CONVEXITY ANALYSIS OF HEXAGONALLY SAMPLED IMAGES
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CONTENTS

SUMMARY ix

1. INTRODUCTION 1

1.1 General background: image sampling 1
1.2 General background: convexity 7
1.3 Statement of the problem and outline of the thesis 15

2. IMAGE SAMPLING 19

2.1 Introduction 19
2.2 Sphere packings on lattices 21
2.3 Rectangular sampling lattices 24
2.4 Hexagonal sampling lattices 33
2.5 Triangular sampling lattices 47
2.6 Nonuniform sampling lattices 63
2.7 Concluding remarks 67

3. DISCRETE CONVEXITY 69

3.1 Introduction and preliminary notions 69
3.2 Connectivity and Euler characteristic 76
3.3 Discrete convexity on the square mosaic 91
3.4 Discrete convexity on the hexagonal lattice 111
3.5 Discrete convexity on the hexagonal mosaic 130
3.6 Concluding remarks 142
4. HALF-CELL EXPANSION AND CELLULAR CONVEXITY 145

4.1 Introduction 145
4.2 Duality between hexagonal and triangular complexes 147
4.3 Minimal polygons on hexagonal mosaics 169
4.4 Minimal polygons and half-cell expansion 187
4.5 Concluding remarks 192

5. STRAIGHTNESS AND CONVEXITY 195

5.1 Introduction and preliminary notions 195
5.2 Digital straightness and digital convexity 201
5.3 Cellular straightness and cellular convexity 213
5.4 Nontight straightness and convexity 219
5.5 Concluding remarks 226

REFERENCES 229

SAMENVATTING 239

ACKNOWLEDGEMENT 245
SUMMARY

When a digital computer is programmed to analyze a two-dimensional geometric object, the picture containing the object is usually represented as an array of numbers in the computer’s memory. To digitize the picture, a sampling process is used to extract from the picture a discrete set of real numbers ("samples"), and a quantization process is applied to these samples to yield numbers having a discrete set of possible values. This process of image digitization affects the computer’s ability to analyze the shape of the object. Generally, image digitization generates huge volumes of data. The effort involved in handling this raw information without pre-processing is tremendous. Therefore, this matrix of brightness values of a picture must be transformed into a more compact mathematical structure before we can proceed with any pattern recognition analysis of the two-dimensional geometric object. One way of doing this is to segment the image, a process on the boundary of image processing and pattern recognition, more specifically on the boundary of image sampling and image analysis. The problem is basically one of psychophysical perception: Gestalt principles dictate certain grouping preferences based on features such as proximity and similarity, or based on the discrimination of objects against a background. The proper selection of a segmentation method lies outside the scope of this thesis, since the thesis is restricted to the study of the shape of "silhouettes", i.e. plane objects in already segmented images.

Once the geometric object is converted into a discrete form, we like to define notions or to measure parameters of the components in the discrete space in order to extract shape information from the objects. These notions or parameters are often translated by analogy from the continuous space into the discrete space, which may lead to severe problems associated with the representation of a geometric object by a discrete retina. For instance, consider convex objects such as circles and
triangles which clearly should be judged as being convex after digitization even though their boundaries are jagged in the discrete space. In this thesis the well-known geometric property of convexity is discussed concerning only those blobs (black objects on a white background) which are convex in nature.

An application of convexity analysis can be found in the analysis of erythrocytes and leukocytes, where a convexity constraint has been used to reconstruct the contour of the blob, when only a part of its contour had been found in the previous segmentation step. The knowledge of the shape of the objects can be incorporated into the segmentation operation itself such that it is constrained to find convex objects. Then, starting from an initial connected set of discrete points, points are iteratively aggregated to the set where the convexity of the component under study is tested. This method has been applied to the segmentation of cell images; essentially of blood and bone narrow cells. A quite different application of convexity analysis can be found in the automatic visual inspection of industrial parts passing by on an assembly line. Because of their well-defined shape, these objects (usually nonconvex) can be described neatly by concavity trees which represent the information to fill in local concavities and metaconcavities. The concavity tree is obtained by a process of deleting and adding convex subsets of the object, i.e. convex sets of connected picture elements, in a way that the resulting object is convex.

Last, much work has been done on the decomposition of digital sets, approximated by an arbitrary polygon, into a union of a finite number of (overlapping) convex polygons. Such convex decompositions provide structural analysis of a complex object in terms of a graph that has nodes for intersection sets and primary convex polygons, and an arc between intersecting polygons.

The notion of convexity has been discussed by numerous authors for blobs represented by rectangular digitization schemes. The particularly interesting hexagonal sampling strategy has been excluded explicitly in the theory of convexity analysis, although the hexagonal representation seems highly suitable for the shape analysis of binary blobs since it produces less sampling noise for circular figures than the rectangular grid. The gain in accuracy, with the use of the hexagonal grid with respect to the rectangular grid having an equal sampling density, becomes even more
evident with small isotropically shaped blobs (e.g., tumors, blood cells, calcifications, gall stones, etc.) which frequently occur in biomedical images. Therefore, in this context one should consider the notion of convexity for the hexagonal sampling strategy, which will be accomplished in this thesis.

A connected set of points in the continuous space is said to be convex if the line segment joining any two points of the set is contained in the set. This and similar properties of the notion of convexity are well defined and understood in Euclidean geometry. But it is not quite obvious what is meant by these geometric properties in the discrete space. One approach to digital geometry, i.e. the mathematical study of geometric properties of digital picture subsets, consists of verifying whether properties of the continuous notion of convexity carry over to the discrete space. Such an approach has been applied for the topological notion of connectivity, more precisely for the problem of the discrete topological description of the different tessellations currently used in image analysis, such as the square and hexagonal tessellations. In fact, digital topology is concerned with properties of digital subsets, involving such concepts as adjacency and connectedness, but not size or shape. As with these topological concepts, a geometrical concept such as convexity can be defined on the digital subset as such.

An alternative approach is to map the discrete grid onto the continuous space. This approach, which is used in this thesis concurrently to the above approach, emphasizes the translation of a "mosaic" into a continuous curve. Thus one must define a continuous curve or polygon, for instance, the minimum-perimeter polygon, i.e., a polygon whose size and shape coincide with that of a stretched thread constrained to lie inside the contour of the digital object. Then, once such a polygon is found, the processing can continue within the framework of continuous geometry.

A few definitions of discrete convexity have been proposed in the literature and investigated quite extensively. It turned out that these definitions are virtually equivalent. However, this equivalency is obtained exclusively for objects represented by a set of cells in a rectangular mosaic or by a set of points in a rectangular lattice. The definition of convexity applied in this thesis is effectively equivalent to the existing ones. Never-
theless, the formulation will fit in the hexagonal mosaic, a mosaic explicitly excluded in the theory of convexity analysis which has been published thus far. Therefore, the formulations and proofs are quite different because of the substantial differences between rectangular and hexagonal digitization schemes.

Sklansky and Kibler (1976) derived a theory to analyze convexity and related properties of binary-valued digital images valid for a fairly large class of mosaics. These so-called acute mosaics have in common that every element of the mosaic is a convex polygon or cell and that the union of every pair of adjacent cells forms a set bounded by a convex polygon. Consequently, acute mosaics do not include hexagonal mosaics. Therefore, the general theory of Sklansky and Kibler cannot be utilized for the hexagonal mosaic; a different approach is necessary.

Henceforth, an object represented by a set of cells on a mosaic is denoted as a cellular complex. To circumvent the non-acuteness of the hexagonal mosaic the concept of the half-cell expansion is introduced, thereby enabling a cellular complex on the hexagonal mosaic to be converted to a cellular complex on the acute triangular mosaic. Then the convexity of complexes, i.e. cellular convexity, is defined in terms of a "preimage" of the complex, that is a continuous figure or polygon which could have been an analog continuous blob, or polygon, which generated the complex. More precisely, cellular convexity is defined in terms of a preimage of the half-cell expansion (on the triangular mosaic) of the given cellular complex on the hexagonal mosaic. Thus, if for a given complex on the hexagonal mosaic there exists a convex preimage of the the half-cell expansion (on the triangular mosaic) of the given complex, the complex is said to be cellularly convex.

In the first part of the thesis the hexagonal and triangular lattices are investigated in two ways. First, their sampling theorems are discussed, which obviously specify how close the image samples must be to represent the

J. Sklansky and D.F. Kibler (1976)
"A Theory of Nonuniformly Digitized Binary Pictures,"
input picture unambiguously. The hexagonal sampling theorem is not new, as it has been already covered in the literature, but it is used here as an introduction to the derivation of the triangular sampling theorem. The primary result of this derivation is rather surprising, which is the foremost reason why it is included in this thesis: the triangular sampling strategy is, from a signal-theoretical point of view, nothing more than a special case of the ordinary rectangular sampling strategy.

Second, the notion of connectivity is discussed and the related notion of the "genus" of an image, i.e. the number of components minus the number of holes. Furthermore, a newly derived algorithm is presented to calculate the genus for triangularly sampled images.

In the second part of the thesis the various definitions of convexity and the related geometric properties are considered. As a result a relation is established between discrete convexity and the different geometric properties that can be defined on convex sets of lattice points or on cellular complexes. A relation valid for both the rectangular and hexagonal sampling strategy. Finally, the relation between discrete convexity and straightness - a relation well defined and understood in Euclidean geometry - is discussed for the hexagonal case.
INTRODUCTION

This thesis is concerned with digital geometry, by which we mean the mathematical study of geometrical properties of digital picture subsets. In particular, we apply the theory of digital convexity to the shape analysis of subsets of digital images acquired by sampling the brightness values of the analog input picture at a hexagonal lattice of points. Digital or discrete convexity is well established for rectangularly sampled pictures, and an intriguing question is how to extend this concept to the hexagonal lattice.

The first sections of this introduction contain a general discussion of the background of this study. In the last section a more specific description of the subject of the thesis is presented, together with an outline.

1.1 General background: image sampling

In digital image processing and analyzing systems one usually deals with arrays of numbers obtained by spatially sampled points of a physical picture. (The elements of this array are called pixels, and the value is called its gray level.) Image samples normally represent some physical measurement of a continuous picture, for example, measurements of the brightness, photographic density, or some desired parameter. In the design and analysis of image sampling and reconstruction systems, we regard input pictures as deterministic fields. Let \( f(x_1, x_2) \) denote a continuous, spatially infinite, ideal picture field representing some parameter of a physical picture. In a perfect image-sampling system spatial samples of the ideal picture would, in effect, be obtained by multiplying the ideal picture by a spatial function composed of an infinite array of Dirac delta functions, arranged in a lattice of regular spacing. It is a well-known and fundamental fact that if the sampling is performed by a lattice of
rectangularly spaced Dirac delta functions, the exact reconstruction of a bandlimited image requires a sampling period equal to or smaller than one half the period of the finest detail within the analog input picture (Petersen and Middleton, 1962). This sampling condition is analogous to the one-dimensional sampling theorem for time-varying signals that requires a time-varying signal to be sampled at a rate of at least twice its highest-frequency component (Shannon, 1949). Less well appreciated is the knowledge that there are many strategies by which this sampling can be performed, each of which represents a different generalization of the one-dimensional periodic sampling (Mersereau, 1979). These alternative sampling strategies differ in their assumptions about how the continuous picture is bandlimited, in the number of samples that must be taken, and in the efficiency of the resulting image-processing algorithms. Importantly, rectangular sampling, the most common approach, is generally not the most efficient (Mersereau, 1979). In fact, the hexagonal sampling strategy requires 13.4% fewer samples than square sampling for isotropic band-limited input pictures. Furthermore, Mersereau (1979) showed that hexagonal sampling offers substantial savings of both machine storage and arithmetic computations ranging from 13 to 50% for many signal-processing operations compared with rectangular sampling. The important class of pictures having isotropic spectra is connected to the geometric problem of the densest packing of hyperspheres. The literature on sphere packings is extensive, and is intimately related to the theory of error-correcting codes in Information Theory.

Hexagonal sampling has seen applications in image processing, but the motivation was one of a topological nature, rather than one of computational savings. Each sample location in a rectangular sampling lattice has four axial nearest neighbors and four diagonal neighbors at greater distance. Each sample location in a hexagonal lattice, however, has six nearest neighbors, each of which is at the same distance. In an upcoming paragraph this topological property will be emphasized. First we proceed by analyzing other sampling strategies.

By a straightforward application of Euler’s polyhedral formula it can be shown (Ore, 1963) that only three repetitive planar graph patterns, or lattices, exist; and these can be formed by quadrangles, hexagons, or triangles as the primitives. This last lattice is slightly irregular:
neighboring triangular pixels that share an edge have opposite orientations. The behavior of the triangular lattice, from a signal-theoretical point of view, has not yet been investigated. Moreover, there does not seem to exist in the literature a sampling theorem suited for the triangular lattice. It is noted that the pixels in the triangular and rectangular lattices can be effected by a mesh of infinite lines, in a way which is not possible in the hexagonal case.

Finally we note that the triangular lattice is sometimes (Scholten and Wilson, 1983) referred to as a hexagonal lattice since the interconnections between the lattice points form a hexagonal grid. Similarly, the hexagonal sampling lattice is occasionally referred to as the triangular lattice (Freeman, 1979). In this thesis we will exclusively use the notion of hexagonality for the lattice and "mosaic" based on hexagonally shaped sampling regions, pixels or cells. Similarly, the notion of triangularity is uniquely assigned to the lattice and mosaic based on triangularly shaped sampling regions, pixels or cells.

After the image-sampling process, the second step in image processing is "segmentation". Segmentation is basically a process of assigning the pixels to classes: one simple way of doing this, called "thresholding", classifies the pixels according to whether or not their gray levels exceed a given threshold. Often the value of the threshold is derived from the histogram of all pixel gray levels; see for a review the survey paper of Weszka (1978). These and other methods of segmenting digital images will not be reviewed here; for an introduction to this subject see, e.g., Rosenfeld and Kak (1982).

Once an image has been segmented into subsets, it can be described in terms of properties of these subsets and their inter-relationships. Some of these properties depend on the gray levels of the points that belong to a subset, but others are "geometrical" properties which depend only on the positions of these points. Especially basic are topological properties of the subsets, involving such concepts as adjacency and connectedness, (Rosenfeld, 1970, 1979), whereas examples of geometric properties are convexity, elongatedness, three-lobedness, etc. This thesis develops some basic parts of digital geometry, dealing with connected sets, arcs and curves. Many other topics have been omitted, e.g. area and perimeter, distance and diameter, disks, and geodesics.
An alternative approach is to imagine the digitization of pictures as a many-to-one transformation produced by an artificial retina interposed between a scene and the digital computer. This approach, which has been proposed by Sklansky and co-workers emphasizes the translation of a "mosaic" into a continuous figure (Sklansky, 1970; Sklansky and Kibler, 1976; Fam, 1976). The size, shape, and density of the retinal elements (the artificial analog of rods and cones) affects the computer's ability to analyze the scene. Sklansky and co-workers restricted themselves to binary objects or "blobs", i.e. continuous black figures on a white background, and to binary "cellular images" of the objects, i.e. 1's on a background of 0's. A cell holding a 1 represents a nonempty projection of the object onto the cell. Essentially, when using this approach to image analysis and digital geometry, one must define a continuous curve through, for instance, the cellular boundary of such a cellular image or "cellular complex". Once such a curve is found, given the cellular complex, the processing can continue within the framework of continuous geometry. The choice of the continuous curve or the continuous "preimage" is, in essence, an approximation problem.

With regard to the size, shape and density of the cells in the mosaic, Pavlidis (1980) indicated that there is no finite sampling interval which permits the digitization of blobs without errors. Clearly, Shannon's criterion is not applicable because binary blobs have infinite bandwidths. For the case where the Dirac field approach, that is, the approach of sampling with a lattice of sample points, is applied to binary input pictures, Pavlidis (1980) introduced a new sampling theorem: A binary image and a square sampling lattice with lattice interval \( l \) are said to be compatible if the following condition holds: There exists a number \( d > l/\sqrt{2} \) such that for each boundary point of each region \( R \) of a given set, it is possible to find a circle with diameter \( d \) which is tangent to the boundary and lies entirely within \( R \). The same is also true for the complement of \( R \). For the hexagonal lattice with lattice constant \( h \) the critical number equals \( d > 2/3 h/\sqrt{3} \), the length of a diagonal of an individual cell. In the light of these figures the hexagonal lattice is superior to the square lattice under this sampling theorem. Given a maximum curvature of the figures to be digitized, the lattice constant \( h \) of the hexagonal lattice is related to the lattice constant \( l \) of the square lattice by \( h = 1/2 l/\sqrt{6} \): a gain in "efficiency" of 18.4%. The figure previously
found, based on the theory of Shannon, was 13.4\%, which is in fact
determined by the ratio of the area of a square cell and the area of a
hexagonal cell, both on a lattice with an equal lattice constant. In con-
trast, the 18.4\% figure is determined by the ratio of the radii of the
circumscribed circles of a square and a hexagonal cell, both on a lattice
with an equal lattice constant.

In yet another approach, when comparing the spatial resolution of lattices
with differently shaped cells, the density should be similar (Smeulders
et al., 1980). Then, clearly, $h = l \sqrt[4]{4/3}$. As a consequence, the figures
mentioned above are lowered by a factor of $\sqrt[4]{3/4}$, where in that case
these corrected figures no longer represent a gain in efficiency, but rather
one in spatial resolution, i.e., a gain in the ability to accurately represent
the input picture. In the same study of Smeulders et al. (1980), it is
concluded that with respect to shape analysis of isotropic blobs, the
hexagonal lattice is more isotropic than the rectangular lattice, that is, the
hexagonal lattice produces less quantization noise for circular figures than
the rectangular lattice. The gain upon the use of the hexagonal lattice
becomes more evident when smaller figures are analyzed.

Hexagonal sampling has seen some applications in image processing moti-
vated by the absence of logical complications in the sense of connectivity
(Rosenfeld and Pfaltz, 1968; Golay, 1969; Deutz, 1970, 1972). The
relative complication encountered in establishing logical-connectivity
functions for rectangular lattices stems from the fact that there are two
choices of connectivity: four-neighbor or eight-neighbor. The choice is left
to the user (Rosenfeld, 1970). As a result, if an object is to be eight-neigh-
bor connected, then the background must be four-neighbor connected,
and vice versa. This kind of choice of connectivity does not arise in the
case of hexagonal arrays; for these, both the object and background are
six-neighbor connected. Therefore the hexagonal lattice involves one in
less logical complications, a property exploited by Serra (1982).

One of the earliest applications of the hexagonal logic of Golay (1969) is
found in the analysis of blood cells (Ingram and Preston, 1970), while
Preston (1971) reported on the usage of Golay’s digital logic in a special-
purpose computer system to perform high-speed hexagonal picture
processing.
Historically, at the "Centre de Geostatistique et de Morphologie Mathematique" in Paris, Fontainebleau, the hexagonal lattice has been chosen as it is more appropriate to the concepts of Mathematical Morphology (Matheron, 1975; Serra, 1982). The structure of mathematical morphology is derived from a set of basic principles including the concept of a structuring element, a predefined shape which is employed as a probe to test out the spatial nature of the image being analyzed. Image transformations employing structuring elements are implemented as sequences of logical neighborhood transformations like erosion and dilatation. These operations are most easily realized in a hexagonal logic as has been shown in the "Texture Analyser" of Klein and Serra (1972), including TV-scanning input, sampling and quantizing, logic, programming matrices and output video.

Recently, Rosenfeld (1983) applied the hexagonal lattice in studying topological properties like connectivity of gray-scale images, primarily to avoid complications with the opposite-connectivity situation in rectangular lattices. Parenthetically, it is interesting to note that Rosenfeld explicitly ignored the special problems involved in defining convexity on a discrete grid.

In two other recently published papers, the hexagonal cell is used as a generator pattern for a hierarchy or pyramid of hexagonal arrays. Only those trees are considered in which the generator pattern is the same at each level, and which covers the plane, without overlap. First, Crettez and Simon (1982) used a hexagonal shape to model the structure of the receptive fields (RFs) of the cells in the visual primary cortex. The cell receptive field may be defined as the small region of the visual space over which we can influence the firing or the slackening of the cell activity by light or dark stimuli. The RFs must be considered as two-dimensional structures with probably a circular rather than a rectangular form (Crettez and Simon, 1982). The assumed circular symmetry of the ocular vision, and also the regular distribution of the ganglion RFs assumed in each RF layer, suggest a central and hexagonal tessellation which minimizes the gaps between circular RF centers of the ganglion cells. Under the assumption of a subdivision of the set of RFs into different layers, these layers are hierarchically constructed by repeated application of an elementary hexagonal connecting process.
Second, Burt (1980) introduced a pyramid structure for coding hexagonally sampled binary images. The structure is analogous to the quadtree which has recently been developed for rectangularly sampled images. Burt conjectures that the hexagonal lattice may provide the largest compact tile of any admissible pyramid structure defined on either hexagonal or rectangular lattices.

1.2 General background: convexity

The properties of convexity of continuous plane figures are well defined and understood, but what is meant by convexity of a cellular complex is not quite obvious. Two examples of continuous connected plane figures, which are obviously convex and concave, are shown in Fig. 1.1(a) and (b), respectively. Two cellular complexes are shown in Fig. 1.2. It is illustrated in Fig. 1.3 that the complexes in Fig. 1.2 are the digitized images of the corresponding figures of Fig. 1.1. When considered as continuous plane figures, both complexes are concave. In fact, every complex is concave, except for the single-pixel complex, and in the case of a rectangular mosaic, those of rectangular shape are exceptions. However, we want the definition of convexity of complexes to be such that the complex of Fig. 1.2(a) is convex, and the complex of Fig. 1.2(b) is concave. Hence, it is desirable to have a definition of convexity of complexes which satisfies our intuitive notion of convexity.

A few definitions of convexity of complexes have been proposed by Unger (1959), Feder (1968), Minsky and Papert (1969) and Sklansky (1970) and investigated quite extensively by Montanari (1970b), Sklansky (1970), Sklansky et al. (1972) and Chassery (1982). Recently, Kim published in rapid order a number of papers concerning convexity and straightness (Kim, 1981, 1982a, 1982b; Kim and Rosenfeld, 1980, 1982a; Kim and Sklansky, 1981, 1982). It turned out that the three independent definitions of convexity of complexes defined in the literature are virtually equivalent. Hence, the concept of convexity seems well defined in the discrete case. Many corresponding properties of convex regions in Euclidean geometry have been shown to hold in digital geometry for convex digital regions. However, these results have been obtained exclusively for the rectangular lattice and mosaic. It is not quite obvious whether these results apply to other lattices and mosaics, e.g. the hexagonal or triangular
Fig. 1.1  Two continuous connected plane figures (blobs).
(a) Convex blob. (b) Concave blob.
Fig. 1.2  Two cellular complexes on the hexagonal mosaic.
Fig. 1.3  Two continuous plane figures and their corresponding cellular complexes.
lattices and mosaics.

A more general theory of convexity developed by Sklansky and Kibler (1976) is applicable to a fairly large class of mosaics, i.e. the class of "acute" mosaics. An acute mosaic is defined as follows: every element of the mosaic is a convex polygon or cell, every finite region of the plane contains a finite number of cells, the cells are only overlapping at their boundaries, the union of every pair of adjacent cells forms a set bounded by a convex polygon, and the union of all the cells covers the entire plane (Sklansky and Kibler, 1976). Clearly, the triangular mosaic, in which an individual cell has the shape of an isosceles triangle, is included in the class of acute mosaics, and the hexagonal mosaic, in which an individual cell has the shape of a hexagon, is not.

The digitization of arbitrary blob-like objects on an acute mosaic may be accurately represented by "minimum-perimeter polygons", which usually contain many fewer vertices than the number of cells in the boundaries of the digitized blobs. (A minimum-perimeter polygon of a cellular complex is a polygon of the shortest perimeter among the polygons whose images coincide with the given cellular complex.) Sklansky (1970) and Montanari (1970b) have shown how the minimum-perimeter polygon of a "regular" cellular image can be an instrument for recognition of convexity on a rectangular mosaic. This result has been extended by Sklansky and Kibler (1976) to regular complexes on acute mosaics. The restriction of regularity requires that a complex have no peninsular protusion that is one cell wide, a condition in accordance with the usage in piecewise linear topology (Hudson, 1969). Although this is a weak condition, it is still significant, particularly in noisy images.

The restriction of regularity can be eliminated by using the concept of half-cell expansion, introduced by Sklansky (1972b). By sacrificing a small amount of accuracy, any simply connected cellular complex can be converted into a regular complex, thereby making it possible to apply the theory of minimum-perimeter polygons to all simply connected complexes. In this thesis the half-cell expansion will be explored to convert the non-acute hexagonal mosaic into the acute triangular mosaic, in such a way that the theory of convexity for acute mosaics can be extended to the hexagonal mosaic.
Fig. 1.4 A cellular complex, its convex hull and minimum-perimeter polygon; the concavities are represented by the shaded areas; the concavities within the concavities by the dotted cell edges.
Before analyzing a cellular complex, one should remove undesired noise from the given cellular complex, using, for example, the known techniques of edge sharpening, edge smoothing, and hole filling, gap filling, shrinking, etc.; see, e.g. Rosenfeld and Kak (1982). Montanari (1970a) has shown how even the minimum-perimeter polygon provides a means of smoothing the contour of a cellular complex. A hole-filling algorithm can be applied to convert all the given complexes to simply connected complexes. In other applications the holes are treated as meaningful components of the image and the shape of the holes can be described by the available concavity-measuring techniques described in Sklansky (1972) and Montanari (1970b).

Cellular blobs of arbitrary shape often occur in the analysis of chromosomes, blood cells, tumors, and aircraft silhouettes, (Mendelsohn et al., 1968; Ingram and Preston, 1970; Bacus and Gose, 1972; Ballard and Sklansky, 1973). Several techniques of analyzing digitized blobs are currently available. Among them are skeletons or medial axis transformation (Rosenfeld and Pfaltz, 1966; Blum, 1973), bending energy (Young et al., 1974), Fourier descriptors (Zahn and Roskies, 1972), and minimum-perimeter polygons (Sklansky, 1972). Among these techniques, the latter is particularly well suited to the analysis and description of concavities. The minimum-perimeter polygon can produce in co-operation with the convex hull a tree describing the hierarchic arrangement of concavities, concavities within concavities, etc. in a complex (Sklansky, 1972). This description is called a "cellular concavity tree". Fig. 1.5 shows the cellular concavity tree of the complex shown in Fig. 1.4. A related approach based solely on the convex hull is reported by Batchelor (1970a, 1970b).

Sklansky (1975) established the mathematical theory for concavity-filling techniques constructed from local operations that find the concavities of cellular blobs directly. Furthermore, Sklansky and co-workers (Sklansky et al., 1974, 1976), showed how any minimum-perimeter polygon may be computed by an iterative sequence of these local operations in a parallel-structured computer.

Another technique, frequently used, is the decomposition of cellular blobs into primary convex subsets (Feng and Pavlidis, 1975), a technique which is based on the assumption that shape perception is a hierarchical
Fig. 1.5 The cellular concavity tree of the complex shown in Fig. 1.4.
process (Pavlidis, 1968, 1972a, 1972b). The minimum-perimeter polygon is once again highly useful in establishing an appropriate representation of the blob under consideration in terms of a polygonal approximation of the original object. The result of decomposition has a number of desirable features (Pavlidis, 1977), of which we mention here two relevant ones:
1) They are translation and rotation invariant and insensitive to registration.
2) To a large extent they are size invariant. Problems may occur only when some of the objects in a picture are so small as to be of the same order of magnitude as what is considered noise for others.

A related approach is the analysis of convex blobs by computing dominant points which tend to dominate our perception (Rosenberg, 1972, 1974; Langridge, 1972). Rosenberg (1972) suggests that non-convex blobs can be analyzed by decomposing them into convex blobs and then computing the dominant points for each subset. However, an important aspect of the effect of using this form of convex/concave is that we do not obtain any relationship at all between the constituent parts. Davis (1977) noted that the approach of Rosenberg is not generally applicable and has shortcomings that preclude using it as a basis for a general theory of shape. Nevertheless, for the class of blobs which are convex in nature, the Rosenberg approach is highly suitable.

Finally, we indicate an early application of the notion of convexity in character recognition by Munson (1968), where the concavities lying in between a segmented handprinted character and its convex hull are used as attributes in a pattern classification system.

1.3 Statement of the problem and outline of the thesis

The subject to which this thesis is devoted is the study of convexity of blobs obtained by a hexagonal sampling procedure. The aim of the study is twofold:
1) The characterization in signal-theoretical terms of sampling procedures other than the rectangular sampling strategy.
2) The investigation into the extendibility of the concept of convexity to hexagonally represented blobs.
The first part of this objective requires an investigation into the performance of sampling procedures. With respect to the second part of the objective, the concept of convexity seems well defined and understood for blob-like objects represented on rectangular lattices and mosaics, mainly due to the papers recently published by Kim and co-workers. It is an intriguing question whether these concepts can be carried over to the hexagonal lattice and mosaic.

In this thesis we will only consider those blobs which are convex in nature, and if they are not constant in brightness, it is assumed that an appropriate segmentation algorithm has been applied. It follows that the images under consideration will be binary. The convexity hypothesis is well supported by the fact that most of the blobs which are considered in particle science (TiO\textsubscript{2} powder particles, carbon black, etc.) and in biomedicine (nuclei, nucleoli, lymphocytes, blood cells, etc.) have a convex shape (Arcelli and Levialdi, 1973). As a consequence of the fact that our picture is binary, all the information obtainable from the blob is present on its contour (Attneave and Arnoul, 1956). Then, one possible definition of a convex object is that of one having no points of negative curvature along its boundary. Thus, given the importance of curvature in shape perception (Attneave, 1954), it is natural to place special emphasis on convex objects as prototypes of simplicity. The many desirable mathematical properties of convex sets, are another factor (Valentine, 1964; Benson, 1966). Generally speaking, it is much easier to recognize the shape of convex objects than that of arbitrary objects, and there exist a number of techniques applicable only to this class of shapes (Pavlidis, 1977). Many of the simple shape descriptors give far more meaningful results or convex sets than on nonconvex. This is true for measures of elongation, symmetry, roundness, etc. even though they can be defined on arbitrary sets.

Although there is a growing interest in three-dimensional digital geometry, and several papers have been published characterizing 3D-convexity and other geometric properties, we will not treat the concept of convexity in three-dimensional polyhedral geometry (Kim and Rosenfeld, 1982; Kim, 1983, 1984). We note that there exist in general five admissible parallelohedra to tessellate the three-dimensional space without gaps and overlaps: cube, hexagonal prism, rhombic dodecahedron,
elongated dodecahedron, and truncated octahedron. In the sense of sphere packings the truncated octahedron is clearly the best parallelohedron and it is very likely that it is the optimal polyhedron (Gersho, 1979).

Outline of the thesis

The thesis is organized as follows. In Chapter 2, the two-dimensional sampling theorem as an extension of Shannon's one-dimensional sampling theorem for time-varying signals is reviewed, in order to derive two-dimensional sampling theorems for hexagonally and triangularly sampled pictures. Next, these regular mosaics are extended to a more general random tessellation.

In Chapter 3, the problem of defining convexity on cellular blobs is dealt with. First, the well-known Euler relation is evaluated for the three mosaics, i.e., rectangular, hexagonal, and triangular. Next, the several approaches to defining convexity on the rectangular mosaic are reviewed, in order to establish a consistent and sound theory for the analysis of convexity on the hexagonal lattice and mosaic. An initial attempt to define convexity on the hexagonal mosaic as a straightforward extension of the rectangular mosaic will be shown to fail due to the "non-acuteness" of the hexagonal mosaic.

In Chapter 4, the half-cell expansion is introduced as a means to circumvent the non-acuteness of the hexagonal mosaic. A new definition is introduced for the convexity of cellular complexes on hexagonal mosaics, which is proved to be equivalent to virtually all the geometric properties. Convexity is defined in terms of an "artificial" preimage which could have been an analog continuous blob, or polygon, that generated the half-cell expansion on the triangular mosaic of the complex under consideration on the hexagonal mosaic. When gray-scale input pictures are subject to shape analysis, this artificial preimage can be considered as the object obtained in a segmentation step, in the process of digitizing, segmenting and analyzing blob-like objects. By geometric properties we mean those properties of cellular complexes defined on the complex as such without reference to an originating preimage; that is, geometric properties of complexes within the framework of digital geometry and topology.
Next, the theory of minimal polygons is applied to cellular complexes on
the hexagonal mosaic, and finally this theory is extended to the half-cell
expansion on the triangular mosaic of the complex on the hexagonal
mosaic. As a result we establish a “hexagonal” relation between the
different definitions of discrete convexity and the different geometric
properties defined on sets of lattice points and cellular complexes.

In Chapter 5, the relation between convexity and straightness is discussed.
A well-known property of Euclidean convexity is that the straight line
segment joining any interior point $A$ of a convex figure to any other point
$B$ of the figure consists entirely (with the possible exception of the point
$B$) of interior points of the convex figure. It has been shown by Kim and
Rosenfeld (1982a) that such a relation exists in digital geometry based on
notions of convexity and straightness defined on the rectangular lattice
and mosaic. In this thesis it will be shown that such a relation carries over
to the hexagonal and triangular lattices and mosaics.

Remark 1: Although some writers insist on using the phrase “if and only
if” when stating a definition, we prefer to use the more classical “if”.
Should this displease the reader, we suggest that he replace the word “if”
by the alternative “iff” in the definitions.

Remark 2: For reasons of graphical clarity, continuous figures will be
depicted by a closed solid line and by a labeled arrow pointing to its
boundary only, instead of a totally black region on a white background
and an arrow pointing to the region.
2.1 Introduction

Pictures can be considered as continuous two-dimensional functions, the gray level being a continuous function of the position in the picture. The aim of sampling is to represent this two-dimensional signal by a finite string or array of numbers, called samples. Then, the only constraint on the numbers is that it should be possible to reconstruct the picture from them. An alternative approach is to imagine the digitization process as a many-to-one transformation produced by an artificial retina interposed between a scene and a digital computer. The size, shape, and density of the retinal elements (the artificial analog of human rods and cones) affect the computer's ability to analyze the scene. In both concepts the retinal elements or pixels are assumed to form a lattice.

The most common type of array is the rectangular lattice. This is not surprising, for conventional scanning techniques, operating in a line-by-line fashion, readily suggest this type of lattice and the subsequent form of processing. Furthermore, it is the most natural extension of the well-known Whittaker-Kotelnikov-Shannon sampling theorem for bandlimited one-dimensional signals. Two other regular lattices are the hexagonal and triangular lattice in which the individual pixels have the polygonal shape of a hexagon and a triangle, respectively. The fact that none of these lattices follows a strict row-and-column arrangement is of no real consequence, for given a fixed rectangular lattice, the others can always be derived from it, see, e.g., Laž (1984).

In general, the sensor elements may have an odd shape or may be non-uniformly distributed. Such nonuniform lattices, while presently uncommon, are likely to be used more frequently in the future for several reasons. Sklansky and Kibler (1976) have given the following arguments: 1) Variable resolution for acquiring fine detail in selected areas can be
achieved by lattices with spatially varying lattice intervals.

2) The need for detecting special shapes, such as circles, angles, crosses, straight lines, or concavities, may lead to the use of special nonuniform lattices specially designed for the analysis of these shapes.

3) As the technology for arrays of light-sensitive devices progresses, the elements of these arrays may become so small that it may become impossible to accurately place these elements in, say, rectangular lattices. Consequently, the lattices of future hardware realizations of artificial retinas may be similar to the nonuniform lattices found in human retinas (Lindsay and Norman, 1972).

4) Nonuniform lattices are appropriate for modeling human and other natural retinas.

The theory of uniform sampling lattices and their spatial frequency behavior was treated in a classic paper by Petersen and Middleton (1962). The case of a rectangular sampling lattice has also been given by Bracewell (1956). The case of a hexagonal sampling lattice with emphasis on the aliasing phenomenon has been treated by Montgomery (1975). Petersen and Middleton (1962) showed that the rectangular sampling lattice is a special case of a more general sampling lattice using a skewed periodic lattice in two (or even more) dimensions provided the analog input picture is bandlimited, i.e. its Fourier transform vanishes outside some region in the frequency domain. Hexagonal sampling can also be covered by this general theory. The use of a generalized approach, rather than the development of a different theory for each specific sampling lattice, is deliberate; the advantage of this is that the relative merits of each lattice can be assessed more easily. Otherwise, factors pertaining to the individuality of each approach would have been taken into account.

Of interest will be the relative efficiency of each lattice. A lattice is said to be an efficient lattice if it uses a minimum number of samples to achieve an exact reproduction of the bandlimited picture. The efficiency of the lattice is dependent upon the tiling of the Fourier space in elementary tiles, the form of which depends on the band region of the analog input picture. The elementary band region that embeds the band region of the input picture with the smallest area and thereby completely covers the Fourier space without gaps or overlaps leads to a sampling lattice with the lowest possible density. An important class of pictures having isotropic
spectra - isotropic in the sense of describing a spectrum which cuts off at the same frequency in all spatial directions, i.e. one whose band region is a circle - is connected to the geometric problem of the densest packing of hyperspheres. The literature on sphere packings is extensive, and the subject is intimately related to the theory of error-correcting codes in Information Theory. For more information the reader is referred to Leech and Sloane (1971), Rogers (1964), Schrijver (1979) and Sloane (1977). A few relevant parameters of sphere packings will be presented in Section 2.2.

By a straightforward application of Euler's polyhedral formula it can be shown (Ore, 1963) that there exist only three repetitive planar graph patterns or lattices; these can be formed by quadrangles, hexagons, or triangles as the primitives. Each of the three lattices is dealt with in turn in the following sections. The shape of the band region and the tiling of Fourier space as a result of a two-dimensional sampling theorem are then developed. Finally, the relative efficiency of each of the three lattices is determined assuming a circular band region of the analog input picture. Nonuniform lattices will be treated in the last section.

2.2 Sphere packings on lattices

In vector quantization, also known as block or multi-dimensional quantization, an ordered set of k samples (i.e. a k-dimensional vector in Euclidean space) is mapped onto one or a finite set of representative or "output" vectors. The coder identifies the region of space in which the input vector lies and assigns a corresponding code word. Note that the coder is essentially defined by a partition of the k-dimensional space of vectors or points into a finite number of regions. The geometry of this partition is of fundamental interest in the theory of optimal quantization.

It has been shown (Gersho, 1979) that an optimal k-point quantizer that minimizes distortion in the mean-squared-error sense has a Voronoi partition. Furthermore, it is necessary that each output point be the centroid of the region in which it lies.

Quantizers whose points lie on a regular lattice are treated by Sloane
(1977) in a comprehensive review of the literature on lattices, sphere packings, and the use of binary codes to generate lattices. Sloane (1981) also gives a useful set of tables containing several important lattices, of which the two-dimensional lattices deserve attention in relation to digital image processing.

In the following we will present some useful properties of lattices and sphere packings.

A regular lattice, that is a homogeneous, periodic arrangement of points in the plane, is defined by any non-singular 2x2 matrix $M$ of linearly independent base vectors which span the lattice. The matrix $M$ is called the generator matrix of the lattice $L$. Note that the rows of $M$ are points of the lattice and that $L$ consists of all integer combinations of the rows of $M$. The determinant of $L$ is

$$\det L = (\det MM^T)^{\frac{1}{2}} = |\det M|.$$  \hfill (2.2.1)

Three useful properties of a lattice are the density, the kissing number, and the theta function.

The **density** $\Delta$ of any sphere packing is, loosely speaking, the fraction of the space that is covered by nonoverlapping spheres centered about the lattice points. The density of a plane lattice is given by

$$\Delta = \frac{\pi \rho^2}{\det L},$$  \hfill (2.2.2)

where $\rho$ is the radius of $L$, i.e. $2\rho$ is the minimum distance between two lattice points.

The **kissing number** $\tau$ is the number of spheres that touch a given sphere, where the spheres are centered about the lattice points and have a maximum radius without overlapping each other.

The **theta function** $\Theta_L$ of a lattice gives the number of centers at each distance from a lattice point $l$. The theta function is defined as

$$\Theta_L(z) = \sum_{x \in L} q^{\|x\|},$$  \hfill (2.2.3)
where $q = e^{\pi i z}$. The theta function is a holomorphic function of $z$ for $\text{Im}(z) > 0$ (Gunning, 1962). If $N_m$ denotes the number of centers $x \in L$ with $\|x\| = m$, i.e. at a squared Euclidean distance of $m$ from the lattice point $I$, then (2.2.3) can be rewritten as

$$
\Theta_L(z) = \sum_{m=0}^{\infty} N_m q^m,
$$

(2.2.4)

where $m$ runs through all the values of $\|x\|$ for $x \in L$. The first two terms of $\Theta_L(z)$ are 1 and $\tau q^4 \rho^2$, where $\rho$ and $\tau$ are defined as above.

The theta functions of many lattices can be specified concisely in terms of the classical Jacobi theta functions $\theta_2$ and $\theta_3$, which are defined as follows:

$$
\theta_2(z) = 2 \sum_{m=0}^{\infty} q^{(m + \frac{1}{2})^2} = 2q^{1/4} + 2q^{9/4} + 2q^{25/4} + \ldots,
$$

$$
\theta_3(z) = 1 + 2 \sum_{m=1}^{\infty} q^{m^2} = 1 + 2q + 2q^4 + 2q^9 + \ldots.
$$

The integral quadratic form associated with $L$ is defined as

$$
\xi(x_1, x_2) = \sum_{i,j,k=1}^{2} x_i M_{ij} M_{kj} x_k
$$

or in vector notation

$$
\xi(x) = x M M^T x^T.
$$

(2.2.6)

The theta function can then be rewritten as

$$
\Theta_L(z) = \sum_{x \in L} q^{x M M^T x}.
$$

(2.2.7)
A very old problem in number theory poses the question of the number of ways an integer \( m \) can be expressed as the sum of squares. A moment's thought shows that the general answer is given by the coefficient of \( q^m \) in the expansion of \( \Theta_3(z)^4 \) (Sloane, 1981). This link with number theory makes it possible to apply the vast literature on diophantine equations and modular forms to the study of lattices (Gunning, 1962; Dickson, 1966; Hardy and Wright, 1954; Ranking, 1977).

The Voronoi region \( V(x) \) (Voronoi, 1909) of a lattice point is defined as the region consisting of all points which are closer to that point than to any other. More precisely, \( V(x) \) is defined as the closed set:

\[
V(x) = \{ y \in \mathbb{R}^2 : \|x - y\| \leq \|x - y_j\| \text{ for all } j \neq i \} .
\]  

(2.2.8)

Voronoi regions are also called Dirichlet regions, Brillouin zones, Wigner-Seitz cells, or nearest-neighbor regions.

The area of \( V(x) \) can be expressed in the standard parameters of the lattice:

\[
\text{Area } V(x) = \frac{\pi p^2}{\Delta} = \det L .
\]  

(2.2.9)

Finally, \( \beta \) denotes the number of Voronoi cells meeting at each of its vertices and \( \tau \), the kissing number, is equal to the number of sides of the Voronoi region.

2.3 Rectangular sampling lattices

Let the two-dimensional matrix of real numbers \( g(x_1, x_2) \) represent a discrete monochromatic image. The variable (or sample) \( g(x_1, x_2) \) represents the intensity value of the analog input picture \( f(x_1, x_2) \) at the spatial coordinate \( (x_1, x_2) \), where \( x_1 \) and \( x_2 \) are the vertical and horizontal position variables, respectively. It will be convenient in subsequent developments to use \( x_1 \) and \( x_2 \) as either discrete or continuous variables, depending on the context of the discussion. More realistically, the variable \( g(x_1, x_2) \) represents an integrated value about a small neighborhood of the coordinate \( (x_1, x_2) \), whose region depends on the
physical properties of the sensor - or retinal-element. The image representation \( g(x_1, x_2) \) is stored for computer processing purposes as an array of numbers.

**Vectorial approach**

Intuitively, it is clear that if the samples are taken sufficiently close to each other, the sampled data are an accurate representation of the analog picture in the sense that \( f(x_1, x_2) \) can be reconstructed with considerable accuracy by simple interpolation. For a particular class of images (known as bandlimited images) the reconstruction can be accomplished exactly, providing only that the interval between samples is not greater than a certain limit. This result was first pointed out by Petersen and Middleton (1962) for N-dimensional signals sampled on a skewed (i.e. nonorthogonal) sampling grid.

An analog picture \( f(x_1, x_2) \) whose Fourier spectrum \( F(\omega_1, \omega_2) \) vanishes over all but a bounded region \( R \) of the spatial frequency plane can be reproduced from its values taken over a regular lattice provided the lattice intervals \( l_1 \) and \( l_2 \) are small enough to ensure nonoverlapping of the spectrum \( R \) with its copies on the regular reciprocal lattice with lattice radii \( b_1 \) and \( b_2 \) with \( l_1 \leq \pi/b_1 \) and \( l_2 \leq \pi/b_2 \). The constant \( l_1 \) denotes the interline distance and \( l_2 \) is the sample interval on a scan line, while \( b_1 \) and \( b_2 \) are the bandwidths of \( R \) in the \( w_1 \) and \( w_2 \) direction, respectively.

The base vectors \( w_1 \) and \( w_2 \) spanning the reciprocal lattice in the Fourier plane are uniquely derived from the base vectors \( x_1 \) and \( x_2 \) of the spatial sampling lattice by

\[
x_i \cdot w_j = 2\pi \delta_{ij},
\]

(2.3.1)

where \( i \) and \( j \) independently take on the values 1 and 2, \( \delta_{ij} \) stands for the Kronecker delta and \( \cdot \) denotes the ordinary dot product.

In Fig. 2.1(a) the rectangular sampling lattice is shown with the black dots as the actual sample points, together with the boundaries of the sensor elements (pixels), indicated by the solid lines. The discrete image can now be written as a sequence of samples of the
Fig. 2.1 Rectangular sampling. (a) Sampling locations in the \((x_1', x_2')\)-plane and pixel-boundary grid. (b) Rectangular band region \(R\) in the \((\omega_1, \omega_2)\)-plane.
analog picture taken in two directions:

\[ g(m_1, m_2) = f(m_1 l_1, m_2 l_2) , \quad (2.3.2) \]

where \( m_1 \) and \( m_2 \) take the discrete values 0, ± 1, ± 2, ....

According to the two-dimensional sampling theorem, for \( f(x_1, x_2) \) to be exactly recoverable from \( g(x_1, x_2) \), it must be bandlimited over the rectangular region \( R \) shown in Fig. 2.1(b).

**Analytical approach**

The sampling process for two-dimensional pictures can be formulated mathematically by making use of the two-dimensional impulse function \( \delta(x_1, x_2) \), which is defined by

\[ f(m_1, m_2) = \iint_{-\infty}^{\infty} f(x_1, x_2) \delta(x_1 - m_1 l_1, x_2 - m_2 l_2) \, dx_1 \, dx_2 . \quad (2.3.3) \]

A two-dimensional sampling function consists of a train of impulses separated \( l_1 \) units in the \( x_1 \)-direction and \( l_2 \) units in the \( x_2 \)-direction located at the black dots as in Fig. 2.1(a). This field of impulses is denoted as \( s(x_1, x_2) \):

\[ s(x_1, x_2) = \sum_{m_1 = -\infty}^{\infty} \sum_{m_2 = -\infty}^{\infty} \delta(x_1 - m_1 l_1, x_2 - m_2 l_2) \quad (2.3.4) \]

and its Fourier transform is denoted by \( S(\omega_1, \omega_2) \):

\[ S(\omega_1, \omega_2) = \frac{4\pi^2}{l_1 l_2} \sum_{m_1 = -\infty}^{\infty} \sum_{m_2 = -\infty}^{\infty} \delta(\omega_1 - \frac{2\pi m_1}{l_1}, \omega_2 - \frac{2\pi m_2}{l_2}) . \quad (2.3.5) \]

Given \( f(x_1, x_2) \), where \( x_1 \) and \( x_2 \) are continuous, a sampled version is obtained by forming the product \( s(x_1, x_2) f(x_1, x_2) \). The equivalent operation in the spatial frequency domain is a convolution of \( S(\omega_1, \omega_2) \) and \( F(\omega_1, \omega_2) \), where according to (2.3.5) \( S(\omega_1, \omega_2) \) is a field of impulses with separation \( 2\pi/l_1 \) and \( 2\pi/l_2 \) in the \( w_1 \) and \( w_2 \) directions, respectively. If \( f(x_1, x_2) \) is bandlimited the result of convolving might look like the case shown in Fig. 2.1(c). Note that the function shown is
Fig. 2.1  (Continued.) Rectangular sampling. (c) Periodic extension of the transform of a continuous bandlimited signal, bandlimited within band region $R$ of (b).
periodic in two dimensions. However, when \( f(x_1, x_2) \) is also space-limited by using a two-dimensional rectangular window \( h(x_1, x_2) \) the problem is that the transform of the sampled image is distorted by the convolution of \( H(\omega_1, \omega_2) \) and \( S(\omega_1, \omega_2) \ast F(\omega_1, \omega_2) \). This distortion, which is due to the spatially limited nature of digital images, precludes complete recovery of \( f(x_1, x_2) \) from its samples. To avoid mathematical contradiction between space-limitedness and band-limitedness we will consider the original image to be periodic horizontally and vertically (Dudgeon and Mersereau, 1984).

Another phenomenon, frequently referred to as aliasing, precludes complete recovery of an under-sampled image, i.e. sampling with greater intervals than specified by the sampling theorem. The convolution of \( S(\omega_1, \omega_2) \) and \( F(\omega_1, \omega_2) \) will result in overlapping copies of the band region \( R \). Unfortunately, this overlapping is irreversible and the reconstructed image will not be equal to the analog input picture.

**Efficiency and spatial neighborhood**

The efficiency of a sampling lattice is defined as the ratio between the area of the true band region of the analog input picture and the area of the region \( R \) with which the Fourier plane is tessellated. In general, the area of \( R \) is determined by the parallelogram spanned by the two reciprocal lattice vectors \( w_1 \) and \( w_2 \). It is important to note here that this parallelogram is not the only repetitive figure with \( \{ w_1, w_2 \} \) as a base. In fact, an infinite variety of spectra may exactly be reproduced under a given sampling lattice. If the band region of the input picture does not constitute a tiling of the Fourier plane without gaps or overlaps then the associated region \( R \) is sought such that \( R \) tessellates the Fourier plane and encloses the band region of the input picture.

The most common appearance of the rectangular sampling lattice is the square sampling lattice with sampling interval

\[
i = l_1 = l_2 .
\]  

(2.3.6)

In this case, the analog input picture is assumed to be bandlimited within a square shaped band region \( R \) with limiting frequency \( \omega = \pi / l \) in both spatial-frequency directions.
Fig. 2.2  The square sampling lattice. The first five shells around a lattice point contain 1, 4, 4, 4 and 8 points, respectively.
The generator matrix of this sampling lattice is then given by

\[ M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \]  
(2.3.7)

with \( \rho = l/2 \), \( \text{det} = l^2 \) and \( \tau = 4 \).

The associated quadratic form of the lattice follows from (2.2.6):

\[ \xi(x_1, x_2) = x_1^2 + x_2^2 \]  
(2.3.8)

leading to the theta function

\[ \Theta_L(z) = \sum_{x_1, x_2 = -\infty}^{\infty} q^{x_1^2 + x_2^2} \]  
(2.3.9)

\[ = \left[ \theta_3(z) \right]^2 \]

\[ = 1 + 4q + 4q^2 + 4q^4 + 8q^5 + \ldots \]  

The coefficients of \( q \) denote the number of lattice points on consecutive shells around a lattice point, as shown in Fig. 2.2. The first five shells contain 1, 4, 4, 4, 8 lattice points, respectively. So each pixel has four nearest neighbors, four second-nearest neighbors, four third-nearest neighbors and so on.

An analog input picture having a square band region with limit frequency \( \pi/l \) will be sampled on a square lattice with sampling interval \( l \) with 100% efficiency. However, the efficiency for circular bandlimited pictures is significantly less than 100%. For this class of input pictures the density of the reciprocal lattice in terms of the theory of sphere packings expresses the efficiency of the sampling lattice. With the generator matrix \( M \) of the reciprocal lattice

\[ M = \begin{bmatrix} 2\pi/l & 0 \\ 0 & 2\pi/l \end{bmatrix} \]  
(2.3.10)

with \( \text{det} = 4\pi^2/l^2 \), the density follows from

\[ \Delta = \pi/4 = 0.785 \]  
(2.3.11)
Fig. 2.3 Hexagonal sampling. (a) Sampling locations in the $(x_1, x_2)$-plane determined by the vectors $u_1$ and $u_2$ which comprise a skewed sampling raster. (b) Sampling locations in the $(\omega_1, \omega_2)$-plane determined by the reciprocal vectors $v_1$ and $v_2$. 
leading to an efficiency of the square sampling lattice for circular band-limited pictures of 78.5%, which is, loosely speaking, the fraction of the square band region of the lattice that is covered by the inscribed circle of the circular bandlimited input picture.

2.4 Hexagonal sampling lattices

The coordinate system of the hexagonal lattice has in principal three directions given by the vectors \( u_1, u_2 \) and \( u_3 \), as shown in Fig. 2.3(a). However, two of these vectors, for instance the vectors \( u_1 \) and \( u_2 \), should be sufficient to form a base in the image plane. Given any lattice point \( u \), we can reach \( u \) from the origin by making an integer number of steps \( p_1 \) (positive, negative or zero) in the direction of \( u_1 \) and then an integer number of steps \( p_2 \) in the direction of \( u_2 \); then \( (p_1, p_2) \) are the coordinates of \( u \). Henceforth, these oblique coordinates of a point on the hexagonal lattice will always be given with respect to the \( u_1 \) and \( u_2 \) axes.

**Vectorial approach**

The hexagonal sampling lattice is then most easily visualized as being generated by all linear combinations of the two base vectors \( u_1 \) and \( u_2 \); that is, each lattice point \( u \) is given by the position vector \( u \):

\[
 u = p_1 u_1 + p_2 u_2 .
\]  

(2.4.1)

Fig. 2.3(a) shows the hexagonal sampling lattice together with the boundary of the pixel at the origin, and the parallelogram spanned by \( u_1 \) and \( u_2 \) forming a skewed sampling raster. The constant \( r_1 \) denotes the interline distance and \( r_2 \) the sampling interval on a line.

To construct the reciprocal lattice in the spatial-frequency domain that corresponds to the hexagonal lattice we first construct according to (2.3.1) a vector \( v_1 \) perpendicular to \( u_2 \). Its length is determined by the condition that the dot product \( u_1 \cdot v_1 \) must equal \( 2\pi \). Similarly, \( v_2 \) is constructed perpendicular to \( u_1 \), so that the dot product \( u_2 \cdot v_2 \) equals \( 2\pi \). The reciprocal lattice is then generated by taking all linear combinations of \( v_1 \) and \( v_2 \); that is, each reciprocal lattice point is given by the position
Fig. 2.4  The Solomon’s seal band region formed by two interlaced triangles. Note that $b = b_1 = b_2$.

Fig. 2.5  The limit quadrangles of the family of hexagonal band regions; the common points are indicated by open circles.
vector $v$:

$$v = q_1 v_1 + q_2 v_2.$$ \hspace{1cm} (2.4.2)

In Fig. 2.3(b) the reciprocal lattice is shown by the black dots together with the base vectors $v_1$ and $v_2$ and the reciprocal parallelogram spanned by these base vectors.

The reciprocal lattice is uniquely determined by the sampling lattice, the shape of the band region, however, is rather arbitrary as long as the Fourier plane is tessellated by the band region and its periodic copies are without overlap. In fact, an infinite variety of spectra may exactly be reproduced under this sampling lattice.

Out of the family of hexagonal band regions we have selected the Voronoi region $V(o)$ about the origin, having the same area as the parallelogram spanned by $v_1$ and $v_2$, as shown in Fig. 2.3(b). It is a space-filling region, i.e. its periodic copies about each reciprocal lattice point cover the entire $(w_1, w_2)$-plane and its construction depends only on the reciprocal lattice. To prevent aliasing due to undersampling, the analog input picture, when hexagonally sampled according to (2.4.1), should be bandlimited within a band region $R$ equal to this Voronoi region.

We could have selected as band region the enclosed polygon of the 6-pointed star, sometimes referred to as Solomon’s seal, i.e. the convex polygon formed by two interlaced triangles each having two equal sides. This polygon, having the same area as $V(o)$, is shown in Fig. 2.4.

Finally, the two band regions limiting the family of hexagonal band regions are shown in Fig. 2.5. These limit polygons have the shape of a rectangle and a diamond, as degenerates of a hexagon. The hexagons of the family, including these unit polygons, have six points in common, three positioned midway between the two base vectors $v_1$ and $v_2$ and midway between the vector $v_3$ perpendicular to $u_3$, and three positioned on these vectors in opposite direction. The six points are indicated in Fig. 2.5 by open circles.

Furthermore, the area of each polygon is given by the area of the
parallelogram spanned by $v_1$ and $v_2$: 

$$\text{area } V(o) = \det L.$$ 

**Analytical approach**

A formal proof of the form of the reciprocal lattice will be outlined in terms of the Fourier transform of the sampling lattice using the Dirac-field approach. Mersereau (1979) showed that a hexagonal lattice can be expressed as the superposition of two rectangular two-dimensional fields of impulses, one of which is a shifted copy of the other, both having lattice interval $2r_1$ in the vertical $x_1$-direction and $r_2$ in the horizontal $x_2$-direction. Using (2.3.4) yields 

$$s(x_1, x_2) = \sum_{m_1 = -\infty}^{\infty} \sum_{m_2 = -\infty}^{\infty} \delta(x_1 - m_1 2r_1, x_2 - m_2 r_2) +$$

$$+ \sum_{m_1 = -\infty}^{\infty} \sum_{m_2 = -\infty}^{\infty} \delta(x_1 - m_1 2r_1 - r_1, x_2 - m_2 r_2 - \frac{1}{2} r_2).$$

(2.4.3)

The Fourier transform of $s(x_1, x_2)$ is given by 

$$S(\omega_1, \omega_2) = \frac{4\pi^2}{2r_1 r_2} \sum_{m_1 = -\infty}^{\infty} \sum_{m_2 = -\infty}^{\infty} \delta(\omega_1 - \frac{\pi m_1}{r_1}, \omega_2 - \frac{2\pi m_2}{r_2}).$$

$$\cdot [1 + \exp(-j(\omega_1 r_1 + \omega_2 r_2/2))]$$

(2.4.4)

which leads to 

$$S(\omega_1, \omega_2) = \frac{4\pi^2}{2r_1 r_2} \sum_{m_1 = -\infty}^{\infty} \sum_{m_2 = -\infty}^{\infty} \delta(\omega_1 - \frac{\pi m_1}{r_1}, \omega_2 - \frac{2\pi m_2}{r_2}).$$

$$\cdot (1 + (-1)^{(m_1 + m_2)})$$

(2.4.5)

in which the term $(-1)^{(m_1 + m_2)}$ is the Fourier transform of the shifting operation of one of the fields of impulses in the spatial domain with respect to the other field.
The last factor between parenthesis in (2.4.5) equals zero if \( m_1 + m_2 \) is odd, so obviously \( m_1 + m_2 \) should be set even, e.g.

\[
m_1 = n_1 \quad \text{and} \quad m_2 = 2n_2 - n_1. \tag{2.4.6}
\]

In this case it follows that

\[
S(\omega_1, \omega_2) = \frac{4\pi^2}{r_1 r_2} \sum_{n_1 = -\infty}^{\infty} \sum_{n_2 = -\infty}^{\infty} \delta(\omega_1 - \frac{\pi n_1}{r_1}, \omega_2 - \frac{2\pi (2n_2 - n_1)}{r_2}),
\]

which is the form for the hexagonal lattice with line interval \( \pi/r_1 \) in the \( w_1 \)-direction and an interval of \( 4\pi/r_2 \) in the \( w_2 \)-direction, as shown in Fig. 2.3(b).

The orthogonal band region that belongs to the orthogonal expression (2.4.7) is just the rectangular limit polygon as given in Fig. 2.5. The proper sampling intervals can be derived in a straightforward manner, from the sizes of this band region \( (\pi/r_1 \text{ and } \pi/r_2) \).

Assuming a Voronoi band shape, the Fourier plane is tessellated in Voronoi regions with a shape identical to \( V(\sigma) \), as shown in Fig. 2.6. Then, the function \( S(\omega_1, \omega_2) \) consists of single impulses located at the nucleus of each of these Voronoi regions. This periodic extension of the spectrum of the analog input picture can be determined from its Fourier transform through convolution:

\[
F_p(\omega_1, \omega_2) = F(\omega_1, \omega_2) * S(\omega_1, \omega_2). \tag{2.4.8}
\]

If \( f_p(x_1, x_2) \) denotes the inverse Fourier transform of \( F_p(\omega_1, \omega_2) \), then \( f(x_1, x_2) \) can be recovered by filtering \( f_p(x_1, x_2) \) with a low-pass filter with a unit frequency response over region \( R \) and a zero response elsewhere. To make this argument more quantitative a formal derivation will be given of this theorem.

Let the bandwidths of the hexagonal band region \( R \) be given by \( b, b_1 \), and \( b_2 \) as indicated in Fig. 2.3(a). As before, we write the hexagonal lattice as the superposition of two rectangular fields of impulses to perform the inverse Fourier transform.
Fig. 2.6 Periodic extension of the band region given in Fig. 2.3(b).
\[
S(\omega_1, \omega_2) = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \delta(\omega_1 - n_1, 2b_1, \omega_2 - n_2(2b_2 + b)) + \\
\quad + \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \delta(\omega_1 - n_1, 2b_1 - b_1, \omega_2 - n_2(2b_2 + b)) \cdot \frac{2b_2 + b}{2},
\]

where the incremental variables \(n_1\) and \(n_2\) do not necessarily take the same values for a specific lattice point as \(n_1\) and \(n_2\) do in equation (2.4.7).

The inverse Fourier transform yields
\[
s(x_1, x_2) = \frac{1}{2b_1(2b_2 + b)} \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \delta(x_1 - \frac{2\pi n_1}{2b_1}, x_2 - \frac{2\pi n_2}{2b_2 + b}) + \\
\quad + \frac{1}{2b_1(2b_2 + b)} \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \delta(x_1 - \frac{2\pi n_1}{2b_1}, x_2 - \frac{2\pi n_2}{2b_2 + b}) \cdot (n_1 + n_2),
\]
in which the last factor is again due to the shifting operation. This factor equals zero if \(n_1 + n_2\) is odd, so obviously \(n_1 + n_2\) should be set even, e.g.
\[
n_1 = m_1 \quad \text{and} \quad n_2 = 2m_2 \cdot m_1,
\]
where \(m_1\) and \(m_2\) are not equal to \(m_1\) and \(m_2\) defined in equation (2.4.3) by the same argument as above.

Substituting (2.4.11) in (2.4.10) gives
\[
s(x_1, x_2) = \frac{h_1 \cdot h_2}{4\pi^2} \sum_{m_1=-\infty}^{\infty} \sum_{m_2=-\infty}^{\infty} \delta(x_1 - m_1, x_2 - \frac{2m_2 \cdot m_1}{2} h_2),
\]
where for notational convenience \(h_1\) and \(h_2\) are defined by
\[
h_1 = \frac{\pi}{b_1} \quad \text{and} \quad h_2 = \frac{4\pi}{2b_2 + b}.
\]

Analysing (2.4.12) and (2.4.13) reveals that \(h_1\) and \(h_2\) are the maximum sampling intervals, that is, \(h_1\) is the interline distance equal to \(r_1\) and \(h_2\) is
the sample interval on a line equal to \( r_2 \). Therefore, (2.4.12) describes the hexagonal lattice (2.4.3).

The inverse Fourier transform of (2.4.8) is given by

\[
f_p(x_1, x_2) = (4\pi)^2 f(x_1, x_2) s(x_1, x_2) .
\]  

(2.4.14)

Substituting (2.4.12) in (2.4.14) gives

\[
f_p(x_1, x_2) = h_1 h_2 \sum_{m_1 = \infty}^{\infty} \sum_{m_2 = \infty}^{\infty} f(m_1 h_1, \frac{2m_2 - m_1}{2} h_2) .
\]

\[
\cdot \delta(x_1 - m_1 h_1, x_2 - \frac{2m_2 - m_1}{2} h_2) .
\]  

(2.4.15)

Consequently, it is possible to recover \( f(x_1, x_2) \) from \( f_p(x_1, x_2) \) by low-pass filtering over a hexagonal band region \( R \). To demonstrate this, the impulse response of such a filter is substituted in (2.4.15) to give

\[
f(x_1, x_2) = h_1 h_2 \sum_{m_1 = \infty}^{\infty} \sum_{m_2 = \infty}^{\infty} f(m_1 h_1, \frac{2m_2 - m_1}{2} h_2) .
\]

\[
\cdot d(x_1 - m_1 h_1, x_2 - \frac{2m_2 - m_1}{2} h_2) ,
\]  

(2.4.16)

where

\[
d(x_1, x_2) = \frac{1}{4\pi^2} \int_{R} \int \exp[i(x_1 \omega_1 + x_2 \omega_2)]d\omega_1 d\omega_2 .
\]  

(2.4.17)

This interpolation function allows the values of \( f(x_1, x_2) \) to be reconstructed at points in between the sample locations given by the hexagonal sampling lattice.

Formula (2.4.16) shows clearly that restricting the Fourier transform of the analog input picture \( f(x_1, x_2) \) to a hexagonal band region \( R \) and computing the inverse transform followed by a low-pass filter over that band region exactly recovers the hexagonally sampled analog picture as a function of its sample values.

It is clear that if the analog picture is not hexagonally bandlimited,
sampling with the intervals given by (2.4.13) will result in overlapping of the shifted copies of the region \( R \). Similar to the rectangular sampling process, this overlapping is irreversible and results in a dramatic distortion in the reconstructed image.

**Shape of the lattice**

Let \( \varphi \) denote the angle between the \( u_j \) base vector and the positive \( x_j \)-direction as indicated in Fig. 2.3(a). Then the resulting sampling lattice as a function of \( \varphi \) and \( h \) is described by the generator matrix

\[
M = \begin{bmatrix}
    h & -h \\
    \frac{2 \tan \varphi}{2} & \frac{2 \tan \varphi}{h} \\
    0 & h
\end{bmatrix}, \quad (2.4.18)
\]

where for notational simplicity \( h \) is set according to \( h = h_2 = r_2 \), the sample interval on a line. The corresponding generator matrix of the reciprocal lattice yields

\[
N = \begin{bmatrix}
    \frac{4\pi \tan \varphi}{h} & 0 \\
    \frac{2\pi \tan \varphi}{h} & \frac{2\pi \tan \varphi}{h}
\end{bmatrix}. \quad (2.4.19)
\]

The choice of the reciprocal generator matrix \( N \) determines the sampling generator matrix \( M \) since \( M \) and \( N \) are related by equation (2.3.1), or written in matrix notation:

\[
N^T M = 2\pi I, \quad (2.4.20)
\]

where \( I \) is the \( 2 \times 2 \) identity matrix. The choice of \( N \) is not unique, in general, an adequate density of samples in the spatial domain can represent any bandlimited signal with several sampling geometries. However, it is often desirable to represent \( f(x_1, x_2) \) with as few samples as possible. Since \( |\text{det } M| \) equals, according to equations (2.2.9) and (2.2.1), the area of the parallelogram spanned by the base vectors, the density of
Fig. 2.7  Regular hexagonal sampling ($\varphi = 30^\circ$). (a) Spatial sampling locations and regular hexagon as Voronoi region. (b) Reciprocal lattice and rotated regular hexagon as Voronoi region.
samples per unit area is given by $1/|\text{det } M|$. Minimizing this quantity is equivalent to minimizing $|\text{det } N|$. Therefore, to provide an efficient sampling scheme for a bandlimited picture, the generator matrix $N$ is sought, which has the smallest value of $|\text{det } N|$ but which nevertheless avoids aliasing for the particular shape of the band region of the analog input picture.

Among the numerous possibilities for the generator matrix as a function of the angle $\varphi$, only two sampling lattices are commonly used. First, the lattice that consists of regular hexagons, regular in the sense that the hexagon has an enclosed angle of 120° and has six equal sides. This lattice is spanned by the base vectors $u_1$ and $u_2$ having the same length and also forming an angle of 120°, which results in $\varphi = 30°$. Then, the distances from a pixel centroid to its six neighboring centroids are all equal to the lattice interval $h$. This lattice corresponds to the generator matrix

\[
M = \begin{bmatrix}
\frac{hV3}{2} & \frac{h}{2} \\
0 & h
\end{bmatrix}
\]  
(2.4.21)

and to the reciprocal matrix

\[
N = \begin{bmatrix}
\frac{4\pi}{hV3} & 0 \\
\frac{2\pi}{hV3} & \frac{2\pi}{h}
\end{bmatrix}
\]  
(2.4.22)

These two lattices and their Voronoi regions are shown in Figs. 2.7(a) and 2.7(b). It should be noted that the regular hexagon in the Fourier domain is a scaled and rotated version of the regular hexagon in the spatial domain.

Second, the other commonly used sampling lattice is the so-called staggered array described by Rosenfeld and Pfaltz (1968), which results in $\varphi = 45°$. In contrast to the rectangular lattice only alternate rows of this
lattice are identical, while the odd-indexed rows are staggered at one half sample intervals with respect to the even numbered rows. In fact, rather than do the actual shifting, sampling can take place with a square lattice having doubled density \((l_1 = l_2 = h/2)\) and only using even column points in the even rows and odd column points in the odd rows. In Rosenfeld and Pfaltz (1968) this lattice is regarded as "hexagonal", since each sample point is said to have six neighbors, two on its own row and two each on the rows above and below it, but in fact each sample point has eight neighbors, four diagonal neighbors at distance \(h/\sqrt{2}\), two horizontal neighbors at distance \(h\) and two vertical neighbors likewise at distance \(h\). The diamond shape of the Voronoi region about each lattice point leads to a rotated diamond in the Fourier domain, as shown in Fig. 2.5. The bandwidths of this diamond are \(b_1 = b_2 = 2\pi/h\).

A lattice not commonly in use is the rotated hexagonal lattice with \(\varphi = 60^\circ\) and regular hexagons as Voronoi regions in both domains. It will be used in subsequent developments in the next section to obtain an upper bound for the triangular lattice.

**Efficiency and spatial neighborhood**

The spatial orientation of a pixel and its neighboring elements can be described neatly by the associated quadratic form of the hexagonal lattices. Substituting the matrix given by equation (2.4.18) in (2.2.5) yields

\[
\xi(x_1, x_2) = \frac{1}{4} \left( \frac{1}{\tan^2 \varphi} + 1 \right) x_1^2 - x_1 x_2 + x_2^2,
\]

where for notational simplicity \(h\) is set equal to one. For the most common hexagonal lattice \((\varphi = 30^\circ)\) this leads to the theta function:

\[
\Theta_L(z) = \sum_{x_1, x_2 = -\infty}^{\infty} q^{x_1^2 - x_1 x_2 + x_2^2}.
\]

Sloane (1981) gives a derivation of (2.4.24) in terms of the standard Jacobi functions:
\[ \Theta_L(z) = \theta_3(z) \theta_3(3z) + \theta_2(z) \theta_2(3z) \]
\[ = 1 + 6q + 6q^3 + 6q^4 + 12q^7 + \ldots \]
\hspace{1cm} (2.4.25)

Thus, the first five shells around each pixel centroid contain 1, 6, 6, 6 and 12 pixel centroids, respectively, as illustrated in Fig. 2.8. Here again it is shown that each pixel has six neighbors, this time as a result of applying the theory of sphere packings on a sampling lattice.

An analog picture having a hexagonal band region will rarely be found. However, due to spherical aberration or defocusing of the camera system a circular band region is rather common. Representing these circular band-limited pictures by hexagonal sampling will lead to a considerable gain in efficiency with respect to the square sampling strategy, as will be shown in the next paragraph.

Substituting the matrix of the reciprocal regular hexagonal lattice in the definition of the density given by (2.2.2) yields
\[ \Delta = \frac{\pi}{2\sqrt{3}} = 0.907, \]
\hspace{1cm} (2.4.26)

which is again, loosely speaking, the fraction of the hexagonal band region of the lattice that is covered by the inscribed circle of the circular band-limited input picture.

It has been proved by Rogers (1964) that this result is the highest attainable density for packing circles in the plane. In Information Theory block quantizers for uniformly distributed inputs are based on the theory of sphere packings. Recently, the classic Bell Laboratories Memorandum of Newman, entitled "The Hexagon Theorem" was published (Newman, 1982) giving a new and simple proof of the optimality of the hexagonal lattice for encoding uniform input.

Along quite different lines Petersen and Middleton (1962) have also proved that there is no more efficient sampling scheme for circularly bandlimited signals than hexagonal sampling. By taking the ratio of equations (2.3.11) and (2.4.26) it follows that hexagonal sampling requires 13.4\% fewer samples than rectangular sampling to represent the
Fig. 2.8 The hexagonal sampling lattice. The first five shells around a lattice point contain 1, 6, 6, 6 and 12 points, respectively.
same circularly bandlimited analog picture. Thus, the area of a hexagonal pixel can be $2/\sqrt{3}$ times the area of a square pixel yet can have the same spatial-frequency resolution.

In Sharp (1961) it is shown that by arranging the elements of a beam-scanning planar antenna array in a hexagonal pattern rather than a rectangular pattern, the number of elements needed in the array is reduced. This reduction is 13.4% by arranging the elements in a pattern of equilateral triangles rather than in a square pattern, a result strictly equivalent to ours.

Another commonly used sampling lattice with the diamond shaped Voronoi region ($\varphi = 45^\circ$) has a density according to (2.2.2) in the Fourier domain of

$$\Delta = \frac{\pi}{4} = 0.785.$$  

This result clearly shows that no gain in spatial-frequency resolution is attained in relation to the ordinary square lattice. This is yet another reason to discard the diamond lattice.

The rotated hexagonal lattice ($\varphi = 60^\circ$) obviously has a density of

$$\Delta = \frac{\pi}{2\sqrt{3}} = 0.907.$$  

2.5 Triangular sampling lattices

By a straightforward application of Euler's polyhedral formula it can be shown (Ore, 1963) that only three repetitive planar graph patterns, or lattices, exist; and these can be formed by quadrangles, hexagons, or triangles as the primitives. The rectangular and hexagonal lattices have been treated in Sections 2.3 and 2.4, and the triangular lattice will be dealt with in this section.

**Vectorial approach**

In the triangular sampling lattice an individual pixel has the shape of an
Fig. 2.9 Triangular sampling, isosceles triangle as Voronoi region. In the origin of the lattice a pixel is shown surrounded by its three edge neighbors. The dashed hexagon represents the originating hexagon of the lattice.
isosceles triangle. The lattice is most easily visualized by connecting the lattice points of a hexagonal lattice by straight line segments. The centers of the circumscribed circles of the isosceles triangles thus formed define the triangular lattice. These circumcenters are merely the corner points of the hexagonal pixels in the hexagonal lattice. In Fig. 2.9 the triangular lattice is indicated by the black dots and an originating hexagon is shown by the dashed lines. In the origin of the lattice a triangular pixel is shown together with three of its neighbors.

Let \( \{u_1, u_2, u_3\} \) be a set of three principal vectors. Because of the lack of symmetry about the origin, the lattice cannot be generated by only two vectors. Consequently, each lattice point \( u \) is given by the position vector \( u \):

\[
 u = \rho_1 \ u_1 + \rho_2 \ u_2 + \rho_3 \ u_3 . \tag{2.5.1}
\]

A more symmetric lattice is given in Fig. 2.10 where the shape of each pixel is an equilateral triangle with the nice property that the centroid coincides with the center of the circumscribed circle, that is the point defining the triangular pixel as a Voronoi region. This lattice can be described by any pair of base vectors out of the set of three. Let \( \{u_1, u_2\} \) be such a base; then any lattice point is given by the position vector \( u \):

\[
 u = \rho_1 \ u_1 + \rho_2 \ u_2 . \tag{2.5.2}
\]

However, careful inspection of the lattice reveals that these base vectors - or any other pair from the \( \{u_1, u_2, u_3\} \)-set - also generate points in the plane which are not occupied by lattice points of the triangular lattice. For instance, \((1, 1)\) is a point generated by \( \{u_1, u_2\} \), but not part of the lattice. In fact, all corner points of the triangular pixels, obviously not present in the triangular lattice, are also covered by \( \{u_1, u_2\} \). These corner points are merely the centroids of the hexagonal pixels which originated the triangular lattice.

Note that the combination of the triangular lattice and the sub-lattice defined by the corner points of its pixels forms a hexagonal lattice uniquely generated by \( \{u_1, u_2\} \) with triple density with respect to the originating hexagonal lattice.
Fig. 2.10  Triangular sampling, equilateral triangle as Voronoi region ($\varphi = 30^\circ$). In the origin of the lattice a pixel is shown surrounded by its three edge neighbors. The dashed hexagon represents the originating hexagon of the lattice of the same shape and size as the regular hexagon of Fig. 2.7(a). Note that $t_1 = t/\sqrt{3}$ and $t_2 = 2t/\sqrt{3}$.
On the other hand, the second set of vectors, denoted by \( \{ \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \} \), that can be defined on the lattice is inadequate too, since any base formed by a subset of two vectors cannot uniquely generate the lattice either. If we let \( \{ \mathbf{v}_1, \mathbf{v}_2 \} \) be such a base then the lattice points with position vectors \( \mathbf{u}_1, \mathbf{u}_2 \) or \( \mathbf{u}_3 \), for instance, cannot be covered by \( \{ \mathbf{v}_1, \mathbf{v}_2 \} \).

The impossibility of defining a proper generating base for the triangular lattice makes a global treatment based on the reciprocity of lattices according to (2.3.1) inapplicable. An analytical approach similar to the approach used for the hexagonal lattice will be shown to fail as well.

**Analytical approach**

As in Section 2.4 the triangular lattice can be expressed as the superposition of four rectangular two-dimensional fields of impulses, each one a shifted copy of the other. In Fig. 2.11 the four fields are shown together with their base vectors and origin. Let \( r_1 \) and \( r_2 \) be the lattice constants of the base rectangular field in the \( x_1 \)-direction and the \( x_2 \)-direction, respectively. Then, \( s(x_1, x_2) \) is given by

\[
\begin{align*}
  s(x_1, x_2) &= \sum_{m_1=-\infty}^{\infty} \sum_{m_2=-\infty}^{\infty} \delta(x_1 - 2r_1 m_1, x_2 - r_2 m_2) + \\
  &+ \sum_{m_1=-\infty}^{\infty} \sum_{m_2=-\infty}^{\infty} \delta(x_1 - 2r_1 m_1 - 2/3 r_1, x_2 - r_2 m_2) + \\
  &+ \sum_{m_1=-\infty}^{\infty} \sum_{m_2=-\infty}^{\infty} \delta(x_1 - 2r_1 m_1 - 1/2 r_2 m_2) + \\
  &+ \sum_{m_1=-\infty}^{\infty} \sum_{m_2=-\infty}^{\infty} \delta(x_1 - 2r_1 m_1 - 5/3 r_1, x_2 - 2r_2 m_2 - 1/2 r_2).
\end{align*}
\]

(2.5.3)

The Fourier transform of \( s(x_1, x_2) \) equals
Fig. 2.11  The four rectangular fields of impulses in the $(x_1, x_2)$-plane with their respective base vectors and origin.
\[ S(\omega_1, \omega_2) = \frac{4\pi^2}{2r_1 r_2} \sum_{m_1 = -\infty}^{\infty} \sum_{m_2 = -\infty}^{\infty} \delta(\omega_1 - \frac{2\pi m_1}{2r_1}, \omega_2 - \frac{2\pi m_2}{r_2}) \times \]

\[ \cdot \left[ 1 + \exp(-j\pi(2/3 m_1)) + \exp(-j\pi(m_1 + m_2)) + \exp(-j\pi(5/3 m_1 + m_2)) \right], \]

(2.5.4)

which leads to

\[ S(\omega_1, \omega_2) = \frac{4\pi^2}{2r_1 r_2} \sum_{m_1 = -\infty}^{\infty} \sum_{m_2 = -\infty}^{\infty} \delta(\omega_1 - \frac{2\pi m_1}{2r_1}, \omega_2 - \frac{2\pi m_2}{r_2}) \times \]

\[ \cdot \frac{2/3 m_1}{(1 + (-1)} + \frac{m_1 + m_2}{(-1)} + \frac{5/3 m_1 + m_2}{(-1)} \right). \]

(2.5.5)

Only those Dirac impulses contribute to the sum for which the following set of conditions holds:

\[ 2/3 m_1 = \text{even} \quad m_1 + m_2 = \text{even} \quad 5/3 m_1 + m_2 = \text{even}. \]

It is, therefore, convenient to set

\[ m_1 = 3n_1 \quad \text{and} \quad m_2 = 2n_2 \cdot n_1, \]

(2.5.6)

which results in

\[ S(\omega_1, \omega_2) = \frac{8\pi^2}{r_1 r_2} \sum_{n_1 = -\infty}^{\infty} \sum_{n_2 = -\infty}^{\infty} \delta(\omega_1 - \frac{3\pi n_1}{r_1}, \omega_2 - \frac{2\pi}{r_2} (2n_2 \cdot n_1)). \]

(2.5.7)

This is the form of a hexagonal field of Dirac impulses with a spacing of \(3\pi/r_1\) in the \(w_1\)-direction and \(4\pi/r_2\) in the \(w_2\)-direction with alternate rows \(2\pi/r_2\) shifted in the \(w_2\)-direction. These positions of the impulses are indicated in Fig. 2.12 by the black dots. This lattice is exactly the
Fig. 2.12  The rotated reciprocal hexagonal lattice in the \((\omega_1, \omega_2)\)-plane \((\varphi = 60^\circ)\). The "sample" interval on a line is equal to \(4\pi/r_2 = 4\pi/h\) and the "interline" distance is equal to \(3\pi/r_1 = 2\pi\sqrt{3}/h\) which is a triple "interline" distance with respect to Fig. 2.7(b). Note that Fig. 2.12 is reduced in scale with a factor two with respect to Fig. 2.7(b).
reciprocal lattice given by the generator matrix (2.4.19) when \( \varphi \) equals 60\(^\circ\):

\[
N = \begin{bmatrix}
4\pi \sqrt{3} & 0 \\
\hbar & 0 \\
2\pi \sqrt{3} & 2\pi \\
\hbar & \hbar
\end{bmatrix}
\] (2.5.8)

The corresponding spatial lattice has a triple density in the \( x'_1 \)-direction with respect to (2.4.21) and Fig. 2.7(a) since from Figs. 2.7(a) and 2.10 follows that

\[
t = \hbar \quad \text{and} \quad 1/2 t_1 = 1/3 \cdot 1/2 \hbar \sqrt{3} \quad .
\] (2.5.9)

Here we see that apparently in the transformation of \( s(x_1, x_2) \), based on four rectangular fields, the two fields containing the pixel corner points are implicitly also taken into account. This additional field is written as

\[
s'(x_1, x_2) = \sum_{m_1 = -\infty}^{\infty} \sum_{m_2 = -\infty}^{\infty} \delta(x_1 - 2r_1 m_1 - 1/3 r_1, x_2 - r_2 m_2 - 1/2 r_2) + \\
+ \sum_{m_1 = -\infty}^{\infty} \sum_{m_2 = -\infty}^{\infty} \delta(x_1 - 2r_1 m_1 - 4/3 r_1, x_2 - r_2 m_2) ,
\] (2.5.10)

which after transformation contributes to the Fourier sum if

\[
1/3 m_1 + m_2 = \text{even} \quad \text{and} \quad 4/3 m_1 = \text{even} .
\]

These conditions are fulfilled by (2.5.6), so no additional constraints are imposed by \( s'(x_1, x_2) \) in case the triangular lattice is extended to a hexagonal lattice by combining \( s(x_1, x_2) \) and \( s'(x_1, x_2) \).

Consequently, the shape of the Fourier transform of the triangular lattice
Fig. 2.13  Recurrent nonuniform sampling for \( N = 2 \).
is equivalent to the shape of the Fourier transform of the triangular lattice extended to a hexagonal lattice by the addition of $s'(x_1, x_2)$. These additional sample points make the lattice regular in the sense that it can be described by a single set of two base vectors, such that the theorem defining the reciprocity of lattices can be used. Without these additional sample points a global treatment is inapplicable and incorrect.

**Recurrent sampling theorem**

In the following we will introduce a one-dimensional nonuniform sampling theorem known in Information Theory which will be extended to two-dimensional nonuniform sampling lattices like the triangular lattice, thereby uncovering the true limiting frequencies.

Shannon (1949) mentioned that the samples needed to specify a one-dimensional bandlimited signal need not be equally spaced, although if there is a considerable bunching, the samples must be known very accurately in order to adequately reconstruct the signal. The reconstruction process is also more involved with unequal sampling. Yen (1956) formalized this idea in four generalized sampling theorems on four special nonuniform sampling processes. One of these theorems is of special interest in relation to the triangular sampling lattice. We will restate this theorem here in terms of a one-dimensional function. Next this theorem will be extended to the triangular lattice.

Let a sampling scheme consisting of groups of $N$ sample points with a recurrent group periodicity of $N/2W$ seconds be called a recurrent nonuniform sampling scheme, where $W$ is the bandwidth of the signal to be sampled. Fig. 2.13 gives an example of such a sampling configuration. In one period the points are denoted by $t_p, p = 1, 2, ..., N$. The complete set of sample points are written as $t_{p,m} = t_p + mN/2W, m = ... -1, 0, 1, ....$ For this sequence the following theorem has been proved (Yen, 1956):

A bandwidth-limited signal is uniquely determined by its values at a set of recurrent sample points, $t = t_{p,m} = t_p + mN/2W, p = 1, 2, ..., N; m = ..., -1, 0, 1, ....$

The reconstruction function is:
\[ f(t) = \sum_{m=-\infty}^{\infty} \sum_{p=1}^{N} f(t_{pm}) \psi_{pm}(t), \] (2.5.11)

where

\[ \psi_{pm}(t) = \frac{\prod_{q=1}^{N} \sin \left( \frac{2\pi W}{N} (t-t_{q}) \right)}{\prod_{q=1, q \neq p}^{N} \sin \left( \frac{2\pi W}{N} (t_{p}-t_{q}) \right)} \cdot \frac{(-1)^{mN}}{2W}. \] (2.5.12)

The Fourier spectrum \( S_{pm}(\omega) \) of the composing function \( \psi_{pm}(t) \) is broken into \( N \) discontinuous segments but limited in the bandwidth \( |\omega| \leq 2\pi W \).

The triangular lattice of Fig. 2.9 uniformly samples the analog input picture along the scan lines, the rows of the lattice. However, along the columns, the picture is nonuniformly sampled by a recurrent sampling procedure with \( N=2 \). Let \( 2\varphi \) denote the top angle of the isosceles triangle about the origin and \( t=r_{2} \) the uniform sampling period along a scan line; then the alternating sampling intervals in the \( x_{i} \)-direction are given by

\[ t_{1} = \frac{t(\cos^{2}\varphi \cdot \sin^{2}\varphi)}{\sin 2\varphi} \quad \text{and} \quad t_{2} = \frac{t}{\sin 2\varphi}, \] (2.5.13)

which adds up to

\[ t_{1} + t_{2} = \frac{t}{\tan \varphi}. \] (2.5.14)

Then, from the recurrent nonuniform sampling theorem applied to the columns, it follows that the picture should be bandlimited in the \( x_{i} \)-direction by
$$|\omega_1| \leq 2\pi \tan \varphi / t,$$

where $t/\tan \varphi$ is the recurrent period. In this case the interpolation function given by (2.5.12) allows the reconstruction of the values of $f(x_1, x_2)$ at points in between the recurrent sample locations. Furthermore, the input picture should obviously be bandlimited in the $x_2$-direction by

$$|\omega_2| \leq 2\pi / t.$$

It should be noted that the angle $\varphi$ introduced above is equal to the generating angle of the originating hexagon, as illustrated in Fig. 2.9.

As a result, the triangular sampling lattice with sampling interval $t$ and alternating line distances $t_1$ and $t_2$ has the same frequency resolution as a uniform rectangular sampling lattice, with a sampling period of $l_1 = t/2$ and $l_2 = (t_1 + t_2)/2$, as shown in Fig. 2.1(a). By imaginary replacing of the recurrent column sample points by uniformly spaced points, that is, shifting alternating samples in each column such that only even rows remain, as shown in Fig. 2.14, a rectangular lattice with equivalent spatial-frequency resolution is created.

**Shape of the lattice**

The particularly interesting triangular lattice we dealt with in our attempt to set up the framework for the classical global Fourier analysis is the one with $\varphi = 30^\circ$, which is the dual of the regular hexagonal lattice. As has been noted before, the pixels have the shape of an equilateral triangle and the lattice is rather symmetric. It can even be described by any set of two base vectors as a subset of $\{u_1, u_2, u_3\}$, although the lattice is obviously still not complete. If the missing locations in the lattice are occupied by additional sample points, making the lattice complete, then this triangular lattice degenerates to a regular hexagonal lattice with $\varphi = 60^\circ$. The Voronoi regions in both domains of this extended lattice are regular hexagons rotated 90 degrees relative to the standard hexagonal lattice. The limiting frequencies of the orthogonal band region that determine the sampling intervals follow from (2.4.19) where $\varphi = 60^\circ$: 
Fig. 2.14  Imaginary shifting of the recurrent sample locations to uniform locations, shown for $\phi = 30^\circ$. Recurrent locations are indicated by $\bullet$, uniform locations by $x$. 
\[ \omega_1 = 2\pi \sqrt{3}/h = 2\pi \sqrt{3}/t, \quad \text{and} \]  
\[ \omega_2 = \pi/h = \pi/t. \]  

(2.5.17)

The orthogonal band region for the triangular lattice \((\varphi = 30^\circ)\) is determined by the recurrent sampling theorem (2.5.15) and (2.5.16):

\[ \omega_1 = 2\pi/t \sqrt{3} = 2\pi/3t, \quad \text{and} \]  
\[ \omega_2 = 2\pi/t = 2\pi/t_1 \sqrt{3}; \]  

(2.5.18)

where \(t = t_1 \sqrt{3}\), since \(\varphi = 30^\circ\) in (2.5.13).

The values of \(\omega_1\) and \(\omega_2\) of (2.5.14) and (2.5.15) clearly show the increase in spatial frequency resolution by 3/2 due to the addition of sample points at the free positions in the triangular lattice.

Another interesting lattice is the one with \(\varphi = 45^\circ\), where \(\omega_1 = \omega_2\). In this lattice the two recurrent sample points are bunched together, that is, the two interleaved sequences of equispaced sample points in the \(x_1\)-direction coincide. This lattice has been dealt with in Section 2.4, where it was shown that the lattice had in both the spatial and frequency domains, no advantages over the ordinary square lattice. Nevertheless, Rosenfeld and Pfaltz (1968) treated this "staggered array" lattice as spatially hexagonal.

**Spatial neighborhood and efficiency**

The theory of sphere packings and the associated quadratic form cannot be applied to the triangular lattice, since there is no formal description in terms of the generator matrices as defined in Section 2.2. The sphere packing on an incomplete lattice must be defined in a more complicated way, but will be omitted here; see, for example, Rogers (1964).

The efficiency for circular bandlimited pictures is equal to the efficiency of the associated rectangular lattice having the same spatial-frequency resolution. The generator matrix for this reciprocal lattice yields
Fig. 2.15  Nuclei and growth process. (a) Square lattice. (b) Hexagonal lattice. (c) Triangular lattice. The three lattices have the same density defined by an equal pixel area, see Fig. 2.16.
\[ N = \begin{bmatrix} \frac{2\pi}{t\sqrt{3}} & 0 \\ 0 & \frac{2\pi}{t} \end{bmatrix}. \] (2.5.19)

The density follows from
\[ \Delta = \frac{\pi\sqrt{3}}{12} = 0.453, \]
leading to an efficiency of the triangular sampling lattice for circular bandlimited pictures of 45.3%. Note that the area of a pixel of the triangular lattice and the area of a pixel of the associated rectangular lattice are clearly the same.

2.6 Nonuniform sampling lattices

The three regular lattices and their tessellations of the image plane can be viewed as special cases of a general theory to tessellate the plane. The rectangular, hexagonal and triangular tessellations may be produced by a growth process. The nodes of the lattices are the nuclei of growing cells. Cells will grow unimpeded in a circular fashion until they reach the tightly packed states as shown in Fig. 2.15. At this moment each circle has four, six or three points of contact with its neighbors. Fig. 2.15(c) clearly shows that each sextet of circles encloses a region of the plane not covered by a cell. This region can exactly contain such a circle; however, the triangular lattice then degenerates to a regular hexagonal lattice, as has been shown in Section 2.5. As the cells continue to grow, the point of contact between the circles expands into straight line segments and the circles are transformed into rectangles, hexagons or triangles, defining the area of a pixel. See Fig. 2.16.

Each of these tessellations has a dual form, determined by the straight lines connecting the growth centers of neighboring cells, as is shown in Fig. 2.16.

These regular tessellations can be extended to more general random
Fig. 2.16 Elementary regions (pixels) and their dual form indicated by the dashed lines. (a) Square lattice. (b) Hexagonal lattice. (c) Triangular lattice. The areas of the square, hexagonal and triangular pixels are identical.
tessellations: Poisson line, Voronoi, and Delaunay tessellation, respectively. The definitions of these tessellations will now briefly be summarized; for details see Roach (1968), Schachter and Ahuja (1979), and Ahuja and Schachter (1983).

In the Poisson line model, a Poisson process chooses points in the infinite strip $0 \leq \theta \leq \pi$, $-\infty < \rho < \infty$. Each of these points defines a line \( x_1 \cos \theta + x_2 \sin \theta - \rho = 0 \), and these lines define a tessellation of the plane.

In the Voronoi model, a Poisson process chooses points (called “nuclei”) in the plane. Each nucleus defines a Voronoi region consisting of all the points in the plane that are nearer to it than to any other nucleus.

In the Delaunay model, the tessellation is obtained by joining all pairs of nuclei whose Voronoi cells are adjacent.

Fig. 2.17 gives an impression of the appearance of these tessellations, where the Voronoi and Delaunay tessellation are plotted in one figure by duality.

A regular lattice is characterized by its kissing number and its tessellation by the number of cells meeting at each of its vertices. The values for the three lattices are listed below:

<table>
<thead>
<tr>
<th>kissing number ( \tau )</th>
<th># cells meeting at each vertex ( \beta )</th>
<th>resulting tessellation</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>6</td>
<td>triangular</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>quadrangular</td>
</tr>
<tr>
<td>6</td>
<td>3</td>
<td>hexagonal</td>
</tr>
</tbody>
</table>

Table 2.1
Characteristics of regular lattices
Fig. 2.17 Random tessellations. (a) Poisson line tessellation. (b) Voronoi tessellation denoted by the solid lines and its dual Delaunay tessellation denoted by the dashed lines. Nuclei of the Voronoi tessellation are indicated by the black dots; Nuclei of the Delaunay tessellation by the open circles.
A property of the nonuniform tessellations considered here is the property that the expected values for these parameters are the same for each cell in such a tessellation. Furthermore, for the Poisson line, Voronoi, and Delaunay models, these values are the same as those for the rectangular, hexagonal, and triangular tessellations, respectively. One may expect that the growth process in the above tessellations will have similar characteristics. Without a complete mathematical description of the growth process, the identity of some of the characteristics cannot be proved exactly. However, we may expect that the equal numbers of positive and negative variations in the values of these parameters for different cells will cancel out in large tessellations.

2.7 Concluding remarks

Image sampling has been discussed in signal-theoretical terms, that is, an analog input picture is considered as a two-dimensional bandlimited signal. In particular, we discussed the topic of sampling density with emphasis on the hexagonal and triangular lattices. We have extended the recurrent sampling theorem for one-dimensional bandlimited signals in such a way that it can be applied to two-dimensional bandlimited signals. The derivation of the recurrent sampling theorem was inspired by a statement of Shannon (1949), in which he states that the samples needed to specify a one-dimensional bandlimited signal need not be equally spaced, as is the case in a triangular lattice. So in fact, it turned out that the triangular sampling strategy is a special case of the well-known rectangular sampling strategy.

In general, Shannon-based sampling allows the analog picture to be recovered at every point from just its sampled values. It is often not possible to properly sample a picture. Usually the number of samples that can be taken, stored or processed in some fashion is very limited. So if the picture is not properly bandlimited before sampling, aliasing problems crop up. When the picture is circularly bandlimited, the hexagonal sampling strategy is highly suitable, as it can handle higher frequencies, given a fixed number of samples, than the two other uniform sampling strategies. When the picture does not satisfy, one way or another, the Shannon condition, analog prefiltering may lead to minimization of
aliasing problems. In fact, convolutional filtering may be used to satisfy the Shannon condition closely enough to greatly improve the output image. This kind of filtering may be implemented by making each sample point represent a finite area in the scene rather than an infinitesimal spot. Thus the edge of, for instance, a geometrical object would only occupy a part of such a small area, causing the intensity of the corresponding dot in the output image to be computed as a weighted average of the brightness of the object and its local background. Making each sample represent a finite area, which is, in fact, exactly the "alternative approach" mentioned in the first paragraph of Section 2.1, has the effect of applying a convolutional filter before the picture is sampled.

Image segmentation is the next step up to image analysis. It provides the primitives on which the feature extraction is based. Segmentation is 

basically a process of assigning the pixels to classes: one simple way of doing this, called "thresholding", classifies the pixels according to whether or not their gray values exceed a given threshold. A more sophisticated approach to segmentation includes the region-growing techniques that incorporate domain-dependent semantics. In such an approach semantic labels can guide the merging process by using a priori knowledge concerning the regions to be found. For example, shape features of the components as roundness or ellipticity can be tested in the different steps of the iterative segmentation procedure. In two recent papers of Chassery and Garbay (1983, 1984) a criterion of convexity is applied to segment convex, possibly overlapping, components. This is motivated by the convex shape of cells encountered in exfoliative cytology, namely leukocytes and erythrocytes.

Given the accuracy with which isotropically shaped blobs, like cells, can be represented by a hexagonal sampling strategy with respect to the square and triangular sampling strategies having an equal sampling density, it goes without saying that one should consider in this context the notion of convexity for the hexagonal sampling strategy, which will be accomplished in the remainder of this thesis.
3.1 Introduction and preliminary notions

A figure $q$ in the continuous space $\mathbb{IR}^2$ is said to be convex if it entirely contains all segments connecting any two of its points. The regions enclosed by a circle, a semicircle and an ellipse are examples of convex figures. All triangles are convex. A straight line is a convex figure since, if $A$ and $B$ are any of its two points, it contains all of the segment $AB$. In the same way, any ray and any segment of the line are convex figures. Every convex part of the line (different from the entire line) is either a segment or a ray. The line, the ray, and the segment are called one-dimensional convex figures. The remaining planar convex figures are said to be two-dimensional.

The following theorems have been proved for planar continuous figures in Euclidean geometry (Lyusternik, 1963).

**Theorem 3.1.1**
If a convex figure $q$ contains three points $A$, $B$ and $C$ which do not lie on one straight line, then $q$ contains all of the triangle $ABC$.

**Theorem 3.1.2**
The intersection of two convex figures, if it is not empty, is a convex figure (zero-, one-, or two-dimensional).

**Theorem 3.1.3**
A segment joining any interior point $A$ of a convex figure $q$, to any other point $B$ of $q$ consists entirely (with the possible exception of the point $B$) of interior points of $q$.
Theorem 3.1.4
If from a point $O$ lying inside a bounded convex figure $q$ a ray $OL$ is drawn, then this ray intersects the boundary of $q$ at one and only one point.

In image analysis, when converting continuous two-dimensional figures into a discrete form, we wish to define notions or to measure parameters on objects in the discrete space. These notions or parameters are very often translated by analogy from the continuous space into the discrete space.

One approach shows the convergence of the discrete notion towards the continuous one when the pixel size tends to 0. Examples of this approach can be found when using metric parameters such as area, perimeter or location of the centroid.

A second approach consists of verifying all properties of the continuous notion of convexity translated into the discrete space. Such an approach has been applied for the topological notion of connectivity. In this thesis, a similar approach is proposed for the notion of convexity. This notion has been discussed by numerous authors, mainly restricted to the rectangular mosaic. Our definition is effectively equivalent to existing ones. Nevertheless, our formulation will fit in the hexagonal mosaic, a mosaic explicitly excluded in the theory of convexity analysis published so far. Therefore, our formulations and proofs are quite different because of the substantial differences between the digitization schemes.

As an introduction we will first formally treat the two different notions of a representation scheme: the lattice representation and the covering representation, a terminology inspired by Serra (1982).

Lattice representation for blobs

Our attention is focused on analog pictures consisting of black figures on a white background and the digital representation of these figures. A point $(x_1, x_2)$ on an analog picture, where $x_1$ and $x_2$ take on real values, is considered as a figure point if $f(x_1, x_2)$ is "black", and as a background
point otherwise. Consider a regular lattice $L$ superimposed on the analog picture. Then, the lattice representation of the input picture consists of those points $(x_1, x_2)$, where $x_1$ and $x_2$ are discrete, for which $f(x_1, x_2)$ has the value "black". An example of a continuous figure on the square lattice and its lattice representation is shown in Fig. 3.1(a), whereas the hexagonal and triangular lattice representations of the same figure are shown in Fig. 3.1(b) and (c), respectively.

Formally, the lattice representation is defined as follows:

**Definition 3.1.1**
Lattice representation for blobs
A set of lattice points $D$ is said to be the digital image of a plane figure $q$, and $q$ a preimage of $D$ if

i) $D \subseteq q$ and

ii) $\overline{D} \cap q = \emptyset$

where $\overline{D}$ denotes the complement of $D$, that is the set of lattice points not in $D$.

This definition is the binary analogon of the Dirac-field approach to gray-scale images in the sense that only the lattice points themselves instead of a spatial neighborhood make up the representation.

**Covering representation for blobs**

Consider a regular lattice $L$ superimposed on the analog input picture. Associated with each lattice point is an elementary cell, the pixel. If the support of a black figure meets the open cell centered at $(x_1, x_2)$, the cell $(x_1, x_2)$ is part of the covering representation of the input picture. That is, the covering representation of an analog input picture consists of those cells that have a nonempty projection of the continuous figures into the cells. The example of Fig. 3.1 is shown in the covering representation in Fig. 3.2 on the square, hexagonal and triangular mosaics.

Formally, the covering representation is defined as follows:

**Definition 3.1.2**
Covering representation for blobs
A set of cells $C$ is said to be the cellular image of a plane figure $q$, and $q$ a preimage of $C$ if
Fig. 3.1 A continuous connected plane figure (blob) and its lattice representation on the three lattices. (a) The square lattice with lattice constant \( l \). (To be continued on the next page.)

Fig. 3.2 The continuous figure of Fig. 3.1 and its covering representation on the three mosaics. The three mosaics have the same density as in Fig. 3.1, such that the area of the three respective pixels are identical \( h = l \sqrt{4/3} \) and \( t = h \sqrt{2} \). Note that the lattice points inside the continuous figure coincide with the corresponding points of Fig. 3.1. (a) The square mosaic. (To be continued on the next page.)
Fig. 3.1  (Continued.) A continuous connected plane figure (blob) and its lattice representation on the three lattices. (b) The hexagonal lattice with lattice constant $h$, where $h = l \sqrt{4/3}$. (To be continued overleaf.)

Fig. 3.2  (Continued.) The continuous figure of Fig. 3.1 and its covering representation on the three mosaics. (b) The hexagonal mosaic. (To be continued overleaf.)
Fig. 3.1  (Continued.) A continuous connected plane figure (blob) and its lattice representation on the three lattices. (c) The triangular lattice with lattice constant $t$, where $t = h \sqrt{2}$.

Fig. 3.2  (Continued.) The continuous figure of Fig. 3.1 and its covering representation on the three mosaics. (c) The triangular mosaic.
i) \( q \subseteq s(C) \), where \( s(C) \) denotes the space of \( C \), and

ii) for each cell \( e \) of \( C \), \( e^o \cap q^o \neq \emptyset \), where \( e^o \) and \( q^o \) are the interiors of \( e \) and \( q \).

The set of cells or cellular complex is the unique representation of the figure \( q \), because the mosaic is defined to consist of open cells. Therefore, the representation in accordance with Definition 3.1.2 is referred to as the "open-cell covering representation". Referring to an unknown publication of Freeman (1971?), Serra (1982) essentially used the same definition for the covering representation and mentioned its commonly usage in image analysis to digitally code curves, which is in fact not true. In Freeman (1961, 1970, 1974) the Grid Intersection Quantization scheme is discussed, a scheme that indeed is frequently used for curve representations, but which differs considerably from the covering representation.

Sklansky (1970) defined the covering representation on a different mosaic:

**Definition 3.1.3 (Sklansky, 1970)**
A set of cells \( C \) is said to be the closed-cell cellular image of a plane figure \( q \), and \( q \) a preimage of \( C \) if

i) \( q \subseteq s(C) \), where \( s(C) \) denotes the space of \( C \), and

ii) for each cell \( e \) of \( C \), \( e \cap q \neq \emptyset \).

The requirement in ii) is \( e \cap q \neq \emptyset \) rather than \( e^o \cap q^o \neq \emptyset \). Thus, the cellular image of a figure may not be unique according to this definition. For example, the image of a cell is either the cell itself or the union of the cell with any subset of its neighbors.

In this digitization scheme a cell is part of the representation if the support of the analog figure meets the closed cell, that is the common boundary of two neighboring cells is regarded as belonging to both of them. In fact, if this boundary is assigned to just one of the cells, a sequence of figures which passes through the other cell, closer and closer to its boundary, then the digitization of the limiting figure of the sequence would not be the same as the digitization of the figures in the sequence.
Representations which do not go to the limit are unacceptable in cases where a standard figure having a given image has to be defined, e.g. the minimum-perimeter polygon. Achieving the minimal perimeter of a sequence of polygons involves a passage to the limit. In fact, in this case the standard figure always has an ambiguous image. In many cases, however, the cellular image of a given figure is either unique or is understood from the context.

In the next section we will present the notion of connectivity defined on the three lattices and some simple formulas for object counting and determination of holes in an image, two parameters needed for convexity analysis in the upcoming sections. In Section 3.3 we will give an overview of the concept of convexity as it has been treated in the literature, which is mainly restricted to the rectangular mosaic. In Section 3.4 the problem of convexity is formulated on the hexagonal lattice related to the notion of convexity of the continuous figure as the preimage of the discrete representation. It will be verified that the theorems which hold in the continuous space will hold in the discrete space under the definition of convexity for the hexagonal lattice.

3.2 Connectivity and Euler characteristic

Introduction

To describe the digital equivalents of such notions as connectivity, straightness or convexity, the digital representation by means of a set of points on a lattice is too poor. Knowing just the points that make up \( S \), the set of lattice points, and \( \overline{S} \) its complement, is not enough. Since the lattice is the support on which the digital images are drawn up, according to a specific representation, the notions of connectivity and neighborhood, which relate points on the lattice, have to be defined on the lattice.

In dealing with patterns on rectangular lattices it has been pointed out by Rosenfeld (1970) that there is a choice between two connectivity situations. The pattern may be either 4-neighbor connected - using axial neighbors on the lattice only - or 8-neighbor connected - using any of the eight peripheral neighbors on the lattice. The choice is left to the user.
Correspondingly, the complement or background must, mathematically, be either 8-neighbor or 4-neighbor connected. This statement can be verified by the theorem of Euler, defined originally in Euclidean geometry as part of the theory of polyhedra, particularly convex polyhedra (Lyusternik, 1963). This theorem relates the number of vertices of a polyhedron to the number of edges and regions. For any convex polyhedron, the Euler characteristic or Euler number is equal to 2. In formula, this is written as

\[ m + n - l = 2, \] (3.2.1)

in which \( m \) is the number of vertices, \( n \) the number of faces and \( l \) the number of edges of the polyhedron. On the surface of a polyhedron we have a natural net in which the vertices are the nodes, the faces the regions, and the edges the lines. A closed polygon in the plane is a net consisting of \( k \) nodes (the vertices of the polygon), and \( k \) lines (the sides of the polygon). Such a closed polygon divides the plane into two regions, the regions of our net, i.e. the interior region and the background.

A disconnected net is composed of \( s \) connected nets. For example, the net consisting of two isolated polygons consists of two connected nets, where \( s = 2 \). The Euler characteristic for such a net in the plane is:

\[ m + n - l = s + 1. \] (3.2.2)

The theorem of Euler can now be defined for a sampling lattice with its corresponding pixel-boundary grid. Each component in the image plane is considered as one polygon consisting of several subpolygons, the individual pixels. Consequently the number of regions found in the image plane is equal to the number of pixels plus one for the background region. Normally the background consists of one region; however, in case components have one or more holes the number of background regions is incremented by the same amount. The Euler relation is then given by:

\[ v + p + 1 + h - l = c + 1, \] (3.2.3)

where \( v \) is the number of corner points (vertices) of the pixels of the pattern, \( p \) the number of pixels in the pattern, the unit term 1 the back-
Fig. 3.3 Hexagonal connectivity. Six object pixels, shown by the shaded regions, enclosing one background pixel in the hexagonal mosaic.

Fig. 3.4 The two orientations of a triangular pixel in the triangular mosaic. (a) Top-up orientation. (b) Top-down orientation.
ground, \( h \) the number of holes, \( l \) the number of edges (lines) of the pixels of the pattern, \( c \) the number of components and where the unit term on the right corresponds to the unit term in eq. (3.2.2). The Euler characteristic is the simplified and rearranged equation

\[
c - h = v + p - l.
\]  

(3.2.4)

The number \( c - h \) is also called the genus of the pattern, i.e. the number of components minus the number of holes (Rosenfeld and Kak, 1982).

A hole is defined here as the number of components of the background (complement pattern) exceeding 1.

**Euler characteristic in the three lattices**

The characteristic of (3.2.4) is applicable for each of the lattices treated so far. Therefore it is a powerful mathematical tool which can be applied in defining connectivity on these lattices as part of a generalized approach. In dealing with patterns on rectangular lattices it has been pointed out that pattern and background should be connected oppositely.

The hexagonal lattice deals symmetrically in connectivity with \( S \) and \( \bar{S} \) and it does not have ambiguous diagonal configurations. Both object and background must be 6-connected, so there is no choice in connectivity. This is verified by using the Euler formula on the pattern shown in Fig. 3.3. Assuming 6-connectivity, the genus is equal to zero by virtue of the Euler formula. Since both the number of components and holes is equal to one, the genus has the value zero. Thus, both values agree.

The topological structure of the triangular lattice is much more complicated than either structure discussed above. Figs. 3.4(a) and (b) clearly show that the triangular pixels have alternating orientations in the lattice. Consequently, the arrangement of neighboring pixels will vary accordingly. Here too it will be necessary to establish a different neighborhood connectivity for both pattern and background. For each of the orientations of a central pixel, the nearest neighbors are arranged in a different way. It can be observed from Fig. 3.5 that the nearest neighbors can be divided into three disjunct sets such that all pixels in one set are equidistant to the central pixel. The first set contains the triangular
Fig. 3.5 Triangular connectivity. The ordering of neighboring pixels in the triangular mosaic. The pixels indicated by "1" are the 3-neighbors of the central pixel, those indicated by "2" together with the 1's make up the set of 9-neighbors and the 1's, 2's and 3's make up the set of 12-neighbors of the central pixel. An equivalent ordering exists for a top-down-oriented central pixel.
neighbors whose centroids are at a distance of \( t_1 = t / \sqrt{3} \) away from the central pixel’s centroid. The lattice constants \( t_1 \) and \( t \) have been defined as the interline distance and sample interval on a scan line in Section 2.5. The second set contains six pixels, each centroid of which is at a distance of \( t \) away, and the third set contains three pixels at a distance of \( 2t_1 = 2t / \sqrt{3} \). Note that all pixels are included as neighbors which either share an edge or a vertex with the central pixel. This suggests a neighborhood connectivity of 3, 6 or 12. Fig. 3.5 gives the topological relations between the central pixel and its neighbors, numbered according to the sets to which they belong. It has been found (Deutz, 1972) that the Euler relation (3.2.4) is satisfied only if the pattern is 12-connected and the background 3-connected or the reverse.

Apart from the mathematical properties of the Euler characteristic in relation to the connectivity defined on the lattice, the genus is of particular interest from a topological point of view. The Euler number is topologically invariant, and is locally countable. Furthermore, the Euler number is of interest in characterizing an image by its convex components. This has been discussed in depth by Sklansky (1972) in terms of the concavity tree as a structure for describing the hierarchical arrangement of concavities, concavities within concavities etc. in a simply connected object.

**Euler algorithms**

Gray (1971) derived simple methods for computing the genus of an image by counting local configurations of various types in an image, defined on the rectangular or hexagonal lattice. Because of the complexity of the connectivity relation on the triangular lattice, Gray did not discuss the triangular lattice. In the following we briefly review the general idea and extend the algorithm to triangular lattices.

**Rectangular lattice**

Consider an arbitrary set of object pixels forming an image \( S \) on the rectangular lattice. Let \( Q_1 \) be the number of quads, i.e. 2x2 arrays of pixels, in the image that contain just one object pixel, \( Q_2 \) the number of quads that contain just three of them, and \( Q_D \) the number of quads that
Fig. 3.6  
(a) The configurations of hexagonal pixels contributing to $Q_1$.  
(b) The configurations of hexagonal pixels contributing to $Q_2$.  
Object pixels are shown by the shaded regions.
contain just two diagonally adjacent object pixels. Then the genus of $S$ is given by

$$g(S) = \frac{1}{4} (Q_1 \cdot Q_3 \pm Q_D),$$

(3.2.5)

where the plus sign holds for the 4-connectivity of $S$ and 8-connectivity of $S$, and the minus sign for the reverse connectivity situation. This formula can be interpreted as the number of convex corners of $S$ minus its number of concave corners. Indeed, each $Q_1$ pattern has a convex corner at its center point; each $Q_3$ pattern has a concave corner; and each $Q_D$ pattern has two concave corners if diagonally adjacent object pixels are connected, and two convex corners if they are not.

An interesting fact about this Euler number algorithm is that only 10 of a total of 16 quad patterns are relevant for the genus and consequently for convexity analysis.

**Hexagonal lattice**

For the hexagonal lattice the genus formula is most easily derived. Let $Q_1$ be the number of triplets formed by hexagonal pixels that contain just one object pixel, and $Q_2$ the number that contain just two of them. Then the genus of $S$ is

$$g(S) = \frac{1}{6} (Q_1 \cdot Q_2).$$

(3.2.6)

Fig. 3.6(a) shows the six possible configurations that may contribute to $Q_1$ and similarly in Fig. 3.6(b) those that may contribute to $Q_2$. By virtue of the symmetry of the hexagonal lattice and the absence of ambiguous diagonal configurations, only two triplets are sufficient for calculating the genus. Any of the six pairs of complementary triplets will be adequate. Then, equation (3.2.6) simplifies to

$$g(S) = Q_1' \cdot Q_2',$$

(3.2.7)

where $Q_1'$ and $Q_2'$ denote the selected pair of complementary configurations.
<table>
<thead>
<tr>
<th>Index i</th>
<th>configuration</th>
<th># different patterns</th>
<th>coefficient 12-connected</th>
<th>coefficient 3-connected</th>
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<td>+ 2</td>
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</tbody>
</table>

Table 3.1 The different disjunct groups of patterns with a representative pattern of each group and the coefficients for 12-neighbor connectivity and 3-neighbor connectivity for \( S \) and oppositely connected \( \bar{S} \). Object pixels are shown by the shaded regions.
Clearly, $g(S)$ can be interpreted as the number of convex corners of $S$ minus its number of concave corners, since the pattern contributing to $Q_1$ forms a convex corner of an object and the complementary pattern contributing to $Q_2$ a concave corner. As a result only 2 of a total of 14 triplets are relevant for the genus calculation.

**Triangular lattice**

For the triangular lattice there are in total 64 types of patterns formed by six adjacent triangular pixels. These 64 configurations can be divided into 14 distinct groups. In each group the patterns are formed by merely rotating the patterns. Table 3.1 gives a representative of each of the 14 groups and the number of distinct similar patterns in the group that can be formed by rotation. Clearly, this column adds up to 64. The first set of 32 patterns is formed by the first 7 groups in which there is at most a single set of adjacent object pixels. A second set of 32 patterns is formed by the remaining patterns, which are characterized by two or three sets of adjacent object pixels. If $Q_i$ denotes the number of times a pattern of group $i$ ($i = 0, ..., 13$) is found in an arbitrary set of triangular object pixels $S$, then the genus of $S$ is given by

$$g(S) = \frac{1}{6} (2Q_1 + Q_2 \cdot Q_4 \cdot 2Q_5 \cdot 6Q_7 \cdot 3Q_8 \cdot 3Q_9 \cdot 4Q_{10} \cdot 2Q_{11} \cdot 2Q_{12} \cdot 4Q_{13})$$

(3.2.8)

if 12-connectivity is assumed for $S$ and 3-connectivity for $\bar{S}$.

For the reverse connectivity situation the following formula holds:

$$g(S) = \frac{1}{6} (2Q_1 + Q_2 \cdot Q_4 \cdot 2Q_5 + 6Q_7 + 3Q_8 + 3Q_9 + 2Q_{10} + 4Q_{11} + 4Q_{12} + 2Q_{13})$$

(3.2.9)

The coefficients in (3.2.8) and (3.2.9) can be interpreted as the number of convex and concave corners in a pattern of a distinct group. The acute corners are counted twice, and the double corners once. Indeed, the $Q_1$ and $Q_2$ patterns have an isolated convex corner at their center points; the $Q_4$ and $Q_5$ patterns have concave corners; the diagonal configurations
$Q_7$ to $Q_{13}$ have concave corners in case $S$ is 12-connected and convex corners if $S$ is 3-connected, where $\overline{S}$ is in either case connected oppositely.

The above formulas for the genus of $S$ are obviously complicated by the diagonal configurations with the rather untransparent coefficients. Considering the single and double concavities in these diagonal configurations, the coefficients can easily be derived by counting the single concavities twice and the doubles once. Denoting the number of isolated single non-object pixels in the hexagon by $c_1$ and the number of isolated non-object pixel pairs by $c_2$, the genus yields

$$g(S) = \frac{1}{6} (2Q_1 + Q_2 \cdot 2c_1 - c_2).$$  \hspace{1cm} (3.2.10)

Finally, defining $d_1$ as the number of isolated single object pixels in the hexagon and $d_2$ as the number of isolated object pixel pairs, the genus for the reverse connectivity situation yields

$$g(S) = \frac{1}{6} (2d_1 + d_2 \cdot q_4 \cdot 2Q_5).$$  \hspace{1cm} (3.2.11)

Because of both the lack of symmetry in the triangular lattice, as also has been shown in Section 2.5, and the complexity of the connectivity relations as well as the complicated diagonal configurations, there is no simple formula for the genus as derived for the hexagonal lattice.

**Euler characteristic and convexity**

Before analyzing a digitized figure one should remove undesired noise from the given digital region or cellular complex, using, for example, the known techniques of edge sharpening, edge smoothing, and hole filling, gap filling, shrinking, etc. The Euler number is a powerful instrument for detecting the number of holes to be filled before the convexity analysis can be applied. In other applications one may wish to treat the holes as components of the picture, and analyze the shapes of the holes by the available concavity techniques, described in the literature thus far only for the rectangular lattice and mosaic. In any case, convexity analysis can only be applied to simply connected objects, i.e. connected objects with no holes, since otherwise such elementary properties of convexity as
described by Theorems 3.1.1 to 3.1.5 can not be carried over to the
discrete space.

To conclude this section, we will give some preliminary notions defined
on the three lattices and mosaics, which we need in the remainder of the
thesis to treat discrete convexity.

Lattices and digital regions

Because of the Euler characteristic and the connectivity choice to be made
we restrict ourselves to digital regions, that is to 8, 6 or 12 simply con-
nected sets of lattice points, sometimes referred to as connected com-
ponents or, for short, objects.

Definition 3.2.1
An interior point of a digital region is a lattice point \( d \) whose 4, 6 or 3
neighbors are included in the digital region in the case of rectangular
hexagonal or triangular lattice, respectively.
A boundary point of a digital region is a lattice point \( d \) such that at least
one of its 4, 6 or 3 neighbors is not in the digital region in the case of the
three lattices, respectively.

*  

Definition 3.2.2
The boundary of a digital region, denoted as \( \delta D \), is the sequence of
boundary points such that every two successive points are 8-, 6- or
12-connected in the respective lattices.

*  

Note that every boundary point of \( D \) is on \( \delta D \) and no interior point of \( D \)
is on \( \delta D \). In Fig. 3.7 the interior points of the digital regions of Fig. 3.1
are indicated by black dots, the boundary points by asterisks and the
boundary \( \delta D \) by a solid line.

The complement of \( D \), that is the set of lattice points not in \( D \), is denoted
as \( \bar{D} \).
Fig. 3.7 A connected component in the lattice representation of the continuous figure of Fig. 3.1 on the three lattices. The density of the lattices is according to Fig. 3.1. Interior points are indicated by the black dots; boundary points by the asteriks; the boundary $\delta D$ by the solid line. (a) The square lattice. (b) The hexagonal lattice.
Fig. 3.7  (Continued.) A connected component in the lattice representation of the continuous figure of Fig. 3.1 on the three lattices.
(c) The triangular lattice.
Mosaics and cellular complexes

In convexity analysis, complexes on the three mosaics have to be simply-and edge-connected, that is, 4-connected on the rectangular, 6-connected on the hexagonal and 3-connected on the triangular mosaics. It is easy to see that otherwise, given a complex, there does not exist a convex pre-image of the complex in the covering representation. In other words, an 8- or 12-connected complex with non-edge connected cells cannot have a convex preimage.

Definition 3.2.3
An interior cell of a complex is a cell whose 4, 6 or 3 neighbor elements are included in the complex defined on the three respective mosaics. A boundary cell of a complex is a cell at least one whose 4, 6, or 3 neighbor elements is not in the complex defined on the respective mosaics. A corner cell of a complex is a boundary cell of which at least 2 or 3 connected neighboring cells are not in the complex defined on the rectangular or hexagonal mosaic, respectively.

In other words an interior cell is a cell having no points of its edges (the end points included) belonging to the boundary of the complex.

Definition 3.2.4
The space of a cellular complex $C$, denoted as $s(C)$, is the union of all points belonging to the cells of $C$. The boundary of $s(C)$ is referred to as the boundary of the complex and is denoted as $\delta C$. The cellular boundary of a complex $C$ is the set of cells that meet $\delta C$ (i.e., that have a non-empty intersection with $\delta C$).

Given a complex $C$, $\lambda(C)$ denotes the set of centers of the cells of the complex. Thus, $\lambda(C)$ is the set of lattice points uniquely associated with the complex $C$. The complement of $C$, that is the set of cells not in $C$, is denoted as $\overline{C}$. Similarly, the complement of $\lambda(C)$ is denoted as $\overline{\lambda(C)}$. 
3.3 Discrete convexity on the square mosaic

Introduction

In order to get insight into the notion of convexity as defined in the discrete space, we will now review the approaches which led to several definitions of convexity for the rectangular (or square) sampling lattice. In Section 3.1 two ways of representing continuous two-dimensional binary objects or “blobs”, i.e. black objects on a white background, have been considered. The method of representing a gray-scale image on the various lattices according to Shannon’s criterion and the subsequent segmentation of that image into binary parts is quite different from the method of representing continuous blobs, although the notion of convexity will be similar for both concepts. In fact, continuous blobs have an infinite bandwidth so that there is no Shannon-based sample interval to represent the blob without errors. Pavlidis (1980) discussed choosing of the proper sampling lattice so that shape information is preserved.

Approaches to discrete convexity

Unger

The first known approach to formalizing the notion of convexity in the discrete space was Unger’s algorithm for detecting concavities on a square grid with his “spatial computer” (Unger, 1959). A complex $C$ is said to be vertically concave if there exists no triplet $(c_1, c_2, c_3)$ of points forming a strictly vertical straight line segment such that $c_1$ and $c_3$ are center points of $C$ and $c_2$ is a center point of $\overline{C}$. Similar definitions apply to horizontally concave complexes, and to $+45^\circ$ and $-45^\circ$ concave complexes.

Then, a complex obtained by the covering representation is said to be convex if no concavity in either the horizontal, vertical or diagonal direction can be detected. Since only these directions were considered as surely straight, it is easy to draw a concave blob that is convex in all directions. On the other hand, if one wants to generalize such a definition by considering digital straight line segments (Rosenfeld, 1974), connecting $c_1$ and $c_3$ via $c_2$, the way of constructing such a digital arc is
Fig. 3.8 The "example figure" and the "example complex". A concave continuous connected figure $q$ on the square mosaic in the covering representation. Convex complex by Unger.
not univocally defined (Freeman, 1970) and no further improvement along these discrete lines can be obtained. However, an elegant solution is given by Minsky and Papert (1969) and Kim and Sklansky (1982) considering real straight lines in any direction between discrete points.

Fig. 3.8 gives an example of a concave figure q digitized on a square mosaic by the covering representation. This figure and its representation will be used in subsequent analysis to compare the various definitions of convexity. It will be referred to as “the example figure” or “the example complex”.

The complex is uniquely determined by the figure in the open-cell covering representation. Let $e_1$ and $e_2$ be the cells which enclose the north-west side of the complex and $b_1$ and $b_2$ their bottom corner points on the interior side. Any line segment between a point $x$ inside cell $e_1$ and a point $y$ inside cell $e_2$ will intersect the cells $e_3$ and $e_4 \in \overline{C}$. Thus in the open-cell representation, there is no convex preimage of the given complex. However, in the closed-cell representation there is a limit figure which consists in the north-west side of a straight line segment between $b_1$ and $b_2$, which is obviously a convex side of this figure. Hence, there exists, as will be proved later, a convex preimage of the complex in the closed-cell representation. Nevertheless, any preimage of the complex in the open-cell representation is concave.

We conclude that according to the definition of discrete convexity by Unger the example complex is convex.

*Minisky and Papert*

Minisky and Papert (1969) defined the convexity of digital regions in terms of a geometric property: A set of points $X$ on a planar continuum $IR^2$ is convex if and only if there exists no triplet $(x_1, x_2, x_3)$ of collinear points in $IR^2$ such that $x_1$ and $x_3$ are in $X$ and $x_2$ is not. This definition is extended by Kim and Sklansky (1982) to sets of collinear lattice points. First the definition of the line property is required.

*Definition 3.3.1*

A set of lattice points $D$ is said to have the line property if there is no
Fig. 3.9  The example complex. Concave complex by Minsky and Papert and Kim due to the line property. The dashed line connects two points of the complex via two points of its complement.
triplet \((d_1, d_2, d_3)\) of collinear (in the Euclidean sense) lattice points such that \(d_1\) and \(d_3\) are points of \(D\) and \(d_2\) is a point of \(\overline{D}\).

Then, in Kim and Sklansky (1982) a set of lattice points is digitally convex if and only if it has the line property.

Sets of lattice points being convex according to this definition will be referred to as sets having the line property. Note that the line property and digital convexity are defined over the set of lattice points \(D\) rather than over the cellular complex \(C\):

**Definition 3.3.2**
A cellular complex \(C\) is said to be convex if its associated set of lattice points \(\lambda(C)\) has the line property.

This extended Minsky and Papert definition is illustrated in Fig. 3.9, showing that the example complex is concave.

**Sklansky**

Sklansky (1970) defined convexity of complexes in terms of the continuous figure of which the complex is an image.

**Definition 3.3.3**
A cellular complex is said to be convex if there exists at least one continuous convex figure \(q\) of which the given set of cells is an image.

Since this definition is given in terms of preimages, it depends on the representation scheme employed. As stated before, Sklansky used the closed-cell covering representation to obtain the cellular complex. Then, it is possible to relate convexity of complexes to a special preimage: the minimum-perimeter polygon. First we give a few preliminary notions.

**Definition 3.3.4**
An element of a cellular complex \(C\) is said to be isolated if it shares only
Fig. 3.10 The example complex. Convex complex by Sklansky due to convex preimage in the closed-cell covering representation.

Fig. 3.11 An irregular complex with in (a) a convex preimage in the closed-cell covering representation and in (b) its concave MPP. Example taken from Sklansky et al. (1972).
one edge with other cells of the complex. A cellular complex is said to be *regular* if it has no isolated elements, otherwise a complex is said to be *irregular*.

*Definition 3.3.5*

A *minimum-perimeter polygon* (MPP) of a cellular complex $C$ is any polygon $q$ having $C$ as an image, such that there is no polygon of a shorter perimeter also having $C$ as an image.

*With these definitions the following theorem is proved:*

*Theorem 3.3.1* (Sklansky, 1970)

A regular complex $C$ on a rectangular mosaic is *convex* if and only if the MPP of the complex is convex.

*To determine whether a regular complex is convex, it is sufficient to find its MPP and test its convexity. For, if the MPP is convex, a convex preimage is found and by Definition 3.3.3 the complex is convex. It should be noted that convexity is defined in terms of a convex preimage and that the MPP is only a means of detecting convexity!*

The example complex is shown in Fig. 3.10 together with the MPP, which is a convex preimage of the complex in the closed-cell covering representation.

To determine whether an irregular complex is convex, heuristics must be used to search for a convex preimage. This search is not a practical test for convexity, however, because even after an indefinitely long unsuccessful search such a preimage may still exist. An irregular complex may have a convex polygon as its preimage, but its MPP may be concave, a case which is shown in Fig. 3.11 taken from Sklansky *et al.* (1972). In this figure, the concave polygon ABCBEA is the MPP of the complex, while the complex is a cellular image of the convex polygon ABDEA.
Fig. 3.12  The example complex. Concave complex by Feder due to concave chain-code polygon.
Feder

Feder (1968) proposed that a cellular complex be considered convex if the sequence of chain codes (Freeman, 1961) of the boundary forms a convex polygon. The chain code of the boundary is the piecewise linear curve connecting the centers of those connected boundary cells of the complex $C$ that share an edge with $\delta C$. By this definition of convexity a regular cellular image of any triangle on a rectangular mosaic is concave. Since all triangles are convex, this result contradicts our intuitive requirements for convexity of complexes. In Fig. 3.12 the example complex is shown to be concave by Feder.

Hodes

Hodes (1970) modified the midpoint property of the complexes first used by Minsky and Papert (1969), and related that geometric property to the minimum-perimeter polygon approach to convexity of Sklansky et al. (1972). First, a few preliminary notions are required. Given any two cells, $e_1$ and $e_2$, the cell $e$ is the midpoint cell of $e_1$ and $e_2$ if and only if $e$ contains the midpoint of the real line segment between a point in $e_1$ and a point in $e_2$.

Definition 3.3.6

The midpoint cover of two cells $e_1$ and $e_2$, denoted by $M(e_1, e_2)$, is the smallest set of midpoint cells, such that for every segment between a point of $e_1$ and a point of $e_2$ there is a midpoint cell included.

A midpoint cover of $e_1$ and $e_2$ contains in the rectangular mosaic, one, two or four cells, depending on the relative position of $e_1$ and $e_2$.

Definition 3.3.7

A complex $C$ is said to have the midpoint property if for any pair of cells $e_1$ and $e_2$ of $C$, at least one of the cells of the midpoint cover $M(e_1, e_2)$ is an element of $C$.

As a result Hodes shows that the midpoint property is actually stronger
Fig. 3.13  A cellular complex with a convex MPP. (a) With midpoint property. (b) Without midpoint property.
than the property of having a convex MPP. Hodes claims that the midpoint property is a reasonable criterion for convexity of complexes because it includes the MPP as well. That is, if a complex has the midpoint property, its MPP will be convex. The reverse is, surprisingly, not true. In Fig. 3.13(a) a complex is shown which has the midpoint property and therefore the MPP of the complex is convex. The reverse case is shown in Fig. 3.13(b), where the complex has a convex MPP but the midpoint property is violated by the cell pair $e_1$ and $e_2$.

The fact that a (regular) complex $C$ satisfies the midpoint property is no guarantee that one can find a convex preimage of $C$ in the open-cell representation, since the midpoint property includes the property of having a convex MPP which in turn includes a convex preimage in the closed-cell representation. To preclude this ambiguity in properties on the different digitization schemes, Hodes proved the following theorem:

**Theorem 3.3.2 (Hodes, 1970)**
A regular cellular complex has a convex preimage in the open-cell covering representation if and only if the MPP does not touch the boundary of the complex.

In Fig. 3.14 the example complex is shown to have the midpoint property and the convex MPP touching the boundary. Hence, there is no convex preimage in the open-cell representation.

In fact Theorem 3.3.2 contains a paradox, since it relates a property of preimages digitized by the open-cell covering representation to the property of having an MPP which is exclusively defined on the closed-cell representation. For, achieving the minimum involves a passage to the limit, which is only defined in a closed-cell representation.

The class of convex cellular complexes determined by the convexity of the MPP is somewhat larger than the class of convex complexes suggested by the extended Minsky and Papert definition and the Hodes criterion of not touching the boundary by the MPP and also larger than the class of convex complexes suggested by the definition of Feder. As shown in Fig. 3.10, the complex has a convex MPP and therefore a convex preimage
Fig. 3.14  The example complex. Concave complex by Hodes due to the MPP touching the boundary. The complex has the midpoint property.
in the closed-cell representation and thus is convex by the Sklansky
definition. Fig. 3.9 showed the Minsky and Papert criterion formalized by
Kim in the line property indicating the concavity of the complex. Since
the MPP touches the boundary of the complex, the complex is also
concave by the Hodes definition. In fact, as has been shown by Kim (1981),
the Minsky and Papert notion of convexity is equivalent to the Hodes
criterium of the MPP not touching the boundary for the class of regular
complexes. Finally, the complex is concave by the definition of Feder, as
shown in Fig. 3.12. The Unger definition is too simple to be referred to
anymore.

Algorithms for convexity analysis

Algorithms for detecting concavities in cellular complexes have been
developed by Sklansky et al. (1972) in terms of the MPP, by Arcelli and
Cordella (1974) in terms of the Hodes not-touching definition and by
Gaafar (1977) for the same criterion. The Sklansky algorithm will be used
in Section 4.3 and will be referred to as algorithm M. We will briefly
review the two other algorithms which are typical examples of both
parallel and sequential approaches to the detection of the convexity of
complexes. First we need a few preliminary notions.

Definition 3.3.8
A chain is any finite sequence of cells, each of which, except for the first,
is an edge neighbor of its predecessor. A chain is regular if the predecessor
and successor of every element of the chain are distinct, otherwise a chain
is said to be irregular. A chain is said to be closed if the first and the last
elements are edge neighbors.

Definition 3.3.9
A boundary chain of C is any closed chain of cells in the cellular
boundary such that every cell of the cellular boundary is an element
of the chain.

Definition 3.3.10
The spinal path of a chain is the piecewise linear curve obtained by
Fig. 3.15  The example complex. Concave complex by Gaafar due to the absence of a convex polygon inside the cellular boundary.
connecting the centers of the successive cells of the chain by straight line segments.

* Arcelli and Cordella

Arcelli and Cordella (1974) applied Definition 3.3.3 and Theorem 3.3.2 to derive an algorithm based on the chain code of the spinal path of the boundary chain. The edges of the spinal path, expressed in sequences of equal codes, identify, taken two by two, an internal angle \( a \), whose value is a multiple of \( \pi/4 \) radians. When the lengths of successive edges measured in equal chain code elements are both greater than one and the difference in chain code \( i \) and code \( i + 1 \) equals one, i.e. an internal angle of \( 5/4 \pi \), a concavity is detected, where the spinal path is traversed counterclockwise. All other local configurations indicating a concavity are identified by a triplet of successive edges such that, following the spinal path counterclockwise, the difference between the codes of the first and the third edge is 2 (modulo 8) and the central edge has the intermediate code. However, configurations on which a "gentle" concavity is present are not detected by this parallel procedure, which is a serious drawback.

The examples given in Fig. 3.13(a) and (b) are both detected as concave since there is a triplet of successive edges of the spinal path with the chain codes 4, 3 and 2, in that order, satisfying the second condition for detecting a concavity. The code sequence of the spinal-path polygon of the example complex, as shown in Fig. 3.12, equals 0 1 2 1 2 4 5 4 5 4 5 6 0, where the edges which are greater than one code element are underlined. Here the gentle concavity on the north-west side of the complex is not detected, whereas according to Theorem 3.3.2 the complex does not have a convex preimage.

* Gaafar

Gaafar (1977) developed a sequential algorithm capable of determining any concavity defined by Definition 3.3.3 in a cellular complex obtained by the open-cell covering representation. Therefore, the result is equivalent to Theorem 3.3.2.
Fig. 3.16 The example complex. Concave complex by Kim due to the area property. The shaded regions constitute the polygons of $P(C; c_1, c_2)$. 
The algorithm scans the boundary chain and consequently calculates the slope of the real line segment lying entirely in the chain of cells having an equal code, i.e. a block of equal codes. For convexity to hold, the slopes should increase when traversing the boundary chain counterclockwise.

The algorithm detects the concavity in the example complex as shown in Fig. 3.15, since there is no set of straight line segments with an increasing slope strictly inside the north-west part of the boundary chain.

**Additional approach to discrete convexity**

*Kim*

In contrast to the previous definitions of convexity of complexes given in Unger (1959), Feder (1968), Minsky and Papert (1969), Sklansky (1970) and Hodes (1970), Kim defines convexity in terms of a geometric property of the complex (Kim, 1981). The cellular convexity of any complex may be determined without considering any of its preimages. It is proved that for the case of regular complexes, the definition of Sklansky (1970), given in terms of convex preimages, is equivalent to Kim’s definition. Before defining the area property, we first give a few preliminary notions.

Let $c_1$ and $c_2$ be two centers of cells of a simply connected complex $C$, then $P(C; c_1, c_2)$ denotes the set of polygons whose boundaries consist of parts of $c_1, c_2$ and $\delta C$, and whose interiors are subsets of the complement of $s(C)$. An example is given in Fig. 3.16 in which $P(C; c_1, c_2)$ is the shaded area. Then we have the following:

**Definition 3.3.11**

A cellular complex $C$ is said to have the *area property* if for any two centers $c_1$ and $c_2$ in $C$, there is no cell which is an element of $C$ and whose center is in $P(C; c_1, c_2)$.

Then by Kim (1981), a complex is said to be *cellularly convex* if and only if it has the area property. This definition is illustrated in Fig. 3.16,
Fig. 3.17 The example figure and its lattice representation. Concave digital region by Kim due to the line property. (a) The dashed line connects two points of the digital region via a point of its complement.
showing that the example complex is concave. Complexes being convex according to this definition are referred to as complexes having the area property.

The main result is the proved equivalence of the extended Minsky and Papert definition - i.e. the line property of the associated set of lattice points - and Kim's definition - i.e. the area property of the complex.

For the case of regular complexes, Kim used, without notification, the result of Hodes (1970) to relate the area property to the MPP:

**Theorem 3.3.3**
A regular complex $C$ is cellular convex - i.e. has the area property - and has a convex preimage in the open-cell covering representation if and only if its MPP does not touch the boundary of $C$.

*Concluding remarks*

Kim (1981) introduced a new definition of convexity of complexes in terms of a geometric property, the area property. It was shown in Kim (1981) that convexity of a complex $C$ in terms of the area property is equivalent to the convexity of the associated set of lattice points $\lambda(C)$ in terms of the line property. The latter definition is an extension of the Minsky and Papert approach to convexity. Furthermore, for regular complexes Kim proved the equivalence between the area property and the definition of convexity of Sklansky (1970) in terms of a convex preimage. Then, the convexity of regular complexes, i.e. the area property, is shown by Hodes (1970) to be equivalent to a convex minimum-perimeter polygon of the complex not touching the boundary of the complex.

Hence, it turned out that for the case of regular complexes in the open-cell representation, the line property, the area property, the convexity of a preimage and the convex MPP not touching the boundary of the complex are equivalent, leading to a virtual equivalence of the definitions.

This result is obtained for cellular complexes defined on rectangular or square mosaics. On a hexagonal mosaic every complex, considered as
(b) The lattice representation of the example figure. (Continued.) Each point of the digital region is represented by a digitization cross. The auxiliary line shows that there does not exist a convex preimage.
a continuous plane figure, is concave, except for the one-cell complex. Therefore, it is not obvious whether the proofs of the theorems which lead to the virtual equivalence of the various approaches to convexity will hold for hexagonal mosaics.

The above results were obtained on complexes under the covering representation of blobs. Kim (1982b) proved that under the lattice representation of blobs the definition of convexity by Sklansky in terms of preimages is completely equivalent to the geometric properties.

Fig. 3.17(a) shows the concave figure digitized by the lattice representation and an example of a line segment between lattice points which violate the line property. If each lattice point of the digital region $D$ is represented by its open digitization cross, the configuration shown in Fig. 3.17(b) arises. The point $d$ is a lattice point of $\bar{D}$ and should lie strictly outside the boundary of the continuous figure. By virtue of the auxiliary line drawn in Fig. 3.17(b) it is clear that there is no convex preimage of the region $D$, nor a straight line segment bounding the preimage in the north-west direction.

As a final result Kim states the following theorem:

**Theorem 3.3.4** (Kim, 1982b)
Given a digital region $D$, the following statements are equivalent:
1) $D$ has a convex preimage in the lattice representation.
2) $D$ has the line property.
3) $D$ has the chord property.
4) The set of polygons $P(D; d_1, d_2)$ contains no point of $\bar{D}$.
5) The convex hull of $D$ contains no point of $\bar{D}$.

This theorem will be shown to hold for the hexagonal lattice under the lattice representation as well.

3.4 Discrete convexity on the hexagonal lattice

The results in earlier papers about convexity were obtained on cellular complexes and digital regions defined on rectangular or square lattices and
mosaics. In this section we will derive a theory to analyze convexity of digital images on the hexagonal lattice. The next section will treat the convexity on the hexagonal mosaic.

As stated before continuous figures can be represented by a set of lattice points, referred to as a digital region, when the lattice representation for digitizing blobs is used, or by a set of cells, referred to as a cellular complex, when the covering representation for digitizing blobs is used. Associated with a cellular complex is a set of points being the centers of the cells constituting the complex. These points lie on a regular lattice. However, given an arbitrary continuous figure, the set of lattice points obtained by the lattice representation essentially differs from the set of lattice points associated with the cells of the cellular complex obtained by the covering representation of the same continuous figure.

Let convexity of digital regions and cellular complexes be defined in terms of the convexity of their preimages in the corresponding representations. Then, what is meant by convexity of the set of lattice points associated with a cellular complex is not obvious. In general, the preimage of a cellular complex according to the covering representation is not a preimage of the associated set of centers according to the lattice representation. Thus, given a cellular complex, two distinct convex preimages ought to be sought: one for verifying the convexity of the complex and one for the convexity of the digital region associated with the complex.

First, we will introduce the definitions and notations that are used in this section to prove the various lemmas, which in turn are used to prove the theorems on convexity. Some of the terminology and notation has been introduced before in Section 3.3, primarily defined on the rectangular mosaic. Next, the lemmas that relate the geometric properties of digital regions will be proved. Finally, it will be shown that under the definition of digital convexity these geometric properties are equivalent, i.e. the result is similar to Theorem 3.3.4 obtained by Kim (1982b) for the rectangular lattice.

Preliminary notions and geometric properties

Let us consider the set of lattice points of a hexagonal lattice as defined
by the generator matrix given by eq. (2.4.21), the regular hexagonal lattice. Once again, we assume that the discrete representations on the lattices and complexes are simply connected unless explicitly stated otherwise.

In addition to the Definitions 3.2.1 to 3.2.4 we note that a cell of a cellular complex $C$ is sometimes referred to as an element $e$ of the complex and its lattice point or center by a corresponding $c$.

**Definition 3.4.1**
The space of a digital region $D$, denoted as $s(D)$, is the subset of the plane interior to $\delta D$, the boundary of $D$.

Some of the geometric properties of digital regions on rectangular lattices were treated in Section 3.3. These and other properties will be defined here formally on the hexagonal lattice, although most definitions are similar to the rectangular ones given in the previous section.

**Definition 3.4.2**
Area property of a digital region
For any two lattice points $d_1$ and $d_2$ of $D$, $P(D; d_1, d_2)$ denotes the set of polygons whose boundaries consist of non-empty subsegments of $d_1 d_2$ and $\delta D$, and whose interiors are subsets of the complement of $s(D)$. The digital region $D$ is said to have the area property if there are no two points $d_1$ and $d_2$ of $D$ such that $P(D; d_1, d_2)$ has a polygon, containing a point of $\overline{D}$. This was defined for square digital regions by Kim (1982b).

**Definition 3.4.3**
Area property of a complex
Let $c_1$ and $c_2$ be the centers of two cells of complex $C$. Let $P(C; c_1, c_2)$ denote the set of polygons whose boundaries consist of non-empty parts of $c_1 c_2$ and $\delta C$, and whose interiors are subsets of the complement of $s(C)$. The complex $C$ is said to have the area property if for any two centers $c_1$ and $c_2$ of $C$ there is no cell which is an element of $\overline{C}$ and whose center is in $P(C; c_1, c_2)$. This property was originally defined by Kim (1981) for complexes on square mosaics.
Note that for any two lattices points $d_1$ and $d_2$, $P(C; d_1, d_2)$ is a subset of $P(D; d_1, d_2)$, where $D$ is the associated set of lattice points $\lambda(C)$.

**Theorem 3.4.1**
A cellular complex $C$ has the area property if and only if the associated set of lattice points $\lambda(C)$ has the area property.

*Proof:*
Let's first suppose $C$ does not have the area property; then there are two elements $e_1$ and $e_2$ of $C$ with their centers $c_1$ and $c_2$, such that $P(C; c_1, c_2)$ contains the center $c$ of a cell $e$ not in $C$. By Definition 3.4.3 $c$ is a lattice point of the complement of $\lambda(C)$. Since $P(C; c_1, c_2)$ is included in $P(\lambda(C); c_1, c_2)$, $c$ is a point of $P(\lambda(C); c_1, c_2)$. Hence, $\lambda(C)$ does not have the area property.

Let's now suppose $C$ does have the area property; then for any two elements $e_1$ and $e_2$ of $C$, $P(C; c_1, c_2)$ does not contain the center of any element $e$ not in $C$ nor any lattice point of the complement of $\lambda(C)$. The interior of $P(\lambda(C); c_1, c_2)$ outside $P(C; c_1, c_2)$ is a subset of $s(C)$, and therefore does not contain a point of the complement of $\lambda(C)$. Hence, $\lambda(C)$ has the area property.

This proves the theorem for both the rectangular and hexagonal case.

\[\square\]

It should be noted that Theorem 3.4.1 has not been proved before to hold for the rectangular case, although Kim (1982b) referred to the result, enabling him to prove Theorem 3.3.4.

**Definition 3.4.4**
Line property
A digital region $D$ or a cellular complex $C$ is said to have the **line property**
if there exists no triplet of collinear lattice points \((d_1, d_2, d_3)\) or \((c_1, c_2, c_3)\) respectively, such that \(d_1\) and \(d_3\) are points of \(D\) or \(c_1\) and \(c_3\) are the centers of cells \(e_1\) and \(e_3\) of \(C\) and \(d_2\) is a point of \(\overline{D}\) or \(c_2\) is the center of cell \(e_2\) of \(\overline{C}\).

*Definition 3.4.5*

Midpoint property
A digital region \(D\) or a cellular complex \(C\) is said to have the **midpoint property** if for any pair of lattice points \(d_1\) and \(d_2\) of \(D\) or centers \(c_1\) and \(c_2\) of cells \(e_1\) and \(e_2\) of \(C\), at least one of its midpoints belongs to \(D\) or \(C\), respectively.

*If for every pair of lattice points \(d_1\) and \(d_2\) of \(D\) or centers \(c_1\) and \(c_2\) of cells \(e_1\) and \(e_2\) of \(C\) for which the single midpoint exists, this midpoint belongs to \(D\) or \(C\), respectively; then the digital region \(D\) or the cellular complex \(C\) is said to have the single midpoint property. Note that the single midpoint property bears the greatest resemblance to the midpoint property of Minsky and Papert (1969), since it yields their order measure of three, whereas the midpoint property may have order six. Furthermore, the single midpoint property is a weaker version of the midpoint property since only pairs of lattice points that have a single midpoint are considered.*

Let \(d_1 = (p_1, p_2)\) and \(d_2 = (q_1, q_2)\) be two lattice points of a digital region \(D\) or the centers of the cells \(e_1\) and \(e_2\) of a cellular complex \(C\), given in oblique coordinates with respect to the \(u_1\) and \(u_2\) axes. Further, let \(z = (x_1, x_2)\) be a point such that \(x_1 = (p_1 + q_1)/2\) and \(x_2 = (p_2 + q_2)/2\). We will state a result concerning the topology of midpoints in the discrete space.

*Theorem 3.4.2*

The midpoint cover of two lattice points of a hexagonal region consists of one or four midpoints, depending on the orientation of the points.

*Proof:*
If \(z\) is a lattice point, then the locus of the midpoints of the straight line
Fig. 3.18   Midpoint property; (a) $p_1 q_1$ is even and $p_2 q_2$ is odd; (b) $p_1 q_1$ is odd and $p_2 q_2$ is even.
segment between any (real) point in \( e_1 \) and any real point in \( e_2 \) is merely the hexagonal cell centered at \( z \). If \( z \) is not a lattice point, then \( z \) is a point on an edge of some cell of the cellular mosaic; in fact \( z \) is a midpoint of such an edge.

First, suppose without loss of generality that \( d_1 \), \( d_2 \) is in the direction of the principal axis \( u_2 \). Then, if \( p_2 \cdot q_2 \) is even and therefore the number of cells in between \( d_1 \) and \( d_2 \) is odd, \( z \) is a lattice point, a case considered above. If \( p_2 \cdot q_2 \) is odd, and therefore the number of cells in between \( d_1 \) and \( d_2 \) is even, the cellular chain with spinal path \( d_1 \), \( d_2 \) can be split into two chains having an equal number of cells. The common edge of these chains contains the point \( z \), positioned halfway along the edge.

Let's next suppose that \( d_1 \), \( d_2 \) has an arbitrary direction. Without loss of generality we assume that \( d_1 \) is located above and to the right of \( d_2 \). Then, if \( p_1 \cdot q_1 \) is even, and therefore the number of cells in between \( d_1 \) and \( d_2 \) in the \( u_1 \)-direction is odd, there is a row of lattice points exactly in between the row of \( d_1 \) and the row of \( d_2 \), in which \( z \) is positioned (see Fig. 3.18(a)). Let's suppose further that \( p_2 \cdot q_2 \) is even; then \( z \) is a lattice point, a case which is treated above. However, if \( p_2 \cdot q_2 \) is odd, \( z \) is a point halfway in between two lattice points and therefore \( z \) lies on the common edge of two neighboring cells in the row of \( z \). Since \( z \) is on the ray through the lattice points of the row, \( z \) is the midpoint of the edge.

Let's next consider the case when \( p_1 \cdot q_1 \) is odd; then \( z \) is not in a row of lattice points but exactly in between two rows. The ray through \( z \) in the \( u_2 \)-direction intersects the edges of the cells in the rows below and above \( z \) precisely at their midpoints. The point \( z \) itself is such an intersection point by virtue of the following argument. Since \( p_1 \cdot q_1 \) is odd, the points \( d_1 \) and \( d_2 \) are in rows that are shifted half a lattice unit toward each other in the \( u_2 \)-direction and therefore \( z \) is shifted to the right a fourth part of a lattice unit with respect to the lattice points in the row of \( d_1 \) if \( p_2 \cdot q_2 \) is odd, and with respect to the lattice points in the row of \( d_2 \) if \( p_2 \cdot q_2 \) is even (see Fig. 3.18(b)). Moreover, the intersection points of the ray through \( z \) and the cell edges are alternately shifted with respect to \( d_1 \) and \( d_2 \) and therefore \( z \) is such an intersection point.

Since \( z \) is the midpoint of a cell edge and since the locus of midpoints of
Fig. 3.19 The midpoint cover in the hexagonal mosaic. The locus of the midpoints is covered by one or four hexagonal cells.

Fig. 3.20 The elementary region about a lattice point such that the interior of the region is the locus of points with $d_{hex} < 1$ with respect to the lattice point.

Fig. 3.21 The domain of a digital region $D$ to check the chord property.
The straight line segment between any (real) point in \( e_1 \) and any (real) point in \( e_2 \) is a hexagon centered at \( z \) of the same shape and size of a cell, four cells of the cellular mosaic are symmetrically covered, and therefore the midpoint cover contains four cells. Note that in case there is a single midpoint, the midpoint cover only contains that single cell.

The midpoint cover of two lattice points of a square cellular mosaic may consist of one, two or four lattice points, since the locus is a square that can be covered by two adjacent square cells. In the case of a hexagonal cellular mosaic a hexagon can only be covered by itself or by a quartet of cells, as shown in Fig. 3.19. The coverage with a triplet is excluded since \( z \) cannot be a corner point of a cell, as has been proved above.

Let \( d_1 = (p_1, p_2) \) and \( d_2 = (q_1, q_2) \) once again be two lattice points of a digital region \( D \) or the centers of the cells \( e_1 \) and \( e_2 \) of a cellular complex \( C \), given in oblique coordinates. Let \( z = (x_1, x_2) \) be the point such that \( x_1 = (p_1 + q_1)/2 \) and \( x_2 = (p_2 + q_2)/2 \). Then we define the following:

**Definition 3.4.6**

Let \( d_1 = (p_1, p_2), d_2 = (q_1, q_2) \) and \( z = (x_1, x_2) \) be defined as above. If \( z \) is a lattice point it is said to be the **median point** of \( d_1 \) and \( d_2 \). If \( z \) is not a lattice point, the two lattice points on \( d_1, d_2 \) which are nearest to \( z \) on opposite sides of \( z \) are the **median points** of \( d_1 \) and \( d_2 \).

By definition, the single midpoint of \( d_1 \) and \( d_2 \), if it exists, is also the median point of \( d_1 \) and \( d_2 \).

From Definition 3.4.8 we define the following property:

**Definition 3.4.7**

Median-point property

A digital region \( D \) or cellular complex \( C \) is said to have the **median-point property** if for any pair of lattice points \( d_1 \) and \( d_2 \) of \( D \) or centers \( c_1 \) and \( c_2 \) of cells \( e_1 \) and \( e_2 \) of \( C \), at least one of the median points is in \( D \) or \( C \), respectively.
To proceed with the chord property for hexagonal lattices, we need the notion of distance on the hexagonal lattice.

**Definition 3.4. 8**
Let \( d_1 = (p_1, p_2) \) and \( d_2 = (q_1, q_2) \) be two lattice points or two centers of the cells \( e_1 \) and \( e_2 \), given in oblique coordinates with respect to the \( u_1 \) and \( u_2 \) axes. The *hexagonal distance* between \( d_1 \) and \( d_2 \) is given by

\[
d_{\text{hex}} ((p_1, p_2), (q_1, q_2)) = \max \left\{ |p_1 - q_1|, |p_2 - q_2| \right\}, \text{ if } (p_1 - q_1) \text{ and } (p_2 - q_2) \text{ have the same sign}
\]

\[
= |p_1 - q_1| + |p_2 - q_2|, \text{ if they have opposite signs.}
\]

In one formula

\[
d_{\text{hex}} ((p_1, p_2), (q_1, q_2)) = \max \left\{ |p_1 - q_1|, |p_2 - q_2|, |p_1 - q_1 + p_2 - q_2| \right\}
\]

The proof of which is given in Luczak and Rosenfeld (1976), using a slightly different coordinate grid.

The locus of the points \( x = (x_1, x_2) \) about a lattice point \( d = (p_1, p_2) \) such that \( d_{\text{hex}} ((x_1, x_2), (p_1, p_2)) < 1 \) is just the interior of a hexagon with diameter \( 2n \) centered at \( (p_1, p_2) \). This elementary region is essentially the standard 6-neighbor distance function generalized to real arguments.

Fig. 3.20 depicts such an elementary region centered about a lattice point.

Now we can define the notion of nearness:

**Definition 3.4. 9**
Let \( d_1 \) and \( d_2 \) be two lattice points of a digital region \( D \) or two centers of the cells \( e_1 \) and \( e_2 \) of a cellular complex \( C \). The straight line \( d_1 d_2 \) is said to lie near \( D \) if for any real point \( x = (x_1, x_2) \), given in oblique coordinates, there exists a lattice point \( d' = (d_1, d_2) \) of \( D \) such that \( d_{\text{hex}} ((x_1, x_2), (d_1, d_2)) < 1 \).

From this the chord property can be defined, a property which was

**Definition 3.4.10**

Chord property

A digital region $D$ or a cellular complex $C$ is said to have the chord property if for any pair of lattice points $d_1$ and $d_2$ of $D$ or centers $c_1$ and $c_2$ of cells $e_1$ and $e_2$ of $C$, the chord $d_1 d_2$ or $c_1 c_2$, respectively, lies near $D$ or $C$, respectively.

Fig. 3.21 shows the domain of a digital region $D$ composed of elementary hexagons with diameter $2h$ centered about the lattice points of $D$, inside of which any real point should be on the straight line segment $d_1 d_2$ between any two points $d_1$ and $d_2$ of $D$ in order for $D$ to possess the chord property.

In Rosenfeld (1974) it was shown for the square lattice under the standard grid intersection quantization scheme (Freeman, 1970) that the chord property is a necessary and sufficient condition for a digital arc to be a digital straight line segment. In this section we will relate the chord property to the convexity of digital regions on the hexagonal lattice. In the next section the relation will be extended to cellular complexes. The relation between straightness and convexity will be treated in Chapter 5.

Finally, we state the last geometric property to be defined for a digital region:

**Definition 3.4.11**

Convex-hull property

A digital region $D$ or cellular complex $C$ is said to have the convex-hull property if the convex hull of $D$ or $\lambda(C)$, respectively, denoted as $CH(D)$ or $CH(C)$, respectively, does not contain a point of $\overline{D}$ or $\overline{\lambda(C)}$, respectively.

Before relating the convexity of digital regions to the geometric properties we will state several lemmas that relate the geometric properties
Fig. 3.22  A digital region $D$ with (a) a bulge and (b) a trough.
introduced above to each other, which leads to the main theorem that the geometrical properties are virtually equivalent to each other.

Kim (1981) proved the equivalence between the area property of a cellular complex and the line property of the associated set of lattice points, both properties defined on the square cellular mosaic. Here, we will make use of Kim’s result in order to prove the equivalence of both the area property of a digital region and the line property of a digital region on a hexagonal lattice.

Lemma 3.4.3
A digital region \( D \) has the area property if and only if it has the line property.

*Proof:*
Suppose that \( D \) has the area property. Then for any pair of points \( d_1 \) and \( d_2 \) of \( D \), \( P(D; d_1, d_2) \) does not contain a point not in \( D \). Hence, there is no collinear triplet \( (d_1, d, d_2) \) with \( d_1, d_2 \in D \) and \( d \in \overline{D} \) and therefore \( D \) has the line property.

Now suppose that \( C \) does not have the area property. Then there are two points \( d_1 \) and \( d_2 \) such that \( P(D; d_1, d_2) \) contains a point \( d \) of \( \overline{D} \). The problem is to determine whether a triplet \( (d_1, d, d_2) \) is collinear.

If \( d_1, d_2 \) is in the direction of one of the principal axes \( u_1, u_2 \) or \( u_3 \), then \( d_1, d_2 \) contains a lattice point \( d \) not in \( D \). Hence, \( (d_1, d, d_2) \) is the desired triplet.

Next, suppose that \( d_1, d_2 \) has an arbitrary direction. Without loss of generality assume that \( d_1 \) is located above and to the right of \( d_2 \), and if \( d_1 = (p_1, p_2) \) and \( d_2 = (q_1, q_2) \) that \( |p_1 - q_1| > |p_2 - q_2| \). If \( \delta D \) between \( d_1 \) and \( d_2 \) has a bulge or a trough as shown in Fig. 3.22(a) and (b), then the triplet \( (d', d, d'') \) satisfies the condition. In all other cases \( \delta D \) between \( d_1 \) and \( d_2 \) is composed of line segments having only two directions, in our assumption the \( u_1 \) and \( u_2 \) directions, as shown in Fig. 3.23(a).

These configurations are sometimes referred to as regions having the shape of steps. It is easy to see that the topology of the region between
Fig. 3.23  A digital region $D$ of which $\delta D$ is in the shape of steps. (a) In
the directions of $u_1$ and $u_2$ only. (b) Rotation of the $u_1$-axis
over $30^\circ$ such that $\delta D$ is in the shape of steps in the directions
of $x_1$ and $x_2$ only.
$d_1$ and $d_2$ is, if in the shape of steps, preserved when the hexagonal lattice is "de-slanted" into a square lattice. The transformation is performed by rotating the $u_1$-axis 30° counterclockwise, whereas the $u_2$-axis remains fixed, as shown in Fig. 3.23(b). Under this rotation collinear triplets $(d_1, d, d_2)$ maintain their collinearity.

For configurations on square lattices having the shape of steps, Kim (1981) proved the existence of collinear triplets which violate the line property of the associated digital region of a cellular complex not having the area property. By virtue of Theorem 3.4.1 the area property of a cellular complex is equivalent to the area property of the associated digital region. Hence, by applying the "de-slanting"-transform and Kim’s result, our proof is completed.

\[
\square
\]

**Lemma 3.4.4**

If a digital region $D$ has the line property, then it also has the midpoint property.

\[
\star
\]

**Proof:**

Let’s suppose $D$ does have the line property. Then for every pair of lattice points $d_1$ and $d_2$ of $D$, there is no triplet $(d_1, d, d_2)$ of collinear points such that $d$ is in $\bar{D}$. If $d_1$ and $d_2$ have only one midpoint, it is on $d_1d_2$ and therefore in $D$. Now let’s suppose the midpoint cover $M(d_1, d_2)$ contains four lattice points. Then not all of them lie on the same side of $d_1d_2$, and therefore $D$ contains a point of $M(d_1, d_2)$.

\[
\square
\]

The converse of Lemma 3.4.4 is not true, that is, a digital region may have the midpoint property while there is a collinear triplet violating the line property, as the example in Fig. 3.24 shows. The digital region $D$ does not have the line property due to the triplet $(d_1, m_1, d_2)$ or $(d_1, m_2, d_2)$ yet $m_4$ provides the midpoint property of $D$.

**Lemma 3.4.5**

A digital region $D$ has the median-point property if and only if it has the
Fig. 3.24 A digital region $D$ with the midpoint property, and without the line property.

Fig. 3.25 A digital region $D$ that violates the line property; the median points of $d_1$ and $d_2$ are denoted by $m_1$ and $m_2$ and the median points of $d'_1$ and $d'_2$ by $m'_1$ and $m'_2$. 
line property.

*Proof:
Suppose the digital region $D$ has the line property. Then for every pair of lattice points $d_1$ and $d_2$ of $D$, there is no collinear triplet $(d_1, d, d_2)$ such that $d$ is a point of $\overline{D}$. Therefore, every median point of $d_1$ and $d_2$ will be a point of $D$, since the median points are by definition on $d_1 d_2$, and so $D$ has the median-point property.

Now suppose $D$ does not have the line property. Then there is a triplet of collinear points $(d_1, d, d_2)$ such that $d_1$ and $d_2$ are points of $D$ and $d$ is not. Let $d'_1$ be the lattice point of $D$ on $d_1 d$ nearest $d$. Such a point exists, since $d_1$ itself is such a point, as shown in Fig. 3.25. Similarly, let $d'_2$ be the lattice point of $D$ nearest to $d$ on the other side of $d$ on $d d_2$. Then all lattice points on $d'_1 d'_2$ except for $d'_1$ and $d'_2$ themselves are points of $\overline{D}$. Therefore, the median points of $d'_1 d'_2$ are all points of $\overline{D}$. Hence, $D$ does not have the median-point property.

□

Lemma 3.4.6
A digital region $D$ has the chord property if and only if it has the line property.

*Proof:
Suppose $D$ does not have the chord property; then there exist two lattice points $d_1$ and $d_2$ of $D$ and a point $z$ on $d_1 d_2$ such that $z$ is outside the region composed of elementary hexagons with diameter $2h$ about each lattice point of $D$. The center point of the cell associated with $z$ is not in $D$, otherwise $z$ would not violate the chord property. If the center point of this cell is in $P(D; d_1, d_2)$, then $D$ does not have the area property. If the center point is not in $P(D; d_1, d_2)$, then there exists a neighboring cell whose center is in $P(D; d_1, d_2)$ and is a point of $\overline{D}$. Therefore, $D$ does not have the area property and by virtue of Lemma 3.4.3 $D$ does not have the line property.

Now suppose $D$ does not have the line property. Then there exists a triplet
Lemma 3.4.7
A digital region \( D \) has the convex-hull property if and only if \( D \) has the area property.

Proof:
Suppose \( CH(D) \) contains a point \( d \) of \( \overline{D} \). Then there exists an edge of \( CH(D) \) between two successive vertices \( d_1 \) and \( d_2 \) of \( CH(D) \) such that the polygon \( P(D; d_1, d_2) \) contains \( d \). Hence, \( D \) does not have the area property.

Next, suppose \( CH(D) \) does not contain a point of \( \overline{D} \). Since the set of polygons \( P(D; d_1, d_2) \) of any pair of points \( d_1, d_2 \) of \( D \) is completely contained in \( CH(D) \), there is no \( P(D; d_1, d_2) \) that contains a point of \( \overline{D} \), and so \( D \) has the area property.

The geometric properties defined on the hexagonal lattice are restricted to digital regions which have several nice global properties such as a smooth contour \( \delta D \) or the property of having no outside point inside its convex hull. Similar geometric properties hold for continuous convex figures, in Euclidean space. Therefore, it would be convenient if these geometric properties were to carry over to digital regions determined by the lattice representation of a continuous convex figure.

In the following we will give a simple and elegant definition of convexity of digital regions that will be proved to be completely equivalent to the geometric properties.

Definition 3.4.14
A digital region \( D \) is said to be digitally convex if there exists a convex figure \( q \) whose lattice representation is \( D \). If a digital region is not digitally
convex, then it is *digitally concave*.

The definition of digital convexity is similar to the cellular one of Sklansky (1970), who defined convexity of cellular blobs obtained with the closed-cell covering representation in terms of the convexity of the preimage. It is important to note here that this definition of convexity depends strongly on the digitization scheme, whereas the geometric properties are defined on the digital regions or cellular complexes alone, i.e., considered without reference to a preimage that originated the region or complex.

**Lemma 3.4.8**
A digital region $D$ is digitally convex if and only if $D$ has the convex-hull property.

**Proof:**
Suppose that $CH(D)$ contains a point $d$ of $\bar{D}$. Any convex figure that contains $D$ will contain $d$, since $CH(D)$ is the smallest of these convex figures. Hence, $D$ cannot be the lattice representation of any convex figure and therefore $D$ is not digitally convex.

Next, suppose that $CH(D)$ does not contain a point $d$ of $\bar{D}$. Since $CH(D)$ is convex, and $CH(D)$ contains $D$, the digital region $D$ has a convex preimage and is therefore digitally convex.

The Lemmas 3.4.3 to 3.4.8 combine to yield the following theorem:

**Theorem 3.4.9**
Given a digital region $D$, the following statements are equivalent:
1) $D$ is digitally convex.
2) $D$ has the line property.
3) $D$ has the area property.
4) $D$ has the median-point property.
5) $D$ has the chord property.
6) $D$ has the convex-hull property.
By virtue of Theorem 3.4.9 the geometric properties are equivalent to the convexity of the digital region obtained by the lattice representation on the hexagonal lattice of continuous blobs. However, the geometric properties of the set of lattice points associated with the cellular complex obtained by the covering representation are not equivalent to convexity defined in terms of preimages of the cellular complex under the covering representation. Convexity of complexes on the hexagonal mosaic will be treated in the next section.

As a consequence of Theorem 3.4.9 gray-scale images segmented into binary objects are open for convexity analysis. Here, there is no reference to a continuous figure $q$ as a preimage for an object. Nevertheless, digital regions representing a single object in the segmented image are said to be digitally convex if one of the geometric properties hold and therefore by virtue of Theorem 3.4.9 all geometric properties hold. Then one can imagine a continuous convex binary figure $q$ which could have been a preimage of the object.

3.5 Discrete convexity on the hexagonal mosaic

Introduction

In the previous section a theory was derived for convexity analysis of digital regions as connected subsets of lattice points on hexagonal lattices. Convexity was defined in terms of a preimage under the lattice representation and was shown to be equivalent to the geometric properties defined on the lattice. Since the definition of convexity depends strongly on the digitization scheme, a cellular version of the definition of convexity now based on the covering representation has a different impact on the geometric properties.

Sklansky and Kibler (1976) derived a theory to analyze convexity and related properties of binary-valued digital images on a fairly large class of nonuniform as well as uniform mosaics. These so-called acute mosaics are characterized by the following properties: every element of the mosaic is a convex polygon or cell, the cells are only overlapping at their boundaries and the union of every pair of adjacent cells (edge neighbors) is a convex polygon. Consequently, acute mosaics do not include hexagonal mosaics.
Therefore, the theory of Sklansky and Kibler cannot be utilized for the hexagonal mosaic; a different approach is necessary.

**Preliminary notions**

By virtue of Theorem 3.4.1 and Theorem 3.4.9 the geometric properties of a cellular complex $C$ and the geometric properties of the associated set of lattice points $\lambda(C)$ are equivalent. Therefore, given a cellular complex $C$ a cellular version of the definition of convexity should be in accordance with that result to confirm the soundness of the theory.

**Supposition 3.5.1**

A cellular complex $C$ is said to be **cellularly convex** if there exists a convex figure $q$, whose covering representation is $C$.

If cellular convexity as defined in Supposition 3.5.1 is proved to be equivalent to one of the geometric properties of a complex $C$, then cellular convexity is by virtue of Theorem 3.4.9 equivalent to all geometric properties defined on $C$ and $\lambda(C)$. The strategy will be to relate the property of having a convex hull not containing points of the complement and the property of having a convex preimage in the covering representation. First we will give a few definitions.

**Definition 3.5.1**

A **transverse** edge of any chain $k$ is the edge shared by some two-element sub-chain of $k$.

**Definition 3.5.2**

The **core** of a complex $C$ is the union of all cells, edges and vertices of the elements of $C$ which do not contain a point of $\delta C$.

The core of a hexagonal complex can have protuberances of one cell-edge length, as shown in Fig. 3.26.

**Convex hull and convexity**

**Lemma 3.5.1**

If a cellular complex $C$ does not have the convex-hull property, then there is no convex plane figure whose covering representation is the complex $C$. 
A cellular complex and its core, shown by the dashed region. Note the protuberances of one cell-edge length.

A cellular complex $C$ with the convex-hull property yet not having a convex preimage.
Proof:
Let $C$ be a cellular complex whose convex hull $CH(C)$ contains a point of the complement of $\lambda(C)$. Then there are two vertices $c_1$ and $c_2$ of $CH(C)$ such that $P(C; c_1, c_2)$ contains the center $c$ of a cell $e$ not in $C$. Let $u$ be the corner point of $e$ farthest from $c_1$ and $c_2$ and inside $P(C; c_1, c_2)$ and let $u_1$ and $u_2$ be the corresponding corner points of the cells $e_1$ and $e_2$ such that $u_1 u_2$ is parallel to $c_1 c_2$. Then the distance from $u$ to $c_1 c_2$ is not less than $1/3 h \sqrt{3}$ directed inside the complex. Suppose that there exists a convex preimage $q$ of the complex $C$; then $q$ contains the straight line segment between two points of $q$, one being inside $e_1$, and the other inside $e_2$. The distance from any point on this straight line segment to $c_1 c_2$ is less than $1/2 h \sqrt{3}$, directed inwards or $1/6 h \sqrt{3}$ directed outwards. This implies that $u$ is contained inside $q$ and therefore that cell $e$ is part of the covering representation of $q$, which is a contradiction. Hence, there is no convex preimage of $C$.

The converse of Lemma 3.5.1 is not true, that is, a cellular complex $C$ may have a convex hull $CH(C)$ which does not contain a point of the complement of $\lambda(C)$ and yet not have a convex preimage. An example is given in Fig. 3.27.

According to the covering representation at least one interior point of each of the cells $e_1$, $e_2$ and $e_3$ of $C$ must be in a preimage $q$. For a convex preimage $q$ with a convex edge between the cells $e_2$ and $e_3$, such a point in $e_3$ should be interior and to the right of the midpoint of the edge of $e_3$ between $e_1$ and the rest of the blob. On the other hand, the point in $e_3$ should be interior and to the left of the midpoint to enable a convex edge between the cells $e_1$ and $e_3$. Hence, there is no convex preimage $q$ such that the complex $C$ is the covering representation of $q$. By virtue of Definition 3.3.4 the complex of Fig. 3.27 is not regular since the cells $e_2$ and $e_3$ are isolated cells.

**Lemma 3.5.2**
If a cellular complex $C$ is regular and has a convex preimage then $C$ has the convex-hull property.

*
Fig. 3.28  An illustration of the proof of Lemma 3.5.2.
Proof:
Let $q$ be a convex preimage of a cellular complex $C$; then $q$ traverses every cell of the cellular boundary. Let the edge shared by two neighboring cells of the cellular chain be denoted as the transverse edge. Since $C$ is regular, $q$ intersects every transverse edge once. Let $p$ be the convex polygon with a minimal perimeter which is contained inside the cellular boundary of $C$ and whose interior contains the core of the complex. Thus, $p$ is a convex preimage of $C$. The vertices of $p$ are end points of the transverse edges of the boundary chain at the interior side of the edge since otherwise $p$ would not have a minimal length. Let $v_i$ be a vertex of $p$. The two centers of the cells sharing the transverse edge at $v_i$ are denoted as $a_i$ and $b_i$. Drop a perpendicular from $a_i$ and $b_i$ to the line through $v_i, v_{i+1}$. Let the length of the longer of these two perpendiculars be denoted by $d_i$ and the corresponding center $a_i$ or $b_i$ by $c_i$. In case $v_i, v_{i+1}$ is parallel to one of the main axes the perpendiculars at $v_i$ have equal lengths and both $a_i$ and $b_i$ are selected. For each $v_i$ obtain the center(s) $c_i$. Now let the polygon whose $k$ vertices are $c_0, c_1, ..., c_{k-1}$ be $r$. We claim that $r$ is the convex hull of $C$ and that $r$ does not contain a point of $\lambda(C)$. (See Fig. 3.28.)

It is obvious that $c_i, c_{i+1}$ is parallel to $v_i, v_{i+1}$ for $0 \leq i \leq k-1$ where the subscripts are modulo $k$, or $c_i, c_{i+1}$ is a part of the spinal path of the boundary chain if two or more points $c_i$ at vertex $v_i$ coincide or the opposite if at $v_i$ two points $c_i$ arise from $v_i$ due to equal distances. Therefore, $L c_{i-1}, c_i, c_{i+1} = L v_{i-1}, v_i, v_{i+1}$ and $r$ is convex because $q$ is.

The edges of $r$ are obtained by a parallel translation of the corresponding edges of $p$ outwards by $d_{\max} = \max_i \{ d_i \} < 1/2 h$. The distance from any (real) point on $q$ to a point of the complement of $\lambda(C)$ is at least $1/2 h$. Hence the translated polygon does not contain a point of the complement of $\lambda(C)$. When an edge of $r$ is formed by a part of the spinal path between the adjacent cells of the boundary chain this edge obviously does not contain a point of the complement of $\lambda(C)$ either. Thus, $r$ is the convex hull of $C$ and does not contain any point of the complement of $\lambda(C)$.

$\square$

Lemma 3.5.3
If the convex hull of a regular cellular complex $C$ is not contained
Fig. 3.29 A convex-hull edge passes through a center of a cell \( e \in Q \).

Fig. 3.30 A cell \( e \in Q \) traversed by a convex-hull edge. (a) Three corners of \( e \) inside \( CH(C) \). (b) One corner of \( e \) inside \( CH(C) \). The length of the portion of the diagonal inside \( CH(C) \) is denoted by \( d_m \). A sextant of the hexagonal cell is indicated in (a).
completely inside $s(C)$ and does not contain a point of the complement of $\lambda(C)$ then the cellular complex may not have a convex preimage.

* 

Proof:
Suppose that $CH(C) \notin s(C)$; then some cell $e$ not in $C$ is traversed by an edge of $CH(C)$, or in other words, $e^o \cap CH^o(C) \neq \emptyset$. Let $Q = \{ e \mid e \in C \land e^o \cap CH^o(C) \neq \emptyset \}$. The edge of $CH(C)$ passing through $e$ in $Q$ is not parallel to one of the axes, $u_1$, $u_2$ or $u_3$. Suppose it has the $u_2$-direction. Since the convex hull edge is the straight line segment between the centers $c_1$ and $c_2$ of two cells of $C$, these cells are in line with $e$. Thus, the edge passes through the center $c$ of $e$ and $CH(C)$ contains $c$, (see Fig. 3.29). This is a contradiction. Similarly the edge of $CH(C)$ passing through $e \in Q$ cannot be parallel to $u_1$ or $u_3$.

Now suppose the edge of $CH(C)$ passing through $e$ has an arbitrary direction and does not pass through the center $c$ of a cell $e \in Q$. We apply the shrinking procedure of Kim to transform the convex hull such that it is completely contained in $s(C)$. Because the centers of the elements $e$ of $Q$ are outside $CH(C)$, there are at most three corners of any $e$ which are inside $CH(C)$. Draw the diagonals from the center $c$ to these corners. These diagonals intersect an edge of $CH(C)$, as can be seen in Fig. 3.30. Consider the portions of these line segments inside $CH(C)$. Drop from each corner of $e$ inside $CH(C)$ a perpendicular to the edge of $CH(C)$ traversing $e$. Let the length of the portion of the diagonal inside $CH(C)$ corresponding to the longest perpendicular be denoted by $d_m$. If there is only one corner of $e$ inside $CH(C)$, $d_m$ is the length of this line segment. Let $d_{max} = \max_e \{ d_m \mid e \in Q \} < 1/3 \sqrt{3}$, the length of a diagonal.

Let $c_i$ be a vertex of $CH(C)$. The two edges of $CH(C)$ incident to $c_i$ do not lie strictly in two adjacent sextants of $e_j$. Let's suppose that they do lie in two adjacent sextants (see Fig. 3.31). This is possible only if $e_j$ is an isolated element since if $e_j$ is not isolated the edges are parallel to the main axes $u_1$, $u_2$ or $u_3$. But $C$ is regular and therefore has no isolated element. So in between two edges of $CH(C)$ lies at least one sextant of $e_j$.

The angle at vertex $c_i$ of $CH(C)$ is less than $180^\circ$ and at least $60^\circ$ since $C$ is regular. Therefore there are one or two sextants of $e_j$ which are included
Fig. 3.31 The two convex-hull edges incident to $c_j$ in adjacent sextants of $e_f$. 
entirely in $CH(C)$. Suppose that there is only one sextant included. Then we denote the corner of $e_i$ on the diagonal in this sextant by $u_i$.

Now suppose there are two sextants of $e_i$ which are included entirely in $CH(C)$. We consider the following two possible cases:

**Case 1**

The two edges of $CH(C)$ incident to $c_i$ do not lie in the same half of the cell $e_i$ enclosing the two sextants, as shown in Fig. 3.32(a). Let the sextants and edges be denoted as indicated. The cell sides crossed by edge 1 and edge 2 are parallel. Select the sextant whose diagonal is parallel to these cell sides. Let the corner of $e_i$ on this diagonal be denoted by $u_i$. Suppose only one-sextant vertices and vertices of case 1 occur; then, obtain for each $e_i$ and its vertex $c_i$, the point $v_i$ which is $d_{max}$ away from $c_i$ on the diagonal $c_i u_i$ (see Fig. 3.33). Now let the polygon whose $k$ vertices are $v_0, v_1, \ldots, v_{k-1}$ be denoted by $p$. We claim that $p$ is convex and that $p$ is a preimage of $C$.

For two reasons, the edge $v_i v_{i+1}$ is parallel to $c_i c_{i+1}$ for $0 \leq i \leq k-1$, where the subscripts are modulo $k$. First, because the edge $c_i c_{i+1}$ enters the cell $e_{i+1}$ by crossing a cell side which is parallel to the crossed cell side of $e_i$ in leaving $e_i$, and second because the translation of $c_i c_{i+1}$ is in the direction of such a cell side. It should be noted, however, that $v_i v_{i+1}$ does not necessarily cross parallel cell sides of the cells $e_i$ and $e_{i+1}$ because $v_i v_{i+1}$ is not an edge between cell centers. Therefore, $L v_{i-1}, v_i, v_{i+1} = L c_{i-1}, c_i, c_{i+1}$ and $p$ is convex because $CH(C)$ is convex.

Next we show that $p$ is a preimage of $C$. As shown above, in neither of the three directions $u_1, u_2$ or $u_3$ does an edge of $CH(C)$ pass through a cell $e$ of $\bar{C}$. The edges of $p$ in the $u_1, u_2$ or $u_3$ directions are obtained by parallel translation of corresponding edges of $CH(C)$ inward by $d_{max} < 1/3 h \sqrt{3}$, where $1/3 h \sqrt{3}$ is the length of a diagonal. All the other edges of $p$ are obtained from corresponding edges of $CH(C)$ by parallel translation in the direction of the diagonal inside the sextant by $d_{max}$. Hence, they do not pass through the interior of any cell in $\bar{C}$. Furthermore, $e^* \cap p^* \neq \emptyset$ for any $e$ in $C$ since its center is in $CH(C)$. Thus, $p$ is a preimage of $C$. 
Fig. 3.32  (a) The two convex-hull edges incident to $c_i$ in different halves of $e_i$. Note that the edges 1 and 2 are both translated in the direction of diagonal 2.
Fig. 3.32  (Continued.) (b) The two convex-hull edges incident to $c_i$ in the same half-cell $e_i$. 
Case 2

The two edges of \(CH(C)\) incident to \(c_i\) do lie in the same half of
the cell \(e_i\) enclosing the two sextants, as shown in Fig. 3.32(b).
The construction used above to shrink the convex hull to a convex
preimage is doomed to failure since the sides of the cell \(e_i\) which are
intersected by a convex-hull edge incident to \(c_i\) are not parallel to each
other. Then, if the convex-hull edges are translated in the direction of
the corresponding cell side, the vertex \(v_i\) of the convex polygon thus
constructed may be outside cell \(e_i\) (see Fig. 3.32(b)). This is possible only
if \(d_{\text{max}}\) is close to its maximal value, the length of a diagonal. Hence,
the constructed polygon \(p\) is not a preimage of \(C\).

This proves the lemma.

\[\square\]

Lemmas 3.5.1 to 3.5.3 combine to yield the following theorem:

**Theorem 3.5.4**
The property of cellular convexity of a cellular complex \(C\) is not equi-
valent to the geometric properties defined on \(C\) and its associated lattice
\(\lambda(C)\) regardless of whether or not the complex is regular.

Here we derived a relationship which will keep us from transforming
Supposition 3.5.1 into a definition of cellular convexity based on a sound
theory.

### 3.6 Concluding remarks

Digital convexity has been discussed for a digital region defined on the
hexagonal lattice and an attempt has been made to define cellular
convexity for a cellular complex defined on the hexagonal mosaic. Both
approaches were inspired by the vast amount of literature that exists on
the problem of convexity, exclusively defined on the rectangular lattice
and mosaic. It turned out that digital convexity of a digital region is
virtually equivalent to the geometric properties, namely, the line, area,
median-point, chord and convex-hull property. In the following diagram
we illustrate the relations proved thusfar between the notion of convexity, geometric properties and the representation scheme used.

Convex preimage $\xrightarrow{\text{Lattice representation;}}$ Digital region
  digital region is $\xleftrightarrow{\text{has the geometric}}$
Convex figure $\xrightarrow{\text{digitally convex}}$
properties

What we need is a similar relation for the hexagonal mosaic. In fact, what is missing is a definition of cellular convexity, such that given a convex figure $q$, its lattice representation, possessing the geometric properties, is digitally convex, and its covering representation, also possessing the geometric properties, is cellularly convex. This would then result in the diagram given below. Only under these conditions is the set of lattice points of the digital region identical to the set of lattice points associated with the cellular complex.

Lattice representation;
digital region is
digitally convex $\xleftrightarrow{\text{has the geometric}}$
Digital region has
properties

Convex preimage

Convex figure

Covering representation;
cellular complex is
cellularly convex $\xleftrightarrow{\text{associated set of lattice}}$
Cellular complex and
points has the geo-
metric properties

Then the concept of discrete convexity is well defined and sound in the sense that many equivalent properties of convex regions in Euclidean geometry carry over to the discrete space.

In the next chapter we will propose a new definition of cellular convexity for the hexagonal mosaic to achieve this goal.
HALF-CELL EXPANSION AND CELLULAR CONVEXITY

4.1 Introduction

In Chapter 3 we derived a theory for determining whether or not a digitized blob represented by a finite set of lattice points (a digital region on a hexagonal lattice) can be the digitization of a convex region (a convex preimage). We failed to derive a similar theory for convexity of the digitized blob represented by a finite set of cells (a cellular complex on a hexagonal mosaic) such that cellular convexity is equivalent to digital convexity. In this chapter we will revise the definition of cellular convexity using the concept of the half-cell expansion of a complex, introduced by Sklansky (1972), to convert any simply connected cellular complex into a regular complex. With this revised definition digital convexity, cellular convexity and the geometric properties will be shown to be equivalent, a result that reinforces the soundness of our approach to convexity analysis for the hexagonal mosaic.

In Section 4.2 we will formalize the notion of the half-cell expansion of a hexagonal cellular complex. Then, a new definition of cellular convexity, compatible with the definition of digital convexity defined on a set of lattice points, will be introduced. In Section 4.3 the theory of minimal polygons will be extended to the hexagonal mosaic in relation to the convexity of a preimage, whereas in Section 4.4 the new approach to cellular convexity will be related to the theory of minimal polygons. Lastly, an already-known algorithm for finding the minimum-perimeter polygon on a rectangular mosaic will be shown to be applicable for the hexagonal case too.
The continuous plane figure of Figs. 3.1 and 3.2 represented by a nontight cellular complex on the hexagonal mosaic.
4.2 Duality between hexagonal and triangular complexes

Introduction

In Section 3.1 we introduced the covering representation of a plane figure essentially as the smallest set of cells covering the plane figure. A more accurate representation can be obtained by inserting the notion of tightness into Definition 3.1.2 of a cellular image.

Definition 4.2.1
Extended covering representation for blobs
A set of cells $C$ is said to be the cellular image of a plane figure $q$, and $q$ a preimage of $C$ if

i) $q \subseteq s(C)$,

ii) for each cell $e$ of $C$, $e^\circ \cap q^\circ \neq \emptyset$, where $e^\circ$ and $q^\circ$ are the interiors of $e$ and $q$, and

iii) each edge of an element $e$ of $C$ is part of $\delta C$ if $r^* \cap q \neq \emptyset$, where $r^*$ is the edge $r$ excluding its end points.

Fig. 4.1 shows the planar figure of Fig. 3.2 now digitized on the hexagonal mosaic by the covering representation of Definition 4.2.1. Some of the transverse edges in the cellular boundary are now in fact part of the boundary of the complex due to condition iii).

Definition 4.2.2
The inner boundary of a complex is the set of transverse edges of the elements of the complex originated by condition iii) of Definition 4.2.1.

Definition 4.2.3
A cellular complex is said to be tight if it has an empty inner boundary, otherwise the complex is nontight.

In our previous attempt to establish a theory for cellular convexity we restricted ourselves to tight complexes. Here we will explore the notion of
Fig. 4.2  A cellular complex $C$ on the hexagonal mosaic and the half-cell expansion $H(C)$ on the dual triangular mosaic. In $H(C)$ the cellular boundary is depicted by the solid triangular cells.
tightness in order to derive a relation between the geometric properties and the property of having a convex preimage in the covering representation.

**Half-cell expansion of complexes**

In Chapter 2 it has been shown that the hexagonal mosaic and the triangular mosaic are very much related. The latter mosaic is even constructed from the first since it is formed by the hexagonal coordinate grid. Let $H$ denote the hexagonal cellular mosaic. Construct the triangular lattice $T$ by connecting the lattice points of the hexagonal lattice (i.e. the centers of the cells of $H$) by straight line segments. Let $C$ be a cellular complex on $H$ and $\delta C$ its boundary. Then we define:

**Definition 4.2.4**

The half-cell expansion of $C$, denoted by $H(C)$, is the triangular complex on $T$ such that the spinal path of its boundary chain is $\delta C$, the boundary of $C$ on $H$.

In Fig. 4.2 the complex $C$ is shown with its half-cell expansion $H(C)$ and the cellular boundary of $H(C)$. The half-cell expansion $H(C)$ is a finite set of triangular cells defined on the triangular cellular mosaic $T$. If $C$ is a simply connected complex on $H$, then $H(C)$ is a simply connected half-cell expansion on $T$.

If the lattice constant of the hexagonal mosaic is denoted by $h$, as has been proposed in Chapter 2, then by construction the lattice constant of the triangular mosaic is also equal to $h$. This distance $h$ corresponds to the distance between two equally oriented triangular cells, e.g. to adjacent cells in the same row. In Fig. 3.5 the cells at a distance $h$ from the central pixel are indicated by the number 2.

**Regularity of complexes**

In Section 3.3 we briefly brought up the notion of regularity. Here we give a more detailed description of regularity. First a definition of a simple closed planar curve.
Definition 4.2.5 (Valentine, 1964)
A simple closed planar curve is a homeomorphic mapping $f$ (i.e. a 1-1 mapping) from $I \rightarrow \mathbb{R}^2$, where $I = [0,1]$ denotes the closed unit interval and where $f(0) = f(1)$.

Definition 4.2.6 (Montanari, 1970b)
A complex $C$ is regular if for every cell in it, the preceding and following cells are different from each other.

This definition is attributed to Montanari (1970b). In other words, a complex $C$ is regular if every pair of elements of $C$ lies on a regular chain (Definition 3.3.8) belonging entirely to $C$. Hence, the spinal path of any boundary chain of $C$ is a simple closed curve. Thus, every boundary chain of $C$ is regular and therefore, the spinal path of each of these boundary chains is a simple closed curve.

A second definition of regularity is attributed to Sklansky et al. (1972). First a few preliminary definitions:

Definition 4.2.7 (Sklansky et al., 1972)
A cellular complex $C$ is chained if every pair of elements of $C$ lies on a chain belonging entirely to $C$.

Definition 4.2.8 (Sklansky et al., 1972)
A cellular complex $C$ is said to be simply chained if it is chained and $\delta C$ is a simple closed curve.

Then we have the following definition of regularity:

Definition 4.2.9 (Sklansky et al., 1972)
A cellular complex $C$ is said to be regular if it is simply chained, and if there exists a boundary chain of which the predecessor and successor of every element of the chain are distinct.

In a later paper, Sklansky and Kibler (1976) mentioned that Definition
4.2.9 is equivalent to the formulation of Montanari, which is, in fact, not the case. Moreover, Sklansky and Kibler defined the notion of a normal complex: A complex $C$ is normal if $C$ is simply connected and $C$ has no end cells, i.e. cells sharing only a single edge with $C$. Clearly, every Sklansky-normal complex is a Montanari-regular complex, and every Sklansky-regular complex is both Sklansky-normal and Montanari-regular.

According to Definition 4.2.9, a complex $C$ is regular if every pair of elements of $C$ lies on a (not necessarily regular) chain belonging entirely to $C$, though the boundary chain should be a regular chain. Moreover, $\delta C$ should be a simple closed curve, a condition not included in the definition of Montanari.

When only tight complexes are considered, both definitions agree. In the general case, however, Sklansky’s definition is too stringent, a conclusion implicitly adopted by Sklansky in his work on the half-cell expansion, as will be shown by the following example in Fig. 4.3 of a cellular complex $C$ and its half-cell expansion $H(C)$. The complex $C$ in Fig. 4.3 is simply connected and has a simple closed boundary $\delta C$. However, there is no regular chain connecting any pair of cells due to the hexagonal cells depicted by an $e$. Since the simple curve $\delta C$ is the spinal path of a boundary chain of $H(C)$, $H(C)$ has a regular boundary chain, although this chain brushes past itself. Furthermore, since $C$ is simply connected $H(C)$ is regular by the definition of Montanari. Because $\delta H$ is not a simple closed curve $H(C)$ is not simply chained and therefore $H(C)$ is not regular by the definition of Sklansky. Nevertheless, Sklansky (1972) states that similar nontight half-cell expansions on the square mosaic are regular by Definition 4.2.9, thereby violating his own definition of regularity. More specifically, Sklansky stated that on a rectangular mosaic, the half-cell expansion of any simply 8-connected complex is Sklansky-regular. Henceforth, we will only refer to Montanari’s definition of regularity.

**Theorem 4.2.1**
The half-cell expansion $H(C)$ of a simply connected tight cellular complex $C$ is regular, whether or not $C$ is regular.
Fig. 4.3 An example of a connected nonregular cellular complex $C$, its half-cell expansion $H(C)$ with a regular boundary chain and a non-simple closed boundary $\delta H$.

Fig. 4.4 Illustration of the proof of Lemma 4.2.2.
Proof:
Consider a simply connected tight complex \( C \), regular or not; then \( \delta C \) is a simple closed curve. Since \( \delta C \) is the spinal path of the boundary chain of \( H(C) \), this chain is regular. Thus, \( H(C) \) has no peninsulated elements, which proves the theorem.

In the dual hexagonal and triangular mosaics every regular half-cell expansion is even simply cored, that is \( H(C) \) has a non-empty core and there exists a "nonrepeating" (closed) boundary chain: Nonrepeating means that all elements of the chain are distinct (Sklansky et al., 1972) a property not present in square mosaics. In the case of irregular tight 8-connected complexes with a one-cell-thick arm on square lattices, the boundary \( \delta C \) is only closed, not simply closed, and has intersections at some isolated points. Hence, the half-cell expansion of such a complex will not be simply cored, for its boundary chain will be repeating.

Cellular convexity

We are now ready to state a revised definition of cellular convexity.

Definition 4.2.10
A cellular complex \( C \) is said to be cellularly convex if there exists a convex plane figure \( q \), such that the half-cell expansion of \( C \) is the covering representation of \( q \).

Note that \( C \) is a complex on the hexagonal mosaic and \( H(C) \) the covering representation of \( q \) on the triangular mosaic. The main result of this section is that if a cellular complex \( C \) is cellularly convex by a convex preimage \( q \) of the half-cell expansion \( H(C) \) under the covering representation, then the set of lattice points \( \lambda(C) \) is digitally convex by the same preimage \( q \) under the lattice representation, where \( q \) is then a convex preimage of \( \lambda(C) \).

The line property of a set of lattice points given in Definition 3.3.1 carries over to the half-cell expansion \( H(C) \) with the restriction that only triplets of center points which belong to cells of \( H(C) \) of equal orientation are to
Fig. 4.4 (Continued.) Illustration of the proof of Lemma 4.2.2.
be considered.

**Definition 4.2.11**

A triangular complex $H$ is said to have the line property if there is no triplet $(t_1, t, t_2)$ of equally oriented cells whose centers are collinear and such that $t_1$ and $t_2$ are elements of $H$ and $t$ is an element of $\overline{H}$.

**Lemma 4.2.2**

The associated set of lattice points $\lambda(C)$ of a cellular complex $C$ does not have the line property if and only if the half-cell expansion $H(C)$ is either nontight or does not have the line property.

**Proof:**

Suppose $\lambda(C)$ does not have the line property; then there exists a triplet $(c_1, c, c_2)$ of collinear lattice points such that $c_1, c_2 \in \lambda(C)$ and $c \notin \lambda(C)$. Let $e_1$, $e_2$ and $e$ be the elements of which $c_1$, $c_2$ and $c$ are the centers. Suppose further that a pair of opposite edges of $e$ belongs to $\delta C$ and that only one edge of $e$ does not belong to $\delta C$. Then the line segment which passes through $c$ perpendicular to that single edge is part of the inner boundary of $H(C)$. Hence $H(C)$ is nontight (see Fig. 4.4(a)).

If two adjacent edges of $e$ do not belong to $\delta C$, the configuration of Fig. 4.4(b) may occur. The elements $e_1$ and $e_2$ originate the parts $uv$, $vc$ and $cw$, $wx$ of $\delta H$, respectively. Note that the triangular cell $t_1$ bounded by $uv$, $vc_1$ and $c_1u$ is a cell of $H(C)$ because $H(C)$ is regular by virtue of Theorem 4.2.1. The same argument applies to the cell $t_2$ bounded by $wx$, $xc_2$ and $c_2w$. Since the triangular cells $t$, $t_1$ and $t_2$ have the same orientation in the triangular mosaic their centers are collinear, while $t_1$ and $t_2$ are elements of $H(C)$ and $t$ is an element of $H(C)$. Hence, $H(C)$ does not have the line property.

Any other configuration of two or more edges of $e$ that do not belong to $\delta C$ is covered by the cases considered above.

Next, suppose that neither pair of opposite edges of $e$ belongs to $\delta C$. Then there are possibly three non-adjacent edges of $e$ which do not belong to
Fig. 4.5 Illustration of the proof of Lemma 4.2.2.
\( \delta C \), a configuration in which the half-cell expansion is trivially either non-tight or does not have the line property. The other possible configuration of edges of the element \( e \) has three adjacent edges that do not belong to \( \delta C \). Without loss of generality, assume that these edges of \( e \) are the two top edges and the edge on the righthand side of the center \( c \), as shown in Fig. 4.4(c). Let the triangular cells generated by the expansion of the cells \( e_1, e \) and \( e_2 \) in the direction of the top of these hexagonal cells be denoted as \( t_1, t \) and \( t_2 \). Then the set of centers of these triangular cells is collinear, since the triplet \( (c_1, c, c_2) \) is collinear. The cells \( e_1 \) and \( e_2 \) are elements of \( C \) and \( e \) is an element of \( \overline{C} \). Therefore, \( t_1 \) and \( t_2 \) are cells of \( H(C) \) and \( t \) is not. Hence, \( H(C) \) does not have the line property.

Next, we suppose that \( H(C) \) is either nontight or does not have the line property. First we consider the case in which \( H(C) \) is nontight. There exists a part of the inner boundary of \( H(C) \) that starts from the outer boundary of \( H(C) \) at point \( c \), the center of the triplet \( (c_1, c, c_2) \), and has at least unit length. In Fig. 4.5 the line segment \( cc' \) represents this unit-length portion of the inner boundary. The two possible cases are considered:

1) The two unit-length parts of the outer boundary of \( H(C) \) at point \( c \) have an enclosing angle of \( \alpha = 60^\circ \) (see Fig. 4.5(a)).

2) Their enclosing angle \( \alpha \) equals \( 120^\circ \) (see Fig. 4.5(b)).

The angle \( \alpha = 180^\circ \) is not considered since this leads to a concavity and not to a nontightness, as will be shown in the last part of the proof.

**Case 1**

The parts of the outer boundary are formed by the half-cell expansions of the edges of two elements \( e_1 \) and \( e_2 \) of \( C \) separated by the cell \( e \) of \( \overline{C} \). Then, since the triplet \( (c_1, c, c_2) \) is the desired triplet, \( \lambda(C) \) does not have the line property.

**Case 2**

Let \( e_1 \) be the cell of \( C \) next to the cell \( e \). The ray from \( c_1 \) passing through \( c' \), the point on the inner boundary on unit distance from \( c \), passes through the center of each cell in that direction. At least one of these cells, say \( e_2 \), is an element of \( C \), otherwise \( C \) would not be connected. Hence, \( (c_1, c, c_2) \) is the desired triplet and therefore \( \lambda(C) \) does not have the line property.
Fig. 4.6 Illustration of the proof of Lemma 4.2.2

Fig. 4.7 A nontight triangular complex $H$. 
Second, let’s consider the case where $H(C)$ does not have the line property. Suppose there is a triplet of equally oriented triangular cells $(t_1, t, t_2)$, whose centers are collinear, and where $t_1$ and $t_2$ are elements of $H(C)$ and $t$ is not. Let $e$ be a hexagonal cell not in $C$ such that the center of $t$ is a corner point of $e$ and such that the center $c$ of $e$ is the farthest away from the ray through $t$ defining the triplet. The lattice points $c_1$ and $c_2$ are the centers of the cells $e_1$ and $e_2$ in $C$ whose centers of $t_1$ and $t_2$ are corner points. The elements $e_1$ and $e_2$ are chosen in the direction similar to $e$. These cells exist because $H(C)$ is regular. (See Fig. 4.6.) Then the triplet $(c_1, c, c_2)$ is collinear and $c_1, c_2 \in \lambda(C)$ and $c$ is not. Hence, $\lambda(C)$ does not have the line property.

Lemma 4.2.3

If a triangular complex $H$ has a convex preimage, it is tight.

Proof:

Suppose $H$ is nontight. Then it has a nonempty inner boundary. Let $t_1$ and $t_2$ be two elements of $H$ sharing an edge which is a part of the inner boundary of $H$, as is shown in Fig. 4.7. Let $q$ be any preimage of $H$. Since $t_1$ and $t_2$ are elements of $H$ there are interior points $x$ and $y$ of $t_1$ and $t_2$, respectively, that are points of $q$. Since the points of the edge shared by $t_1$ and $t_2$, except possibly the end points, are not in $q$, $xy$ contains a point not in $q$. Hence, $q$ is not convex, and so $H$ does not have any convex preimage.

Note that there is no equivalent lemma for the hexagonal mosaic. A nontight hexagonal complex may have a convex preimage, as shown in Fig. 4.8, a property which will be exploited in Chapter 5.

Lemma 4.2.4

A regular tight triangular complex $H$ has the line property if and only if it has a convex preimage whose covering representation is $H$.

Sketch of the proof: we first consider the case that the complex $H$ does not have the line property; then we prove that there is no convex preimage of $H$. Second, we consider the opposite case; then we construct, by considering in a shrinking procedure all possible orientations of the edges of the convex hull of $H$, a convex polygon, which will be proved to be a convex preimage of $H$. 
Fig. 4.8 A nontight cellular complex $C$ on the hexagonal mosaic yet a convex preimage $q$. 
Proof:
Suppose that $H$ is a complex that does not have the line property. Then there exists a triplet of collinear triangular lattice points whose corresponding triangular cells $(t_1, t, t_2)$ have the same orientation, while $t_1, t_2 \in H$ and $t \in \overline{H}$. Let $u$ be the corner point of $t$ furthest from the real straight line segment $l$ between the centers of the cells $t_1$ and $t_2$ in the direction of the complex. Let $u_1$ and $u_2$ be the corresponding corner points of the cells $t_1$ and $t_2$, such that $u_1 u_2$ is parallel to $l$. The distance from $u_1 u_2$ to $l$ is at least $1/6h\sqrt{3}$ or $1/3h\sqrt{3}$ depending on the orientation of the cells, where the triangular lattice constant is set equal to the hexagonal lattice constant by construction of the half-cell expansion. Suppose that there exists a convex preimage $q$ of the complex $H$, then $q$ contains a straight line segment between a point of $t_1$ and a point of $t_2$. The distance from any point on this straight line segment to $l$ is less than $1/6h\sqrt{3}$ or $1/3h\sqrt{3}$. This implies that $u$ is contained inside $q$ and thereby the cell $t$ is part of the covering representation of $q$, which is a contradiction. Hence, there does not exist a convex preimage.

Now, suppose that $H$ is a complex that does have the line property. Let $CH(H)$ be the (Euclidean) convex hull over the set of triangular lattice points belonging to the cells of $H$, given by the set of vertices $v_0, v_1, ..., v_{k-1}$. Suppose that $CH(H)$ is contained in $s(H)$; then $CH(H)$ is completely inside $\delta H$ and obviously every lattice point of $H$ is inside $CH(H)$. Hence, $CH(H)$ is a convex preimage of $H$. Now suppose that $CH(H)$ passes through a cell $t$ which is an element of $\overline{H}$. Let $Q$ denote the set of all such elements which are traversed by an edge of $CH(H)$ and which belong to $\overline{H}$. The edge of $CH(H)$ passing through $t$ in $Q$ is neither horizontal nor has one of the $\pm 60^\circ$, or $\pm 120^\circ$ directions relative to the horizontal. Suppose it is horizontal. Then two possible orientations of the cells $t_1$ and $t_2$ exist, whose center points define the horizontal edge $l$.

First, $t_1$ and $t_2$ are top-up oriented, as shown in Fig. 4.9(a). Then the edge $l$ passes through the center of any element in the row between $t_1$ and $t_2$.

Since $H$ has the line property, these points on $l$ are the center points of cells of $H$. This is not possible, because $H$ is regular. Second, suppose $t_1$ and $t_2$ are top-down oriented. Then $l$ passes through the center of $t_1$ which violates the line property. (See Fig. 4.9(b).) Similarly, the edge $l$ through $t$ cannot have one of the other directions. Note that edges of $CH(H)$ having $\pm 30^\circ$, $\pm 90^\circ$ or $\pm 150^\circ$ directions relative to the horizontal, are contained inside $s(H)$. Because these edges only pass through centers of cells, and because $H$ has the line property, these edges do not pass through any cell in $Q$. 
Fig. 4.9 Illustration of the proof of Lemma 4.2.4.
Fig. 4.10 Illustration of the proof of Lemma 4.2.4.

Fig. 4.11 Illustration of the proof of Lemma 4.2.4.
Fig. 4.12 Illustration of the proof of Lemma 4.2.4.

Fig. 4.13 The six sectors of the hexagon formed two interlaced triangular cells.

Fig. 4.14 Illustration of the proof of Lemma 4.2.4.
Now suppose the edge of $CH(H)$ passing through a cell $t$ in $Q$ has an arbitrary direction. Because the center of $t$ is outside of $CH(H)$, there are at most two corners of any $t$ in $Q$ which are inside $CH(H)$. If two corners are inside $CH(H)$, the cell $t$ has two 3-neighbor cells $t_1$ and $t_2$ in $Q$ on either side. Each of these cells has only one corner inside $CH(H)$, as shown in Fig. 4.10.

We next draw the line segments from the centers of $t_1$ and $t_2$ to their corner inside $CH(H)$. These line segments intersect an edge of $CH(H)$. Let’s then consider the portions of these line segments inside $CH(H)$. Let $l_t$ denote the length of the longer of the two. If only one corner of $t$ is inside $CH(H)$, $l_t$ is the length of this line segment. Since $H$ is regular, however, these one-corner cells will not appear isolated. Suppose there is an isolated cell $t$ in $Q$ surrounded by two 3-connected neighbors $t_1$ and $t_2$ in $H$. Then by virtue of the regularity of $H$ a few more cells, e.g. $t_3$ and $t_4$, are required, as can be seen in Fig. 4.11. Obviously the triplet of the centers of the cells $t_3$, $t$ and $t_4$ violates the line property. Hence a regular complex $H$ which has the line property has no isolated cells in $Q$.

In Fig. 4.12 an example is given of a pair of “single-corner” cells $t_1$ and $t_2$ in $Q$ through which $CH(H)$ traverses. As will be proved the two cells $t_i$ and $t_{i+1}$, of which the two vertices $v_i$ and $v_{i+1}$ are the centers, have equal orientation, and therefore only the length $l_t$ of the cell of the pair $t_1$, $t_2$, which has the same orientation as $t_i$ and $t_{i+1}$, will be considered in the subsequent analysis. The length of the longest of such line segments of the cells in $Q$ is denoted as $l_{max}$.

Let the hexagon formed by two interlaced triangular cells be divided into six sectors numbered counterclockwise from 1 to 6, as shown in Fig. 4.13. A sector pair is any combination of two adjacent sectors. Top-up cells have three sector pairs, namely $(2,3)$, $(4,5)$ and $(6,1)$, whereas the top-down cells have the $(1,2)$, $(3,4)$ and $(5,6)$ sector pairs.

Let $v_i$ be a vertex of $CH(H)$. The two edges of $CH(H)$ incident on $v_i$ do not lie strictly in the same sector pair of the cell $t_i$ of which $v_i$ is the center. Suppose that they do lie in the same sector pair (see Fig. 4.14). This is possible only if $t_i$ is an isolated element of $H$, since otherwise one of the edges should be in the direction of a sector-pair boundary. But $H$ is
Fig. 4.15 Illustration of the proof of Lemma 4.2.4.

Fig. 4.16 Illustration of the proof of Lemma 4.2.4.
regular, and so has no isolated element. Then since \( t_1 \) is not an isolated cell it must have two 3-connected neighbors in \( H \). Consider the cell \( t_i \) of Fig. 4.15. Suppose that the cell on the righthand side and the one on the bottom are elements of \( H \). Then it is obvious that the \( CH(H) \) should enclose the centers of these cells and therefore \( CH(H) \) must have an edge in the sector pair \((2,3)\) and in the sector pair \((4,5)\) of cell \( t_j \). Hence, there is always exactly one sector pair of \( t_j \) which is included entirely in \( CH(H) \). The angle at \( v_i \) is less than \( 180^\circ \) and by the argument above at least \( 120^\circ \), where \( 120^\circ \) corresponds to the case with adjacent edges strictly on the boundaries of a sector pair.

Let \( v_i \) and \( v_{i+1} \) be two adjacent vertices of \( CH(H) \) which are not neighbors to each other. Then \( t_i \) and \( t_{i+1} \) are equally oriented. Suppose they are not. Without loss of generality assume that \( v_i \) is located above and to the right of \( v_{i+1} \) and that \( t_i \) is top-up oriented and \( t_{i+1} \) top-down. Then the edge \( v_i v_{i+1} \) may leave \( t_i \) in sector 5, as shown in Fig. 4.16. The two incident edges of \( CH(H) \) at vertex \( v_i \) should enclose a sector pair in the interior angle; so the edge \( v_{i-1} v_i \) has to leave \( t_i \) in the opposite sector, that is in sector 2. Thus at vertex \( v_i \) the sector pair \((1,6)\) is enclosed by the edges of \( v_i \). The edge \( v_i v_{i+1} \) enters the cell \( t_{i+1} \) in sector 2. Again the incident edges at vertex \( v_{i+1} \) should enclose a sector pair, this time the sector pair \((3,4)\), and therefore the edge \( v_{i+1} v_{i+2} \) is in sector 5. Enclosing opposite sector pairs by adjacent vertices is possible only if \( v_{i-1} v_i \) and \( v_{i+1} v_{i+2} \) are part of a concave polygon, which is a contradiction. Hence, \( v_i \) and \( v_{i+1} \) are equally oriented.

Let the corner of the cell \( t_i \) included in \( CH(H) \) at each vertex \( v_i \) be denoted as \( u_i \). Note that \( u_i \) is in the sector pair of \( v_i \) which is enclosed by the incident edges at \( v_i \). For each cell \( t_i \) of the vertices \( v_i \) of \( CH(H) \) obtain the point \( p_i \) which is \( l_{max} \) away from \( v_i \) on the line segment \( v_i u_i \). Now let the polygon whose \( k \) vertices are \( p_0, p_1, ..., p_{k-1} \) be \( P(H) \). Then \( P(H) \) is convex and a preimage of \( H \) according to the covering representation.

It is obvious that \( p_i p_{i+1} \) is parallel to \( v_i v_{i+1} \) for \( 0 \leq i \leq k-1 \), where the subscripts are modulo \( k \), since the cells \( t_i \) and \( t_{i+1} \) are pairwise equally oriented and therefore \( v_i u_i \) is parallel to \( v_{i+1} u_{i+1} \). When \( t_i \) and \( t_{i+1} \) are 3-connected neighbors they have a common corner point \( u_i \) and therefore \( p_i p_{i+1} \) is parallel to \( v_i v_{i+1} \). By these arguments, \( \angle p_{i-1} p_i p_{i+1} = \)
$L v_{i-1} v_i v_{i+1}$, and $P(H)$ is convex because $CH(H)$ is.

Now we show that by construction $P(H)$ is a preimage of $H$. As shown above, none of the edges of $CH(H)$ that have a direction which is a multiple of $30^\circ$ passes through a cell of $H$. The corresponding edges of $P(H)$ are obtained from the edges of $CH(H)$ by parallel translation in the direction of the corners $u_i$ of the cells $t_i$ inside $CH(H)$ over a distance $l_{max}$, which is strictly less than the length of the diagonal. Therefore, these edges of $P(H)$ remain passing through the respective elements of $H$. All the other edges of $P(H)$ are obtained from corresponding edges of $CH(H)$ by parallel translation in the direction of the corners $u_i$ over a distance $l_{max}$. Hence, these edges of $P(H)$ do not pass through the elements of $\bar{Q}$ nor through any other element of $H$. So, $P(H)$ is contained in $s(H)$. Furthermore, each cell $t$ of $H$ is intersected by an edge of $P(H)$ or is completely inside $P(H)$ since the center of cell $t$ is in $CH(H)$. That is, $t^o \cap P^o(H) \neq \emptyset$ for any $t$ in $H$. Thus, $P(H)$ is a preimage of $H$. This proves the lemma.

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**Theorem 4.2.5**
A cellular complex $C$ is cellularly convex if and only if the set of lattice points $\lambda(C)$ is digitally convex.

**Proof:**
Suppose $C$ is cellularly convex. Then by definition the half-cell expansion $H(C)$ of $C$ has a convex preimage according to the covering representation. By reason of Lemma 4.2.3 $H(C)$ is tight. Because of Theorem 4.2.1 $H(C)$ is regular. So $H(C)$ has the line property because of Lemma 4.2.4. Hence, by virtue of Lemma 4.2.2 $\lambda(C)$ has the line property. Therefore, $\lambda(C)$ is digitally convex by reason of Theorem 3.4.9.

Now suppose that $\lambda(C)$ is digitally convex. Then $\lambda(C)$ has the line property and area property because of Theorem 3.4.9. Hence, based on Lemma 4.2.2, $H(C)$ is tight and has the line property. $H(C)$ is regular because of Theorem 4.2.1. Therefore by virtue of Lemma 4.2.4 $H(C)$ has a convex preimage and thus $C$ is cellularly convex.
Discussion

The half-cell expansion of Sklansky which has been introduced to convert any simply connected cellular complex into a regular complex has been used here to obtain an equivalence between the convexity of the complex and that of its set of centers. Convexity is now defined in terms of a preimage of the half-cell expansion of the complex and therefore a convex preimage whose half-cell expansion is the covering representation, has to be sought. By construction this particular preimage is also a convex preimage of the set of lattice points associated with the originating complex according to the lattice representation. Thus, if for a given cellular complex there exists a convex preimage $q$ of its half-cell expansion, then the complex is cellurally convex and its associated set of lattice points is digitally convex by the very same preimage $q$. The problem that remains to be solved is the search for such a convex preimage. Merely searching for such a preimage is not a practical test for convexity, however, because even after an indefinitely long unsuccessful search such a preimage may still exist. What we need is an algorithm for constructing a figure $q$ such that $H(C)$ or $\lambda(C)$ is the representation, and such that if $q$ is concave, every other figure whose image is $H(C)$ or $\lambda(C)$ will necessarily be concave, too. We will show in Section 4.4 that the minimum-perimeter polygon answers this need for the triangular mosaic. First, in Section 4.3 we will extend the theory of limit properties to the hexagonal mosaic in order to obtain an instrument for the detection of a convex preimage of the cellular complex as it is defined on the hexagonal mosaic.

4.3 Minimal polygons on hexagonal mosaics

Introduction

The theory of minimal polygons is closely related to the theory of discrete convexity. Earlier papers have shown that the minimum-perimeter polygon (MPP) of a cellular complex can be an instrument in automating the recognition of convexity (Sklansky, 1970; Montanari, 1970b). In order to guarantee the existence and uniqueness of the MPP, although uniqueness does not seem essential to its utility, the cellular complex must satisfy certain constraints of connectedness, and the boundary of the
Fig. 4.17 The hexagonal mosaic composed of half-open hexagonal cells.
complex must satisfy certain constraints of smoothness. Furthermore, to be able to define a minimum at all, it is necessary that the individual cells of the mosaic be closed. The closed-cell restriction will be made explicit along the lines of the theory of limit properties by Montanari (1970b).

The definition of the MPP given in Definition 3.3.5 is attributed to Sklansky (1970). In Sklansky and Kibler (1976) it has been shown that there exists an MPP of any complex on an arbitrary closed-cell mosaic, but that the MPP is unique only for regular complexes on acute mosaics. These acute mosaics are characterized by the property that the union of every pair of adjacent cells (edge neighbors) is a convex polygon. Consequently, acute mosaics do not include hexagonal mosaics. Furthermore, in Sklansky and Kibler (1976) a cellular complex on an acute mosaic was proved to have a convex preimage if and only if the MPP of the complex is convex. Since the hexagonal mosaic is not included in the class of acute mosaics, the uniqueness of the MPP and its relevance for the detection of a convex preimage of a cellular complex on a hexagonal mosaic will be investigated in detail.

Closed-cell mosaics

In Section 3.1 the covering representation of continuous figures was defined by the covering of the figure with open cells centered on the lattice points, which implies that the cells are non-overlapping. Here it seems to be necessary to require the cells to be closed, and the common edge of two neighboring cells to be regarded as belonging to both of them, which implies that some representations are ambiguous. However, this ambiguity will only be met in case the boundary of a figure coincides with a cell edge. If, on the other hand, the cell edges are uniquely assigned to only one cell, i.e. with half-open cells, one could have a sequence of figures passing through the neighboring cells of a particular cell \( e \), closer and closer to the edges of \( e \), where the representation of the limit curve of the sequence would not be the same as the representation of the figures in the sequence. Fig. 4.17 gives an example of a mosaic with half-open hexagonal cells.

Rosenfeld and Kim (1982) have treated other advantages and disadvantages of the various definitions of open, half-open and closed-cell
digitization schemes. However, they restricted themselves to the rectangular lattice and mosaic.

Representations which do not go to the limit are unacceptable in cases where a minimal figure like the MPP, having a given representation, has to be defined, since achieving the minimum perimeter involves a passage to the limit. In fact, the MPP always has an ambiguous representation. Thus, if ambiguous representations were not allowed, the minimum would not exist.

**Minimum-perimeter polygons**

The minimum-perimeter polygon was originally introduced by Sklansky (1970) for use on the rectangular mosaic (see Definition 3.3.5). In two theorems, Sklansky stated and proved the uniqueness of the MPP of a cellular complex on a rectangular complex:

**Theorem 4.3.1** (Theorem 1 in Sklansky (1970))

If every minimum-perimeter polygon of a cellular complex $C$ contains the core of $C$, then $C$ has precisely one minimum-perimeter polygon. *

**Theorem 4.3.2** (Theorem 4 in Sklansky (1970))

On a rectangular mosaic, every minimum-perimeter polygon of a cellular complex contains the core of the complex. *

It follows from Theorems 4.3.1 and 4.3.2 that a cellular complex on a rectangular mosaic has a unique MPP. In a subsequent paper Sklansky and Kibler (1976) treated the more general case of acute mosaics.

**Definition 4.3.1**

A mosaic is said to be acute if

i) every element of the mosaic is a convex polygon or cell;

ii) every finite region of the plane contains a finite numbers of cells;

iii) the cells are overlapping only at their boundaries;

iv) the union of every pair of adjacent cells forms a set bounded by a convex polygon;

v) the union of all the cells covers the entire plane. *
As a result Sklansky proved the following theorem:

**Theorem 4.3.3 (Theorem 3.10 in Sklansky and Kibler (1976))**

If $C$ is a regular complex on an acute mosaic, then $C$ has exactly one minimum-perimeter polygon.

In proving this result Sklansky had to restrict himself to acute mosaics because he needed the property that the straight line segment between any point of a cell to any point on an adjacent edge-neighbor cell is completely included in both cells, which is not the case in non-acute mosaics. As a result of the definition of a cellular complex and the arguments used in proving Theorem 4.3.3 the non-uniqueness of Sklansky's MPP shows up for those cellular complexes defined on non-acute mosaics, where the MPP cuts off corners of the core of the complex or even passes through several adjacent cells of the most exterior core peel. Fig. 4.18 gives an example of an MPP on the hexagonal mosaic. The MPP shown cuts off corners and segments of the core in several places, so that Theorems 4.3.1 and 4.3.2 are not applicable for the hexagonal mosaic.

On the other hand, if we insert the concept of tightness into the closed-cell covering representation, the MPP can be shown to be unique for the hexagonal mosaic as well. The tightness extension has been introduced in Definition 4.2.1. Then, as a consequence of the extended covering representation, the polygon shown in Fig. 4.18 is no longer a preimage of the complex but a preimage of the nontight complex shown in Fig. 4.19. And therefore the polygon cannot be the MPP of the complex given in Fig. 4.18. If the definition of a cellular image is applied without the additional tightness condition, that is Definition 3.1.3, then the polygon shown in Fig. 4.18 is indeed a preimage and an MPP of the complex shown in Fig. 4.18.

It should be noted that given a set of cells it is important to know with which definition of the covering representation it is obtained. An empty inner boundary of a complex obtained by the extended covering representation describes a different set of preimages than the intrinsic empty inner boundary of a complex obtained by standard covering representation. In other words, a tight complex obtained by the extended definition is not
Fig. 4.18  A regular cellular complex on the non-acute hexagonal mosaic and its minimum-perimeter polygon.

Fig. 4.19  The MPP of Fig. 4.18 as a preimage of a nonregular and non-tight cellular complex having the same outer boundary as the complex of Fig. 4.18.
equivalent to a complex obtained by the standard definition, although the complexes are composed of the same set of cells and defined by the same preimage.

In the following we will prove the existence and uniqueness of the MPP, given a cellular complex with an empty inner boundary obtained by the extended covering representation.

Theory of limit properties

The idea behind the existence and uniqueness proof of the MPP of a regular, tight cellular complex on an arbitrary mosaic is to regard perimeter as a function from a suitable space of polygons into the real numbers. If the space is compact and if perimeter is a continuous function, then since every continuous function on a compact set attains its minimum, the polygon at which perimeter is a minimum is the MPP. Unfortunately, the space of preimages is not compact, since a curve is the limit of its polygonal approximations, so the space of preimages is not closed. Moreover, perimeter is not continuous because for any straight line there is a very jagged polygonal line with an arbitrary long length (Mandelbrot, 1982). In order to avoid the defects of the space of preimages, we define a subclass of equivalent figures which is compact and on which perimeter is continuous.

First we define the cellular image of a planar curve.

Definition 4.3.2
Covering representation for curves
A set of cells $C$ is the closed-cell cellular image of a planar curve $f$, and $f$ a preimage of $C$ if
i) $f \subseteq s(C)$, and
ii) for every element $e$ of $C$, $e \cap f \neq \emptyset$.

Then, the cellular image of a figure $q$ bounded by a closed planar curve $f$ is the union of the cells inside its boundary together with the set of cells representing the boundary. Now we define several properties of planar curves. When reference is made to the corresponding properties of the boundary, these properties also hold for the corresponding figures.
Fig. 4.20  Two weakly equivalent curves $f'$ and $f''$. 
Definition 4.3.3 (Montanari, 1970b)
Given a cellular complex, two curves \( f' \) and \( f'' \) are said to be strongly equivalent if they have the same representation sequence without ambiguity, and weakly equivalent if they have at least one common representation sequence.

As an example consider the curves \( f' \) and \( f'' \) in Fig. 4.20. The representation sequence of \( f' \) is given by

\[\{e_1\}, \{e_1, e_2\}, \{e_2, e_3\}, \{e_3, e_4\}, \{e_4\}, \{e_4, e_5\}, \ldots,\]

whereas \( f'' \) is represented by the sequence

\[\{e_1\}, \{e_1, e_2\}, \{e_2, e_3, e_10\}, \{e_3, e_{10}\}, \{e_3, e_4, e_{10}\}, \{e_4\}, \{e_4, e_6\}, \ldots.\]

Thus, \( f' \) and \( f'' \) and therefore \( q' \) and \( q'' \) are weakly equivalent because \( e_1, e_2, e_3, e_4, \ldots \) is a possible representation sequence of both \( f' \) and \( f'' \).

Note that strong equivalence is an equivalence relation, while weak equivalence is not, because it does not have the symmetry and transitivity properties.

Definition 4.3.4 (Montanari, 1970b)
A curve \( f \) is said to be well formed if the cardinality of every set of cells in the representation sequence is two or less, and every set of cardinality two in this sequence is adjacent to two sets of cardinality one and is equal to their union.

If a curve is well formed, the sets of cardinality two in its sequence are in a sense redundant; in fact, the representation in sets of cardinality one is sufficient and unique. The most important property, however, is that in this case the representation corresponds to the representation of the curve on the similar mosaic with open instead of closed cells. With regard to Fig. 4.20, \( f' \) is well formed, whereas \( f'' \) is not.

In this framework, Montanari (1970b) derived several results concerning well-formed plane curves which also apply to the corresponding figures.
Here we reproduce some of the relevant theorems to be used to derive a modified result which fits the hexagonal mosaic.

**Theorem 4.3.4** (Theorem 2 in Montanari (1970b))
Given a closed-cell covering representation and a well-formed figure $q$, a Cauchy sequence of well-formed figures strongly equivalent to $q$ can be found such that the limit figure is non-well-formed and therefore is not strongly but weakly equivalent to $q$.


**Theorem 4.3.5** (Theorem 3 in Montanari (1970b))
Let $R_k$ ($k = 1, 2, \ldots$) be any Cauchy sequence of representations of figures $q_k$ ($k = 1, 2, \ldots$) and let $R$ be the limit representation of the figure $q$. If all the figures of the sequence are weakly equivalent by the common representation $C$ in a closed-cell mosaic, then $C$ is also a representation of $q$.

By virtue of Theorem 4.3.5 the subspace of the figures weakly equivalent to a given figure is always compact, and by Theorem 4.3.4 there exists a limit figure. If the Cauchy sequence of equivalent figures is limited to polygonal figures having at most two vertices interior to any edge of a cell and no vertices interior to a cell, then it is obvious that perimeter is a continuous function defined on that Cauchy sequence. The limit figure is always only weakly equivalent to the given figure and non-well formed, so that it has an ambiguous representation. On the other hand, if only well-formed figures were allowed, the minimal figure would not exist.

**Theorem 4.3.6** (Theorem 5 in Montanari (1970b))
Given a well-formed figure with a regular cellular complex on a closed-cell mosaic, the minimal polygon weakly equivalent to the given figure is unique.

*Minimal polygon and convexity*

Given a cellular complex $C$, it is useful for convexity analysis to reconstruct a well-formed plane figure $q$, which is a preimage of $C$, and such
hat if \( q \) is concave then every other preimage of \( C \) will necessarily be concave too. Sklansky and Kibler (1976) showed for the acute case that if the minimal polygon is concave, there exists no convex figure weakly equivalent to the minimal polygon.

\textbf{Theorem 4.3.7}

A regular complex \( C \) defined on an acute mosaic has a convex preimage if and only if the minimum-perimeter polygon of the complex is convex.

A similar result is proved by Montanari (1970b):

\textbf{Theorem 4.3.8}

If \( f \) is a well-formed convex closed curve with a regular complex, then the minimal polygon of \( f \) is convex.

* 

A careful comparison of these theorems reveals that there is a subtle but important difference in the use of the minimal polygon. Sklansky relates the polygon to the complex, whereas Montanari relates it via a Cauchy sequence to a well formed known preimage of the complex. In the latter case ambiguities may arise with complexes defined on non-acute mosaics:

Consider the closed-cell hexagonal mosaic. A single cell is a convex polygon, and given any pair of cells they are either disjoint or they share a straight line segment, the transverse edge. An end point of a transverse edge between two cells \( e_1 \) and \( e_2 \) belongs in the hexagonal case also to a third cell \( e_3 \), an edge neighbor of both \( e_1 \) and \( e_2 \). In the interior of the complex, an end point of a transverse edge of the boundary chain even belongs to the core of the complex. As has been shown, well-formed figures cannot pass through the end points of the transverse edges. It is easy to see that, given a well-formed figure, all figures strongly equivalent to the given one intersect the same set of transverse edges and in the same order, while all figures (strongly or weakly) equivalent to the given one intersect the same closed transverse edges (now including the end points) and in the same order. The minimal polygon weakly equivalent to the given figure is a polygon with its vertices on the transverse edges. In fact, given two consecutive intersection points between the minimal polygon
Fig. 4.21 A single complex with two neither strongly nor weakly equivalent preimages leading to two different limit figures.
and the transverse edges, the straight line segment joining these points belongs entirely to the cell defined by the transverse edges, since the cell is convex. Thus, the part of the minimal polygon between these points must be just this straight line segment, for otherwise it would not be of minimal length.

Summarizing the above arguments, we conclude that given only the complex obtained by the standard covering representation, that is only the set of cells without any knowledge of an inner boundary, several non-equivalent preimages exist, each of which defines a Cauchy sequence of strongly equivalent figures and a weakly equivalent limit figure. As an example, consider the complex shown in Fig. 4.21(a) and (b), having different well-formed preimages \( q' \) and \( q'' \), respectively. The representation sequence of the corresponding boundary curve given in a non-redundant form is

\[
e_1, e_2, e_3, e_4, e_5, e_6, e_7, \ldots, e_{13}, e_{14}
\]

and

\[
e_1, e_2, e_3, e_4, e_{17}, e_5, e_{18}, e_6, e_7, e_8, e_9, e_{20}, e_{10}, e_{21}, e_{11}, e_{12}, e_{22}, e_{13}, e_{14}
\]

Thus, \( q' \) and \( q'' \) are neither strongly nor weakly equivalent to each other, because of the cells \( e_5 \) and \( e_6 \) which are not in the representation of \( q' \). Each of the figures \( q' \) and \( q'' \) defines a Cauchy sequence of well-formed figures strongly equivalent to the given figure. The limit figure weakly equivalent to a given figure \( q \) intersects the closed transverse edges between any pair of adjacent cells of the representation sequence of the boundary of \( q \). Since the representation sequences of the boundaries of \( q' \) and \( q'' \) are different, the sets of transverse edges are different and likewise the limit figures as well. However, the set of strongly equivalent figures to which \( q' \) belongs is of particular interest. Given only the complex, \( q' \) belongs to the set of preimages of the complex with an empty inner boundary now obtained by the extended covering representation. In fact, this is the only set of preimages enclosing the complete core of the complex. Hence, we can apply the work of Sklansky on acute mosaics, which leads to the main theorem of this section:
Theorem 4.3.9
A regular tight complex defined on a closed-cell mosaic has a unique minimal polygon which is convex if and only if there exists a convex preimage.

By the equivalence theory this minimal polygon is weakly equivalent to the convex preimage. If only well-formed preimages are considered, which is the case in practical implementations of the covering representation with open cells and non-ambiguous representations, then the following theorem applies for tight complexes obtained by the extended covering representation:

Theorem 4.3.10
A regular tight complex $C$ on a hexagonal mosaic has a unique minimal polygon which is convex and does not touch the boundary $\partial C$ if and only if there exists a convex well-formed preimage.

Proof:
Let $C$ be a tight cellular complex that does not have a well-formed convex preimage in the extended covering representation. Then $C$ does not have a well-formed convex preimage either, if $C$ is obtained by the standard covering representation and if the preimage intersects each transverse edge of the cells forming the cellular boundary. However, $C$ can have a non-well-formed convex preimage $q$ and therefore by reason of Theorem 4.3.9 a convex MPP. It remains to be shown that this non-well-formed MPP does touch the boundary. It is obvious that in case $C$ does not have a non-well-formed convex preimage it does not have a convex MPP either, whether or not it touches the boundary of $C$. Suppose that the end points of the transverse edges between two adjacent cells of the cellular boundary which make $q$ non-well-formed are all located on the boundary of the core of $C$. Then, the figure intersecting the same set of transverse edges (not including the end points) as $q$ is also contained in the cellular boundary and is convex, which is a contradiction. Therefore, there is at least one end point of a transverse edge on $q$ which is located on $\partial C$. Since $q$ is convex this end point is on the straight line segment between two vertices $v_1$ and $v_2$ of $q$. The vertices $v_1$ and $v_2$ are points of the core of $C$, since
otherwise there is no need for the straight line segment to touch $\delta C$. The part of the minimal polygon between the two transverse edges of which $v_1$ and $v_2$ are end points must be precisely this straight line segment, for otherwise it would not be of minimal length. Hence, the MPP touches the boundary.

Now suppose the MPP is concave. Let $v_1$, $v$ and $v_2$ be three consecutive vertices of the MPP such that $v$ is a concave vertex (see Fig. 4.22). The distance between the line through $v_1$ and $v$ and the edge of the MPP $v_1 v_2$ at point $v_2$ is at least a quarter of a cell diagonal, which is equal to $1/4 \times 2/3 h\sqrt{3} = 1/6 h\sqrt{3}$. If there is a convex preimage $q$ of the complex, then it should have a convex vertex at $v$ or two consecutive collinear edges at $v$. In the first case, the distance between $q$ and the MPP at $v$ should be strictly more than $1/6 h\sqrt{3}$, and in the second case, this distance is precisely $1/6 h\sqrt{3}$. The minimal polygon, that is the non-well-formed preimage of $C$ with least perimeter is the only preimage $q$ which has two consecutive collinear edges at $v$ if the concavity of the MPP is only $1/6 h\sqrt{3}$ deep. Hence, the minimal polygon is convex at $v$. Any other preimage that passes through both $e_1$ and $e_2$ will be concave at $v$ too. Since the minimal polygon intersects the cells $e_1$ and $e_2$ only at their respective bottom corner points, this minimal polygon is not a well-formed preimage of $C$. Since it is the only convex polygon between $e_1$ and $e_2$ in $C$, $C$ does not have a well-formed convex preimage.

Now suppose the MPP touches the boundary of $C$. Then the MPP can be concave, a case considered above, or the MPP can be convex. Let $v$ be the point at which the MPP meets $\delta C$. Then there are two vertices $v_1$ and $v_2$ of the MPP such that $v_1$ and $v_2$ are on the boundary of the core of $C$ and $v_1$, $v$ and $v_2$ are collinear with $v$ between $v_1$ and $v_2$. For $q$ to be a well-formed convex preimage of $C$, $v_1$ and $v_2$ must be interior points of $q$ and $v$ must be an interior point of $q$ since $v$ is a point of a cell not in $C$. Thus, $q$ is not convex and therefore $C$ does not have a well-formed convex preimage. This completes the proof.

The previous theorem applies for tight cellular complexes obtained by the extended covering representation. It also applies to a specific subset of preimages not cutting off corners of the core of the cellular complex.
Fig. 4.22 Illustration of the proof of Theorem 4.3.10; the shaded region denotes the core of the complex. Note the protuberance of one cell-edge length.
obtained by the standard covering representation. However, in the latter case, given only the set of cells, without any knowledge of whether the generating preimage did in fact intersect all transverse edges of adjacent cell pairs forming the cellular boundary, it is not possible to determine a unique minimal polygon. Several minimal polygons may exist, neither weakly nor strongly equivalent to each other, each defining a different set of preimages. Each of these minimal polygons uniquely defines a set of intersected transverse edges of the cellular boundary which in turn define the set of preimages. One of these sets is of particular interest, namely the set completely enclosing the core of the complex. This set corresponds to the set of preimages defined by the complex viewed as the tight complex in the extended covering representation. If the minimal polygon of this set of preimages is concave, this set will not contain a convex preimage. However, a convex preimage of the complex may still exist in the standard covering representation, which cuts off one or more corners of the core of the complex. This ambiguity is due to the phenomenon that the hexagonal mosaic is not acute. In passing through two neighboring cells a straight line segment may cross a third cell, adjacent to both.

**Minimal polygon algorithm**

An algorithm for finding the minimum-perimeter polygon has been proposed by Sklansky (1970) and has been developed by Sklansky *et al.* (1972) and Sklansky and Kibler (1976). Montanari (1970a) stated the problem as a nonlinear programming problem. However, taking into account the peculiarity of the MPP, a fast iterative method can be used. The approach of Sklansky and his co-workers can be summarized as follows: Imagine a rubber band constrained to lie in the cellular boundary of a complex. The rubber band will then take the shape which minimizes its length, with the resulting polygon approximating the original boundary with maximum error less than the lattice constant. The shape taken by the rubber band is polygonal if the individual cells of the mosaic are themselves polygonal, which obviously is the case here.

We proceed now to describe a method for computing the MPP. The algorithm of Sklansky and Kibler (1976), called "algorithm M", which is proved to construct the MPP of a regular complex on an acute mosaic, will be adapted for use on a non-acute mosaic like the hexagonal mosaic.
Theorem 4.3.11
If \( C \) is a regular complex on a non-acute mosaic, then algorithm M
(Sklansky and Kibler, 1976) constructs the minimum-perimeter polygon of the complex.

Proof:
At an initial vertex \( v_i \) of the MPP, such as a vertex of the spinal path which is in fact a center point of a cell in the cellular boundary, a straight line \( l \) is constructed inside the boundary chain such that it passes through as many cells of the chain as possible. Let the last cell of the boundary chain intersected by the straight line \( l \) be denoted as \( e_j \). Then, the line \( l \) is inside a cone bounded by two lines, one touching a corner point of \( e_j \) at the righthand side, and another at the lefthand side, relative to the direction in which the boundary chain is traversed. If the center of the cell \( e_{j+1} \), that is the next cell in the chain, is to be right of \( l \), then let \( l \) be the right limiting line of the cone. Otherwise \( l \) is the left limiting line. Then, the line \( l \) between \( v_i \) and the corner point of a cell \( e_j \) is an edge of the MPP, where \( e_j \) is the cell that actually bounds the cone nearest to the end point of \( l \). If the procedure is repeated at each new vertex until all vertices are traversed by an edge of the polygon, then a second pass through the polygon found thus far may be needed to remove the initial vertex \( v_i \) if \( v_i \) is not a vertex found in the last step. This second pass through the boundary chain stops if two adjacent vertices remain unchanged. This construction gives the MPP of the complex \( C \) since each of the edges of the polygon is contained in the boundary \( \partial C \) and each cell of the cellular boundary was visited such that the core is contained in the polygon. Its length is minimal by construction via the cone defined at each vertex. Therefore, the polygon is the MPP.

Concluding remarks

In Section 3.5 we focused upon the topic of convexity on the hexagonal mosaic and its relation to the geometric properties defined on the corresponding lattice. We rejected the supposition that hexagonal cellular convexity should be synonymous with the property of having a convex preimage in the covering representation. The main reason is the proved
inconsistency between the geometric properties and the property of having a convex preimage. By the same argument there will be no equivalency between the minimal polygon of a hexagonal complex and the geometric properties of the associated set of lattice points. Furthermore, the main result is only applicable for tight and regular complexes. The latter restriction can be circumvented by means of the theory of half-cell expansion. Therefore, we derive in the next section a relationship between the minimal polygon defined on the triangular half-cell expansion of a hexagonal complex, on the one side, and the convexity of a preimage of the half-cell expansion and the geometric properties defined on the associated lattice, on the other side.

4.4 Minimal polygons and half-cell expansion

In this section we extend the theory of minimal polygons to the half-cell expansion of a hexagonal cellular complex in order to derive a sound connection between the geometric properties, cellular convexity and minimal polygons. The main result is that a complex $C$ is cellularly convex if and only if the MPP of its half-cell expansion $H(C)$ does not touch the boundary $\partial H$.

Since the half-cell expansion of a hexagonal complex is defined on a triangular mosaic, the theory of minimal polygons on acute mosaics can be utilized. By virtue of Theorem 4.2.1 the half-cell expansion of a cellular complex is regular and by reason of Theorem 4.3.3 the minimal polygon of the half-cell expansion is unique. In order to derive a relationship between minimal polygons and the theory of convexity as it was developed in Chapter 3, we first analyze the relation between digital regions and minimal polygons.

**Definition 4.4.1**
The *minimum-perimeter polygon* of a digital region $D$ is any polygon $p$ having $D$ as an "image", and such that there is no polygon of shorter perimeter having $D$ as an image.

In Fig. 4.23 an example is given of a digital region and its MPP. It is
Fig. 4.23  A digital region $D$ and its MPP; the larger black dots make up $D$; the open circles are points of $\overline{D}$ on the MPP.

Fig. 4.24  The digital region $D$ of Fig. 4.23 as an illustration of the proof of Lemma 4.4.1.
obvious that the MPP has an ambiguous digitization since otherwise the minimum would not exist. Note that every point \( d \) of \( D \) is inside or on the MPP and no point of \( \overline{D} \) is inside the MPP. Furthermore, the polygon is composed of straight line segments between lattice points only.

**Lemma 4.4.1**

If \( p \) is the MPP of a digital region \( D \) then every convex vertex of \( p \) occurs at a point of \( D \) and every concave vertex of \( p \) occurs at a point of \( \overline{D} \).

*Proof:*

Suppose \( p \) has a convex vertex which is a point of \( \overline{D} \); then we construct a polygon \( p' \) whose perimeter is strictly less than the perimeter of \( p \) and which is a preimage of \( D \). Let \( d_v \) be the convex vertex and a point of \( \overline{D} \) (see Fig. 4.24). Then construct \( p' \) such that \( d_v \) is replaced by a sequence of corner points of \( D \) and such that \( p' \) does contain every point of \( D \) and none of \( \overline{D} \) in its interior. Then \( p' \) is a preimage of \( D \) and has shorter perimeter than \( p \), which is a contradiction. Now suppose that \( d_v \) is a concave vertex and a point of \( D \). Then by an argument similar to the above we can obtain a polygon \( p'' \) which is an MPP and whose perimeter is less than that of \( p \).

\[ \square \]

**Lemma 4.4.2**

The MPP of a digital region \( D \) is unique.

*Proof:*

Suppose the MPP is not unique, and let \( p_1 = (a_1, \ldots, a_m) \mod m \) and \( p_2 = (b_1, \ldots, b_n) \mod n \) be two minimum-perimeter polygons. Let \( a_1 = b_1 \) be a common starting point of the two polygons and let \( s \) be the first point along the polygons such that \( a_s \neq b_s \). Now let \( x \) be the first point along \( p_1 \) and \( p_2 \) at which the polygons intersect after \( a_{s-1} = b_{s-1} \). Such a point \( x \) exists, since \( p_1 \) and \( p_2 \) meet again at \( a_1 = b_1 \). Since \( p_1 \) and \( p_2 \) are both minimum-perimeter polygons the vertices of \( p_1 \) and \( p_2 \) between \( a_{s-1} \) and \( x \) are points of \( D \) and \( \overline{D} \) and hence convex and concave, respectively, or the reverse, by virtue of Lemma 4.4.1. But this is a contradiction.

\[ \square \]
Fig. 4.25 The digital region $D$ of Fig. 4.23 shown on the dual triangular mosaic. The MPP shown is the MPP of both $D$ and $H(C)$ where $D = \lambda(C)$. The larger black dots are points of $D$; the open circles points of $\overline{D}$. 
Theorem 4.4.3
A digital region $D$ has the line property if and only if no point of $\overline{D}$ is a point of the MPP of $D$.

*Proof:
Suppose that there is no vertex of the MPP of $D$ which is a point of $\overline{D}$. Then there is no triplet of collinear points $(d_1, d, d_2)$ such that $d_1, d_2 \in D$ and $d \in \overline{D}$. Hence, $D$ has the line property.

Now suppose that a vertex of the MPP is a point of $\overline{D}$. Then this point, denoted as $d$, is either a concave vertex or is on the straight line between two vertices at each side of $d$. Let $d_1$ and $d_2$ be the vertices of MPP such that they are points of $D$ and closest to $d$ at either side of $d$. These points $d_1$ and $d_2$ exist since there are at least three convex vertices in each polygon and each convex vertex of the MPP is a point of $D$. Obviously, the triplet of collinear points $(d_1, d, d_2)$ violates the line property, which proves the theorem.

As before, we can replace $D$ by $\lambda(C)$ and reformulate Definition 4.4.1 and the accompanying lemmas and theorems in terms of the associated set of lattice points of a cellular complex $C$. Fig. 4.25 illustrates the duality between the lattice representation $D$ and the half-cell expansion of a cellular complex $C$ where $D = \lambda(C)$.

Theorem 4.4.4
The MPP of a regular triangular half-cell expansion $H(C)$ of a cellular complex $C$ and the MPP of the associated set of lattice points $\lambda(C)$ are identical.

*Proof:
Consider a regular half-cell expansion $H(C)$; then the MPP of $H(C)$, denoted by $p$, is a preimage of $H(C)$ and therefore contains every point of $\lambda(C)$, and its interior does not contain any point of the complement of $\lambda(C)$. Moreover, $p$ has a minimal length. By reason of Theorem 4.3.3 $p$ is unique and by Lemma 4.4.2 the MPP of $\lambda(C)$ is unique. This proves the theorem.
Theorem 4.4.5
A cellular complex $C$ is cellularly convex if and only if there is no cell of
$\bar{C}$ such that its center lies in the MPP of $H(C)$, the half-cell expansion of $C$.

Proof:
By Theorem 4.4.4 the MPP of $H(C)$ is identical to the MPP of $\lambda(C)$. Thus, the proof can immediately be obtained from Theorems 4.2.5 and 4.4.3.

The impact of Theorem 4.4.5 is that we have created a single instrument for automating the recognition of the convexity of a binary-valued image regardless of whether the image is digitized on a hexagonal lattice or on a hexagonal mosaic. This is due to the fact that the minimal polygon of the half-cell expansion turned out to be a preimage of the associated set of lattice points of the originating cellular complex. In terms of digitization, the figure generating the cellular complex in a closed-cell triangular mosaic is identical to the figure generating the associated hexagonal digital region by the lattice representation.

4.5 Concluding remarks

Cellular convexity has been discussed for a cellular complex defined on the cellular mosaic. It has been shown that the convexity of a preimage $q$ of the half-cell expansion of a cellular complex $C$ is equivalent to the digital convexity of the associated set of lattice points $\lambda(C)$. Furthermore, this set of lattice points is shown to be the digital region $D$ obtained by digitizing the very same preimage $q$ by the lattice representation.

The theory of limit properties has been used to study the notion of minimal polygons in relation to the existence of a convex preimage for a cellular complex on the hexagonal mosaic, a mosaic explicitly excluded in the literature on minimal polygons. It turned out that the minimal polygon can be an instrument for detecting a convex preimage for the class of tight complexes defined by the extended covering representation. However, given only a set of cells obtained by the standard covering representation without knowledge of whether the generating preimage
intersected the transverse edges of the cells forming the cellular boundary, it is not possible to determine a unique minimal polygon. Furthermore, since there is no unique minimal polygon which cuts off all corners of the core, there is no straightforward solution to define the minimal polygon for cellular complexes obtained by the standard covering representation on the hexagonal mosaic.

The concept of minimal polygons has been shown to be applicable to detecting cellular convexity defined by the convexity of a preimage of the half-cell expansion. Since the half-cell expansion is defined on the triangular mosaic, an acute mosaic, the results of Sklansky can be utilized. It has been shown that the minimal polygon of the associated set of lattice points $\lambda(C)$ of the complex $C$ is identical to the minimal polygon of the half-cell expansion $H(C)$. In the following “hexagonal” diagram we illustrate the proved relations between the notions of convexity, the geometric properties, the representation schemes and the minimal polygon.

Now the concept of discrete convexity is well defined, and sound in the sense that many equivalent properties of convex regions in Euclidean geometry carry over to the discrete space.
5.1 Introduction and preliminary notions

The notion of convexity of a figure $f$ in the continuous planar space is based on the property that the straight line segment joining any interior point $A$ of a convex figure $q$ to any other point $B$ of $q$, consists entirely (with the possible exception of the point $B$) of interior points of $q$, as it was stated before in Theorem 3.1.3. So it is natural to consider the discrete representation of straight line segments in relation to discrete convexity. Rosenfeld (1974) established the following result:

**Theorem 5.1.1** (Rosenfeld, 1974)
A digital arc is the digitization of a straight line segment if and only if it has the chord property.

The chord property was defined by Definition 3.4.12, whereas the notion of an arc is given as follows:

**Definition 5.1.1** (Rosenfeld, 1974)
A digital arc $D$ is a digital region in which every point except two has exactly two neighbors in $D$, and the exceptional two, called end points, each have exactly one neighbor in $D$, where neighborhood depends on the connectivity defined on the lattice.

Rosenfeld's result relates a digitization to a geometric property and thus depends on the digitization scheme employed. The digitization scheme used in this connection is essentially the standard grid intersection quantization method of digitizing curves, as was introduced by Freeman (1970).
Fig. 5.1  The square lattice, 8-neighbor connectivity. (a) Grid intersect quantization. (b) Object boundary quantization. (c) Chain-code directions used to code (a) and (b).
Definition 5.1.2 (Freeman, 1970)
Grid intersect quantization (G.I.Q.)
Whenever the curve to be digitized crosses a grid line of the coordinate grid of lattice points superimposed on the curve, the lattice point nearest to the crossing is a point of the curve's digitization.

When the crossing is exactly midway between two lattice points, the one with smaller coordinates is chosen. The arc thus obtained is usually coded by the Freeman chain-code string (Freeman, 1961). In the rectangular lattice the string is composed of two standard elements differing in length by a factor of $\sqrt{2}$, i.e. one is the side and one is the diagonal of a small square of the lattice. The hexagonal lattice uses a single standard distance for each of the six code directions. Note that this is the only grid utilizing a single standard distance. The triangular grid employs even three different distances in a ratio $1 : \sqrt{3} : 2$ to each other. Figs. 5.1 to 5.3 illustrate the various G.I.Q. and coding methods on the three different lattices.

The relation between the convexity of a digital region and the digital straight line segment connecting a pair of its points has been established by Kim and Rosenfeld (1982a). They showed that a digital region is convex by virtue of the line property if and only if every pair of points in the digital region is connected by a digital straight line segment lying in it. In deriving this result, however, two different digitization schemes were used, the standard G.I.Q. method for curves (Definition 5.1.2) and the lattice representation for digitizing regions (Definition 3.1.1). The reason was that the scheme for digitizing curves could not be used to digitize regions. Two options for coping with this problem are open.

First, a new digitization scheme can be defined for digitizing curves such that the new scheme can be used to digitize regions as well. In fact, the digitization of a curve under the new scheme should be the same as the digitization of a region obtained by thickening the curve infinitesimally on one side. Obviously, this new scheme is the cellular representation of a planar curve, as was defined in Definition 4.3.2. In fact, Kim (1982a) used this scheme to derive a similar relation between straightness and convexity, as was derived by Kim and Rosenfeld using G.I.Q. and lattice representation. However, both results are obtained exclusively on the
The hexagonal lattice. (a) Grid intersect quantization. (b) Object boundary quantization. (c) Chain-code directions used to code (a) and (b).
Fig. 5.3  The triangular lattice, (a) to (c) 12-neighbor connectivity.
(a) Grid intersect quantization. (b) Object boundary quantization. (c) The two sets of chain codes used to code both (a) and (b). Note that there are 12 different chain-code directions but 18 different chain-code elements.
Fig. 5.3 (Continued.) The triangular lattice, (d) to (f) 3-neighbor connectivity. (d) Grid intersect quantization. (e) Object boundary quantization. (f) Chain-code directions, having unit length, used to code (d) and (e).
rectangular lattice. In Section 5.3 we will try to establish a similar result with the covering representation scheme on hexagonal mosaics.

The second option is to adopt the digitization scheme for digitizing curves that connects the lattice points nearest the line segment when going along the intersected column boundaries in the direction of the positive x-axis (Vossepoel and Smeulders, 1982). This scheme is also referred to as the digital image of an arc or lattice representation to the right (Kim, 1982b) and as the object boundary quantization (O.B.Q.) (Dorst and Smeulders, 1984). This scheme can be used to digitize regions as well. Kim (1982b) used it to derive a relation between digital convexity and digital straightness valid only for the rectangular lattice. However, this relation can easily be extended to the hexagonal lattice, as will be shown in the next section.

5.2 Digital straightness and digital convexity

To be able to handle the relation between digital straightness and digital convexity of a digital region we first give a definition of the digitization scheme for curves.

**Definition 5.2.1**

Lattice representation for curves

Consider a planar curve \( f \) with a coordinate grid of lattice points superimposed on the curve. A set of lattice points \( D \) is said to be the digital image of the curve \( f \), and \( f \) a preimage of \( D \) if

i) whenever \( f \) passes through a lattice point, the lattice point is a point of \( D \);

ii) whenever \( f \) crosses a grid line although not at a lattice point, the lattice point nearest the crossing point and to the right of \( f \) is a point of \( D \).

Then, the digital image of a figure \( q \) bounded by a closed planar curve \( f \) traversed clockwise is the union of the lattice points representing the curve \( f \) together with the lattice points internal to these. Therefore, the representation is sometimes referred to as the object boundary quantization (Dorst and Smeulders, 1984). Note that Definition 5.2.1 could have
Fig. 5.4 Illustration of the column concept on the square lattice used for curve representation (Vossepoel and Smeulders, 1982). The columns are alternately shaded. (a) 4-Neighbor connectivity. (b) 8-Neighbor connectivity.
been formulated the other way around. Then, the curve as a boundary would have to be traversed in the opposite direction (counterclockwise) to obtain the lattice representation of the enclosed object.

The following lemmas will be proved only for the hexagonal lattice, since the rectangular lattice has been treated in the literature, and the triangular lattice will be shown to be inconsistent in adopting the notion of nearness.

Lemma 5.2.1
The O.B.Q. representation of a straight line segment is a digital arc and has the line property.

*Proof:*
We first adopt the concept of a "column" as introduced by Vossepoel and Smeulders (1982). Figs. 5.4 to 5.6 illustrate the columns by the shaded regions for the three lattices. Note that the coordinate grid of the hexagonal lattice is a triangular mosaic, whereas the triangular lattice has a hexagonal "coordinate" grid. The problem here of defining a proper coordinate system has been dealt with in extenso in Chapter 2. Let $D$ be the digitization of a straight line segment $l$. If $l$ is in the direction of one of the axes of the lattice, then $D$ is a string of lattice points, as illustrated in Fig. 5.7. It is obvious that $D$ is a digital arc and has the chord property.

Now suppose $l$ is not in a direction of one of the main axes. Without loss of generality assume that $l$ has a slope $\tan \alpha, 0 \leq \alpha \leq \pi/3$; then, $l$ will intersect each column boundary once. Subsequently, there exists nearest the intersection in the preferred (right) direction a lattice point on that column boundary. The sequence of lattice points thus found forms a string which is obviously a digital arc. It remains to be shown that $D$ has the line property. To do so, we suppose $D$ does not have the line property. Then there exists a triplet $(d_1, d, d_2)$ of collinear points with $d_1, d_2 \in D$ and $d \in \overline{D}$. If $l$ passes through $d_1$ and $d_2$, then obviously $l$ would pass through $d$, which is a contradiction. Without loss of generality suppose $l$ passes through $d_1$ and not through $d_2$, then because $d_2 \in D$, $d_2$ is on the right side of $l$. Since the triplet $(d_1, d, d_2)$ is collinear the point $d$ is also on the right side of $l$. The distance from $l$ along a column boundary to $d_2$ is less than the lattice constant $h$; therefore, the distance from $l$ to $d$ along its column boundary is also less than $h$. Hence, $d$ is a point of $D$, which is
Fig. 5.5 Illustration of the column concept on the hexagonal lattice used for curve representation (Vossepoel and Smeulders, 1982). The columns are alternately shaded. 6-Neighbor connectivity.
Fig. 5.6 Illustration of the column concept on the triangular lattice used for curve representation. The columns are alternately shaded. (a) 3-Neighbor connectivity. (b) 12-Neighbor connectivity.
Fig. 5.7 Three straight lines $l$, each in the direction of one of the three main axes.
a contradiction.

Now suppose \( l \) has an arbitrary direction. Then, both \( d_1 \) and \( d_2 \) are on the right side of \( l \) within a distance \( h \) along their respective column boundary. Since the triplet \((d_1, d, d_2)\) is collinear the lattice point \( d \) is also within distance \( h \) to \( l \) and therefore a point of \( D \), which is a contradiction. Hence, there is no collinear triplet of lattice points to violate the line property. Finally, if \( l \) passes neither through \( d_1 \) nor through \( d_2 \), a straight line segment \( l' \) can be obtained by parallel translation of \( l \) in the preferred direction, which passes through \( d_1 \) or \( d_2 \). This proves the lemma.

\[ \square \]

**Lemma 5.2.2**

If a digital arc has the chord property, then it is a digital straight line segment.

\[ * \]

**Proof:**

Let \( D \) be a digital arc that has the chord property. Then by reason of Theorem 5.1.1 there is a straight line segment \( l \) of which \( D \) is the G.I.O. Although this theorem is proved only by Rosenfeld (1974) for the rectangular lattice, it is easily proved to hold for the hexagonal lattice too. Since a digital arc is in the shape of steps, the topology is preserved when the hexagonal lattice is "de-slanted" into a square lattice. This transformation has been utilized before in the proof of Lemma 3.4.3.

Now assume without loss of generality that \( l \) has a slope \( \tan \alpha \) where \( 0 \leq \alpha \leq \pi/3 \), and that when \( l \) crosses a column boundary in the G.I.O. between two lattice points, the one nearest \( l \) becomes a point of \( D \). If a point of \( D \) lies to the right of \( l \), then its distance along the column boundary to \( l \) is less than \( 1/2 \ h \), and if a point of \( D \) lies to the left of \( l \), then its distance along the column boundary is less than or equal to \( 1/2 \ h \). Let \( g \) be the straight line segment obtained by parallel translation of \( l \) outwards, that is, in the opposite direction with respect to the preferred (right) direction, in the direction of the columns over a distance of \( 1/2 \ h \). Then \( g \) is a straight line segment of which \( D \) is the O.B.O. Hence, \( D \) is a digital straight line segment.

\[ \square \]
Fig. 5.8  Elementary leaf-shaped region used to define the notion of nearness on the triangular lattice.

Fig. 5.9  The domain of a digital region $D$ to check the chord property on the triangular lattice; the larger black dots constitute $D$; the open circles are points of $\overline{D}$ inside the domain of $D$.
Lemma's 5.2.1, 5.2.2 and 3.4.6 combine to yield the following theorem:

**Theorem 5.2.3**  
A digital arc has the chord property if and only if it is a digital straight line segment.

**Theorem 5.2.4**  
A digital arc is a digital straight line segment if and only if it is digitally convex.

**Proof:**  
By definition a digital arc is a digital region and therefore if a digital arc has one of the geometric properties, then it has by virtue of Theorem 3.4.9 all the geometric properties and is digitally convex. By reason of Theorem 5.2.3 a digital arc has the chord property, which proves the theorem.

**Theorem 5.2.5**  
A digital region $D$ is digitally convex if and only if for any two points $d_1$ and $d_2$ of $D$, there is a digital straight line segment in $D$ completely contained in $D$.

**Proof:**  
Suppose that $D$ is digitally convex. Let $d_1$ and $d_2$ be any two points of $D$. Consider the case that $d_1, d_2$ lies within the region $s(D)$ bounded by $\partial D$. Then the digital image in the O.B.Q. consists of points of $D$ only, and hence there is a digital straight line segment between $d_1$ and $d_2$. Next consider the case when some segments of $d_1, d_2$ lie outside $s(D)$. Then $P(D; d_1, d_2)$ does not contain a point of $\bar{D}$ because $D$ has the area property by virtue of Theorem 3.4.9. Thus, whenever $d_1, d_2$ crosses a column boundary, the lattice point that lies in the preferred direction and nearest the crossing point is a lattice point of $D$. Hence, there is a digital straight line segment in $D$ between $d_1$ and $d_2$. Next, suppose that $D$ is not digitally convex. Then by reason of Theorem 3.4.9 $D$ does not have the line property. Then there is a triplet of collinear points $(d_1, d, d_2)$ such that $d_1, d_2 \in D$ and $d \in \bar{D}$. Thus, the O.B.Q. of $d_1, d_2$ does contain the
Fig. 5.10 Three examples of cellular complexes of straight line segments; (a) and (b) are no cellular arcs.
point $d$ of $\overline{D}$ and is therefore not contained completely in $D$. Hence, there is no digital straight line segment inside $D$ between the two points.

This result corresponds to that of Kim (1982b) obtained for the rectangular lattice. Even though the results are analogous the proofs are here and there different because of differences between the digitization schemes and lattices.

It should be noted that Lemma 5.2.2 is proved here only to hold for the hexagonal case. So all subsequent lemmas and theorems that refer to Lemma 5.2.2 are only valid for the hexagonal lattice. Due to the irregular nature of the triangular lattice, in the sense that not all integer positions in the coordinate grid are occupied by lattice points, there is no feasible definition of nearness which is consistent with the geometric properties. Nearness is defined with respect to the connectivity defined on the lattice. For a straight line segment between two lattice points to be near a digital region $D$, any real point on the line segment should be near a point of $D$ in the sense of 12-neighbor distance. The set of points in the plane about a triangular lattice point within a 12-neighbor distance generalized to real arguments has the shape of a leaf composed of three generating hexagons. Fig. 5.8 shows the two different orientations of this leaf-shaped region. To investigate whether the definition of nearness can be used to define the chord property in such a way that the chord property is equivalent to the other geometric properties, we will give an illustrative example. In Fig. 5.9 a boundary curve $f$ is digitized on the triangular lattice by the O.B.Q. scheme. The resulting string of lattice points is assumed to be 12-connected. The domain about the set of lattice points $D$ such that each point inside the region is near a lattice point of $D$ consists of a set of leaf-shaped regions centered about each lattice point of $D$. In contrast to the square and the hexagonal lattice the domain about $D$ contains lattice points in its interior that are not points of $D$. In fact, this is due to the phenomena of the two different structuring elements in which there can be found center points of structuring elements inside the region that are not points of $D$. One of these points, indicated by $d$ in Fig. 5.9, forms a collinear triplet with two points of $D$: $(d_1, d, d_2)$. Since $d \in \overline{D}$ this triplet violates the line property of $D$. However, any chord between two lattice points of $D$ is inside the region defined by the nearness condition. Hence,
Fig. 5.10 (Continued.) (c) Cellular arc.
$D$ has the chord property. Furthermore, it is easy to see that there is no convex preimage of $D$ in the O.B.Q. because any convex curve would enclose point $d$, which is a contradiction. Hence, the digital arc of Fig. 5.9 has the chord property but is not a digital straight line segment, thereby violating Lemma 5.2.2.

Our previous work on the triangular lattice regarding convexity was restricted to the cellular triangular mosaic as the half-cell expansion of a cellular complex on the hexagonal mosaic. Here we demonstrate that there is no way of deriving a consistent theory of convexity based on geometric properties for the triangular lattice. Even digital straightness on the triangular lattice does not seem to lead to a straightforward and transparent theory. The only practical utility of the triangular mosaic in convexity analysis will be in the duality relation to the hexagonal mosaic in half-cell expansion analysis.

5.3 Cellular straightness and cellular convexity

In Chapter 4 we formulated the definition for the cellular image of a planar curve $f$ in such a way that the covering representation of a figure $q$ bounded by a closed curve $f$ is given by the union of the cells internal to its boundary together with the set of cells representing the boundary. Thus the scheme for digitizing curves by their cellular images may be considered equivalent to the scheme for digitizing regions.

**Definition 5.3.1**

A cellular arc $C$ is a cellular complex in which every cell except two has exactly two neighbors in $C$, and the exceptional two, called end cells, each have exactly one neighbor in $C$, where neighborhood depends on the connectivity defined on the mosaic.

Kim (1982a) derived a result relating the straightness of cellular arcs and the chord property for the case of cellular straight line segments. This result is identical to the classical result of Rosenfeld (1974) (cf. Theorem 5.1.1) obtained for digital straight line segments. Both results apply only on the rectangular grid. Rosenfeld’s result has been extended for the hexagonal lattice in Section 5.2. In this section we try to extend Kim’s
Fig. 5.11 A straight line \( l \) in the \( u_2 \)-direction; (a) is a cellular arc; (b) is not.
result to the hexagonal mosaic. To illustrate the difficulties we will
inevitably meet, Fig. 5.10 gives some examples of cellular images of
straight line segments. The following lemma follows directly from these
examples:

**Lemma 5.3.1**
The cellular image of a straight line segment is not necessarily a cellular arc.

**Lemma 5.3.2**
The cellular image of a straight line segment has the chord property
whether or not it is a cellular arc.

**Proof:**
Kim (1982a) proved this lemma for rectangular cellular arcs. We will adopt
his strategy to extend the result to the hexagonal mosaic without the
arcness condition. Let \( l \) be a straight line segment and \( C \) its cellular image
on the hexagonal mosaic. If \( l \) is parallel to one of the three axes, then \( C \) is
either a sequence of cells forming a cellular arc or two adjacent chains of
cells without the arc property. (See Fig. 5.11.) In both cases, however, it
is obvious that \( C \) has the chord property.

Now suppose that \( l \) is not parallel to one of the main axes. Without loss of
generality assume that \( l \) has a slope \( \tan \alpha, -\pi/6 \leq \alpha \leq +\pi/6 \). Let \( c_1 \) and \( c_2 \)
be the centers of the elements \( e_1 \) and \( e_2 \) of \( C \) and \( r_1 \) and \( r_2 \) the upper
corner points and \( s_1 \) and \( s_2 \) the lower corner points of \( e_1 \) and \( e_2 \),
respectively (see Fig. 5.12). Then \( l \) lies between the line segments \( r_1 r_2 \)
and \( s_1 s_2 \). Let \( z \) be any point of \( c_1 c_2 \) and \( e \) the cell containing \( z \). If \( e \) is
an element of \( C \) then \( c_1 c_2 \) is near \( C \). Suppose \( e \) is not an element of \( C \).
Note that \( c \) is not on \( c_1 c_2 \), since if it were, \( l \) would pass through \( c \) and \( e \)
would be an element of \( C \). Assume without loss of generality that \( c \) lies
above \( c_1 c_2 \). Then the cell \( e_3 \) just below and to the right of \( e \) is an element
of \( C \), that is, \( z \) has a smaller \( u_3 \) coordinate then \( c \), and therefore \( z \) is near
the elementary hexagon with diagonal \( 2h \) about \( c_3 \). Thus \( c_1 c_2 \) lies near \( C \)
and therefore \( C \) has the chord property.

\qed
Fig. 5.12  Illustration of the proof of Lemma 5.3.2.

Fig. 5.13  The proof of Lemma 5.3.3.
Lemma 5.3.3
If a cellular arc has the chord property, then it is not necessarily a cellular-
straight line segment.

The proof follows directly from Fig. 5.10(c) and Fig. 5.13. If a straight
line segment / has an arbitrary direction and if its cellular image is a
 cellular arc, there is a good chance that its image will consist of several
"runs" of equal chain-code cells. However, due to the geometry of the
mosaic, the run length of an individual run cannot exceed five. In the
example of Fig. 5.13, the cell \( e_j \) adjacent to the last cell of the first run
should be included in the complex, thereby violating the arcness condi-
tion. However, a single element, which doubles the adjacency between
consecutive runs is not enough if the run length exceeds nine, as shown in
the table given below. The table indicates the number of overlapping cells
for a given run length needed for a complex to be a cellular straight line
segment, i.e. to have a straight line segment as a preimage.

<table>
<thead>
<tr>
<th>run length</th>
<th>overlap</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 - 5</td>
<td>1</td>
</tr>
<tr>
<td>6 - 9</td>
<td>2</td>
</tr>
<tr>
<td>10 - 13</td>
<td>3</td>
</tr>
<tr>
<td>14 - 17</td>
<td>4</td>
</tr>
</tbody>
</table>

In the table the overlap is given by: \( 1/4 \, (\text{maximal run length} + 1) - 1/2 \),
where overlap is defined as the number of neighboring cells between two
adjacent runs.

Combining Lemmas 5.3.2 and 5.3.3, we cannot obtain a necessary and
sufficient geometric condition for a cellular complex to be a cellular
straight line segment. Even if we drop the arcness condition, the chord
property, and with it the other geometric properties, are not equivalent to
the property of having a straight line segment as a preimage. Therefore, it
makes no sense to relate cellular straightness to cellular convexity - in the
sense of Supposition 3.5.1 - such that for any pair of points of a cellular
complex there exists a cellular straight line segment. Moreover, cellular
convexity itself has been proved to be inconsistent in its relation to the
geometric properties (Theorem 3.5.4).
Fig. 5.14 Two examples of nontight cellular complexes. (a) A curved figure. (b) A polygonal figure. Note that these figures clearly exceed the maximum-curvature limit of the sampling theorem of Pavlidis (1980).
Both inconsistencies are due to the same phenomenon, namely that the hexagonal mosaic is not acute: a straight line segment, in passing through two neighboring cells, may cross a third cell adjacent to both. Hence, a cellular image of a straight line segment may have overlapping runs of cells, whereas in the convexity analysis, problems arise with figures that cut off corners of the core of the complex. To overcome the non-acuteness of the mosaic, the half-cell expansion has been introduced for the convexity analysis. The result was that cellular convexity turned out to be equivalent to digital convexity. Then, the relation between digital convexity and digital straightness derived in Section 5.2 provides the connection between convexity and straightness.

A quite different approach to relating cellular straightness and convexity on the hexagonal mosaic will be presented in the next section.

5.4 Nontight straightness and convexity

A cellular complex was said to be tight in Definition 4.2.3 if it had an empty inner boundary. Fig. 5.14 shows two examples of nontight images of plane figures. In considering nontight versions of the digitization schemes for figures and planar curves we might be able to find a relation between nontight straightness and nontight convexity.

In Section 4.3 we introduced the covering representation of a planar curve such that the definition is consistent with that of the cellular image of the plane figure $q$ bounded by the planar curve $f$. A definition that explores the notion of tightness consistent with Definition 4.2.1 is given by

Definition 5.4.1 Extended covering representation for curves
A set of cells $C$ is said to be the cellular image of a planar curve $f$, and $f$ a preimage of $C$ if
\begin{enumerate}
\item $f \subseteq s(C)$;
\item for each cell $e$ of $C$, $e^o \cap f \neq \emptyset$, where $e^o$ is the interior of $e$;
\item each edge of an element $e$ of $C$ is part of $\delta C$ if $r^* \cap q \neq \emptyset$, where $r^*$ is the edge $r$ excluding its endpoints.
\end{enumerate}
Fig. 5.15  A planar curve and its nontight cellular image.

Fig. 5.16  Nontight versions of the straight line segments taken from previous Figs; (a) Fig. 5.10 (a).
Then, the cellular image of a figure $q$ bounded by a closed planar curve $f$ is the union of the cells internal to its boundary together with the set of cells representing the boundary and the cell sides forming the inner boundary external to the curve $f$. Fig. 5.15 shows a planar curve $f$ and its nontight cellular complex. The examples of cellular complexes of straight line segments given in Fig. 5.10(a) and (b) and Fig. 5.11(b) are depicted in Fig. 5.16 by a nontight cellular complex. We note from Fig. 5.16 that the cellular image according to Definition 5.4.1 of a straight line segment is clearly a cellular arc, a property whose proof is omitted here. However, the problem encountered in Section 5.3, illustrated by Fig. 5.13, is not solved by inserting the notion of tightness. Indeed a nontight cellular arc having the chord property may still not have a straight line segment as its preimage. In essence, the digitization scheme has not changed much: the same set of cells is involved. The only advantage is that the overlap between two adjacent runs is better specified by virtue of the inner boundary.

We will continue here with cellular convexity of nontight complexes.

**Definition 5.4.2**

An element of a nontight complex $C$ is said to be a *virtual cell* if it is a boundary cell and at least one of its sides belongs to the inner boundary of $C$.

Note that $\delta C$ is defined as the boundary of $C$, including the inner boundary, whereas the symbol $\delta s(C)$ is used for the outer boundary as such. A lattice point of the associated digital region $\lambda(C)$ whose cell is a virtual cell is called a *virtual point*.

As a start we will redefine one of the geometric properties such that it incorporates the notion of tightness.

**Definition 5.4.3**

A digital region $D$ has the *line property* if and only if

i) there exists no triplet $(d_1, d, d_2)$ of collinear points such that $d_1$ and $d_2$ are points of $D$ and $d$ is a point of $\overline{D}$;

ii) there exists no triplet $(d_1, d, d_2)$ of collinear points of $D$ among which
Fig. 5.16 (Continued.) (b) Fig. 5.10 (b); (c) Fig. 5.11 (b).
$d$ is a virtual point of $D$.

* 

**Supposition 5.4.1**

A nontight cellular complex is said to be *cellularly convex* if there exists a convex figure $q$ whose covering representation is $C$.

* 

It should be noted that this supposition is only applicable on non-acute mosaics, since whenever a complex on an acute mosaic has a non-empty inner boundary there cannot exist a convex preimage by construction. For the cellular convexity as meant in Supposition 5.4.1 to be equivalent to the geometric properties it should be equivalent to at least one of them, for example, the line property given by Definition 5.4.3. First we need a lemma.

**Lemma 5.4.1**

If a nontight complex $C$ has a convex preimage, then there are no two nonvirtual (corner) elements in the same row or column of $C$ on either side of any virtual cell of $C$ in that row or column.

* 

**Proof:**

Let the cells $e_1$ and $e_2$ be two nonvirtual (corner) elements of a nontight complex $C$. Without loss of generality assume that $e_1$ and $e_2$ are in the same row as the virtual element $e$ of $C$ and that $e_1$ is to the left of $e_2$. The cell $e'_1$ immediately to the right of $e_1$ does not share a part of the inner boundary $\delta C - \delta s(C)$ of $C$ with $e_1$, because $e_1$ is a nonvirtual element. In addition, the cell $e'_2$ immediately to the left of $e_2$ does not share a part of the inner boundary with $e_2$ because of the same argument. Suppose there is a convex plane figure $q$ such that $p$ is a preimage of $C$. Then there are points $x$ and $y$ of $q$ which are interior points of $e_1$ and $e_2$, respectively. These points can be chosen such that $x$ intersects the edge between $e_1$ and $e'_1$, and the edge between $e_2$ and $e'_2$. The part of the inner boundary of $C$ between virtual element $e$ and its virtual neighbor on the same row is also intersected by $x$ $y$. However, since $q$ is convex the line segment $x$ $y$ is entirely inside $q$ and hence in $C$. But in case $e$ is a virtual element this is a contradiction. This proves the lemma. See for an
Fig. 5.17  Illustration of the proof of Lemma 5.4.1.

Fig. 5.18  Illustration of the proof of Theorem 5.4.2.
As a result the top row of cells of the complex given in Fig. 5.14(a) precludes the complex from being convex, whereas the opposite holds for the top row of the complex given in Fig. 5.14(b). Nevertheless, the latter complex is concave due to the inner boundary segment in the lower part of the complex.

**Theorem 5.4.2**

If a nontight complex has a convex preimage, then it may not have the line property.

*Proof:

Suppose a nontight complex $C$ has a convex preimage $q$. Let $e_1$ and $e_2$ be two nonvirtual (corner) elements of $C$. If $e_1$ and $e_2$ are in the same row or column, then by virtue of Lemma 5.4.1 there is no triplet of collinear points $(c_1, c, c_2)$ to violate the line property of $\lambda(C)$.

Next, consider the case where $e_1$ and $e_2$ are neither in the same row nor in one of the columns. Without loss of generality assume $e_1$ is above $e_2$. If the run lengths of the runs between $e_1$ and $e_2$ defined by the straight edge between $e_1$ and $e_2$ of the convex polygonal preimage are larger than four then $C$ has a virtual cell in between $e_1$ and $e_2$. Fig. 5.18 gives an example of such a complex and its convex preimage. Let $u_1$ and $u_2$ be the vertical and horizontal distances between $c_1$ and $c_2$ along the basic directions $u_1$ and $u_2$. Then since $u_2/u_1$ is an integer in Fig. 5.18, $c_1c_2$ passes through the center of a cell $e$ in each row in between the rows of $e_1$ and $e_2$. Because the run length of the run in the row of $e_1$ is assumed to be at least four this cell $e$ may be a virtual cell, as shown in Fig. 5.18. Hence, there may be a triplet $(c_1, c, c_2)$ of collinear points of $\lambda(C)$ of which $C$ is a virtual point of $\lambda(C)$. This proves the theorem.

The main result is that cellular convexity of nontight complexes is not equivalent to the geometric properties. Therefore, it does not make much sense to relate this kind of convexity to cellular straightness of nontight complexes.
5.5 Concluding remarks

The relation between discrete convexity and straightness - a relation well defined and understood in Euclidean geometry - has been discussed for both the hexagonal lattice and mosaic. It turned out that digital convexity is equivalent to the property that any two points of a digital region can be connected by a digital straight line segment inside the region. This result adds another property to the list of properties already shown in Section 3.4 to be equivalent to digital convexity.

However, we failed to derive a similar relation between cellular convexity and cellular straightness defined on the hexagonal mosaic. This failure is due to the same phenomenon we met before: the non-acuteness of the hexagonal mosaic.

Hence, we derived another argument against cellular convexity based on Supposition 3.5.1, a definition of convexity which has been proved in Section 3.5 to be inconsistent with the geometric properties. Therefore, it is not surprising that cellular straightness, which is closely related to one of the geometric properties, the chord property, is not equivalent to the cellular convexity of Supposition 3.5.1. In Section 4.2 we introduced the half-cell expansion to cope with the non-acuteness of the hexagonal mosaic. Then, the relation between digital convexity and digital straightness provides the connection between the convexity of the hexagonal complex by its half-cell expansion and the straightness of its associated lattice. In order to exploit the non-acuteness of the hexagonal mosaic we introduced the notion of tightness in a final attempt to relate cellular convexity and cellular straightness without making use of the half-cell expansion. However, in doing so, we did not even succeed in relating cellular convexity of nontight complexes, a concept only applicable for non-acute mosaics, to the geometric properties.

Freeman (1970) formulated the linearity conditions for digital straight line segments on the rectangular lattice, and chain-code strings made up of 8-neighbor connected chain codes. It has been shown (Vossepoel and Smeulders, 1982) that these linearity conditions can be generalized to e.g. the hexagonal lattice or to other connectivities. Kim (1982a) investigated cellular straightness for rectangular mosaics using the standard linearity
conditions. However, for the hexagonal mosaic such a result has not been obtained yet. In Section 5.3 we showed that the arcness property of a cellular complex is not equivalent to the chord property of that complex, both properties forming the base for the linearity conditions. It is an open question whether the overlapping of adjacent runs, a concept introduced in Section 5.3, can be formulated as an additional linearity condition, or whether it suffices to utilize the linearity conditions on the set of lattice points associated to the cellular complex with or without overlap defined on it.
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SAMENVATTING

Indien een digitale computer geprogrammeerd is voor het analyseren van een tweedimensionaal meetkundig object dat deel uitmaakt van een beeld, dan wordt het beeld gewoonlijk voorgesteld door een rij getallen in het geheugen van de computer. Hiertoe dient het beeld ruimtelijk bemonsterd te worden, d.w.z. dat de intensiteitswaarde van het beeld bepaald wordt in een eindig aantal discrete punten, “monsters” genoemd, die vervolgens gekwantiseerd worden tot een eindig aantal discrete niveaus. Dit digitaliseren van beelden heeft invloed op het vermogen van de computer om de vorm van het object te analyseren. In het algemeen levert het digitaliseren van beelden een gigantisch volume aan gegevens op. Er is een reusachtige inspanning nodig om deze ruwe informatie zonder voorbewerking te verwerken. Voordat vervolgens het tweedimensionale object met behulp van patroonherkenningsmethoden geanalyseerd kan worden, moet de matrix met helderheidswaarden omgezet worden in een meer compacte vorm. Eén manier om dit te doen is het segmenteren van het beeld in zijn samenstellende delen, een proces op de overgang van beeldverwerking naar patroonherkenning, of beter gezegd, op de grens van beeldbemonstering en beeldanalyse. Het probleem komt in wezen overeen met problemen uit de psycho-fysische perceptie: Gestalt-principes bepalen zekere criteria voor samenvoeging zoals nabijheid en gelijkensoortigheid of criteria gebaseerd op het onderscheid van objecten tegen een achtergrond. De juiste keuze van een segmentatiemethode valt buiten het kader van dit proefschrift, dat zich beperkt tot de vormstudie van “silhouetten”, d.w.z. tot tweedimensionale objecten in reeds gesegmenteerde beelden.

Is het meetkundige object eenmaal beschikbaar in discrete vorm, dan ontstaat er behoefte aan het definiëren van begrippen en aan het meten van parameters van de discrete componenten ten behoeve van het extraheren van vorminformatie van het object. Deze begrippen en parameters worden vaak direct vertaald van de continue ruimte naar de discrete ruimte, een
aanpak die vaak tot ernstige problemen leidt, veroorzaakt door de repre-
sentatie van een object door middel van een discreet netvlies. Zo dienen
eenvoudige convexe figuren, zoals cirkels en driehoeken, ook na digitalisa-
tie, als convex beschouwd te worden, hoewel in de discrete ruimte de rand
van zulke figuren nogal getand is. In dit proefschrift wordt een onderzoek
beschreven naar de bekende meetkundige eigenschap van convexiteit van
juist die “blobs” (zwarte objecten tegen een witte achtergrond) die van
nature convex zijn.

Een toepassing van convexiteitsanalyse kan gevonden worden in de
analyse van rode en witte bloedlichaampjes, waarbij een convexiteits-
criterium gebruikt wordt om de omtrek van gedeeltelijk overlappende
cellen in reeds gesegmenteerde beelden te reconstrueren. De kennis over
de vorm van de objecten kan ook opgenomen worden in de segmentatie-
methodes zelf, waardoor deze beperkt wordt tot het vinden van convexe
objecten. Dan zullen, uitgaande van een verzameling verbonden punten, in
elke stap van het algoritme beeldelementen samengevoegd worden op
grond van de convexiteit van het te vormen object. Deze methode bleek
zeer bruikbaar voor het scheiden van bloed- en beenmergcellen van de
achtergrond. Een geheel andere toepassing van de convexiteitsanalyse
vindt men in de herkenning van industriële onderdelen, die tijdens het
voorbijgaan op een lopende band gadegeslagen worden door een inspectie-
systeem. Deze objecten kunnen vanwege hun niet-convexe vorm en hun
vormvastheid uitstekend beschreven worden door concaviteitsbomen, die
de informatie bevatten om de plaatselijke concaviteiten en de concavi-
teiten in deze plaatselijke concaviteiten op te vullen. Deze concaviteits-
boom wordt verkregen door van het onderhavige object, convexe
deelverzamelingen te verwijderen en/of er aan toe te voegen, zodanig
dat het resulterende object convex is. Tenslotte kan verwezen worden
naar het werk verricht op het gebied van de ontbinding van een door
een veelhoek benaderde figuur in een samenspel van een eindig aan-
tal, mogelijk elkaar overlappende, convex veelhoeken. Zulke convexe
ontbindingen verschaffen een structurele analyse van een ingewikkeld
object in termen van een grafische beschrijving met knooppunten voor
doorsneden en primaire convexe veelhoeken, en een boog tussen over-
lappende veelhoeken.

Het begrip convexiteit is door vele onderzoekers beschreven, echter alleen
voor die binaire blobs verkregen via een rechthoekig bemonsteringschema. Het intrigerende hexagonale bemonsteringsschema wordt zelfs expliciet uitgesloten in de theorie van de convexiteitsanalyse, hoewel het zeer geschikt lijkt voor de vormanalyse van binaire blobs. Het hexagonale raster veroorzaakt namelijk bij cirkelvormige figuren minder bemonsteringsruis dan het rechthoekige raster. De winst in nauwkeurigheid bij het gebruik van het zeshoekige raster, vergeleken met dat van het rechthoekige raster, bij een gelijke bemonsteringsdichtheid, is juist opmerkelijk bij kleine isotrope blobs, zoals tumors, bloedcellen, verkalking, galstenen, etc., die veelvuldig voorkomen in biomedische beelden. Het spreekt dan ook vanzelf, dat in dit proefschrift ruime aandacht besteed wordt aan het begrip convexiteit in het hexagonale raster.

Een verzameling verbonden punten in de continue ruimte heet convex, indien het lijnstuk tussen twee willekeurige punten van de verzameling geheel binnen die verzameling valt. In het andere geval spreekt men van concaaf. Deze en soortgelijke eigenschappen van het begrip convexiteit zijn goed gedefinieerd in de Euclidische meetkunde. Maar hoe deze eigenschappen zich gedragen in de discrete ruimte, d.w.z. de ruimte waarin het meetkundige object na bemonstering voorgesteld wordt, is niet duidelijk.

Eén benadering voor digitale meetkunde, ofwel de wiskundige studie van meetkundige eigenschappen van digitale deelverzamelingen van een beeld, bestaat uit het onderzoeken of de eigenschappen van het continue convexiteitsbegrip overdraagbaar zijn naar de discrete ruimte. Een dergelijke benadering is toegepast voor het topologische begrip van connectiviteit, of beter gezegd, het probleem van de discrete topologische beschrijving van de verschillende mozaïeken, die tegenwoordig gebruikt worden in de beeldanalyse, zoals het rechthoekige en zeshoekige mozaïek. In feite houdt de digitale topologie zich bezig met concepten als nabijheid en verbondenheid, maar niet met omvang of vorm. Net als bij deze topologische concepten kan een meetkundig concept, zoals convexiteit van een meetkundig object, gedefinieerd worden op de digitale representatie als zo danig.

Een alternatieve benadering is het afbeelden van het discrete raster in de continue ruimte. Deze benadering, die in dit proefschrift in samenspel met de bovengenoemde methode gebruikt wordt, benadrukt de vertaling van
een mozaïek in een continue kromme. Men moet dan een continue
kromme of veelhoek specificeren, bijvoorbeeld de veelhoek met minimale
omtrek (dat is de veelhoek waarvan de omvang en vorm samenvallen met
die van een gespannen draad, die binnen de rand van het digitale object
moet liggen). Dan kan, wanneer een dergelijke veelhoek is gevonden, de
verwerking ervan plaatsvinden binnen het raam van de continue meet-
kunde.

In de literatuur zijn enkele definities van discrete convexiteit voorgesteld
en tamelijk uitvoerig onderzocht. Het kwam er op neer dat deze definities
nagenoeg equivalent zijn. Deze gelijkwaardigheid is echter alleen vastge-
steld voor het geval dat de objecten voorgesteld worden door een verzame-
ling cellen in een rechthoekig mozaïek of door een verzameling punten in
een rechthoekig raster. De definitie van convexiteit gehanteerd in dit
proefschrift is in feite equivalent aan de bestaande definities, de formu-
leringen en bewijzen echter zijn ook geschikt voor het hexagonale
mozaïek. In de tot nog toe gepubliceerde literatuur over de theorie van
convexiteitsanalyse wordt het hexagonale mozaïek expliciet uitgesloten.
Wegens de wezenlijke verschillen tussen de rechthoekige en zeshoekige
bemonsteringsschema’s wijken de formuleringen en bewijsvoeringen
daardoor nogal af.

Sklansky en Kibler (1976) hebben een theorie opgesteld voor de analyse
van convexiteit en verwante eigenschappen voor binaire digitale beelden,
bruikbaar voor een redelijk grote klasse van mozaïeken. Deze zogeheten
acute mozaïeken hebben met elkaar gemeen, dat elk element van het
mozaïek een convexe veelhoek of cel is, en dat de vereniging van elk paar
naburige cellen een verzameling vormt, die begrensd wordt door een con-
vexe veelhoek. De consequentie is dat het zeshoekige mozaïek niet
behoort tot deze klasse van acute mozaïeken. Daardoor kan deze
algemene theorie van Sklansky en Kibler niet gebruikt worden voor het
hexagonale mozaïek; een andere benadering is nodig.

Indien een object voorgesteld wordt door een verzameling cellen, dan

J. Sklansky en D.F. Kibler (1976)
"A Theory of Nonuniformly Digitized Binary Pictures,"
spreekt men van een celgroep. Ter vermijding van het "non-acute" zijn van het hexagonale mozaïek wordt het concept van de halve-celexpansie geïntroduceerd, dat ons in staat stelt een celgroep op het zeshoekige mozaïek te converteren naar een celgroep op het acute driehoeksmozaïek. Er kan dan, op basis van een "genererend figuur" van de celgroep, een definitie voor de convexiteit van celgroepen (ook wel cel-convexiteit genoemd) gevormd worden. Een dergelijk genererend figuur is een continue figuur of veelhoek, die de analoge continue blob of veelhoek geweest kan zijn, die de celgroep voortgebracht heeft. Aangescherpt komt het er op neer, dat cel-convexiteit gedefinieerd wordt via de convexiteit van een genererend figuur van de halve-celexpansie (op het driehoeksmozaïek) van de gegeven celgroep op het zeshoekige mozaïek. Met andere woorden, indien er voor een gegeven celgroep op het zeshoekige mozaïek een convex genererend figuur bestaat voor de halve-celexpansie (op het driehoeksmozaïek) van de gegeven celgroep, dan heet de celgroep cel-convex.

In het eerste deel van dit proefschrift worden twee belangrijke aspecten van de zeshoekige en driehoekige rasters behandeld. Eerst wordt ingegaan op de bemonsteringstheorema's, die duidelijk beschrijven hoe dicht de bemonsteringen genomen moeten worden om het beeld ondubbelzinnig weer te geven. Het hexagonale bemonsteringstheorema is niet nieuw en is reeds in de literatuur behandeld. Het wordt hier gebruikt als inleiding voor de afleiding van het driehoeks-bemonsteringstheorema. Het resultaat van die afleiding is nogal verrassend. Dit was de belangrijkste reden om het op te nemen in dit proefschrift: het driehoeks-bemonsteringstheorema is, gezien vanuit een signaaltheoretische invalshoek, een speciaal geval van het bekende rechthoekige bemonsteringstheorema.

Vervolgens wordt het begrip connectiviteit en het verwante begrip "genus" (dat is het aantal componenten verminderd met het aantal gaten) besproken en wordt een nieuw ontwikkeld algoritme voor de berekening van het genus op het driehoeksmozaïek afgeleid.

In het tweede deel wordt uitgebreid aandacht gegeven aan de verschillende definities van convexiteit en de verwante meetkundige eigenschappen. Als belangrijkste resultaat wordt een relatie vastgesteld tussen discrete con-
vexiteit enerzijds en de verschillende meetkundige eigenschappen, gedefiniëerd op convexe digitale puntenverzamelingen en op celgroepen, anderzijds, een relatie die geldig is voor zowel de rechthoekige als de zeshoekige bemonstering. Tenslotte wordt voor het hexagonale geval de relatie tussen discrete convexiteit en discrete rechtheid van liijnstukken beschreven in analogie met relaties beschreven in de Euclidische meetkunde.
I am very grateful to all those who have contributed to this thesis. Without them this thesis would never have gotten finished.

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STELLINGEN

behorende bij het proefschrift van H.P.A. Haas

Delft, 4 juni 1985
1. Het verkeerd interpreteren van het digitalisatieschema van Hodes en het vervolgens negeren van de behaalde resultaten, stelt Kim in staat een "nieuw" digitalisatieconcept te definiëren en vergelijkbare resultaten te behalen.


2. De veelhoek met minimale omtrek is niet definieerbaar als een genererend figuur voor een binair digitaal object in een niet-dubbelzinnig digitalisatieschema, zoals is aangetoond in paragraaf 4.3 van dit proefschrift.

3. Regelmatische figuren van Montanari blijken normaal te zijn voor Sklansky en Kibler.


4. De analyse in grootte en aantal van een verdeling van poederdeeltjes aan de hand van een tweedimensionale binaire afbeelding wordt aanzienlijk vergemakkelijkt door te veronderstellen dat de deeltjes een convex vorm en constante helderheid bezitten.

5. De regelmatige zeshoek als elementaire bouwvorm biedt zowel bouwtechnisch als stedebouwkundig voordelen boven een vierkant of rechthoek.

6. De uitspraak dat een leven gewijd aan het lezen van boeken, qua hoeveelheid aangeboden informatie in bits, overeenkomt met een halffuur televisie kijken, gaat voorbij aan de werkelijke kracht van de intelligentie: het vermogen te vergeten.
7. Het feit dat men enige jaren geleden ook op de universiteiten en hogescholen zeer kritisch stond tegenover multinationale ondernemingen terwijl deze tegenwoordig juist zeer gewaardeerd worden als bronnen ter financiering van het wetenschappelijk onderzoek, wijst op een sterk vermogen te vergeten.

8. Het idee is de drager van de innovatie, de techniek vervult slechts een ondersteunende rol.

9. Het gebruik van de uitdrukking "op de eerste plaats" dan wel "in de eerste plaats" berust meer op religieuze dan op dialectische gronden.

10. Het telkenmale betogen door de fiscus dat alcoholica niet tot de elementaire behoeften behoren, berust op een uiterst nuchtere visie.

1. Al staat het venster op de concertzaal verder open dan ooit tevoren, de Compact Disc brengt de concertzaal niet bij u in huis.

2. De hartelijke ontvangst van de jonge doctor door zijn omgeving na zijn (a-) sociale functioneren tijdens de totstandkoming van het proefschrift, wijst op een grote vergevingsgezindheid.