Buckling of anisotropic cylindrical shells subjected to combined axial compression, normal pressure, bending and shear loading

December 1989

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Summary

This dissertation deals with the stability behaviour of anisotropic cylindrical shells subjected to compressive stresses. It is an extension of the work done by Meyer (Ref 1). He investigated the stability behaviour of orthotropic cylinders. First an analytical derivation of the stability matrix is performed. This stability matrix will be used to calculate the critical buckling load of a perfect cylinder. Next computer programs are developed to calculate the critical buckling load. Finally, to explain the working of this software and to illustrate some problems and their solution, a numerical calculation is performed for an anisotropic cylinder.
Acknowledgement

I wish to express my gratitude towards all the members of vakgroep C, in particular towards Prof. dr. J. Arbocz for his support and constructive criticism during my research. Further I want to thank Ir. J. Hol, who was always there to answer my questions about the development of the software.
List of symbols

\(N_a\)  - buckling load for pure axial compression
\(N_b\)  - buckling load for pure bending
\(N_{bv}\)  - buckling load for pure cantelever shear
\(N_a\)  - axial component of combined buckling load
\(N_b\)  - bending component of combined buckling load
\(N_{bv}\)  - bending buckling load due to cantelever load
\(p\)  - normal external pressure

\[ N_{cl} = \frac{E_{11} t^2}{R \sqrt{3(1-v_{12}v_{21})}} \]  - (equivalent isotropic) classical buckling load

\(R\)  - radius of cylinder
\(l\)  - length of cylinder
\(t\)  - skin thickness
\(E_{11}\)  - Young's modulus in i-direction
\(v\)  - Poisson's ratio
\(x, \theta\)  - non-dimensionalized shell coordinates
\( (\cdot)_x \) - partial differentiation with respect to \( x \)

\( F = 2\pi R N_a \) - axial compressive force

\( M = \pi R^2 N_\theta \) - beam type end moment

\( V = \pi R V_0 = \frac{\pi R^2 N_{bV}}{L} \) - end shear

\( \varepsilon_x, \varepsilon_\theta, \gamma_{x\theta} \) - strains of reference surface

\( \chi_x, \chi_\theta, \chi_{x\theta} \) - curvature changes of reference surface

\( \kappa_x = -\chi_x \)

\( \kappa_\theta = -\chi_\theta \)

\( \kappa_{x\theta} = -2\chi_{x\theta} \)

\( N_x, N_\theta, N_{x\theta} \) - stress resultants

\( M_x, M_\theta, M_{x\theta} \) - stress couple resultants

\( N_x^0, N_\theta^0, N_{x\theta}^0 \) - prebuckling stress resultants

\( u_i, v_i, w_i \) - displacement functions

\[
[A] = \begin{bmatrix}
A_{11} & A_{12} & A_{13} \\
A_{12} & A_{22} & A_{23} \\
A_{13} & A_{23} & A_{33}
\end{bmatrix}
\] - extensional stiffness matrix
\[ [B] = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{12} & B_{22} & B_{23} \\ B_{13} & B_{23} & B_{33} \end{bmatrix} \quad \text{- bending stretching coupling matrix} \]

\[ [D] = \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ D_{12} & D_{22} & D_{23} \\ D_{13} & D_{23} & D_{33} \end{bmatrix} \quad \text{- flexural stiffness matrix} \]

\( h_k \)

\( \vec{Q}_{ij} \)

\( (\vec{Q}_{ij})_k \)

\[ \bar{M} = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix} \quad \text{- matrix of linear operators} \]

\[ \bar{N} = \begin{bmatrix} N_{11} & N_{12} & N_{13} \\ N_{21} & N_{22} & N_{23} \\ N_{31} & N_{32} & N_{33} \end{bmatrix} \quad \text{- matrix of buckling operators} \]

\( \lambda \)

\( \bar{L} = \bar{M} - \lambda \bar{N} \)

\( \hat{p} = \frac{p}{\lambda} \); \( \hat{N_a} = \frac{N_a}{\lambda} \); \( \hat{N_b} = \frac{N_b}{\lambda} \); \( \hat{V_0} = \frac{V_0}{\lambda} R \)

\( dA = R^2 \text{dxd}\theta \) \text{- cross sectional area of a small element dxd}\theta

\( v \)
\[ \bar{A}_{rs} = \int_A \begin{bmatrix} u_r M_{11} u_s & u_r M_{12} v_s & u_r M_{13} w_s \\ v_r M_{21} u_s & v_r M_{22} v_s & v_r M_{23} w_s \\ w_r M_{31} u_s & w_r M_{32} v_s & w_r M_{33} w_s \end{bmatrix} \]

\[ \lambda \bar{E}_{rs} = \int_A \begin{bmatrix} u_r N_{11} u_s & u_r N_{12} v_s & u_r N_{13} w_s \\ v_r N_{21} u_s & v_r N_{22} v_s & v_r N_{23} w_s \\ w_r N_{31} u_s & w_r N_{32} v_s & w_r N_{33} w_s \end{bmatrix} \]

\[ X_s = \begin{bmatrix} A_s \\ B_s \\ C_s \end{bmatrix} \]

- eigenvector components

\[ A_{mn}, A'_{mn}, B_{mn}, B'_{mn}, C_{mn}, C'_{mn} \]

- components used in the Fourier series solution

\[ C_{ji} \]

- shell coordinate function

\[ R_{ji}, S_{ji}, T_{ji}, T'_{ji}, F_{ji}, G_{ji}, K_{ji}, K'_{ji}, O_{ji}, P_{ji}, Q_{ji}, U_{ji}, V_{ji}, W_{ji}, X_{ji}, Y_{ji} \]

- integer functions defined by integrals

\[ I_1, I_2, I_3, \ldots, I_{12} \]

- relief integrals defined to determine \( R_{ji}, S_{ji}, T_{ji} \) etc.

\[ \bar{R}_{ji} = \frac{2}{\pi RL} R_{ji} \]

- non-dimensionalized integral \( R_{ji} \)
$$\bar{S}_{ji} = \frac{2}{\pi R_L} S_{ji}$$  \hspace{1cm} \text{non-dimensionalized integral } S_{ji}

etc.

$$\delta_{ij}$$  \hspace{1cm} \text{Chronecker delta : 1 for } i=j \text{ \hspace{1cm} 0 for } i=j$$

$$\lambda_m = \frac{m \pi R}{L}$$  \hspace{1cm} \text{wave parameter}

$$\lambda_q = \frac{q \pi R}{L}$$  \hspace{1cm} \text{wave parameter}

$$m, n, p, q$$  \hspace{1cm} \text{wave integers}

$$\bar{\varepsilon}_{ji}$$  \hspace{1cm} \text{element of normalized stability matrix}

$$[Z] = \begin{bmatrix}
  z_{11} & z_{12} \\
  z_{21} & z_{22} \\
  z_{31} & z_{32} \\
  z_{41} & z_{42}
\end{bmatrix}$$  \hspace{1cm} \text{coupling matrix for}\begin{bmatrix}
  A_{mn} \\
  A_{mn} \\
  B_{mn} \\
  B_{mn}
\end{bmatrix} \text{ and }\begin{bmatrix}
  C_{mn} \\
  C_{mn}
\end{bmatrix}$

$$a_x, a_\theta$$  \hspace{1cm} \text{spacing of stringers, frames}

$$b_x, b_\theta$$  \hspace{1cm} \text{thickness of stringers, frames}

$$a, b, d$$  \hspace{1cm} \text{rib spacing, width, depth for } +45^\circ$ 
  \hspace{1cm} or \hspace{1cm} $0^\circ$-$60^\circ$ stiffening

$$A_x, A_\theta$$  \hspace{1cm} \text{cross-sectional area of stringers, frames}

$$P_i$$  \hspace{1cm} \text{axial force in } i^{th} \text{ stringer}
$T_i$  - torque in $i^{th}$ stringer
$G_i J_i$  - twisting rigidity of $i^{th}$ stringer
$r_i$  - specific twist of $i^{th}$ stringer
e_i  - normal strain in stringer
e_{xx}, e_{\theta \theta}, e_{x\theta}$  - components of strain tensor
$a_i^\alpha$  - direction cosine
$I_0$  - moment of inertia of the stringer with regard to the reference surface
$l_1, l_2, e$  - excentricity terms of stringers for additional stiffness calculation
1. Introduction

One of the most common structural configurations used to transmit loads is the circular cylindrical shell. When this circular cylinder is subjected to tensile stresses, the calculation of its structural strength is rather simple and the discrepancies between analytically predicted allowable loads and test values of failure loads are small. On the other hand if the cylinder is subjected to compressive stresses it is rather difficult to predict the failure loads. This difficulty lies in the stability behavior of the cylinder under compressive load. For compressive stresses it is no longer permissible to neglect small changes in the equilibrium equations due to the deformation and, as a consequence, the governing equations are more complex and difficult to solve.

In 1972 Meyer (Ref 1) developed software to calculate the buckling load of isotropic and orthotropic cylinders subjected to compressive stresses. In the field of aerospace materials though research didn’t stop and experts came up with new materials eg. the composite materials. These are built-up as a laminated sheet, consisting of thin layers of high strength aluminum alloys and strong fibres embedded in a resin adhesive. These composite materials are nearly always anisotropic.

In order to make it possible to perform stability calculations for these anisotropic cylinders Meyer’s theoretical analysis (Ref. 1) has to be extended for anisotropic cylinders and the requisite software must be developed.
2. The buckling equations

The buckling equations used here are those derived by Fliigge (Ref 2), which assume membrane condition of prebuckling stress in the cylinder. Any prebuckling bending due to edge constraint at the ends of the cylinder, to end closures in actual usage or to the loading fixture under test conditions is neglected.

The circular cylinder is subjected to the following external forces (fig 1).

F = applied axial load
M = applied end moment
V = applied shearing load
p = uniform internal pressure

The maximum values of internal stress resultants will appear for θ=0°. The applied axial load F will yield an internal stress resultant constant with respect to x and θ.

\[ N_a = \frac{F}{2\pi R} \quad (2.1) \]

The end moment M will result in a maximum internal stress resultant constant with respect to x.

\[ N_b = \frac{M}{\pi R^2} \quad (2.2) \]

The shear load V finally yields a compressive stress for θ=0° of the form

\[ N_{bv} = V_0 x \quad x \in [0, L/R] \quad (2.3) \]

Where for x = L/R we must have

\[ N_{bv} = \frac{VL}{\pi R^2} \quad (2.4) \]

For V0 this yields

\[ V_0 = \frac{V}{\pi R} \quad (2.5) \]

Looking at the chosen sign convention (fig 2) we will end up with the following membrane prebuckling stress resultants

\[ N_x^0 = -N_a - N_b \cos \theta - V_0 x \cos \theta \quad (2.6a) \]
\[ N_\theta^0 = -pR \quad (2.6b) \]

\[ N_{\theta x}^0 = V_0 \sin \theta \quad (2.6c) \]

These stress resultants will be found to satisfy the membrane equilibrium equations for circular cylinders.

\[ N_{xx}^0 + N_{x\theta x}^0 = 0 \quad (2.7a) \]

\[ N_{\theta x x}^0 + N_{\theta \theta x}^0 = 0 \quad (2.7b) \]

By considering a bending perturbation from the membrane prebuckling condition, Flugge (Ref 2) derived the following buckling equations which are expressed in the notation of this thesis.

\[ N_{xx} + N_{x\theta x} + N_\theta(u_{xx} + w_{x}) + N_{x\theta xx} + 2N_{x\theta x}u_{x\theta} = 0 \quad (2.8a) \]

\[ N_{\theta x} + N_{x\theta x} + (-M_{x\theta x} - M_{\theta x})/R + N_{\theta}(v_{\theta x} - w_{x}) + N_{x\theta xx} + 2N_{x\theta x}(v_{\theta x} - w_{x}) = 0 \quad (2.8b) \]

\[ (M_{xx} + M_{\theta xx} + 2M_{x\theta x})/R + N_\theta + N_\theta(-u_{x} + v_{x}) + w_{\theta x} + N_{x\theta xx} + 2N_{x\theta x}(v_{x} + w_{x}) = 0 \quad (2.8c) \]

In the above equations the small differences between the terms \( N_{x\theta} \) and \( N_{\theta x} \) as well as between \( M_{x\theta} \) and \( M_{\theta x} \) have been neglected.

With the exception of the pressure terms the other terms \( N_{xx}, N_{x\theta} \) and \( N_\theta \) may be functions of the spatial coordinates \( x \) and \( \theta \). The number of dependent variables in equations (2.8a-c) exceeds the number of equations. Therefore additional relations must be introduced to properly define the problem.

For that purpose the constitutive equations found in Ref 3 will be used.
\[
\begin{bmatrix}
N_x \\
N_y \\
N_{xy} \\
M_x \\
M_y \\
M_{xy}
\end{bmatrix} =
\begin{bmatrix}
A_{11} & A_{12} & A_{13} & B_{11} & B_{12} & B_{13} \\
A_{12} & A_{22} & A_{23} & B_{12} & B_{22} & B_{23} \\
A_{13} & A_{23} & A_{33} & B_{13} & B_{23} & B_{33} \\
B_{11} & B_{12} & B_{13} & D_{11} & D_{12} & D_{13} \\
B_{12} & B_{22} & B_{23} & D_{12} & D_{22} & D_{23} \\
B_{13} & B_{23} & B_{33} & D_{13} & D_{23} & D_{33}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy} \\
\kappa_x \\
\kappa_y \\
\kappa_{xy}
\end{bmatrix}
\]  

(2.9)

where

\[
A_{ij} = \sum_{k=1}^{n} (\overline{Q}_{ij})_k (h_k - h_{k-1})
\]  

(2.10a)

\[
B_{ij} = \frac{1}{2} \sum_{k=1}^{n} (\overline{Q}_{ij})_k (h_k^2 - h_{k-1}^2)
\]  

(2.10b)

\[
D_{ij} = \frac{1}{3} \sum_{k=1}^{n} (\overline{Q}_{ij})_k (h_k^3 - h_{k-1}^3)
\]  

(2.10c)

In these equations \(\overline{Q}_k\) is the stiffness matrix of each layer \(k\).

We must remember that, because of our sign convention the layers are numbered from the outer surface inward (fig 3). For rib stiffened shells, additional stiffness terms must be introduced in the stiffness matrix. Three kinds of stiffening can be considered namely 0°-90° rib stiffening (rings and stringers), ±45° rib stiffening and 0°±60° rib stiffening (isogrid). The calculation of the additional stiffness terms for these three kinds of stiffening is dealt with in appendix A.

Together with the strain-displacement relations (2.11a-b) of Love-Reisner (Ref 4) the number of dependent variables in the three equations may be reduced to the three components of the displacement vector \((u,v,w)\).

\[
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_\theta \\
\gamma_{x\theta}
\end{bmatrix} =
\begin{bmatrix}
u'_x \\
v'_\theta \\
v'_x + u'_\theta
\end{bmatrix}
\]  

(2.11a)
\[
\begin{bmatrix}
x_x \\
x_\theta \\
x_{x\theta}
\end{bmatrix}
= \frac{1}{R}
\begin{bmatrix}
-w_{,xx} \\
-w_{,\theta\theta} - v_{,\theta} \\
-2w_{,\theta\theta} - v_{,x}
\end{bmatrix}
\] (2.11b)

Since these partial differential equations (2.11a-b) are linear, it is convenient to define them in terms of the operator \( M \) of the linear shell and the operator \( N \) of the buckling terms.

Thus

\[
\begin{bmatrix}
u \\
v \\
w
\end{bmatrix}
- \lambda
\begin{bmatrix}
u \\
v \\
w
\end{bmatrix}
= \{0\}
\] (2.12)

where

\[
M \quad \overset{\text{def}}{=} \begin{bmatrix}
M_{11} & M_{12} & M_{13} \\
M_{21} & M_{22} & M_{23} \\
M_{31} & M_{32} & M_{33}
\end{bmatrix}
\] (2.13a)

\[
N \quad \overset{\text{def}}{=} \begin{bmatrix}
N_{11} & N_{12} & N_{13} \\
N_{21} & N_{22} & N_{23} \\
N_{31} & N_{32} & N_{33}
\end{bmatrix}
\] (2.13b)

and \( \lambda \) is the buckling parameter.

The components of the \( M \)-operator acting on \((u,v,w)\) are

\[
\begin{bmatrix}
M_{11}u \\
M_{12}v \\
M_{13}w
\end{bmatrix}
= \begin{bmatrix}
A_{11}u_{,xx} + 2A_{13}u_{,x\theta} + A_{33}u_{,\theta\theta} \\
(A_{13} - B_{13}/R)v_{,xx} + (A_{12} - B_{12}/R + A_{33} - B_{33}/R)v_{,x\theta} \\
-A_{12}w_{,x} - A_{23}w_{,\theta} - B_{11}w_{,xxx}/R - 3B_{13}w_{,xx\theta}/R \\
- (B_{12}/R + 2B_{33}/R)w_{,x\theta} - B_{23}w_{,\theta\theta}/R
\end{bmatrix}
\] (2.14a)
\[
\begin{bmatrix}
M_{21}^u \\
M_{22}^v \\
M_{23}^w
\end{bmatrix}
= \begin{bmatrix}
(A_{13} - B_{13}/R)u,_{xx} + (A_{12} + A_{33} - B_{12}/R - B_{33}/R)u,_{x\theta} + (A_{23} - B_{23}/R)u,_{\theta\theta} \\
(A_{33} - 2B_{23}/R - D_{33}/R^2)v,_{xx} + (2A_{23} - 4B_{23}/R + 2D_{23}/R^2)v,_{\theta\theta} \\
(-A_{23} + B_{23}/R)w,_{xx} + (-A_{22} + B_{22}/R)w,_{\theta\theta} + (-B_{13}/R + 2D_{13}/R^2)w,_{\theta\theta} \\
+ (2D_{23}/R^2)w,_{x\theta} + (3B_{23}/R + 3D_{23}/R^2)w,_{x\theta\theta} + (-B_{22}/R + D_{22}/R^2)w,_{\theta\theta\theta}
\end{bmatrix}
\]

(2.14b)

\[
\begin{bmatrix}
M_{31}^u \\
M_{32}^v \\
M_{33}^w
\end{bmatrix}
= \begin{bmatrix}
A_{12}u,_{x} + A_{23}u,_{\theta} + B_{11}u,_{xx}/R + 3B_{13}u,_{xx\theta}/R + (B_{12}/R + 2B_{33}/R)u,_{x\theta\theta} + B_{23}/Ru,_{\theta\theta\theta} \\
(A_{23} - B_{23}/R)v,_{x} + (A_{22} - B_{22}/R)v,_{\theta} + (-D_{13}/R^2 + B_{12}/R - D_{12}/R^2 + 2B_{33}/R)w,_{x\theta\theta} \\
-A_{22}w,_{xx}/R - 2B_{12}w,_{xx\theta}/R - 2B_{22}w,_{\theta\theta}/R - 4B_{23}w,_{x\theta}/R - D_{11}w,_{xx\theta\theta}/R - 4D_{13}w,_{xx\theta\theta\theta}/R - (2D_{12}/R^2 + 4D_{33}/R^2)w,_{x\theta\theta\theta}
\end{bmatrix}
\]

(2.14c)

In like manner, the components of the buckling operator with the buckling parameter, \(\lambda N\), acting on the displacement vector \((u, v, w)\) are

\[
\begin{bmatrix}
N_{11}^u \\
N_{12}^v \\
N_{13}^w
\end{bmatrix}
= \begin{bmatrix}
\hat{\delta}Ru,_{\theta\theta} + (\hat{N}_a + \hat{N}_b \cos \theta)u,_{xx} + \hat{V}_0 x \cos \theta u,_{xx} \\
0 \\
\hat{\delta}Rw,_{x}
\end{bmatrix}
\]

(2.15a)
\[
\begin{bmatrix}
N_{21}u \\
N_{22}v \\
N_{23}w
\end{bmatrix}
= \lambda
\begin{bmatrix}
0 \\
\hat{p}Rv,_{\theta} + (\hat{N}_a + \hat{N}_b \cos\theta)v,_{xx} + \hat{v}_0 \cos\theta v,_{xx} \\
-\hat{p}Rw,_{\theta} + 2\hat{v}_0 \sin\theta w,_{x} \\
-2\hat{v}_0 \sin\theta w,_{x}\theta
\end{bmatrix}
\tag{2.15b}
\]

\[
\begin{bmatrix}
N_{31}u \\
N_{32}v \\
N_{33}w
\end{bmatrix}
= \lambda
\begin{bmatrix}
-\hat{p}Rw,_{x} \\
\hat{p}Rv,_{\theta} - 2\hat{v}_0 \sin\theta v,_{x} \\
\hat{p}Rw,_{\theta} + (\hat{N}_a + \hat{N}_b \cos\theta)w,_{xx} + \hat{v}_0 \cos\theta w,_{xx} \\
-2\hat{v}_0 \sin\theta w,_{x}\theta
\end{bmatrix}
\tag{2.15c}
\]

where \(\lambda > 0\) and

\[
\hat{p}R = pR/\lambda ; \hat{N}_a = N_a/\lambda ; \hat{N}_b = N_b/\lambda ; \hat{v}_0 = V_0/\lambda
\]

such that \(p : N_a : N_b : V_0\) are in a preassigned ratio.

One now inquires whether the equation

\[
[M] \begin{bmatrix} u \\ v \\ w \end{bmatrix} - \lambda [N] \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\tag{2.16}
\]

possesses a solution for non-trivial values of \(\lambda\). Considered in this way the solutions \(\lambda\) are called eigenvalues and the corresponding mode shapes \((u, v, w)\) are called eigenfunctions.

Alternate formulations of the problem may be obtained if one considers some of the loads \(p, N_a, N_b\) and \(V_0\) as given and then determines the remaining load or loads by equation (2.16). In the formulation adopted in this thesis three of the loads must be assumed as given e.g. \(p, N_b\) and \(V_0\). In this case these three loads are considered to belong to \(M\). \(N\) only contains the \(N_a\)-terms. Now the critical load \(N_a\) corresponding with the three given loads can be determined.
3. The Galerkin formulation of solution

Since the N-operator has variable coefficients, it is not possible to find a set of elementary functions which satisfy the differential equations. Instead, one is forced to employ infinite series of \( u, v \) and \( w \). If these series do not satisfy the differential equations, conditions for an approximate solution with a finite number of terms may be conveniently obtained by the method of Galerkin. If, in addition, these series are mathematically complete and convergent, the approximate solutions approach the exact solution in the limit as the number of terms approach infinity.

The variational formulation and conditions for Galerkin’s method in the non-linear theory is given in Ref 5. Assuming this the development proceeds as follows.

\[
\left[ I \right] = \left[ M \right] - \lambda \left[ N \right] \tag{3.1}
\]

A variational procedure yields

\[
\int_A \begin{bmatrix} \delta u & 0 & 0 \\ 0 & \delta v & 0 \\ 0 & 0 & \delta w \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} \cdot dA = 0 \tag{3.2}
\]

where

\[
dA = R^2 dx d\theta \tag{3.3}
\]

Let \( u, v \) and \( w \) be expanded in series

\[
\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \sum_{i=1}^{\infty} \begin{bmatrix} A_i u_i(x, \theta) \\ B_i v_i(x, \theta) \\ C_i w_i(x, \theta) \end{bmatrix} \tag{3.4}
\]

Substitution of (3.4) in equation (3.2) yields
\[
\int \begin{bmatrix}
\delta u & 0 & 0 \\
0 & \delta v & 0 \\
0 & 0 & \delta w
\end{bmatrix} \begin{bmatrix}
A_i u_i \\
B_i v_i \\
C_i w_i
\end{bmatrix} dA = \{0\}
\] (3.5)

Since \( L \) is linear we get
\[
\int \sum_{i=1}^{\infty} \begin{bmatrix}
\delta u & 0 & 0 \\
0 & \delta v & 0 \\
0 & 0 & \delta w
\end{bmatrix} \begin{bmatrix}
A_i u_i \\
B_i v_i \\
C_i w_i
\end{bmatrix} dA = \{0\}
\] (3.6)

The operations \( \int \) and \( \Sigma \) do commute. If the series are uniformly convergent this condition will certainly be satisfied. The variations of \( u, v \) and \( w \) are
\[
\begin{bmatrix}
\delta u \\
\delta v \\
\delta w
\end{bmatrix} = \sum_{j=1}^{\infty} \begin{bmatrix}
\delta (A_j u_j) \\
\delta (B_j v_j) \\
\delta (C_j w_j)
\end{bmatrix} = \sum_{j=1}^{\infty} \begin{bmatrix}
u_j \delta A_j \\
v_j \delta B_j \\
w_j \delta C_j
\end{bmatrix}
\] (3.7)

Since these are linearly independent, the coefficients of \( \delta A_j, \delta B_j \) and \( \delta C_j \) in equation (3.6) must vanish separately. This gives \( j \) sets of series
\[
\sum_{i=1}^{\infty} \int \begin{bmatrix}
u_j & 0 & 0 \\
\delta v & 0 \\
0 & 0 \end{bmatrix} \begin{bmatrix}
A_i u_i \\
B_i v_i \\
C_i w_i
\end{bmatrix} dA = \{0\}
\] (3.8)

The series of equation 3.8 will be truncated at
\[i, j = N\]

For our problem the single series (3.4) is actually a double series.
\[
\begin{bmatrix}
u \\ v \\ w \end{bmatrix} = \left( \begin{array}{cc}
M_1 & N_1 \\
\Sigma & \Sigma \\
m=1 & n=0
\end{array} \right) \begin{bmatrix}
A_{mn} u_{mn}(x, \theta) \\
B_{mn} v_{mn}(x, \theta) \\
C_{mn} w_{mn}(x, \theta)
\end{bmatrix}
\]  

(3.9)

In order to cast these series into the form of equation (3.8) it is necessary to reorder the double series (3.9) as a single series (3.4). This operation will not affect the convergence if the series are absolutely convergent. It may be accomplished by partial row, column or diagonal summation. Working out equation (3.8) yields

\[
\sum_{i=1}^{N_A} \int_{A} \begin{bmatrix}
u_{ij} & 0 & 0 \\
0 & v_{ij} & 0 \\
0 & 0 & w_{ij} \end{bmatrix} \begin{bmatrix}
A_i u_i \\
B_i v_i \\
C_i w_i
\end{bmatrix} \, dA = \]

\[
\sum_{j=1,2,3,\ldots,N} \int_{A} \begin{bmatrix}
u_{ij}L_{11} u_i A_i + u_{ij}L_{12} v_i B_i + u_{ij}L_{13} w_i C_i \\
v_{ij}L_{21} u_i A_i + v_{ij}L_{22} v_i B_i + v_{ij}L_{23} w_i C_i \\
w_{ij}L_{31} u_i A_i + w_{ij}L_{32} v_i B_i + w_{ij}L_{33} w_i C_i
\end{bmatrix} \, dA = \{0\}
\]

(3.10)

Now define the 3x3 algebraic submatrices

\[
\overline{A}_{rs} = \int_{A} \begin{bmatrix}
u_{r11} u_s & u_{r12} v_s & u_{r13} w_s \\
v_{r21} u_s & v_{r22} v_s & v_{r23} w_s \\
w_{r31} u_s & w_{r32} v_s & w_{r33} w_s
\end{bmatrix} \, dA
\]

(3.11a)

\[
\overline{B}_{rs} = \int_{A} \begin{bmatrix}
u_{r11} u_s & u_{r12} v_s & u_{r13} w_s \\
v_{r21} u_s & v_{r22} v_s & v_{r23} w_s \\
w_{r31} u_s & w_{r32} v_s & w_{r33} w_s
\end{bmatrix} \, dA
\]

(3.11b)
and the eigenvector

\[
X_s = \begin{bmatrix}
A_s \\
B_s \\
C_s
\end{bmatrix}
\]

(3.12)

With these submatrices (3.11a-b) and eigenvector (3.12) equation (3.10) becomes

\[
\sum_{s=1}^{N} \left( \begin{array}{c}
\lambda \bar{A}_{rs} \\
-\lambda \bar{B}_{rs}
\end{array} \right) X_s = \{0\}
\]

\[r = 1, 2, 3, \ldots, N\]

(3.13)

The solution of the problem has now been reduced to the algebraic homogeneous equations

\[
\begin{bmatrix}
A_{11} - \lambda B_{11} & A_{12} - \lambda B_{12} & \cdots & A_{1N} - \lambda B_{1N} \\
A_{21} - \lambda B_{21} & A_{22} - \lambda B_{22} & \cdots & A_{2N} - \lambda B_{2N} \\
& \ddots & \ddots & \ddots \\
& & & & \ddots \\
A_{N1} - \lambda B_{N1} & A_{N2} - \lambda B_{N2} & \cdots & A_{NN} - \lambda B_{NN}
\end{bmatrix}
\begin{bmatrix}
X_1 \\
X_2 \\
\vdots \\
X_N
\end{bmatrix} = \{0\}
\]

(3.14)

Symbolically

\[
\bar{A}X - \lambda \bar{B}X = 0
\]

(3.15)

The conditions for a non-trivial solution of equation (3.12) to exist is that

\[
\det (\bar{A} - \lambda \bar{B}) = 0
\]

(3.16)

A physical interpretation of $\bar{A} - \lambda \bar{B}$ may be made as the system rigidities in terms of the coordinate functions $\{u, v, w\}$. For real values of $\lambda$ the smallest one defines the buckling load in terms of the allowed degrees of freedom. When the load increases from zero and reaches the value $\lambda_{\min}$ the displacements tend to grow without bound. One says that the system has "buckled". As the degrees of freedom are increased the sequence of values of $\lambda_{\min}(N)$ decreases monotonically, since the constraints enforcing
\[
\begin{bmatrix}
  u_i \\
  v_i \\
  w_i 
\end{bmatrix}
= \begin{bmatrix} 0 \end{bmatrix} \quad \text{for } i > N
\]  

(3.17)

are being relaxed. Mathematically speaking, the space of allowable functions defining \( \lambda_{\text{min}} \) is being enlarged so that smaller minimums are possible.

For complete sets of functions and convergent series we get

\[
\lim_{N \rightarrow \infty} \lambda_{\text{min}}(N) = \lambda_{\text{cr}}
\]  

(3.18)

where \( \lambda_{\text{cr}} \) defines the physical buckling load of a perfect cylinder. If we found \( \lambda_{\text{cr}} \) we also can find the eigenvector.

\[
X(\lambda_{\text{cr}}) = \begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  \vdots \\
  x_N 
\end{bmatrix}, \quad X_i(\lambda_{\text{cr}}) = \begin{bmatrix}
  A_i(\lambda_{\text{cr}}) \\
  B_i(\lambda_{\text{cr}}) \\
  C_i(\lambda_{\text{cr}}) 
\end{bmatrix}
\]  

(3.19)

which is obtained from equation (3.14) by solving for \( \lambda = \lambda_{\text{cr}} \). The functions

\[
\begin{bmatrix}
  u(\lambda_{\text{cr}}) \\
  v(\lambda_{\text{cr}}) \\
  w(\lambda_{\text{cr}}) 
\end{bmatrix}
= \sum_{i=1}^{\infty}
\begin{bmatrix}
  A_i(\lambda_{\text{cr}})u_i \\
  B_i(\lambda_{\text{cr}})v_i \\
  C_i(\lambda_{\text{cr}})w_i 
\end{bmatrix}
\]  

(3.20)

are the eigenfunctions of the problem. They describe the buckled shape of the cylinder and are defined up to a constant multiplier since equation (3.15) is homogeneous.
4. Solution of the buckling equations

Next the Fourier series solution functions have to be introduced. For the orthotropic case Meyer found (Ref 1)

\[
    u = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} A_{mn} \cos \theta \cos \lambda_m x \\
    v = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} B_{mn} \sin \theta \sin \lambda_m x \\
    w = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} C_{mn} \cos \theta \sin \lambda_m x
\]

\[\lambda_m = \frac{m \pi R}{L}\]

(4.1)

For the isotropic case there are either combinations of even or combinations of uneven derivatives in the separate elements of the \(\bar{M}\) and \(\bar{N}\)-operator e.g. \(u_{xx}\) and \(u_{\theta \theta}\) (even), \(w_{\theta}\) and \(w_{x \theta}\) (uneven) etc. Therefore with these Fourier series solution functions (4.1) the number of dependent variables will be equal to the number of equations. For the anisotropic case though we also have to deal with combinations of even and uneven derivatives e.g. \(u_{xx}\) and \(u_{x \theta}\) (eq. 2.14a). The solution functions of equation (4.1) now will yield two times as many equations as independent variables. This forces us to add other Fourier series to these solution functions in order to double the number of variables. This yields

\[
    u = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} A_{mn} \sin \theta \sin \lambda_m x + \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} A'_{mn} \cos \theta \cos \lambda_m x \\
    v = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} B_{mn} \sin \theta \sin \lambda_m x + \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} B'_{mn} \cos \theta \cos \lambda_m x \\
    w = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} C_{mn} \cos \theta \sin \lambda_m x + \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} C'_{mn} \sin \theta \cos \lambda_m x
\]

\[\lambda_m = \frac{m \pi R}{L}\]

(4.2)

These functions are complete in the \(x\) and \(\theta\) coordinates over the fundamental domain
\[ 0 \leq x \leq L/R \quad , \quad -\pi \leq \theta \leq \pi \] (4.3)

The only disadvantage of these functions is that they don't satisfy the homogeneous boundary conditions for the additional loads and displacements due to buckling.

\[
N_x(0) = N_x(L/R) = 0 \\
M_x(0) = M_x(L/R) = 0 \\
v(0) = v(L/R) = 0 \\
w(0) = w(L/R) = 0
\] (4.4)

This will cause some trouble, especially for the case where shear forces are involved. A further discussion of this problem will be described in chapter 8.

4.1 Elements of the M-operator

The next thing to do is to substitute the functions (4.2) into the M-operator (2.13-15). First introduce the shell coordinate function stiffnesses \( C_{ij} \) are introduced:

\[
C_{11} = -A_{11}\lambda_m^2 - A_{33}n^2 \\
C_{12} = 2A_{13}n\lambda_m \\
C_{13} = (-A_{13} + B_{13}/R)\lambda_m^2 + (-A_{23} + B_{23}/R)n^2 \\
C_{14} = (A_{12} - B_{12}/R + A_{33} - B_{33}/R)\lambda_mn \\
C_{15} = A_{23}n - 3B_{13}\lambda_m^2n/R - B_{23}n^3/R \\
C_{16} = A_{12}\lambda_m - B_{11}\lambda_m^3/R - (B_{12}/R + 2B_{33}/R)\lambda_mn^2 \\
C_{31} = (-A_{13} + B_{13}/R)\lambda_m^2 + (-A_{23} + B_{23}/R)n^2 \\
C_{32} = (A_{12} + A_{33} - B_{12}/R - B_{33}/R)\lambda_mn \\
C_{33} = (-A_{33} - D_{33}/R^2 + 2B_{33}/R)\lambda_m^2 + (-A_{22} + 2B_{22}/R - D_{22}/R^2)n^2
\]
\[ C_{34} = (2A_{23} - 4B_{23}/R + 2D_{23}/R^2)\lambda_m n \]

\[ C_{35} = (A_{22} - B_{22}/R)n + (-B_{12}/R - 2B_{33}/R + 2D_{33}/R^2 + D_{12}/R^2)\lambda_m^2 n \]
\[ + (-B(2,2)/R + D(2,2)/R^2)n^3 \]

\[ C_{36} = (A_{23} - B_{23}/R)\lambda_m + (-B_{13}/R + D_{13}/R^2)\lambda_m^3 \]
\[ + (-3B_{23}/R + 3D_{23}/R^2)\lambda_m n^2 \]

\[ C_{51} = A_{23}n - 3B_{13}\lambda_m^2 n/R - B_{23}n^3/R \]

\[ C_{52} = -A_{12}\lambda_m^2 + B_{11}\lambda_m^3 + (B_{12}/R + 2B_{33}/R)\lambda_m n^2 \]

\[ C_{53} = (A_{22} - B_{22}/R)n + (-B_{12}/R + D_{12}/R^2 - 2B_{33}/R + 2D_{33}/R^2)\lambda_m^2 n \]
\[ + (-B_{22}/R + D_{22}/R^2)n^3 \]

\[ C_{54} = (-A_{23} + B_{23}/R)\lambda_m + (B_{13}/R - D_{13}/R^2)\lambda_m^3 \]
\[ + (3B_{23}/R - 3D_{23}/R^2)\lambda_m n^2 \]

\[ C_{55} = -A_{22} + 2B_{12}\lambda_m^2/R + 2B_{22}n^2/R - D_{11}\lambda_m^4/R^2 \]
\[ + (-2D_{12}/R^2 - 4D_{33}/R^2)\lambda_m^2 n^2 - D_{22}n^4/R^2 \]

\[ C_{56} = 4B_{23}\lambda_m n/R - 4D_{13}\lambda_m^3 n/R^2 - 4D_{23}\lambda_m^2 n^3/R^2 \]

(4.1.1)

Now with the use of these shell coordinate function stinesses the M-operator can be determined.

\[
\begin{bmatrix}
M_{11}^u_i \\
M_{12}^v_i \\
M_{13}^w_i \\
\end{bmatrix} = \begin{bmatrix}
C_{11}^A_{mn} + C_{12}^{A'}_{mn} \\
C_{13}^B_{mn} + C_{14}^{B'}_{mn} \\
C_{15}^C_{mn} + C_{16}^{C'}_{mn} \\
\end{bmatrix} - \sin \lambda_m x \sin \theta + \begin{bmatrix}
C_{12}^A_{mn} + C_{11}^A_{mn} \\
C_{14}^B_{mn} + C_{13}^B_{mn} \\
-C_{16}^C_{mn} - C_{15}^C_{mn} \\
\end{bmatrix} \cos \lambda_m x \cos \theta
\]

(4.1.2a)

15
\[
\begin{bmatrix}
M_{11}^{u_1} \\
M_{22}^{v_1} \\
M_{33}^{w_1}
\end{bmatrix} = 
\begin{bmatrix}
C_{31}A_{mn} + C_{32}A'_{mn} \\
C_{33}B_{mn} + C_{34}B'_{mn} \\
C_{35}C_{mn} + C_{36}C'_{mn}
\end{bmatrix} 
- \sin\lambda_m x \sin\theta + 
\begin{bmatrix}
C_{32}A_{mn} + C_{31}A'_{mn} \\
C_{34}B_{mn} + C_{33}B'_{mn} \\
-C_{36}C_{mn} - C_{35}C'_{mn}
\end{bmatrix} 
\cos\lambda_m x \cos\theta
\]  
(4.1.2b)

\[
\begin{bmatrix}
M_{31}^{u_1} \\
M_{32}^{v_1} \\
M_{33}^{w_1}
\end{bmatrix} = 
\begin{bmatrix}
C_{51}A_{mn} + C_{52}A'_{mn} \\
C_{53}B_{mn} + C_{54}B'_{mn} \\
C_{55}C_{mn} + C_{56}C'_{mn}
\end{bmatrix} 
- \sin\lambda_m x \cos\theta + 
\begin{bmatrix}
-C_{52}A_{mn} - C_{51}A'_{mn} \\
-C_{54}B_{mn} - C_{53}B'_{mn} \\
C_{56}C_{mn} + C_{55}C'_{mn}
\end{bmatrix} 
\cos\lambda_m x \sin\theta
\]  
(4.1.2c)

For convenience of further development in the Galerkin process, the following functions of the integers \((m,n;p,q)\) will be defined.

\[
R_{ji} = \int_A (\cos\theta \cos\theta \cos\lambda_m x \cos\lambda_q x) \, dA
\]

\[
S_{ji} = \int_A (\sin\theta \sin\theta \sin\lambda_m x \sin\lambda_q x) \, dA
\]

\[
T_{ji} = \int_A (\cos\theta \cos\theta \sin\lambda_m x \sin\lambda_q x) \, dA
\]

\[
T'_{ji} = \int_A (\sin\theta \sin\theta \cos\lambda_m x \cos\lambda_q x) \, dA
\]  
(4.1.3)

These functions are evaluated in appendix B. Rewriting equations (4.1.2a-c) yields

\[
M_{11}^{u_1} = M_{11}^{u_1} + M_{11}^{u_2}
\]

\[
M_{12}^{v_1} = M_{12}^{v_1} + M_{12}^{v_2}
\]

\[
M_{13}^{w_1} = M_{13}^{w_1} + M_{13}^{w_2}
\]

\[
M_{21}^{u_1} = M_{21}^{u_1} + M_{21}^{u_2}
\]

\[
M_{22}^{v_1} = M_{22}^{v_1} + M_{22}^{v_2}
\]

\[
M_{22}^{v_1} = M_{22}^{v_1} + M_{22}^{v_2} \quad \text{etc.}
\]  
(4.1.4)
where

\[
\begin{align*}
M_{11}u_1 &= A_{mn} \left( C_{11}\sin_m x\sin\theta + C_{12}\cos_m x\cos\theta \right) \\
M_{11}u_2 &= A'_{mn} \left( C_{12}\sin_m x\sin\theta + C_{11}\cos_m x\cos\theta \right) \\
M_{12}v_1 &= B_{mn} \left( C_{13}\sin_m x\sin\theta + C_{14}\cos_m x\cos\theta \right) \\
M_{12}v_2 &= B'_{mn} \left( C_{14}\sin_m x\sin\theta + C_{13}\cos_m x\cos\theta \right) \\
M_{13}w_1 &= C_{mn} \left( C_{15}\sin_m x\sin\theta - C_{16}\cos_m x\cos\theta \right) \\
M_{13}w_1 &= C'_{mn} \left( C_{16}\sin_m x\sin\theta - C_{15}\cos_m x\cos\theta \right) \\
M_{21}u_1 &= A_{mn} \left( C_{31}\sin_m x\sin\theta + C_{32}\cos_m x\cos\theta \right) \\
M_{21}u_2 &= A'_{mn} \left( C_{32}\sin_m x\sin\theta + C_{31}\cos_m x\cos\theta \right) \\
M_{22}v_1 &= B_{mn} \left( C_{33}\sin_m x\sin\theta + C_{34}\cos_m x\cos\theta \right) \\
M_{22}v_2 &= B'_{mn} \left( C_{34}\sin_m x\sin\theta + C_{33}\cos_m x\cos\theta \right) \\
\end{align*}
\] etc. \hspace{1cm} (4.1.5)

Splitting up the terms \( u_j, v_j \) and \( w_j \) yields

\[
\begin{align*}
u_j &= u'_1 + u'_2 \\
v_j &= v'_1 + v'_2 \\
w_j &= w'_1 + w'_2 \\
\end{align*}
\] \hspace{1cm} (4.1.6)

where

\[
\begin{align*}
u'_1 &= \sin_q x\sin\theta \\
u'_2 &= \cos_q x\cos\theta \\
v'_1 &= \sin_q x\sin\theta \\
\end{align*}
\]
\[ v_2' = \cos\theta_x \cos\varphi \]
\[ w_1' = \sin\theta_x \cos\theta \]
\[ w_2' = \cos\theta_x \sin\varphi \]  
(4.1.7)

Using these definitions matrix \( \overline{A}_{ji} \) can be defined as follows.

\[
\overline{A}_{ji} = \int_A \begin{bmatrix}
u'_1 M_{11} u_1' & u'_1 M_{12} v_1' & u'_1 M_{13} w_1' & u'_1 M_{14} d' \\
u'_2 M_{11} u_1' & u'_2 M_{12} v_1' & u'_2 M_{13} w_1' & u'_2 M_{14} d' \\
u_1' M_{21} v_1' & v_1' M_{22} v_1' & v_1' M_{23} w_1' & v_1' M_{24} d' \\
u_2' M_{21} v_1' & v_2' M_{22} v_1' & v_2' M_{23} w_1' & v_2' M_{24} d'
\end{bmatrix} dA
\]  
(4.1.8)

We now get for some of the elements of \( \overline{A}_{ji} \)

\[
A_{11} = \int_A u_1' M_{11} u_1' dA = \int_A u_m' \sin \theta_x \sin \psi \sin \theta_m \sin \varphi \sin \theta m dA
\]

\[
= A_m^m \int_A \sin \psi \sin \psi \sin \theta_m \sin \varphi \sin \theta m dA
\]

\[
= A_m^m C_{11} S_{ji}
\]

\[
A_{12} = \int_A u_1' M_{11} u_2' dA = \int_A u_1' \sin \phi \sin \theta_m \sin \varphi \sin \theta m dA
\]

\[
= A_m^m C_{12} \int_A \sin \psi \sin \psi \sin \theta_m \sin \varphi \sin \theta m dA
\]

\[
= A_m^m C_{12} S_{ji}
\]

\[
A_{15} = \int_A u_1' M_{13} w_1' dA = \int_A \cos \theta \sin \theta_m \sin \varphi \sin \theta m dA
\]

\[
= A_m^m C_{15} \int_A \sin \psi \sin \psi \sin \theta_m \sin \varphi \sin \theta m dA
\]

\[
= A_m^m C_{15} S_{ji}
\]
\[ A_{16} = \int_{A} u'_{1} M_{2} w_{2} dA = \int_{A} \sum_{m}^{n} \sin \lambda \sin \phi (C_{16} \sin \lambda \sin \theta - C_{15} \cos \lambda \cos \phi) dA \]
\[ = c'_{mn} \int_{A} \sin \theta \sin \phi \sin \lambda \sin \lambda \cos \lambda \cos \lambda \cos \lambda \cos \phi \cos \lambda \cos \lambda \cos \lambda \cos \phi dA \]
\[ = c'_{mn} c_{15} \int_{A} \sin \theta \sin \phi \sin \lambda \sin \lambda \cos \lambda \cos \lambda \cos \lambda \cos \phi \cos \lambda \cos \lambda \cos \lambda \cos \phi dA \]
\[ = c'_{mn} c_{15} s_{ji} \]
\[ A_{25} = \int_{A} u'_{2} M_{1} w_{1} dA = \int_{A} \sum_{m}^{n} \sin \lambda \cos \phi (C_{15} \sin \lambda \sin \theta - C_{16} \cos \lambda \cos \phi) dA \]
\[ = c'_{mn} c_{15} \int_{A} \sin \theta \cos \phi \sin \lambda \sin \lambda \cos \lambda \cos \lambda \cos \lambda \cos \phi \cos \lambda \cos \lambda \cos \lambda \cos \phi dA \]
\[ = c'_{mn} c_{15} r_{ji} \]
\[ A_{26} = \int_{A} u'_{2} M_{1} w_{2} dA = \int_{A} \sum_{m}^{n} \sin \lambda \cos \phi (C_{16} \sin \lambda \sin \theta - C_{15} \cos \lambda \cos \phi) dA \]
\[ = c'_{mn} c_{15} \int_{A} \sin \theta \cos \phi \sin \lambda \sin \lambda \cos \lambda \cos \lambda \cos \lambda \cos \phi \cos \lambda \cos \lambda \cos \lambda \cos \phi dA \]
\[ = c'_{mn} c_{15} r_{ji} \]
\[ A_{55} = \int_{A} w'_{1} M_{2} w_{2} dA = \int_{A} \sum_{m}^{n} \sin \lambda \cos \phi (C_{55} \sin \lambda \cos \theta + C_{56} \cos \lambda \sin \theta) dA \]
\[ = c'_{mn} c_{55} \int_{A} \cos \theta \cos \phi \sin \lambda \sin \lambda \cos \lambda \cos \lambda \cos \lambda \cos \phi \cos \lambda \cos \lambda \cos \lambda \cos \phi dA \]
\[ = c'_{mn} c_{55} t_{ji} \]
\[ A_{56} = \int_{A} w'_{2} M_{1} w_{2} dA = \int_{A} \sum_{m}^{n} \sin \lambda \cos \phi (C_{56} \sin \lambda \cos \theta + C_{55} \cos \lambda \sin \theta) dA \]
\[ = c'_{mn} c_{56} \int_{A} \sin \phi \cos \phi \sin \lambda \sin \lambda \cos \lambda \cos \lambda \cos \lambda \cos \phi \cos \lambda \cos \lambda \cos \lambda \cos \phi dA \]
\[ = c'_{mn} c_{56} t_{ji} \]
\[ A_{65} = \int_{A} w'_{2} M_{1} w_{1} dA = \int_{A} \sum_{m}^{n} \cos \lambda \sin \phi (C_{55} \sin \lambda \cos \theta + C_{56} \cos \lambda \sin \theta) dA \]
\[ = c'_{mn} c_{55} \int_{A} \cos \theta \sin \phi \sin \lambda \sin \lambda \cos \lambda \cos \lambda \cos \lambda \cos \phi \cos \lambda \cos \lambda \cos \lambda \cos \phi dA \]
\[ = c'_{mn} c_{56} t_{ji} \]
\[ A_{66} = \int w_2 N_{13} w_2 dA = \int_{A} C_{mn} \cos \theta \sin \theta (C_{56} \sin \lambda x \cos \theta + C_{55} \cos \lambda x \sin \theta) dA \]

\[ = C'_{mn} C_{56} \int \cos \theta \sin \theta \sin \lambda x \cos \lambda x dA + C_{mn} C_{55} \int \sin \theta \sin \theta \cos \lambda x \cos \lambda x dA \]

\[ = C'_{mn} C_{55} T'_{ji} \quad (4.1.9) \]

Finally for \( \bar{A}_{ji} \), we get

\[
\bar{A}_{ji} = \begin{bmatrix}
S_{ji} & 0 & 0 & 0 & 0 \\
0 & R_{ji} & 0 & 0 & 0 \\
0 & 0 & S_{ji} & 0 & 0 \\
0 & 0 & 0 & R_{ji} & 0 \\
0 & 0 & 0 & 0 & T_{ji}
\end{bmatrix} \begin{bmatrix}
C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\
C_{12} & C_{11} & C_{14} & C_{13} & -C_{16} & -C_{15} \\
C_{13} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\
C_{14} & C_{31} & C_{34} & C_{33} & -C_{36} & -C_{35} \\
C_{15} & C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\
-C_{52} & -C_{51} & -C_{54} & -C_{53} & C_{56} & C_{55}
\end{bmatrix}
\]

\[ (4.1.10) \]

4.2 Elements of the N-operator

In the same way as for the M-operator the Galerkin integrals of the N-operator will be

\[
-\lambda \bar{B}_{ji} = \int_{A} \begin{bmatrix}
u_{11} N_{11} u_{11} & \nu_{11} N_{11} u_{12} & \nu_{12} N_{12} v_{1} & \nu_{11} N_{12} v_{2} & \nu_{11} N_{13} w_{1} & \nu_{11} N_{13} w_{2} \\
u_{12} N_{11} u_{21} & \nu_{12} N_{11} u_{12} & \nu_{12} N_{12} v_{2} & \nu_{12} N_{12} v_{2} & \nu_{12} N_{13} w_{1} & \nu_{12} N_{13} w_{2} \\
v_{21} N_{21} w_{11} & v_{21} N_{21} w_{12} & v_{22} N_{22} v_{1} & v_{22} N_{22} v_{2} & v_{23} N_{23} w_{1} & v_{23} N_{23} w_{2} \\
v_{22} N_{21} w_{21} & v_{22} N_{21} w_{12} & v_{22} N_{22} v_{2} & v_{22} N_{22} v_{2} & v_{23} N_{23} w_{1} & v_{23} N_{23} w_{2} \\
w_{13} N_{31} w_{11} & w_{13} N_{31} w_{12} & w_{13} N_{32} v_{1} & w_{13} N_{32} v_{2} & w_{13} N_{33} w_{1} & w_{13} N_{33} w_{2} \\
w_{23} N_{31} w_{21} & w_{23} N_{31} w_{12} & w_{23} N_{32} v_{2} & w_{23} N_{32} v_{2} & w_{23} N_{33} w_{1} & w_{23} N_{33} w_{2}
\end{bmatrix} dA
\]

\[ (4.2.1) \]

where

\[
N_{11} u_{1} = (pN_{a_1}^{2} + N_{b_1}^{2} \lambda^{2} + N_{d_1}^{2} \cos \theta + V_{0}^{2} \lambda^{2} x \cos \theta) \sin \theta \sin \lambda x
\]

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\[ N_{11}^{u_2} = (pr^n^2 + N_a^2 m + N_b^2 m \cos \theta + V_0^2 m \cos \theta) \sin \theta \cos \lambda_m x + 2nV_0^m \lambda_m \sin \theta \sin \theta \cos \lambda_m x \]

\[ N_{12}^{v_1} = 0 \]

\[ N_{12}^{v_2} = 0 \]

\[ N_{13}^{w_1} = -pR_m \cos \theta \cos \lambda_m x \]

\[ N_{13}^{w_2} = pR_m \sin \theta \sin \lambda_m x \]

\[ N_{21}^{v_1} = 0 \]

\[ N_{21}^{v_2} = 0 \]

\[ N_{22}^{v_1} = (pr^n^2 + N_a^2 m + N_b^2 m \cos \theta + V_0^2 m \cos \theta) \sin \theta \sin \lambda_m x + 2V_0^m \lambda_m \sin \theta \cos \theta \cos \lambda_m x \]

\[ N_{22}^{v_2} = (pr^n^2 + N_a^2 m + N_b^2 m \cos \theta + V_0^2 m \cos \theta) \cos \theta \cos \lambda_m x + 2V_0^m \lambda_m \sin \theta \sin \theta \sin \lambda_m x \]

\[ N_{23}^{w_1} = -pR \sin \theta \sin \lambda_m x - 2V_0^m \lambda_m \sin \theta \cos \theta \cos \lambda_m x \]

\[ N_{23}^{w_2} = pR \cos \theta \cos \lambda_m x + 2V_0^m \lambda_m \sin \theta \sin \theta \sin \lambda_m x \]

\[ N_{31}^{u_1} = pR_m \sin \theta \cos \lambda_m x \]

\[ N_{31}^{u_2} = -pR_m \cos \theta \sin \lambda_m x \]

\[ N_{32}^{v_1} = -pR \cos \theta \sin \lambda_m x + 2V_0^m \lambda_m \sin \theta \sin \theta \cos \lambda_m x \]

\[ N_{32}^{v_2} = pR \sin \theta \cos \lambda_m x - 2V_0^m \lambda_m \sin \theta \cos \theta \sin \lambda_m x \]

\[ N_{33}^{w_1} = (pr^n^2 + N_a^2 m + N_b^2 m \cos \theta + V_0^2 m \cos \theta) \sin \theta \cos \lambda_m x - 2V_0^m \lambda_m \sin \theta \sin \theta \cos \lambda_m x \]

\[ N_{33}^{w_2} = (pr^n^2 + N_a^2 m + N_b^2 m \cos \theta + V_0^2 m \cos \theta) \sin \theta \cos \lambda_m x - 2V_0^m \lambda_m \sin \theta \cos \theta \sin \lambda_m x \]

\[ (4.2.2) \]

These will include the following integrals.

\[ F_{ji} = \int_A \cos \theta (\cos \theta \cos \phi \cos \lambda_m x \cos \lambda_q x) \, dA \]
\[
\begin{align*}
G_{ji} &= \int_A \cos(\theta \sin \theta \sin \lambda_m \sin \lambda_q \cos \lambda x) \, dA \\
K_{ji} &= \int_A \cos(\cos \theta \cos \theta \cos \lambda_m \sin \lambda_q \cos \lambda x) \, dA \\
K'_{ji} &= \int_A \cos(\sin \theta \sin \theta \sin \lambda_m \cos \lambda_q \sin \lambda x) \, dA \\
O_{ji} &= \int_A \cos(\cos \theta \cos \theta \cos \lambda_m \cos \lambda_q \cos \lambda x) \, dA \\
P_{ji} &= \int_A \cos(\sin \theta \sin \theta \sin \lambda_m \cos \lambda q \sin \lambda x) \, dA \\
Q_{ji} &= \int_A \cos(\cos \theta \cos \theta \sin \lambda_m \sin \lambda_q \sin \lambda x) \, dA \\
U_{ji} &= \int_A \cos(\sin \theta \sin \theta \cos \lambda_m \cos \lambda_q \cos \lambda x) \, dA \\
V_{ji} &= \int_A \sin(\sin \theta \cos \theta \sin \lambda_m \cos \lambda_q \sin \lambda x) \, dA \\
W_{ji} &= \int_A \sin(\cos \theta \sin \theta \sin \lambda_m \sin \lambda_q \sin \lambda x) \, dA \\
X_{ji} &= \int_A \sin(\sin \theta \cos \theta \cos \lambda_m \sin \lambda_q \sin \lambda x) \, dA \\
Y_{ji} &= \int_A \sin(\cos \theta \sin \theta \sin \lambda_m \cos \lambda_q \cos \lambda x) \, dA
\end{align*}
\] 

These integrals are evaluated in appendix B.

For ease of calculation the elements of the \( -\lambda B_{ji} \) -matrix will be calculated for each loading condition separately.
Due to pressure, $p$

\[-\lambda B_{11} = -\lambda B_{33} = \int_A u_1'N_{11}u_1\,dA = \int_A pRn^2\sin\theta\sin\lambda_q\sin\theta\sin\lambda_m\,xdA\]

\[= pRn^2S_{ji}\]

\[-\lambda B_{22} = -\lambda B_{44} = \int_A u_2'N_{11}u_2\,dA = \int_A pRn^2\cos\theta\cos\lambda_q\cos\theta\cos\lambda_m\,xdA\]

\[= pRn^2R_{ji}\]

\[-\lambda B_{66} = \int_A w_2'N_{33}w_2\,dA = \int_A pRn^2\sin\theta\cos\lambda_q\sin\theta\cos\lambda_m\,xdA\]

\[= pRn^2T_{ji}\]

\[-\lambda B_{55} = \int_A w_1'N_{33}w_1\,dA = \int_A pRn^2\cos\theta\sin\lambda_q\cos\theta\sin\lambda_m\,xdA\]

\[= pRn^2T_{ji}\]

\[-\lambda B_{16} = \int_A u_1'N_{13}w_2\,dA = \int_A pR\lambda_m\sin\theta\sin\lambda_q\sin\theta\sin\lambda_m\,xdA\]

\[= pR\lambda_mS_{ji}\]

\[-\lambda B_{25} = \int_A u_2'N_{13}w_1\,dA = \int_A pR\lambda_m\cos\theta\cos\lambda_q\cos\theta\cos\lambda_m\,xdA\]

\[= -pR\lambda_mR_{ji}\]

\[-\lambda B_{35} = \int_A v_1'N_{33}w_2\,dA = \int_A -pRn\sin\theta\sin\lambda_q\sin\theta\sin\lambda_m\,dA\]

\[= -pRnS_{ji}\]

\[-\lambda B_{46} = \int_A v_2'N_{23}w_1\,dA = \int_A pRn\cos\theta\cos\lambda_q\cos\theta\cos\lambda_m\,xdA\]

\[= pRnR_{ji}\]

\[-\lambda B_{52} = \int_A w_1'N_{31}u_2\,dA = \int_A -pR\lambda_m\cos\theta\sin\lambda_q\cos\theta\sin\lambda_m\,xdA\]

\[= -pR\lambda_mT_{ji}\]
\[ -\lambda_{61} = \int_A w_2 N_{31} u_1 dA = \int_A p R^t_m \sin \theta \cos \lambda q x \sin \theta \cos \lambda m x dA \]
\[ = p R^t_m T^t_{ji} \]
\[ -\lambda_{53} = \int_A w_1 N_{32} v_1 dA = \int_A -p R \cos \theta \sin \lambda q x \cos \theta \sin \lambda m x dA \]
\[ = -p R T_{ji} \]
\[ -\lambda_{64} = \int_A w_2 N_{32} v_2 dA = \int_A -p R \sin \theta \cos \lambda q x \sin \theta \cos \lambda m x dA \]
\[ = p R T^t_{ji} \]

All other terms of \(-\lambda_{ji}\) are zero. In short, due to the pressure terms:

\[
-\lambda_{ji} = p R \begin{bmatrix}
S_{ji} & 0 & 0 & 0 & 0 & 0 \\
0 & R_{ji} & 0 & 0 & 0 & 0 \\
0 & 0 & S_{ji} & 0 & 0 & 0 \\
0 & 0 & 0 & R_{ji} & 0 & 0 \\
0 & 0 & 0 & 0 & T_{ji} & 0 \\
0 & 0 & 0 & 0 & 0 & T^t_{ji}
\end{bmatrix}
\begin{bmatrix}
n^2 & 0 & 0 & 0 & 0 & \lambda_m \\
0 & n^2 & 0 & 0 & -\lambda_m & 0 \\
0 & 0 & n^2 & 0 & -n & 0 \\
0 & 0 & 0 & n^2 & 0 & n \\
0 & -\lambda_m & -n & 0 & n^2 & 0 \\
\lambda_m & 0 & 0 & n & 0 & n^2
\end{bmatrix}
\]

Due to bending and axial compression, \(N_b\) and \(N_a\)

\[ -\lambda_{11} = -\lambda_{33} = \int_A u_1' N_{11} u_1 dA \]
\[ = \int_A \sin \theta \sin \lambda q x (N a_m^2 + N b_m^2 \cos \theta) \sin \theta \sin \lambda m x dA \]
\[ = \int_A N a_m^2 \sin \theta \sin \lambda q x \sin \theta \sin \lambda m x dA + \int_A N b_m^2 \cos \theta \sin \theta \sin \lambda q x \sin \theta \sin \lambda m x dA \]
\[ = N a_m^2 S_{ji} + N b_m^2 G_{ji} \]
\[ -\lambda_{22} = -\lambda_{44} = \int_A u_2' N_{11} u_2 dA \]
\[-\lambda_{B_{55}} = \int_A w_1' N_{33} w_1 dA \]
\[= \int_A \cos\phi \cos\lambda x (N_{a_m} + N_{b_m} \cos\theta) \cos\theta \cos\lambda \, dA \]
\[= \int_A N_{a_m} \cos\phi \cos\lambda x \cos\theta \cos\lambda \, dA + \int_A N_{b_m} \cos\phi \cos\lambda x \cos\theta \cos\lambda \, dA \]
\[= N_{a_m} \lambda_{T_ji}^2 + N_{b_m} \lambda_{K_ji}^2 \]

\[-\lambda_{B_{66}} = \int_A w_2' N_{33} w_2 dA \]
\[= \int_A \sin\phi \cos\lambda x (N_{a_m} + N_{b_m} \cos\theta) \sin\theta \cos\lambda \, dA \]
\[= \int_A N_{a_m} \sin\phi \cos\lambda x \sin\theta \cos\lambda \, dA + \int_A N_{b_m} \sin\phi \cos\lambda x \sin\theta \cos\lambda \, dA \]
\[= N_{a_m} \lambda_{T_{ji}'} + N_{b_m} \lambda_{K_{ji}'} \]

(4.2.7)

All other terms are zero. This yields \(-\lambda\mathbf{B}_j\) due to pure bending and axial compression.

\[-\lambda\mathbf{B}_j = \lambda_{N_a}^2 \begin{bmatrix} S_{ji} & 0 & 0 & 0 & 0 & 0 \\ 0 & R_{ji} & 0 & 0 & 0 & 0 \\ 0 & 0 & S_{ji} & 0 & 0 & 0 \\ 0 & 0 & 0 & R_{ji} & 0 & 0 \\ 0 & 0 & 0 & 0 & T_{ji} & 0 \\ 0 & 0 & 0 & 0 & 0 & T_{ji}' \end{bmatrix} + \lambda_{N_b}^2 \begin{bmatrix} G_{ji} & 0 & 0 & 0 & 0 & 0 \\ 0 & F_{ji} & 0 & 0 & 0 & 0 \\ 0 & 0 & G_{ji} & 0 & 0 & 0 \\ 0 & 0 & 0 & F_{ji} & 0 & 0 \\ 0 & 0 & 0 & 0 & K_{ji} & 0 \\ 0 & 0 & 0 & 0 & 0 & K_{ji}' \end{bmatrix} \]

(4.2.8)
Shear loading terms, $V_0$

\[ -\lambda B_{11} = -\lambda B_{33} = \int_A u_1^N u_1^N dA = \int_A \sin \theta \sin \lambda x (V_0^2 \lambda^2 \cos \theta \sin \theta \sin \lambda x + 2V_0^2 \lambda \sin \theta \cos \theta \cos \lambda x) dA \]

\[ = \int_A \lambda \sin \lambda x \cos \lambda x dA \]

\[ = V_0^2 \lambda^2 q_{ji}^x + 2V_0^2 \lambda n q_{ji}^x \]

\[ -\lambda B_{22} = -\lambda B_{44} = \int_A u_2^N u_2^N dA = \int_A \cos \theta \cos \lambda x (V_0^2 \lambda^2 \cos \theta \cos \theta \cos \lambda x + 2V_0^2 \lambda \sin \theta \cos \theta \cos \lambda x) dA \]

\[ = \int_A \lambda \cos \lambda x \cos \lambda x dA \]

\[ = V_0^2 \lambda^2 q_{ji}^y + 2V_0^2 \lambda n q_{ji}^y \]

\[ -\lambda B_{55} = \int_A u_3^N u_3^N dA = \int_A \sin \theta \sin \lambda x (V_0^2 \lambda^2 \cos \theta \sin \theta \cos \lambda x - 2V_0^2 \lambda \sin \theta \sin \theta \cos \lambda x) dA \]

\[ = \int_A \lambda \sin \lambda x \cos \lambda x dA \]

\[ = V_0^2 \lambda^2 q_{ji}^z - 2V_0^2 \lambda n q_{ji}^z \]

\[ -\lambda B_{66} = \int_A u_4^N u_4^N dA = \int_A \cos \theta \cos \lambda x (V_0^2 \lambda^2 \sin \theta \cos \theta \cos \lambda x - 2V_0^2 \lambda \sin \theta \cos \theta \cos \lambda x) dA \]

\[ = \int_A \lambda \cos \lambda x \cos \lambda x dA \]

\[ = V_0^2 \lambda^2 q_{ji}^x - 2V_0^2 \lambda n q_{ji}^x \]

\[ -\lambda B_{35} = \int_A w_1^N w_3^N dA = \int_A \sin \theta \sin \lambda x (-2V_0^2 \lambda \sin \theta \sin \theta \cos \lambda x) dA \]

\[ = \int_A -2V_0^2 \lambda \sin \theta \sin \theta \cos \lambda x dA \]

\[ = -2V_0^2 \lambda n^x \]

\[ -\lambda B_{46} = \int_A w_2^N w_2^N dA = \int_A \cos \theta \cos \lambda x (2V_0^2 \lambda \sin \theta \sin \theta \sin \lambda x) dA \]

\[ = \int_A 2V_0^2 \lambda \sin \theta \sin \theta \sin \lambda x dA \]
\[ \int 2V_{0m} \lambda \sin \theta \cos \theta \sin \theta \cos \lambda q \sin \lambda x dA \]
\[ = -2V_{0m} \lambda v \]

\[ -\lambda B_{53} = \int w_i N_{32} v_i dA = \int \cos \theta \sin \theta \cos \theta \sin \lambda x (2V_{m0} \lambda \sin \theta \sin \theta \cos \lambda x) dA \]
\[ = \int 2V_{0m} \lambda \sin \theta \cos \theta \sin \theta \sin \lambda x \cos \lambda x dA \]
\[ = 2V_{0m} \lambda X_{ji} \]

\[ -\lambda B_{64} = \int w_i N_{32} v_i dA = \int \sin \theta \cos \lambda x (-2V_{m0} \sin \theta \cos \theta \sin \theta \sin \lambda x) dA \]
\[ = \int -2V_{0m} \lambda \sin \theta \sin \theta \cos \theta \cos \lambda x \sin \lambda x dA \]
\[ = -2V_{0m} \lambda Y_{ji} \]

(4.2.9)

All other terms are zero. For \(-\lambda B_{ji}\) due to shear loading this yields

\[ \lambda B_{ji} = V_{0m} \]

\[ \begin{bmatrix}
\lambda_{mji} + 2nW_{ji} & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda_{mji} + 2nV_{ji} & 0 & 0 & 0 & 0 \\
0 & 0 & 2nW_{ji} & 0 & -2W_{ji} & 0 \\
0 & 0 & 0 & \lambda_{mji} + 2nW_{ji} & 0 & 2V_{ji} \\
0 & 0 & 0 & 0 & \lambda_{mji} - 2nX_{ji} & 0 \\
0 & 0 & 0 & -2Y_{ji} & 0 & \lambda_{mji} - 2nY_{ji}
\end{bmatrix} \]

(4.2.10)

4.3 Complete definition of the stability matrix

The complete stability matrix elements appear as follows
Now we can regard three of the four loads as given and search for the fourth.
As a further convenience, each element of the stability matrix will be reduced by the constant

\[ 4.3.1 \]
factor \pi RL/2. This yields

\[ \widetilde{S}_{ji} = \frac{2}{\pi RL} S_{ji} ; \quad \widetilde{R}_{ji} = \frac{2}{\pi RL} R_{ji} ; \quad \widetilde{T}_{ji} = \frac{2}{\pi RL} T_{ji} \quad \text{etc.} \]

(4.3.2)

For n=0 three of these six equations do vanish. This can be seen as follows. For n=0 we get:

\[ \widetilde{R}_{ji} = \widetilde{T}_{ji} = 2\delta_{ij} \]

\[ \widetilde{S}_{ji} = \widetilde{T}_{ji} = 0 \]

\[ \widetilde{F}_{ji} = \widetilde{K}_{ji} = 0 \]

\[ \widetilde{G}_{ji} = \widetilde{K}_{ji} = 0 \]

\[ \widetilde{F}_{ji} = \widetilde{W}_{ji} = 0 \]

\[ \widetilde{U}_{ji} = \widetilde{V}_{ji} = 0 \]

\[ \widetilde{X}_{ji} = \widetilde{V}_{ji} = 0 \]

\[ c_{12} = c_{14} = c_{15} = c_{32} = c_{34} = c_{35} = c_{51} = c_{53} = c_{56} = 0 \]

(4.3.3)

Substitution now yields

\[ \bar{A}_{ji} - \lambda B_{ji} = 2\delta_{ji} \]

\[ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & c_{11} & 0 & c_{13} & -c_{16} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & c_{31} & 0 & c_{33} & -c_{36} & 0 \\ 0 & c_{52} & 0 & c_{54} & c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \]

\[ + 2pR 8 \]

\[ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\lambda_m & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\lambda_m & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \]

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\[ + 2 \lambda_m^2 a_{ij} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \lambda_m^2 \delta_{ji} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \]

\[ + \lambda_m \nu_0 \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_m \delta_{ji} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_m \delta_{ji} & 0 \\ 0 & 0 & 0 & 0 & \lambda_m \delta_{ji} \end{bmatrix} \]

(4.3.4)

The \( n=0 \) case implies the identical satisfaction of the circumferential equation of equilibrium.

Equation (4.3.4) may be brought into more compact form by eliminating the columns and rows of zeros that occur in the \( n=0 \) and the \( p=0 \) case. To do this simply omit the first, the third and the sixth column of the \( \bar{A}_{ji} \) \( - \lambda \bar{B}_{ji} \) matrix when \( n=0 \) and the first, the third and the sixth row of this matrix when \( p=0 \), where \( p \) is the Galerkin integer corresponding to \( n \).

In accordance with the discussion preceeding equations (3.7) and (3.8), where the double series in \( m \) and \( n \) are reordered into single series in \( i \), partial summation over \( n \) will be followed by summation over \( m \). With this knowledge the \( \bar{A}_{ji} \) matrix is identified as in appendix C. The matrix \( -\lambda \bar{B}_{ji} \) has the same form as the \( \bar{A}_{ji} \) matrix.

In order to see the composition of the \( \bar{A}_{ji} \) and \( -\lambda \bar{B}_{ji} \) submatrices more clearly, the integer functions developed in appendix B will be displayed in the format of the \( \bar{A}_{ji} \) matrix. This is also done in appendix C.
5. The reduced stability matrix

When the shear load is included in the loading, rather large order matrices are required for convergence and the computer time involved in solving the algebraic eigenvalue problem defined in appendix C becomes excessive. Therefore it will be necessary to consider some means of reducing the order of the matrix by approximation.

One way to do this is by dropping out the buckling terms in the axial and circumferential equations of equilibrium. This process was already used by Flügge (Ref 2) for the case of axial compression plus torsion, where the justification was based upon order of magnitude relations. For this approximation, the N-operator of equation (2.12) becomes

\[
\mathbf{N} = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
N_{31} & N_{32} & N_{33}
\end{bmatrix}
\]  \hspace{1cm} (5.1)

The first two equations of (3.10) reduce to

\[
\sum_{i=1}^{N} \int_{A} \begin{bmatrix}
0 & 0 \\
0 & 0 \\
0 & 0
\end{bmatrix} \begin{bmatrix}
M_{11} & M_{12} & M_{13} \\
M_{21} & M_{22} & M_{23} \\
M_{31} & M_{32} & M_{33}
\end{bmatrix} \begin{bmatrix}
A_{i}u_{i} \\
B_{i}v_{i} \\
C_{i}w_{i}
\end{bmatrix} dA = (0)
\]  \hspace{1cm} (5.2)

Continuation of this reduction in terms of the 6x6 matrix \( \mathbf{A}_{ji} \) yields

\[
\begin{bmatrix}
A_{11} & A_{12} & A_{13} & A_{14} & A_{15} & A_{16} \\
A_{21} & A_{22} & A_{23} & A_{24} & A_{25} & A_{26} \\
A_{31} & A_{32} & A_{33} & A_{34} & A_{35} & A_{36} \\
A_{41} & A_{42} & A_{43} & A_{44} & A_{45} & A_{46}
\end{bmatrix}
\begin{bmatrix}
A_{mn} \\
A'_{mn} \\
B_{mn} \\
B'_{mn} \\
C_{mn} \\
C'_{mn}
\end{bmatrix} = (0)
\]  \hspace{1cm} (5.3)
With equation (5.3) $A_{mn}$, $A'_{mn}$, $B_{mn}$ and $B'_{mn}$ can be expressed in terms of $C_{mn}$ and $C'_{mn}$.

\[
\begin{bmatrix}
A_{11} & A_{12} & A_{13} & A_{14} \\
A_{21} & A_{22} & A_{23} & A_{24} \\
A_{31} & A_{32} & A_{33} & A_{34} \\
A_{41} & A_{42} & A_{43} & A_{44}
\end{bmatrix}
-1
\begin{bmatrix}
A_{15} & A_{16} \\
A_{25} & A_{26} \\
A_{35} & A_{36} \\
A_{45} & A_{46}
\end{bmatrix}
= -
\begin{bmatrix}
A'_{mn} \\
B_{mn} \\
A'_{mn} \\
B_{mn}
\end{bmatrix}
= -
\begin{bmatrix}
C_{mn} \\
C'_{mn}
\end{bmatrix}
\] (5.4)

Now

\[
[A][X] = -[B][Y] \quad \Rightarrow \quad [X] = -[A]^{-1}[B][Y]
\] (5.5)

yields for equation (5.4)

\[
\begin{bmatrix}
A_{mn} \\
A'_{mn} \\
B_{mn} \\
B'_{mn}
\end{bmatrix}
= -\begin{bmatrix}
A_{11} & A_{12} & A_{13} & A_{14} \\
A_{21} & A_{22} & A_{23} & A_{24} \\
A_{31} & A_{32} & A_{33} & A_{34} \\
A_{41} & A_{42} & A_{43} & A_{44}
\end{bmatrix}
^{-1}
\begin{bmatrix}
A_{15} & A_{16} \\
A_{25} & A_{26} \\
A_{35} & A_{36} \\
A_{45} & A_{46}
\end{bmatrix}
= \begin{bmatrix}
C_{mn} \\
C'_{mn}
\end{bmatrix}
\] (5.6)

Substitution of (5.6) into the last two equations of (4.3.1) yields

\[
(\bar{A}_{ji} - \lambda \bar{B}_{ji})_{\text{red}}
\]

\[
= \begin{bmatrix}
\bar{T}_{ji} & 0 \\
0 & \bar{T}'_{ji}
\end{bmatrix}
\begin{pmatrix}
C_{51}Z_{11} + C_{52}Z_{21} + C_{53}Z_{31} + C_{54}Z_{41} + C_{55} \\
-C_{52}Z_{11} - C_{51}Z_{21} + C_{54}Z_{31} + C_{53}Z_{41} + C_{55}
\end{pmatrix}
\]

\[
+ \begin{pmatrix}
\bar{T}_{ji} & 0 \\
0 & \bar{T}'_{ji}
\end{pmatrix}
\begin{pmatrix}
-n^2Z_{21} - nZ_{31} + \lambda \frac{Z_{22}}{n} \\
\lambda \frac{Z_{11} + nZ_{41}}{m} + n^2
\end{pmatrix}
+ \lambda^2 \begin{pmatrix}
\bar{T}_{ji} & 0 \\
0 & \bar{T}'_{ji}
\end{pmatrix}
\begin{pmatrix}
\frac{\lambda^2}{m^2} \\
\frac{\lambda^2}{m}
\end{pmatrix}
\]

\[+ \lambda^2 \begin{pmatrix}
\bar{K}_{ji} & 0 \\
0 & \bar{K}'_{ji}
\end{pmatrix} + \lambda \bar{V}
\begin{pmatrix}
Z_{31} + \lambda \frac{Z_{21}}{m} \\
\frac{Z_{11} + nZ_{41}}{m} - n\bar{V}_{ji}
\end{pmatrix}
\]

\[+ \lambda^2 \begin{pmatrix}
\bar{K}_{ji} & 0 \\
0 & \bar{K}'_{ji}
\end{pmatrix} + \lambda \bar{V}
\begin{pmatrix}
2Z_{31} + \lambda \frac{Z_{21}}{m} \\
-2\bar{Y}_{ji}Z_{41}
\end{pmatrix}
\]

\[+ \lambda^2 \begin{pmatrix}
\bar{K}_{ji} & 0 \\
0 & \bar{K}'_{ji}
\end{pmatrix} + \lambda \bar{V}
\begin{pmatrix}
2Z_{31} + \lambda \frac{Z_{21}}{m} \\
-2\bar{Y}_{ji}Z_{41} + \lambda \frac{Z_{11} + nZ_{41}}{m} - 2n\bar{Y}_{ji}
\end{pmatrix}
\] (5.7)
Now the rank of the matrix is reduced by a factor 3. The rank of the submatrices $\begin{pmatrix} \bar{A}_{ji} & \lambda \bar{B}_{ji} \end{pmatrix}_{\text{red}}$ is equal to 2. For $n=0$ the second row vanishes and for $p=0$ the second column disappears. The agreement between solutions found with the full matrix and solutions found with the reduced matrix is close for $m$ is larger than 2. Meyer proved this agreement to be within 1.5% for the combined bending-axial compression case (Ref 1). For $m=1$ and $m=2$ some corrections may be necessary and probably it is better to use the full matrix for these buckling modes. In chapter 8 for some load cases a full and a reduced matrix calculation is performed and the solutions confirm the above statement.
6. Convergence

It is necessary to check the stability matrix for convergence whether a sequence of approximations based upon the Galerkin formulation converges to a unique solution of the problem. This has already been done by Meyer (Ref 1). He used von Koch's convergence rule and proved the convergence of several cases.

case 1: \( N_b = V_0 = 0 \) full matrix

case 2: \( V_0 = 0 \) full matrix

case 3: reduced matrix

Since in this report the same cases are treated and the same kind of solution functions are used the convergence has already been proved. However this proof is only of theoretical interest. Now one is concerned with the following question: "What is the required number of terms to accurately approximate the critical buckling load?"

Some insight is provided by the following considerations. Since the equations are homogeneous, each row can be normalized by dividing it by its largest element. These largest elements lie along the principal diagonal of the matrix. One of them, the smallest, represents the critical eigenvalue for pure compression. For values of \( m \) and \( n \), away from the ones for which the critical eigenvalue for pure compression is found, the diagonal elements will increase rapidly. After normalization the other elements in rows with large principal diagonal elements will get smaller and smaller. This yields that for principal diagonal elements, away from the critical element, the off-diagonal elements approach zero.

When, for a certain block of elements, all off-diagonal elements are zero, this block is uncoupled from the other elements. In this way eigenvector components for regions of the stability matrix away from the critical block, defining the lowest eigenvalue, will be zero.

This behaviour can be illustrated as follows. Let \( \tilde{\varepsilon}_{ji} \) be an element of the new normalized matrix. For the purpose of illustration imagine that \( \tilde{\varepsilon}_{ji} \) is a continuous function of the column index \( i \). Looking at figure 4 it can be seen that the closer the non-critical rows approach a "curve", which has the value 1.00 on the principal diagonal and is zero elsewhere, the closer the corresponding eigenvalue components approach zero. These eigenvector components define the mode shape. This yields that away from the critical row the definition of the mode shape becomes better and better and the corresponding eigenvalue approaches the unique solution of the problem.

This figure also learns us another thing. If we have to deal with shear loading intervals of \( m \) and \( n \) have to be introduced in order to find the critical shear force. Looking for a solution with a certain reasonable accuracy we will end up with a rather large matrix. Suppose that the critical shear force appears for \( m=2-6 \) and \( n=5-15 \). For an anisotropic shell this leaves us at least with a 110x110 matrix. On the other hand if we would start at \( m=1 \) and \( n=0 \) and then build up the matrix to reach the wished accuracy we would find almost the same critical shear force for \( m=1-6 \) and
\( n=0-15 \). This would be a matrix of rank 186 and the computer time needed to reach the same accuracy would increase considerably.

Therefore it is better to search for the critical interval of \( m \) and \( n \) first for which the lowest eigenvalue will be found. For the above case for instance this could be the interval \( m=3-5 \) and \( n=8-12 \). Then the matrix can be enlarged by including the \( m \)'s and \( n \)'s around this critical interval in order to reach the accuracy mentioned above. The question how to find this critical interval will be discussed in the next chapter.
Now the software for calculating the critical buckling load can be developed. Before doing so one has to think of a way to find these critical loads.

In order to find the critical loads large matrices are involved. The determinant of these matrices has to be set equal to zero. This zero-point can be found as follows. First substitute the material properties and the given loads into the stability matrix. Then a starting load, which is lower than the critical load you expect to find, has to be chosen and a calculation of the determinant can be performed. Now we are especially interested in the sign of the determinant. If the starting load is increased with a certain step-size the value of the determinant will approach zero and suddenly the sign of the determinant changes. This implies that between the last two loads there has to be a zero-point. The next thing to do is to reduce the step-size and search for the zero-point between the last two loads. The step size can be reduced again and again until the zero-point is found with a certain previously chosen accuracy. Finally the zero-point is determined more exactly through a linear interpolation within the last interval.

This way of working involves a few difficulties we have to be aware of. The computer programs are developed for calculating the critical loads of anisotropic cylindrical shells. A calculation of the critical loads for isotropic and orthotropic material is also possible, but especially then one has to be careful.

The first problem appears if one wants to calculate the critical loads of a nearly isotropic or orthotropic shell. From experience it is known that for this case the zero-points are very close to each other. One now runs the risk of passing two zero-points at the same time and the sign change of the determinant won’t be detected. Therefore attention has to be payed to the size of the determinant. If the determinant decreases and then suddenly starts to increase one must be aware of the fact that there is a chance that two zero-points have been passed at once. In this region a search has to be started for an eventual zero-point with a reduced step-size.

The second problem appears if you have to deal with isotropic material. Now it is necessary to neglect the anisotropic terms in the displacement functions to find the zero-points. This is caused by symmetry of the stability matrix for isotropic or orthotropic material. In appendix C an example of a stability calculation for these two kinds of material is given and the above statement is proven. A positive consequence is that the rank of the matrix is reduced by a factor 2 and therefore the computer time needed to perform a stability calculation will decrease enormously.

For the calculation of the critical loads the total problem can be separated into three cases.

case 1 : only axial compression (and normal pressure)
\[ N_0 = 0 \; ; \; V_0 = 0 \]

case 2 : only pure bending (and normal pressure, axial compression)
\[ V_0 = 0 \]
case 3 : shear loading (and normal pressure, axial compression , pure bending)

These three cases will be discussed separately in the next paragraphs.

Finally the way of writing an input file for the developed software has to made clear to anyone, who wants to perform a stability calculation with the use of this software. Therefore an outlay of the input file has been made in appendix E and a description of every input variable is given there.

7.1 Axial compression (and normal pressure)

For the axial compression (and normal pressure) case one has the choice between the full or the reduced matrix method. The full matrix method deals with a 6x6 matrix or for the isotropic case with a 3x3 matrix (see chapter 6). On the other hand the reduced matrix method deals with a 2x2 matrix or for the isotropic case a "1x1" matrix.

The developed program has the possibility to calculate the critical eigenvalue for one m (number of half waves in axial direction) and for several values of n (number of waves in circumferential direction) simultaneously. For instance if one chooses m=1 and one takes five values of n starting at n=1 the calculation will go as follows. The starting point for $N_B/N_{cl}$ is 0.9 and the step-size is 0.1. In table 1 only the sign of the determinant is given. The first change of sign appears for n=5. Now the eigenvalue for n=5 can be determined more accurately.

This example has been chosen to show that one has to be aware of two difficulties. First one can see that for this example a change of sign will appear for n=5. We are left with the possibility that for n=6 the eigenvalue is even smaller than for n=5. This implies that one has to search on for values of n beyond n=5 to find the absolute minimum eigenvalue. The second problem is caused by the fact that only one value of m is considered. Therefore one also has to search for the critical value of m.

To eliminate these two problems the following automatic search procedure has been developed. First the n-interval will be replaced until on both sides of the critical value of n there are at least two more values of n for which the eigenvalue is larger than the one for the critical n. Then m is enlarged by 1 and the whole calculation must be repeated. This whole procedure will go on until two succeeding values of m are found for which the critical eigenvalue is larger than the absolute critical eigenvalue found for the preceeding value of m. This rather difficult explanation can be simplified by an example.

suppose : m=1 n=5 local minimum

m=2 n=7 absolute minimum

m=3 n=8 local minimum
m = 4 n = 10 local minimum

n-interval = 5

The automatic search procedure now will at least consider those combinations of m and n given in table 2.

In this way the absolute critical eigenvalue is surrounded by its nearest "neighbours" with respect to the size of the eigenvalue and the critical buckling load can be found.

7.2 Pure bending (and normal pressure, axial compression)

Now one has the choice either to calculate \( N_b \) for a given value of \( N_a \) and p or Na for a given value of \( N_b \) and p. For these two alternatives again we have the possibility to work with the full or the reduced matrix method. For \( m \) larger than 2 Meyer proved that the solution found with the reduced matrix method is in good agreement with the one found with the full matrix method and therefore the use of the full matrix method can be reduced to the cases where \( m = 1 \) or \( m = 2 \) (chapter 5).

How do we search for the lowest eigenvalue? The first method is a non-automatic search. A certain \( m \) and a n-interval must be chosen. The n-interval will always start at \( n = 0 \). The program calculates the eigenvalue for the chosen \( m \) and n-interval and then stops. The next thing to do now is to enlarge the n-interval by 1 and make a new calculation for the eigenvalue. One can go on enlarging the n-interval until a certain required accuracy is reached.

This whole procedure can also be done in an automatic way. A value of \( m \), a n-interval to start with and a required accuracy have to be entered then. First the eigenvalue for this \( m \) and n-interval is calculated. Then the n-interval is enlarged by 1 and the next eigenvalue is calculated. Before going on with a third calculation a check is made whether the difference between the two found eigenvalues is smaller than the required accuracy. If so the program stops. If this check turns out to be negative, the n-interval is enlarged by 1, a new calculation for the eigenvalue will be carried out and a check whether the accuracy is reached will be made again. If this accuracy is reached the program stops. Otherwise the speed of convergence will be determined from the three found eigenvalues and a new n-interval for which the accuracy will be reached with respect to this speed of convergence will be calculated. Now the whole procedure repeats itself until the required accuracy is reached.

We are still left with one problem. The chosen \( m \) doesn't have to be the \( m \) for which the lowest critical buckling load will be found. The solution to this problem can be found with the example in figure 5. In this figure the size of the determinant is plotted as a function of the bending load. This is done for several values of \( m \). We can see that the determinant corresponding with the \( m \), for which the absolute critical buckling load is found, has the smallest value in the
region preceding this critical buckling load. By calculating the determinant for a n-interval and several values of m in this region, the critical m can be found. After this search for the critical value of m, one can go on determining the eigenvalue and the n-interval for which the eigenvalue satisfies a required accuracy.

7.3 Shear loading (and normal pressure, axial compression, pure bending)

For the shear loading case three possible variables are to be calculated, namely $V_0$, $N_b$, and $N_a$. Only the reduced matrix method is used because of the fact that the rank of the matrix would be very large if one would use the full matrix method. The computation time needed to find the lowest eigenvalue of this matrix would be excessive.

The first difficulty to deal with is that it is unknown for which m- and n-interval the lowest eigenvalue will appear. Therefore a procedure was developed that will search for the critical m- and n-interval. One has to choose an interval for the minimum value of m and n and also the size of the m- and n-interval one wants to consider. Then a value for $N_{bN}/N_{cl}$ (or $N_{bN}/N_{cl}$ or $N_{aN}/N_{cl}$) has to be selected that is at least near the critical buckling load one expects to find. Now the program will calculate the determinant for all combinations of $m_{min}$ and $n_{min}$ one puts in. The m- and n-interval, for which the determinant reaches a minimum, is the critical one and has to be subjected to a closer examination.

The last thing one can do now is enlarge the m- and n-interval around this region of critical wave numbers to improve the accuracy of the found solution. Suppose the size of the m- and n-interval is set equal to 3 resp. 5. The region of critical wave numbers found then is e.g. m=8-11 and n=4-8. Now the accuracy will improve if one takes for instance m=7-12 and n=2-10. In this way one has to try different combinations of numbers of m and n in order to find the absolute critical buckling load.
8. Numerical results

Before performing a numerical calculation the material properties and the geometrical shape of the used anisotropic cylinder have to be defined. In order to make it possible to compare several reports the same kind of cylinder must be dealt with. Some frequently used shells are defined by e.g. Khot and Booton. Now is chosen for the Booton-shell, a glass-epoxy stiffened cylinder built-up of three layers. The properties of the cylinder are as follows.

\[ R = 2.670 \text{ inch} \]
\[ L = 3.776 \text{ inch} \]
\[ t = 0.0267 \text{ inch} \]

three layers

\[ E_{11} = 5.83 \times 10^6 \text{ psi} \]
\[ E_{22} = 2.42 \times 10^6 \text{ psi} \]
\[ v_{12} = 0.363 \]
\[ G_{12} = 0.668 \times 10^6 \text{ psi} \]
\[ h = \frac{t}{3} = 0.0089 \text{ inch} \]

\[ \eta_i \text{ ( from outer to inner layer )} \]
\[ \eta_1 = -30^\circ \]
\[ \eta_2 = 0^\circ \]
\[ \eta_3 = 30^\circ \]

To define the positive direction of the angle \( \eta_i \) one of the layers is presented in figure 6.

First a combined pure bending-axial compression calculation and a plot of the critical buckling load as a function of the axial compression load was performed. Then the influence of internal and external pressure on this curve was examined. Finally a combined shear-axial compression calculation was performed and these results were compared with those found in the combined pure bending-axial compression calculation.
8.1 Combined pure bending-axial compression

The first calculation made was a combined pure bending-axial compression calculation without internal or external pressure. The results of this calculation are presented in table 3 and the curve is plotted in figure 7. Not only the final results are printed, but also, for the pure compression case, the buckling load for the "neighbouring" values of m and, for the combined pure bending-axial compression case, the results for the search of the critical value of m in order to show the working of the numerical procedure. The calculation was performed with the reduced matrix method because m is larger than 2. In order to show the agreement between the reduced and the full matrix method also a full matrix calculation of the buckling load for \( N_a = 0 \) was performed.

Looking at the curve in figure 7 it can be seen that it is an almost straight line. There is a small deflection of the curve near the pure compression point. This deflection will be discussed in the next paragraph. Further we have to pay attention to the fact that the critical value of m is not equal along the whole curve. In the region \( N_a = 200-300 \) lbf/in this critical value of m changes from 7 to 6 and consequently there is a slight change of the buckling mode.

8.2 Combined pure bending-axial compression-normal pressure

First an internal pressure in the shell is introduced. The results of the whole calculation are presented in table 4 and figure 8. The calculation for \( p = -30 \) psi was performed with the reduced matrix method. For the three external pressure cases the full matrix method was applied because of the low value of m. In order to show the disagreement between the full and the reduced matrix method for these three cases a reduced matrix calculation was performed for \( p = 14 \) psi and \( N_a = 0 \). The difference appears to be 11.63 %.

In figure 8 one can see that a rather large internal pressure (\( p = -30 \) psi) has to be assumed to get a slight improvement of the buckling properties. The deflection near the pure compression point has decreased compared to the deflection for the curve without any internal or external pressure, the so-called zero pressure curve. The change of the buckling mode in axial direction in the interval \( N_a = 200-300 \) lbf/in has disappeared and m is equal to 7 along the whole curve.

Introduction of an external pressure has a more profound influence on the buckling curve. For an external pressure \( p = 6 \) psi the deflection near the pure compression point has increased. The buckling mode shape for the pure compression point has changed also. For a decreasing axial compression part of the total loading the curve will move towards the zero pressure curve and for \( N_a = 300-0 \) lbf/in they will coincide. From this one may conclude that external pressure has more influence on buckling caused by axial compression than on buckling caused by pure bending. If the
external pressure is enlarged now the whole curve will move away from the zero pressure curve. The critical $m$ will be equal to 1 along the whole curve. To illustrate this the critical buckling load is plotted as a function of $m$ and $p$. In figure 8 this is done for axial compression only and in figure 9 for pure bending only. In both figures one can see that the buckling load for $m=1$ is influenced most by a change of pressure. For a certain external pressure the buckling load for $m=1$ will become the absolute minimum and thus decisive.

8.3 Combined shear-axial compression

In the combined shear-axial compression case we will come across some difficulties that need a further explanation. These difficulties are caused by the fact that the critical buckling load can not be found in the "normal" way. In order to make it possible to find these eigenvalues some assumptions have to be made that will be discussed in the next few paragraphs.

8.3.1 The problem of determining the critical shear load

Before calculating the critical shear load one has to determine the $m$- and $n$-interval, for which this critical shear load will appear. First the $m$-interval is set equal to 3 and the $n$-interval equal to 5. The critical interval search is performed now for $N_a=0$, 100, 200, 300, and 350 lbf/in. Next the critical shear load for these values of $N_a$ has been tried to find. The results of this calculation are printed in table 5. How to determine the large intervals of $m$ and $n$ to calculate the critical buckling load for $N_a=300$ lbf/in and $N_a=350$ lbf/in will be discussed later.

Looking at this table one can draw three striking conclusions.

1. For $N_a=0$ lbf/in and $N_a=100$ lbf/in no eigenvalues are found.

2. If, for $N_a=200$ lbf/in, the $m$- and $n$-interval are extended the eigenvalue suddenly disappears.

3. For $N_a=300$ lbf/in and $N_a=350$ lbf/in there are no problems to find the critical buckling load.

Why is it possible to determine the critical buckling load for $N_a=300$ lbf/in and $N_a=350$ lbf/in while for lower values of $N_a$ no critical buckling load can be found any more? The answer
to his question can be found in the second column of table 5. For \( N_a = 300 \text{ lb/ft} \) we see that the critical interval appears to be \( m=5-7 \) and \( n=5-9 \). If we now look at \( N_a = 200 \text{ lb/ft} \) we see that this critical interval changed into \( m=1-3 \) and \( n=6-10 \). Previous work showed that the critical compressive force was found for \( m=6 \) and \( n=6 \). The critical shear interval for \( N_a = 300 \text{ lb/ft} \) includes these values of \( m \) and \( n \). The buckling mode found for \( N_a = 300 \text{ lb/ft} \) thus is a so-called "axial compression mode". From work done by Schröder (Ref 6) one can learn that for a relative short cylinder subjected to transverse shear loading the critical \( m \)-interval will include low values of \( m \). Because of the fact that the critical interval for \( N_a = 200 \text{ lb/ft} \) includes \( m=1-3 \) one can say that the buckling mode has changed into a "shear mode".

For \( N_a = 0 \text{ lb/ft} \) and \( N_a = 100 \text{ lb/ft} \) no eigenvalues can be found at all. For \( N_a = 200 \text{ lb/ft} \) eigenvalues can be found for small intervals of \( m \) and \( n \), but these eigenvalues do vanish if the intervals of \( m \) and \( n \) are extended. To form a picture of what happens here a few figures were performed, in which the determinant of the stability matrix was plotted as a function of the bending load \( N_{by} \) caused by transverse shear for \( N_a = 200 \text{ lb/ft} \). In figure 11 one can see that for \( m=1-3 \) and \( n=7-9 \) there are four zero-points for the determinant. Figure 12, where \( m=1-4 \) and \( n=6-10 \), shows that the first and second zero-point have disappeared and if one goes on enlarging the \( m \)- and \( n \)-interval to \( m=1-4 \) and \( n=5-11 \) all the zero-points have disappeared (figure 13). From chapter 3 though one learned that, if the degrees of freedom are increased, the critical buckling load has to decrease, because the constraints enforcing are being relaxed. The reason for this different behaviour has to do with the influence of the boundary conditions on the solution.

8.3.2 Influence of the boundary conditions on the detectability of the critical shear load

For the combined pure bending-axial compression case no problem was encountered in determining the critical buckling load. For the combined shear-axial compression case though the critical shear load could not be found for decreasing axial compression.

Buckling, caused by axial compression or pure bending, will appear in the middle of the cylinder (figure 14). Buckling though, caused by shear loading, will occur in an area near one of the boundaries of the cylinder and that is what causes trouble. In the isotropic case for \( u, v \) and \( w \) Fourier series were introduced, that satisfied the homogeneous boundary conditions stated below.

\[
v(0) = v(L/R) = w(0) = w(L/R) = 0
\]

\[
N_x(0) = N_x(L/R) = M_x(0) = M_x(L/R) = 0
\]

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For the anisotropic case other Fourier series had to be added to the ones used in the isotropic case to bring the number of variables into line with the number of equations again (chapter 4). By introducing these Fourier series the homogeneous boundary conditions are no longer satisfied. Because axial compression and pure bending cause buckling in the middle of the cylinder the boundary conditions won’t have much influence on these buckling loads. Shear loading though causes buckling near one of the boundaries and the boundary conditions will have enough influence on the buckling mode to disturb the convergence of the critical buckling load.

This explanation can be supported by looking at table 5. For $N_a = 300 - 389.5$ lb/in we don’t have any trouble to determine the critical shear load, because the buckling mode is an "axial compression mode". This was explained in paragraph 8.3. For this load case the cylinder buckles away from the boundary and the influence of the boundary conditions on the critical buckling load is insignificant. If $N_a$ is less than 300 lb/in though the location where buckling takes place will move towards one of the boundaries. The eigenvalue becomes unreliable and suddenly vanishes. The influence of the boundary conditions will grow and the buckling mode will be a "shear mode". In figure 15 also a few plots were performed of the displacement $w$ as a function of $x$ and $\theta$ for $N_a = 200$ lb/in, $m = 1 - 4$ and $n = 6 - 10$. One can see that for $x = 0$ and $x = L/R$ the displacement $w$ is not equal to zero and thus some negative influence of the boundary on the solution can be expected. Now we found a reason for not finding the right eigenvalues for the combined shear-axial compression case and one has to think of another way to find them.

8.3.3. The equivalent isotropic way of solution

From paragraph 8.3.1 one learned that for a large axial compression part the critical buckling load can be found. If the shear loading part increases, the eigenvalue suddenly vanishes and no critical buckling load can be found any more. The stability problem has to be made solvable for this case too.

The additional Fourier series in equation (4.2) are the cause for not fulfilling the boundary conditions. They had to be introduced because of the fact that the $\mathbf{B}$-matrix was not equal to zero. Whitney (Ref. 7) met the same problem and he suggested to omit the bending-stretching coupling. This causes the $\mathbf{B}$-matrix to be equal to zero and allows us to use the Fourier series for the isotropic and orthotropic case. In this way the boundary conditions can be fulfilled again and the approximate critical buckling load can be determined. This solution is called the equivalent isotropic solution.

To prove the rightness of this equivalent isotropic solution the critical shear load of a cylinder with the same material properties as the Booton cylinder was calculated in the anisotropic and the equivalent isotropic way. Only the length of the cylinder was varied so that $L/R$ was between 0.5 and 3.25. For the cases where the eigenvalue couldn’t be found the determinant was minimised. The results are presented in table 7. In this table also the step-size was printed, needed
to find the critical shear load. If no step-size is printed the eigenvalue couldn't be found and the determinant was minimised.

The first thing we can see is that for \( L/R \geq 2.5 \) the anisotropic solution could be found. Apparently the detectability of the critical buckling load in the anisotropic way also depends on the ratio of \( L/R \). The smaller the ratio of \( L/R \), the larger the problem to find the anisotropic solution. To find the equivalent isotropic solution the step-size has to be diminished also if the ratio of \( L/R \) is reduced, but a solution is always found. In figure 16 the critical buckling load is plotted as a function of \( L/R \) and one can see that for \( L/R \geq 1.7 \) the two solutions are in good agreement. For smaller values of \( L/R \) there is a slight difference between the solutions found with the two different methods. For this area we must remember that the anisotropic solution is only an estimation of the real solution, because of the fact that the real solution couldn't be located.

The calculation of the critical buckling load for the combined shear-axial compression case has to be performed as follows. First try to calculate the critical buckling load in the anisotropic way, starting with axial compression only and then moving towards an increasing shear loading part. As long as this critical buckling load is found without any problem the anisotropic way of calculating must be continued. If, for certain ratios of \( L/R \), the eigenvalue can't be found any more for an increasing shear loading part, one has to step over to the equivalent isotropic way of calculating. In figure 17 two examples of possible curves are plotted. Curve 1 is an example of a combined shear-axial compression curve where there are no problems to find all eigenvalues in the anisotropic way. For case 2 it was not possible to find all eigenvalues in the anisotropic way and the equivalent isotropic way of calculating had to be applied. We can see that curve 2 consists of two parts, the anisotropic part and the equivalent isotropic part. In this figure the total equivalent isotropic curve is presented to show that, if the eigenvalue can be found in the anisotropic way the equivalent isotropic way of calculating may not be applied. The closer you come to the pure compression point, the larger the difference of the two solutions. From this one can learn, that, for the axial compression (and pure bending) case, the equivalent isotropic solution is too rigorous and the additional Fourier series for the displacements \( u, v \) and \( w \) of the anisotropic cylinder may not be neglected. This is caused by the fact that for these two kinds of loading the cylinder will buckle somewhere in the middle (figure 14) and leaving out the additional Fourier series will have more negative influence on the solution than not fulfilling the homogeneous boundary conditions.

8.3.4 Final solution of the combined shear axial compression case

After this theoretical explanation of the method for calculating the critical buckling load, this calculation can be performed for the Booton cylinder. The results of this calculation are printed in tables 8-11. According to table 5 the anisotropic way of solution was successful only for \( N_a = 300 \text{ lbf/in} \) and \( N_a = 350 \text{ lbf/in} \). The equivalent isotropic way of calculating is performed for \( N_a = 0, 100, 200 \) and \( 300 \text{ lbf/in} \). In e.g. table 8 one can see how the large intervals of m's and n's,
for which the absolute lowest eigenvalue appears, is found. First the upper border of the m-interval is moved and the difference between the last two eigenvalues is regarded. If this difference becomes very small the moving of the upper border is stopped. The same procedure can be repeated for the lower border of the m-interval and the upper and lower border of the n-interval. Finally the large intervals of m and n can be chosen and the critical buckling load can be calculated. The resulting curves are plotted in figure 18. The mainly outside curve is the combined pure bending-axial compression curve we previously found. The mainly inside curve is the combined shear-axial compression curve.

At the abscissa, where \( N_a = 0 \) lbf/in, the presence of shearing stresses have reduced the bending allowables to only 62% of the load/inch corresponding to pure bending by an end couple. Because of the fact that the cylinder is relatively short \((L/R=1.4142)\) this is not an unexpected result. The shearing stresses are relatively high compared with the bending stresses \((V_0/N_{by}=0.707)\). In figure 16 it is proven, that for longer cylinders, where \( V_0 \) is much less than \( 0.707\cdot N_{by} \), one may expect a less significant reduction in strength.

The combined shear-axial compression curve in figure 18 will approach the real curve very close. At the area where the transfer is made from the anisotropic to the equivalent isotropic method of calculating one may expect a small deviation. This is caused by the fact that the curve found in the anisotropic way had to be extrapolated to "close" it.
Meyer (Ref 1) presented in his work an analysis for stiffened cylinders, subjected to combined axial compression, uniform normal pressure, pure bending and transverse shear. In this report the same analysis is presented for anisotropic shells. Computer programs were developed to calculate the critical buckling loads for isotropic, orthotropic and anisotropic cylindrical shells. The software was checked out on the basis of the results found by Meyer for orthotropic cylinders. Finally a calculation of the critical buckling load was made for three combined loading cases to show the working of the developed software and to allow a rough comparison with the results found by Meyer. As anisotropic cylinder the Booton shell was used (chapter 8).

To make it possible to calculate the critical buckling load for anisotropic cylinders the Fourier series solution functions for the displacements u, v and w had to be extended with additional Fourier series, that don't satisfy the homogeneous boundary conditions. For axial compression and pure bending the fact that the homogeneous boundary conditions are not satisfied won't have much influence on the critical buckling load, because of the fact that buckling will occur in the middle of the cylinder, away from the boundaries (figure 14). For transverse shear though buckling will occur near one of the boundaries and this has a negative influence on finding the critical buckling load for relative short cylinders. A solution to this problem is presented in paragraph 8.3.3.

Looking at the three curves (figure 7,8,18) the conclusion may be that the shape of these curves roughly agrees with the corresponding curves found by Meyer. In figure 7 we can see that the curve for the combined pure bending-axial compression case is an almost straight line. If the pure bending part of the total load is almost zero there is a small deflection of the curve. Meyer also found this small deflection and compares his results with the results of Seide and Weingarten, who did find a complete straight line.

In the presence of external pressure the axial load/inch is reduced significantly more by the pressure than the bending load/inch (figure 8). Internal pressure has a relatively insignificant effect upon the combined bending-axial compression curve. Again this is in good agreement with the results found by Meyer.

In figure 18 the curve for the combined shear-axial compression case is plotted. In order to make a comparison the bending-axial compression curve is plotted too. The transverse shear load produces a significant reduction of the critical bending load. This reduction though depends on the ratio of L/R as can be seen in figure 16. The longer the cylinder, the lesser the reduction of the critical bending load.

Finally the conclusion may be that, as far as a rough comparison of anisotropic and orthotropic cylinders is considered, this report is in good agreement with the results found by Meyer. We can see the same kind of influence of normal pressure, length and transverse shear on the combined bending-axial compression curve. Further some knowledge of the rough shape of the searched curve may help to locate the lowest eigenvalue of the stability matrix and reduces the chance of missing this lowest eigenvalue during the search.
Literature

1. Meyer, Robert Richard. "Buckling of stiffened cylindrical shells subjected to combined axial compression, normal pressure, bending and shear loading", University of California, Los Angeles, 1972


<table>
<thead>
<tr>
<th>$\frac{N_a}{N_{cl}}$</th>
<th>n</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
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<td>+</td>
<td>+</td>
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<td>+</td>
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<td>+</td>
<td>+</td>
<td>+</td>
</tr>
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<td></td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>-</td>
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Table 1: Sign of the determinant for increasing values of $\frac{N_a}{N_{cl}}$

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<th>n</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
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<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
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<tbody>
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<td>x</td>
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<td>x</td>
<td>x</td>
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<td></td>
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<td></td>
</tr>
<tr>
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<td></td>
<td>x</td>
<td>x</td>
<td>cr</td>
<td>x</td>
<td>x</td>
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<td></td>
<td></td>
<td></td>
<td></td>
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</tr>
<tr>
<td>2</td>
<td></td>
<td>x</td>
<td>x</td>
<td>0</td>
<td>x</td>
<td>x</td>
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Table 2: Combinations of m and n considered to find the critical buckling load

- x : combination of m and n considered
- 0 : combination of m and n for which a local minimum appears
- cr : combination of m and n for which the critical buckling load appears
<table>
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<th>$N_b$ (lbf/in)</th>
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<th>$N_a$ (lbf/in)</th>
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<td>389.5</td>
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<table>
<thead>
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<th>det</th>
<th>calculation of $N_b$ (lbf/in)</th>
<th>accuracy = 1.4254 lbf/in</th>
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<td></td>
<td></td>
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<td>0.000066</td>
<td>m=6; n=0-12 10.1</td>
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</tr>
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<td></td>
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<td>16.3</td>
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<td>35.6</td>
<td>5</td>
<td>0.039200</td>
<td>m=6; n=0-12 50.4</td>
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<td>300.0</td>
<td>99.8</td>
<td>5</td>
<td>0.034600</td>
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<td>210.5</td>
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<tr>
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<td>8</td>
<td>0.048200</td>
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<td>6</td>
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<td>314.6</td>
</tr>
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<td></td>
<td>7</td>
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<tr>
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<td>392.0</td>
<td>6</td>
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<td>418.2</td>
</tr>
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<td></td>
<td>7</td>
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</tr>
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<td></td>
<td>8</td>
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<td></td>
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</tr>
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</table>

(full matrix: $N_a=0.0$ lbf/in, $m=7$, $n=0-12$ : $N_b=416.4$ lbf/in)

Table 3: Results for the combined axial compression-pure bending load case for the Booton cylinder (reduced matrix method)
<table>
<thead>
<tr>
<th>pressure p (psi)</th>
<th>$N_a$ or $N_b$ (lbf/in)</th>
<th>m and n</th>
<th>$N_a$ or $N_b$ (lbf/in)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-30.0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>reduced matrix</td>
<td>$N_b = 0.0$</td>
<td>$m=7$; $n=5$</td>
<td>$N_a = 400.5$</td>
</tr>
<tr>
<td></td>
<td>$N_a = 370.0$</td>
<td>$m=7$; $n=0-10$</td>
<td>$N_b = 38.4$</td>
</tr>
<tr>
<td></td>
<td>$N_a = 300.0$</td>
<td>$m=7$; $n=0-10$</td>
<td>$N_b = 114.0$</td>
</tr>
<tr>
<td></td>
<td>$N_a = 0.0$</td>
<td>$m=7$; $n=0-10$</td>
<td>$N_b = 428.0$</td>
</tr>
<tr>
<td>6.0</td>
<td>$N_b = 0.0$</td>
<td>$m=1$; $n=7$</td>
<td>$N_a = 358.3$</td>
</tr>
<tr>
<td>full matrix</td>
<td>$N_a = 340.0$</td>
<td>$m=1$; $n=0-10$</td>
<td>$N_b = 47.0$</td>
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<tr>
<td></td>
<td>$N_a = 320.0$</td>
<td>$m=1$; $n=0-10$</td>
<td>$N_b = 78.2$</td>
</tr>
<tr>
<td></td>
<td>$N_a = 300.0$</td>
<td>$m=6$; $n=0-10$</td>
<td>$N_b = 104.5$</td>
</tr>
<tr>
<td>10.0</td>
<td>$N_b = 0.0$</td>
<td>$m=1$; $n=7$</td>
<td>$N_a = 252.8$</td>
</tr>
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<td>full matrix</td>
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<td>$m=1$; $n=0-10$</td>
<td>$N_b = 51.5$</td>
</tr>
<tr>
<td></td>
<td>$N_a = 200.0$</td>
<td>$m=1$; $n=0-10$</td>
<td>$N_b = 104.7$</td>
</tr>
<tr>
<td></td>
<td>$N_a = 100.0$</td>
<td>$m=1$; $n=0-10$</td>
<td>$N_b = 233.6$</td>
</tr>
<tr>
<td></td>
<td>$N_a = 0.0$</td>
<td>$m=1$; $n=0-10$</td>
<td>$N_b = 354.0$</td>
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<tr>
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<tr>
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<td>$N_b = 159.6$</td>
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<tr>
<td></td>
<td>$N_a = 0.0$</td>
<td>$m=1$; $n=0-10$</td>
<td>$N_b = 221.9$</td>
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</tbody>
</table>

(reduced matrix: $N_a = 0$ lbf/in, $p = 14.0$ psi, $m = 1$, $n = 0-10$ : $N_b = 247.7$ lbf/in)

Table 4: Results for the combined axial compression-pure bending-normal pressure load case for the Booton cylinder
<table>
<thead>
<tr>
<th>$N_{bv}$ (lbf/in)</th>
<th>critical shear interval for certain value of $N_{bv}$ (lbf/in)</th>
<th>critical buckling load</th>
<th>interval</th>
<th>$N_{bv}$ (lbf/in)</th>
</tr>
</thead>
</table>
| 0.0  | $N_{bv} = 228.1$  
m = 1-3 ; n = 7-11 | no critical buckling load found |  |  |
| 100.0 | $N_{bv} = 213.8$  
m = 1-3 ; n = 7-11 | no critical buckling load found |  |  |
| 200.0 | $N_{bv} = 142.5$  
m = 1-3 ; n = 6-10 | m = 1-3 ; n = 7-9  
m = 1-4 ; n = 6-10  
m = 1-4 ; n = 5-11 | 207.5  
190.8  
no critical buckling load found |  |
| 300.0 | $N_{bv} = 114.0$  
m = 5-7 ; n = 5-9 | m = 1-8 ; n = 3-11 | 108.1 |  |
| 350.0 | $N_{bv} = 57.0$  
m = 5-7 ; n = 5-9 | m = 3-8 ; n = 3-11 | 56.0 |  |

Table 5: Results for the combined axial compression-shear load case
Table 6: Stiffness matrix for the Booton cylinder

Table 7: Results of a calculation of $N_{bv0}$ in the equivalent isotropic and the anisotropic way for a Booton cylinder with variable length
<table>
<thead>
<tr>
<th>m-interval</th>
<th>n-interval</th>
<th>$\frac{N_{bv}}{N_{cl}}$</th>
<th>diff (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5-7</td>
<td>5-9</td>
<td>0.08559748</td>
<td>1.54</td>
</tr>
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<td>5-9</td>
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<td>3.24</td>
</tr>
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<td>3-7</td>
<td>5-9</td>
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<td>3-9</td>
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<td>2-9</td>
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Table 8: Results of the search for the critical m- and n-interval for the combined axial compression-shear load case. The anisotropic way of calculation is used and for the final m- and n-interval $N_{bv}$ is calculated ($N_a = 3501bf/in$)
<table>
<thead>
<tr>
<th>m-interval</th>
<th>n-interval</th>
<th>$\frac{N_{bv}}{N_{Cl}}$</th>
<th>$N_{Cl} = 712.7$ lbf/in</th>
<th>diff (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5-7</td>
<td>5-9</td>
<td>0.1748156</td>
<td></td>
<td>2.16</td>
</tr>
<tr>
<td>5-8</td>
<td>5-9</td>
<td>0.1710238</td>
<td></td>
<td>0.52</td>
</tr>
<tr>
<td>5-9</td>
<td>5-9</td>
<td>0.1701344</td>
<td></td>
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<td>5-9</td>
<td>0.1748156</td>
<td></td>
<td>5.17</td>
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<tr>
<td>4-7</td>
<td>5-9</td>
<td>0.1657773</td>
<td></td>
<td>1.32</td>
</tr>
<tr>
<td>3-7</td>
<td>5-9</td>
<td>0.1635971</td>
<td></td>
<td>0.73</td>
</tr>
<tr>
<td>2-7</td>
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<td></td>
<td>0.49</td>
</tr>
<tr>
<td>1-7</td>
<td>5-9</td>
<td>0.1616024</td>
<td></td>
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<td>5-9</td>
<td>0.1748156</td>
<td></td>
<td>1.41</td>
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<td>5-7</td>
<td>3-9</td>
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<td>0.06</td>
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<tr>
<td>5-7</td>
<td>2-9</td>
<td>0.1712304</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$m=1-8 \quad n=3-11 \quad \frac{N_{bv}}{N_{Cl}} = 0.15172883 \quad N_{bv} = 108.1$ lbf/in

Table 9: Results of the search for the critical m- and n-interval for the combined axial compression-shear load case. The anisotropic way of calculation is used and for the final m- and n-interval $N_{bv}$ is calculated ($N_a = 300$ lbf/in)
<table>
<thead>
<tr>
<th>m-interval</th>
<th>n-interval</th>
<th>( \frac{N_{\text{dv}}}{N_{\text{cl}}} )</th>
<th>( N_{\text{cl}} = 712.7 \text{ lbf/in} )</th>
<th>diff (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-3</td>
<td>6-10</td>
<td>0.19250038</td>
<td></td>
<td>1.86</td>
</tr>
<tr>
<td>1-3</td>
<td>6-11</td>
<td>0.18891476</td>
<td></td>
<td>0.06</td>
</tr>
<tr>
<td>1-3</td>
<td>6-12</td>
<td>0.18879823</td>
<td></td>
<td>0.80</td>
</tr>
<tr>
<td>1-3</td>
<td>6-13</td>
<td>0.18725780</td>
<td></td>
<td>0.00</td>
</tr>
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<td>1-3</td>
<td>6-14</td>
<td>0.18725638</td>
<td></td>
<td>0.00</td>
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<tr>
<td>1-3</td>
<td>6-15</td>
<td>0.18725705</td>
<td></td>
<td>0.30</td>
</tr>
<tr>
<td>1-3</td>
<td>5-15</td>
<td>0.18667797</td>
<td></td>
<td>0.00</td>
</tr>
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<td>4-15</td>
<td>0.18666049</td>
<td></td>
<td>0.29</td>
</tr>
<tr>
<td>1-4</td>
<td>4-15</td>
<td>0.18116800</td>
<td></td>
<td>0.00</td>
</tr>
<tr>
<td>1-4</td>
<td>4-16</td>
<td>0.18116809</td>
<td></td>
<td>0.00</td>
</tr>
<tr>
<td>1-5</td>
<td>4-15</td>
<td>0.18116787</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[ m=1-4 \quad n=4-15 \quad N_{\text{dv}}/N_{\text{cl}} = 0.18116800 \quad N_{\text{dv}} = 129.1 \text{ lbf/in} \]

**Table 10**: Results of the search for the final m- and n-interval for the combined axial compression-shear load case. The equivalent isotropic way of calculation is used and the final m- and n-interval \( N_{\text{dv}} \) is calculated ( \( N_a = 300 \text{ lbf/in} \))

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<table>
<thead>
<tr>
<th>(N_a \text{ or } N_{bv}) (lbf/in)</th>
<th>(m) and (n) or (m)- and (n)-interval</th>
<th>(N_a) or (N_{bv}) (lbf/in)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(N_{bv} = 0.0)</td>
<td>(m = 7; n = 0)</td>
<td>(N_a = 489.0)</td>
</tr>
<tr>
<td>(N_a = 300.0)</td>
<td>(m=1-4; n=4-15)</td>
<td>(N_{bv} = 129.1)</td>
</tr>
<tr>
<td>(N_a = 200.0)</td>
<td>(m=1-4; n=4-15)</td>
<td>(N_{bv} = 175.1)</td>
</tr>
<tr>
<td>(N_a = 100.0)</td>
<td>(m=1-4; n=4-15)</td>
<td>(N_{bv} = 218.4)</td>
</tr>
<tr>
<td>(N_a = 0.0)</td>
<td>(m=1-4; n=4-15)</td>
<td>(N_{bv} = 260.4)</td>
</tr>
</tbody>
</table>

Table 11: Results for the combined axial compression–shear load case where the calculation is performed in the equivalent isotropic way.
figure 1: The loading the cylinder is subjected to

figure 2: The sign-convention for the stress resultants and the stress couples
Figure 3: Nomenclature for the stacking sequence of the layers.

Figure 4: Elements of the normalized stability matrix as a function of row $j$ and column $i$. 

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Figure 5: The determinant of the stability matrix as a function of the bending load and the number of half waves in axial direction.

Figure 6: Local and global coordinate system for one layer.
Bootton cylinder dimensions

\[ R = 2.67 \, \text{in} \]
\[ L = 3.776 \, \text{in} \]
\[ t = 0.0267 \, \text{in} \]

3 layers

\[ h_{\text{layer}} = 0.0089 \, \text{in} \]
\[ E_{11} = 5.83 \times 10^6 \, \text{psi} \]
\[ E_{22} = 2.42 \times 10^6 \, \text{psi} \]
\[ v_{12} = 0.363 \]
\[ G_{12} = 0.668 \times 10^6 \, \text{psi} \]
\[ \eta_i = -30^\circ, 0^\circ, 30^\circ \]

\[ N_a = \frac{F}{2 \pi R} \quad ; \quad N_b = \frac{M}{\pi R^2} \]

Figure 7: Combined bending-axial compression for the Bootton cylinder
Boothon cylinder dimensions

\[ \begin{align*}
R &= 2.67 \text{ in} \\
L &= 3.776 \text{ in} \\
t &= 0.0267 \text{ in}
\end{align*} \]

3 layers

\[ \begin{align*}
n_{\text{layer}} &= 0.0089 \text{ in} \\
E_{11} &= 5.83 \times 10^6 \text{ psi} \\
E_{22} &= 2.42 \times 10^6 \text{ psi} \\
\nu_{12} &= 0.363 \\
G_{12} &= 0.668 \times 10^6 \text{ psi} \\
\eta_i &= -30^\circ, 0^\circ, 30^\circ
\end{align*} \]

\[ \begin{align*}
N_a &= \frac{F}{2\pi R} ; N_b = \frac{M}{\pi R^2}
\end{align*} \]

figure 8: Combined normal pressure-bending-axial compression for the Boothon cylinder
Figure 9: Buckling load for axial compression as a function of the pressure and the number of half waves in axial direction.

Figure 10: Buckling load for pure bending as a function of the pressure and the number of half waves in axial direction.
Figure 11: The determinant of the stability matrix as a function of the bending load $N_{BV}$ caused by transverse shear $m = 1-3$; $n = 7-9$

Figure 12: The determinant of the stability matrix as a function of the bending load $N_{BV}$ caused by transverse shear $m = 1-4$; $n = 6-10$
figure 13: The determinant of the stability matrix as a function of the bending load $N_{bV}$ caused by transverse shear $m = 1-4$; $n = 5-11$

figure 14: Critical buckling areas of a cylinder for different kinds of loading
Figure 15: The displacement w as a function of x and θ for:

\[ N_a = 200 \text{ lbf/in} \]
\[ m = 1-4 \]
\[ n = 6-10 \]
figure 16: Comparison of the buckling load, found in the equivalent isotropic and the anisotropic way of calculating, as a function of L/R
figure 17: Two possible combined shear - axial compression curves
Boothen cylinder dimensions

\[ R = 2.67 \text{ in} \]
\[ L = 3.776 \text{ in} \]
\[ t = 0.0267 \text{ in} \]

3 layers

\[ h_{\text{layer}} = 0.0089 \text{ in} \]
\[ E_{11} = 5.83 \cdot 10^6 \text{ psi} \]
\[ E_{22} = 2.42 \cdot 10^6 \text{ psi} \]
\[ \nu_{12} = 0.363 \]
\[ G_{12} = 0.668 \cdot 10^6 \text{ psi} \]
\[ \eta_i = -30^\circ, 0^\circ, 30^\circ \]

\[ N_a = \frac{F}{2\pi R}; N_{bv} = \frac{VL}{\pi R^2} \]

Figure 18: Comparison of the combined shear - axial compression curve and the combined bending - axial compression curve
Appendix A

Additional terms for rib stiffened cylinders

The theory how to add the additional stiffness terms due to the rib stiffening of a cylinder to the terms already existing has been treated by Meyer (Ref 1) and will not be repeated again here. The additional stiffness terms will be calculated for three different kinds of stiffening

1. $0^\circ$-$90^\circ$ rib stiffening

2. $\pm45^\circ$ rib stiffening

3. $0^\circ$-$+60^\circ$ rib stiffening (isogrid)

First we have to define the specific twist of a bar.

The total bar torque is defined by the torsional rigidity, $GJ$, and the specific twist of the bar, $\tau_i$, as can be seen in equation A.1.

$$T_i = G_i J_i \tau_i$$  \hspace{1cm} (A.1)

When the bars are in the coordinate directions the matter is straightforward. Otherwise we have to transfer the twist of the bar into the coordinate axes. First set $\chi = \chi x$ and $\chi = \chi \theta$. Looking at figure 19 we can see that

$$\chi_{12}(i) = a_1^{\alpha \beta} a_2^{\alpha \beta}$$  \hspace{1cm} (A.2)

Working out equation A.2 yields

$$\chi_{12}(i) = a_1^{\chi} a_2^{\chi} + a_1^{\chi \theta} a_2^{\chi \theta} + a_1^{\theta \chi} a_2^{\theta \chi} + a_1^{\theta \chi \theta}$$  \hspace{1cm} (A.3)

where

$$a_1^x = \cos \eta_1$$

$$a_2^x = -\sin \eta_1$$

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\[ a_1^\theta = \sin \eta_i \]
\[ a_2^\theta = \cos \eta_i \]  

(A.4)

Substitution of equation A.4 in A.3 now yields

\[ \chi_{12}(i) = \sin \eta_i \cos \eta_i \left( \chi_{i\theta} - \chi_{xx} \right) + (\cos^2 \eta_i - \sin^2 \eta_i) \chi_{x\theta} \]  

(A.5)

The physical twist is twice \( \chi_{12} \).

\[ \tau_i = 2 \left[ \sin \eta_i \cos \eta_i \left( \chi_{i\theta} - \chi_{xx} \right) + (\cos^2 \eta_i - \sin^2 \eta_i) \chi_{x\theta} \right] \]  

(A.6)

1. 0° - 90° rib stiffening

This kind of stiffening is formed by a set of stringers in axial direction and a set of rings or frames in circumferential direction. First define axial bar loads \( P_1 \) and \( P_2 \) and torque loads \( T_1 \) and \( T_2 \) (fig 20). \( M_{x\theta} \) and \( M_{\theta x} \) can be calculated now.

\[ P_1 = b_x E_x \epsilon_x \]
\[ P_2 = b_\theta E_\theta \epsilon_\theta \]

\[ \eta_1 = 0^\circ \]
\[ \tau_1 = 2 \chi_{x\theta} \]
\[ T_1 = G_x J_x \tau_1 = 2 G_x J_x \chi_{x\theta} \]
\[ M_{x\theta} = - \frac{T_1}{a_x} = - \frac{G_x J_x}{a_x} 2 \chi_{x\theta} \]

\[ \eta_2 = 90^\circ \]
\[ \tau_2 = -2 \chi_{x\theta} \]
\[ T_2 = G_\theta J_\theta \tau_2 = -2 G_\theta J_\theta \chi_{x\theta} \]
\[ M_{\theta x} = \frac{T_2}{a_\theta} = - \frac{G_\theta J_\theta}{a_\theta} 2 \chi_{x\theta} \]  

(A.7)

The "smearing-out" of \( P_1 \) and \( P_2 \) yields

\[ \sigma_x = \frac{P_1}{a_x} = \frac{b_x E_x}{a_x} \epsilon_x \]
\[ \sigma_\theta = \frac{P_2}{a_\theta} = \frac{b_\theta E_\theta}{a_\theta} \epsilon_\theta \]  

(A.8)
For the additional stiffness terms we have the following formulas (Ref 1)

\[ A_{i j}^a = \int z \sigma_{i j} dz \]
\[ B_{i j}^a = \int z \sigma_{i j} z dz \]
\[ D_{i j}^a = \int z \sigma_{i j} z^2 dz \]  \hspace{1cm} (A.9)

The stiffness terms due to the torque load (A.7) have to be added to the stiffness terms found with equation (A.9). As reference surface the midsurface of the skin is chosen. For the 0\(^\circ\)-90\(^\circ\) rib stiffened shell this yields

\[ A_{11}^a = \frac{E_x}{a_x} \int b_x dz = \frac{E_x A_x}{a_x} \]
\[ A_{22}^a = \frac{E_\theta}{a_\theta} \int b_\theta dz = \frac{E_\theta A_\theta}{a_\theta} \]

\[ A_{12}^a = A_{33}^a = 0 \]

\[ B_{11}^a = \frac{E_x}{a_x} \int b_x z dz = \frac{1}{2} \frac{E_x A_x}{a_x} \]
\[ B_{22}^a = \frac{E_\theta}{a_\theta} \int b_\theta z dz = \frac{1}{2} \frac{E_\theta A_\theta}{a_\theta} \]

\[ B_{12}^a = B_{33}^a = 0 \]

\[ D_{11}^a = \frac{E_x}{a_x} \int b_x z^2 dz = \frac{E_I x^3}{a_x} \]

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\[ D_{22}^a = \frac{E_\theta}{a_\theta} \int b_\theta z^2 \, dz = \frac{E_\theta I_{0x}}{a_\theta} \]

\[ D_{12}^a = 0 \]

\[ D_{33}^a = \frac{1}{2} \left[ \frac{G_x J_x}{a_x} + \frac{G_\theta J_\theta}{a_\theta} \right] \]  \hspace{1cm} (A.10)

Note that the eccentricity terms \( l_1 \) and \( l_2 \) are positive for inside ribs and negative for outside ribs. Further we can see that \( D_{33}^a \) is positive in contrast with \( M_{x0} \) and \( M_{\theta x} \). This is caused by the fact that \( \kappa_{x0} = -2 \chi_{x0} \).

2. +45° rib stiffening

This pattern consists of sets of stringers at + 45° to the longitudinal axis (figure 21). The grid lines are not in the coordinate directions. Therefore we have to transfer strains from the bar axes into the coordinate axes.

Let \( e_i \) be the uniaxial rib strain for the \( i^{th} \) rib and let \( e_{xx}, e_{\theta \theta} \) and \( e_{x\theta} \) be components of the strain tensor in the \((x, \theta)\) coordinate system. Now \( \eta_i \) is the angle between the \( i^{th} \) bar coordinate and the x-axis. For the strain component \( e_1 \) we now get

\[ e_1 = e_{xx} \cos^2 \eta_i + 2 e_{x\theta} \cos \eta_i \sin \eta_i + e_{\theta \theta} \sin^2 \eta_i \]  \hspace{1cm} (A.11)

For the angles that the bar coordinates make with the x-axis we have \( \eta_1 = 45^\circ \) and \( \eta_2 = 135^\circ \). For \( e_1 \) and \( e_2 \) we now get

\[
\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 2 & 1 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} e_{xx} \\ e_{x\theta} \\ e_{\theta \theta} \end{bmatrix}
\]  \hspace{1cm} (A.12)

Now we can go on as for the 0°-90° stiffening.
\[ P_1 = \text{bee}_1 \]
\[ P_2 = \text{bee}_2 \]
\[ \eta_1 = -45^\circ \quad \eta_2 = 45^\circ \]
\[ \tau_1 = \chi_x - \chi_\theta \quad \tau_2 = \chi_\theta - \chi_x \]
\[ T_1 = GJ\tau_1 = GJ \left( \chi_x - \chi_\theta \right) \quad T_2 = GJ\tau_2 = GJ \left( \chi_\theta - \chi_x \right) \]

This yields for \( M_{x_\theta} \), \( M_{\theta x} \), \( M_x \), and \( M_\theta \) due to torsion

\[ M_{x_\theta} = \frac{-\left(T_1 + T_2\right) \cos 45^\circ}{a \left(2a\right)} = 0 \]

\[ M_{\theta x} = \frac{\left(T_1 + T_2\right) \cos 45^\circ}{a \left(2a\right)} = 0 \]

\[ M_x = \frac{\left(T_2 - T_1\right) \cos 45^\circ}{a \left(2a\right)} = \frac{GJ}{a} \left(\chi_\theta - \chi_x\right) \]

\[ M_\theta = \frac{\left(T_1 - T_2\right) \cos 45^\circ}{a \left(2a\right)} = \frac{GJ}{a} \left(\chi_x - \chi_\theta\right) \]

(A.13)

The "smearing-out" of \( P_1 \) and \( P_2 \) yields

\[ \sigma_x = \sigma_\theta = \frac{\left(P_1 + P_2\right) \cos 45^\circ}{a \left(2a\right)} = \frac{P_1 + P_2}{2a} = \frac{bE}{2a} \left(e_{xx} + e_{\theta\theta}\right) \]

\[ \tau_{x\theta} = \tau_{x\theta} = \frac{\left(P_1 - P_2\right) \cos 45^\circ}{a \left(2a\right)} = \frac{P_1 - P_2}{2a} = \frac{bE}{a} e_{x\theta} = \frac{bE}{2a} \chi_{x\theta} \]

(A.14)

Substitution of equation A.15 in A.8 now yields, together with the stiffness terms due to torsion (equation A.14)

\[ A_{11}^a = \int_0^a b dz = \frac{EA}{2a} \]

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\[ A_{12}^a = \frac{E}{2a} \int b dz = \frac{EA}{2a} \]

\[ A_{22}^a = \frac{E}{2a} \int b dz = \frac{EA}{2a} \]

\[ A_{33}^a = \frac{E}{2a} \int b dz = \frac{EA}{2a} \]

\[ B_{11}^a = \frac{E}{2a} \int b dz = \frac{eEA}{2a} \]

\[ B_{12}^a = \frac{E}{2a} \int b dz = \frac{eEA}{2a} \]

\[ B_{22}^a = \frac{E}{2a} \int b dz = \frac{eEA}{2a} \]

\[ B_{33}^a = \frac{E}{2a} \int b dz = \frac{eEA}{2a} \]

\[ D_{11}^a = \frac{E}{2a} \int b z^2 dz + \frac{GJ}{a} = \frac{EbI_0}{2a} + \frac{GJ}{a} \]

\[ D_{12}^a = \frac{E}{2a} \int b z^2 dz - \frac{GJ}{a} = \frac{EbI_0}{2a} - \frac{GJ}{a} \]

\[ D_{22}^a = \frac{E}{2a} \int b z^2 dz + \frac{GJ}{a} = \frac{EbI_0}{2a} + \frac{GJ}{a} \]

\[ D_{33}^a = \frac{E}{2a} \int b z^2 dz = \frac{EbI_0}{2a} \]  \hspace{1cm} (A.16)

Note that again e is positive for inside ribs and negative for outside ribs.
3. 0° - +60° rib stiffening

First we have to transfer stresses and strains from the bar axes to the coordinate axes. With \( \eta_1=0^\circ, \eta_2=60^\circ \) and \( \eta_3=120^\circ \) this yields

\[
\begin{bmatrix}
e_1 \\
e_2 \\
e_3
\end{bmatrix} = \frac{1}{4}
\begin{bmatrix}
4 & 0 & 0 \\
1 & 2\sqrt{3} & 3 \\
1 & -2\sqrt{3} & 3
\end{bmatrix}
\begin{bmatrix}
e_{xx} \\
e_{x\theta} \\
e_{\theta\theta}
\end{bmatrix}
\]  
(A.17)

According to figure 22 we now get

\[
P_1 = bEe_1 \\
P_2 = bEe_2 \\
P_3 = bEe_3
\]

\[
\eta_1 = 0^\circ \\
\eta_2 = -60^\circ \\
\tau_1 = 2x_{x\theta} \\
\tau_2 = \frac{3(x_x-x_{x\theta})}{2} - x_{x\theta} \\
T_1 = GJ\tau_1 = 2GJx_{x\theta} \\
T_2 = GJ\tau_2 = GJ \left[ \frac{3(x_x-x_{x\theta})}{2} - x_{x\theta} \right] \\
\eta_3 = 60^\circ \\
\tau_3 = \frac{3(x_{x\theta}-x_x)}{2} - x_{x\theta} \\
T_3 = GJ\tau_3 = GJ \left[ \frac{3(x_{x\theta}-x_x)}{2} - x_{x\theta} \right]
\]  
(A.18)

This yields for \( M_{x\theta}, M_{\theta x}, M_x \) and \( M_\theta \) due to torsion

\[
M_{x\theta} = -\frac{2T_1 + (T_2 + T_3)\cos 60^\circ}{a\sqrt{3}} = -\frac{3GJ}{2h} x_{x\theta}
\]

\[
M_{\theta x} = \frac{(T_2 + T_3)\cos 30^\circ}{a} = \frac{3GJ}{2h} x_{x\theta}
\]

\[
M_x = \frac{(T_3 - T_2)\sin 60^\circ}{a\sqrt{3}} = \frac{3GJ}{4h} (x_{\theta} - x_x)
\]

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\[ M_0 = \frac{(T_2 - T_3) \sin 30^\circ}{a} = \frac{3GJ}{4h} (\chi_x - \chi_\theta) \]  

(A.19)

The "smearing-out" of \( P_1 \) and \( P_2 \) yields

\[ \sigma_x = \frac{2P_1 + (P_2 + P_3) \cos 60^\circ}{a \sqrt{3}} = \frac{4P_1 + P_2 + P_3}{2a} \frac{3}{2} = \frac{9bE}{8h} \left( \frac{1}{3} e_{xx} + e_{\theta \theta} \right) \]

\[ \sigma_\theta = \frac{(P_2 + P_3) \sin 60^\circ}{a \sqrt{3}} = \frac{3(P_2 + P_3)}{2a} = \frac{9bE}{8h} \left( \frac{1}{3} e_{xx} + e_{\theta \theta} \right) \]

\[ \tau_{x\theta} = \tau_{\theta x} = \frac{(P_2 - P_3) \sin 60^\circ}{a \sqrt{3}} = \frac{P_2 - P_3}{2a} \frac{3bE}{4h} e_{x\theta} = \frac{3bE}{8h} \gamma_{x\theta} \]  

(A.20)

For the additional stiffness terms we now get, after substitution of equation A.19 in A.8 and addition of the stiffness terms due to torsion (equation A.18)

\[ A_{11}^a = \frac{9E}{8h} \int z \, bdz = \frac{9EA}{8h} \]

\[ A_{12}^a = \frac{9E}{24h} \int z \, bdz = \frac{9EA}{24h} \]

\[ A_{22}^a = \frac{9E}{8h} \int z \, bdz = \frac{9EA}{8h} \]

\[ A_{33}^a = \frac{3E}{8h} \int z \, bdz = \frac{3EA}{8h} \]

\[ B_{11}^a = \frac{9E}{8h} \int z \, bdz = \frac{9eEA}{8h} \]

\[ B_{12}^a = \frac{9E}{24h} \int z \, bdz = \frac{9eEA}{24h} \]

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\[
B_{22}^a = \frac{9E}{8h} \int_b z dz = \frac{9eEA}{8h}
\]
\[
B_{33}^a = \frac{3E}{8h} \int_b z dz = \frac{3eEA}{8h}
\]
\[
D_{11}^a = \frac{9E}{8h} \int_b z^2 dz + \frac{3GJ}{4h} = \frac{9EI_0}{8h} + \frac{3GJ}{4h}
\]
\[
D_{12}^a = \frac{9E}{24h} \int_b z^2 dz - \frac{3GJ}{4h} = \frac{9EI_0}{24h} - \frac{3GJ}{4h}
\]
\[
D_{22}^a = \frac{9E}{8h} \int_b z^2 dz + \frac{3GJ}{4h} = \frac{9EI_0}{8h} + \frac{3GJ}{4h}
\]
\[
D_{33}^a = \frac{3E}{8h} \int_b z^2 dz + \frac{3GJ}{4h} = \frac{3EI_0}{8h} + \frac{3GJ}{4h}
\]

(A.21)

Here again we must remember that \(e\) is positive for inside and negative for outside rib stiffening.
figure 19: Rotation of coordinates

figure 20: Element of a 0°-90° rib grid
figure 21 : Element of a ±45° rib grid

figure 22 : Element of a 0°±60° rib grid (isogrid)
Appendix B

Evaluation of the integrals

Before calculating the integrals $R_{ji}$, $S_{ji}$, $T_{ji}$, $F_{ji}$ etc., a calculation of smaller integrals is made before composing the larger ones. The integrals have to be put into a standard form first. For $R_{ji}$ for instance we have

$$R_{ji} = \int_A \cos \theta \cos \phi \cos \lambda_m x \cos \lambda_q x \, dA \tag{B.1}$$

where

$$dA = R^2 \, dx \, d\theta$$

$$x = \begin{cases} \frac{L}{R} & ; \quad \theta = \frac{2\pi}{0} \\ 0 & ; \quad \theta = 0 \end{cases} \tag{B.2}$$

Now we introduce a new variable $y$.

$$x = \frac{L}{\pi R} y$$

$$dA = \frac{L}{\pi R} \, d\theta \, dy ; \quad \lambda_m x = my ; \quad \lambda_q x = qy \tag{B.3}$$

For equation (B.1) this yields

$$R_{ji} = \int_0^{2\pi} \int_0^\pi \cos \theta \cos \phi \cos \mu \cos q \, dy \, d\theta \tag{B.4}$$

In this way the $\lambda$-terms in the sin and cos are eliminated. Now the smaller integrals of which the total ones are composed will be calculated.
\[
I_1 = \int_0^{2\pi} \cos n\theta \cos n\theta \, d\theta = \begin{cases} 
2 & ; \ n=p=0 \\
1 & ; \ n=p\neq 0 \\
0 & ; \ otherwise
\end{cases}
\]

\[
I_2 = \int_0^{\pi} \cos m\theta \cos q\theta \, dy = \frac{\pi}{2} \begin{cases} 
1 & ; \ m=q \\
0 & ; \ otherwise
\end{cases}
\]

\[
I_3 = \int_0^{2\pi} \sin n\theta \sin n\theta \, d\theta = \pi \begin{cases} 
1 & ; \ n=p=0 \\
1 & ; \ n=p\neq 0 \\
0 & ; \ otherwise
\end{cases}
\]

\[
I_4 = \int_0^{\pi} \sin m\theta \sin q\theta \, dy = \frac{\pi}{2} \begin{cases} 
1 & ; \ m=q \\
0 & ; \ otherwise
\end{cases}
\]

\[
I_5 = \int_0^{2\pi} \cos n\theta \cos m\theta \, d\theta \\
= \frac{1}{2} \int_0^{2\pi} \left[ \cos (n-1)\theta + \cos (n+1)\theta \right] \cos m\theta \, d\theta \\
= \frac{1}{2} \left( K_1 + K_2 \right)
\]

\[
K_1 = \int_0^{2\pi} \cos (n-1)\theta \cos m\theta \, d\theta = \pi \begin{cases} 
2 & ; \ n-1=p=0 \\
1 & ; \ n-1=p\neq 0 \ or \ p-1=n=0 \\
0 & ; \ otherwise
\end{cases}
\]

\[
K_2 = \int_0^{2\pi} \cos (n+1)\theta \cos m\theta \, d\theta = \pi \begin{cases} 
1 & ; \ n+1=p\neq 0 \\
0 & ; \ otherwise
\end{cases}
\]

(\ n+1=p=0 \ does \ not \ occur \ since \ n \geq 0 \ )

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\[ I_5 = \int_0^{2\pi} \cos \theta \cos n \theta \cos p \theta \, d\theta = \frac{\pi}{2} \begin{cases} 2 ; & n-1=p=0 \text{ or } p-1=n=0 \\ -1 ; & n-1=p \neq 0 \text{ or } p-1=n \neq 0 \\ 0 ; & \text{otherwise} \end{cases} \]

\[ I_6 = \int_0^{2\pi} \cos \theta \sin n \theta \sin p \theta \, d\theta = \frac{1}{2} \int_0^{2\pi} \left[ \sin (n+1) \theta + \sin (n-1) \theta \right] \sin p \theta \, d\theta \]

\[ = \frac{1}{2} \left( A_1 + A_2 \right) \]

\[ A_1 = \int_0^{2\pi} \sin (n+1) \theta \sin p \theta \, d\theta = \frac{\pi}{2} \begin{cases} 1 ; & n+1=p \neq 0 \\ 0 ; & \text{otherwise} \end{cases} \]

\[ A_2 = \int_0^{2\pi} \sin (n-1) \theta \sin p \theta \, d\theta = \frac{\pi}{2} \begin{cases} 1 ; & n-1=p \neq 0 \\ -1 ; & n=0 \text{ or } p=1 \\ 0 ; & \text{otherwise} \end{cases} \]

\[ I_6 = \int_0^{2\pi} \cos \theta \sin n \theta \sin p \theta \, d\theta = \frac{\pi}{2} \begin{cases} 1 ; & (n+1=p \text{ or } p+1=n) \text{ and } n \neq 0, p \neq 0 \\ 0 ; & \text{otherwise} \end{cases} \]

(The comma is to be read as "and")

\[ I_7 = \int_0^\pi y \cos m \theta \cos q y \, dy \]

\[ = \frac{1}{2} \int_0^\pi y \left[ \cos (m+q) y + \cos (m-q) y \right] \, dy \]

Now we have to consider two cases
\[ 2I_7 = \int_0^\pi y \, dy + \int_0^\pi yc\cos^2my \, dy \]

\[ = \left[ \frac{y^2}{2} \right]_0^\pi + \left[ \frac{1}{2m} \sin 2my \right]_0^\pi - \int_0^\pi \sin 2my \frac{dy}{2m} \]

\[ = \frac{\pi^2}{2} + \left[ \frac{1}{(2m)^2} \cos 2my \right]_0^\pi = \frac{\pi^2}{2} \]

\[ I_7 = \frac{\pi^2}{4} \]

\[ 2I_7 = \left[ \frac{y}{m+q} \sin (m+q)y \right]_0^\pi + \left[ \frac{y}{m-q} \sin (m-q)y \right]_0^\pi \]

\[ - \int_0^\pi \sin (m+q)y \frac{dy}{m+q} - \int_0^\pi \sin (m-q)y \frac{dy}{m-q} \]

\[ = \left[ \frac{1}{(m+q)^2} \cos (m+q)y \right]_0^\pi + \left[ \frac{1}{(m-q)^2} \cos (m-q)y \right]_0^\pi \]

\[ = - \frac{2}{(m+q)^2} \begin{cases} 1 & \text{if } m+q \text{ odd} \\ 0 & \text{if } m+q \text{ even} \end{cases} - \frac{2}{(m-q)^2} \begin{cases} 1 & \text{if } m-q \text{ odd} \\ 0 & \text{if } m-q \text{ even} \end{cases} \]

Note that \( m+q \text{ odd} \rightarrow m-q \text{ odd} \)

\[ m+q \text{ even} \rightarrow m-q \text{ even} \]

\[ I_7 = - \frac{2(m^2+q^2)}{(m^2-q^2)^2} \begin{cases} 1 & \text{if } m+q \text{ odd} \\ 0 & \text{otherwise} \end{cases} \]

This yields for all \( m \) and \( q \)
\[ I_7 = \int_0^\pi y \cos \sin \cos^2 \theta \, dy = \begin{cases} 1 \frac{2}{\pi} & ; \ m = q \\
\frac{2(\frac{m^2 + q^2}{m^2 - q^2})^{\frac{1}{2}}}{\frac{1}{2}(m^2 - q^2)} & ; \ m + q \text{ odd, } m \neq q \\
0 & ; \text{ otherwise} \end{cases} \]

\[ I_8 = \int_0^\pi y \sin \sin \sin \theta \, dy = \frac{1}{2} \int_0^\pi y \left[ \cos (m-q) y - \cos (m+q) y \right] \, dy \]

Now we have to consider two cases again:

\[ m = q \]

\[ 2I_8 = \int_0^\pi y \, dy - \int_0^\pi y \cos 2m \, dy \]

\[ = \left[ \frac{y^2}{2} \right]_0^\pi - \left[ \frac{y}{2m} \sin 2my \right]_0^\pi + \int_0^\pi \sin 2my \frac{dy}{2m} \]

\[ = \frac{\pi^2}{2} + \left[ - \frac{1}{(2m)^2} \cos 2my \right]_0^\pi \]

\[ I_8 = \frac{\pi^2}{4} \]

\[ m \neq q \]

\[ 2I_8 = \left[ \frac{y}{m-q} \sin (m-q) y \right]_0^\pi - \left[ \frac{y}{m+q} \sin (m+q) y \right]_0^\pi \]

\[ - \int_0^\pi \sin (m-q) y \frac{dy}{m-q} + \int_0^\pi \sin (m+q) y \frac{dy}{m+q} \]

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\[ \begin{align*} 
&= \left[ \frac{1}{(m-q)^2} \cos(m-q)y \right]_0^\pi - \left[ \frac{1}{(m+q)^2} \cos(m+q)y \right]_0^\pi \\
&= -\frac{2}{(m-q)^2} \begin{cases} 
1 & \text{if } m+q \text{ odd} \\
0 & \text{if } m+q \text{ even} 
\end{cases} + \frac{2}{(m+q)^2} \begin{cases} 
1 & \text{if } m+q \text{ odd} \\
0 & \text{if } m+q \text{ even} 
\end{cases} \\
I_8 &= -\frac{4mq}{(m^2-q^2)^2} \begin{cases} 
1 & \text{if } m+q \text{ odd} \\
0 & \text{otherwise} 
\end{cases} 
\end{align*} \]

This yields for all \(m\) and \(q\)

\[ I_8 = \int_0^\pi y \sin y \sin y \ dy = -\frac{\pi^2}{4} \begin{cases} 
\frac{\pi^2}{4} & \text{if } m=q \\
\frac{4mq}{(m^2-q^2)^2} & \text{if } m+q \text{ odd, } m \neq q \\
0 & \text{otherwise} 
\end{cases} \]

\[ I_9 = \int_0^{2\pi} \sin \theta \sin \theta \cos \theta \ d\theta \]

\[ = \frac{1}{2} \int_0^{2\pi} \left( \sin(p+1)\theta - \sin(p-1)\theta \right) \sin \theta \ d\theta \]

\[ = \frac{1}{2} \left( B_1 - B_2 \right) \]

\[ B_1 = \int_0^{2\pi} \sin(p+1)\theta \sin \theta \ d\theta = \pi \begin{cases} 
1 & \text{if } p+1=0 \\
0 & \text{otherwise} 
\end{cases} \]

\[ B_2 = \int_0^{2\pi} \sin(p-1)\theta \sin \theta \ d\theta = \pi \begin{cases} 
1 & \text{if } p-1=0, \ n=1 \\
-1 & \text{if } p=0, \ n=0 \\
0 & \text{otherwise} 
\end{cases} \]

This yields for \(I_9\)
\[ I_9 = \int_0^{2\pi} \sin \theta \sin n \theta \cos p \theta \, d\theta = \frac{\pi}{2} \begin{cases} 2 & n-1=p=0 \\ 1 & n+1=p\neq0, \ p\neq0 \\ -1 & p-1=n\neq0 \\ 0 & \text{otherwise} \end{cases} \]

\[ I_{10} = \int_0^{2\pi} \sin \theta \cos n \theta \sin p \theta \, d\theta = \frac{1}{2} \int_0^{2\pi} \left[ \sin (n+1) \theta - \sin (n-1) \theta \right] \sin p \theta \, d\theta \]

\[ = \frac{1}{2} \left( C_1 - C_2 \right) \]

\[ C_1 = \int_0^{2\pi} \sin (n+1) \theta \sin p \theta \, d\theta = \pi \begin{cases} 1 & n+1=p\neq0 \\ 0 & \text{otherwise} \end{cases} \]

\[ C_2 = \int_0^{2\pi} \sin (n-1) \theta \sin p \theta \, d\theta = \pi \begin{cases} 1 & n-1=p\neq0 \\ -1 & n=0, \ p=1 \\ 0 & \text{otherwise} \end{cases} \]

This yields for \( I_{10} \)

\[ I_{10} = \int_0^{2\pi} \sin \theta \cos n \theta \sin p \theta \, d\theta = \frac{\pi}{2} \begin{cases} 2 & p-1=n=0 \\ 1 & n+1=p\neq0, \ n\neq0 \\ -1 & p-1=n\neq0 \\ 0 & \text{otherwise} \end{cases} \]

\[ I_{11} = \int_0^\pi \sin \theta \cos q \theta \, d\theta = \frac{1}{2} \int_0^\pi \left[ \sin (q+m) \theta - \sin (q-m) \theta \right] \, d\theta \]

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\[ I_{11} = \int_0^\pi \sin \gamma \cos q \gamma \, dy = \begin{cases} \frac{2m}{m^2-q^2} & ; m+q \text{ odd} \\ \frac{m^2-q^2}{m} & ; \text{otherwise} \end{cases} \]

\[ I_{12} = \int_0^\pi \cos \gamma \sin q \gamma \, dy = \frac{1}{2} \int_0^\pi \left( \sin (m+q) \gamma - \sin (m-q) \gamma \right) \, dy \]

\[ = \frac{1}{2} \left[ \begin{array}{c} \frac{1}{m+q} \cos (m+q) \gamma + \frac{1}{m-q} \cos (m-q) \gamma \\ \frac{1}{m+q} \end{array} \right]_0^\pi \]

\[ = \frac{1}{m+q} \begin{cases} 1 & ; m+q \text{ odd} \\ 0 & ; m+q \text{ even} \end{cases} - \frac{1}{m-q} \begin{cases} 1 & ; m-q \text{ odd} \\ 0 & ; m-q \text{ even} \end{cases} \]

With these twelve integrals we can calculate the integrals we are looking for

\[ R_{ji} = \int_A \left( \cos \theta \cos \theta \cos \lambda_m x \cos \lambda_q x \right) \, dA \]

\[ = \frac{RL}{\pi} \int_0^\pi \cos \gamma \cos q \gamma \, dy \int_0^{2\pi} \cos \theta \cos \theta \, d\theta \]

\[ = \frac{RL}{\pi} \cdot I_2 \cdot I_1 \]

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\[ R_{ji} = \frac{\pi RL}{2} \begin{cases} 2 & ; m=q, \ n=p=0 \\ -1 & ; m=q, \ n=p\neq0 \\ 0 & ; \text{otherwise} \end{cases} \]

\[ S_{ji} = \int_A ( \sin\theta \sin\phi \sin\lambda_m x \sin\lambda_q x ) dA \]
\[ = \frac{RL}{\pi} \int_0^\pi \sin y \sin y \ dy \int_0^{2\pi} \sin\theta \sin\phi \ d\theta \]
\[ = \frac{RL}{\pi} \cdot I_4 \cdot I_3 \]

\[ S_{ji} = \frac{\pi RL}{2} \begin{cases} 1 & ; m=q, \ n=p\neq0 \\ 0 & ; \text{otherwise} \end{cases} \]

\[ T_{ji} = \int_A ( \cos\theta \cos\phi \sin\lambda_m x \sin\lambda_q x ) dA \]
\[ = \frac{RL}{\pi} \int_0^\pi \sin y \sin y \ dy \int_0^{2\pi} \cos\theta \cos\phi \ d\theta \]
\[ = \frac{RL}{\pi} \cdot I_4 \cdot I_1 \]

\[ T_{ji} = \frac{\pi RL}{2} \begin{cases} 2 & ; m=q, \ n=p=0 \\ -1 & ; m=q, \ n=p\neq0 \\ 0 & ; \text{otherwise} \end{cases} \]

\[ T'_{ji} = \int_A ( \sin\theta \sin\phi \cos\lambda_m x \cos\lambda_q x ) dA \]
\[ = \frac{RL}{\pi} \int_0^\pi \cos y \cos y \ dy \int_0^{2\pi} \sin\theta \sin\phi \ d\theta \]
\[ T'_{ji} = \frac{\pi RL}{2} \begin{cases} 1 & \text{if } m=q, \ n=p \neq 0 \\ 0 & \text{otherwise} \end{cases} \]

\[ F_{ji} = \int_A \cos \theta \left( \cos \theta \cos \phi \cos \lambda \cdot x \cos \lambda \cdot y \right) dA \]
\[ = \frac{RL}{\pi} \int_0^\pi \cos \theta \cos \phi \cos \lambda \cdot x d\phi \int_0^{2\pi} \cos \theta \cos \phi \cos \lambda \cdot y d\theta \]
\[ = \frac{RL}{\pi} \cdot I_2 \cdot I_5 \]

\[ F_{ji} = \frac{\pi RL}{4} \begin{cases} 2 & \text{if } m=q, \ (n-1=p=0 \text{ or } p-1=n=0) \\ 1 & \text{if } m=q, \ (n-1=p \neq 0 \text{ or } p-1=n \neq 0) \\ 0 & \text{otherwise} \end{cases} \]

\[ G_{ji} = \int_A \sin \theta \sin \phi \sin \lambda \cdot x \sin \lambda \cdot y dA \]
\[ = \frac{RL}{\pi} \int_0^\pi \sin \theta \sin \phi \sin \lambda \cdot x d\phi \int_0^{2\pi} \cos \theta \sin \phi \sin \lambda \cdot y d\theta \]
\[ = \frac{RL}{\pi} \cdot I_4 \cdot I_6 \]

\[ G_{ji} = \frac{\pi RL}{4} \begin{cases} 1 & \text{if } m=q, \ (n+1=p \text{ or } p+1=n), \ n \neq 0, \ p \neq 0 \\ 0 & \text{otherwise} \end{cases} \]

\[ K_{ji} = \int_A \cos \theta \left( \cos \theta \cos \phi \sin \lambda \cdot x \sin \lambda \cdot y \right) dA \]
\[ = \frac{RL}{\pi} \int_0^\pi \sin \theta \sin \phi \sin \lambda \cdot x d\phi \int_0^{2\pi} \cos \theta \cos \phi \cos \lambda \cdot y d\theta \]
\[ K_{ji} = \begin{cases} \frac{\pi R L}{4} & m=q, \ (n-1=p=0 \text{ or } p-1=n=0) \\ 1 & m=q, \ (n-1=p\neq0 \text{ or } p-1=n\neq0) \\ 0 & \text{otherwise} \end{cases} \]

\[ K'_{ji} = \int_A \cos \theta \ (\sin \theta \sin \phi \cos \lambda_m \cos \lambda_q \ x) \ dA \]
\[ = \frac{RL}{\pi} \int_0^{\pi} \cos \phi \cos \theta \ y \ d\theta \]
\[ = \frac{RL}{\pi} \cdot I_2 \cdot I_6 \]

\[ K'_{ji} = \begin{cases} \frac{\pi R L}{4} & m=q, \ (n+1=p \text{ or } p+1=n), \ n\neq0, \ p\neq0 \\ 0 & \text{otherwise} \end{cases} \]

\[ O_{ji} = \int_A \cos \theta \ (\cos \theta \cos \phi \cos \lambda_m \cos \lambda_q \ x) \ dA \]
\[ = \frac{L^2}{\pi} \int_0^{\pi} \cos \phi \cos \theta \ y \ d\theta \]
\[ = \frac{L^2}{\pi} \cdot I_7 \cdot I_5 \]

\[ O_{ji} = \begin{cases} \frac{L^2}{2\pi} & m=q, \ (n-1=p=0 \text{ or } p-1=n=0) \\ \frac{\pi}{4} & m=q, \ (n-1=p\neq0 \text{ or } p-1=n\neq0) \\ -\frac{4(m^2+q^2)}{(m^2-q^2)^2} & m\neq q, \ m+q \text{ odd}, \ (n-1=p=0 \text{ or } p-1=n=0) \\ -\frac{2(m^2+q^2)}{(m^2-q^2)^2} & m\neq q, \ m+q \text{ odd}, \ (n-1=p\neq0 \text{ or } p-1=n\neq0) \\ 0 & \text{otherwise} \end{cases} \]
\[ P_{ji} = \int_{A} x \cos \theta (\sin \theta \sin \phi \sin \lambda_m \sin \lambda_q x) \, dA \]
\[ = \frac{L^2}{\pi^2} \int_{0}^{\pi} \int_{0}^{2\pi} y \sin \phi \sin \phi \, dy \int_{0}^{2\pi} \cos \theta \sin \theta \sin \phi \, d\theta \]
\[ = \frac{L^2}{\pi^2} \cdot I_8 \cdot I_6 \]

\[ P_{ji} = \frac{L^2}{2\pi} \begin{cases} 
\frac{\pi^2}{4} ; & m=q, \ (n+1=p \text{ or } p+1=n), \ n\neq 0, \ p\neq 0 \\
-\frac{4mq}{(m^2-q^2)^2} ; & (m\neq q, \ m+q \text{ odd}), \ (n+1=p \text{ or } p+1=n), \ n\neq 0, \ p\neq 0 \\
0 ; & \text{otherwise}
\end{cases} \]

\[ Q_{ji} = \int_{A} x \cos \theta (\cos \phi \cos \phi \sin \lambda_m \sin \lambda_q x) \, dA \]
\[ = \frac{L^2}{\pi^2} \int_{0}^{\pi} \int_{0}^{2\pi} y \sin \phi \sin \phi \, dy \int_{0}^{2\pi} \cos \theta \cos \phi \cos \phi \, d\theta \]
\[ = \frac{L^2}{\pi^2} \cdot I_8 \cdot I_5 \]

\[ Q_{ji} = \frac{L^2}{2\pi} \begin{cases} 
\frac{\pi^2}{4} ; & m=q, \ (n-1=p=0 \text{ or } p-1=n=0) \\
\frac{\pi^2}{4} ; & m=q, \ (n-1=p\neq 0 \text{ or } p-1=n\neq 0) \\
-\frac{8mq}{(m^2-q^2)^2} ; & (m\neq q, \ m+q \text{ odd}), \ (n-1=p=0 \text{ or } p-1=n=0) \\
-\frac{4mq}{(m^2-q^2)^2} ; & (m\neq q, \ m+q \text{ odd}), \ (n-1=p\neq 0 \text{ or } p-1=n\neq 0) \\
0 ; & \text{otherwise}
\end{cases} \]
\[ U_{ji} = \int_{A} x \cos \theta \left( \sin \theta \sin \phi \cos \lambda_{m} x \cos \lambda_{q} x \right) dA \]
\[ = \frac{L^2}{\pi^2} \int_{0}^{\pi} \frac{2\pi}{y \cos \phi \cos \lambda_{m} y} dy \int_{0}^{2\pi} \cos \theta \sin \phi \sin \phi d\theta \]
\[ = \frac{L^2}{\pi^2} \cdot I_{7} \cdot I_{6} \]

\[ V_{ji} = \int_{A} \sin \theta \left( \sin \theta \cos \phi \sin \lambda_{m} x \cos \lambda_{q} x \right) dA \]
\[ = \frac{RL}{\pi} \int_{0}^{\pi} \sin \theta \cos \phi \cos \phi \sin \lambda_{m} y \cos \lambda_{q} y dy \int_{0}^{2\pi} \sin \phi \sin \theta \cos \phi \sin \phi d\theta \]
\[ = \frac{RL}{\pi} \cdot I_{11} \cdot I_{9} \]

\[ V_{ji} = \frac{RL}{2} \left[ \frac{2m}{m^2 - q^2} \right]_{0}^{\infty} \left[ \begin{array}{ll}
2 & ; m+q \text{ odd}, n-1=p=0 \\
1 & ; m+q \text{ odd}, p+1=n \neq 0, p \neq 0 \\
-1 & ; m+q \text{ odd}, p-1=n \neq 0 \\
0 & ; \text{otherwise}
\end{array} \right] \]

\[ W_{ji} = \int_{A} \sin \theta \left( \cos \theta \sin \phi \cos \lambda_{m} x \sin \lambda_{q} x \right) dA \]
\[ = \frac{RL}{\pi} \int_{0}^{\pi} \cos \phi \sin \alpha \cos \lambda_{m} y \sin \lambda_{q} y dy \int_{0}^{2\pi} \sin \theta \cos \theta \sin \phi \sin \phi d\theta \]
\[ = \frac{RL}{\pi} \cdot I_{12} \cdot I_{10} \]
\[ W_{ji} = \frac{RL}{2} \frac{2q}{m^2 - q^2} \begin{cases} 
2 & ; m+q \text{ odd }, \ p-1=n=0 \\
1 & ; m+q \text{ odd }, \ n+1=p\neq0 , \ n\neq0 \\
-1 & ; m+q \text{ odd }, \ n-1=p\neq0 \\
0 & ; \text{otherwise} 
\end{cases} \]

\[ X_{ji} = \int_A \sin\theta \left( \sin n \cos \phi \sin m \cos \lambda \right) dA \]

\[ = \frac{RL}{\pi} \int_0^{\pi} \sin m \cos \phi \cos \lambda \, dy \int_0^{2\pi} \sin \theta \sin n \cos \phi \, d\theta \]

\[ = \frac{RL}{\pi} \cdot I_{12} \cdot I_9 \]

\[ X_{ji} = \frac{RL}{2} \frac{2q}{2^2 - m^2} \begin{cases} 
2 & ; m+q \text{ odd }, \ n-1=p=0 \\
1 & ; m+q \text{ odd }, \ p+1=n\neq0 , \ p\neq0 \\
-1 & ; m+q \text{ odd }, \ p-1=n\neq0 \\
0 & ; \text{otherwise} \end{cases} \]

\[ Y_{ji} = \int_A \sin\theta \left( \cos n \sin \phi \sin m \cos \lambda \right) dA \]

\[ = \frac{RL}{\pi} \int_0^{\pi} \sin m \cos \phi \sin \lambda \, dy \int_0^{2\pi} \sin \theta \cos n \sin \phi \, d\theta \]

\[ = \frac{RL}{\pi} \cdot I_{11} \cdot I_{10} \]

\[ Y_{ji} = \frac{RL}{2} \frac{2m}{m^2 - q^2} \begin{cases} 
2 & ; m+q \text{ odd }, \ p-1=n=0 \\
1 & ; m+q \text{ odd }, \ n+1=p\neq0 , \ n\neq0 \\
-1 & ; m+q \text{ odd }, \ n-1=p\neq0 \\
0 & ; \text{otherwise} \end{cases} \]

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Appendix C

The evaluated integrals in the format of the $\overline{A}_{ji}$ matrix

Before displaying the integer functions, developed in appendix B, the format of the $\overline{A}_{ji}$ matrix is given below

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<td>0 1 2 .. N-1</td>
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\[
\begin{array}{c|cccc|cccc}
1 & 1 & 0 & A_{11} & A_{12} & A_{13} & .. & A_{1N} & .. & .. \\
2 & 1 & 1 & A_{21} & A_{22} & A_{23} & .. & A_{2N} & .. & .. \\
3 & 1 & 2 & A_{31} & A_{32} & A_{33} & .. & A_{3N} & .. & .. \\
.. & .. & .. & .. & .. & .. & .. & .. & .. & .. \\
N & 1 & N-1 & A_{N1} & A_{N2} & A_{N3} & .. & A_{NN} & .. & .. \\
N+1 & 2 & 0 & .. & .. & .. & .. & .. & .. & .. \\
N+2 & 2 & 1 & .. & .. & .. & .. & .. & .. & .. \\
N+3 & 2 & 2 & .. & .. & .. & .. & .. & .. & .. \\
.. & .. & .. & .. & .. & .. & .. & .. & .. & .. \\
2N & 2 & N-1 & .. & .. & .. & .. & .. & .. & .. \\
2N+1 & 3 & 0 & .. & .. & .. & .. & .. & .. & .. \\
2N+2 & 3 & 1 & .. & .. & .. & .. & .. & .. & .. \\
2N+3 & 3 & 2 & .. & .. & .. & .. & .. & .. & .. \\
\end{array}
\]

Each $\overline{A}_{ji}$ submatrix is of rank 6, except for the matrices where $p=0$ or $n=0$. These submatrices consist of 3 rows where $p=0$ and 3 columns where $n=0$.

All the integrals, that were evaluated in appendix B, are displayed in the format of this $\overline{A}_{ji}$ matrix on the next pages.

95
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$$\overline{R}_{ji} = \overline{T}_{ji} = \begin{cases} 2 ; m=q, n=p=0 \\ 1 ; m=q, n=p\ne0 \\ 0 ; otherwise \end{cases}$$
\[ S_{ji} = T_{ji}^* = \begin{cases} 1 & ; m=q, n=p \neq 0 \\ 0 & ; \text{otherwise} \end{cases} \]
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\[
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2 & ; m=q, \ (n-1=p=0 \ or \ p-1=n=0) \\
1 & ; m=q, \ (n-1\neq0 \ or \ p-1\neq0) \\
0 & ; \text{otherwise}
\end{cases}
\]

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\[
\overline{G}_{ji} = \overline{K}_{ji} = \begin{cases} 
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0 & : \text{otherwise}
\end{cases}
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\frac{R}{L} \cdot \sigma_{ij} = \begin{cases} 
\frac{1}{4} & 2; m=q \quad \text{or} \quad n-1=p=0 \\
\frac{2(m^2-q^2)}{x^2(m^2-q^2)^2} & 1; m=q \quad \text{or} \quad p-1=n\neq0 \\
2; m+q \text{ odd} \quad \text{or} \quad n-1=p=0 \\
1; m+q \text{ odd} \quad \text{or} \quad p-1=n\neq0 \\
0; \text{ otherwise}
\end{cases}
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\[
R_{ji} = \begin{cases} 
\frac{1}{4} & : m=q, \ (n+1=p \ or \ p+1=n), \\
\frac{4mg}{2(m^2-q^2)} & : m+q \ odd, \ (n+1=p \ or \ p+1=n), \\
0 & : otherwise \\
\end{cases}
\]
\[
\begin{array}{cccccccccccc}
  \text{i} & \text{m} & \text{n} & \text{j} & \text{q} & \text{p} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
 1 & 1 & 0 & 0 & 1/2 & 0 & -16/9\pi^2 & \\
 2 & 1 & 1 & 1/2 & 0 & 1/4 & -16/9\pi^2 & 0 & -8/9\pi^2 & \\
 3 & 1 & 2 & 1/4 & 0 & 1/4 & -8/9\pi^2 & 0 & -8/9\pi^2 & \\
 4 & 1 & 3 & 1/4 & 0 & 1/4 & -8/9\pi^2 & 0 & -8/9\pi^2 & \\
 5 & 1 & 4 & 1/4 & 0 & -8/9\pi^2 & 0 & \\
 6 & 2 & 0 & 0 & -16/9\pi^2 & 0 & 1/2 & \\
 7 & 2 & 1 & -16/9\pi^2 & 0 & -8/9\pi^2 & 1/2 & 0 & 1/4 & \\
 8 & 2 & 2 & -8/9\pi^2 & 0 & -8/9\pi^2 & 1/4 & 0 & 1/4 & \\
 9 & 2 & 3 & -8/9\pi^2 & 0 & -8/9\pi^2 & 1/4 & 0 & 1/4 & \\
 10 & 2 & 4 & -8/9\pi^2 & 0 & -8/9\pi^2 & 1/4 & 0 & \\
\end{array}
\]

\[
\begin{align*}
\frac{R}{L} Q_{ji} &= \begin{cases} 
\frac{1}{4} & 2; m=q, (n-1=p=0 \text{ or } p-1=n=0) \\
1 & 1; m=q, (n-1=p\neq0 \text{ or } p-1=n\neq0) \\
\frac{4mc}{\pi^2(m-q)^2} & 2; m+q \text{ odd}, (n-1=p=0 \text{ or } p-1=n=0) \\
-\frac{1}{4} & 1; m+q \text{ odd}, (n-1=p\neq0 \text{ or } p-1=n\neq0) \\
0 & \text{otherwise}
\end{cases}
\end{align*}
\]
\[
\begin{array}{cccccccccccc}
& i & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\hline
m & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 \\
j & 0 & 1 & 2 & 3 & 4 & 0 & 1 & 2 & 3 & 4 \\
p & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
p & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 \\
q & 0 & 0 & 1 & 4 & 0 & 0 & 10/9π² & 0 & 10/9π² & 0 \\
q & 0 & 0 & 0 & 1/4 & 0 & 0 & 10/9π² & 0 & 10/9π² & 0 \\
q & 0 & 1/4 & 1/4 & 0 & 0 & 1/4 & 0 & 10/9π² & 0 \\
q & 0 & 1/4 & 0 & 0 & 0 & 1/4 & 0 & 10/9π² & 0 \\
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q & 0 & 0 & 0 & 0 & 0 & 1/4 & 0 & 1/4 & 0 & 10/9π² & 0 \\
\end{array}
\]

\[
\frac{R_i U_{ji}}{L} = \begin{cases} 
\frac{1}{4}, & \text{if } m=q, \ (n+1=p \text{ or } p+1=n), \quad n \neq 0, \ p \neq 0 \\
\frac{-2(m^2+q^2)}{\pi^2(m-q^2)}, & \text{if } m+q \text{ odd}, \ (n+1=p \text{ or } p+1=n), \quad n \neq 0, \ p \neq 0 \\
0, & \text{otherwise}
\end{cases}
\]

103
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\[
\bar{V}_{ij} = \frac{2m}{\pi(m^2 - q^2)}
\]

- \(2; m+q \text{ odd }, n-1=p=0\)
- \(1; m+q \text{ odd }, p+1=n\neq0, p\neq0\)
- \(-1; m+q \text{ odd }, p-1=n\neq0\)
- \(0; \text{ otherwise}\)
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\[
\bar{w}_{ji} = \frac{2a}{\pi(q^2 - m^2)}
\]

- \begin{cases} 2; & m+q \text{ odd}, \; p-l=n=0 \\ 1; & m+q \text{ odd}, \; n+1=p=0, \; n\neq0 \\ -1; & m+q \text{ odd}, \; n-1=p=0 \\ 0; & \text{otherwise} \end{cases}

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\[
\bar{X}_{ji} = \frac{2 \alpha}{\pi (q_e^2 - m^2)} \begin{cases} 
2 & m+q \text{ odd} \land n-1=p=0 \\
1 & m+q \text{ odd} \land p+1=n \neq 0,\ p \neq 0 \\
-1 & m+q \text{ odd} \land p-1=n \neq 0 \\
0 & \text{otherwise}
\end{cases}
\]

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\[
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\text{m} & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 \\
\text{n} & 0 & 1 & 2 & 3 & 4 & 0 & 1 & 2 & 3 & 4 \\
\text{j} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\text{q} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\text{p} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{cccccccccccc}
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21 & 1 & 8/3\pi & 0 & -4/3\pi \\
31 & 2 & 0 & 4/3\pi & 0 & -4/3\pi \\
41 & 3 & 0 & 4/3\pi & 0 & -4/3\pi \\
51 & 4 & 0 & 4/3\pi & 0 & 0 \\
62 & 0 & 0 & 0 & 0 & 0 \\
72 & 1 & -4/3\pi & 0 & 2/3\pi \\
82 & 2 & -2/3\pi & 0 & 2/3\pi & 0 \\
92 & 3 & -2/3\pi & 0 & 2/3\pi \\
10 & 2 & 4 & -2/3\pi & 0 & 0 \\
\end{array}
\]

\[
\bar{y}_{ji} = \frac{-2m}{\pi(m^2-q^2)} \begin{cases} 
2 & \text{m+q odd } p-1=n=0 \\
1 & \text{m+q odd } n+1=p\neq0, \ n\neq0 \\
-1 & \text{m+q odd } n-1=p\neq0 \\
0 & \text{otherwise}
\end{cases}
\]

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Appendix D

The stability calculation for an isotropic or orthotropic cylinder

If the software, developed to perform stability calculations for anisotropic cylinders is used to calculate the critical buckling load of an isotropic or orthotropic cylinder one must be aware of the fact that the isotropic stability matrix has to be applied to find the critical buckling load. This is caused by the fact that there has to be a sign change, if, during the search procedure, an eigenvalue is passed. This change of sign will not be found if the additional Fourier series for the anisotropic cylinder are not neglected. This assertion will be proved with an example.

The first calculation of the determinant is made without leaving out the additional Fourier series for the anisotropic cylinder. The rank of the matrix will be equal to 6. In this matrix (equation D.1) $d_1$ is a real element of the stability matrix.

\[
[L] = \begin{bmatrix}
  d_1 & 0 & 0 & d_2 & 0 & d_3 \\
  0 & d_1 & d_2 & 0 & -d_3 & 0 \\
  0 & d_2 & d_4 & 0 & d_5 & 0 \\
  d_2 & 0 & 0 & d_4 & 0 & -d_5 \\
  0 & -d_3 & d_5 & 0 & d_6 & 0 \\
  d_3 & 0 & 0 & -d_5 & 0 & d_6 \\
\end{bmatrix}
\] (D.1)

If the determinant of this matrix is calculated we will get the following expression

\[
\text{det} = d_1^2d_5^4 + d_2^4d_6^2 + d_3^4d_4^2 + d_1^2d_4^2d_6^2 + 4d_2^2d_3^2d_5^2 \\
+ 4d_2d_3^3d_4d_5 - 4d_1d_2d_3^3d_5 + 4d_2^3d_3d_5d_6 \\
- 2d_1d_3^2d_4d_5^2 - 2d_1d_2^2d_5^2d_6 + 2d_1^2d_4^2d_5^2d_6 + 2d_2^2d_3^2d_4d_6 \\
- 2d_1^2d_4^2d_6^2 - 2d_1d_3^2d_4^2d_6 - 4d_1d_2d_3d_4d_5d_6
\] (D.2)

The second calculation is performed with the Fourier series solution functions for the isotropic
cylinder only. Therefore one must take the second, third and fifth rows and columns form the matrix in equation D.1

$$
\begin{bmatrix}
  d_1 & d_2 & -d_3 \\
  d_2 & d_4 & d_5 \\
  -d_3 & d_5 & d_6 \\
\end{bmatrix}
$$  \hspace{1cm} (D.3)

This yields for the determinant

$$
\text{det}_{iS} = d_1 d_5^2 - d_2^2 d_6 - d_3^2 d_4 + d_1 d_4 d_6 - 2 d_2 d_3 d_5
$$  \hspace{1cm} (D.4)

Our way of searching the eigenvalue is based on the fact that there is a sign change if an eigenvalue is passed. For the matrix of eq. D.3, where we consider the Fourier series solution functions for the isotropic cylinder only, the sign change can be found easily and the eigenvalue can be determined. On the other hand if the additional Fourier series for the anisotropic cylinder are not neglected we are left with a 6x6 matrix (equation D.1) and we will not find any sign change. A closer examination of equation D.4 learns that, if this equation is squared, we will get the same expression as the one in equation D.2. Therefore not neglecting the additional Fourier series for the anisotropic cylinder yields that, for a stability calculation for an isotropic cylinder, all the eigenvalues can be found two times. If one eigenvalue is passed now during our search procedure we actually pass two eigenvalues at once and there will be no sign change. This explains why the additional Fourier series have to be neglected if we are dealing with isotropic or orthotropic cylinders.
Appendix E

The input data

The easiest way of explaining how to make an input data file for a stability calculation is by means of a small input program. In this small program you are directed to the next input line with the help of "if" and "goto" statements. You won't have much trouble making an input file if you go through this program carefully. The meaning of each variable is described below the input program.

Before making an input file I must point you at the necessity of reading the whole report first. You must be aware of the problems you can be faced with (e.g. you can miss the critical buckling load, because of the fact that your step-size is too large). Further you can find a kind of strategy in this report how to tackle a stability problem.

Finally on the last few pages of this appendix the name of the main program and the used subroutines is given. In a few lines the purpose of each subroutine is described.

INPUT PROGRAM
INTEGER FORMAT 10I5
REAL FORMAT 6D12.5
INTEGER (I-N)
REAL (A-H,O-Z)
read ISHORT,INTSRH
read R,L,T,p
read NCASE
read NLOAD
if (NCASE=1) goto 10
   if (NCASE=2 and NLOAD=1) read FBEND
   if (NCASE=2 and NLOAD=2) read FNORM
   if (NCASE=3 and NLOAD=1) read FBEND,FSHEAR
   if (NCASE=3 and NLOAD=2) read FNORM,FSHEAR
   if (NCASE=3 and NLOAD=3) read FNORM,FBEND
   if (INTSRH=0) goto 10
read BLOAD

if (NCASE=3) goto 5
read NTRMS0
read MS,MMAX
goto 60

5  read NS,NTRMS0,NMAX
read MS,NUMBM,MMAX
read ISOEQ
goto 60

10 read BLOAD,STEP0,BLMAX
read ITERM

if (NCASE=3) goto 50
read NAUTO

if (NCASE=2) goto 15
read NFULL,NRED
read NS,NTRMS0
if (NAUTO=0) read MS
if (NAUTO=1) read MS,MMAX
goto 60

15 read NTRMS0
read MS
if (NAUTO=0) read NFULL,NRED
if (NAUTO=1) read EPS
goto 60

50 read NS,NTRMS0
read MS,NUMBM
read ISOEQ
read IPLOT
if (IPLOT=0) goto 60
read NPLT,KINDPL
read (XPLT(I),I=1,NPLT)
goto 60

60 read ISTIF
read LAYS
do 65 I=1,LAYS
    read E11(I),E22(I),ANU12(I),G12(I),TL(I),THETA(I)
65 continue
read RREF,HREF,EREF,ANUREF
if (ISTIF=0) read IOUT
if (ISTIF=1,2,3) read EX,AX,BX,DX,XNUX,TWX
if (ISTIF=1) read EPHI,APHI,BPHI,DPHI,XNPHI,TWPHI
end

Explanation of the input variables

ISHORT 0: maximum printout
         1: minimum printout

INTSRH 0: calculation of the critical buckling load
         1: interval search - critical m for pure bending case
         - m- and n-interval for transverse shear case

R      radius of shell (inch)
L  length of shell (inch)

T  skin thickness (inch)

p  normal pressure (psi), positive for external pressure

NCASE  1 : only axial compression (and normal pressure)
        2 : only pure bending (and axial compression, normal pressure)
        3 : transverse shear (and pure bending, axial compression, normal pressure)

NLOAD  1 : search for the critical buckling load due to axial compression
        2 : search for the critical buckling load due to pure bending
        3 : search for the critical buckling load due to transverse shear

FNORM  compressive load caused by axial compression (lbf/in)

FBEND  maximum compressive load (at $\theta=0^\circ$) caused by pure bending (lbf/in)

FSHEAR  maximum compressive load (at $\theta=0^\circ$ and $x=L/R$) caused by transverse shear (lbf/in)

LOAD  if INTSRH=0 : non-dimensionalized starting point for eigenvalue search

$N_{\text{search}}/N_c$
if NTSRH=1 : non-dimensionalized value of \( N_{\text{chosen}} / N_{\text{cl}} \) for which the interval search will be performed

**STEP0** non-dimensionalized step-size for the search procedure \( d(N_{\text{search}} / N_{\text{cl}}) \)

**BLMAX** non-dimensionalized value of \( N_{\text{search}} / N_{\text{cl}} \) for which the searching procedure will stop

**NS** if NCASE=1 : lowest value of n considered

if NCASE=3 : smallest element of the n-interval

**NTRMS0** if NCASE=1 : number of n's considered starting with NS (NTRMS0 10)

if NCASE=2,3 : size of the n-interval

**NMAX** maximum value of the smallest element of the n-interval during the interval search

**MS** if NCASE=1 and NAUTO=0 : value of m considered

if NCASE=1 and NAUTO=1 : lowest value of m considered

if NCASE=2 and INTSRH=0 : value of m considered

if NCASE=2 and INTSRH=1 : lowest value of m considered

if NCASE=3 and INTSRH=0 : smallest element of the m-interval

if NCASE=3 and INTSRH=1 : starting point for the smallest element of the
m-interval

**NUMBM**  size of the m-interval

**MMAX**  if NCASE=1,2 : maximum value of m considered
          if NCASE=3 : maximum value of the smallest element of the m-interval

**ISOEQ**  0 : anisotropic way of calculating
            1 : equivalent isotropic way of calculating

**ITERM**  number of reductions of the step-size

**NAUTO**  0 : non-automatic search procedure
            1 : automatic search procedure

**NFULL**  0 : reduced matrix method
            1 : full matrix method

**NRED**  0 : full matrix method
            1 : reduced matrix method

**EPS**  wished accuracy \(\frac{(N_i-N_{i-1})}{N_{cl}}\)

**IPlot**  0 : no plot of the buckling mode is made
            1 : plot of the buckling mode is made (only possible for NCASE=3)

**NPlot**  number of plots made

**KINDPL**  0 : plot of the circumferential buckling mode
            1 : plot of the axial buckling mode

**XPlot(i)**  if KINDPL=0 : x-coordinate for the \(i^{th}\) plot
              \(x \in [0,L/R] : x\) is non-dimensionalized
if KINDPL=1 : y-coordinate for the $i^{\text{th}}$ plot (inch)

attention : $y = 0$-R

**ISTIF**

0 : no rib stiffening
1 : $0^\circ$-$90^\circ$ rib stiffening
2 : $+45^\circ$ rib stiffening
3 : $0^\circ$-$+60^\circ$ rib stiffening

**LAYS**

number of layers (for isotropic material NLAYS=1!) (for isotropic material : E11=E22)

**E11(I)**

Young’s modulus of $i^{\text{th}}$ layer in fibre direction

**E22(I)**

Young’s modulus of $i^{\text{th}}$ layer perpendicular to the fibre direction (for isotropic material : E11=E22)

**ANU12(I)**

Poisson’s ratio of $i^{\text{th}}$ layer

**G12(I)**

shear modulus of $i^{\text{th}}$ layer

**T(I)**

thickness of $i^{\text{th}}$ layer

**THETA(I)**

angle between fibre direction and x-axis

**RREF**

0.000000D+00 added to the input in order to use the already existing subroutine

**HREF**

"STIF"; not used further during the stability calculation

**ANUREF**

**IOUT**

0 : inside rib stiffening
1 : outside rib stiffening

**EX**

if ISTIF=1 : Young’s modulus of the $0^\circ$-stringers
if ISTIF=2,3 : Young’s modulus of the stringers
AX  stringer spacing
BX  stringer width
DX  stringer depth
TWX  torsion constant J of the stringers
     ( TWX = DX·BX^3/3 )
EPHI Young's modulus of the rings or frames
APHI  ring spacing
BPHI  ring width
DPHI  ring depth
TWPHI  torsion constant of the rings or frames
       ( TWPHI = DPHI·BPHI^3/3 )
Used programs and subroutines

**CYLINDER**
- main program

**STIF**
- subroutine to calculate the global stiffness matrix

**ORTHO**
- subroutine to calculate the additional stiffness terms due to rib stiffening (0°-90°, ±45° or 0°-±60° rib stiffening)

**JUSTNF**
- subroutine to calculate the critical buckling load of a cylinder subjected to axial compression (and normal pressure)

**REDMAT**
- subroutine to calculate the reduced stability matrix, where only the stiffness terms are considered

**INVER4**
- subroutine to invert a 4x4 matrix

**INCBEN**
- subroutine to calculate the critical buckling load of a cylinder subjected to pure bending (and axial compression, normal pressure) with the help of the reduced matrix method

**INCSHE**
- subroutine to calculate the critical buckling load of a cylinder subjected to shear loading (and axial compression, pure bending, normal pressure) with the help of the reduced matrix method

**NORMAL**
- subroutine to add the elements of the full or reduced stability matrix due to axial compression to this stability matrix

**PURBF**
- subroutine to add the elements of the reduced stability matrix due to pure bending to this stability matrix

**SHEAR**
- subroutine to add the elements of the reduced stability matrix due to shear loading to this stability matrix
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<td>subroutine to calculate the critical buckling load of a cylinder subjected to pure bending (and axial compression, normal pressure) with the help of the full matrix method</td>
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<td>DETERC</td>
<td>subroutine to calculate the determinant of a matrix</td>
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<td>subroutine to solve a matrix equation</td>
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