A k-MODEL FOR STABLY STRATIFIED NEARLY HORIZONTAL TURBULENT FLOWS

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A k-model is formulated that consists of the turbulent kinetic energy equation and an algebraic expression for the mixing length taking into account the influence of stratification. Applicability of the model is restricted to shallow, nearly horizontal flows. For local-equilibrium flows the model reduces to the well-known mixing-length hypothesis including a damping function depending on the gradient Richardson number. The model is applied to stratified turbulent Poiseuille flow between two parallel plates.
1. Introduction

At present turbulent models of various degrees of complexity are available for stably stratified flows. A widely used model in engineering is the second-order closure k-ε model (e.g., Rodi, 1980). In this model velocity and length scales of the energy containing eddies are calculated from a modelled form of the transport equation for turbulent kinetic energy k and a transport equation for the dissipation rate ε that is of a more empirical nature. Local eddy viscosities and diffusivities can be determined once the model has been calibrated by tuning several empirical model constants.

The stability and accuracy of numerical schemes based on these turbulence models usually require relatively small mesh sizes and time steps so that the extent of the computations involved may be prohibitive for flows on a geophysical scale. Therefore, there continues to be a need of relatively simple models for flows of this kind. In this report a k-model is formulated that consists of the turbulent kinetic energy equation and an algebraic expression (rather than a transport equation) for the length scale taking stratification into account. The k-model is the simplest model that can account for non-equilibrium conditions. The semi-empirical length scale formulation suggested is limited to quasi-stationary, shallow flow over a nearly horizontal bed. For convenience, the attention is restricted herein to vertical exchange of mass and momentum in two-dimensional flows. Extension to three dimensions is straightforward.

The model is formulated in sections 2 and 3. The special case of local equilibrium is examined in section 4. Sections 5 and 6 give estimates of model coefficients and an application to stratified flow between parallel plates.

2. Turbulent kinetic energy equation

According to the gradient-transport hypothesis the vertical turbulent transports of momentum and buoyancy are given by

$$\frac{\overline{u'w'}}{\nu_e} = -\frac{\partial \overline{k}}{\partial z}$$  \hspace{1cm} (2.1)
where $u$ and $w$ are horizontal and vertical velocity components, $v_t$ and $k_t$ are eddy viscosity and diffusivity, $z$ is a vertical coordinate (positive in upward direction). The buoyancy $b$ is given by

$$b = g \frac{\rho - \rho_0}{\rho_0}$$

(2.3)

where $g$ is the acceleration of gravity, $\rho$ the density and $\rho_0$ a reference density.

The exchange coefficients $v_t$ and $k_t$ are proportional to the product of a velocity scale and a length scale of the turbulent motions, that is,

$$v_t = C_v k^{1/2} l$$

and

$$k_t = C_t k^{1/2} l$$

(2.4a,b)

where $k = \frac{1}{3} \frac{\overline{u_i' u_i'}}{k}$ is the mean turbulent kinetic energy, $l$ a length scale of the energy containing eddies, $C_t$ the turbulent Prandtl or Schmidt number, and $C_v$ an empirical model coefficient.

The one-dimensional turbulent kinetic energy equation when modelled as usually in second-moment turbulence closures is (e.g., Rodi, 1980)

$$\frac{Dk}{Dt} = \frac{1}{2z} \left( \frac{\overline{v_k} \frac{dk}{dz}}{\rho_k} - \overline{u'w'} \frac{\partial u}{\partial z} + \overline{b'w'} - C_d k^{3/2} \frac{k}{l} \right)$$

(2.5)

where $C_k$ is coefficient similar to the turbulent Prandtl number and $C_d$ is another model constant. The terms on the right-hand side of (2.5) represent internal redistribution (diffusion), shear production, destruction by buoyancy transport, and dissipation of turbulent kinetic energy. The dissipation $\varepsilon$ is given by the assumption of spectral equilibrium,

$$\varepsilon = C_d k^{3/2} \frac{k}{l}$$

(2.6)

In the case of local-equilibrium flows (2.5) simplifies to

$$- \overline{u'w'} \frac{\partial u}{\partial z} + \overline{b'w'} - C_d k^{3/2} \frac{k}{l} \approx 0$$

(2.7)
by definition.

Simple and physically acceptable boundary conditions for the turbulent kinetic energy equation are given by Rodi (1980). If buoyancy transport through the bed is absent, local equilibrium near the bed will give, also see (4.7),

\[
\frac{k}{u_*^2} = \frac{1}{\sqrt{c_{vv}c_{dd}}} = \frac{1}{c_p^2}
\]

where \(\rho_0 u_*^2\) is the bed shear stress. This boundary condition is to be applied at some (small) distance above the bed proper. The boundary condition suggested by Rodi for a free surface (subscript \(S\)) is

\[
\frac{\partial k}{\partial Z} = 0 \text{ if } \frac{k_S}{u_*^2} > \frac{1}{c_p^2} \text{ and } \frac{k_S}{u_*^2} = \frac{1}{c_p^2} \text{ otherwise.}
\]

3. Length scale

In first approximation, the vertical length scale \(L\) of the large, energy containing eddies is conceived to be determined by

- the presence of the boundaries (e.g., bed and free surface), and
- density stratification.

Semi-empirical expressions representing the influence of the boundaries in unstratified flow are available, e.g. a mixing length \(L_n\) according to a parabolic distribution,

\[
L_n = \chi \frac{Z(1 - \frac{Z}{\alpha})}{1 - \frac{Z}{\alpha}}
\]  

(3.1)

where \(\chi\) is the Von Karman constant, \(\alpha\) the water depth, and \(Z = 0\) at the bed. The restricting influence of the boundaries on the length scale \(L\) is nonlocal in character.

Stratification also restricts the size of the large eddies. Stillinger et al. (1983) find that in buoyancy dominated turbulence the vertical length scale (or "overturning" scale) of the turbulent motions is proportional to a length scale introduced by Ozmidov (1965),

\[
L_S \propto \frac{\varepsilon^{1/2}}{(\partial \theta/\partial Z)^{3/4}}
\]

(3.2)
Eliminating the dissipation between (2.6) and (3.2) gives, with

$$l_s = c_s \left( \frac{k}{\partial b/\partial z} \right)^{1/2}$$  

(3.3)

where $c_s$ is a coefficient. The result (3.3) can also be derived from a simple energy argument. Substituting from (2.2) it is easily shown that the length scale $l_s$ also is closely related to the Monin-Obukov length (e.g., Turner, 1973).

It may be concluded on the basis of the experiments of Stillinger et al. (1983) and Itsweire (1984) that the length scale is equal to $l_s$, if $l_s << l_n$ (and $c_s$ is chosen accordingly). If, on the other hand, $l_s >> l_n$ the stratification is so weak that $l_s \approx l_n$. In intermediate situations interpolation as shown in figure 1 is suggested here. The interpolation formula proposed can be formally written as

$$\frac{l}{l_n} = f \left( \frac{l_s}{l_n} \right)$$  

(3.4)

Suitable formulae of this type are discussed in section 4.

Equation 3.3 can be interpreted in a different way. The ratio of buoyancy transport to dissipation can be written as

$$\frac{-\bar{b} \bar{w}}{\varepsilon} = \frac{c_v}{u' \bar{c}_D} \left( \frac{l^2}{k} \right) \frac{\partial \bar{b}}{\partial z}$$  

(3.5)

This ratio is in fact more suitable to characterize the stability of the flow than the commonly used flux and gradient Richardson numbers, since $-\bar{b} \bar{w} / \varepsilon$ fluctuates much less. It is always between zero and a maximum of order one, whereas both Richardson numbers can become large in non-equilibrium turbulence with small or zero shear stress and mean-velocity gradient (e.g., Gartrell, 1980). Empirical evidence indicates that $-\bar{b} \bar{w} / \varepsilon$ remains greater than zero in near-collapsing turbulence (e.g., Gregg and Briscoe, 1979; also see Imberger and Hamblin, 1982),

$$\left( \frac{-\bar{b} \bar{w}}{\varepsilon} \right)_c = c_B > 0$$  

(3.6)

where the subscript $c$ refers to the critical near-collapsing state of the turbulence. The coefficient $c_B$ is identical to a parameter $R_{crit}^*$ defined by Gibson and Launder (1976).

Equations 3.5 and 3.6 give, with $l = l_s$,
Comparing this result with (3.3) gives

\[ C_s^2 = \frac{C_B C_D C_{\text{lc}}}{C_V} \tag{3.7} \]

The length scale formulation suggested could be refined in various ways. This will not be attempted however, since the simplicity of the k-model would then be lost.

4. Special case of local equilibrium

If local equilibrium is assumed, equations 2.1, 2.2, 2.4 and 2.7 will give

\[ \left( \frac{1}{R_i} - \frac{1}{C_t} \right) \frac{\delta \bar{b}}{\delta z} = \frac{C_D}{C_V} \left( \frac{k}{l^2} \right) \tag{4.1} \]

where \( R_i \) is the gradient Richardson number,

\[ R_i = \frac{\delta \bar{b} / \delta z}{(\delta \bar{u} / \delta z)^2} \tag{4.2} \]

Substituting from (3.3) and (3.7), equation 4.1 gives a value for the critical Richardson number \( R_{iC} \) at which local-equilibrium turbulence would collapse,

\[ R_{iC} = C_{\text{lc}} C_B \left( \frac{1}{1 + C_B} \right) \tag{4.3a} \]

The critical flux Richardson number \( R_{fC} = R_{iC} / C_{\text{lc}} \) is given by

\[ R_{fC} = \frac{C_B}{1 + C_B} \tag{4.3b} \]

In the case of the equilibrium assumption, \( R_i \) is confined to the range \( 0 \leq R_i \leq R_{iC} \). Actually, however, \( R_i \) (and \( R_f \)) can vary from zero to infinity in the full equation 2.5.

Substituting from (3.3), (3.4) and (4.1), the eddy viscosity and length scale \( l \) are found to be given by

\[ \nu_t = C_V \left( \frac{l^2}{C_D} \left( 1 - \frac{R_i}{R_t} \right) \right) \left( \frac{\delta \bar{u}}{\delta z} \right) \tag{4.4} \]
and

\[ \frac{L}{L_n} = \mathcal{F} \left[ C_s \frac{L}{L_n} \sqrt{\frac{C_p}{C_D}} \left( \frac{1}{R_i} - \frac{1}{\sigma_t} \right) \right] \]  \hspace{1cm} (4.5)  

In principle \( L \) can be eliminated between (4.4) and (4.5) to give an expression of the form (assuming that \( \sigma_t \) is a function of \( R_i \) only)

\[ \nu_t = \left[ \frac{1}{\nu} \right] \mathcal{F}(R_i) \]  \hspace{1cm} (4.6)

which is the well-known mixing-length formulation taking the effect of stratification into account by introducing a damping function \( \mathcal{F}(R_i) \).

In unstratified flow the length scale \( L \) equals \( L_n, R_i = 0 \) and \( \mathcal{F}(R_i) = 1 \). Equations 4.4 and 4.6 then give

\[ C_D = C_N \]  \hspace{1cm} (4.7)

The function \( \mathcal{F}(R_i) \) is given by [(4.5) shows that \( L/L_n \) is a function of \( R_i \)]

\[ \mathcal{F}(R_i) = \left( \frac{L}{L_n} \right)^2 \sqrt{1 - \frac{R_i}{\sigma_t}} \]  \hspace{1cm} (4.8)

The damping function \( \mathcal{F}(R_i) \) depends on the interpolation expression \( f(L/L_n) \) assumed. The approach pursued here is to choose this expression so as to obtain realistic functions \( \mathcal{F}(R_i) \). Figure 2 shows damping functions \( \mathcal{F} \) calculated for near-wall turbulence assuming the following interpolation expressions:

\[ \frac{1}{L} = \frac{1}{L_n} + \frac{1}{L_s} \]  \hspace{1cm} (4.9)

\[ \frac{1}{L^2} = \frac{1}{L_n^2} + \frac{1}{L_s^2} \]  \hspace{1cm} (4.10)

and

\[ \frac{L}{L_n} = 1 - \frac{4}{(1 + \frac{1}{2} \frac{L_s}{L_n})^2} \]  \hspace{1cm} (4.11)

It was further assumed in figure 2 that \( C_b = 0.18 \) and \( \sigma_t = 1 \) (see section 5). The expressions 4.9, 4.10 and 4.11 are all seen to give decreasing functions \( \mathcal{F} \), as required. The result derived from
(4.10) is close to the Monin-Obukov relationship.

There is no agreement in the literature as to the precise shape of the damping function $F$ for a particular situation. Therefore a similar problem exists with the interpolation formula.

5. **Estimates of coefficients**

It seems consistent with the simplicity of the k-model to assign constant values to the various model coefficients as much as possible, thus disregarding their dependence on stratification. Launder and Spalding (1972) suggest $c_v c_D \approx 0.08$ for unstratified flow; together with (4.7) one then obtains $c_v \approx 0.53$ and $c_D \approx 0.15$. Equation 4.3b shows that the constant $c_D$ is directly related to the critical flux Richardson number for local-equilibrium conditions. Realistic values of $c_D$ should range from 0.18 to 0.25 (Gartrell, 1980; Gregg and Briscoe, 1979; Imberger and Hamblin, 1982; Osborn, 1980; Turner, 1973). According to Launder and Spalding (1972) $\bar{c}_k \approx 1.0$ in unstratified flow, although Sonin (1983) obtains $\bar{c}_k \approx 0.74$ for turbulence produced by an oscillating grid.

The turbulent Prandtl number $\bar{G}_t$ is close to one in wall-generated turbulence (e.g., Arya, 1972). However, its dependence on stratification cannot be neglected in free shear flows. In unstratified free shear flows $\bar{G}_t$ is less than one. Assuming that free shear turbulence at local equilibrium is in a near-collapsing state when $\bar{R}_t = \bar{R}_c = 1$ (Abarbanel et al., 1984), equation 4.3a together with the $c_B$ values mentioned will give $\bar{G}_t = 5$ to 7. The measurements of Ueda et al. (1981) seem to indicate an even larger increase in $\bar{G}_t$ near $\bar{R}_t = 1$. A variable Prandtl number can be introduced by assuming that it is a (decreasing) function of $L/l_n$. This approach is equivalent to a Richardson dependence of $\bar{G}_t$ under local-equilibrium conditions, as indicated by (4.5).

6. **Application: stratified turbulent Poiseuille flow between two parallel plates**

The length-scale formulation of section 3 is applied to the fully developed, stationary turbulent flow between two horizontal parallel plates (figure 3). The upper plate is heated uniformly, and the lower plate is kept at constant temperature so
that the heat flux generated is constant in space and time. The resulting buoyancy effect reduces the intensity and length scale of the turbulence, in particular in the core region. In this region the production of turbulent kinetic energy is relatively small, and (assuming a linear equation of state) even vanishes at the centre-line. Including the diffusion of turbulent kinetic energy in the turbulence model employed is therefore essential here.

In view of the above discussion and (3.6), it can be expected that a critical buoyancy flux exists for which the turbulence near the centre-line is completely suppressed. One of the aims of the analysis is to calculate this critical buoyancy flux.

Using (2.1), (2.4a) and (4.7), and substituting \( \overline{w'w'} = (z/h) u_k^2 \) the \( k \)-equation (2.5) can be written for this flow as

\[
\frac{d}{dz} \left( \frac{c_v}{\theta_k} k^{1/2} \frac{dk}{dz} \right) + \frac{u_k^4}{c_v k^{1/2} L} \left( \frac{z}{h} \right)^2 + \overline{b'w'} - c_v^3 \frac{k^{3/2}}{\nu L} = 0 \tag{6.1}
\]

Here \( 2h \) is the distance between the plates, and \( \overline{b'w'} \) is a known (negative) constant. The mixing length \( l_h \) for unstratified flow is assumed to be given by Nikuradse's formula

\[
l_h = 0.14 - 0.08 \left( \frac{z}{h} \right)^2 - 0.06 \left( \frac{z}{h} \right)^{1/4} \tag{6.2}
\]

Substituting from (2.2), (2.4b), (3.7) and (4.7), the expression for the length scale \( l_s \) becomes

\[
l_s^2 = c_B c_v^3 \frac{\sum \Delta_c}{c_v} \frac{k^{3/2} L}{-b'w'} \tag{6.3}
\]

The results presented below are for interpolation formula 4.10 to calculate the length scale \( l \). Expressions 4.9 and 4.11 produce slightly different results.

The boundary conditions derive from the zero diffusive transports of turbulent kinetic energy at the plates, \( z = \pm h \). This transport also vanishes at \( z = 0 \) because of symmetry. On integration (6.1) therefore gives

\[
\int_0^{\pm h} \left[ \frac{u_k^4}{c_v k^{1/2} L} \left( \frac{z}{h} \right)^2 + \overline{b'w'} - c_v^3 \frac{k^{3/2}}{\nu L} \right] dz = 0 \tag{6.4}
\]

This condition was used to check numerical calculations. The length scale \( l \) tends to zero at \( z = \pm h \). To avoid divergence of the
integral in (6.4), one must require that

\[ k = \frac{u_x^2}{c_V^2} \text{ at } z = \pm h \]  

(6.5)

Since \( \frac{dk}{dz} \) was found to remain finite at \( z = \pm h \), the condition of zero diffusive transports of turbulent kinetic energy at the plates was satisfied. The condition at the plane of symmetry is

\[ \frac{dk}{dz} = 0 \text{ at } z = 0 \]  

(6.6)

Calculations were carried out for half of the region

\[-h \leq z \leq h.\]

Introducing dimensionless variables according to

\[ z = \frac{h}{\tilde{z}} \]
\[ k = \frac{u_x^2}{\tilde{k}} \]
\[ L = \frac{h}{\tilde{L}} \]

the dimensionless buoyancy flux \( B \) becomes

\[ B = \frac{-b w^1 h}{u_x^2} \]  

(6.7)

This quantity can also be conceived as an overall flux Richardson number. Defining a vertical transport velocity \( w \) according to

\[ w = \frac{-b w^1}{\Delta b} \]

where \( \Delta b \) is a characteristic buoyancy difference across the layer of fluid (figure 3), equation 6.7 can be written as

\[ \frac{w}{u_x} = BF_x \]  

(6.8)

Here \( F_x = \frac{u_x^2}{(\Delta b h)} \) is an overall internal Froude number. Equation 6.8 is used later for comparison with certain experimental results.

The value of the constant \( c_V \) assumed was 0.53 as indicated
in section 5. Since the turbulence in the flow examined is generated at the plates, the Prandtl number \( \frac{\nu}{\kappa} \) was taken constant (so that \( \frac{\nu}{\kappa} = \frac{\nu}{\kappa} \)). Note that the length \( L_c \) given by (6.3) then is independent of the value of \( \frac{\nu}{\kappa} \). However, the mean-buoyancy distribution does depend on \( \frac{\nu}{\kappa} \). The coefficients \( \frac{\nu}{\kappa} \) and \( \frac{\nu}{\kappa} \) were varied in the computations.

Equations 4.10, 6.1, 6.2 and 6.3 were solved numerically together with the conditions 6.5 and 6.6 using a quasi-linearization method.

Figure 4 shows a comparison for the unstratified case of calculated results (assuming \( \frac{\nu}{\kappa} = 1 \)) with Laufer's (1951) experiments. The prediction of the turbulent kinetic energy near the plane of symmetry \( z = 0 \), which is rather sensitive to variations in \( \frac{\nu}{\kappa} \), is quite satisfactory.

Figure 5 shows predictions (with \( \frac{\nu}{\kappa} = 1 \), \( \frac{\nu}{\kappa} = 0.2 \) and \( \frac{\nu}{\kappa} \) depending linearly on \( \frac{\nu}{\kappa} \)) of turbulent kinetic energy, length scale, mean buoyancy and mean velocity for various values of the dimensionless buoyancy transport \( B \). Stratification is seen to influence the turbulence mainly in the core region. As a result, changes in the buoyancy transport in the wall regions would hardly influence the results. In this instance, the critical value \( B_c \) of the buoyancy transport with complete suppression of the turbulence at \( z = 0 \) is about 0.17. Figure 6 shows the calculated critical value \( B_c \) as a function of \( \frac{\nu}{\kappa} \) and \( \frac{\nu}{\kappa} \).

Kranenburg (1986) carried out experiments on strongly stratified flow in a closed annular channel, and found a relationship equivalent to (6.8) with \( B \approx 0.17 \) for \( 0 < F^* < 0.13 \). The calculations with \( \frac{\nu}{\kappa} = 1 \), \( \frac{\nu}{\kappa} = 1 \) and \( \frac{\nu}{\kappa} = 0.2 \) give \( B = B_c \approx 0.17 \) for \( F^* = 0 \) and \( B \approx 0.15 \) for \( F^* \approx 0.13 \). The experimental buoyancy profiles do not agree well with the calculated profiles, the observations showing smaller gradients \( \partial b / \partial z \) near the plates and near the centre-line. Part of this discrepancy is caused by the different distributions of the buoyancy transport. Since the channel was relatively narrow, the presence of sidewalls may also have played a role.

* Some calculations were carried out with \( \frac{\nu}{\kappa} \) depending linearly on \( l / L_n \) (see section 5). Increasing the critical Prandtl number from one to two was found to increase the critical buoyancy transport discussed below by some fifteen percent.
Simpson and Hunter (1974) define a mixing efficiency \( \eta \) for tidal flow in the sea according to

\[
\eta = \frac{1}{2} \frac{\bar{b}w'/a}{c \langle |U|^3 \rangle}
\]  

(6.9)

where \( U \) is the tidal velocity, the brackets indicate a tidally averaged value, and \( c \) is a drag coefficient for bottom friction. Using the present notation and assuming \( a \equiv h \), (6.9) can be written as

\[
\eta = \frac{1}{2} c^{1/2} B
\]

Assuming \( c = 25.10^{-4} \) and \( B_c = 0.17 \), complete suppression of the turbulence at the free surface would occur for \( \eta \approx 4.10^{-3} \). However, stratification would become appreciable already at \( \eta \approx 10^{-3} \) (e.g., figure 5b). The observations of Simpson and Hunter indicate the same critical value of \( \eta \) to an order of magnitude.

7. Concluding remarks

Equations 3.3 and 3.4 essentially represent the length scale formulation employed. These equations imply that the ratio of local length scale to mixing length for unstratified flow \( (L/L_h) \) is made dependent on a "turbulence Richardson number" defined by

\[
\frac{\partial b/\partial z}{k/L^2}
\]

which replaces the commonly used dependence on the gradient Richardson number. It is shown in section 4, that the two formulations are equivalent in the case of local equilibrium.

Obviously this formulation can only be approximate. Only the influence of local stratification is taken into account. Nonlocal effects of stratification are ignored completely. Furthermore, the advection of turbulence structures affecting the length scale is also neglected, which restricts applications to gradually varying flows.

Although an application of the model was presented herein, more tests are needed to examine the validity and limitations of the various hypotheses put forward.
REFERENCES


W. Rodi, 1980, Turbulence models and their application in hydraulics - a state of the art review, IAHR.


NOTATION

\[ a \]
- water depth

\[ b, b', \bar{b} \]
- buoyancy, fluctuation and mean value

\[ B \]
- dimensionless buoyancy transport

\[ c_b, c_d, c_s, c_v \]
- model constants

\[ F, f, F_k \]
- drag coefficient

\[ g, h, k, K_t \]
- functions

\[ F_r, R_i, t \]
- internal Froude number

\[ l, l_2, l_n \]
- acceleration of gravity

\[ R_f, R_i, t \]
- half the distance between plates

\[ u, u', \bar{u}, \tilde{u} \]
- turbulent kinetic energy

\[ u_t, \bar{u} \]
- eddy diffusivity

\[ \lambda, \lambda_k, T \]
- length scales

\[ w, w' \]
- flux and gradient Richardson numbers

\[ w' \]
- time

\[ u, u' \]
- velocity component, and fluctuation

\[ \bar{u} \]
- mean horizontal velocity component

\[ w, w' \]
- friction velocity

\[ U \]
- tidal velocity

\[ w' \]
- fluctuation of vertical velocity component

\[ w' \]
- vertical transport of volume

\[ z \]
- vertical coordinate

\[ \Delta b \]
- vertical coordinate

\[ \epsilon, \eta \]
- characteristic buoyancy difference

\[ \epsilon \]
- dissipation rate

\[ \eta \]
- mixing efficiency

\[ \kappa \]
- Von Karman constant

\[ \nu_t, \nu_t' \]
- eddy viscosity

\[ \rho, \rho_0 \]
- density, and reference density

\[ C_t, C_s \]
- turbulent Prandtl (Schmidt) number

\[ C_k \]
- model coefficient

Subscripts

\[ i \]
- 1, 2, or 3

\[ c \]
- critical

\[ s \]
- surface
Fig. 1. Suggested relationship between length scales $l$ and $l_s$ defined by equation 3.3.
Fig. 2. Damping functions for local equilibrium, $c_B = 0.18$, $\sigma_t = \sigma_{tc} = 1$. 
Fig. 3. Schematic of stratified Poiseuille flow between parallel plates.
Fig. 4. Comparison for unstratified flow between calculations and Laufer's (1951) experimental data.
Fig. 5. Calculated results for $\sigma_k=1$, $c_b=0.2$, $\sigma_t=1$;

a. turbulent kinetic energy; b. length scale.

Curves: A B C D E F

B = 0 0.04 0.08 0.12 0.16 0.17
Fig. 5. Calculated results for $c_k=1$, $c_B=0.2$, $c_t=1$;
c. mean buoyancy; d. mean velocity.

Curves: A B C D E G

$B=$ 0 0.04 0.08 0.12 0.16 0.165
Fig. 6. Calculated critical buoyancy transport as a function of $c_B$ and $\sigma_k$ for $\sigma_t = \sigma_{tc}$. 