x^3 y^2 + o x^2 y^3 + p x^3 y^3
$$

are very suitable for this purpose. The 16 coefficients a to p must then be accurately determined from redundant data of $\Delta$, x and y of a large number of reference points distributed regularly over the whole area. The limits of an area for which a given combination of coefficients a to p is valid, are shown by the remaining discrepancies in the reference points.

These discrepancies must be acceptable for plotting on a scale of 1 : s and must not, for example exceed a tolerance $t = s \times 0.1$ mm. The size of the area of application of the method is consequently also dependent on the scale of the chart. The plotting of a spheroidal pattern must therefore be preceded by analysis of a division of the area into blocks in order to establish the coefficients of the adaptation formulae necessary for determining the corrections $\Delta X'$ and $\Delta Y'$ for each sub-division of the area. These coefficients are constants for the formulae that are used for an extension of the programme for plane hyperbolae. Certain additional modifications must also be made to the programme to adapt the recalculated part of the pattern.

The mathematical department of I.T.C. developed these adaptation formulae in the form of third order polynomials to derive a spherical or spheroidal pattern from a plane hyperbolic pattern. This is a special application of the general problem to make a complex relationship between the dependent variables c and d and the independent
variables $a$ and $b$ more manageable in terms of computation technology in accordance with $c=f(a,b)$ and $d=g(a,b)$. Approximation $c'$ and $d'$ for $c$ and $d$ according to less complex relations are taken as the starting point: $c'=f'(a,b)$ and $d'=g'(a,b)$ so that

$$c = c' + \Delta c' \quad \text{and} \quad d = d' + \Delta d'$$

with $\Delta c'$ and $\Delta d'$ as the new unknowns; the relationships between $\Delta c'$ and $\Delta d'$ and $c'$ and $d'$ respectively can in practice be assumed to be in accordance with third order polynomials. A precondition however is that exact values for $\Delta c'$ and $\Delta d'$ can be calculated for $n$ chosen values for $a$ and $b$. The adaptation formulae obtained can then be used for incidental values of $a$ and $b$. An article will be published on this subject, describing experience with applications in a wide range of fields. The first application is the one described in this article.
6. Matching the hyperbolic chart pattern to the transmitted pattern

Model refinement of the chart pattern by replacing distances in the plane by spherical or spheroidal distances in the formula for the lane number leads to a distinct increase in the accuracy of position fixing. Since this refinement is based on mathematical principles it can obviously be programmed. The previous chapter was devoted to a solution of this problem. It is, however, also necessary to ensure that the chart pattern matches the 'transmitted pattern' as closely as possible. The transmitted pattern is a phenomenon of which the equation for lane number \( L' \) can be written as follows:

\[
L' = \int_0^a l_1(s) ds + \int_0^b l_2(s) ds - \int_0^c l_3(s) ds + k
\] (21)

In this formula, \( s \) represents the distance travelled by the wave front. The multiplicative constant \( l \) depends on this distance because \( l \) is related to the velocity of wave propagation. This velocity varies on the one hand with the electro-magnetic conductivity of the terrain over which the waves travel and on the other hand with prevailing atmospheric conditions. If we assume \( l(s) \equiv l \) we obtain formulae (19) and (20), on which the chart pattern was based in the previous chapters.

If the exact position of the ship’s antenna can be determined, using a more accurate method of position fixing, and expressing these positions in rectangular or geographical coordinates, the lane numbers can be derived from it in accordance with the chart pattern used (see chapter 5.2). Comparing these lane numbers with the values of the transmitted pattern read off from the Decometers, discrepancies will occur which can be explained by differences between formulae (19) or (20) and formula (21). The question is how to integrate these differences into the chart pattern, or in other words whether model refinement can be programmed on account of physical phenomena. An obvious step is to attempt to keep the discrepancies as small as possible. This process, which necessitates appropriate adjustment of the pattern constants \( l \) and \( k \), is known as calibration of the pattern. The mathematical techniques involved are described in the following chapter.

The differences between calculated and observed lanes, which remain after calibration, are shown in the histograms in figure 15. These differences are expressed in hundredths of a lane and refer to 223 calibration points of the Hi-Fix pattern for Groningen. In general these discrepancies are positive and negative for both the East and West patterns in the sectors shown in figure 16.

The distribution and location of the differences observed suggest physical causes.
Figure 15. Histogram of discrepancies between chart pattern and transmitted pattern for Groningen Hi-Fix Chain
The matching of the chart pattern obtained after calibration with the transmitted pattern for the Groningen Hi-Fix Chain was done in the manner described below. This method was introduced when the introduction of separate pattern constants \( l \) and \( k \) for each chartsheet proved unsatisfactory in areas where adjoining sheets, on the scale used, failed to link up properly with one another.

The method employed could be described as a variant of the method dealt with in chapter 5.2. The plane hyperbolic pattern is maintained, but the grid is modified. This modification is achieved by the 'Anblock' method as used in photogrammetry and automated using existing programmes of the Survey Department of Rijkswaterstaat. The principles of the method were developed by C. M. A. Van Den Hout (see bibl. 5).

A brief review of these principles in relation to their application for our particular purpose will be useful at this point.

For each calibration point the position is determined in two ways: firstly by theodolite measurements from the shore, giving the terrestrial coordinates \( T_x \) and \( T_y \) and secondly by Decometer readings \( L_1 \) and \( L_2 \) taken on the ship. A transformation of the latter in the plane hyperbolic patterns gives the coordinates \( P_x \) and \( P_y \). A distinction must therefore be made between a terrestrial system \( T_{x,y} \) and a pattern system \( P_{x,y} \).
If the transmitted patterns coincide with the chart patterns for a calibration point, both systems give the same coordinates. A relationship must now be established between the \( T \) system and the \( P \) system, by means of which the grid intersections in the \( T \) system can be transformed into the \( P \) system. This relationship must be flexible and correspond as accurately as possible to local conditions.

The area to be covered is sub-divided into sub-areas (models), having, for example, a size of \( 1 \times 1 \text{ km} \). The coordinate lines of the \( T \) system having round values in km can serve as boundaries of this sub-area system. A block of models \( (15 \times 15 \text{ km}) \) to be adjusted as a whole has 225 models and \( 16 \times 16 = 256 \) model angle points. On a scale of \( 1 : 10000 \) these angle points give a 10 cm grid. Each model has 4 angle points, expressed in \( T_x \) and \( T_y \) coordinates. A calibration point in such a model has two sets of coordinates \( T_x, T_y \) and \( P_x, P_y \). The discrepancy between these coordinates must be eliminated as far as possible. This is done by using linear conversion formulae for each model. For a particular model \( m \) we then have the conversion formulae:

\[
P_x = a_m T_x + b_m T_y + c_m \\
P_y = -b_m T_x + a_m T_y + d_m
\]

(22)

In a block of 225 models there are therefore \( 225 \times 4 = 900 \) unknowns in the form of the transformation constants \( a, b, c \) and \( d \).

\( P_x \) and \( P_y \) are known at the calibration points, but not at the angle points of the model. There are therefore \( 256 \times 2 = 512 \) unknowns in the form of the coordinates \( P_x \) and \( P_y \) for the angle points of the models. This gives a total of \( 900 + 512 = 1412 \) unknowns. For each model 8 equations (22) must be written for the angle points, so that there are in all \( 225 \times 8 = 1800 \) such equations. If there are 40 calibration points in the block, a further \( 40 \times 2 = 80 \) equations are formed. In our example we therefore have 1880 linear equations with 1412 unknowns.

This adjustment problem results in 1412 normal equations with 1412 unknowns, of which only the 512 coordinates \( P_x, P_y \) of the model angle points need to be included in the output of the automation system to enable plottings of these points by the automatic plotter.

For the 58 charts scale \( 1 : 10000 \) and 77 charts scale \( 1 : 5000 \) containing patterns of the Groningen Hi-Fix Chain, 9 blocks were processed in the manner described above. Detailed results obtained for a 1 : 2500 chart used for constructing the dam closing the Lauwerszee are subsequently discussed here as an example (see figure 17). For the area concerned \( (2000 \times 1750 \text{ m}) \) a separate block was formed with models of \( 250 \times 250 \text{ m} \) in accordance with the coordinate intersections indicated in figure 17. This gave us 56 models, which contained 7 calibration positions, indicated by point numbers.

After adjustment by the Anblock method, the differences between the calculated and observed lane numbers proved to have been reduced to an acceptable level. This is shown in the following table. In this table lane number characteristics are: \( P = \) position...
Figure 17. Illustration of Anblock adjustment measured on Decometer, \( T \) = position calculated on normal grid, and \( A \) = position calculated on modified grid after adjustment by the Anblock method.

### West pattern

<table>
<thead>
<tr>
<th>Point</th>
<th>( P )</th>
<th>( T )</th>
<th>( T-P )</th>
<th>( A )</th>
<th>( A-P )</th>
</tr>
</thead>
<tbody>
<tr>
<td>183</td>
<td>494.704</td>
<td>494.716</td>
<td>+0.012</td>
<td>494.702</td>
<td>−0.002</td>
</tr>
<tr>
<td>203</td>
<td>494.947</td>
<td>494.918</td>
<td>−29</td>
<td>494.950</td>
<td>+3</td>
</tr>
<tr>
<td>213</td>
<td>499.958</td>
<td>499.961</td>
<td>+3</td>
<td>499.948</td>
<td>−10</td>
</tr>
<tr>
<td>263</td>
<td>504.900</td>
<td>504.925</td>
<td>+25</td>
<td>504.905</td>
<td>+5</td>
</tr>
<tr>
<td>273</td>
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<td>504.961</td>
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<td>504.950</td>
<td>+12</td>
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<tr>
<td>283</td>
<td>509.950</td>
<td>509.948</td>
<td>−2</td>
<td>509.941</td>
<td>−9</td>
</tr>
<tr>
<td>293</td>
<td>510.006</td>
<td>510.027</td>
<td>+21</td>
<td>510.002</td>
<td>−4</td>
</tr>
</tbody>
</table>
East pattern

<table>
<thead>
<tr>
<th>Point</th>
<th>P</th>
<th>T</th>
<th>T - P</th>
<th>A</th>
<th>A - P</th>
</tr>
</thead>
<tbody>
<tr>
<td>183</td>
<td>64.054</td>
<td>64.106</td>
<td>+0.052</td>
<td>64.052</td>
<td>−0.002</td>
</tr>
<tr>
<td>203</td>
<td>67.935</td>
<td>68.002</td>
<td>+67</td>
<td>67.932</td>
<td>−3</td>
</tr>
<tr>
<td>213</td>
<td>64.041</td>
<td>64.110</td>
<td>+69</td>
<td>64.043</td>
<td>+2</td>
</tr>
<tr>
<td>263</td>
<td>63.946</td>
<td>64.015</td>
<td>+69</td>
<td>63.948</td>
<td>+2</td>
</tr>
<tr>
<td>273</td>
<td>66.195</td>
<td>66.272</td>
<td>+77</td>
<td>66.192</td>
<td>−3</td>
</tr>
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<td>283</td>
<td>61.849</td>
<td>61.896</td>
<td>+47</td>
<td>61.849</td>
<td>0</td>
</tr>
<tr>
<td>293</td>
<td>64.031</td>
<td>64.103</td>
<td>+72</td>
<td>64.035</td>
<td>+4</td>
</tr>
</tbody>
</table>

In general it can be stated that, as is the case with any system of adjustment, some discrepancies will remain in the calibration points \((A - P)\), but their order of magnitude is much smaller than the original discrepancies \((T - P)\).

In effect the method described above can be considered as a special kind of chart projection in which the transmitted pattern is represented by a flat, hyperbolic pattern. The grid no longer consists of a network of squares but has been modified, which means that data from other charts must locally be fitted in accordingly.

The extent to which the method described in chapter 5.3 can be employed has not yet been studied. Since the mathematical laws are not generally applicable, it may well be that the use of adaptation formulae is restricted to a more limited area.

An alternative to the method discussed above would again be the use of a correction chart. Repetition of calibration measurements for Hi-Fix chains has furthermore shown, that the corrections established in this way are not constant. This must be attributed to fluctuations in the propagation velocity of wave fronts due to changes in ground conductivity and atmospheric conditions. If these factors are to be taken into account, it will be necessary to use a monitor in the vicinity of the working area.
7. Pattern calibration

For plotting the pattern constants \( n \) and \( k \) in formulae (19) and (20) must be known. If we work with assumed values the matching of the chart pattern with the transmitted pattern becomes doubtful. The discrepancies would prove to be too great.

A simple method of determining \( n \) and \( k \) consists in observing the minimum and maximum lane numbers in the pattern. If \( a+b-c=0 \), we have \( L=k \) and if \( a+b-c=2b \) we obtain \( L=n+k \). The Decometer readings must then be taken on the base line extension at the master side to determine \( k \) and at the slave side to determine \( n \). This is part of the system employed for Decca chains; aircraft are used for base crossing, and the minimum and maximum Decometer values then derived. For such observations to be made from ships, it would be necessary for both base extensions to extend over the sea, which is rarely the case.

In particular in the case of Hi-Fix chains, with their in general much shorter range and narrower lane width than Decca chains, one is dependent on extra calibration positions, in which the position of the antenna can be fixed simultaneously, using the position patterns and other, more accurate methods.

The latter include theodolite sightings from fixed points on the shore and electromagnetic distance measurements to fixed points. This results in coordinates of the calibration position in the local system of coordinates.

The distances \( a_i \) and \( c_i \) in formula (1), i.e. \( L_i = l (a_i + b - c_i) + k \) can then be derived, so that the coefficient \( d_i = a_i + b - c_i \) is known; the value read off on the Decometer is called \( L' \).

One calibration position therefore gives one equation with two unknowns, \( l \) and \( k \), for each pattern; two calibration positions would theoretically be sufficient to determine \( l \) and \( k \), but it is obvious that the latter is calculated via adjustment using redundant values.

The correction formulae for \( n \) calibration points are then as follows:

\[
\begin{align*}
v_1 &= d_1 l + k - L_1' = L_1 - L_1' \\
v_n &= d_n l + k - L_n' = L_n - L_n'
\end{align*}
\]

in which \( v \) represents the discrepancies at the calibration points between the chart pattern \( L \) and the transmitted pattern \( L' \).

An adjustment reduces \( \Sigma vv \) to a minimum, so that the normal equations are as follows:

\[
\begin{align*}
\Sigma d_i d_i . l + \Sigma d_i . k - \Sigma d_i L_i' &= 0 \\
\Sigma d_i . l + nk - \Sigma L_i' &= 0
\end{align*}
\]
the solution being:

\[
\Sigma d_iL'_i - \frac{\Sigma d_i \Sigma L'_i}{n} \]

\[
\frac{\Sigma d_i d_i}{n} = \frac{\Sigma d_i \Sigma d_i}{n} - \]

and

\[
k = \frac{\Sigma L'_i}{n} - \frac{\Sigma d_i}{n} \cdot l
\]

This calculation process has been automated for working from the coordinates of the calibration points and the Decometer readings \(L'\). In addition to the results \(l\) and \(k\) the output gives the discrepancies \(v\) for the calibration points and also \(\Sigma vv\).

The latter result enables us to determine through \(\frac{\Sigma vv}{n-2}\) the standard deviation in matching the chart pattern with the transmitted pattern.

For the Groningen Hi-Fix chain with 223 calibration positions a value of \(\sigma = 0.080\) has been calculated for the West pattern and \(\sigma = 0.072\) for the East pattern. These results illustrate the need for a method such as that described in Chapter VI.

Details of the arrangements for calibration measurements on land in this area, are described in the article by R. F. Veenendaal (see bibl. 6).

The coordinates were determined by intersection from 3 or more theodolite positions, using a computer. In every case the programme included a combination of 2 directions with an angle of intersection lying between minimum and maximum introduced values. The coordinates obtained in this way were averaged. The differences in relation to these mean values were calculated from the standard deviations \(m_x\) and \(m_y\) for each point, with the following weights: 1 for 2, 2 for 3, 3 for 4 and 4 for 5 calculated intersections. This gave a standard deviation in the coordinate determination for the whole process of \(m_x = 31\) cm and \(m_y = 29\) cm, showing that the method followed is reliable for our purposes.

The standard deviation in the Decometer readings of course affects the calibration results. The change in the frequency of the Europoort Hi-Fix Chain in 1969 from 1900.60 kc/s (LF) to 2876 kc/s (HF) made it possible to study this problem when the LF and HF patterns could simultaneously be transmitted and received on board of the ship. The LF and HF Decometers, installed side by side, were photographed at 49 positions in the pattern concerned. The photographs were read off afterwards. Taking into account the differences in position of the LF and HF antennae, the following relationship can be established.

Calling the readings in the low frequency pattern \(L'\), those in the high frequency pattern \(H'\), the corresponding discrepancies \(v\) and \(w\), the multiplicatory constants \(l_i\) and \(l_h\) and the additive constants \(k_i\) and \(k_h\), we have:
\[ L_i' + v_i = l_i d_i + k_i \]

and

\[ H_i' + w_i = l_h d_i + k_h \]

By equating \( d_i \) we obtain the relation:

\[
\frac{L_i' - k_i + v_i}{l_i} = \frac{H_i' - k_h + w_i}{l_h}
\]

If we introduce \( \frac{l_h}{l_i} = f \) into the expression as a new unknown, we obtain the following correction equations with \( f \) and \( k_h - f k_i \) as unknowns.

\[ A = w_i - f v_i = f L_i' + (k_h - f k_i) - H_i' \]

Now

\[ \sigma_A^2 = \frac{\sum A \Delta}{n-2} = \sigma^2 + f^2 \sigma^2 \]

or

\[ \sigma^2 = \frac{\sigma_A^2}{1 + f^2} \]

where \( \sigma \) is the standard deviation in a Decometer reading, attributable to instrument and reading errors.

The standard deviation for the stability of this transmitted pattern without reference to a chart pattern was found to be \( \sigma = 0.022 \) lane for the South pattern and \( \sigma = 0.020 \) lane for the North pattern.

Calibration of the chart pattern for the Hi-Fix Europoort area at 176 calibration points gave \( \sigma = 0.049 \) lane for the South pattern and \( \sigma = 0.039 \) lane for the North pattern. These results were much better than those of the Hi-Fix Groningen. Even so, it can be seen that approximately half of these \( \sigma \) values are primarily attributable to pattern instability.

Calibration of the chart pattern in the operating area of the Wadden Chain gave better results. Siting the master and 3 slave stations I, II and III in favourable positions and avoiding wave-front paths over land, gave the following results: \( \sigma_1 = 0.02 \) \( \sigma_{11} = 0.015 \) and \( \sigma_{111} = 0.015 \).

If Decca chains are to be calibrated not only at the baseline extensions but also at incidental points in the pattern, the method described can no longer be used because the ship’s position is too far away from the shore. Consideration is being given to the
use of the two-range Hi-Fix or Sea-Fix system, with which the distances between the ship’s position and the slave stations on the coast could be determined with a precision that is adequate. The geographical coordinates $\varphi$ and $\lambda$ can be derived for the calibration points from these values, thus giving $a_i$ and $b_i$ in accordance with formula (20).
Caption to figures

Fig. 1 Types of position patterns
Fig. 2 Principal directions in automatic plotting
Fig. 3 Stepwise plotting of a straight line
Fig. 4 The terrestrial and chart coordinate system
Fig. 5 The parameters $\xi$ and $\eta$ for points on a lane
Fig. 6 Possible variations in the number of intersections between lanes and the edges of the chart
Fig. 7 Points to be plotted in a lane
Fig. 8 Possible variations in the number of intersections between circles and the edges of the chart
Fig. 9 Arcs of circles to be plotted on the chart
Fig. 10 Approximation of plotting arcs
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Fig. 13 The transmitter positions of Master and Slaves
Fig. 14 Discrepancy between plane chart pattern and spherical, or spheroidal, chart pattern
Fig. 15 Histogram of discrepancies between chart pattern and transmitted pattern for Groningen Hi-Fix Chain
Fig. 16 General chart of Groningen Hi-Fix patterns
Fig. 17 Illustration of Anblock adjustment
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