THE DYNAMICS OF BREAKER BARS

Considered as a Diffusion Process

A barred cross-shore profile
Terschelling coast, 1989

M.Sc. Thesis
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Department of Civil Engineering
Section of Coastal Engineering

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An Analytical Approach

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PREFACE

During a short stay at Rijkswaterstaat I met ir. W.Th. Bakker. His enthusiasm and my interest in coastal morphology finally resulted in this thesis. The subject was defined in the scope of NOURTEC, an EEC project involving beach and foreshore supplies.

This report forms the theoretical part of a study about cross-shore sediment transports linked to breaker bars and foreshore supplies. A more practical continuation of this study will be performed in Denmark.

It has not been easy to combine analytical mathematics with dynamics in coastal morphology. Sometimes the link from mathematics to physics was hard to recognise.

I think that this is due to a double abstraction performed in this study.

First morphologic processes are simplified "by calling them q". This leads to the method of line-modelling, which predicts coastal behaviour without addressing sediment transport processes in detail.

The next step is to let "q" govern a mathematical diffusion process. This provides a new point of view on describing cross-shore profile shapes.

I first thank ir. W.Th. Bakker for not only being inventive, but for his enthusiasm and his help with analytics as well.

I thank dr.ir. J.A. Roelvink for his help in linking analytics to physics, and for his suggestion and support to develop a numerical computer program to conclude this study.

Furthermore, I wish to thank dr.ir. J.v.d.Graaff and prof.dr.ir. J.A. Battjes for their supervision and comments.

I gratefully acknowledge ing. M.Z. Voorendt, who helped me after a computer crash, and dr.ir. W.T.v. Horssen (Technical Mathematics and Informatics) for his help and advise with analytics.

I would like to mention M.A.F. Knaapen, graduate student at the faculty of Technical Mathematics and Informatics, and his supervisor prof.dr.ir. A.W. Heemink for their pleasant cooperation.

I thank my father for his help with the English language, and I specially thank Roosmarijn Jakma, my fiancée, for her compassion and her patience.

Delft, 1st February 1995
Stephan van der Biezen
ABSTRACT

In the scope of NOURTEC, an EEC project, efforts are made to simulate the behaviour of executed foreshore supplies with the aid of mathematical computer models. One of the models used are line models.

On behalf of the DUT two studies contribute to NOURTEC line modelling. One concentrates on the longshore transport, and the present study focuses on cross-shore sediment transport. Both studies try to improve the applicability of line models in evaluating and forecasting the behaviour of the coast, specially after a nourishment.

The cross-shore profiles supplied and evaluated in the scope of NOURTEC, and many other cross-shore profiles as well, are characterised by one or more breaker bars. Foreshore supplies very often have the same length scales as these breaker bars. At the Terschelling coast, for instance, a nourishment was executed by filling the trough between the outer two bars. This illustrates that, studying cross-shore morphology after a supply, one should include the phenomenon of breaker bars in the study as well.

This report contains a study about both breaker bars and supplies. Concentrating on the cross-shore profile, in this study breaker bars and supplies are interpreted as harmonic or instantaneous disturbances of an equilibrium profile respectively. An equilibrium profile is a profile shape for which there is no sediment transport. The essence of line modelling is that a cross-shore profile, for given wave and sediment parameters, tends towards an equilibrium shape. Using this concept, breaker bars can be schematised by a harmonic boundary condition and supplies by an initial surplus of sediment in the equilibrium profile.

In this study, the equilibrium profile shape and the consequences of a disturbance are described by two expressions. One is the well known continuity equation, and the other is a sediment transport equation. The latter describes the magnitude and the direction of sediment transports as a function of the profile height and the profile slope. Transports due to wave asymmetry, undertow and the gravity force are included in this expression. In case the profile height and the profile slope meet the equilibrium profile shape for a certain location in the profile, the sediment transport equals zero. Thus it follows that, in case of a disturbance of the equilibrium profile slope, these two expressions describe a diffusion process. In this report, the diffusion process is analyzed both analytically and numerically.

The analytical part merely concentrates on the derivation of scale rules, not on the accuracy of the outcome. A scale parameter was defined for the diffusion process due to the gravity force. A more extended analysis including wave asymmetry and undertow as well did not lead to significant different results. The analytical part is concluded with a proposal for further study.

In the numerical part a computer program is developed based on the sediment transport equation and the continuity equation. As could be expected from the analytical results, no spontaneous increase of any disturbance was found. It appears that breaker bars can only be generated at the upper boundary of the profile. In that case propagation in seawards direction is found, together with a strong dissipation in the upper part of the profile.
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Chapter 1. Introduction

1.1. Context of this study

NOURTEC

The NOURTEC-project is an EEC-research project in the scope of the program for Marine Science and Technology (MAST II). NOURTEC stands for Innovative Nourishment Techniques Evaluation.

For this purpose, and also for practical reasons, the national coastal authorities of three different countries on the North Sea have carried out shoreface supplies and a combination of shoreface and beach nourishment. These supplies can be seen as full scale experiments, and an extensive monitoring program forms an essential part of this project. Hindcast as well as forecast computer modelling will take place. One type of models used for the coastal forecast and the evaluation of sand supplies are line models.

The supplies have been executed at the coasts of Denmark, Germany and the Netherlands. In the Netherlands a shoreface supply has been carried out. Both in Germany and in Denmark a combination of a shoreface and a beach nourishment is realised. The coasts of Denmark and the Netherlands in particular are characterised by breaker bars.

A short description of the supplies is given below.

Denmark

In May and June 1993 two separate supplies have been executed on the West coast of Denmark near Torsminde.

Both supplies consist of an amount of 250,000 m$^3$ sand. One supply is placed on the beach, and the other is placed on the shoreface. Each of the supplies has a longshore dimension of approximately 1 km, and between the supplies is a longshore distance of 2 km.

Purpose of this is to be able to compare the effects of a nourishment on the beach with a nourishment on the inshore. The shoreface nourishment is located at the outer -4 meter depth contour.

The Danish coast at Torsminde is quite appropriate for using line models, because of its almost parallel depth contours.

A map of the site is given in Figure 1.1.

Germany

In 1992 an amount of 450,000 m$^3$ sand has been supplied on the west point of the German Wadden island of Nordeney. This point is protected by groynes, revetments and seawalls. The profile seems to have its equilibrium shape which is, however, affected by current as well. The seaward limit of the nourishment coincides with the 5 meter depth contour. A map of the Nordeney site is given in Figure 1.2.
Figure 1.2 Nordeney supply

The Netherlands

In spring and summer of 1993 an underwater supply of 2 million m$^3$ has been carried out on the Wadden island of Terschelling. The sand is supplied to the shoreface of the central part of the island over a stretch of 4.4 km.

The inshore of Terschelling is characterised by the existence of three breaker bars. These bars are moving seawards during a period of some 10 years, with a distance between the bars of 500 meters. The propagation velocity of the bars is ca. 50 meters per year.

The nourishment is positioned in the trough between the outer two breaker bars.

Concentrating on cross-shore features, this study will mainly be focused on Terschelling and partly on the coast of Torsminde, as at the site of Nordeney the amount of longshore sediment transport is too large to neglect.

Figure 1.3 shows the Terschelling site.
In case of beach supplies, line methods provide a handsome tool to forecast profile changes. In particular coasts with almost parallel depth contours, like Torsminde and Terschelling, can quite well be schematised according to the line-model theory. The presence of breaker bars, however, complicates the schematisation of the cross-shore profile.

1.2. Scope, cooperation and workplan

Scope

The scope of this study is to describe morphologic changes of a cross-shore profile in a schematic way. The schematisation is based on an equilibrium profile, derived from line-modelling, and uses the Bailard formula. The validity of this schematisation is studied both analytically and numerically. A spin-off of this study could be a better insight into the applicability of line-models in case of bar-dominated cross-shore profiles or foreshore supplies. In this way the present study can contribute to NOURTEC.

Cooperation

This study, together with the M.Sc. study of Nico Kersting (1994) [9] on longshore transport, will contribute to NOURTEC. Both studies are dedicated to the method of line-modelling. Line models are used in the scope of NOURTEC to predict and evaluate sand supplies. As breaker bars play an important role at the Terschelling site (the supply has been positioned between the outer two breaker bars), this study focuses on cross-shore transport in relation to breaker bars. Constantly it is tried to implement the basics of line-modelling in the theory, e.g. by starting the analysis of breaker bars with an equilibrium profile.

Furthermore, this study has been an analytical contribution to a study of Michiel Knaapen [10] handling profile developments numerically. In his study, which has a more statistical character, it is tried to find relations between measured coastal forms and forms, which
are calculated using mathematical deterministic theories. Mathematical filter techniques are applied to process the measurements. His study is performed at the Faculty of Technical Mathematics and Informatics, under supervision of prof.dr.ir. A.W. Heemink.

The outcome of his study was very helpful to understand the shortcomings of the analytical approach. The possibility for both of us to verify our results turned out to be most useful.

The study is performed at the TUD, under supervision of W.T. Bakker.

**Workplan**

The following activities have been carried out:

1. Literature study on cross-shore processes and cross-shore models.
2. Study of the thesis of V.d. Kerk [8], and adaptation of his computer program in order to obtain the data specifically needed in this study.
3. Derivation of expressions for the functions $F_2$ and $F_3$, representing respectively the combined influence of wave asymmetry and undertow, and the influence of the gravity force on cross-shore sediment transport. $F_2$ and $F_3$ determine the cross-shore equilibrium profile.
4. Analysis of the function $F_3$, treating $F_3$ as a diffusion coefficient in calculating profile disturbances, and comparison of these results with numerical outcome provided by Michiel Knaapen [10].
5. Analysis of both $F_2$ and $F_3$, trying to describe the growing process of breaker bars as well.
6. Comparison of the analytical results to numerical results, based on the same $F_2$ and $F_3$. For this goal a numerical program was developed.

**1.3. Scientific interest of the present study**

In this study, in particular barred profiles are subject to investigation. Characteristic for this study is that the coastal processes are treated like diffusion processes. Breaker bars in barred profiles are interpreted as being disturbances of the equilibrium profile. This implies a strong link towards line-modelling.

V.d. Kerk already studied the development of an equilibrium profile. The outcome of his computer program was used for further study into the subject. In this thesis the diffusion process after disturbance of the equilibrium profile was analyzed. In the analysis, the functions $F_2$ and $F_3$ mentioned in the previous section play an important role. These functions determine the diffusion process.
In terms of morphology, \( F_3 \) links sediment transport to the profile slope and \( F_2 \) yields sediment transports in case of a horizontal slope. In analytical terms, \( F_3 \) acts primarily like a diffusion coefficient and \( F_2 \) causes possibly instability.

Using the functions \( F_2 \) and \( F_3 \), the following questions were addressed in this study.

1. In which way, morphological as well as analytical, do \( F_2 \) and \( F_3 \) describe an equilibrium profile.

2. Interpreting breaker bars as harmonic disturbances of the equilibrium profile, to which extent is it possible to use \( F_2 \) and \( F_3 \) to describe those breaker bars.

3. Is it possible to evaluate a supply using \( F_2 \) and \( F_3 \) by schematising the supply as an initial disturbance of the equilibrium profile.

4. Is there any disturbance of the equilibrium profile which is supported by \( F_2 \) and \( F_3 \). If so, in morphology this means that initiation of breaker bars can be described using \( F_2 \) and \( F_3 \), in mathematics this means that growing sequences in the diffusion process occur.

After completion of the theory, the analytical results are verified using numerical outcome.

1.4. Layout of this report

In chapter 2 of this report, an orientating literature study is presented on existing cross-shore models, focusing on process-based models in particular. Furthermore, two recent studies about barred profiles are treated, and the validity of the present study is explained. Thereafter, an introduction to the phenomenon of breaker bars at the Terschelling site is included. Finally assumptions made in the present study are formulated.

Chapter 3 is dedicated to the study of V.d. Kerk [8]. In the introduction of chapter 3, the reason for using V.d. Kerk’s results for further study is outlined. Furthermore, after a description of the meaning and importance of the coastal constant \( s_y \), the calculation of a non-dimensional equilibrium profile is discussed. Finally, the results of V.d. Kerk of most interest to the present study, referred to as the functions \( F_2 \) and \( F_3 \), are explained both theoretically and physically.

In chapter 4, further analysis on breaker bars using the functions \( F_2 \) and \( F_3 \) is performed. An equation is derived describing the sediment transport due to a vertical profile disturbance, including \( F_2 \) and \( F_3 \) as a function of the profile height \( z \). After discussion of all terms this equation consists of, the term representing the influence of the gravity force is further analyzed.

In addition to chapter 4, in chapter 5 the analysis using \( F_2 \) and \( F_3 \) is continued. Because the equation describing sediment transport derived in chapter 4 is difficult to analyze, a transformation is performed. Breaker bars are schematised as horizontal profile disturb-
ances instead of vertical profile disturbances. This results in another sediment transport equation, which is easier to work with. Using this equation, a kinematic equation is found.

In chapter 6, this kinematic equation is analyzed for an initial disturbance. Gradients in sediment transport are found, indicating an erosional or an accretional profile location. The initial horizontal disturbance is executed indirectly via a disturbance in the z-coordinates of the equilibrium profile. This is explained in detail in chapter 5.

A more general way of analyzing the kinematic equation derived in chapter 5 is presented in chapter 7. Applying "division of variables" some properties of the kinematic equation are presented. Further possibilities for analytical analysis are proposed.

In chapter 8, a simple numerical computer program is presented and discussed that calculates profile developments after an initial disturbance of the equilibrium profile. The program uses the functions $F_2$ and $F_3$ to calculate sediment transports.

Finally, in chapter 9 conclusions on this study as a whole are formulated and recommendations for further study are given.
Chapter 2. Overview on cross-shore modelling and breaker bars

2.1. Introduction

In this chapter first an overview on existing cross-shore profile models is presented. From these models, in particular the process based models are discussed in section 2.3. A welcome guide has been the publication of Roelvink and Bröker (1993), "Cross-shore profile models" [15].

In the sections 2.4 and 2.5 a discussion on the latest theories about breaker bars is presented. For this goal use is made of Roelvink [14] and of Lippmann and Holman [11]. These sources of information are chosen because two different aspects of the analysis of breaker bars are highlighted.

Roelvink merely concentrates on the aspect of cross-shore modelling, and the description of cross-shore processes that play a role in the generation of breaker bars. L&H concentrate more on the aspect of the morphodynamics of breaker bars. Breaker bars appear to go through different stages, correlated to changing incident wave conditions.

In section 2.6 the present study will briefly be discussed in the perspective of the range of efforts performed in analysing breaker bars.

Furthermore, in section 2.4 a short introduction to the phenomenon of breaker bars at the Terschelling site is included. Finally in section 2.5 some assumptions are formulated.

2.2. State-of-the-art in cross-shore modelling

Roelvink and Bröker [15] divide the available cross-shore evolution models into four classes, which will be mentioned and briefly described below.

1. Descriptive models
2. Equilibrium profile models
3. Empirical profile evolution models
4. Process-based models

ad 1. Descriptive models
These models are based on observations of beaches and the various morphologic stages beaches can change to. After having collected information on a wide range of natural beaches, it is possible to make classifications and to find the most important parameters determining the type of beach stage and the way it develops. Short (1978) [18] states that the direction of beach change (erosion or accretion) depends on the variations in the breaker wave power. Wright and Short (1984) [25] defined a parameter "omega" that governs the beach changes:

\[ \Omega = \frac{H_b}{w_s \cdot T} \]  

(2.1)

in which \( w_s \) is the fall velocity of the sediment at the bar crest, \( H_b \) is the wave height of the waves the moment before breaking and \( T \) is the wave period. It can be seen that \( \Omega \) contains sediment properties and incident wave conditions. A certain value of \( \Omega \) corresponds to a certain type of beach stage. Lippmann and Holman [11] (1990) find that the motion of the outer breaker bar seems to correlate with the incident wave height.
Furthermore, one of their conclusions is that more than 75% of the change in the amplitude of the bar crest is explained by cross-shore motions of the bar.

Descriptive models provide for some important insight into profile-changing mechanisms. Unfortunately, these descriptive models usually cannot cope with artificial profile changes such as beach nourishments.

ad 2. *Equilibrium profile models*

These models are generally based on the idea that profiles tend towards an equilibrium state, depending on given conditions. This concept has proven to be useful in case of sandy beaches and dunes, provided that longshore sediment transport can be neglected and that the conditions are sufficiently stationary in time.

The shape of the equilibrium profile was described by Bruun (1954) [5] as follows:

\[ h = Ax^{2/3} \]

which was confirmed by Dean (1974) [7]. The theory was used to predict dune erosion due to storm surges, by fitting profile form vertically to the storm surge level, and horizontally until the sediment balance of the profile is correct.

ad 3. *Empirical profile evolution models*

Another type of models, in which the equilibrium profile also plays an important role, is empirical profile evolution models. The difference between these models and the models described above is the schematisation of the beach profile into depth contours.

Bakker (1968) suggested that the motion of the depth contours, i.e. the profile development, is proportional to the deviation of the present profile from the equilibrium profile. This proportionality is expressed by empirical constants. These empirical constants need to be determined by calibration for a specific project, or by numerical computation using process-based models. In this way, a time-dependent model is found, that converges towards the equilibrium profile, for instance the one defined by Bruun (1954) and Dean (1977), see equation (2.2).

Line models are an example of empirical profile evolution models.

ad 4. *Process-based models*

The most recent type of models is the process-based model, or deterministic model. This type of models explicitly treats the processes which cause a profile to change. The models consist of different components, each component governing a typical process. Although much experience is built-up with these models over the last years, still a lot about the processes involved and the interactions between the different components remains unknown.
The fact that these models try to describe the processes that play a role in nature makes them interesting and important to study. For this reason the next section will be dedicated to this type of models.

Another reason is that the model of V.d. Kerk (1987), which he wrote as a part of his M.Sc. thesis [8] and which will be used in this study, provides a transition between the process-based models and the empirical profile evolution models.

2.3. Process-based models

The processes that can be distinguished in process-based models, concentrating on the cross-shore profile, are:

1. The translation of incoming waves and currents (tidal currents) towards currents and water movements close to the bottom. Breaking of waves is involved, as well as the present bathometry.

2. The translation of these currents into sediment transport, calculating sediment concentrations. Sediment properties are involved.

3. Translation of the sediment transport into changes in bottom level. The continuity equation for the sediment volume is used.

After calculation of the adapted profile, the procedure can be repeated.

In the second process, it is important to distinguish between different timescales. The total volume of sediment transport is built-up by contributions of the time-averaged transport, the transport due to low frequency waves, the transport due to high frequency waves and the transport caused by turbulent fluctuations. Each of those has its importance and magnitude. It depends on the kind of problem that has to be solved which contribution is relevant, and which can be neglected.

Roelvink and Bröker [15] discuss different types of transport models which are used in process-based models. Amongst others the so-called “time-varying total load models” are explained.

The model of V.d. Kerk [8] is to a large extent based on the cross-shore profile model of Stive (1986) [20], which model uses a time-varying total load model to describe the sediment transport as well.

These models reduce the vertical integral of the sediment flux:

$$q(x,t) = \int u(x,z,t) \cdot c(x,z,t) dz$$  \hspace{1cm} (2.3)

to a simpler product of total load and current velocity at a reference level:

$$q(x,t) = u_{ref}(x,t) \cdot L(u_{ref})$$ \hspace{1cm} (2.4)

This way of handling the sediment transport is justified if the layer in which most of the sediment transport occurs has an almost uniform vertical velocity distribution.
Assuming that most of the transport takes place near the bed, where the sediment concentration can adapt very quickly to changes in the near bed velocity, it can be stated that the total load is a function of the instantaneous velocity at a reference level. In this case, equation (2.4) can be reduced to:

\[ \bar{q}(x) = u_{ref}(x) \cdot L(u_{ref}) \].

Although it seems that \( \bar{q} \) is time averaged, the essence is that a time varying sediment transport is averaged.

Choosing a simple power function to describe the relation between the total load and the mean bed velocity, and using Bagnold’s (1966) formulae for bed load and suspended transport, Bowen (1980) first applied equation (2.5) to cross-shore transport. Bailard re-derived the formulations used by Bowen, and presented the notation that is used nowadays.

In the Bailard formula transport is described by a linear combination of velocity moments, sediment parameters and the bottom slope. The Bailard formula, reduced to the cross-shore profile, is implemented in the computer program of V.d. Kerk. Most process-based models use the Bailard formula to link near-bottom currents to sediment transport.

Furthermore, in Roelvink and Bröker [15] five process-based morphodynamic profile models are tested and compared. One of the models is UNIBEST, developed by Delft Hydraulics.

An interesting result is found after calculating a slope exposed for 4.3 hours to regular waves with a period of 6 seconds and a wave height of 1.5 meter. All five models calculate a bar at the location of the breaker line, which is in accordance with the experimental results. The bar is generated because of the change from onshore transport in front of the breaker line, where no waves break, to offshore transport after the breaker line due to the undertow. In case of irregular waves, the bar formation is less pronounced.

It may be clear there is still much to find out about breaker bars, as in reality bars are present coastward from the location where the waves start to break as well. Furthermore, often multiple bar systems are observed. This illustrates that the processes involved are very complicated to fit into a computer model.

A detailed description of possible bar configurations is presented in the next section, based on Lippmann and Holman [11], further referred to as L&H.

2.4. The spatial and temporal variability of sand bar morphology

A first remark made by L&H is that the breaker bars, because of their nearshore position and the amount of sediment the bars consist of, are important features in studying beach stability.

L&H mention that, in literature, diverse forms of natural beach systems have been reported extensively. At first it was assumed that bar forms are related to summer and winter wave climates. During months characterised by a higher wave energy, the bars are located more offshore. Later it was observed by Sonu (1973) that some bar forms are a
response to beach cycles on a shorter time scale, e.g. a series of storms. Many authors afterwards observed that the growth of waves causes a transition of the profile towards a more eroded stage, and that subsidence of waves causes accretional transitions.

L&H distinguish between two lines of interpretation of field data. They mention the equilibrium models on the one side, and the sequential models on the other side. The first type of models may be referred to as empirical profile evolution models, and the second type of models as process based models. Both types were described in section 2.2. The essence of equilibrium models is that these models calculate with non-varying incident wave conditions, whereas sequential models calculate with variable incident wave conditions.

**Equilibrium models**

An interesting concept of an equilibrium evolution theory is based on the antinode or nodal locations of standing waves in the cross-shore zone, e.g. reflected incident waves or progressive edge waves. From this theory results a simple expression predicting the cross-shore length scale of the bar:

\[ x = \frac{g \cdot \beta \cdot X}{\sigma^2} \]  

(2.6)

where \( \beta \) denotes the beach slope, \( X \) is a constant and \( \sigma \) is the radian frequency of standing waves.

Another more complex standing wave model is proposed for three-dimensional bar forms. These models can predict cross-shore length scales together with longshore length scales associated with periodic three-dimensional bars.

**Sequential models**

It was mentioned above that the first sequential studies focused on the seasonal cycle in erosion and accretion of the cross-shore profile, neglecting longshore variation. L&H call these models the first type of sequential models, and they present some examples.

In the second type of sequential models, attention was paid to the sequence of different morphologic states in the shape and position of nearshore bars. The underlying idea is that there is a limited set of possible morphologic states a profile can adapt itself to. An attempt was made to identify these states, and furthermore to correlate these states with wave height and wave power. This yields a qualitative understanding of the conditions most closely connected to each possible beach state. This type of sequential models is referred to as descriptive models in section 2.2. The parameter \( \Omega \) was mentioned which provides a link between external conditions and beach stage. L&H furthermore present the classification scheme used by Wright and Short [25] that specifies six beach stages. They remark that most previous long term studies were primarily based on visual observations only. From the pattern of breaking waves, a lot of information can be obtained about the location and the length scales of the breaker bars. This information, however, is restricted to merely qualitative observations. The more subtle features of the bar structures are not revealed.
The study of L&H has been to qualify the temporal and spatial variability of sand bar morphology. Especially the effect of high energy storm events on the three-dimensional morphology is subject of investigation. For this, use is made of a video technique that processes morphologic data over a 2 year period.

The study of L&H can be divided into two distinct lines. First, in addition to Wright et al (1985 and 1986), observed morphologic stages were classified, resulting in an eight state model based on four classification criteria. This means that two more beach stages are added to the classification scheme by Wright et al.

Second, the bar crest positions in time are decomposed into two-dimensional and three-dimensional components. In this study, the first line will be concentrated on in order to obtain an overview on most previous insights in breaker bar behaviour. L&H present a scheme of classification criteria, with which it is quite simple to classify an observed bar into one of the eight distinguished beach stages. The scheme is presented below:

![Flow chart classification of criteria](image)

Figure 2.1 Flow chart classification of criteria

Before specifying the beach stages A to H, the four classification criteria are explained.
**Characteristic 1. Existence or absence of a bar.**
This characteristic is quite simple to recognise in a given morphologic situation. The existence of a bar is defined by the existence of two separate breaking patterns. One pattern is due to the breaking of waves over the bar crest, the other is due to the shore break.

**Characteristic 2. The dominant bar scaling.**
The dominant length scales of the bars, cross-shore as well as longshore, can be either incident or infragravity. Incident bar scales are approximately equal to the length scales of the incident waves (in the order of meters). Infragravity bar scales are much larger (in the order of hundreds of meters). In case there are no bars present in the profile, the width of the surf zone is used to estimate the representative length scales.

**Characteristic 3. Longshore variability.**
Three possibilities are given for this characteristic. Absence of longshore variability (linear two dimensional bar system), a rhythmic longshore variability or a non-rhythmic longshore variability. A rhythmic longshore variability indicates the presence of crescentic bars, welded bars, or regularly spaced rip channels.

**Characteristic 4. The trough.**
The trough between the upper bar (in case of a multiple bar system) and the shore can be either continuous or discontinuous. The trough is considered discontinuous when the bar is attached to the shoreline. At this location there is a continuous wave breaking from the bar to the shoreline.

The beach stages A to H, resulting from the flow chart, are described below.

<table>
<thead>
<tr>
<th>Bar type H. Dissipative (unbarred; infragravity scaled surf zone)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bar type G. Infragravity scaled 2D bar (no longshore variability)</td>
</tr>
<tr>
<td>Bar type F. Non-Rhythmic 3D bar (longshore variable; continuous trough; infragravity scaled)</td>
</tr>
<tr>
<td>Bar type E. Offshore Rhythmic bar (longshore rhythmicity; continuous trough; infragravity scaled)</td>
</tr>
<tr>
<td>Bar type D. Attached Rhythmic bar (longshore rhythmicity; discontinuous trough; infragravity scaled)</td>
</tr>
<tr>
<td>Bar type C. Non-rhythmic, attached bar (discontinuous trough; infragravity scaled)</td>
</tr>
<tr>
<td>Bar type B. Incident scaled bar (little or no longshore variability; may be attached; incident scaled)</td>
</tr>
<tr>
<td>Bar type A. Reflective (unbarred, incident scaled surf zone)</td>
</tr>
</tbody>
</table>

Table 2.1 Classification scheme
The bar types at the top of the classification scheme, see Table 2.1, are called higher bar types, those at the bottom are called lower bar types. It is remarked by L&H that further classification is possible, e.g. distinction between incident scaled bars with varying degrees of longshore periodicity, or including the number of bars in the cross-shore profile into the classification scheme. For single barred systems, however, or in case the outer bars are neglected, L&H state that the set of criteria is complete and unique.

Testing the reliability of the classification method (which is the probability that one chooses a wrong bar type) it was found that some 78% of the bar types was classified correctly. This illustrates the robustness of the criteria. Furthermore, L&H define a parameter that denotes the probability of occurrence of each beach state. Table 2.2 shows these probabilities. The probabilities of occurrence are determined by counting the number and the duration of occurrences of a certain bar type during a monitoring period of two years. The coast was exposed to different wave climates, including storm surges. For local sediment parameters is referred to the original literature.

<table>
<thead>
<tr>
<th>Bar Type</th>
<th>Pi(%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>H</td>
<td>-</td>
</tr>
<tr>
<td>G</td>
<td>6.7</td>
</tr>
<tr>
<td>F</td>
<td>13.8</td>
</tr>
<tr>
<td>E</td>
<td>24</td>
</tr>
<tr>
<td>D</td>
<td>43.6</td>
</tr>
<tr>
<td>C</td>
<td>3.3</td>
</tr>
<tr>
<td>B</td>
<td>8.6</td>
</tr>
<tr>
<td>A</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 2.2 Probabilities of occurrence

It can be seen that lower bar types, with the exception of type C, seem to persist longer, which would mean that these types are more stable.

L&H mention that upstate transitions occur very rapidly, whereas transitions to lower bar types were more often gradual, especially for the lowest bar types. Bar type D was the most commonly observed bar type. Another remark by L&H is that, for up-state transitions, the final state after transition upwards is relatively independent of the initial state, whereas for down state transitions the previous state is more important. From comparison from bar types with wave conditions it follows that higher bar types can generally be associated with higher wave parameters. As all defined wave parameters show the same behaviour, for the sake of simplicity instead of the wave parameters the wave height can be used.
Furthermore, L&H conclude that lower $H_s$ lead to down transitions. As lower state bars are generally located closer to the shoreline, these states represent accretional sequences. Higher $H_s$ and increasing wave energy lead to up-state transitions, which consequently indicate erosional sequences.

From further analysis it follows that the linear state, as well as the non-rhythmic bars, may be considered relatively unstable, while rhythmic states seem much more stable. Rhythmic bars increase in stability as the bars become attached to the shoreline or move closer to the shoreline. At the Terschelling site, the breaker bars are attached to the shoreline, which could indicate a stable bar configuration.

Up-state transitions appear to be more dependent on $H_s$ than down-state transitions. As dependency on $H_s$ signifies a tendency towards an equilibrium state in case $H_s$ does not change, this could indicate that up-state transitions (erosional) are in fact equilibrium dominated transitions, and that downstate transitions (accretional) tend to be sequential (not depending on incident wave conditions).

This is an interesting assumption, as this could mean that the method of line modelling (profile behaviour directly determined by the difference between equilibrium state and actual state) is to be applied in case the site is characterised by up-state (erosional) transitions.

Note that the terms erosional and accretional are used to indicate onshore and offshore bar movement, and do not refer to any cross-shore transport. This new definition of the terms erosional and accretional is consistent with L&H, but inconsistent with the rest of this report. The terms are used in this review, however, to indicate onshore and offshore bar movement.

A final remark is made about the stability of the bar configurations. It is observed by L&H that linear bar systems (no variation longshore) only tend to occur during storm surges. After storm surges, as the wave energy declines rapidly, the linear two-dimensional bar system rapidly develops towards a three-dimensional bar system. This indicates that linear bars are quite unstable.

In the next section, Roelvink [14] is discussed briefly on breaker bars.

2.5. Surfbeat and its effect on cross-shore profiles

One of the conclusions of L&H [11] is that long term records are invaluable in order to assess the morphologic behaviour of sand bars. The reason is that incident wave parameters do not suffice in predicting bar development. Many other influences play a role, which cannot be measured by studying only the incident waves analysing the breaking wave pattern.

To address bar genesis, according to L&H short term experiments that investigate other fluid motions like edge waves are necessary. This conclusion provides a link to Roelvink [14].

Roelvink [14] analyses the effect of surfbeat on cross-shore profiles. Surfbeat is a general name for all kinds of long-wave motions in the surfzone. These motions are generated by
variations in the radiation stress. Every incident short wave has its contribution to the radiation stress. Variations in the radiation stress occur because of variations in the wave height of the incoming short waves. The variations in the radiation stress create long waves, known as infragravity waves.

Roelvink states that these infragravity waves have a significant influence on the cross-shore profile, as not only the net currents are influenced by the long wave motion but the sediment concentrations as well. This leads to a sediment transport contribution which has the same order of magnitude as the sediment transport due to the undertow or wave asymmetry. The fact that the length scales of breaker bars have the same order of magnitude as the length scales of the long waves supports this hypothesis.

As long waves in the surfzone play an important role, it is attempted to describe the generation of long waves according to Roelvink.

First it is necessary to define a so-called transition region. Because the wave height of the incident waves is not the same for every incoming wave, the waves do not break at the same location. Their breakpoints move forwards and backwards, within the transition zone. Therefore, in the transition region there is a gradient in the radiation stress, as waves are breaking in this region. And this radiation stress gradient varies in time due to incoming wave groups. The variation generates long wave motions, both in onshore as in offshore direction. Furthermore, the onshore directed long wave is reflected off the beach, and interferes with the offshore directed wave. In this way, a resulting offshore wave radiates from the transition zone. This is a free wave, which is no longer attached to a wave group. In case the incident waves approach the coast obliquely, the propagation direction of the free waves coming from the transition zone can be bend towards the coast by refraction. These "trapped" long waves are called edge waves.

The relative importance of edge waves compared to the influence on cross-shore long wave motions is not quite clear. The question that will be further concentrated on is: what is the influence of surf beat on the morphology?

Roelvink concludes that the effect of the interaction between short wave orbital velocity fluctuations and the long wave velocity must be dominant in influencing cross-shore profiles. He focuses on describing this interaction.

Furthermore, from two examples indicating the sensitivity of the sediment transport distribution over the profile to profile changes, it follows that two aspects determine the dynamic morphological behaviour. One aspect is the above described interaction, the other is the sensitivity of the transport pattern to profile changes.

In order to analyze both aspects, first the basic transport pattern on an unbarred equilibrium profile was calculated. The sensitivity of the long wave-short wave interaction term to the configuration of longshore bars was studied in relation to this basic transport pattern. In this way the sensitivity of the transport pattern due to the interaction between short and long waves to profile changes was studied.

Second, the importance of the interaction between long waves and short waves was studied separately by investigating the sensitivity of the results for a typical barred profile to incident wave conditions.
The main conclusion of Roelvink [14] is that bar formation due to the presence of long waves is not found in his study. Long waves play an important role in bar evolution, though only a destructive role. Roelvink remarks, however, that only cross-shore long waves were calculated with. Alongshore propagating edge waves may change his conclusions.

2.6. The present study in its perspective

The present study will focus on the sensitivity of the sediment transport pattern due to incident short waves to profile changes. This is done in a very schematised manner, in order to enable an analytical approach. The changes in the sediment transport pattern are linked to a bottom update via the continuity equation.

It is likely that possible bar systems found in this study will be incident scaled, see L&H, as only incident short waves are calculated. The infragravity scaled bars are most likely due to the influence of long wave motions.

Another consequence of the restriction to short waves only is that diffusion possibly plays an important role on profile configurations with large length scales compared to the wavelength of short waves.

In the next section, a short qualitative introduction to breaker bars at the Terschelling site is presented.

2.7. Introduction to breaker bars at Terschelling

In draft work documents, Van Vessem [22] and Ruessink [17] have analyzed the morphology of the Terschelling coast. The Terschelling coast is characterised by breaker bars.

In his analysis Ruessink [17] concentrates on the morphology of the coast related to sediment transport between foreshore and inshore, coastal behaviour as a consequence of changing hydrodynamic conditions and the link between morphology and coastal erosion or accretion.

Van Vessem [22] concentrates on the underwater sand supply at Terschelling, and studies the morphology to determine the position of the nourishment and the nourishment quantity.

Their remarks concerning breaker bars are outlined below.

Van Vessem [22] states that a detailed analysis of breaker bars is necessary to answer the question where to position the supply. Therefore, for every year of which measurements are available, in every profile the crests of the breaker bars were located. From this it can be concluded that each year three breaker bars are present in front of the Terschelling coast. In general the breaker bars are not exactly parallel to the coast, but show a small angle with the RSP-line. The outer breaker bar shows the largest angle.

The RSP-line is a line marked by so-called beach piles. The RSP-line is used as a reference for sounding lines perpendicular to the coast.
Note: The NAP-line denotes the Dutch reference water line. The NAP-line is the waterline at the coast for a water level equal to NAP, which is the middle between the tide induced mean high water level and mean low water level.

Furthermore, Van Vessem concludes that a longshore motion of the breaker bars is hard to detect. It seems that the cross-shore motion is dominant.

From comparison of the yearly measurements it follows that the breaker bars show a seawards migration. Only in some cases the bars move in coastward direction. The velocity with which the bars move is not constant in time. Concerning a seawards moving breaker bar, the propagation speed of the bar close to the coast line is small compared to the propagation speed of a more seaward breaker bar. The propagation speed increases as the offshore distance increases. As soon as the outer bar is dissipated, the propagation speed of the adjacent bar at a higher level increases in seawards direction. This illustrates that the distance between the bar crests is will not be constant in time.

The magnitude of the propagation speed of the outer bars varies between 27 to 60 meters per year. At sounding line 1700 for instance, which is the centre of the Terschelling supply, the propagation speed equals 50 meter per year. It is not clear whether the supply will influence the propagation of the bars.

A new breaker bar is generated as soon as the outer bar has disappeared. The generation of a new breaker bar occurs close to the shoreline. The mean level of the crests of the breaker bars amounts -5.3 meter, varying between -2.5 and -7.3 meter (NAP). The level of the troughs of the bars varies between -5.5 and -9.1 meter.

Ruessink [17] used the JARKUS-data, together with extra measurements (TAW sounding lines) for the morphological analysis.

From a comparison between the bathometry of sounding lines 10 to 24 of year 1975 and year 1980, Ruessink concludes that the morphology has a repetition period of about 13 to 14 years. This is the period between generation of a breaker bar, and its disappearance offshore.

It is remarked that the migration speed close to the shoreline can also be negative (temporary migration coastward). As the offshore distance increases, the migration speed also increases and remains positive (migration seawards). Ruessink suggests that the bar migration speed is not constant, or slowly decreasing and increasing, but is changing over the year. Especially in autumn and winter, the bars migrate in seaward direction. In these seasons the number of storms and the total storm duration is largest. In summer usually small coastward migrations occur. These small coastward migrations, however, are negligible compared to the long-term behaviour of breaker bars. The migration speed seems to be related to the storm duration in a season, and also to the disappearance of the outer breaker bar.

From the above brief description of the breaker bars, it can be concluded that the morphological processes involved will indeed be complex, as the behaviour of the breaker bars appears to be very unpredictable.
2.8. Assumptions

From the above section it has become clear that assumptions are necessary to simplify the problem enabling an analytical approach. Some assumptions were already made by V.d. Kerk. These assumptions are:

1. In studying breaker bars, the gradients of longshore transport can be neglected compared to the influence of cross-shore transport. In theory the (general) seawards migration of the bars could possibly result from a longshore translation of the bars, which show an angle with the coastline. The connection between the disappearance of the outer breaker bar and the propagation speed of the bar next in line in the same cross-shore profile indicates, however, that cross-shore processes are dominant in the behaviour of breaker bars.

2. The Bailard formula, describing the total load sediment transport and adapted by V.d. Kerk to the cross-shore section, is assumed to be sufficiently accurate and reliable. This assumption seems correct, as the Bailard formula provides good results in time-averaged total load models. At this moment, however, some effort is made to derive a more accurate formulation in the scope of UNIBEST.

3. The wave height is assumed to be only depending on the water depth, leaving out the wave energy. This is an important assumption, as this is not true behind a breaker bar. Behind breaker bars the water depth increases, which, according to this assumption, will result in the same wave height behind a breaker as at the seaside of a breaker bar. Due to energy losses near the bar crest (breaking), however, in reality this is not the case. For this reason, only the part of the profile seaward of the outer breaker bar can be calculated correctly. The pros and contras of this assumption will be outlined in chapter 3.

Some extra assumptions were made in this report. These are listed below.

1. A logical first assumption to make is that the study of V.d. Kerk, apart from his assumptions, is scientifically correct. It is assumed that the equilibrium profile calculated by V.d. Kerk is really only a function of \(k_0 h, s_0\) and \(w_r T/H\).

2. In setting up the theory some assumptions about the wave asymmetry, the under­

In the next chapter, the study of V.d. Kerk is discussed.
Chapter 3. A cross-shore equilibrium profile according to the Crostran concept

3.1. Introduction

The choice for V.d. Kerk's results.

In line-models, many parameters governing the coastal behaviour are represented by so-called coastal constants. In section 2.2, line-models were already referred to as an example of empirical evolution models.

Concentrating on the cross-shore profile in particular, the coastal constant \( s_y \) represents many cross-shore sediment transport properties. In section 3.2, the coastal constant \( s_y \) will be explained.

Applying line models to forecast cross-shore coastal behaviour, reliable values of \( s_y \) are crucial for the reliability of the forecast. V.d. Kerk [8] attempted to develop a theoretical basis for \( s_y \). This chapter is dedicated to this attempt, for the following reasons.

Based on the CROSTRAN concept V.d. Kerk wrote a computer program that calculates a non-dimensional equilibrium profile. CROSTRAN (CROSs-shore TRANsport) is a dynamic cross-shore model, developed by Stive [20] that uses the Bailard formula [3].

Since the components that calculate the equilibrium profile are all expressed as a function of the non-dimensional water depth \( k_i h \) (and other deep water wave and sediment parameters), these components can be used to determine the reaction (in terms of sediment transport) of the profile to a disturbance in the equilibrium profile (change in \( h \)). The timescale in which a disturbance disappears provides an indication for the value of \( s_y \).

This is the main reason why V.d. Kerk is used for further study, as in the present study breaker bars will be schematised as harmonic disturbances of the equilibrium profile. Another reason is the relative simplicity of the model of V.d. Kerk, compared to more advanced computer models.

The layout of this chapter.

This chapter is to a large extent based upon the M.Sc. thesis of V.d. Kerk [8]. A BASIC computer program, derived from a PASCAL computer program by V.d. Kerk, is used to calculate the equilibrium profile.

In this chapter, first the coastal constant \( s_y \) is discussed, to obtain an impression of its meaning and importance. Furthermore, in section 3.3 the Crostran concept is explained and in section 3.4 a review is presented on V.d. Kerk's computer program. In section 3.5 expressions for the functions \( F_2 \) and \( F_3 \) are derived, which form the basis for further study on breaker bars. The physics of \( F_2 \) and \( F_3 \) are presented in section 3.6. Finally, in section 3.7 the results of some computations with the computer program by V.d. Kerk are discussed, and in section 3.8 some conclusions on this chapter are formulated.
In this chapter, most of the derivations and argumentations made by V.d. Kerk will be ignored. For more details about V.d. Kerk’s theory the reader is referred to the original literature.

3.2. The coastal constant $s_y$ explained

The coastal constant $s_y$ is originally connected to so-called "line-models". The essence of these models is that a continuous beach profile is schematised into a discontinuous profile, see Figure 3.1.

A schematisation of the beach profile by one line (or one "step") was first made by Pelnard-Considère (1954). He presented a relation between the position of the coastline and the angle of wave incidence, schematising the whole profile by one vertical line. This schematisation was extended by Bakker (1968) to a two-line schematisation of the beach profile, which was again improved by Swart (1974).

In order to build up a theory based on this schematisation, two assumptions are made by both Bakker and Swart:

1. Under given wave conditions and sediment parameters an equilibrium profile will be formed, characterised by a cross-shore transport equal to zero.
2. The rate of cross-shore transport, and consequently the motion of the depth contours, is proportional to the deviation of the existing profile from the equilibrium profile.

This "proportionality" is expressed by $s_y$. In formula-form this results in the following expression:

$$S_y = s_y (W - (L2-L1))$$

(3.1)
where

\[(L2 - L1) = \text{present distance between two contourlines}\]
\[W_y = \text{equilibrium distance between the contourlines}\]
\[s_y = \text{proportionality constant (m/y)}\]
\[S_y = \text{sediment transport in m}^3 \text{ per year per meter profile}\]

So \(s_y\) contains a lot of information about wave-climate, type of sediment, and other influences.

A higher value of \(s_y\) results in a more rapidly adapting profile. It is known that the upper part of a beach profile changes more rapidly than the lower part, so it is likely that \(s_y\) will not be constant over a profile. In practice, however, one value is used for the complete profile.

It is important to remark that the constant \(s_y\) only has a meaning in connection with an equilibrium state, as a consequence of its definition. As \(S_y\) can be computed time-dependently, \(s_y\) will be constant in time, unless there is a slow variation in average wave-climate or sediment parameters in time.

Now that the definition of \(s_y\) is explained with a short description of its physical meaning, the present knowledge about the value of \(s_y\) is summarised below.

As already mentioned, \(s_y\) contains a lot of information about important parameters nearshore. But very little is understood theoretically. This means that \(s_y\) still is more or less a "black box". A value is assigned to it, which is mostly based on empirical data. Methods used to obtain such a value are described below.

Swart (1974) states that \(s_y\) depends on the wave height, wave period, local depth and size of sediment particles:

\[s_y = f_1(H, T, h, D_{50})\] (3.2)

By means of "best fit" curves, he derives empirical formulas, 8 in total containing 28 constants to determine a value for \(s_y\). This illustrates the black box character of \(s_y\).

Brüning [6] determined \(s_y\) using measurements of a sand supply on Sylt. His method is based upon the availability of these data. By integrating (3.1) in longshore direction he derived the following formula:

\[s_y = \frac{[-\Delta(O_1) \ h_1]}{[\Delta(t) \ (O_1 - O_2)]}\] (3.3)

where:

\[\Delta(O_1) = O_1(t + \Delta(t)) - O_1(t)\] This is the difference between the remaining amount of supplied sand on the beach at \(t + \Delta(t)\) and at \(t\)

\[h_1 = \text{the height of the beach part, determining the division of beach and foreshore}\]

\[\Delta(t) = \text{timestep between two adjacent measurements}\]

\[O_1 = \text{amount of supplied sand on the beach}\]

\[O_2 = \text{amount of supplied sand on the foreshore}\]
The resulting values for $s_y$, however, are very inaccurate.

Swain (1986) based his theory partly on the theory of Swart (1974). Swart introduces time-constants to describe the values for $(L2-L1)$ in time:

$$(L2 - L1)_{\text{theor.}} = W_y - A e^{-t/T_e} - B e^{-t/T_t} - C e^{-t/T_e}$$

where:

- $W_y$ = the equilibrium distance between L2 and L1
- $T_0 = h_1 h_2 / (s_y (h_1 + h_2))$
- $h_1$ = height onshore profile, or beach profile
- $h_2$ = height offshore profile, or foreshore profile
- $A, B, C$ = constants, depending on $s_y, s_t, s_y$ and the profile location
- $s_e$ = a proportionality constant for the upper boundary condition
- $s_t$ = a proportionality constant for the lower boundary condition

The time constants $T_e$ and $T_t$ are connected to respectively the upper boundary and the lower boundary of the schematised profile. $T_0$ is defined between these boundaries, and depends on $s_y$ as given before:

$$T_0 = \frac{h_1 h_2}{s_y (h_1 + h_2)}$$

According to Swart, $h_1$ and $h_2$ depend on the deep water wave height, the wave period and the size of the sediment particles:

$$h_1 = f_2(H_0, T, D_{50})$$

$$h_2 = f_3(H_0, T, D_{50})$$

So $T_0$ can be written as follows:

$$T_0 = f_4(H_0, H, T, h, D_{50})$$

Swain (1986) reduces this expression and states:

$$\frac{1}{T_0} = f_5(H)$$

Furthermore, he presents values for $1/T_0$, based on measurements, as a function of $H$.

V.d. Kerk, however, concludes that these values are not valid for prototypes. Consequently, a comparison between model tests and prototypes cannot be performed using this relation.

So none of the methods to determine a value for $s_y$ is completely satisfying. In his thesis V.d. Kerk derives values for $T_0$, as mentioned before, from a small disturbance of the equilibrium profile, which is computed according to the Crostran concept. This concept is explained in more detail in the following section.
In this paper, two methods are presented to determine $s_y$:

1. Investigation of the formal coastal behaviour. Kalman Filtering is a sophisticated means in treating the data. Also a kind of CAD was used to visualise the natural coastal behaviour and the model-lines on the same screen. By changing the coastal constants, the correct coastal constants were determined visually.

2. Calculation of an equilibrium profile which depends on wave climate and sediment parameters. The timescale at which a non-equilibrium profile restores itself to the equilibrium profile provides information on $s_y$ (V.d. Kerk, 1987).

The values of $s_y$ for sand supplies appear to vary from 1.5 meter per year (at Sylt) to 3 meter per year (at Cadzand). For natural morphological processes, smaller values (order 0.5 meter per year) are found.

3.3. The Crostran concept.

This section treats Crostran qualitatively. In section 3.4 more detailed information about the physical background of Crostran will be provided.

Crostran is a computer model, developed by Stive, Wind and De Vriend, which is based on a theory from Stive (1986) [20]. Crostran computes profile changes in time, according to the following scheme (see Figure 3.2.):

![Figure 3.2. Calculation scheme Crostran](image)

The basic idea of Crostran, the Crostran concept, is described below.
From measurements of sediment transport under influence of irregular waves on a two dimensional beach profile, it can be concluded that the transport of sediment can be described in a satisfying way by assuming that the transport is caused by a time-averaged seaward directed current near the bottom, which is mainly caused by breaking waves. Apart from this, also asymmetry of waves (causing edge waves), and non-breaking waves behind a breaker bar cause sediment transport, which is directed onshore.

In Crostran these hydrodynamic mechanisms are modelled, and the current velocity near the bottom is calculated. A next logical step is to couple the near-bed velocities to sediment transport. This link is expressed in the following formula, assuming that the sediment transport is proportional to some power of the orbital velocity:

\[ q(t) = A \cdot U(t) \cdot |U(t)|^n \]

where

- \( q(t) \) = sediment transport
- \( U(t) = U + U_m \cos(\omega t) + \) higher order terms
- \( U_m(t) \) = orbital velocity just outside the boundary layer
- \( \omega \) = angular frequency

Bailard (1981) derived an expression based on this principle, which is used in Crostran. His expression treats transport in the bottom layer and suspended transport separately, and also takes the bottom slope into account.

The formula is reduced for application in the cross-shore section and expresses the time-averaged total load sediment transport:

\[ <i> = <i_b> + <i_s> = \]

\[ = \frac{\rho C_f \epsilon_b}{\tan\phi} \cdot [ < |u|^2 u > - \left( \frac{\tan\beta}{\tan\phi} \right) < |u|^3 > ] + \]

\[ + \frac{\rho C_f \epsilon_s}{W} [ < |u|^3 u > - \left( \frac{\epsilon_s}{W} \right) \tan\beta < |u|^3 > ] \]

where:

- \( u = \bar{u} + \tilde{u} \) where \( \bar{u} \) is the time averaged current velocity near the bottom \( \tilde{u} \) is the variable component of the velocity
- \( <i> \) = the total time-averaged "immersed weight" cross-shore transport
- \( <i_s> \) = suspended transport
- \( <i_b> \) = bottom transport
- \( \rho \) = density of water
- \( C_f \) = shear coefficient (shear = \( C_f u^2 \)) equal to 0.5 \( f_w \). \( C_f \) is related to bottom roughness and stroke length; see for instance Swart (1974)
- \( \phi \) = internal shear angle of the sediment
- \( \tan\beta \) = slope of the profile
- \( W \) = fall velocity of the sediment in water

\( \epsilon_b \) and \( \epsilon_s \) are efficiency factors, which indicate the amount of wave energy that is used for sediment transport.
To calculate \( \bar{u} \), it is assumed that the time averaged current is generated by gradients in the radiation stresses and by a mass-flux towards the coast due to breaking waves. The value of \( \bar{u} \) is determined by adding the contribution of the fraction non-breaking waves to the contribution of the fraction of breaking waves. The fraction breaking waves is determined by a wave dissipation model.

To calculate \( \ddot{u} \), the linear wave theory is applied. To include the effects of asymmetric waves, a second-order Stokes estimation is used to describe the waves.

Using the result of the above mentioned modelling of the current velocities, from (3.11) the time-averaged cross-shore transport can be calculated. This cross-shore transport results in a changed profile, which can be used in a next computation. Thus, the evolution in time of the cross-shore profile can be simulated. So Crostran is a process based model, that computes the evolution of a profile in time.

### 3.4. Calculation of the equilibrium slope

In the following, the equilibrium slope is defined as being the profile shape with no cross-shore transport: \( S_y = 0 \). So this profile is adapted to the wave climate in a way that all transport generating processes are in balance.

The different components of Crostran are mentioned and described qualitatively below. First the wave propagation and dissipation model is discussed, followed by the average undertow and the Bailard formula respectively.

In the following theory all variables are treated non-dimensional. This choice was made by V.d. Kerk for two reasons:

1. The number of variables involved to describe the time constant \( T_0 \) is reduced from 4 (\( H, T, h, D_{50} \)) to 3 (\( k, h, s_0, H/wT \)), where \( s_0 \) denotes the deep water wave steepness.
2. A non-dimensional expression for the time constant can be used for model situations as well as for prototype situations.

#### 3.4.1. Wave propagation and dissipation model

The component that treats wave propagation and dissipation should provide for a relation between \( H/h \) (dimensionless wave height), \( kh \) and \( s_0 \), which can be described as follows:

\[
\frac{H}{h} = f(kh, s_0)
\]  
(3.12)

Because of the, in reality, irregular waves, this becomes:

\[
\frac{H_{rms}}{h} = f(kh, s_0)
\]  
(3.13)

where \( H_{rms} \) = root-mean-square wave height
In Crostran, a Battjes-Janssen model (1978) is used, which is based on an analogy with a constant rolling wave. In this model the following relation is derived for $H_{\text{rms}}$:

$$H_{\text{rms}} = f(H_0, T, h, y)$$

(3.14)

This model is able to treat complex beach profiles, like barred profiles, which is an interesting characteristic of this model. The model is also quite reliable. V.d. Kerk however rejects this model in his thesis, because it appears to be not feasible to derive relation (3.13) from (3.14). He gives two reasons for this:

first $x$ cannot be written explicitly and cannot be eliminated from (3.14), and
secondly, practical problems arise with solving

$$\left(\frac{H_{\text{rms}}}{H_m}\right)^2$$

So another dissipation model is used by V.d. Kerk.

Battjes (1974) in his doctor's thesis describes another dissipation model. The same breaking criterion is used as is in the above described model:

$$H_m = \frac{0.88}{k} \tanh\left(\frac{\gamma}{0.88}kh\right).$$

(3.15)

First a fictive wave height $H_f$ is calculated, taking into account effects like shoaling, refraction and bottom shear stress. In this calculation the linear wave theory is used. Furthermore, assuming that $H_f$ has a Rayleigh distribution, the probability of $H_f$ exceeding the breaking criterion is estimated. This provides the fraction of breaking waves, $Q_b$.

The following relation for non-breaking waves is derived from Battjes:

$$\frac{(H_{\text{rms}})^2}{h} = (1 - Q_b) \frac{s_0^2 (2\pi)^2}{(kh\tanh(kh))^2 2 n \tanh(kh)}$$

(3.16)

where:

$s_0 =$ the deep water wave steepness,

$$n = 0.5 + \frac{kh}{\sinh(2kh)},$$

(3.17)

$$Q_b = \exp\left(-\frac{H_m^2}{h^2} (kh\tanh(kh))^2 2 n \tanh(kh)\right),$$

(3.18)

and the breaking criterion
This model, however, is not suited for calculating barred profiles, which is a serious drawback. In reality, near a breaker bar, the value for $H^\frac{1}{h}$ will be larger in front of the bar than behind, although the value for $kh$ is the same. The described model however computes the same $H^\frac{1}{h}$. This means that only for a coastward decreasing depth this model can be used.

From comparison of Battjes' model with experiments, the dissipation calculated with the model appears to be systematically too small. V.d. Kerk chooses, as an alternative of the model, another breaking criterion, to introduce more dissipation into the model. He replaces (3.15) by:

$$\left(\frac{H_{rms}}{h}\right) = f(kh, s_0)$$ (3.13 repeated)

which originates from Svasek.

This adapted dissipation model forms the first component used by V.d. Kerk.

3.4.2. The average undertow

The second component in calculating the equilibrium slope consists of the average undertow. "Average" means that the orbital movements of the water particles are levelled out, describing only the resulting current near the bottom.

Before treating the undertow theoretically, a short qualitative description of the phenomenon is presented first.

Bagnold (1940) was the first to observe a circulating current in the cross-shore profile. Later on, the mechanism of this circulating current was described qualitatively by M. Dyhr-Nielsen and Torben Sørensen (1970), and more mathematically by Hansen and Svendsen (1984).

One of the most important conclusions is that this circulating current plays a dominant role in the development of breaker bars.

A breaker bar is created on the breaker line, which is the location where two cells border each other, see Figure 3.3.

Translation of the breaker line disturbs the profile: the position of the bar is closely connected to the position of the breaker line.

The mechanism responsible for the undertow is described by J. Buhr Hansen and I.A. Svendsen (1984). Their first remark is that there must be a strong driving mechanism for the undertow to exist. They point out that, due to a substantial amount of turbulence present at all levels between bottom and surface, a current like the undertow will generate large shear stresses. The driving mechanism must be stronger than these shear stresses.
To define the driving mechanism, the depth and time averaged momentum balance is considered. This balance reads:

\[
\frac{\partial S_{xx}}{\partial y} + \rho g (h_0 + \xi) \frac{\partial \xi}{\partial y} + \tau_b = 0 ,
\]  

(3.20)

where:

- \( \xi \) = setup,
- \( \tau_b \) = mean shear stress at the bottom,
- \( S_{xx} \) = radiation stress,
- \( h_0 \) = depth in case of undisturbed water level

which is defined as:

\[
S_{xx} = \int_{z_h}^{\eta} (\rho u^2 + p_D) dz - \frac{1}{2} \rho g \eta^2 ,
\]  

(3.21)

where \( \eta \) is the surface elevation and \( p_D \) is the dynamic pressure:

\[
p_D = \rho g (z - \xi) + p .
\]  

(3.22)

In this balance, the contributions of \( S_{xx} \), \( g \frac{\partial \xi}{\partial x} \) and \( \tau_b \) are not equally distributed over depth. This is shown in Figure 3.4.

It is this difference between the local, at any \( z \)-level, distribution of \( S_{xx} \) and \( g \frac{\partial \xi}{\partial x} \) that is driving the undertow.

In the Figure a seaward directed undertow would develop.

The undertow accelerates until \( \tau_b \) equals the driving force. Thus a steady undertow establishes a balance between the three different forces: \( S_{xx} \), \( g (\partial \xi/\partial x) \) and \( \tau_b \).
Quantitative translations of this mechanism into models are performed by e.g. Dally (1980), Svendsen (1984) and by Stive and Wind (1986). In Crostran, this mechanism is modelled as described below (De Vriend and Stive [23], [24]).

First the water column is divided into three layers: a surface layer above the wave trough level, a middle layer and a bottom layer. Only the middle layer is considered. Second, the velocity in the middle layer is solved from the horizontal momentum balance, both for the case of breaking and for the case of non-breaking waves. The integration constants arising in solving this balance are determined by means of boundary conditions.

The upper boundary condition is prescribed by the shear stress at the wave trough level for both cases (breaking and non-breaking), compensating the momentum decay in this layer. The lower boundary condition follows from a zero-stress-approximation at the bottom in case of breaking waves, and the "Longuet Higgins" (1953) conduction solution in case of non-breaking waves. Another condition is that the net undertow must compensate for the mass flux in the surface layer.

To be able to consider a random breaking wave field, both cases for non breaking as well as for breaking waves should be combined. This is done by adding the contributions for both cases using $Q_b$ as a weight factor, where $Q_b$ is the fraction breaking waves ($0 \leq Q_b \leq 1$).

So, with $u_e$ as the contribution for breaking waves, and $u_s$ the contribution for non-breaking waves, the combined undertow can be written as:

$$\bar{u} = Q_b u_e + (1 - Q_b) u_s$$

The expressions for $u_e$ and $u_s$ are described below and the calculation of $Q_b$ was outlined in section 3.4.1. For a more detailed description the reader is referred to the relevant literature.
Breaking waves

V.d. Kerk uses the model of Stive and Wind (1986), which contains the following formula for $u_e$ between bottom layer and wave trough level:

$$u_e(z) = \frac{1}{2} \left\{ (Zm-1)^2 - \frac{1}{3} \frac{d_t^2}{\rho v_t} \frac{\partial R}{\partial y} + (Zm-\frac{1}{2}) \frac{d_t}{\rho v_t} \tau(\eta_{tr}) + u_r \right\} \quad (3.24a)$$

where:

- $Zm = (z+h)/d_t$ with depth $h$ and height $z$
- $d_t$ = distance between bottom layer and wave trough level
- $v_t$ = turbulent diffusivity coefficient
- $R$ = difference between radiation stress and the set-up pressure gradient, which is the resulting horizontal force acting on the water particles
- $\tau(\eta_{tr})$ = the resulting shear stress at the wave trough level
- $u_r$ = undertow to compensate for the resulting mass flux above wave trough level

Just outside the bottom layer, for $z = -h$, the expression for $u_e$ becomes:

$$u_e(-h) = \frac{1}{3} \frac{d_t^2}{\rho v_t} \frac{\partial R}{\partial y} - \frac{1}{2} \frac{d_t}{\rho v_t} \tau(\eta_{tr}) + u_r \quad (3.24b)$$

with $R$, $\tau(\eta_{tr})$, $u_r$, and $v_t$ according to Stive and Wind (1986):

$$R = \rho <(\bar{u}^2 - \bar{w}^2)> + \rho g \xi \quad (3.25)$$

where $\bar{u}$ and $\bar{w}$ are respectively the horizontal and the vertical variable velocities, $\xi$ is the set-up and $<$ > means averaged in time.

Furthermore:

$$\tau(\eta_{tr}) = -\left(\frac{1}{16} + \frac{0.9}{2\pi} kh\right) \rho g \frac{\partial H_{rms}^2}{\partial y} \quad (3.26)$$

$$u_r = -\frac{1}{10} \left(\frac{g}{h}\right)^{0.5} H_{rms} \quad \text{and} \quad (3.27)$$

$$v_t = 10^{-2} c_h = 10^{-2} \left(\frac{g T}{2\pi}\right) \tanh(kh) \cdot h \quad (3.28)$$

where $c$ is the phase velocity of the waves.

These formulas can be explained as follows. Formula (3.25), as mentioned above, expresses the difference between the radiation stress and the setup pressure gradient. In the calculation of $u_e(-h)$, see (3.24b), the gradient of $R$ in the cross-shore y-direction plays an important role. The term containing $R$ is a positive contribution to $u_e(-h)$.

Formula (3.26), treating the resulting shear stress at the wave trough level, depends strongly on the gradient of $H_{rms}$ in cross-shore direction. A large gradient will result in a large, negative, shear stress.

The value for $u_r$, described in (3.27), is linearly proportional to $H_{rms}$. Higher waves cause a larger mass flux above wave trough level.
The turbulent diffusivity coefficient (3.28) is stated to be linearly proportional to the phase velocity \( c \) of the waves, and the water depth \( h \). So a larger water depth (and higher waves) will result in a larger value for \( v_t \).

V.d. Kerk substitutes (3.25), (3.26), (3.27) and (3.28) in (3.24) and writes the equation in dimensionless form. The result of this is presented below: for a more detailed derivation of the formula reference is made to the original literature.

\[
\frac{u_e(-h)}{gT} = A \left[ \frac{\partial \left( \frac{R}{\rho g h} \right)}{\partial (kh)} \cdot D + \frac{R}{\rho g h} \right] \cdot \frac{1}{(1 + 0.375 \gamma^2)} \tan \beta_e + \\
+ B \left[ \frac{\partial \left( \frac{H_{rmr}}{h} \right)}{\partial (kh)} \cdot D + \frac{H_{rmr}}{h} \right] \cdot \frac{1}{(1 + 0.375 \gamma^2)} \tan \beta_e + C 
\]

(3.29)

where:

\[
A = \frac{1}{3 \cdot 2 \pi} \left( \frac{d}{h} \right)^2 \cdot \frac{kh}{10^{-2}}, \\
B = \frac{1}{(2 \pi)} \left( \frac{d}{h} \right) \cdot \frac{kh}{10^{-2}} \left( \frac{1}{16} + \frac{0.9}{2 \pi kh} \right) \left( \frac{H_{rmr}}{h} \right), \\
C = -\frac{1}{10 \cdot 2 \pi} \sqrt{kh \cdot \tanh(kh)} \left( \frac{H_{rmr}}{h} \right), \\
D = 1 - \frac{\partial \ln(\tanh(kh))}{\partial \ln(kh \cdot \tanh(kh))}, \text{ originating from } \frac{\partial kh}{k \partial h}, \\
\gamma = \text{average } \frac{H_{rmr}}{h} \text{ in the breaker zone; } \gamma \approx 0.39 \text{ is breaking criterion.}
\]

The term \( \frac{R}{\rho g h} \) is derived from (3.25) using the linear wave theory, and yields:

\[
\frac{R}{\rho g h} = \left( \frac{H_{rmr}}{h} \right)^2 \cdot \frac{kh}{4 \sinh(2kh)} + \frac{\xi}{h}
\]

where \( h = h_0 + \xi \); undisturbed water depth plus setup.

Equation (3.29), which is not very transparent, can be written as:

\[
\frac{u_e(-h)}{gT} = f(kh, s_0, \tan \beta_e).
\]

(3.31)
As both \( u_e \) and \( \tan \beta_e \) are unknown variables, this equation must be solved iteratively. V.d. Kerk uses a first guess for \( \tan \beta_e \) in his computer program, and then corrects this first estimate iteratively.

Also V.d. Kerk remarks that the calculated \( u_e \) can only be seen as a first approach towards the real value for \( u_e \). This is mainly due to an inaccurate calculation of the wave height, resulting in an inaccurate setup and \( R/\rho gh \).

**Non-breaking waves**

Using the assumptions of Stive and De Vriend (1987b), which were described in the beginning of this section, the following equation for \( u_e \) is derived (Stive and De Vriend, 1987):

\[
 u_e = \frac{1}{4} \frac{u_m^2}{c} \left\{ 3 - \left[ 2(2 + Z_b) \cos Z_b - 2(1 - Z_b) \sin Z_b - \exp(-Z_b) \exp(-Z_b) \right] \right\} 
\]

(3.32)

where:
- \( u_m \) = the amplitude of the orbital velocity near the bottom
- \( c \) = phase velocity of the waves
- \( Z_b \) = dimensionless vertical coordinate in the bottom layer, comparable to \( Z_m \), (see 3.24a)

With \( Z_b = 1 \) (upper boundary of the bottom layer), and divided by \( gT \), (3.32) yields:

\[
 \frac{u_e}{gT} = \frac{\pi}{2} \frac{u_m}{gT} \tan(hk) \left\{ 3 + e^{-2} - 2e^{-1} \cos(1) \right\} 
\]

(3.33)

The wave-dissipation theory described above forms the second component used by V.d. Kerk.

**3.4.3. The Bailard formula**

The Bailard formula is used in Crostran, and also by V.d. Kerk, as a link between the bottom currents, the sediment transport and the equilibrium slope.

In section 3.3, equation (3.11), the Bailard formula is presented. The equation can be written like:

\[
 <i> = f(u, \tan \beta_e, \text{sedimentparameters}) 
\]

(3.34)

V.d. Kerk chooses \( <i> = 0 \), thus creating a relation describing the equilibrium profile:

\[
 \tan \beta_e = f(u, \text{sedimentparameters}) 
\]

(3.35)

Using the notations introduced by Stive (1986), he derives the following equation:

\[
 \tan \beta_e = \frac{<q_{at}> + <q_{at}^*>>}{<q_{at}^*>/\tan \beta}
\]

(3.36c)

where:
- \( <q_{at}> \) = time averaged sediment transport due to wave asymmetry
\[
<q_{un}> = \text{time averaged sediment transport due to the undertow} \\
<q_{st}> = \text{time averaged sediment transport due to the slope of the profile} \\
\]
The term \(<q_{st}>>\) is assumed to be proportional to \(\tan \beta\). So if \(\tan \beta = \tan \beta_e\) it follows that \(<q_{st}>> + <q_{un}>> = <q_{st}>>.

This formula is used by V.d Kerk as the third component in calculating the equilibrium profile. This third component contains the information about sediment transport, and is directly connected to the behaviour of a disturbance of the equilibrium profile. Therefore, it is important to understand what really happens in this part of the computation.

To achieve this, and to be able to calculate the propagation of disturbances of the equilibrium profile, the Bailard formula is rewritten to:

\[
S_y = F_2 - F_3 \frac{\partial z}{\partial y} ,
\]

where:

\(S_y\) = volume transport of sediment in [m\(^2\)/s] \\
\(F_2\) = a function depending on \(z, T, w, w_s\) and \(H_{rms}\) \\
\(F_3\) = a function depending on \(z, T, w, w_s\) and \(H_{rms}\)

and further developed similar to the V.d. Kerk theory.

3.5. The functions \(F_2\) and \(F_3\)

This section will particularly be dedicated to the equation:

\[
S_y = F_2 - F_3 \frac{\partial z}{\partial y} ,
\]

It is important to remark that \(F_3\) is a positive value. So, keeping this in mind, it can be seen that \(S_y\) increases (which means more offshore transport) as \(\partial z/\partial y\) becomes more negative which is the case with a steeper profile shape. The coordinate system is defined below. This definition will be used throughout this report.

The y-axis is directed positive offshore, the z-axis is directed positive upwards. As \(\tan \beta = \partial z/\partial y\), a negative \(\tan \beta\) signifies a decreasing z seawards, see Figure 3.5. A more precise definition is given in section 3.7.

First the derivation of the expression of \(F_2\) and \(F_3\) will be presented. Thereafter the physical meaning of the two constants will be outlined.

To derive the expressions for \(F_2\) and \(F_3\), the Bailard formula (3.11) is rewritten using the following relations:

\[
S_y = \frac{i}{(\rho_s-\rho)gN} \quad \text{and} \quad \frac{\rho}{\rho_s-\rho} = \frac{1}{\Delta} ,
\]

where \(\rho_s\) is the sediment density, and \(N\) is the local volume concentration of solids.

With this, the Bailard formula becomes:
Figure 3.5 Definition sketch.

\[ <S_y> = <S_{\bar{u}}> + <S_u> = \frac{C_f}{N\Delta g} \left( \frac{\epsilon_b}{\tan \phi} \right) \left[ <|u|^2u> - \left( \frac{\tan \beta}{\tan \phi} \right) <|u|^3> \right] + \]

\[ + \frac{C_f}{N\Delta g} \left( \frac{\epsilon_s}{W} \right) \left[ <|u|^3u> - \left( \frac{\epsilon_s}{W} \tan \beta \right) <|u|^5> \right] \]  

(3.38)

This equation can be rewritten to:

\[ <S_y> = \frac{C_f}{N\Delta g} \left[ \left( \frac{\epsilon_b}{\tan \phi} \right) <|u|^2u> + \left( \frac{\epsilon_s}{W} \right) <|u|^3u> \right] - \]

\[ - \left( \frac{\partial z}{\partial y} \right) \frac{C_f}{N\Delta g} \left[ \left( \frac{\epsilon_b}{\tan^2 \phi} \right) <|u|^3> + \left( \frac{\epsilon_s}{W} \right)^2 <|u|^5> \right] \]  

(3.39)

where:

- \( S_y \) = the total time averaged cross-shore sediment transport
- \( u = \bar{u} + \bar{u} \) where \( \bar{u} \) is the time averaged current velocity near the bottom \( \bar{u} \) is the variable component of the velocity
- \( C_f \) = shear coefficient (shear = \( C_f \ u^2 \)) equal to 0.5 \( f_w \) : \( C_f \) is related to bottom roughness and stroke length; see for instance Swart (1974)
- \( \phi \) = internal shear angle of the sediment
- \( \tan \beta \) = slope of the profile
- \( \Delta = (\rho_s - \rho)/\rho \)
- \( N \) = local volume concentration of solids
- \( W \) = fall velocity of the sediment in water

\( \epsilon_b \) and \( \epsilon_s \) are efficiency factors, which indicate the amount of wave energy that is used for sediment transport.

From this equation, the expressions for \( F_2 \) and \( F_3 \) can be derived:
\[ F_2 = \frac{C_f}{N \Delta g} \left[ \frac{e_b}{\tan \phi} < |u|^2 u > + \frac{e_s}{W} < |u|^3 u > \right] \] (3.40)

\[ F_3 = \frac{C_f}{N \Delta g} \left[ \frac{e_b}{\tan^2 \phi} < |u|^3 > + \frac{e_s}{W} < |u|^5 > \right] \] (3.41)

To be able to obtain values for \( F_2 \) and \( F_3 \), these expressions are rewritten using Stive's notation (1986). After this the expressions are written in non-dimensional form, so that the computer program by V.d. Kerk can be used to calculate \( F_2 \) and \( F_3 \).

The resulting equations are presented below. For more details and for an explanation of the results reference is made to annex A.

\[ F_2 = < q_{as} > + < q_{un} > \] (3.42)

\[ F_3 = \frac{< q_{sl} >}{\partial z / \partial y} \] (3.43)

where:

\(< q_{as} > = \text{sediment transport due to wave asymmetry} \)

\(< q_{un} > = \text{sediment transport due to the undertow} \)

\(< q_{sl} > = \text{sediment transport due to the slope} \)

After rewriting these equations to a non-dimensional form (see annex A), the following equations are derived:

\[ \frac{F_2}{g^2 T^3} = \frac{< q_{as} >_{\text{dimless}}}{< q_{as} >_{\text{dimless}}} + \frac{< q_{un} >_{\text{dimless}}}{< q_{un} >_{\text{dimless}}} \] (3.44)

\[ \frac{F_3}{g^2 T^3} = \frac{< q_{sl} >_{\text{dimless}}}{\partial z / \partial y} \] (3.45)

Substituting these results in (3.37) reads:

\[ \frac{S_y}{g^2 T^3} = \frac{< q_{as} >_{\text{dimless}}}{< q_{as} >_{\text{dimless}}} + \frac{< q_{un} >_{\text{dimless}}}{< q_{un} >_{\text{dimless}}} + \frac{< q_{sl} >_{\text{dimless}}}{< q_{sl} >_{\text{dimless}}} \] (3.46)

In his computer program, V.d. Kerk computes \( < q_{as} >_{\text{dimless}}, < q_{un} >_{\text{dimless}} \)

and \( < q_{sl} >_{\text{dimless}} / (\partial z / \partial y) \). Using this, values for \( \frac{F_2}{g^2 T^3} \) and \( \frac{F_3}{g^2 T^3} \) are plotted, see appendix B (B4 and B3).

These plots are obtained choosing \( s_0 = 0.02 \) and \( W^* = 1.0 \). The latter denotes the non-dimensional fall velocity of the sediment. It is remarked on beforehand that this value is quite large, corresponding to large sediment particles.

3.6. The physical background of \( F_2 \) and \( F_3 \)

Before analysing \( F_2 \) and \( F_3 \) mathematically, it is important to understand the meaning of \( F_2 \) and \( F_3 \) and their importance for the sediment transport.
First it has to be remarked that $F_2$ and $F_3$ are plotted non-dimensional. $F_2$ and $F_3$ both have the dimension of $m^2/s$, or $m^3/s$ per meter. In the plots, however, $F_2$ and $F_3$ are both divided by $g^2T^3$, where $T$ stands for the wave period. For this reason, the magnitude of $F_2$ and $F_3$ in the plots is rather small. The $F_2$-values are of order $10^{-7}$ and the $F_3$-values of order $10^{-6}$. An advantage of plotting $F_2$ and $F_3$ in non-dimensional form is the wider applicability of the plots.

To understand the physical background of $F_2$ and $F_3$, it is important to note that $F_2$ and $F_3$ "cover" three different processes that determine the cross-shore sediment transport, and thus the profile shape. These processes are the wave asymmetry, the undertow and the sediment transport due to the profile slope (a gravity related process). These three processes, and their relative importance, are qualitatively described below.

The wave asymmetry

It is well known that, below the surface, waves cause an orbital movement of water particles. In the bottom layer, this orbital movement is reduced to a horizontal movement, going forwards and backwards periodical with the surface waves. If, in the bottom layer, there is sediment in suspension, the sediment is moved horizontally forwards and backwards as well. Concentrating on the cross-shore profile, only motions perpendicular to the coast are interesting.

This is what happens in case of sinusoidal waves. The sediment is moved forwards and backwards without a net translation. The transport of sediment averaged over a wave period equals zero. Because of the wave asymmetry, however, in reality there is a net sediment transport unequal to zero. The motion forward is different from the motion backward, in sediment concentration and velocity distribution. This results in a net sediment transport in coastward direction. As in case of wave asymmetry the transport depends on instantaneous changes in velocities close to the bottom, a formulation describing the instantaneous transport as a function of the current velocity is needed. Stive [20] adopts some power-function of the velocity.

The undertow

The physics of the undertow was already explained in section 3.4.2 of this chapter. The undertow causes an offshore directed sediment transport. This is the opposite direction of the transport due to the wave asymmetry. It is obvious that the balance between the two components determines the profile shape to a large extent.

The profile slope

Due to the gravity force, in general the sediment particles on a slope will move downwards. Because of this, the profile is flattened out, and possible bars, troughs, ripples etc. in the cross-shore profile will disappear. The gravity force has a diffusive character.
As the sediment needs to be in motion before the gravity force can play a role, the influence of the gravity force will primarily be present in the upper part of the profile. In this part of the profile, the stirring up of sediment takes place.

Comparing the magnitudes of the three different components, their importance for the profile shape can be discussed. In appendix A, the three components are plotted separately (plot A1). The plots are again based on \( s_0 = 0.02 \) and \( W^* = 1.0 \).

Looking at the plots, it is important to note that \( Q_{sl} \) should be multiplied by \( \partial y/\partial z \) to calculate sediment transport. \( Q_{as} \) and \( Q_{un} \) are already expressed in terms of sediment transport and do not need to be multiplied by a factor.

Because of the opposite sign of the sediment transport caused by the wave asymmetry on the one hand, and by the undertow on the other hand, their combination (=\( F_2 \)) results in a value equal to zero for \( F_2 \) in case both processes are in balance. In that case \( S_y \) becomes zero for a horizontal profile slope indicating a breaker bar.

The plot showing only \( Q_{as} \) and \( Q_{un} \) (see also appendix A, plot A2) clearly illustrates that \( Q_{as} \) is directed onshore (negative sign) and \( Q_{un} \) is directed offshore (positive sign). Before the waves break, it can be seen that \( Q_{un} \) becomes a little negative, but due to the breaking of waves the undertow is directed offshore behind the breaker line.

To understand the change in \( F_2 \) and \( F_3 \) in case of a disturbance of the profile, the mathematical functions describing \( F_2 \) and \( F_3 \) are briefly discussed below.

The functions describing \( F_2 \) and \( F_3 \) are:

\[
F_2 = \frac{C_f}{N \Delta g} \left[ \left( \frac{\varepsilon_b}{\tan \phi} \right) \leq |u|^3 u \right] + \left( \frac{\varepsilon_s}{W} \right) \leq |u|^3 u \right] \tag{3.40 repeated}
\]

\[
F_3 = \frac{C_f}{N \Delta g} \left[ \left( \frac{\varepsilon_b}{\tan^2 \phi} \right) \leq |u|^3 u \right] + \left( \frac{\varepsilon_s}{W} \right)^2 \leq |u|^3 u \right] \tag{3.41 repeated}
\]

It can be seen that \( F_2 \) and \( F_3 \) depend on different sediment parameters, some constants and coefficients, and on the current velocity \( u \). The current velocity \( u \) is composed of \( u_e \) (breaking waves) and \( u_s \) (non-breaking waves). Both \( u_e \) and \( u_s \) are calculated by V.d. Kerk as a function of \( k_0 h \) and \( s_0 \). This is important to note, as their dependence on \( h \) means that \( F_2 \) and \( F_3 \) can be applied to disturbances of the equilibrium profile. A disturbance means a local change in the coordinate \( z \), and in the water depth \( h \). This is why \( F_2 \) and \( F_3 \) change in case of a disturbance in the equilibrium profile.

3.7. Discussion on computer results

Some calculations were carried out with the computer program of V.d. Kerk. The results were plotted, the plots are included in appendix B. The results were obtained choo-
sing \( s_0 = 0.02 \) (wave steepness) and \( W^* = 1.0 \) (non-dimensional fall velocity), with \( W^* = W \cdot T / H \), and \( W = 0.2 \, m/s \), \( T = 5 \, s \) and \( H = 1 \, m \).

To understand the meaning of positive or negative values, some definitions concerning the coordinate system are stated below.

V.d. Kerk chooses \( \tan \beta = \frac{\partial h}{\partial x} \),

where \( h \) is the water depth. Furthermore he chooses the x-axis positive towards the coast, and the z-axis positive upwards. So a negative \( \tan \beta \) signifies a decreasing water depth and an increasing profile coastward.

In this report, the x-axis is replaced by the y-axis. So the x-axis becomes the longshore axis, and the y-axis the cross-shore axis. Furthermore, the cross-shore y-axis is defined positive in seawards direction, and

\[ \tan \beta = \frac{\partial z}{\partial y}, \]

where \( z \) is the profile height positive upwards. So, in accordance with V.d. Kerk, a negative \( \tan \beta \) signifies an increasing profile height coastward, as in this case \( \Delta z \) is positive and \( \Delta y \) is negative.

In Figure 4.7 the definition of the coordinate system is presented. The value of \( h_0 \), the water depth at the seawards boundary, depends on \( s_0 \) and \( W^* \). For \( s_0 = 0.02 \) and \( W^* = 1.0 \) the value of \( h_0 \) amounts \( 11.8 \, m \). This value remains constant as the theory is developed for constant \( s_0 \) and \( W^* \). Other wave and sediment parameters would result in another value of \( k_0 \) and thus in another position of the coordinate system.

The notation of \( k_0z \) in this report instead of \( k_0h \) used by V.d.Kerk is chosen to be able to describe profile disturbances by a change in profile height, instead of a change in water depth. The link between the two notations is given by: \( k_0z = k_0h + k_0h_0 \), where \( k_0h \) is negative and \( k_0h_0 = -1.89 \).

In the following, the plots in appendix B are briefly discussed.

The equilibrium profile:

Plot B1 in appendix B shows the equilibrium profile. The equilibrium profile is derived from the equilibrium slope by integrating the slope numerically. The equilibrium slope \( \tan \beta_e \) was calculated iteratively until sediment transports equal zero, see chapter 3. An integration module was added to the original program. The integration is carried out after the calculation of the whole equilibrium slope.

The equilibrium slope is computed iteratively for every point on the profile, starting in deep water and calculating in coastal direction. The slope is corrected iteratively until \( S_e \) equals zero, with an accuracy of 3%. As long as the accuracy is not reached, the slope is corrected and the calculation is repeated. At the upper part of the profile, near the coastline, the computation becomes unstable as the profile slope becomes horizontal.
approaching the crest of the outer breaker bar. At that moment the computation is terminated.

The wave height \( H' \):

The non-dimensional wave height \( H' = \frac{H_{rms}}{h} \) increases towards the coast, because of a decreasing \( h \). V.d. Kerk chooses, as a modified breaking criterion, \( H_b/h = 0.48 \). This value can be seen as the upper limit for \( H_{rms}/h \). The plot of \( H' \) (plot B2) shows an increasing \( H' \) for shallower water. Due to the breaking of waves the line should become horizontal closer to the coastline. This is not found using V.d. Kerk's computer program. V.d Kerk [8] compares his model results to measurements on page 41 of his thesis, and finds the same discrepancy.

The setup:

The setup, which is made non dimensional dividing by \( h \), clearly shows the well known curve near the coast, see plot B3. The setup becomes negative due to the gradient in the radiation stresses perpendicular to the coast, thus creating a pressure gradient. The setup causes a difference in water level of \( \pm 0.03 \cdot h \) in the surf zone. The physics are explained in section 3.4.2.

\( F_2 \):

As mentioned before, \( F_2 \) is a combination of two processes causing a sediment transport. The value of \( F_2 \) over the profile, together with the value of \( F_3 \) over the profile, form an equilibrium combination with the bottom slope in case of an equilibrium profile. The sediment transport component \( F_2 \) consists of a wave asymmetry contribution and an undertow contribution.

According to Nielsen and Sørensen (1970), nearly breaking waves cause an onshore directed current near the bottom, and breaking waves cause an offshore directed current near the bottom (see Figure 3.3). As the direction of the undertow determines the direction of the sediment transport, a coastward directed sediment transport is to be expected outside the breaker zone. This is in accordance with the plot, see plot B3, as a negative \( F_2 \) signifies a coastward directed transport (\( S_y \) is positive seawards, and negative coastward).

As the water depth decreases, the undertow increases, resulting in an increasing offshore transport. At the breaker line, the fraction of breaking waves becomes larger, which causes a smaller coastward directed transport. This is clearly shown in the plot, for \( k_0y \) equal to approximately 5.

\( F_3 \):

As already explained above, \( F_2 \) and \( F_3 \) form an equilibrium combination, both representing sediment transport components. \( F_3 \) can be interpreted as a proportionality constant, expressing the relation between the slope of the profile and the sediment transport. In plot B5 \( F_3 \) is plotted. It can be seen that \( F_3 \) is positive, illustrating that a coastward increasing slope causes a seawards directed gravitational transport of sediment.
In the formula \( S_y = F_2 - F_3 \left( \frac{\partial z}{\partial y} \right) \), it can be seen that an increasing negative \( \frac{\partial z}{\partial y} \), with a positive \( F_3 \), results in a larger \( S_y \) seawards. This is in accordance with what to expect logically, as an increasing negative \( \frac{\partial z}{\partial y} \) describes a steeper profile.

The plot of \( \tan \beta_e \)

The values of \( \tan \beta_e \) are negative over the profile, see plot B6, illustrating that the bed elevation is monotonically increasing in coastal direction. If the plot showed a partly positive \( \tan \beta_e \) a breaker bar would exist in the equilibrium slope.

The plot of \( S_y \)

In theory \( S_y \) should be equal to zero over the profile, according to the definition of the equilibrium profile. In plot B7 of appendix B the values for \( S_y \) have been plotted calculating:

\[
S_y = F_2 - F_3 \cdot \tan \beta_e
\]

to check the computer program. The plot shows that the computation becomes unstable as the slope of the profile becomes more horizontal near the coastline. Probably the instability is due to \( \partial y / \partial z \) approaching to zero. The computer program is terminated when instability occurs. Consequently, the last few data near the vertical z-axis, see Figure 4.7, are not reliable.

3.8. Sensitivity of \( F_2 \) and \( F_3 \)

Because of the importance of \( F_2 \) and \( F_3 \) for further study, the sensitivity of \( F_2 \) and \( F_3 \) for different values of \( W^* \) is briefly investigated. For this goal, \( F_2 \) and \( F_3 \) are plotted for three different values of \( W^* \), namely 1.0, 0.1 and 0.01. The initial wave steepness \( s_0 \) is kept constant at a value of 0.02. As \( W^* = W \cdot T / H_{rms} \), with a constant \( T = 5 \) s and a constant \( H_{rms} = 1.0 \) m, this means that the fall velocity of sediment equals respectively 0.2 m/s, 0.02 m/s and 0.002 m/s. In reality the fall velocity of sediment particles of 200 \( \mu \) amounts approximately 0.01 to 0.02 m/s. The plots (B8 and B9) are included in appendix B

The plot of \( F_2 \) (B8) shows that there is quite a difference in \( F_2 \) values for different values of \( W^* \). It can be seen that larger \( F_2 \) values result from smaller values of \( W^* \), meaning smaller values for the sediment fall velocity. This is to be expected, as smaller sediment particles yield higher sediment transports.

The same can be concluded from the plot of \( F_3 \) (B9) for different values of \( W^* \). Obviously, smaller sediment with a smaller fall velocity causes \( F_2 \) and \( F_3 \) to be larger indicating larger sediment transports.

In this study, \( F_2 \) and \( F_3 \) will be used for \( W^* = 1.0 \) and \( s_0 = 0.02 \) only. Unfortunately, this value was chosen too high and therefore represents large sediment particles.
It is remarked, however, that most results of this study count for other functions of $F_2$ and $F_3$ as well. It would be interesting to compare the results for different values of $W^*$. This could be subject for further investigation.

3.9. Conclusions

From this chapter it can be concluded that:

1. The coastal constant still more or less has a black box character. Although V.d. Kerk succeeded in describing $T_0$ as a function of a disturbance ($\Delta \beta$) of the equilibrium slope, several short-cuts to compute this equilibrium slope were made. In his conclusions, V.d. Kerk states that $T_0$ still hides a lot of information on the exact sediment transport.

2. The wave dissipation module used is incapable of handling breaker bars, which is a major drawback of the theory of V.d. Kerk. This does not mean that the program of V.d. Kerk is useless, because it calculates $F_2$ and $F_3$ correctly towards the coastline, or towards the outer breaker bar in case of breaker bars. As the functions describing $F_2$ and $F_3$ depend on $h$, $F_2$ and $F_3$ can be applied to a barred profile, although assumptions of V.d. Kerk in his wave module do not permit breaker bars to exist in his calculation of the equilibrium profile.

3. The Bailard formula forms an essential part of the theory. At this moment, the accuracy of the sediment transport formula of Bailard is under discussion. New formulas to describe the sediment transport in terms of energy dissipation are currently developed within the scope of UNIBEST.

4. V.d.Kerk only calculates with non-varying short wave conditions. Consequently, the short wave-long wave interaction is not included in the calculation of $F_2$ and $F_3$.

5. The functions $F_2$ and $F_3$ used for further study are only correct in case of an initial wave steepness of $s_0=0.02$ and a non-dimensional fall velocity of sediment in water of $W^*=1.0$. It is simple, however, to repeat all exercises for other $F_2$ and $F_3$. 

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Chapter 4. Analysis of the influence of $F_3$ on a distorted profile

This chapter is partially based on a note of W.T Bakker (16-3-1994).

4.1. Introduction

The amplitude of morphologic changes, described by diffusion equations, could increase because of large gradients in the diffusion coefficients. Like short wave amplitudes increase in shallowing water, the amplitude of disturbances might increase as well due to a decreasing diffusion coefficient. This is analyzed in the previous chapter.

In addition to chapter 3, the functions $F_2$ and $F_3$ will be further focused on. In section 4.2 the physical meaning of the analysis of breaker bars and supplies using $F_2$ and $F_3$ is explained. The link between the present theory and line-modelling is outlined, and a second interpretation of the results for $F_2$ and $F_3$ is presented. In section 4.3 the basic equations for further analysis are derived. An equation describing the sediment transport due to bottom changes is derived, which forms the basis of the theory. The four terms this equation consists of are explained qualitatively in section 4.4.

Furthermore, the term representing the diffusive influence of the gravity force is further analyzed in section 4.5. In section 4.6 the results of the analysis are evaluated, and in section 4.7 the results are compared to numerical outcome.

Finally, in section 4.8 some conclusions on this chapter are formulated.

In the next chapter a more extended analysis of $F_2$ and $F_3$ is presented.

4.2. The use of $F_2$ and $F_3$ in describing breaker bars

In chapter 3 the functions of $F_2$ and $F_3$ defined in this report were calculated over the cross-shore profile (see appendix B) by means of the Bailard theory [3] and V.d. Kerk's [8] investigation and computer program, which is based on the theory of Stive and De Vriend [23], [24]. $F_2$ and $F_3$ were defined as follows (see also chapter 3):

$$ S_y = F_2 - F_3 (\frac{\partial z}{\partial y}) $$

In this and the following chapters, an effort will be made to develop a theory describing the behaviour of breaker bars or foreshore supplies in an analytical way. For this aim, equation (4.1) describing the cross-shore sediment transport is used. Using this equation, the theory will be based upon the method of line modelling. This is explained below.

In chapter 3, the shape of the equilibrium profile was calculated using equations that try to simulate the various physical processes, like undertow and sediment transport, that form a profile. The equations used by V.d. Kerk were treated in chapter 3. This approach to profile changes is clearly process-based.

The method of line modelling, however, uses the difference between the actual state of the profile and the equilibrium profile as the driving force behind profile changes, instead of physical processes. This is a different approach, and a different interpretation of profile changes.
The difference is illustrated using equation (4.1). In this equation, $S_y$ is expressed in $F_2$ and $F_3$ and the bottom slope. $F_2$ and $F_3$ are calculated using physical equations, like the Bailard formula. In other words, $F_2$ and $F_3$ "cover" physical processes (not exactly of course), and from these processes an $S_y$ results. In his computer program, V.d. Kerk calculates the bottom slope, $F_2$ and $F_3$ iteratively until $S_y = 0$. For $S_y = 0$ the equilibrium slope is found.

Another interpretation of the results for $F_2$ and $F_3$ is the following. Considering $F_2$ and $F_3$ as functions that only depend on $z$ for given wave and sediment parameters, a difference in $z$ from the equilibrium profile causes an $S_y$ unequal to zero. In other words: a sediment transport results from a change in $z$. This can be explained as follows.

Another $z$ than the equilibrium $z$ yields other values for $F_2$ and $F_3$. This results in another $\frac{\partial z}{\partial y}$ to make sure that $S_y$ equals zero. So, because of a change in $F_2$ and $F_3$ due to a change in $z$, the bottom slope needs to be adapted to ensure that $S_y$ equals zero. This new slope is again an equilibrium slope, as $F_2$ and $F_3$ were calculated to result in an equilibrium profile. The adaptation of the bottom slope to the new equilibrium slope in reality is performed by sediment transport. This is how sediment transport results from a change in $z$.

Equation (4.1) clearly shows that the sensitivity of $S_y$ to a certain change of $\frac{\partial z}{\partial y}$ depends on $F_3$. A large $F_3$ results in a large $S_y$ for small changes in $\frac{\partial z}{\partial y}$. Consequently, a large $F_3$ is a characteristic for a quickly-adapting profile.

This illustrates the second interpretation of $F_2$ and $F_3$. The functions describe the way a profile develops towards the equilibrium slope after disturbance of the equilibrium profile. $F_2$ and $F_3$ will be different for different sites (Terschelling, Nordeney, Torsminde) as the functions depend on sediment parameters and wave climate. It is obvious that other wave conditions and other sediment parameters yield another equilibrium profile shape.

Following the last interpretation of equation (4.1), apart from being a physical phenomenon, breaker bars and supplies can also be treated as being a disturbance of the equilibrium profile. By means of $F_2$ and $F_3$ the behaviour of the breaker bars in time and place can be calculated.

The problem of describing breaker bars can be divided into two sub-problems: the initiation of breaker bars (how are bars created), and the behaviour of existing bars (why do bars generally move in seawards direction while their amplitude decreases). To both sub-problems the method of line modelling will be applied. Both $F_2$ and $F_3$ have to be involved in answering both sub-problems.

### 4.3. Derivation of the basic equation

Focusing on the periodical behaviour of breaker bars in particular and referring to equation (4.1), it is assumed that:

\[
S_y = S'_y + \bar{S}_y \quad \text{and} \quad z = \bar{z} + \bar{z},
\]

(4.2)  \hspace{1cm} (4.3)
where $z$ is positive upwards. A bar indicates a time-averaged and a tilde a harmonic variable in time. Note that $\tilde{z}$ denotes the equilibrium profile height. The periodical change, expressed by the tilde, has the timescale of the breaker bars, and the time averaging is done over the period of a breaker bar.

It is important to keep this in mind, as many different timescales can be distinguished in cross-shore processes: the timescale of the wave period (seconds), the timescale of the tidal wave (hours) and everything in between (e.g. seiches), the timescale of the changing hydraulic conditions over the seasons (months) and the timescale of the breaker bars (years). Each of those processes with different timescales can cause $F_2$ and $F_3$ to change. The processes are discussed below.

During one wave period, changes in sediment concentrations and current velocities influence $Q_{as}$ (wave asymmetry), $Q_{un}$ (undertow) and $Q_{sl}$ (slope). These fluctuations were already averaged by V.d. Kerk in his computer model. In the Bailard transport formula, used in the computer model, the time-averaged total load sediment transport is calculated. The varying current velocities at the bottom are rewritten to averaged values before being used in the Bailard formula. The details were outlined in chapter 3 and in annex A.

Furthermore, it is very likely that tidal waves have an important influence on $F_2$ and $F_3$, as the increasing water level due to tidal waves changes the location of the breaker line. Consequently, the whole system of breaking waves, undertow and wave asymmetry is shifted onshore and offshore with a period equal to the tidal wave period. Concentrating on a certain location in the profile, $F_2$ and $F_3$ will certainly change in a tidal wave period. This change of $F_2$ and $F_3$, however, is not included in the theory. It is assumed that, compared to the period of breaker bars, the tidal effects can be levelled out. Therefore, only the incoming short wave period is calculated with. Still it is possible that tidal waves, because of the changing water level, have an important influence on the shape of the breaker bars and their propagation.

Finally, during autumn and winter, it is known that the mean wave height is larger than in the rest of the year. This periodical variation in wave conditions will also have an influence on $F_2$ and $F_3$. It would take too much time to include this variation in the analysis: a lot of exercises with different wave parameters would be necessary (using the computer model of V.d. Kerk).

The periodical change of $F_2$ and $F_3$, meant by the tilde, is the change of $F_2$ and $F_3$ because of the periodical change in the vertical coordinate $z$. Note that this change is indirect: there is a non-linear link between a variation of $F_2$ and $F_3$ and a variation in $z$. The reason for this is that $F_2$ and $F_3$ are non-linear functions.

This is illustrated below.

Concentrating on a certain location in the cross-shore profile, $F_2$ and $F_3$ have a certain value connected to the value for $z$. As a breaker bar passes this location (this may take years), the variation in $z$ causes a variation in $F_2$ and $F_3$. Consequently, applying the Taylor expansion, $F_2$ and $F_3$ have to be written as follows:

$$F'_2 = F_2(\tilde{z}) + \tilde{z} \cdot \frac{dF_2(\tilde{z})}{dz} + \tilde{z}^2 \cdot \frac{d^2F_2(\tilde{z})}{dz^2} + \ldots$$

and

$$F'_3 = F_3(\tilde{z}) + \tilde{z} \cdot \frac{dF_3(\tilde{z})}{dz} + \tilde{z}^2 \cdot \frac{d^2F_3(\tilde{z})}{dz^2} + \ldots$$

(4.4)
\[ F_3 = F_3' + F_3'' = F_3(z) + \hat{z} \frac{dF_3(z)}{dz} + \hat{z}^2 \left( \frac{d^2F_3(z)}{dz^2} \right) + \ldots \] (4.5)

The higher order terms of the Taylor expansion are neglected, as \( \hat{z} \) is assumed to be very small. Because of this, the terms containing \( \hat{z}^2 \) can be neglected compared to the first order terms.

In equations (4.4) and (4.5), \( F_2(z) \) and \( F_3(z) \) denote the values for \( F_2 \) and \( F_3 \) for the equilibrium profile. These values, in combination with the equilibrium slope, would result in a sediment transport equal to zero. The last terms in equations (4.4) and (4.5) denote the change in \( F_2 \) and \( F_3 \) for a harmonic change in \( z \). These terms express the non-linear properties of \( F_2 \) and \( F_3 \) as functions of \( z \).

The magnitude of the variation in \( F_2 \) and \( F_3 \) for a variation in \( z \) depends on the location in the profile. Only in the upper part of the profile a variation in \( z \) causes a large variation in \( F_2 \) and \( F_3 \), because of a larger gradient in \( F_2 \) and \( F_3 \). This can be seen in plots B4 and B5 included in appendix B, showing \( F_2 \) and \( F_3 \). The values for \( dF_2/dz \) and \( dF_3/dz \) increase in the upper part of the slope. This illustrates that the magnitudes of \( F_2 \) and \( F_3 \) will be very dependent on the location in the profile.

In case of \( F_3 \), describing the gravity force, this means that the sediment transport due to a variation in the slope of the profile will be larger close to the shoreline than further offshore. The gravity force has more influence close to the shoreline. This could be expected, as close to the shoreline the stirring up of sediment is larger.

From the equations (4.1) to (4.5) it follows that:

\[ S_y = \left\{ F_2(z) + \hat{z} \frac{dF_2(z)}{dz} \right\} - \left\{ F_3(z) + \hat{z} \frac{dF_3(z)}{dz} \right\} \cdot \left( \frac{\partial \hat{z}}{\partial y} + \frac{\partial \hat{z}}{\partial y} \right) \] (4.6)

Writing all terms separately, this equation yields:

\[ S_y = \left\{ F_2(z) + \hat{z} \frac{dF_2(z)}{dz} \right\} - \left\{ F_3(z) \frac{\partial \hat{z}}{\partial y} + F_3(z) \frac{\partial \hat{z}}{\partial y} + \hat{z} \frac{dF_3(z)}{dz} \frac{\partial \hat{z}}{\partial y} + \hat{z} \frac{dF_3(z)}{dz} \frac{\partial \hat{z}}{\partial y} \right\} \] (4.7)

\( S_y \) contains a time-averaged and a harmonic component, see equation (4.2). These two components will be formulated separately below.

Averaging over the period of a breaker bar yields the following:

\[ \frac{\partial}{\partial z} \frac{dF_2(z)}{dz} = 0, \quad \frac{\partial}{\partial z} \frac{dF_3(z)}{dz} = 0 \quad \text{and} \quad \frac{\partial \hat{z}}{\partial y} = 0 \] (4.8)

Substituting this into equation (4.7) yields for the time-averaged component:

\[ S_y' = F_2(z) - F_3(z) \frac{\partial \hat{z}}{\partial y} \] (4.9)

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As already mentioned before, denotes the equilibrium profile height. Consequently, as the equilibrium profile was defined by assuming the cross-shore sediment transport equal to zero, it follows from equation (4.9) that \( \Sigma_y = 0 \).

Concentrating on the harmonic component, equation (4.2) is rewritten to:
\[
\tilde{S}_y = S_y - \Sigma_y.
\]  
Substituting equation (4.7) and (4.9) into (4.10) yields:
\[
\tilde{S}_y = \tilde{\bar{z}} \left[ \frac{dF_2(\tilde{\bar{z}})}{d\bar{z}} - F_3(\tilde{\bar{z}}) \frac{\partial \tilde{\bar{z}}}{\partial y} - \tilde{\bar{z}} \frac{dF_3(\tilde{\bar{z}})}{d\bar{z}} \frac{\partial \tilde{\bar{z}}}{\partial y} - \tilde{\bar{z}} \frac{dF_3(\tilde{\bar{z}})}{d\bar{z}} \frac{\partial \tilde{\bar{z}}}{\partial y} \right].
\]  
This is the expression describing the sediment transport due to the harmonic movement of the bottom level. The equation can be rewritten to:
\[
\tilde{S}_y = \tilde{\bar{z}} \left\{ \frac{dF_2(\tilde{\bar{z}})}{d\bar{z}} - \left[ \frac{dF_2(\tilde{\bar{z}})}{d\bar{z}} \frac{\partial \tilde{\bar{z}}}{\partial y} - \frac{dF_3(\tilde{\bar{z}})}{d\bar{z}} \frac{\partial \tilde{\bar{z}}}{\partial y} \right] - F_3(\tilde{\bar{z}}) \frac{\partial \tilde{\bar{z}}}{\partial y} \right\},
\]  
showing that three terms are directly linked to a change in the bottom level. The last term is not directly linked to \( \tilde{\bar{z}} \), but causes sediment transport via the periodical change in the bottom slope. Another remark is that only the first term of equation (4.12) is independent of the bottom slope.

In this chapter, the contribution of the last term is investigated. In the next chapter, the influence of the other three terms will be concentrated on. Note that the last term is the only linear term. In case of a more linearised approach, leaving out the contribution of \( dF_2/d\bar{z} \) and \( dF_3/d\bar{z} \) in equations (4.4) and (4.5), this term would be the only term resulting from (4.10).

4.4. Explanation of the basic equation

Before concentrating on the last term of equation (4.12), the four terms are briefly discussed. Furthermore, their relative importance to the sediment transport is estimated.

From equation (4.12):
\[
\tilde{S}_y = \tilde{\bar{z}} \left\{ \frac{dF_2(\tilde{\bar{z}})}{d\bar{z}} - \left[ \frac{dF_2(\tilde{\bar{z}})}{d\bar{z}} \frac{\partial \tilde{\bar{z}}}{\partial y} - \frac{dF_3(\tilde{\bar{z}})}{d\bar{z}} \frac{\partial \tilde{\bar{z}}}{\partial y} \right] - F_3(\tilde{\bar{z}}) \frac{\partial \tilde{\bar{z}}}{\partial y} \right\},
\]  
it can be seen that, apart from the last term which will be analyzed in this chapter, three other terms have their influence on \( \tilde{S}_y \) in case of a bottom change as well. The last term results from a completely linearised approach. The three extra terms can be interpreted as representatives of the second-order influences of \( F_2 \) and \( F_3 \).
4.4.1. The first term of the basic equation

The term $\frac{dF_2}{dz}$ contains the derivative of $F_2$. Therefore a brief summary on $F_2$ is presented below:

For calculating sediment transport $F_3$ needs to be multiplied by the bottom slope. $F_2$, however, directly yields sediment transports. Multiplication with the bottom slope is not necessary. This is illustrated by equation (4.1):

$$S_y = F_2 - F_3 \frac{\partial z}{\partial y}. \quad (4.1 \text{ repeated})$$

$F_2$, like $F_3$, depends on the bottom height $z$. Consequently, a disturbance of the equilibrium profile causes $F_2$ to change.

In appendix A, plot A2, the two components of $F_2$ are given. These were briefly discussed in section 3.6. It is assumed that the two components are calculated correctly by V.d. Kerk.

Looking at the plot showing the components of $F_2$, it can be seen that the wave asymmetry causes a negative (coastward directed) contribution to the sediment transport. As the water depth decreases this contribution increases in coastward direction, as could be expected.

The contribution to the sediment transport due to the undertow first shows a negative sign. The curve shows small negative values (coastward directed sediment transport) before becoming positive. This is due to a small coastward directed undertow in case of non-breaking waves in shallow water. As more waves break, the direction of the undertow changes from onshore to offshore. This phenomenon causes breaker bars to grow in cross-shore computer models (like UNIBEST) at the location where waves break, see [15].

In appendix B, a plot of $F_2$ (non-dimensional) is included (plot B4). It is remarked that $F_2$ and $F_3$ always have dimensions m^2 s^{-1}. It will be indicated where $F_2$ and $F_3$ are written in non-dimensional form.

Following $F_2$ from right to left in the plot, from deep water towards the coast, the following can be stated.

In deep water the waves do not influence the profile shape. This results in a cross-shore transport equal to zero. As the water depth decreases, the waves start to have influence on the sediment. The non breaking waves cause a coastward directed sediment transport. The decreasing water depth results in an increasing transport, shown in the plot. $F_2$ becomes more negative very rapidly.

For $F_2$ equal to approximately $-7 \cdot 10^{-8}$ the waves start to break causing an offshore transport by means of the undertow. As more waves break, the offshore transport increases until finally the gradients in offshore and onshore transport are the same for $F_2 = -9 \cdot 10^{-8}$.

At this point, the minimum of $F_2$, the transport contribution due to $F_2$ is the largest, and in onshore direction. The increase of onshore transport due to the decreasing water depth equals the increasing offshore transport due to the growing fraction of breaking waves. After this point, the sediment transport onshore decreases, as the wave energy is dissipated in the surf zone. In the swash zone $F_2$ finally becomes zero.
The computer program calculates $F_2$ until instability occurs as the profile slope becomes horizontal, for $k_0 y$ approaching zero. Extrapolating the plot, it can be seen that $F_2$ will become zero for $k_0 y$ almost equal zero. From (4.1) it can be seen that at this location the sediment transport equals zero provided that the bottom slope equals zero too.

In order to be able to say something about the term $z \frac{dF_2}{dz}$, $F_2$ needs to be known as a function of the profile height $z$. In appendix H, $F_2$ is fitted as a function of $z$, and $dF_2/dz$ is plotted, see plot H1 and H3. The fitting process is described in section 5.5. At this moment a quantitative impression of the term is sufficient.

Looking at the plot showing $\frac{dF_2}{dz}$ some remarks can be made about the influence of the term. To calculate $\tilde{z}$, it is used $T_w = 5s$, resulting in $k_0 \approx 0.16m^{-1}$.

First it can be remarked that, from $\tilde{z} \approx 7m$ going upwards, $dF_2/dz$ becomes negative. In practice this means that a $\tilde{z}$ of 1 meter at the location where $\tilde{z} = 7m$ yields a negative value for $z(dF_z/dz)$, which means an onshore sediment transport. The same can be stated for $\tilde{z} \approx 8m$. At this location, a $\tilde{z}$ of 1 meter would result in an even larger onshore transport of sediment. At $\tilde{z} \approx 9m$, $dF_2/dz$ has its minimum.

Further coastward of $\tilde{z} \approx 9.75m$, $dF_2/dz$ increases rapidly. It means that the offshore sediment transport caused by a $\tilde{z}$ is larger closer to the coastline. At $\tilde{z} = 10m$ for instance, the term $\tilde{z}[dF_z/dz]$ for $\tilde{z} = 1m$ amounts 0.002 m$^2$/s.

But at $\tilde{z} = 10.5m$, a $\tilde{z}$ of 1 meter yields 0.015 m$^2$/s. Note that these sediment transports take place in case of a disturbance only, otherwise the coast would erode rapidly.

The above can be interpreted as follows. Imagine a very small positive initial disturbance of the equilibrium profile. Applying the theory this small initial disturbance ($\tilde{z}$) causes offshore sediment transport. The offshore transport for the same $\tilde{z}$, however, is different for different locations in the profile. This is illustrated in Figure 4.1.

Because of this, the front of the disturbance will become steeper. Other terms in equation (4.12), containing $\partial \tilde{z}/\partial y$, will start to play a role, preventing plunging breaker bars to occur.

In case of a negative $\tilde{z}$, the same process takes place. Instead of offshore transport, an asymmetric onshore transport over the trough will result in a steeper slope of the front of the breaker bar.

It can be concluded that this term causes increasing steepness of profile disturbances. Other dissipative terms become important.

4.4.2. The second term of the basic equation

The term $-\tilde{z} \frac{dF_3}{dz} \frac{\partial \tilde{z}}{\partial y}$ contains the derivative of $F_3$.

To calculate $[dF_z/dz]$, first $F_3$ needs to be fitted as a function of the profile height $z$. This is done in the same way $F_2$ was fitted as a function of $\tilde{z}$. 

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Figure 4.1 Asymmetric cross-shore transport

The fitted function is included in appendix H (plot H2) as well as a plot of $[dF_3/dz]$, see plot H4. The fitting itself is explained in detail in section 5.5.

Looking at the plot of $dF_3/dz$, a quite simple curve is shown. The gradient of $F_3$ increases smoothly for higher values of $z$. This means that the resulting sediment transport from a profile disturbance, going upward in the profile, increases.

From the plot it follows that the value of $dF_3/dz$ is always positive. The value of $\partial z/\partial y$ is always negative, as this term denotes the equilibrium profile slope. In this theory the equilibrium slope itself has no breaker bars.

From this it follows that the sign of the term, telling whether the resulting sediment transport is directed onshore or offshore, depends on the sign of $\partial z$. A positive $\partial z$ results in a positive sign (offshore), a negative $\partial z$ results in a negative sign (onshore). This can be explained as follows.

Concentrating on a certain location in the profile, a $\partial z$ corresponds to another location in the profile. In case of a positive $\partial z$ a more onshore location and in case of a negative $\partial z$ a more offshore location is transferred to (see Figure 4.2).

In case $\partial z = +\partial z_1$, $dF_3/dz$ is larger (see plot). The local slope of the profile at $y = y_1$ is not an equilibrium slope any more, as $dF_3/dz$ has changed. A new equilibrium slope is to be found. This new equilibrium slope is equal to the slope at $y = y^+$, as at this location the slope corresponds to the new value of $dF_3/dz$. The slope at $y = y^+$ is a steeper slope, so the new equilibrium slope at $y = y_1$ will be steeper.

Summarising, the term $-\partial z [dF_3/dz] \partial z/\partial y$ represents the influence of the gravity force in combination with the equilibrium slope. The effect that a disturbance of a steep equilibrium profile will cause larger changes in cross-shore sediment transport than the same disturbance of a less steep equilibrium profile is described by this term as well.
In order to be able to intercompare the magnitudes of the terms in equation (4.12) at the end of this section, it is tried to estimate the value of this term for a \( \tilde{z} \) of 1 meter. As mentioned before, the slope of the equilibrium profile makes quite a difference in the magnitude of this term. To obtain a value for the equilibrium slope, plot B6 of appendix B is used. A maximum slope \( \partial \tilde{z} / \partial y \) of -0.085 is reached for a wave steepness \( s_0 = 0.02 \) and a non dimensional fall velocity \( W^* = 1.0 \). This value is used to estimate the maximum magnitude of this term.

At \( \tilde{z} = 10 \) m, for \( \tilde{z} = 1 \) m this second term yields an offshore contribution to the sediment transport of approximately \( 8.5 \times 10^{-4} \) m\(^2\)/s. This is half of the transport caused by the first term of (4.12), also for \( z = 10 \) m and with a \( \tilde{z} \) of 1 meter. Keeping in mind that the maximum equilibrium slope is concerned, it can be concluded that this second term is less important to the total \( S_y \) than the first term.

4.4.3. The third term of the basic equation

The term \( -\tilde{z} \left[ \frac{dF_3}{dz} \right] \frac{\partial \tilde{z}}{\partial y} \) differs from the previous terms.

The difference is that the third factor in this term includes \( \tilde{z} \) as well. It is important to note that the value of \( \partial \tilde{z} / \partial y \) is assumed to be small. This is a consequence of the shortcomings of the wave module used by V.d. Kerk in his computer model. In section 3.4.1 it was already explained that the wave height behind a breaker bar will not be the same as in front of a breaker bar due to energy dissipation. This phenomenon is not included correctly in the model of V.d. Kerk.

A large positive \( \partial \tilde{z} / \partial y \) would mean that the water depth behind a breaker bar increases, see Figure 4.3.

A positive \( \partial \tilde{z} / \partial y \) is possible, but \( \partial \tilde{z} / \partial y \) should be smaller than the absolute value of \( \partial \tilde{z} / \partial y \).
The resulting shape of the disturbed profile is shown in Figure 4.4.

The meaning of the term \(-z \frac{dF_3}{dz} \frac{\partial \tilde{z}}{\partial y}\) becomes most clear when a horizontal equilibrium profile is chosen. In such a profile, the equilibrium slope equals zero. The slope of the profile is completely determined by the slope of the disturbance, see Figure 4.5.

Concentrating on a positive \(z\), from the term \(-z \frac{dF_3}{dz} \frac{\partial \tilde{z}}{\partial y}\) it can be seen that a negative \(\frac{\partial \tilde{z}}{\partial y}\) results in an offshore transport (positive sign), and a positive \(\frac{\partial \tilde{z}}{\partial y}\) an onshore transport (negative sign).

For negative values of \(z\), however, the directions are reversed, see Figure (4.5). Thus it shows that, due to this term, crests are flattened and troughs are deepened.
Also of this term an assumption of the magnitude will be given. The problem is how to estimate \(\delta z/\delta y\), the slope of a disturbance on a horizontal equilibrium profile. In reality this slope is very different for different locations in the profile. Assuming the maximum slope of a breaker bar on a horizontal slope to be of the same order of magnitude as the maximum slope of the equilibrium profile, the present term would have the same magnitude as the previous term. This seems a fair estimate, as the variations in the magnitude of this term are too large to make a more accurate guess reliable.

4.4.4. The fourth term of the basic equation

Finally the last term of equation (4.12) yielding \(-F_3(z) \frac{\partial z}{\partial y}\) is discussed.

This term is not depending on \(z\) directly, but depends on the slope of the disturbance. It shows the dissipative character of \(F_3\): a negative \(\delta z/\delta y\) (the front of a breaker bar) yields an offshore sediment transport, and a small positive \(\delta z/\delta y\) yields an onshore sediment transport. The effect is dissipative, see Figure 4.6.

The order of magnitude can be estimated also for this term. In accordance with the previous term \(\partial z/\partial y\) is again assumed to be of the order of magnitude of the maximum slope of the equilibrium profile, which is approximately \(-0.085\). Furthermore, with \(F_3(z)=0.017\) at the location of the maximum slope, see plot H2 with \(T=5s\), the value of the term yields 0.00142, which is about \(3/4\) of the order of magnitude of the first term.

Summarising, the magnitudes of the four terms have the following relation:

\[
\]

It is stressed that this is only a very rough estimate. It illustrates, however, that all four terms have a significant magnitude.
A final remark is that the third term of equation (4.12), \( \frac{dF_3(z)}{dz} \cdot \frac{\partial \tilde{z}}{\partial y} \), can be rewritten to \( 1/2 \cdot \left[ \frac{dF_3(z)}{dz} \right] \cdot \frac{\partial \tilde{z}^2}{\partial y} \). Assuming a harmonic disturbance \( \tilde{z} \) as a function of \( y \), this term has a phase difference of 90° compared to the other terms of (4.12). It illustrates that each term has a specific influence on \( \tilde{S}_y \).

In the following section, the last term of equation (4.12) will be analyzed. The reason for this is that the orders of magnitude were determined at a later stage. The results after a less linearised approach showed that three other terms are of significant importance as well. Therefore, the following analysis can be seen as an a-priori analysis.

### 4.5. Analysis treating \( F_3 \) as a function of \( z \)

The equation that is used to investigate the influence of the last term yields:

\[
\tilde{S}_y = -F_3(z) \frac{\partial \tilde{z}}{\partial y} ,
\]

ignoring the other three terms of equation 4.12.

This equation describes the relation between a variation in the sediment transport due to a variation in the bottom slope. It is expected that the dissipative character of \( F_3 \), representing the influence of the gravity force, will be found. It is stressed that the only transport-causing process involved in this investigation is the gravity force in combination with the bottom slope. Treating \( F_3 \) as function of \( \tilde{z} \) means that a linearised approach is chosen and that higher order terms are ignored.

The continuity equation for a breaker bar can be written as follows:

\[
\frac{\partial S_x}{\partial x} + \frac{\partial S_y}{\partial y} + \frac{\partial S_z}{\partial t} = 0
\]

Reducing equation (4.14) to the cross-shore section, and taking only the periodical behaviour into account, equation (4.14) becomes:
\[ \frac{\partial \bar{S}_y}{\partial y} + \frac{\partial \bar{z}}{\partial t} = 0, \]

(4.15) \[ \text{with } \bar{S}_y \text{ according to (4.13).} \]

Before continuing the theory, some remarks have to be made about \( F_3 \). As already mentioned before, \( F_3 \) is a function of \( kh \), and, as \( h \) stands for the water depth, a function of \( z \).
This is illustrated in Figure 4.7.

![Figure 4.7 definition z](image)

By integrating the equilibrium slope, \( F_3 \) is rewritten as a function of \( k_0y \) instead of \( kh \). The dispersion relation was used to transform the wave number \( k \) to \( k_0 \).
Two reasons are given for translating \( kh \) to \( k_0y \):

1. In obtaining a fair impression of the behaviour of \( F_3 \) over the profile, it was needed to have an equidistant scaling of the horizontal axis of the plots. As \( F_3 \) was expressed as a function of \( kh \), this was not the case. The wave number \( k \) is not a constant over the profile, and also \( h \) is not a linear variable.

2. The exchange of results with Michiel Knaapen [11] made it necessary to use \( k_0y \) for the horizontal scaling. In his computer program, the \( y \)-axis denotes the cross-shore distance.

From equation (4.13) it follows that \( F_3 \) should be treated as a function of \( \bar{z} \). This is a consequence of neglecting the other three terms of equation (4.12). It is a less accurate approach compared to treating \( F_3 \) as a function of \( z \). The main reason for choosing this strategy is that the derivation of solutions for equation (4.13) is less complicated compared to equation (4.12). Furthermore, it is interesting to investigate the influence of the last term in equation (4.12) separately.
Substitution of equation (4.13), describing the sediment transport due to the gravity force, into the continuity equation (4.15) yields:

\[-\frac{dF_3(z)}{dz} \frac{\partial z}{\partial y} \frac{\partial z}{\partial y} - F_3(z) \frac{\partial^2 z}{\partial y^2} + \frac{\partial z}{\partial t} = 0 . \quad (4.16)\]

Assuming small equilibrium slopes \( (\partial z/\partial y = 0) \), the first term of this equation can be neglected, reducing equation (4.16) to:

\[- F_3(z) \frac{\partial^2 z}{\partial y^2} + \frac{\partial z}{\partial t} = 0 . \quad (4.17)\]

The consequences of neglecting this term follow from comparison of the results of the analysis to numerical data in section 4.7.

Equation (4.16) can be rewritten to:

\[ \frac{\partial z}{\partial t} = F_3(z) \frac{\partial^2 z}{\partial y^2} . \quad (4.18)\]

This equation looks similar to the diffusion equation

\[ \frac{\partial z}{\partial t} = D \frac{\partial^2 z}{\partial y^2} , \quad (4.19)\]

with the diffusion coefficient equal to \( F_3 \). As \( F_3 \) is a positive function, this would mean that every initial disturbance of \( z \) would disappear for increasing \( y \) and \( t \). But although \( F_3 \) is positive, \( F_3 \) is not a constant. \( F_3 \) decreases rapidly for increasing \( y \). For this reason, equation (4.17) can not be treated like an ordinary diffusion equation. Far out of the coast, however, the waves being of little influence on the profile, \( F_3 \) becomes a constant value. In this area, the diffusion equation could be used. This results in a flattening profile shape at a larger distance from the coastline, which is to be expected.

In some cases it is possible that increasing disturbances result from a rapidly decreasing diffusion coefficient. It would be interesting to find this result analysing \( F_3 \), keeping in mind the question of the initiation of breaker bars and their growing process in the first stages of their existence.

Equation (4.17) is a linear partial differential equation. Searching for a harmonic solution, it is simple to assume the solution to be of the kind:

\[ z = Re(Z e^{i\omega t}) . \quad (4.20)\]

It is remarked that other solutions, apart from harmonic solutions, are possible. These solutions are not addressed, however, as it is assumed that bars in general are best schematised by an harmonic solution.

With this assumption, equation (4.17) becomes (taking the imaginary part of equation (4.20) into account as well):

\[ -F_3(z) \frac{d^2 Z}{dy^2} + i \omega Z = 0 , \quad (4.21)\]

in which \( \omega = \frac{2\pi}{T_s} \) is the angular velocity and \( Z \) the amplitude of the breaker bar.
The details of this transformation are included in annex B, intermezzo 1.

In appendix B the value of \( \frac{F_3(\bar{z})}{g^2T^3} \) is given as a function of \( k_0y \), where \( k_0 \) is the wave number on deep water, and \( y \) the cross-shore distance positive in seawards direction.

Furthermore, \( k_0 = \frac{2\pi}{L_0} \) where \( L_0 \) is the deep water wavelength, and \( L_0 = \frac{gT^2}{2\pi} \).

The values for the initial wave steepness \( s_0 \) and the dimensionless fall velocity \( W^* = \frac{WT}{H_0} \) are chosen to be equal to respectively 0.02 and 1.0, see section 3.8.

To be able to work with \( F_3 \) as a function, a simple mathematical description of \( F_3 \) needs to be formulated. Looking at the plot showing \( \frac{F_3(\bar{z})}{g^2T^3} \) (plot B5) it seems appropriate to assume that \( \frac{F_3(\bar{z})}{g^2T^3} \) is inversely proportional to \( k_0y \):

\[
\frac{F_3(\bar{z})}{g^2T^3} = \frac{A'}{k_0y} \Rightarrow F_3(\bar{z}) = \frac{A}{k_0y},
\]

where \( A = A'g^2T^3 \). \( A \) is a proportionality constant, and \( A > 0 \). The value for \( A' \) can be determined by curve-fitting. The resulting curve is shown in appendix C, plot C1.

With this estimation, the equation describing a breaker bar:

\[
-F_3(\bar{z}) \frac{d^2Z}{dy^2} + i\omega Z = 0 ,
\]

(4.21 repeated)

becomes:

\[
Z'' - \frac{i\omega}{F_3(\bar{z})}Z = 0
\]

(4.23)

The dimension of \( A \) is the same as the dimension of \( F_3 \), which is \([m^2/s]\).

So \( \frac{i\omega}{A} \) is \([m^{-2}]\), and thus the dimensions of equation (4.24) fit.

At this moment, a new independent variable:

\[
u = (\frac{i\omega k_0}{A})^{1/3} y
\]

(4.25)

is introduced.

This variable can be rewritten to:
Using this variable, equation (4.24) yields:

\[ Z'' - uZ = 0 \]  

(4.27)

The details are included in annex B, intermezzo 2.

The independent variable \( u \) can be rewritten to:

\[ u = \frac{e^{i\frac{\pi}{2} \omega k_0}}{A} y = e^{i \frac{\pi}{6} \sqrt{\frac{\omega k_0}{A}}} y. \]  

(4.28)

This formulation of \( u \) illustrates that \( u \) can be seen as a scale parameter. This is an important result, and the description of \( u \) provides for some insight into the physics of breaker bars.

It can be seen, for instance, that the length scale of the breaker bars is inversely proportional to the length scale of the incoming waves. An incoming wave with a larger wavelength results in a breaker bar with a smaller wavelength. This seems reasonable, as it is known that shorter waves in general cause steeper profile forms.

It can also be seen that the length scale of the breaker bars is inversely proportional to a third power root of the time period of the breaker bars, expressed in periods of the incoming waves. This is also a reasonable result, for the same reasons. A long period of the breaker bars (compared to the period of the incoming waves) results in a more compact profile form, with smaller length scales.

Further comments on \( u \) are presented in the evaluation in section 4.3.3.

In intermezzo 3, see annex B, the derivation of equation (4.28) is given. In the following theory, the original formulation for \( u \) is used. Equation (4.28), however, may be a contribution to a better physical understanding of the meaning of \( u \).

Solutions of the differential equation (equation (4.27)) are Airy-functions, \( \text{Ai}(u) \) and \( \text{Bi}(u) \), of which the characteristics of the real parts are pointed out in the Handbook of Mathematical Functions [1], pages 446 to 449.

In order to keep the notation as simple as possible, before continuing with the theory, some definitions are listed below.
The solutions $A_i(u)$ and $B_i(u)$, presented in [1], will be referred to by $Z_A$ and $Z_B$, as these are solutions of the differential equation:

$$Z'' - u'Z = 0 .$$  \hspace{1cm} (4.27 repeated)

Note that $Z_A$ and $Z_B$ are complex functions as $u$ is a complex variable. Consequently, $Z_A$ and $Z_B$ are written as follows:

$$Z_A = z_{A,r} + i z_{A,i}$$ \hspace{1cm} and  \hspace{1cm} (4.29)

$$Z_B = z_{B,r} + i z_{B,i} ,$$ \hspace{1cm} (4.30)

where $z_{A,r}$ and $z_{B,r}$ denote the real parts of the solutions, and $z_{A,i}$ and $z_{B,i}$ the imaginary parts.

Furthermore, referring to equation (4.20),

$$\bar{z} = \Re(ze^{iu})$$ \hspace{1cm} (4.20 repeated)

the final solution for $\bar{z}$ will be given by:

$$\bar{z}_A = \Re(Z_A e^{i\omega t}) = z_{A,r} \cos(\omega t) - z_{A,i} \sin(\omega t)$$ \hspace{1cm} and  \hspace{1cm} (4.31)

$$\bar{z}_B = \Re(Z_B e^{i\omega t}) = z_{B,r} \cos(\omega t) - z_{B,i} \sin(\omega t) .$$ \hspace{1cm} (4.32)

These notations should be kept in mind in order not to "get lost" in the theory.

First the general behaviour of the solutions $Z_A$ and $Z_B$ is investigated below.

A parameter $\xi$ appears to be important:

$$\xi = \frac{2}{3} u^{3/2} \Rightarrow \xi = \frac{2}{3} e^{i\frac{\omega k_0}{A}} y^{3/2} ,$$ \hspace{1cm} (4.33)

and $\xi = \xi_r + i \xi_i$ , \hspace{1cm} (4.34)

in which $\xi_r$ and $\xi_i$ are real functions:

$$\xi_r = \xi_i = \frac{1}{3} \sqrt{2} \left[ \frac{\omega k_0}{A} y^{3/2} .\right.$$ \hspace{1cm} (4.35)

For the details see annex B intermezzo 4.

The behaviour of $Z_A$ and $Z_B$ is illustrated by equations 10.4.59 and 10.4.63 in [1] respectively on pages 448 and 449.

Equation 10.4.59 yields:
\[ Ai(u) \sim \frac{1}{2} \pi^{-1/2} u^{-1/4} e^{-u} \sum_{k=0}^{\infty} (-1)^k c_k u^{-k} , \quad |\arg u| < \pi , \] 

(4.36)

where \( c_k = 1 \). In intermezzo 5, the condition \(|\arg u| < \pi\) is investigated.

Concentrating on the real part of \( \xi , \xi_r \), the following results for \( Z_A (= \text{Ai}(u)) \).

With \( k = 0 \) and \( u \to \infty \) from (4.36) it follows (as \( c_0 = 1 \) :)

\[ \lim_{u \to \infty} Ai(u) = \lim_{u \to \infty} \frac{1}{2\sqrt{\pi}} u^{-1/4} e^{-\xi} = 0 , \quad \text{where} \quad -\xi < 0 . \] 

(4.37)

So equation (4.37) becomes zero for large \( u \), say large \( y \).

Concentrating on the imaginary part of \( \xi \), the same result is found. This leads to the conclusion that solution \( Z_A \) is a stabilising solution of the differential equation. \( Z_A \) approaches to zero for large \( u \).

Furthermore, equation 10.4.63 yields:

\[ \text{Bi}(u) \sim \pi^{-1/2} u^{-1/4} e^{i} \sum_{k=0}^{\infty} c_k u^{-k} , \quad |\arg u| < \frac{1}{3} \pi . \] 

(4.38)

The condition for the argument connected to this solution is also fulfilled, as

\[ |\arg u| = \frac{1}{6} \pi < \frac{1}{3} \pi . \]

It is simple to conclude that this solution, for both \( \xi_r \) and \( \xi_i \), is a rapidly increasing solution of the differential equation, as the exponent for \( e \) is positive.

Summarising, two solutions are found for the differential equation (4.27). These two solutions are further analyzed below, using the series expansion of these solutions presented in [1]. First \( Z_A(u) \) is discussed, afterwards \( Z_B(u) \) is treated.

4.5.1. The solution \( Z_A(u) \)

As already mentioned above, a solution of equation (4.27) is \( Z_A \), with

\[ u = e^{i/6} \sqrt{\frac{\omega k_0}{A}} y . \] 

(4.17 repeated)

Now state that

\[ u = e^{i/6} u_r , \quad \text{so} \] 

(4.39a)

\[ u_r = \sqrt[3]{\frac{\omega k_0}{A}} y . \] 

(4.39b)

Note that \( u_r \) denotes the modulus of \( u \).

According to equation 10.4.2 [1], for \( Z_A \) it can be stated that:
\[ Z_A(u) = c_1 f(u) - c_2 g(u) \]  

where:

\[ f(u) = 1 + \frac{1}{3!} u^3 + \frac{14}{6!} u^6 + \frac{147}{9!} u^9 + \ldots \]  

(4.41a)

\[ g(u) = u + \frac{2}{4!} u^4 + \frac{25}{7!} u^7 + \frac{258}{10!} u^{10} + \ldots \]  

(4.42b)

and, according to equation 10.4.4 and 10.4.5 [1],

\[ c_1 = 0.35502 \quad 80538 \quad 87817 \]
\[ c_2 = 0.25881 \quad 94037 \quad 92807 \]

As in this case \( u \) is a complex value (see 4.25), the solution \( Z_A \) should be written as follows:

\[ Z_A(u) = Re\{Z_A(u)\} + i \cdot Im\{Z_A(u)\} \]  

(4.42)

Using the definitions stated before, this can be rewritten to:

\[ Z_A(u) = z_{A,r} + i z_{A,i} \]  

(4.43)

The expressions for \( z_{A,r} \) and \( z_{A,i} \) are derived for the first four terms of \( f(u) \) and \( g(u) \). The derivation is presented in annex C. Based on the resulting expressions, a FORTRAN computer program is developed computing the expressions. The program is presented in annex D.

Resuming the results until now, \( Z_A = A \iota(u) \) appears to be a complex value. \( Z_A \) consists of a real and an imaginary part, see also annex C.

Because of this, \( Z_A \) can be written as follows:

\[ Z_A(u) = Re\{Z_A(u)\} + i \cdot Im\{Z_A(u)\} \]  

(4.44)

with \( u \) according to equation (4.39a).

This equation is rewritten to:

\[ Z_A = z_{A,r} + i z_{A,i} \]  

(4.45)

Using (4.45), equation (4.20) becomes:

\[ z_A = Re(Z_A e^{i\omega t}) = z_{A,r} \cos(\omega t) - z_{A,i} \sin(\omega t) \]  

(4.46)

The derivation of this equation is included in annex B, intermezzo 6.

As already mentioned above, \( z_{A,r} \) and \( z_{A,i} \) are calculated using a FORTRAN computer program. In order to compute \( z_{A,r} \) and \( z_{A,i} \) accurate enough, the computer program
generates new higher order terms as long as the break-off fault is too large. The number of terms that need to be generated in order to obtain the desired accuracy depends on the value of \( u \). After determination of \( z_{A_r} \) and \( z_{A_i} \), \( z_A \) is easy to compute.

Plots showing \( z_{A_r} \), \( z_{A_i} \) and \( z_A \) are presented in appendix F (F1, F2 and F3).

### 4.5.2. The solution \( Z_B(u) \)

As already mentioned, the second solution of equation (4.27) is:

\[
Bi(u) = Z_B(u) .
\]

(4.47)

Using the same definitions for \( u \) and \( u_r \) (see (4.39a) and (4.39b)), the following analysis can be made.

According to equation 10.4.3, for \( Z_B \) it can be stated that:

\[
Z_B(u) = \sqrt{3} (c_1 f(u) + c_2 g(u))
\]

(4.48)

with \( f(u) \) and \( g(u) \) according to equations (4.41a) and (4.41b), and \( c_1 \) and \( c_2 \) according to 10.4.4 and 10.4.5 [1].

As \( u \) is a complex value (see 4.25), also solution \( Z_B \) should be written as follows:

\[
Z_B(u) = Re\{Z_B(u)\} + iIm\{Z_B(u)\} ,
\]

(4.49)

or, like equation (4.30):

\[
Z_B(iu_r) = z_{B,r} + iz_{B,i} .
\]

(4.50)

Similar to the derivation presented in intermezzo 6 of the previous section, equation (4.20) becomes:

\[
\tilde{z}_B = Re(Z_B e^{i\omega t}) = z_{B,r} \cos(\omega t) - z_{B,i} \sin(\omega t) .
\]

(4.51)

Using the FORTRAN program, also this solution is calculated. The derivation of the solution is presented in annex C, and the plots of the results are included in appendix F (F4, F5 and F6).

In the next section, the plots are briefly discussed.

### 4.6. Evaluation of the results

In this evaluation, the plots included in the appendices will be used as a guide to explain the results derived in this chapter.

The plots B4 and B5 in appendix B showing \( F_2 \) and \( F_3 \) (non-dimensional) and the profile characteristics were already explained in the last section of chapter 3.

Based on these results, the value for \( A' \) was determined. For this determination, the computer program QUATTRO was used. \( A' \) was determined by minimising the sum of the squared differences between \( F_3 / g^2 T^3 \) and \( A' / k_0 y \). In the fitting process, especially
the steep part of $F_3$ is important, because this part is representing the area close to the shoreline where $F_3$ is of major importance. For this reason, the fitting process had to be carried out both visual and numerical. The plot showing the fit is included in appendix C, plot C1.

Using the derived expression for $u_r$ (see 4.28), and also using the determined value for $A'$, some plots were made to illustrate the behaviour of $u_r$ for changing parameters. The constants in the expression for $u_r$ are $A'$, $T_s$ and $T$. The value for $L_0$ is directly dependent of $T$. To investigate the behaviour of $u_r$, one of the constants was changed while the other two constants were not. This was done for all three constants. The three resulting plots are included in appendix D (plots D1, D2 and D3).

The plots show that only a variation of $T$ causes a significant change in the relation between $y$ and $u_r$. Of course also $A'$ and $T_s$ have some influence, but their influences are less important.

Concentrating on plot D3 in which the variation of $T$ is illustrated, it is obvious that a smaller wave period results in a smaller value for $y$ for the same $u_r$. In other words: a certain value for $u_r$ can correspond to different locations in the profile, depending on the value of $T$. This is important, because this illustrates the applicability of the results scaled with $u_r$. The results can be applied to different situations, because $u_r$ can be transformed to $y$ by using the local values for $T$ (and less important $A'$ and $T_s$).

In this way, the local wave climate can be integrated into the theory, as well as the sediment parameters and the behaviour of the breaker bars (via $A'$ resp. $T_s$). For the Terschelling location, a $T$ of 4 to 6 seconds represents most of the waves.

For $y = 1000m$ this results in an $u_r$ of approximately 3 to 6.

With the above description of $u_r$ in mind, and its significance for the theory, now the plots which are scaled with $u_r$ on the horizontal x-axis will be discussed.

In appendix E, plots E1, E2, E3 and E4 show the behaviour of the solutions $Z_A$ and $Z_B$ for large $u_r$. These results are obtained using the FORTRAN computer program. The program calculates with "double precision". Still instability of the calculation process occurs for larger $u_r$ values. This is not relevant, however, as $u_r$ loses its meaning above a value of 10. For such a high value, the cross-shore distance becomes too large to be interesting.

The instability of $Z_b$ for large $u_r$ is difficult to visualise, as the solution itself increases rapidly for larger $u_r$-values, see E2 and E4. It also seems that the solution $Z_A$, being equal to zero for larger $u_r$-values, is more sensitive for numerical instabilities. The reason for this is that the value zero results from a substraction of two large values.

Using the results for $Z_A$ and $Z_B$, the plots for $\tilde{z}_A$ and $\tilde{z}_B$ are presented in appendix F (plots F3 and F6). The plots are scaled with $u_r$ on the horizontal axis making them usable for different local situations.

The plots for $\tilde{z}_A$ and $\tilde{z}_B$ are generated using equations (4.46) and (4.51):

\[
\tilde{z}_A = \text{Re}(Z_A e^{i\omega t}) = z_{A,r} \cos(\omega t) - z_{A,i} \sin(\omega t) 
\]

\[
\tilde{z}_B = \text{Re}(Z_B e^{i\omega t}) = z_{B,r} \cos(\omega t) - z_{B,i} \sin(\omega t),
\]

(4.46 repeated)

(4.51 repeated)
in which $z_{A,r}$ and $z_{A,i}$ denote $\text{Re}\{Z_A(iu)\}$ and $\text{Im}\{Z_A(iu)\}$, and $z_{B,r}$ and $z_{B,i}$ denote $\text{Re}\{Z_B(iu)\}$ and $\text{Im}\{Z_B(iu)\}$.

It is important to remark that the results presented in the plots are not yet adapted to the boundary conditions. Nevertheless, these plots show the length scales of the breaker bars and the way the amplitude of the breaker bars develops in time and place.

Comparing the plots of $\check{z}_A$ and $\check{z}_B$ (F3 and F6) it can be seen that the plot of $\check{z}_A$ shows a decreasing and the plot of $\check{z}_B$ an increasing solution. Both are solutions of the same differential equation, and both solutions are correct. The solution of $\check{z}_A$ is determined by the left boundary condition, and the solution of $\check{z}_B$ is determined by the right boundary condition.

In reality, however, the profile changes in the foreshore are not, or very little, influenced by profile conditions offshore. It is more likely that the breaker zone, with the sediment concentrations being high, will be more important as a boundary condition. For this reason, most attention is paid to solution $\check{z}_A$.

In the next section, this solution will be compared to numerical results. At the same time an explanation of the solution $\check{z}_A$ is presented.

4.7. Comparison to numerical results

In order to be able to check these results, which were obtained by an analytical analysis, comparison was made with results of a numerical calculation.

In his graduate thesis, Michiel Knaapen [10] develops a computer program in which both measurements (the JARKUS data) and deterministic equations are used to describe the profile development. His program uses the measurements to correct the outcome of the equations interactively. Filter techniques are applied to process the measurement data. Without the use of measurements, however, the computer program can calculate profile development in time and place after disturbance of the equilibrium profile initiated by boundary conditions. In calculating the equilibrium profile shape, equations are solved numerically which were also used by V.d. Kerk in an analytical way.

The equation:

$$S_y = F_2 - F_3 \left( \frac{\partial z}{\partial y} \right)$$

is used by Michiel Knaapen to describe the cross-shore sediment transport. The values for $F_2$ and $F_3$ were derived in this report.

At the moment the comparison was made, the contribution of $F_2$ to the sediment transport was not yet implemented in the numerical computer model.

Looking at equation (4.12):

$$S_y = \check{z} \left\{ \frac{dF_2}{dz} - \frac{dF_3}{dz} \frac{\partial \check{z}}{\partial y} - \frac{dF_3}{dz} \frac{\partial \check{z}}{\partial y} \right\} - F_3(z) \frac{\partial \check{z}}{\partial y}$$

(4.12 repeated)
it can be concluded that only the last three terms were included in the numerical calculation. The differences between the analytical and the numerical results are expected to result from the absence of the second and the third term in the analytical analysis.

Choosing the same values for $F_3$ over the profile, and choosing the same boundary conditions, a comparison was made between the analytically calculated $\tilde{z}_A$ and the numerically calculated profile development.

The results of this comparison are plotted in appendix G, plots G1 and G2. In the following an explanation of the plots is presented. First the chosen boundary conditions are given.

The equation describing $\tilde{z}_A$ yields:

$$\tilde{z}_A = Re(Z_A e^{i\omega t}) = z_{A,r} \cos(\omega t) - z_{A,i} \sin(\omega t), \quad(4.46\text{ repeated})$$

with $z_{A,r}$ and $z_{A,i}$ defined in annex C. For $y = 0$, concentrating on the upper boundary, $u_r$ equals zero. For $u_r = 0$, $z_{A,r}$ equals 0.355 ($=c1$) and $z_{A,i}$ equals zero. This implies that equation (4.46) becomes:

$$\tilde{z}_A = 0.355 \cos(\omega t) \quad (4.52)$$

which is used to define the upper boundary condition in the numerical model. Furthermore, other local parameters were chosen:

$$A' = 2.49 \times 10^{-6}, \text{ resulting from the fitted } F_3(\tilde{z}),$$
$$T = 10\text{yr} \text{ and }$$
$$T' = 4.5,6 \text{ sec. This is a fair estimate for the wave climate at the Terschelling location.}$$

It was decided to scale the results with $k_0 y$ on the horizontal axis. Therefore $u_r$ had to be transformed to $k_0 y$. The following equation was used:

$$y = u_r \xi^3 (A'/T, T'/T) \quad (4.53)$$

which is based on equation (4.28).

The plots G1 and G2 show the disturbance of the equilibrium profile, which is initiated at the upper boundary. Both solutions show the same behaviour. The amplitude of the initial disturbance decreases in seawards direction. This is an important result, as the diffusive factor, $F_3$, decreases rapidly over the profile. Because of this an increasing amplitude of the disturbance would also be a possibility. This comparison, however, provides a reasonable security that this is not the case. The question of how to describe the initiation of the breaker bars remains unsolved, only the dissipative character is explained by the investigated term.

Another important similarity is the wavelength of the "disturbance waves". Both the analytical and the numerical solutions show a wave length of about 120 on the $k_0 y$ - axis. With the used parameters, this corresponds to a length of approximately 700m. This order of magnitude is reasonable for breaker bars, see [17].
It is important to remark that originally the plots were scaled with $u_r$ on the horizontal axis. To be able to compare the results, it was more practical to transform $u_r$ to $k_0y$. Doing so, some parameter values had to be chosen making the result less universal. The wavelength of the breaker bars, shown in this plot, is therefore only true for the used parameter values.

There are also some differences between the curves, for instance in the amplitudes. The analytical results show a smaller amplitude than the numerical results. This is primarily due to the absence of two terms in the analytical analysis compared to the numerical analysis. The choice was made to treat $F_3$ as a function of $\bar{z}$, analysing only the last term of equation (4.12). This restriction made it possible to leave three terms out in equation (4.12), making the analysis of the differential equation more simple. In a numerical computer program, however, the whole equation can easily be implemented as was done by Michiel Knaapen [10]. The difference in the amplitudes of the curves may be explained as the contribution of these extra terms.

Another difference between the analytical and the numerical results is the gradient of the functions at the vertical axis (for $k_0y = 0$). For the numerical results, the gradient is almost equal to zero, whereas the gradient for the analytical results is larger.

This difference can be explained by the boundary conditions used by Michiel Knaapen. He defined the derivative of the bottom slope ($\frac{\partial^2 \bar{z}}{\partial y^2}$) at the left boundary equal to zero. This leaves all possibilities open for $\frac{\partial \bar{z}}{\partial y}$. As the diffusion coefficient ($F_3$) approaches to infinity (theoretically) for $k_0y = 0$, also for a small $\frac{\partial \bar{z}}{\partial y}$ a large amount of sediment transport is possible. This can be illustrated by equation (4.13):

$\bar{s}_y = -F_3 \frac{\partial z}{\partial y}$  \hspace{1cm} (4.13 repeated)

The analytical solution is not yet adapted to any boundary condition. A definition of the amount of sediment transport at the upper boundary would certainly result in a smaller gradient of $\bar{z}_A$ at the vertical axis.

4.8. Conclusions

Substituting a reduced expression for the sediment transport into the continuity equation, only containing the influence of the gravity force, a differential equation is obtained that looks like a diffusion equation:

$Z'' - uZ = 0$  \hspace{1cm} (4.27 repeated)

Two solutions are found for the differential equation resulting in a $\bar{z}_A$ and a $\bar{z}_B$. Both $\bar{z}_A$ and $\bar{z}_B$ are functions of $u$ (described in equation 4.28) and time $t$. The variable $u$ is in fact a scale parameter containing the cross-shore distance $y$, and some other profile characteristics.

Note: Rewriting (4.27) to:

$\frac{1}{u} Z'' - Z = 0$  \hspace{1cm} (4.54)
shows that the scale parameter \( u \) can also be seen as a diffusion coefficient. As \( u \) is proportional to \( y \), it is easy to understand that the diffusion coefficient becomes infinite as \( y \) approaches to zero.

The equations for \( z_A \) and \( z_B \) are not yet adapted to any boundary conditions. Solution \( z_B \) is physically less interesting, as this solution is linked to the deep water profile changes. These changes are known to have very little influence on the shoreface profile shape.

The results up to this point show the influence of \( F_3 \), the gravity force contribution, in the cross-shore profile. For a given boundary condition, the propagation of this condition through the profile can be calculated. Length scales of propagating disturbances are obtained, given by the scale parameter \( u \). It is stressed, however, that only the influence of the gravity force is investigated. Wave asymmetry and undertow will also have a significant influence on the behaviour of breaker bars.

Although the equilibrium profile remains the reference starting profile during calculations, \( z_A \) and \( z_B \) are time-dependent. This is a consequence of a time-dependent boundary condition, which is necessary to describe the periodical behaviour of the breaker bars in time. It is important to note that the equilibrium profile remains constant in time, only the disturbances of the equilibrium profile were investigated.

The propagation of the disturbances (breaker bars) is illustrated by the plots of \( z_A \) and \( z_B \) (\( F_3 \) and \( F_6 \)). The plots do not show a standing wave with decreasing amplitude. Propagation of the waves is clearly visible. This propagation is due to the (asymmetric) diffusion coefficient \( F_3 \).

The question remains what \( F_2 \) might contribute to the profile changes, as far as breaker bars are concerned. Apart from this question, there is still a phenomenon uncovered in this theory: the initiation of the breaker bars. It is well known that breaker bars, in general, decrease in amplitude while they move in seaward direction. But the mechanism that causes a breaker bar to grow nearshore is still not well described. Including the function \( F_2 \) in the analysis as well might provide for more insight into the initiation process.

In the next chapter, \( F_2 \) and \( F_3 \) will both be included in the analysis.
Chapter 5. Analysis of profile disturbances using $F_2$ and $F_3$ as functions of $z$

5.1. Introduction

In this chapter, an attempt is made to derive a sediment transport expression that includes both $F_2$ and $F_3$ without having to apply the Taylor expansion on $F_2$ and $F_3$.

The layout of this chapter is as follows. In section 5.2 of this chapter, the reason for applying another schematisation of profile disturbances is outlined. In section 5.3, new definitions of variables and functions adapted to profile disturbances in $y$-direction are presented. In order to understand the consequences of these new definitions, in section 5.4 a detailed explanation on the new approach is included. It is highly recommended to read this section before reading the theory itself.

The following layout of the analytical part of this chapter provides an indication of the contents of the sections 5.5 to 5.7. An explanation of the analytical part of this chapter is included in section 5.4.

Based on the new definitions presented in section 5.3, in section 5.5 an expression for the sediment transport resulting from horizontal disturbances of the equilibrium profile is derived, and in section 5.6 an adapted continuity equation for the cross-shore transport is derived. Substituting the expression for the sediment transport into the continuity equation, in section 5.7 a kinematic equation describing the behaviour of horizontal profile disturbances in time is derived. Two functions containing $F_2$ and $F_3$, that form an important part of the kinematic equation, are analyzed.

Finally, in section 5.8 some conclusions and a set-up for further analysis are included.

5.2. The reason for another schematisation of profile disturbances

In the previous chapter, an equation was derived describing the sediment transport due to the harmonic motion of the bottom level in $z$-direction. The equation yields:

$$
\ddot{z} = - \frac{dF_2(\bar{z})}{dz} - \frac{dF_3(\bar{z})}{dz} \frac{\partial z}{\partial y} - \frac{dF_3(\bar{z})}{dz} \frac{\partial z}{\partial y} - F_3(\bar{z}) \frac{\partial \bar{z}}{\partial y}.
$$

(4.12 repeated)

The derivation of this equation was based on the assumption that second and higher order terms of $F_2$ and $F_3$ can be neglected. The last term of equation (4.12), representing the influence of the gravity force in combination with the bottom slope on the sediment transport, was analyzed in the previous chapter. This term contains $F_3$ as a function of the equilibrium profile height $\bar{z}$.

The results were compared to the output of a numerical computer model which calculates the last three terms of equation (4.12). The first term containing $F_2$ was kept inactive during the computations. The differences are probably due to the absence of the second and the third term in the analytical analysis.
It may be clear that the expression used in chapter 4 to describe the sediment transport covers only a little part of the sediment transport properties. The effects of $F_2$ are ignored, and the approach is linearised using only $F_3$ as a function of $\bar{z}$ without higher order terms.

It would be better to use equation (4.12), instead of (4.13), to describe the sediment transport. Equation (4.12) in combination with the continuity equation introduces the time-dependency of $z$.

Furthermore, in mathematical practice, it is common use to first analyze whether solutions are stable in time neglecting second order terms. Note that, deriving equation (4.12), second order terms of the Taylor expansion were indeed ignored. In case of an unstable solution in time, second order terms are introduced and again stability of the solution is analyzed. This process can be continued until a stable solution is found. Stability indicates that further higher order terms can be neglected.

Analysis of (4.12) in combination with the continuity equation, however, appears to be complex because of the non-linear third term in (4.12). This term makes application of solutions like Bessel functions impossible. After a lot of algebra it was concluded that, in the scope of this study, this analysis would be too complicated.

Another possible way of analysing equation (4.12) is by neglecting the non-linear third term, and performing the iterative analysis of the stability as described above without this term. Unfortunately, this possibility came up at a late stage of this study. Already another less sophisticated way of handling the problem of the non-linear term, and the absence of higher order accuracy, was used. Therefore, this way of analysis using equation (4.12) remains open for further study.

5.3. Introduction to another schematisation of disturbances

The essence of solving the problem of non-linearities, and inaccuracy due to the absence of higher order terms of $F_2$ and $F_3$ in the theory, is that a breaker bar is schematised as a disturbance of the equilibrium profile in $y$-direction instead of in $z$-direction.

First it is assumed that $F_2$ and $F_3$ are known functions of $z$, with $z$ as an independent variable. The functions $F_2$ and $F_3$ are determined by sediment parameters and wave conditions. From $F_2$ and $F_3$, the function $f_e$ is known, as $f_e$ is determined by $F_2$ and $F_3$. The function $f_e$ defines the equilibrium profile. Furthermore, for every $z$, $f_e(z)$ yields a value for $y_e$ resulting in an equilibrium profile (see Figure 5.1).

In Figure 5.1 the coordinate system is shown in another position, to emphasize that $z$ is the independent variable.

At this moment, concentrating on location $z = z_i$, a disturbance $\bar{z}$ is superposed on $z_i$. This yields a $z$-coordinate $z_i' = z_i + \bar{z}$ and, as the equilibrium profile is known, via $f_e$ a $y$-coordinate $y_i'$. This is shown in Figure 5.2.

The consequence of the disturbance is that a new $y$-coordinate, $y = y_i'$, is connected to $z_i'$. The resulting new location $(z_i', y_i')$ is a location on the reshaped profile after
disturbance. Note that a \( \tilde{z} \), superposed on \( z \), in fact causes a disturbance of the profile in y-direction. For other \( z \) - locations the same happens in case of a \( \tilde{z} \) being superposed on \( z \). This does not imply a simple translation of the profile, as \( \tilde{z} \) is different for different values of \( z \). It is again reminded that \( \tilde{z} \) is a function of \( z \), and it is stressed that this is the reason for the profile to deform. The deformation of the profile is a consequence of the different values for \( \tilde{z} \) over the profile. The deformed profile is shown in Figure 5.3.

This is also the reason why the profile slope at \( (z_1, y_1) \) is not equal to the equilibrium slope. The profile is deformed, and therefore the slopes have changed. Before the disturbance takes place, it follows for the sediment transport at \( z = z_1 \) that:

\[
S_s(z_1) = F_2(z_1) - \frac{F_3(z_1)}{(dy_1/dz)} = 0 \tag{5.1}
\]

but after the disturbance, the sediment transport at \( z = z_1 \) follows from:
This is how a cross-shore sediment transport results from a disturbance of the equilibrium profile applying the new schematisation of disturbances. The change in the local bottom slope yields a value for $S_y$. Furthermore, because of the sediment transport caused by the change in the local bottom slope, the $y$-coordinate $y_1$ will change, as a gradient in $S_y$ changes the bathometry. This can be described by application of the continuity equation. The change of $y_1$ will change the bottom slope until the equilibrium slope is reached again.

Applying the continuity equation, which provides the link between sediment transport and bottom change, a kinematic equation is derived in section 5.6. This equation is found to be of the kind of:

\[
\frac{\partial \bar{y}}{\partial t} = f(\frac{\partial \bar{y}}{\partial z}, \frac{\partial^2 \bar{y}}{\partial z^2}).
\]

This chapter is dedicated to the derivation of the kinematic equation. In the next chapters the equation will be analyzed.

Note that there is a difference between the description of cross-shore sediment transport initiated by a disturbance in the equilibrium slope used in this theory and the description used in the theory of chapter 4.

Up to now the $y$-coordinate was chosen to be the independent variable, instead of the $z$ coordinate. Concentrating on a certain value for $y$, the cross-shore distance, a vertical disturbance of the equilibrium profile, in $z$-direction, caused a change in $F_2$ and $F_3$. Furthermore, the cross-shore sediment transport was caused by the change in $F_2$ and $F_3$, and the local profile slope was adapted to the new equilibrium slope connected to the new values for $F_2$ and $F_3$. The change in $F_2$ and $F_3$, however, not only depends on the magnitude of $\frac{dF_2}{dz}$ and $\frac{dF_3}{dz}$ as well. This is a non-linear effect, which causes the analysis to be more complicated.
In the new approach described above, $F_2$ and $F_3$ do not have to change. This is a big advantage, because it simplifies the equations to a large extent. Instead of $F_2$ and $F_3$ causing the sediment transport, it is a change in the bottom slope that represents the disturbance. The change in the bottom slope, in this case, is caused by a horizontal disturbance of the equilibrium profile in $y$-direction.

5.4. New definitions of variables and functions

As was mentioned in section 5.2, treating $F_2$ and $F_3$ as functions of $z$ makes another way of defining the cross-shore sediment transport caused by a disturbance desirable. Otherwise the analysis will become too difficult and the underlying physics can no longer be recognised.

The definitions following from the new schematisation are explained in detail below. The assumptions and decisions that form the basis of the following theory are presented in a logical order.

Treating $F_2$ and $F_3$ as functions of $z$ implies that $z$ is chosen to be the independent variable. To make clear that $z$ is the independent variable, the $y$ and $z$ axis are turned 90 degrees, resulting in the coordinate system shown in Figure 5.4.

![Figure 5.4 New orientation coordinate system](image)

It can be seen that, with this definition of the coordinate system, a change in $z$ on the equilibrium profile causes a change in $y$.

Furthermore, some definitions are formulated. These definitions are necessary to avoid mistakes. It is for instance very important to make clear which variable depends on which variable.

1. At first $z$ is defined as a stable variable. This means that $z$ is an independent variable, and is not changing in time or place. The variable $z$ will be the reference for further analysis.
2. As \( z \) is defined to be an independent variable, the equilibrium profile is given as a function of \( z \). The reason for this is that the equilibrium profile is also independent of time and place. The function describing the equilibrium profile is a non-varying function in time and place, determined by \( F_2 \) and \( F_3 \).

The notation will be as follows:

\[
y_e = f_e(z),
\]

in which:

- \( y_e \) = the y-coordinate of a location on the equilibrium profile, and
- \( f_e(z) \) = the function which defines the equilibrium profile, only depending on \( z \).

3. It was defined above that \( z \) is an independent variable. Furthermore, by means of \( f_e(z) \) a y-coordinate \( y_e \) was connected to \( z \), see equation (5.3). Therefore, every location \((z, y_e)\) denotes a position on the equilibrium profile. In case of a disturbance, however, the profile at some locations is unequal to the equilibrium profile. In order to describe a disturbed y-coordinate, depending on the location in the profile, \( y \) is defined as follows:

\[
y = f_e(z'),
\]

in which:

- \( y \) = the disturbed y-coordinate (uncertain between an upper and an under limit), and
- \( z' \) = the disturbed z-coordinate, depending on the location in the profile.

Note that the same function \( f_e \) is used to describe the relation between \( z \) and \( y_e \) as the relation between \( z' \) and \( y \).

The variable \( z' \) can be defined as follows:

\[
z' = z + \bar{z}
\]

where \( \bar{z} = \bar{z}(z) \).

Summarising, the variable \( z' \) consists of the independent stable variable \( z \) and, added to that, the disturbance \( \bar{z} \) as a function of \( z \). This means that the disturbance in \( z \) varies over the height of the profile. The reason why a disturbance in y-direction in the profile is described by a \( \bar{z} \) is because \( z \) is chosen to be the independent variable. The choice to define \( z \) as the independent variable makes it desirable to describe a disturbance in y-direction indirectly by a \( \bar{z} \).

4. Using the above definitions, it follows for the cross-shore sediment transport in case of the equilibrium profile that:

\[
0 = F_2(z) - \frac{F_3(z)}{(dy_e/dz)},
\]

as \( dy_e/dz \) denotes the equilibrium slope.

Equation (5.6) can be rewritten to:
\[
\frac{d y_e}{d z} = \frac{F_3(z)}{F_2(z)} ,
\]

or, in non-dimensional variables:

\[
\frac{d k_0 y_e}{d k_0 z} = \frac{F_3(k_0 z)}{F_2(k_0 z)} .
\]

Equation (5.8) clearly shows that the equilibrium profile is determined by \(F_2\) and \(F_3\).

The four definitions that are stated above will be used to set up a theory describing the sediment transport due to a disturbance of the equilibrium profile. The procedure in case of a disturbance causing a cross-shore sediment transport was explained in section 5.3. It is remarked that the disturbed profile form should still be monotonic. In the following section, an expression for the sediment transport is derived.

5.5. The sediment transport

As was mentioned in section 5.3, the local cross-shore sediment transport after disturbance of the equilibrium profile is given by:

\[
S_y(z,t) = \frac{F_3(z)}{\frac{\partial y_e}{\partial z}} ,
\]

where \(S_y\), \(F_2\) and \(F_3\) have the dimensions \([m^2/s]\).

In case of an equilibrium profile, the cross-shore sediment transport equals zero. This yields:

\[
\bar{S}_y(z,t) = \frac{F_3(z)}{\frac{\partial y_e}{\partial z}} = 0 .
\]

Subtracting these equations from each other results in the following expression:

\[
\tilde{S}_y(z,t) = S_y(z,t) - \bar{S}_y(z,t) = \frac{F_3(z)}{\frac{\partial y_e}{\partial z}} - \frac{F_3(z)}{\frac{\partial y_e}{\partial z}} .
\]

This expression can be rewritten to:

\[
\tilde{S}_y(z,t) = F_3(z) \left( \frac{\partial y_e}{\partial z} - \frac{\partial y_e}{\partial z} \right) .
\]

In addition to the definition for \(z'\):

\[
z' = z + \tilde{z}
\]

it can be defined that:

\[
y = y_e + \bar{y}
\]

where \(\bar{y}\) is the initial disturbance resulting from \(\tilde{z}\), see Figure 5.4. Using this, equation (5.12) can be rewritten to:
\[
\bar{S}_y(z,t) = F_3(z) \cdot \frac{\left( \frac{\partial y}{\partial z} \right)}{\left( \frac{\partial y_e}{\partial z} \right)^2}.
\] (5.14)

Furthermore it is assumed that \( \partial y / \partial z \ll \partial y_e / \partial z \). This assumption implies that the change of the slope due to a disturbance should be small compared to the slope of the equilibrium profile. This is illustrated in Figure 5.5.

Figure 5.5 Change of profile slope

With \( \partial y / \partial z \ll \partial y_e / \partial z \), it follows for equation (5.14):
\[
\bar{S}_y(z,t) = F_3(z) \cdot \frac{\left( \frac{\partial y}{\partial z} \right)}{\left( \frac{\partial y_e}{\partial z} \right)^2}.
\] (5.15)

Using equation (5.7), this can be rewritten to:
\[
\bar{S}_y(z,t) = \frac{F_2(z)}{F_3(z)} \cdot \left( \frac{\partial y}{\partial z} \right).
\] (5.16)

In order to set up the theory in non-dimensional form, this result is rewritten to:
\[
\bar{S}_y(k_0z,t) = \frac{F_2^*(k_0z)}{F_3^*(k_0z)} \cdot \left( \frac{\partial (k_0y)}{\partial (k_0z)} \right),
\] (5.17)

where:
\[
\bar{S}_y(k_0z,t) = \frac{\bar{S}_y(z,t)}{g^2 T^3},
\] (5.18)
\[
F_2^*(k_0z) = \frac{F_2^*(k_0z)}{g^2 T^3}
\] and
\[
F_3^*(k_0z) = \frac{F_3^*(k_0z)}{g^2 T^3}.
\] (5.19)
This notation will be used to indicate non-dimensional variables; \( dl \) stands for "dimensionless". Equation (5.16) clearly shows the importance of \( F_2 \) and \( F_3 \) for the sediment transport. It shows the larger influence of \( F_2 \) in case of a profile disturbance as well. The advantage of this description compared to equation (4.12) is obvious. This is a consequence of the fact that \( F_2 \) and \( F_3 \) do not change in case of a horizontal disturbance of the equilibrium profile.

In order to link the sediment transport to bottom changes, in the next section the continuity equation is derived according to the new definitions. It was necessary to adapt the continuity equation to the new coordinate system. Furthermore, the continuity equation is rewritten to a non-dimensional form.

### 5.6. The continuity equation

The cross-shore sediment transport resulting from a disturbance will not have the same magnitude for every location in the profile. It is very likely that gradients in the cross-shore transport will occur. Gradients in the sediment transport can cause erosion (a positive gradient) or accretion (a negative gradient). The relation between the gradient in the sediment transport and erosion or accretion is given by the continuity equation. As the \( y \)- and \( z \)-axis have changed position, a short derivation of the continuity equation will be presented below. The sediment balance of an area in the cross-shore profile is concerned, see Figure 5.6.

![Figure 5.6 Sediment balance](image)

Note that \( S_y \) is drawn in \( z \)-direction. This is allowed, as in this theory \( S_y \) is expressed as a function of \( z \). If, over a certain part of the profile, there is a gradient of \( S_y \) in \( y \)-direction, a balance over this part of the profile will show a gradient in the \( z \)-direction as well.

Writing the sediment balance for the section drawn in Figure 5.6 this yields for an interval of duration \( \Delta t \):

\[
\text{incoming volume} - \text{outgoing volume} = \text{change of storage}
\]
in symbols:

\[ S_y \cdot \Delta t - (S_y + \Delta S_y) \cdot \Delta t = \Delta y \cdot \Delta z \]  \hspace{1cm} (5.21)

This equation was already divided by \( \Delta x \), as the longshore sediment transport is not included in this theory.

Equation (5.21) can be rewritten to:

\[ \frac{-\Delta S_y}{\Delta z} = \frac{\Delta y}{\Delta t} \]  \hspace{1cm} (5.22)

From this it follows:

\[ \frac{\partial S_y}{\partial z} + \frac{\partial \bar{y}}{\partial t} = 0 \]  \hspace{1cm} (5.23)

The balance is rewritten to a non-dimensional form. After some calculations, it is found that equation (5.23) can be rewritten to:

\[ \frac{\partial S_y(k_0 z, t/T)_{al}}{\partial (k_0 z)} + \frac{1}{16 \pi^4} \frac{\partial (k_0 \bar{y})}{\partial (t/T)} = 0 \]  \hspace{1cm} (5.24)

which is the continuity equation that will be used in this theory to describe the bottom changes as a function of \( t \). In the next section, the kinematic equation is derived.

### 5.7. The kinematic equation

Substituting equation (5.17), describing the sediment transport due to a change in the bottom slope, into the continuity equation, it is found that:

\[ \frac{1}{16 \pi^4} \frac{\partial (k_0 \bar{y})}{\partial (t/T)} = - \frac{d}{dk_0 z} \left( \frac{F_2^2(k_0 z)_{al}}{F_3(k_0 z)_{al}} \right) \frac{\partial (k_0 \bar{y})}{\partial (k_0 z)} - \left( \frac{F_2^2(k_0 z)_{al}}{F_3(k_0 z)_{al}} \right) \frac{\partial^2 (k_0 \bar{y})}{\partial (k_0 z)^2} \]  \hspace{1cm} (5.25)

The relative simplicity of this kinematic equation lies in the fact that \( F_2 \) and \( F_3 \) remain constant for any \( \bar{y} \).

As \( F_2(k_0 z) \) and \( F_3(k_0 z) \) both only depend on \( k_0 z \), equation (5.25) can be rewritten to:

\[ \frac{1}{16 \pi^4} \frac{\partial (k_0 \bar{y})}{\partial (t/T)} = -Q(k_0 z) \left( \frac{\partial (k_0 \bar{y})}{\partial (k_0 z)} \right) - P(k_0 z) \left( \frac{\partial^2 (k_0 \bar{y})}{\partial (k_0 z)^2} \right) \]  \hspace{1cm} (5.26)

where:

\[ Q(k_0 z) = \frac{d}{dk_0 z} \left( \frac{F_2^2(k_0 z)_{al}}{F_3(k_0 z)_{al}} \right) \]  \hspace{1cm} (5.27)

\[ P(k_0 z) = \left( \frac{F_2^2(k_0 z)_{al}}{F_3(k_0 z)_{al}} \right) \]  \hspace{1cm} (5.28)

Equation (5.26) is a kinematic equation that describes the behaviour of disturbances of the equilibrium profile in time.

The functions \( Q(k_0 z) \) and \( P(k_0 z) \) are investigated below.
In order to be able to say something about \( Q(k_0z) \) and \( P(k_0z) \), first \( F_2 \) and \( F_3 \) need to be known as functions of \( z \). Fitting \( F_2 \) and \( F_3 \) as functions of \( z \) was not done before, only \( F_3 \) as a function of \( y \) was fitted.

To find an expression for \( F_2 \) as a function of \( z \), the original computer output of V.d. Kerk needs to be fitted, without first translating \( kh \) to \( k_0y \) by integrating the equilibrium slope. After translating \( kh \) to \( k_0z \) using the dispersion relation, it is easy to translate \( k_0h \) to \( k_0z \), as \( h = h_0 - z \), where \( h_0 \) denotes the deep water depth (for the definition of \( z \) see Figure 4.7). It is again remarked that the results, as well as the value for \( h_0 \), are based on \( s_0 = 0.02 \) and \( W\cdot T/H = 1.0 \).

First a plot of \( F_2 \) was made, with \( k_0z \) on the horizontal axis (see appendix H plot H1). Keeping in mind the definition of \( z \) the plot is easy to understand. A large value for \( z \) corresponds to a small value for \( h \).

Looking at the curve upside down, the curve seems to be fitted by:

\[
p(q) = C_1 q e^{-C_2 q},
\]

with a maximum at \( q = 1/C_2 \) of \( (C_1/C_2)e^{-1} \). From this \( C_1 \) and \( C_2 \) can be determined. Afterwards the equation is transformed towards the original axis by using:

\[
q = (1.647 - k_0z) \quad \text{and} \quad \frac{F_2}{g^2T^3}.
\]

The resulting fit is plotted in appendix H (H1). The equation describing \( F_2 \) yields:

\[
\frac{F_2}{g^2T^3} = -D'(1.647 - k_0z)e^{-E(1.647 - k_0z)}
\]

where

\[
D' = 2.65 \cdot 10^{-6} \quad \text{and} \quad E = 10.72.
\]

Using equation (5.19) this can be rewritten to:

\[
F_2(k_0z)_{dl} = -D'(1.647 - k_0z)e^{-E(1.647 - k_0z)}
\]

It can be seen that \( F_2 \) equals zero for \( k_0z = 1.647 \), which is true for the chosen wave- and sediment parameters only. In the following theory, this equation will be used to describe \( F_2(k_0z)_{dl} \).

In addition to this, \( F_3 \) is determined as a function of \( z \) as well. This is done in the same way \( F_2 \) was fitted as a function of \( z \).

After fitting the following expression is found for \( F_3 \) as a function of \( z \):

\[
\frac{F_3}{g^2T^3} = B'(k_0z)^C.
\]
where:

\[ B' = 1 \cdot 10^{-3} \quad \text{and} \quad C = 5.5. \]  
\[ (5.37) \quad (5.38) \]

The fitted function is plotted in appendix H (H2).

This equation can be rewritten to:

\[ F_3(k_0 z)_{nl} = B' \cdot (k_0 z)^C, \]  
\[ (5.39) \]

which will be used to describe \( F_3(k_0 z)_{nl} \).

Using the functions describing \( F_3(k_0 z)_{nl} \) and \( F_3(k_0 z)_{el} \), the expressions for \( Q(k_0 z) \) and \( P(k_0 z) \) can be determined.

The expression describing \( P(k_0 z) \) becomes after substitution:

\[ P(k_0 z) = \frac{D' \cdot (1.647 - k_0 z)^2 \cdot e^{-2 \cdot E \cdot (1.647 - k_0 z)}}{B' \cdot (k_0 z)^C}, \]  
\[ (5.40) \]

with \( B' \), \( C \), \( D' \), and \( E \) as defined before. \( P(k_0 z) \) is a non-dimensional function. It can be seen that the function \( P(k_0 z) \) is positive for every value of \( k_0 z \). The function is plotted in appendix I (I).

The expression describing \( Q(k_0 z) \) is more complex. To obtain an expression for \( Q(k_0 z) \), it is necessary to calculate the derivative of \( P(k_0 z) \) to \( k_0 z \). The resulting equation yields:

\[ Q(k_0 z) = \frac{D' \cdot (1.647 - k_0 z) \cdot e^{-2 \cdot E \cdot (1.647 - k_0 z)}}{B' \cdot (k_0 z)^C} \left\{ -2 + 2 \cdot E \cdot (1.647 - k_0 z) - \frac{(1.647 - k_0 z) \cdot C}{k_0 z} \right\}. \]  
\[ (5.41) \]

\( Q(k_0 z) \) is a non-dimensional function as well.

Looking at the equation, and keeping in mind that \( 0 < k_0 z < 1.647 \), it can be seen that the first part of the equation is positive for every value of \( k_0 z \). In other words, the sign of the equation is determined by the part of the equation between braces. The values of \( k_0 z \) for which \( Q(z) = 0 \) follow from:

\[ 1.647 - k_0 z = 0 \quad \text{and} \quad \]  
\[ -2 + 2 \cdot E \cdot (1.647 - k_0 z) - \frac{(1.647 - k_0 z) \cdot C}{k_0 z} = 0, \]  
\[ (5.42) \quad (5.43) \]

where:

\[ E = 10.72 \quad \text{and} \]  
\[ C = 5.5. \]
Equation (5.43) is true for $k_0 z = 0.27$ and $k_0 z = 1.53$. From $k_0 z = 0$ to $k_0 z = 0.27$, $Q(z)$ is negative. Between $k_0 z = 0.27$ and $k_0 z = 1.53$, $Q(k_0 z)$ remains positive, and from $k_0 z = 1.53$ to $k_0 z = 1.647$ $Q(k_0 z)$ is negative again.

The plot showing $Q(k_0 z)$ is included in appendix I (12).

Before continuing with the theory, the kinematic equation is briefly discussed.

The kinematic equation, see equation (5.25), can be written like:

$$\frac{\partial \bar{y}}{\partial t} = f(\frac{\partial \bar{y}}{\partial z}, \frac{\partial^2 \bar{y}}{\partial z^2}) .$$

From this it can be seen that, in case the right term of equation (5.44) is negative, disturbances $\bar{y}$ of the equilibrium profile will decrease in time (a negative gradient in time). In case the right term of equation (5.44) is positive, however, disturbances will increase in time.

The sign of the right term is hard to predict, as the right term contains the functions $P(k_0 z)$ and $Q(k_0 z)$, of which $Q(k_0 z)$ can be positive or negative, depending on $k_0 z$. Furthermore, the slopes of the disturbances can differ for different values of $k_0 z$, and for different values of $t$.

This can be explained by considering an initial disturbance. Via $F_2$ and $F_3$ for $t=0$ in every location $k_0 z$ a sediment transport gradient is calculated. This is the meaning of the right term of the kinematic equation. The left term describes the adaptation of the profile shape in time due to the gradients in the sediment transport for every location in the profile. After adaptation of the profile shape, the procedure can be repeated.

It would be interesting to investigate the sign of the right term of the kinematic equation for different profile shapes. In this way it can be analyzed whether there are conditions for which disturbances increase in time.

5.8. Conclusions

1. In chapter 4, a general expression describing the sediment transport due to an harmonic motion of the bottom in $z$-level was derived ignoring higher order terms of the Taylor expansion, see equation (4.12). Neglecting other terms, only the last term of equation (4.12) was analyzed, leaving out the influence of $F_2$.

In this chapter it was concluded that another approach is necessary in order to include both $F_2$ and $F_3$ in the calculations.

2. Schematising a breaker bar as a monotonic disturbance in $y$-direction instead of in $z$-direction it is possible to derive a kinematic differential equation for disturbances in $y$-direction. This kinematic equation contains all aspects that govern sediment transport as far as included in the theory of V.d. Kerk.

3. It can be concluded that, schematising breaker bars as disturbances in the $y$-direction, it is possible to implement both $F_2$ and $F_3$ as functions of $z$ into the theory. The main reason is that, in order to describe sediment transport due to a disturbance in $y$-direction,
F₂ and F₃ do not have to change. The transport is "generated" by a change in the bottom slope, instead of a change in F₂ and F₃.

The kinematic equation, see (5.25), describes the change in the profile shape k₀y in time due to a disturbance given by (∂k₀y/∂k₀z) and (∂²k₀y/∂(k₀z)²) for t = 0. The kinematic equation, however, is very complicated to analyze analytically.

Although the kinematic equation is a positive result, it may be clear that the applied schematisation of breaker bars remains disputable. In reality the breaker bars will not be monotonic increasing in coastward direction, but have negative as well as positive slopes.

Proceeding with the analysis of breaker bars, it is concluded that there are three different ways to analyze the kinematic equation.

1. Analysis of the right term of the kinematic equation, as proposed in the previous section. This will provide a general insight into the stability of initial profile forms. In chapter 6 this analysis is performed.

2. Solving the kinematic equation by applying separation of variables. This technique yields solutions for the kinematic equation. It is important to define boundary conditions in order to obtain the correct solution. Chapter 7 includes this analysis.

3. A numerical implementation of the kinematic equation into a computer program. This is quite easy to realise. It is even possible to schematise breaker bars as disturbances in the z− coordinate into the computer program. The computer program as well as the numerical output is discussed in chapter 8.
Chapter 6. Investigation of the stability of an initial disturbance

6.1. Introduction

In this chapter the stability of disturbances of the equilibrium profile is investigated. For this goal the sign of the right term of the kinematic equation:

\[ \frac{1}{16 \pi^4} \frac{d}{\partial (t/T)} \left( \frac{\partial (k_0 y)}{\partial (k_0 z)} \right) = - \frac{d}{dk_0 z} \left( \frac{F_2^2(k_0 z)}{F_3(k_0 z)} \right) \left( \frac{\partial (k_0 y)}{\partial (k_0 z)} \right) - \frac{\partial^2 (k_0 y)}{\partial (k_0 z)^2}, \] (5.25 repeated)

or

\[ \frac{1}{16 \pi^4} \frac{d}{\partial (t/T)} \left( \frac{\partial (k_0 y)}{\partial (k_0 z)} \right) = - Q(k_0 z) \left( \frac{\partial (k_0 y)}{\partial (k_0 z)} \right) - P(k_0 z) \left( \frac{\partial^2 (k_0 y)}{\partial (k_0 z)^2} \right), \] (5.26 repeated)

is concentrated on. The functions \( P(k_0 z) \) and \( Q(k_0 z) \) were defined and analyzed in chapter 5. Plots of \( P(k_0 z) \) and \( Q(k_0 z) \) are included in appendix I (11 and 12).

In section 6.2, a general expression is derived describing a profile disturbance in y-direction as a function of \( z(z) \) and the equilibrium profile shape. This expression describes the way in which a horizontal disturbance (in y-direction) of the equilibrium profile is caused by a vertical \( z(z) \).

In section 6.3 this expression is simplified choosing functions for the equilibrium profile and for \( z(z) \). Substituting the resulting simplified expression, describing the horizontal profile disturbance due to a vertical \( z(z,t) \), into the right term of the kinematic equation, in section 6.4 an expression is obtained from which the sign of the right term can be determined for different initial profile shapes. A positive sign indicates an increase in profile height, a negative sign a decrease in profile height.

In section 6.5 this analysis is presented. It is stressed that this analysis can only be performed for the initial situation. It is not known which shape the profile is adapted to after one time step.

Finally, in section 6.6 conclusions on this chapter are formulated.

6.2. A general expression for a horizontal disturbance

In the previous chapter, the functions \( P(k_0 z) \) and \( Q(k_0 z) \) were analyzed already. Looking at the kinematic equation, it can be seen that a description of \( \frac{\partial k_0 y}{\partial k_0 z} \) and \( \frac{\partial^2 k_0 y}{\partial (k_0 z)^2} \) will be most useful to obtain more insight into the equation as well. In this section, an expression for \( \frac{\partial k_0 y}{\partial k_0 z} \) in case of an initial disturbance will be derived. The derivation is presented in dimensional variables. The results are rewritten towards a non-dimensional form afterwards.

The first question is how to describe \( \frac{\partial y}{\partial z} \) as a function of \( z \).

At \((z_1,y_1)\) (see Figure 5.2) it can be stated that:

\[ y_1 = f_s(z_1) = f_s(z_1 + \bar{z}). \] (6.1)

In this equation, \( y_1 \) denotes a distorted y-coordinate due to a \( \bar{z} \).

In general, equation (6.1) yields:
\[ y(z) = f_e(z + \bar{z}(z)) \] (6.2)

From this equation the relation for \( \frac{\partial y}{\partial z} \) can be found. It follows that:

\[ \frac{\partial y(z)}{\partial z} = \frac{df_e(z)}{dz} \cdot \frac{\partial (z + \bar{z}(z))}{\partial z} . \] (6.3)

This equation can be rewritten to:

\[ \frac{\partial y(z)}{\partial z} = \frac{df_e(z)}{d(z + \bar{z}(z))} \cdot (1 + \frac{\partial \bar{z}(z)}{\partial z}) . \] (6.4)

In case \( \frac{\partial \bar{z}(z)}{\partial z} \) would equal to zero, this equation reduces to:

\[ \frac{\partial y(z)}{\partial z} = \frac{df_e(z)}{d(z + \bar{z}(z))} \] (6.5)

This is easy to understand, as \( \frac{\partial \bar{z}(z)}{\partial z} = 0 \) signifies an equal disturbance of the equilibrium profile at every location in the profile. In that case the profile is simply translated, and not deformed. The slope remains the same in every profile location, and no sediment transports are generated.

From equation (6.5) it can be seen that the right term denotes the equilibrium slope at \((z', y)\). In Figure 5.3 it can be seen that this location forms part of the equilibrium profile.

Rewriting equation (5.7), yielding:

\[ \frac{dy_e}{dz} = \frac{df_e(z)}{dz} = \frac{F_3(z)}{F_2(z)} \] (5.7 repeated)

to

\[ \frac{df_e(z + \bar{z}(z))}{d(z + \bar{z}(z))} = \frac{F_3(z + \bar{z}(z))}{F_2(z + \bar{z}(z))} \] (6.6)

it follows for equation (6.5):

\[ \frac{\partial y(z)}{\partial z} = \frac{F_3(z + \bar{z}(z))}{F_2(z + \bar{z}(z))} \cdot (1 + \frac{\partial \bar{z}(z)}{\partial z}) . \] (6.7)

This equation will be used for further analysis. Note that, until so far, no terms have been neglected. Equation (6.7) directly follows from the definitions stated in section 5.2.

Neglecting higher order terms, equation (6.7) can be rewritten to:

\[ \frac{\partial y(z)}{\partial z} = \frac{F_3(z + \bar{z}(z)) \frac{dF_3}{dz}}{F_2(z + \bar{z}(z)) \frac{dF_2}{dz}} \cdot (1 + \frac{\partial \bar{z}(z)}{\partial z} . \] (6.8)

It is remarked that it is justified to neglect the higher order terms, as it is assumed that \( \bar{z} \) is small. Therefore terms containing \( z^2 \) can be neglected.

Equation (6.8) can be rewritten to:
\[ \frac{\partial y(z)}{\partial z} = \frac{F_3(z) \cdot (1 + \frac{\partial \bar{z}(z)}{\partial z})}{F_3(z) \cdot (1 + \frac{\partial \bar{z}(z)}{\partial z})} \cdot (1 + \frac{\partial \bar{z}(z)}{\partial z}) \cdot (1 + \frac{\partial \bar{z}(z)}{\partial z}). \] (6.9)

In the following step it is used that:

\[ \frac{1 + a}{1 + b} \approx 1 + a - b, \] (6.10)

where \( a \) and \( b \) are small.

In addition to this, it follows for equation (6.10):

\[ \frac{\partial y(z)}{\partial z} = \frac{F_3(z) \cdot (1 + \frac{\partial \bar{z}(z)}{\partial z})}{F_3(z) \cdot (1 + \frac{\partial \bar{z}(z)}{\partial z})} \cdot (1 + \frac{\partial \bar{z}(z)}{\partial z}) \cdot (1 + \frac{\partial \bar{z}(z)}{\partial z}). \] (6.11)

Rewriting equation (6.11) to:

\[ \frac{\partial y(z)}{\partial z} = \frac{F_3(z) \cdot (1 + \frac{\partial \bar{z}(z)}{\partial z})}{F_3(z) \cdot (1 + \frac{\partial \bar{z}(z)}{\partial z})} \cdot (1 + \frac{\partial \bar{z}(z)}{\partial z}) \cdot (1 + \frac{\partial \bar{z}(z)}{\partial z}). \] (6.12)

it follows, using equation (5.7), that:

\[ \frac{\partial y(z)}{\partial z} = \frac{\partial y\_e}{\partial z} \cdot (1 + \frac{\partial \bar{z}(z)}{\partial z}) \cdot (1 + \frac{\partial \bar{z}(z)}{\partial z}). \] (6.13)

This equation can simply be rewritten to:

\[ \frac{\partial y(z)}{\partial z} = \frac{\partial y\_e}{\partial z} + \frac{\partial y\_e}{\partial z} \cdot \left( \frac{\partial \bar{z}(z)}{\partial z} + \frac{\partial \bar{z}(z)}{\partial z} \cdot \frac{d}{dz} \left( \ln \frac{\partial y\_e}{\partial z} \right) \right). \] (6.14)

Using equation (5.13) it follows from equation (6.14):

\[ \frac{\partial y(z)}{\partial z} = \frac{\partial y\_e}{\partial z} \cdot \left( \frac{\partial \bar{z}(z)}{\partial z} + \frac{\partial \bar{z}(z)}{\partial z} \cdot \frac{d}{dz} \left( \ln \frac{\partial y\_e}{\partial z} \right) \right). \] (6.15)

Assuming \( \frac{\partial \bar{z}(z)}{\partial z} \) very small compared to 1, which is a logical assumption to make as the disturbances should be small, equation (6.15) finally becomes:

\[ \frac{\partial y(z)}{\partial z} = \frac{\partial y\_e}{\partial z} \cdot \left( \frac{\partial \bar{z}(z)}{\partial z} + \frac{\partial \bar{z}(z)}{\partial z} \cdot \frac{d}{dz} \left( \ln \frac{\partial y\_e}{\partial z} \right) \right). \] (6.16)

In non-dimensional variables this yields:

\[ \frac{\partial (k_0 \hat{y}(z))}{\partial (k_0 z)} = \frac{\partial (k_0 y\_e)}{\partial (k_0 z)} \cdot \left( \frac{\partial (k_0 \hat{z}(z))}{\partial (k_0 z)} + \frac{k_0}{d(k_0 \hat{z}(z))} \cdot \frac{d}{dz} \left( \ln \frac{\partial (k_0 y\_e)}{\partial (k_0 z)} \right) \right). \] (6.17)

This equation describes the disturbance of the profile slope due to a \( \hat{z}(z) \). The equation can be used for substitution in equation (5.26).

Just like equation (6.4), equation (6.16) can be checked in case of a translation of the profile, without deformation. A translation of the profile without deformation is given by:
\[
\frac{\partial z}{\partial z} = 0 \tag{6.18}
\]

which means that \( z \) is the same for every \( z \). If so, the profile is translated without deformation.

Condition (6.18) substituted in equation (6.16) yields:

\[
\frac{\partial z}{\partial z} = \frac{\partial y}{\partial z} \cdot \frac{d}{dz} \left( \ln \frac{\partial y}{\partial z} \right) \tag{6.19}
\]

This can be rewritten to:

\[
\frac{\partial y}{\partial z} = \frac{\partial y}{\partial z} \cdot \frac{d}{dz} \left( \ln \frac{\partial y}{\partial z} \right) \tag{6.20}
\]

\[
\frac{\partial y}{\partial z} = \frac{\partial y}{\partial z} \cdot (1 + \frac{\partial y}{\partial z}) \tag{6.21}
\]

From equation (6.13), going backwards in the derivation to equation (6.7), it can be seen that equation (6.21) reduces to:

\[
\frac{\partial y}{\partial z} = \frac{\partial y}{\partial z} \cdot (1 + \frac{\partial y}{\partial z}) \tag{6.22}
\]

As the right term of this equation does not contain \( z \), this means that the slope does not change due to any \( z \), indicating a translation of the profile without deformation. From this it can be concluded that equation (6.13) is correct on this point.

In the following section, a solution for equation (6.17) is derived. To be able to find a solution, a function has been chosen for the disturbance of the \( z \)-coordinate \( k_0 \tilde{z}(z) \). Substitution of this function at the same time simplifies the equations.

6.3. The initial profile shape for a harmonic disturbance in \( z \)

Resuming equation (6.17):

\[
\frac{\partial (k_0 y(z))}{\partial (k_0 z)} = \frac{\partial (k_0 y)}{\partial (k_0 z)} \cdot \left( \frac{\partial (k_0 \tilde{z})}{\partial (k_0 z)} + k_0 \tilde{z} \cdot \frac{d}{d(k_0 z)} \left( \ln \frac{\partial (k_0 y)}{\partial (k_0 z)} \right) \right) \tag{6.17 repeated}
\]

it can be seen that substitution of this equation into the right term of equation (5.26), in order to find a general expression determining the sign of the right as a function of an initial disturbance term will result in a large expression which is difficult to handle. For this reason it is decided to choose functions describing \( k_0 y(z) \) and \( k_0 \tilde{z}(z) \), in order to simplify the analysis.

The function \( y_\epsilon \), see equation (5.1), is the function that defines the equilibrium profile. Consequently, in order to find this function, the equilibrium profile has to be fitted as a function of \( z \). This is done in appendix J, see plot J1. It is remarked that the equilibrium profile is linked to \( F_2 \) and \( F_3 \); it would be incorrect to choose any other equilibrium profile without calculating other functions for \( F_2 \) and \( F_3 \).

The equilibrium profile is fitted by:
\[ k_0 y_\epsilon = a_1 \cdot e^{(a_1 \cdot k_0 z)} \] (6.23)

where:

\[ a_1 = 134 \quad \text{and} \quad a_2 = -1.9. \]

From equation (6.23) it follows that:

\[ \frac{\partial (k_0 y_\epsilon)}{\partial (k_0 z)} = a_1 \cdot a_2 \cdot e^{(a_1 \cdot k_0 z)}. \] (6.24)

Using equation (6.24), it follows that:

\[
\ln\left(\frac{\partial (k_0 y_\epsilon)}{\partial (k_0 z)}\right) = \ln(a_1 \cdot a_2) + \ln e^{(a_1 \cdot k_0 z)} \\
= \ln(a_1 \cdot a_2) + a_2 \cdot k_0 z,
\] (6.25)

resulting in:

\[ \frac{d}{dk_0 z} \left( \ln\left(\frac{\partial (k_0 y_\epsilon)}{\partial (k_0 z)}\right) \right) = a_2. \] (6.26)

Furthermore, a function describing \( k_0 \tilde{z}(z) \) is defined by:

\[ k_0 \tilde{z}(z,0) = a_3 \cdot \cos(k_d z), \] (6.27)

where:

\[ k_d = \frac{2\pi}{L_d}, \] (6.28)

with \( L_d \) equal to the wavelength of the vertical disturbances in the z-coordinate. The value for \( L_d \) should be chosen in a way that the disturbances are not too large. Choosing this harmonic disturbance, the analysis of the different disturbances becomes less general: only harmonic disturbances are dealt with. Nevertheless this seems an appropriate way of schematising a barred profile.

From equation (6.28) it follows that:

\[ \frac{\partial (k_0 \tilde{z}(z))}{\partial (k_0 z)} = -a_3 \cdot \frac{k_d}{k_0} \cdot \sin(k_d z). \] (6.29)

Substituting the equations (6.24), (6.20) and (6.30) into equation (6.17) yields for \( \partial k_0 \tilde{y}/\partial k_0 z \):

\[ \frac{\partial (k_0 \tilde{z}(z))}{\partial (k_0 z)} = (a_1 \cdot a_2 \cdot a_3 \cdot e^{(a_1 \cdot k_0 z)} \cdot (a_2 \cdot \cos(k_d z) - \frac{k_d}{k_0} \cdot \sin(k_d z))). \] (6.31)

It is remarked that \( a_2 \) is negative. Keeping this in mind it can be seen that, for larger values of \( k_0 \tilde{z} \), an harmonic disturbance of \( k_0 \tilde{z} \) has less influence on the profile slope than the same disturbance for smaller values of \( k_0 \tilde{z} \).

This can be understood looking at the Figures 5.2 and 5.3. In Figure 5.3, a \( \tilde{z} \) at \( z = z_1 \) results in a decrease of \( y_{\epsilon,1} \) to \( y_1 \). The same \( \tilde{z} \) at \( z = z_3 \) would also
result in a decrease of \( y_{z,3} \), but this decrease will be much smaller. This is in accordance with equation (6.31).

In equation (5.26), apart from \( \partial k_{0} \partial k_{z} / \partial k_{0} \), \( \partial^{2} k_{0} \partial (k_{0}z)^{2} \) is needed as well. Differentiating equation (6.31) yields for \( \partial^{2} k_{0} \partial (k_{0}z)^{2} \):

\[
\frac{\partial^{2}(k_{0}y(z))}{\partial (k_{0}z)^{2}} = (a_{1} \cdot a_{2} \cdot a_{3} \cdot e^{(a_{2} k_{0}z)}) \cdot ((a_{2}^{2} - k_{d}^{2}/k_{0}^{2}) \cdot \cos(k_{0}z) - 2 \cdot a_{2} \cdot \frac{k_{d}}{k_{0}} \cdot \sin(k_{0}z)).
\]  

(6.32)

Summarising, the functions (6.31) and (6.32) yield the expressions linking a harmonic disturbance in \( z \) to a horizontal disturbance of the equilibrium profile in \( y \)-direction.

6.4. Calculation of the right term of the kinematic equation

Substituting equations (6.31) and (6.32) into the right term of the kinematic equation (5.26), an expression describing the gradient in the sediment transport for every \( k_{0}z \) in case of an harmonic disturbance is obtained. The kinematic equation becomes after substitution:

\[
\frac{1}{16 \pi^{3}} \cdot \frac{\partial}{\partial (t/T)} = (a_{1} \cdot a_{2} \cdot a_{3} \cdot e^{(a_{2} k_{0}z)}) \cdot \{ (-Q(k_{0}z) a_{2} - P(k_{0}z) (a_{2}^{2} - k_{d}^{2}/k_{0}^{2})) \cdot \cos(k_{0}z) + \}
\]

\[
\frac{1}{16 \pi^{3}} \cdot \frac{\partial}{\partial (t/T)} = (a_{1} \cdot a_{2} \cdot a_{3} \cdot e^{(a_{2} k_{0}z)}) \cdot \{ R(k_{0}z) \cos(k_{d}z) + S(k_{0}z) \sin(k_{d}z) \}
\]  

(6.33)

Keeping in mind the dimensions of the coefficients and of \( P(k_{0}z) \) and \( Q(k_{0}z) \), it can be seen that equation (6.33) is non-dimensional.

Equation (6.33) can be rewritten to:

\[
\frac{1}{16 \pi^{3}} \cdot \frac{\partial}{\partial (t/T)} = (a_{1} \cdot a_{2} \cdot a_{3} \cdot e^{(a_{2} k_{0}z)}) \cdot \{ R(k_{0}z) \cos(k_{d}z) + S(k_{0}z) \sin(k_{d}z) \}
\]  

(6.34)

where \( R(k_{0}z) \) and \( S(k_{0}z) \) are only functions of \( k_{0}z \), defined by:

\[
R(k_{0}z) = -Q(k_{0}z) a_{2} - P(k_{0}z) (a_{2}^{2} - k_{d}^{2}/k_{0}^{2}) \]

(6.35)

and

\[
S(k_{0}z) = Q(k_{0}z) \frac{k_{d}}{k_{0}} + P(k_{0}z) 2 a_{2} \frac{k_{d}}{k_{0}}
\]  

(6.36)

Note that \( R(k_{0}z) \) and \( S(k_{0}z) \) are non-dimensional functions.

From this equation it can be concluded that the right term, representing the gradients in sediment transport for every location \( k_{0}z \) generated by the initial harmonic disturbance, becomes smaller for larger values of \( k_{0}z \) (the exponent \( a_{2} k_{0}z \) is negative). Larger values of \( k_{0}z \) indicate a higher location in the profile, closer to the coastline. The negative exponent prevents morphologic activity from being large very close to the coastline. This is obvious as sediment transport gradients should become zero in the swash zone. In reality, the gradients in sediment transport also decrease in seawards direction. This decrease is represented by the functions \( R(k_{0}z) \) and \( S(k_{0}z) \), as these
functions influence the amplitude of the disturbances generated by the sine and the cosine. The functions $R(k_0 z)$ and $S(k_0 z)$ are included in appendix K, see plot K1.

In the plot $R(k_0 z)$ and $S(k_0 z)$ are printed for different values of $L_d$. $L_d$ denotes the wavelength of the vertical disturbance in the $z-$ coordinate.

It can be seen that a larger $L_d$ reduces the amplitude of $R(k_0 z)$ and $S(k_0 z)$. In other words: a harmonic disturbance with a smaller $L_d$ causes larger gradients in the cross-shore sediment transport.

From equations (6.35) and (6.36) it follows that the value of $k_0$ has a decreasing influence on the amplitude of $R(k_0 z)$ and $S(k_0 z)$. A smaller $L_0$ causes larger values for $R(k_0 z)$ and $S(k_0 z)$, resulting in larger gradients in the sediment transport due to the same harmonic disturbance.

It can be concluded that the relation $k_0 / k_0$ is important for the influence of a disturbance in the vertical $z-$ coordinate on the equilibrium profile. Furthermore, it can be seen that $R(k_0 z)$ and $S(k_0 z)$ become very small for small values of $k_0 z$ as well as for large values of $k_0 z$. This indicates that only in a limited part of the profile, the upper part, a disturbance of the vertical $z$-coordinate has a significant influence on the equilibrium profile. In this part of the profile most of the morphologic activities take place.

In the next section it is analyzed whether the kinematic equation (5.25) derived in this section is a stable equation for different initial disturbances. To investigate this, the right term of the kinematic equation needs to be plotted for every $k_0 z$.

Furthermore, the right term of the kinematic equation is rewritten in a complex form. This method provides an easy way to calculate the modulus of the expression, yielding the maximum sediment transport gradients for every $k_0 z$. The modulus shows the magnitude of the expression for every $k_0 z$. This provides a check on the plots of the expression.

6.5. Analysis of the stability of an initial disturbance

In chapter 5, the kinematic equation for disturbances in $y$-direction:

$$\frac{1}{16 \pi^4} \cdot \frac{\partial (k_0 y)}{\partial (t/T)} = - \frac{d}{d(k_0 z)} \left( \frac{F_2^2(k_0 z)}{F_3(k_0 z)} \cdot \left( \frac{\partial (k_0 y)}{\partial (k_0 z)} \right) \right) - \frac{F_3^2(k_0 z)}{F_3(k_0 z)} \cdot \left( \frac{\partial^2 (k_0 y)}{\partial (k_0 z)} \right)$$  \hspace{1cm} (5.25 repeated)

was derived. After substitution of an harmonic initial disturbance, the equation becomes:

$$\frac{1}{16 \pi^4} \cdot \frac{\partial (k_0 y(z,t))}{\partial (t/T)} = (a_1 \cdot a_2 \cdot a_3 \cdot e^{(a_1 \cdot k_0 z)}) \cdot \{ R(k_0 z) \cos(k_0 z) + S(k_0 z) \sin(k_0 z) \}$$  \hspace{1cm} (6.34 repeated)

In order to be able to say something about the modulus, it is tried to rewrite equation (6.34) to a complex form. This means that equation (6.34) is assumed to be only the real part of the right term of the kinematic equation. In formula form:

$$\text{Re} \left\{ \frac{\partial (k_0 y(z,t))}{\partial (t/T)} \right\} = \frac{\partial (k_0 y(z,t))}{\partial (t/T)} =$$

$$= 16 \pi^4 \cdot (a_1 \cdot a_2 \cdot a_3 \cdot e^{(a_1 \cdot k_0 z)}) \cdot \{ R(k_0 z) \cos(k_0 z) + S(k_0 z) \sin(k_0 z) \}$$  \hspace{1cm} (6.37)

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where \( \frac{\partial (k_\theta \bar{y}(k_\theta z, t))}{\partial (t/T)} \) is the solution in complex form. It is found that \( \frac{\partial (k_\theta \bar{y}(k_\theta z, t))}{\partial (t/T)} \) is given by:

\[
\frac{\partial (k_\theta \bar{y}(k_\theta z, t))}{\partial (t/T)} = 16 \pi^2 \cdot (a_1 \cdot a_2 \cdot a_3 \cdot e^{(a_1 \cdot k_\theta z)} \cdot (R(k_\theta z) - i \cdot S(k_\theta z)) \cdot \{\cos(k_\theta z) + i \cdot \sin(k_\theta z)\} \nonumber
\]

From this it can be seen that the last part of the expression represents the harmonic character of the disturbance, and the first and second part the amplitude of the sediment transport gradients generated by the disturbance. The first term has a decreasing influence on the amplitude for larger values of \( k_\theta \), as \( a_2 \) is negative.

The 'modulus' of the solution is defined by:

\[
| \frac{\partial (k_\theta \bar{y}(k_\theta z, t))}{\partial (t/T)} | = | 16 \pi^2 \cdot a_1 \cdot a_2 \cdot a_3 \cdot e^{(a_1 \cdot k_\theta z)} \cdot \sqrt{R^2(k_\theta z) + S^2(k_\theta z)} | \nonumber
\]

The values for the modulus \( | \frac{\partial (k_\theta \bar{y}(k_\theta z, t))}{\partial (t/T)} | \) are plotted in appendix K, plot K2.

The plots of the modulus first show that the gradients in the sediment transport due to a harmonic disturbance will have a limited magnitude. This means that, for this disturbance, there is no location for which the sediment transport gradient can become infinite. Secondly, it can be seen that the maximum magnitude of the gradients in sediment transport decreases for larger values of \( k_\theta z \) and for smaller values of \( k_\theta \). This means that, for every choice of \( k_\theta \) in the disturbance, there will be a limited interval in the profile where sediment transport gradients are generated.

A third remark about the modulus is, that the over-all magnitude of the modulus decreases for increasing wave length of the initial disturbance. This indicates that initial disturbances with a long wave length and small slopes cause small sediment transport gradients. Such a "smooth" disturbance will not cause large erosional or accretional locations in the profile.

Plotting the right term of the equation (6.34), see plot L1 in appendix L, it can be seen that there are clearly accretional and erosional locations in the profile, indicated by the positive and the negative values for the gradient in sediment transport respectively. The largest gradients occur in the upper part of the profile. The locations where gradients equal to zero are the locations without a disturbance of the equilibrium profile. These are the "nodes" of the harmonic disturbance.

6.6. Conclusions

1. It is possible to analyze the kinematic equation derived in chapter 5 by assuming an initial harmonic disturbance of the \( z \)-coordinate. Linking this disturbance via the equilibrium profile to disturbances in \( y \)-direction, after substitution an expression yielding gradients in the sediment transport for an initial disturbance can be derived.
2. A location in which the sediment transport gradient is positive will increase in height (accretion), a location with a negative sediment transport gradient will consequently decrease in height (erosion). The signs are only known for the initial situation. In time the profile is changed, and new sediment transport gradients should be calculated.

3. The modulus of the gradients in sediment transport after an initial disturbance shows that the upper part of the profile is morphologically the most active part. Furthermore, the gradients decrease in amplitude for disturbances with longer length scales.

4. The plot of the gradients (L1) shows accretional and erosional locations in the profile. From this it can be concluded that the initial disturbance will change; it can not be seen whether some kind of disturbance will increase in amplitude in time. To investigate this a time series should be calculated. It is expected, however, that the gradients for sediment transport will decrease in time due to damping of the initial disturbance.

A more general analysis of the kinematic equation, including time-dependency into the solution, can provide more information on this.
Chapter 7. Analysis of the kinematic equation

7.1. Introduction

In section 5.8, it was mentioned that a second way of analysing the kinematic equation derived in chapter 5, equation (5.25), is by applying "separation of variables". It should be possible to find a solution, as the kinematic equation is linear. A difficulty is, however, that the coefficients are functions of \( k \). This complicates the analysis.

In section 7.2, the variables \( z \) and \( t \) are actually separated, and two separate linear differential equations are obtained. In section 7.3 the boundary conditions for the kinematic equation are defined, and in section 7.4 the function \( Z(z) \) is analyzed. The function \( T(t) \) is analyzed in section 7.5. In section 7.6 some conclusions on this chapter are formulated.

7.2. Separation of variables

For brevity, the notation \( \dot{y} \) instead of \( k \dot{y} \), and \( z \) instead of \( k \dot{z} \), will be used in this section.

Reducing the kinematic equation (5.25) to:

\[
\frac{\partial \dot{y}}{\partial t} + Q(k\dot{z}) \frac{\partial \dot{y}}{\partial z} + P(k\dot{z}) \frac{\partial^2 \dot{y}}{\partial z^2} = 0
\]  

(7.1)

and assuming that the solution can be written as:

\[
\dot{y}(z,t) = Z(z) \cdot T(t)
\]  

(7.2)

it follows after substitution that:

\[
Z \cdot T' + Q \cdot T \cdot Z' + P \cdot T \cdot Z'' = 0
\]  

(7.3)

where:

\[
Z' = \frac{dZ(z)}{dz}, \quad Z'' = \frac{d^2Z(z)}{dz^2} \quad \text{and} \quad T' = \frac{dT(t)}{dt}
\]

After dividing equation (7.3) by \( Z \cdot T \) it is found that:

\[
\frac{Q \cdot Z' + P \cdot Z''}{Z} = -\frac{T'}{T}
\]  

(7.4)

As the functions \( T(t) \) and \( Z(z) \) are independent of each other and at the same time, according to equation (5.89), the terms left and right of the equality sign are equal, the following can be stated:

\[
\frac{Q \cdot Z' + P \cdot Z''}{Z} = -\frac{T'}{T} = -\mu
\]  

(7.5)
with $\mu \in \mathbb{R}$. The constant $\mu$ is called the separation constant. Note that $\mu$ can be both positive and negative.

In equation (7.5) the functions $T(t)$ and $Z(z)$ are described separately. Before analysing these functions, first the boundary conditions are discussed.

### 7.3. Boundary conditions

In reality, the amplitude of the breaker bars decreases in offshore direction. Keeping in mind the definition of the reference axis, see Figure 4.7, it can be stated that:

$$\ddot{y}(0,t) = 0 \quad \text{(7.6)}$$

From this it follows that, using equation (7.2):

$$Z(0) \cdot T(t) = 0 \quad \text{(7.7)}$$

As $T(t) = 0$ would be a trivial solution, it can be concluded that:

$$Z(0) = 0 \quad \text{(7.8)}$$

is a boundary condition. In reality, it can be seen that the amplitude of the breaker bars decreases in onshore direction as well. This means that

$$\ddot{y}(z_{\text{max}}, t) = 0 \quad \text{(7.9)}$$

where $z_{\text{max}}$ is the maximum profile height at the onshore boundary.

Using the same arguments, it follows that

$$Z(z_{\text{max}}) = 0 \quad \text{(7.10)}$$

is the other boundary condition.

From the functions $Z(z)$ and $T(t)$, in the next section first $Z(z)$ will be analyzed.

### 7.4. The function $Z(z)$

From equation (7.5) it follows that:

$$Q \cdot Z' + P \cdot Z'' + \mu Z = 0 \quad \text{(7.11)}$$

This is a linear, second order differential equation with non-constant coefficients. The following analysis is based upon Boyce and DiPrima [4], further referred to as B&D.

On page 83, B&D present the general second order linear equation:

$$P(x) \frac{d^2 y}{dx^2} + Q(x) \frac{dy}{dx} + R(x) \cdot y = G(x) \quad \text{(7.12)}$$

which has the same form as equation (7.11). Equation (7.11) can be obtained by reading $Z(z)$ for $y$, $z$ for $x$, $\mu$ for $R(x)$ and by choosing $G(x) = 0$. 

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B&D state that there is no specific formula for the solution of equation (7.12). This means that there is no general way of solving equation (7.11) in terms of a finite number of elementary functions. Therefore, first some general properties of equation (7.11) are discussed.

**General properties of equation (7.11)**

It is remarked that the functions \( P(k_\phi z) \) and \( Q(k_\phi z) \) are continuous functions, and that \( P(k_\phi z) \) is nowhere zero, see appendix I for plots showing \( P(k_\phi z) \) and \( Q(k_\phi z) \) (11 and 12).

Dividing equation (7.11) by \( P(k_\phi z) \) it follows that:

\[
Z'' + \frac{Q}{P} \cdot Z' + \frac{\mu}{P} \cdot Z = 0 .
\]

(7.13)

Defining:

\[
U(k_\phi z) = \frac{Q(k_\phi z)}{P(k_\phi z)} \quad \text{and} \quad V(k_\phi z) = \frac{\mu}{P(k_\phi z)}
\]

(7.14)

(7.15)

it follows from equation (7.13) that:

\[
Z'' + U \cdot Z' + V \cdot Z = 0
\]

(7.16)

which is in fact a homogeneous equation. It can be shown (B&D theorem 3.3) that, if \( Z = Z_1(z) \) and \( Z = Z_2(z) \) are solutions of equation (7.16), then the linear combination \( Z(z) = c_1 \cdot Z_1(z) + c_2 \cdot Z_2(z) \), where \( c_1 \) and \( c_2 \) are arbitrary constants, is also a solution of equation (7.16). Thus the superposition principle can be applied to equation (7.16). Corresponding to the infinity of the values assigned to \( c_1 \) and \( c_2 \), an infinity of solutions can be constructed for any value of \( \mu \). The value of \( \mu \) is determined by the boundary conditions.

Furthermore, it is shown (B&D theorem 3.4) that every solution \( Z(z) \) of equation (7.16) can be expressed as a combination of \( Z_1(z) \) and \( Z_2(z) \), which means that the solutions \( Z_1(z) \) and \( Z_2(z) \) are fundamental, and include all possible solutions of equation (7.16). Therefore, \( Z(z) = c_1 \cdot Z_1(z) + c_2 \cdot Z_2(z) \) is called the general solution for equation (7.16).

If one solution \( Z_1(z) \) or \( Z_2(z) \) of this homogeneous equation can be found, a second solution can be obtained, in principle, by reducing the order of the equation. The problem in this case is to find the first solution.

The expressions for the coefficients \( U \) and \( V \) yield, using (5.40) and (5.41):

\[
U = 2 \cdot E - \frac{2}{(1.647 - k_{\phi z})} - \frac{C}{k_{\phi z}^2}
\]

(7.17)

and
\[
V = \frac{\mu \cdot B' \cdot (k_0 \varphi)^C}{D'^2 \cdot (1.647 - k_0 \varphi)^2} \cdot e^{2 \cdot E \cdot (1.647 - k_0 \varphi)}
\]  \hspace{1cm} (7.18)

It can be seen that these functions are difficult to deal with, making it impossible to find a first solution in analytical form.

In order to determine solutions for equation (7.16), use has to be made of infinite series. Solutions can not be found as exact formulas, as equation (7.16) contains variable coefficients.

**Analysis using infinite series**

A first remark is that the functions \(U\) and \(V\) should be polynomials in order to find series solutions. This is not yet the case, but it is possible to develop exponential series describing \(U\) and \(V\). Therefore, this method will briefly be discussed. It is likely that this method will be the only method, except for numerical implementation, leading to solutions for \(Z(z)\).

Analysing the function:

\[
Z'' + U \cdot Z' + V \cdot Z = 0 \tag{7.16 repeated}
\]

where:

\[
U = 2 \cdot E - \frac{2}{(1.647 - k_0 \varphi)} - \frac{C}{k_0 \varphi} \quad \text{and} \quad (7.17 \text{ repeated})
\]

\[
V = \frac{\mu \cdot B' \cdot (k_0 \varphi)^C}{D'^2 \cdot (1.647 - k_0 \varphi)^2} \cdot e^{2 \cdot E \cdot (1.647 - k_0 \varphi)} \quad \text{(7.18 repeated)}
\]

it is tried to find a solution of the form of a power series:

\[
Z(z) = \sum_{n=0}^{\infty} a_n \cdot z^n \tag{7.19}
\]

Differentiating this solution term by term yields:

\[
Z'(z) = \sum_{n=1}^{\infty} n \cdot a_n \cdot z^{n-1} \quad \text{and} \quad (7.20)
\]

\[
Z''(z) = \sum_{n=2}^{\infty} n(n-1) \cdot a_n \cdot z^{n-2} \tag{7.21}
\]

Substituting the series (7.20) and (7.21) for \(Z'\) and \(Z''\) in equation (7.16) gives:

\[
\sum_{n=2}^{\infty} n(n-1) a_n \cdot z^{n-2} + U \cdot \sum_{n=1}^{\infty} n a_n \cdot z^{n-1} + V \cdot \sum_{n=0}^{\infty} a_n \cdot z^n = 0 \tag{7.22}
\]

Next, in the first sum \(n\) is replaced by \(n+2\), and in the second sum \(n\) is replaced by \(n+1\). This is called a "shift of index". It follows that:

\[
\sum_{n=2}^{\infty} (n+2)(n+1) a_{n+2} \cdot z^n + U \cdot \sum_{n=0}^{\infty} (n+1) a_{n+1} \cdot z^n + V \cdot \sum_{n=0}^{\infty} a_n \cdot z^n = 0 \tag{7.23}
\]

At this moment, with exponential series describing \(U\) and \(V\), a recurrence relation could be found. From the recurrence relation the coefficients \(a_n\) in equation (7.19) can
be determined. Performing this, however, would become too specialised in the scope of
this study. It remains a possibility for further study, perhaps by a mathematician. It is
remarked that, if the functions $F_2$ and $F_3$ were fitted as polynomials, $U$ and $V$ would be
polynomials as well.

A third theory that can be applied to this problem is the Sturm-Liouville theory.

Analysis using the Sturm-Liouville theory

Keeping in mind that $Q = P'$, equation (7.11) can be rewritten to:

$$P \cdot Z'' + P' \cdot Z' + \mu \cdot Z = 0 .$$  \hspace{1cm} (7.24)

Putting the first two terms together it follows that:

$$(P \cdot Z')' + \mu \cdot Z = 0 .$$  \hspace{1cm} (7.25)

Together with the boundary conditions (7.8) and (7.10) this is a so-called Sturm-Liouville
problem. Some properties of the solutions of the problem are presented in B&D. These
are mentioned below.

A first property is that all of the eigenvalues $\mu_n$ of this problem are real. This follows
from the boundary conditions (7.8) and (7.10). It is shown in B&D, theorem 11.1. A
consequence of this is that one need only look for real eigenvalues in finding eigenvalues
and eigenfunctions of the Sturm-Liouville boundary problem.

Another important theorem is about the orthogonality of the eigenfunctions. Two
functions $u(x)$ and $v(x)$ are said to be orthogonal on an interval if their inner product
vanishes. On the interval $\alpha \leq x \leq \beta$ this means that

$$\int_{\alpha}^{\beta} u(x) \cdot v(x) \, dx = 0 .$$ \hspace{1cm} (7.26)

In theorem 11.2 (B&D) it is proved that, if $\phi_1$ and $\phi_2$ are two eigenfunctions of the
Sturm-Liouville problem, corresponding to eigenvalues $\mu_1$ and $\mu_2$ respectively, and
if $\mu_1 \neq \mu_2$, then

$$\int_{0}^{\pi} \phi_1(z) \cdot \phi_2(z) \, dz = 0 .$$ \hspace{1cm} (7.27)

An harmonic solution is one of the possible orthogonal functions, as it is known that the
functions $\cos(m \pi x / l)$ and $\sin(m \pi x / l)$ are orthogonal (this is used in deriving the
orthogonality equations in the Fourier theory).

A third theorem (B&D theorem 11.3) states that, to each eigenvalue, there corresponds
only one linear independent eigenfunction. Further, the eigenvalues form an infinite
sequence, and can be ordered according to increasing magnitude.

Note that all of the properties stated in theorems 11.1 to 11.3 are indeed exemplified by
the eigenvalues $\mu_n = n^2 \pi^2$ and eigenfunctions $\phi_n(z) = \sin n^2 \pi^2 z$. Therefore, a harmonic
solution for $Z(z)$ is very likely which is in accordance with the harmonic schematisation of breaker bars.

**The sign of the eigenvalues**

From further study it follows that the sign of the eigenvalues, which provides an indication of the stability, depends to a large extent upon the boundary conditions.

Strauss [21] on page 144 treats an equation of the form:

$$-Z'' = \lambda Z \quad (7.28)$$

This equation looks like equation (7.25) for $P=1$. Assuming this, and repeating the boundary conditions:

$$Z(z_{\text{max}}) = Z(0) = 0 \quad (7.29)$$

it is tried to determine the sign of the eigenvalues.

Looking at the boundary conditions (7.29) it can be seen that, for any pair of functions $f(z)$ and $g(z)$ both of which satisfy both boundary conditions, it can be written:

$$f(z)g(z) - f(z)g'(z)\bigg|_{z=0}^{z_{\text{max}}} = 0 \quad (7.30)$$

This means that the boundary conditions are symmetric. Furthermore, theorem 3 on page 120 of [21] states that, if the boundary conditions are symmetric and if

$$f(z) f'(z) \bigg|_{z=0}^{z_{\text{max}}} \leq 0 \quad (7.31)$$

for all real functions $f(z)$ satisfying the boundary conditions, that there is no negative eigenvalue. It can be seen that equation (7.31) is satisfied by the boundary conditions. This means that, for this problem, all $\lambda_n$ are positive. Consequently, all $\mu_n$ of equation (7.25) have to be positive also.

At this moment the function $T(t)$ is analyzed.

**7.5. The function $T(t)$**

From equation (7.5) it follows that:

$$-\mu T + T' = 0 \quad (7.32)$$

Assume that:

$$T(t) = e^{\lambda t} \quad (7.33)$$

Substituting (7.33) into (7.32) yields:

$$\mu = \lambda \quad (7.34)$$
From this it follows that:

\[ T(t) = C_3 \cdot e^{\mu t} \quad (7.35) \]

It can be seen that a positive \( \mu \) causes instability for larger values of \( t \). Former results, however, show stable solutions. This somewhat unexpected result will probably be due to the assumption \( P=1 \), see equation (7.28). The function \( P(k,x) \) represents diffusivity in the system. Choosing \( P=1 \) eliminates the second term in equation (7.24). Without this term, an unstable equation is obtained.

### 7.6. Conclusions

Some conclusions on this chapter are stated below.

1. In principle it is possible to find an analytical solution of the kinematic equation (5.25), derived in chapter 5. However, the solution will be an infinite power series. The accuracy of this type of solutions depends on the number of higher order terms that is included in the solution. Therefore, still a computer is needed to obtain an accurate solution, which makes the solution partly a numerical one. It is remarked that it will be quite difficult to determine this solution, as the non-constant coefficients in the kinematic equation need to be described by power series as well. Otherwise the variable coefficients cannot be used in the derivations.

2. Based on the above it has been concluded that this kind of analytical analysis, which is complicated and at the same time very sophisticated, will not be applied in this project. There are two reasons that may justify this conclusion. First, in the context of this study, a more extended analytical analysis would not "fit-in", as in this report the morphologic changes are most important. Second, it is reminded that in deriving the kinematic equation, and in fitting the functions \( F_2 \) and \( F_3 \) which form the basis of the variable coefficients, inaccuracies were introduced. Therefore it seems that, in case a sophisticated analytical analysis is to be applied, it should have been applied at an earlier stage.

3. Although this chapter does not provide results in terms of solutions, it shows the possibilities of an analytical approach. The advantages of a numerical approach instead of a numerical one are obviously the relative simplicity and the ability of recognising underlying physics. Still, once an accurate analytical solution is obtained, this solution is to be preferred above numerical results. Therefore, further analytical analysis is recommended. Thereby it is preferable to analyze the kinematic equation derived in chapter 4 instead of the kinematic equation derived in chapter 5.

In the next chapter, a numerical approach is presented.
Chapter 8. A numerical computer program using $F_2$ and $F_3$

8.1. Introduction

In section 5.8 a third way of analysing the kinematic equation is proposed: by implementation in a computer program. In this chapter, it is chosen to implement the original kinematic equation, equation (4.12), into a computer program. This should yield the same results as the analytical results of the adapted kinematic equation (5.25) obtained in chapter 6 and 7.

In section 8.2, the equations that form the basis of the numerical computer program and the discretisation of these equations are presented. The boundary conditions and initial conditions, as well as the stability of the numerical scheme, are discussed in section 8.3. In section 8.4 the output of the computer program is discussed for three different calculations. Conclusions are included in section 8.6.

The computer program is listed in annex E.

8.2. The set of equations

Only two differential equations are needed to describe profile changes in time. The first equation describes the sediment transport as a function of the non-dimensional profile slope $d k_6 z / d k_0 y$. The function yields, in non-dimensional form:

$$ S_{y,d}(k_0 z) = F_{2,d}(k_0 z) - F_{3,d}(k_0 z) \cdot \frac{d(k_0 z)}{d(k_0 y)} $$

(8.1)

where:

$$ S_{y,d}(k_0 z) = \frac{S_y(k_0 z)}{g^2 T^3} $$

(8.2)

$$ F_{2,d}(k_0 z) = \frac{F_2(k_0 z)}{g^2 T^3} $$

and

(8.3)

$$ F_{3,d}(k_0 z) = \frac{F_3(k_0 z)}{g^2 T^3} $$

(8.4)

The functions $F_2(k_0 z)$ and $F_3(k_0 z)$ were calculated for an equilibrium profile, choosing an incident wave condition of 5 seconds, a wave steepness $s_0$ of 0.02 and a non-dimensional fall velocity $W^*$ of 1.0. The results were fitted, see chapter 5 section 5.7.

The functions are given by:

$$ \frac{F_2}{g^2 T^3} = -D' \cdot (1.647 - k_0 z) \cdot e^{-E(1.647 - k_0 z)} $$

(5.32 repeated)

where

$$ D' = 2.65 \cdot 10^{-6} $$

(5.33 repeated)
\[ E = 10.72 \]  

and

\[ \frac{F_3}{g^2 T^3} = B' (k_o z)^C \]  

where:

\[ B' = 1 \times 10^{-7} \] and

\[ C = 5.5. \]

The constants are all non-dimensional.

The second equation used in the numerical computer program is the continuity equation for sediment, rewritten in non-dimensional form. The equation yields:

\[
\frac{\partial (S_{y,ax}(k_o z))}{\partial k_o y} + \frac{1}{16} \frac{\partial (k_o z)}{\partial (t/T)} = 0.
\]  

Equation (8.1) and (8.5) will be used to calculate the profile development. Equation (8.1) will be used together with equations (5.32) and (5.36).

**Discretisation**

First equation (8.1) is discretised. Using a "staggered grid" molecule, the sediment transport is calculated as follows:

\[
S_{y,ax[i]} = \frac{1}{2} (F_{2[i]} + F_{2[i+1]}) - \frac{1}{2} (F_{3[i]} + F_{3[i+1]}) \cdot \left[ \frac{k_o z_{i+1} - k_o z_i}{\Delta k_o y} \right]
\]  

and

\[
S_{y,ax[i]} = \frac{1}{2} (F_{2[i-1]} + F_{2[i]}) - \frac{1}{2} (F_{3[i-1]} + F_{3[i]}) \cdot \left[ \frac{k_o z_{i} - k_o z_{i-1}}{\Delta k_o y} \right].
\]

With this, the sediment transport gradient at location \( i \) becomes:

\[
\frac{\partial (S_{y,ax[i]})}{\partial k_o y} = \frac{S_{y,ax[i]} - S_{y,ax[i]}}{\Delta k_o y}
\]  

The calculation molecule is given in Figure 8.1.

Furthermore, equation (8.5) becomes, using Euler's explicit rule:

\[
\frac{S_{y,ax[i]} - S_{y,ax[i+1]}}{\Delta k_o y} + \frac{1}{16} \frac{k_o z_{ax[i]} - k_o z_{ax[i+1]}}{\Delta (t/T)} = 0.
\]

These discretisations are used to develop a numerical computer program.
8.3. Boundary conditions, profile conditions and stability

Boundary conditions

At the upper boundary of the profile, the profile height is assumed to remain constant. This means that the equilibrium slope is "hung-up" at this point. From this point downwards, the equilibrium profile is determined by the equilibrium slope.

At the lower boundary a constant profile height is enforced as well. This means that the gradients in sediment transport equal to zero. It is remarked that still sediment transport through the upper and the lower boundary is possible. The lower boundary is less important than the upper boundary, as there is very little activity further offshore.

Profile conditions

Three kinds of profile conditions are investigated.

1. The equilibrium profile. Choosing the equilibrium profile as initial condition should result in sediment transports equal to zero, as this is the definition of the equilibrium profile shape. This is used to check the stability of the numerical program. It is expected, nevertheless, that small sediment transports will occur, as the equilibrium profile used as initial condition is a fitted profile, which does not precisely equal the equilibrium shape.

2. A harmonic boundary disturbance. This provides a link with chapter 4. In this chapter a harmonic disturbance of the profile height in time at the upper boundary was calculated using $F_3$ only. The numerical computer program offers the possibility of repeating this calculation with the influence of $F_2$ as well.

3. An initially distorted profile. A harmonic initial disturbance with different length scales is used to analyze the sensitivity of the profile for different disturbance shapes. Furthermore, the reaction in different profile locations for a disturbance can be investigated.
It is remarked that foreshore supplies can be interpreted as initial disturbances of the equilibrium profile.

Stability

Convergence implies that the global error, or the difference between the numerical solution and the analytical solution of the partial differential equation, tends to zero if the grid parameters \( \Delta y \) and \( \Delta t \) tend to zero. Furthermore, Lax's equivalence theorem states that stability is a necessary and sufficient condition for convergence, if the consistence condition is satisfied. Consistence means that the difference equation, obtained after discretisation of the differential equation, is exactly equivalent to this differential equation, at each point, in the limiting case that \( \Delta t \to 0 \).

In the following, only the stability of the numerical scheme is analyzed, as the stability analysis requires the assumption that \( F_2 \) and \( F_3 \) are constants. It is clear that this makes consistence impossible. Therefore, consistence and convergence will not be analyzed.

Before the stability condition can be determined, at first all expressions need to be combined to one expression. In the next analysis, for brevity the subscripts \( dl \) will be omitted.

After substitution of equation (8.6) and (8.7) into equation (8.8) it follows that:

\[
S^+_0 - S^-_0 = \frac{1}{2} (F_{2[i+1]} - F_{2[i-1]}) - \frac{1}{2 \cdot d k_y} \left\{ (F_{3[i+1]} + F_{3[i]}) \cdot k_y z_{[n,i-1]} - (F_{3[i-1]} + 2 F_{3[i]} + F_{3[i+1]}) \cdot k_y z_{[n,i]} + (F_{3[i]} + F_{3[i+1]}) \cdot k_y z_{[n,i+1]} \right\}
\]  

Again the problem of variable coefficients arises. The coefficients containing \( F_2 \) and \( F_3 \) in front of \( k_y z \) have index \( i \), which means that these coefficients vary in space. However, in case of this stability analysis which concentrates on the numerical scheme and not on solutions of the equations, it seems allowed to assume that the coefficients are constants. It is expected that this does not affect the stability analysis of the implemented numerical scheme.

Choosing:

\[
F_{2[i+1]} = F_{2[i-1]} = A, \quad \text{and} \quad F_{3[i+1]} = F_{3[i]} = F_{3[i-1]} = B
\]

it follows for (8.10) that:

\[
S^+_0 - S^-_0 = - \frac{B}{\Delta k_y} \cdot (k_y z_{[n,i-1]} - 2 \cdot k_y z_{[n,i]} + k_y z_{[n,i+1]})
\]

Substituting this into the continuity equation (8.9) yields:

\[
- \frac{B}{\Delta k_y^2} \cdot (k_y z_{[n,i-1]} - 2 \cdot k_y z_{[n,i]} + k_y z_{[n,i+1]}) + \frac{1}{16 \pi} \cdot \frac{k_y z_{[n,i+1]} - k_y z_{[n,i-1]}}{\Delta (t/T)} = 0.
\]
This equation is rewritten to:

\[ k_{\varphi[i]}^{n+1} - k_{\varphi[i]}^{n} = \frac{B \cdot 16 \pi^4 \cdot \Delta(t/T_y)}{\Delta k_y^2} \cdot (k_{\varphi[i-1]}^{n} - 2 \cdot k_{\varphi[i]}^{n} + k_{\varphi[i+1]}^{n}) \] 

Defining:

\[ K = \frac{B \cdot 16 \pi^4 \cdot \Delta(t/T)}{\Delta k_y^2} \] 

equation (8.15) becomes:

\[ k_{\varphi[i]}^{n+1} - k_{\varphi[i]}^{n} = K \cdot (k_{\varphi[i-1]}^{n} - 2 \cdot k_{\varphi[i]}^{n} + k_{\varphi[i+1]}^{n}) \] 

This equation looks the same as the equation analyzed in Stelling and Booij [19], page 55. Substituting an harmonic initial disturbance:

\[ \varphi(x) = \sum_{j=-\infty}^{\infty} \text{ampl} \cdot e^{i \frac{2\pi}{L_d} x} \] 

where:

- \( \text{ampl} \) = the amplitude of the harmonic disturbance, and
- \( L_d \) = the non-dimensional wave length

It is found that

\[ \text{ampl}^{n+1} = \left(1 - 2K \cdot \left(1 - \cos \left( \frac{2\pi \Delta k_y}{L_d} \right) \right) \right) \cdot \text{ampl}^n \] 

The relation \( \Delta k_y/L_d \) is the number of points per wave length, its possible minimum value is 2 while there is no maximum. From this it follows that \( 0 \leq \xi \leq \pi \), where \( \xi = \frac{2\pi}{L_d} \cdot (\Delta k_y/L_d) \). Defining:

\[ \left\{1 - 2K \cdot \left(1 - \cos \left( \frac{2\pi \Delta k_y}{L_d} \right) \right) \right\} = r \] 

it can be seen that \( r \) denotes the amplification factor of the numerical method. For stability the condition

\[ |r| \leq 1 \] 

must be satisfied. This yields:

\[ |1 - 2K \cdot (1 - \cos \xi)| \leq 1 \] 

From this it follows that

\[ 0 \leq K \leq \frac{1}{2} \] 

As \( K \) is always positive, it follows after substitution that

\[ \frac{B \cdot 16 \pi^4 \cdot \Delta(t/T)}{\Delta k_y^2} \leq \frac{1}{2} \]
or
\[
\frac{\Delta(t/T)}{\Delta k_0 y^2} \leq \frac{1}{B \cdot 16 \pi^4}
\]  \hspace{1cm} (8.25)

As \( B \) is very small \((F_3 = 10^{-6})\) this can be approximated by:
\[
\frac{\Delta(t/T)}{\Delta k_0 y^2} \leq 10^3.
\]  \hspace{1cm} (8.26)

This condition should be satisfied to prevent numerical instability to occur. It is remarked that at the upper part of the profile \( F_3 \) is largest, leading to the most strict stability condition.

8.4. Discussion on computer output

The numerical output based on the three proposed cases mentioned in section 8.3 is discussed below.
The results are plotted in appendix M.

ad 1. The equilibrium condition.

From plot M1 in appendix M, showing the profile shape in time, it can be concluded that very little change occurs. This is proof of the correct implementation of the equations in the computer program and of the correct fit of the equilibrium profile. From plot M2 in showing the sediment transports it can be seen that in the upper part of the profile sediment transports occur. This is probably due to the deviation between the equilibrium profile determined by \( F_2 \) and \( F_3 \) and the fitted curve, included in appendix J. For the first timesteps the sediment transports that occur are largest. It can be seen that the transports very quickly decrease. After 1000 timesteps, equal to 14 days with \( T_w = 5s \), there remains a general small offshore (positive) sediment transport. This indicates that the equilibrium profile will be reached after a very long time. As the sediment transports have become very small, however, it is assumed that after 1000 timesteps the equilibrium profile is reached.

Note: a non-dimensional sediment transport of \( 1 \cdot 10^{-9} \) equals a sediment transport of \( 1.2 \cdot 10^{-5} \text{m}^2/\text{s} \). This is very small indeed.

ad 2. The harmonic boundary condition.

The equilibrium profile tested in the previous case is used for this exercise as well. At the upper boundary a harmonic disturbance of the profile height is superposed on the equilibrium profile height. The harmonic disturbance reads:
\[
k_0 \zeta = k_{0,\text{equil}} + k_{n} \zeta \cdot \sin \left( \frac{2 \pi n}{n_r} \right)
\]  \hspace{1cm} (8.27)

where:
\[
\begin{align*}
n & \quad \text{number of timesteps}, \\
n_r & \quad \text{period of the harmonic disturbance expressed in timesteps and} \\
k_{n} \zeta & \quad \text{the amplitude of the disturbance.}
\end{align*}
\]
For all calculations it is chosen that $T_d = 10\text{yr}$ and that timestep $\Delta(t/T_w) = 250$. So, with $T_w = 5\text{s}$ it can be seen that a period of 10 years equals 252288 timesteps. The amplitude of the disturbance is chosen equal to 0.355 m, see equation 4.52, in order to be able to compare the results with the results in chapter 4. From this amplitude it follows, with $T_w = 5\text{s}$, that $k_d \varphi = 0.057$.

Plot M3 in appendix M shows the resulting sediment transports for different profile locations for a calculation time of 60 years, which is 6 times the period of the disturbance. It is clearly visible that, compared to the sediment transports that occur without a disturbance, the transports are very small. It must be concluded that the amplitude of the harmonic boundary condition is too small to cause significant sediment transports. The peak at $k_d y = 9$ for $0 < n < 10,000$ can be explained by the difference between the used fitted initial profile and the equilibrium profile.

Because of this, two propagating waves are visible. First a long wave caused by the peak in offshore sediment transport. This wave causes, after some time, a positive sediment transport for locations further offshore. Second, a short wave propagates in the same offshore direction through the profile. This wave, however, is only visible in the upper part of the profile. For $k_d y = 63$ the wave is hardly visible any more.

The fourth plot in appendix M (M4) shows the bottom elevation near the upper boundary for the first few timesteps due to the peak in offshore sediment transport. The fifth (M5) and sixth (M6) plot show the propagation of the large wave through the profile. It can be seen that, due to a decreasing diffusion coefficient, the wave propagation speed decreases further offshore.

A comparison with the results of chapter 4 can not yet be performed, as the initial peak in offshore transport disables a correct comparison. This problem should first be eliminated.

ad 3. The harmonic initial condition.

An initial disturbance of the equilibrium profile can be seen as a schematised supply. A supply is a sudden profile change which has nothing to do with nature. A combination of a discrete disturbance together with a harmonic initial disturbance is chosen to simulate a most natural profile configuration. Using the computer program, for many different types of supply at different locations in the profile the behaviour of the supply can be determined.

A harmonic disturbance offers the possibility to investigate at the same time whether there can be an equilibrium barred profile, supported by the functions $F_2$ and $F_3$.

Plot number 7 in appendix M shows that this is not the case. The initial disturbance used is given by:

$$k_d \varphi = \text{ampl} \cdot e^{0.035 \cdot k_d y} \cdot \sin \left( \frac{2 \cdot \pi \cdot k_d y}{k_d L_d} \right)$$  \hspace{1cm} (8.28)

where

- $\text{ampl}$ = the amplitude of the initial disturbance and
- $k_d L_d$ = the wave length of the initial disturbance.
It can be seen that the initial disturbance disappears very quickly in the first thousand timesteps. Because of the exponential decrease of the initial disturbance some sediment has been added to the equilibrium profile near the upper boundary. This causes an offshore propagating wave. Interpreting this supply of sediment as a supply, from this result it follows that the supply will completely disappear in deep water. It is remarked, however, that this is partly due to the upper boundary condition, which does not permit the upper profile height to increase.

From the eighth plot, see M8, showing the distorted profile compared to the profile after 250,000 timesteps (= 10 yr), two processes can be observed. First the initial harmonic disturbance disappears in time. At the upper part of the profile the disturbance disappears quicker because of the larger diffusion coefficient. At deeper water the disturbance disappears more slowly. The second process is the start of a propagating wave due to the amount of sediment added to the equilibrium profile.

In plot M9 sediment transport gradients are plotted, indicating erosional (negative gradient) or accretional (positive gradient) locations in the profile. Comparing this plot to the plot of some analytical results of chapter 6, plot L1, it can be seen that generally the same results are obtained. Concentrating on the first timestep of the numerical output, and keeping in mind that for \( k \eta < 20 \) the transport gradients in the numerical program will become smaller again due to a smaller initial disturbance, the results match reasonably well. By mistake unfortunately the gradients near the upper boundary were not plotted.

Plots ten, eleven and twelve (M10, M11 and M12) show some results for initial disturbances with different length scales. It was found that for every length scale initial disturbances disappear in time. An important feature can be seen in plot M11. For a profile height equal to 1.647, the fitted function of \( F_2 \) becomes zero. In terms of sediment transport this means that there exists an equilibrium profile for a horizontal profile slope. But it is not determined at which profile height the horizontal bottom should be. The equilibrium is found for any profile height, as long as the profile slope remains horizontal. This is the reason why instability occurs.

According to the theory based on \( F_2 \) and \( F_3 \), breaker bars are likely to be created at this unstable location in the profile. Afterwards the breaker bars propagate through the profile in seawards direction as demonstrated with the harmonic boundary condition.

### 8.5. Proposal for the determination of \( s_y \)

Finally an attempt is made to describe a procedure to determine the magnitude of \( s_y \) over the profile using \( F_2 \) and \( F_3 \). The definition of \( s_y \), included in chapter 3 of this report, is:

\[
S_y = s_y (W - (L2 - L1))
\]

(3.1 repeated)

where

- \((L2 - L1)\) = present distance between two contourlines
- \(W\) = equilibrium distance between the contourlines
- \(s_y\) = proportionality constant \((m/y)\)
- \(S_y\) = sediment transport in \(m^3\) per year per meter profile

The factor \(W - (L2 - L1)\) can be seen as a \( \hat{y} \), a disturbance of the equilibrium profile in \( y \) direction, see also Figure 3.1. Using this, equation (3.1) can be rewritten to:
\[ S_y = s_y \cdot y \]  

(8.29)

Rewriting \( s_y \) explicitly and in non-dimensional form, it follows that:

\[ \frac{s_y}{4 \pi^2 \cdot g T} = \frac{S_y}{g^2 T^3} \cdot \frac{1}{k_y \bar{y}} \]  

(8.30)

or

\[ s_{y,dl} = 4 \pi^2 \cdot \frac{S_{y,dl}}{k_y \bar{y}} \]  

(8.31)

where

\[ s_{y,dl} = \frac{s_y}{g T} \]

From equation (8.31) it can be seen that the value of \( s_y \) could be calculated over the profile in case in every location of the profile \( S_{y,dl} \) due to a constant \( k_y \bar{y} \) would be known. The difficulty is, however, that a constant \( k_y \bar{y} \) implies a translation of the equilibrium profile. The slope remains undisturbed. Consequently, no sediment transports occur.

The only way to find an indication of \( s_y \) using \( F_2 \) and \( F_3 \), instead of calculating \( S_{y,dl} \) for a varying \( k_y \bar{y} \) over the profile, is by introducing another definition:

\[ s_{e,dl} = 4 \pi^2 \cdot \frac{S_{e,dl}}{k_e \bar{z}} \]  

(8.32)

In case the whole profile is "lifted" by a constant \( k_e \bar{z} \), \( F_2 \) and \( F_3 \) as functions of \( k_e \bar{z} \) change while the slope remains constant. The sediment transports calculated after the first timestep, before the profile has been adapted, yield values for \( s_e \) over the profile. Furthermore, assuming that:

\[ s_e = s_y \]  

(8.33)

an indication of \( s_y \) is obtained. This could be an interesting subject for further study.

### 8.6. Conclusions

Some conclusions based on this chapter are given below.

1. A solution of a diffusion equation with variable coefficients is easier to find numerically than analytically.

2. The numerical program can be applied to far more cases than discussed in this study. Different types of supplies can be modelled, although all supplies will be located seawards of the outer breaker bar. Sediment transports and gradients in transport can be studied in detail. After improvement of the program with respect to the initial equilibrium profile and the accuracy of the fitted functions of \( F_2 \) and \( F_3 \), an harmonic boundary condition can be calculated more accurately. It is recommended to implement the program of V.d. Kerk into the present computer program in order to eliminate the extra step of
fitting the functions $F_2$ and $F_3$. This will at the same time make analyses for different 
wave and sediment parameters easier.

3. The most convenient description of the initiation and the behaviour of breaker bars 
based on $F_2$ and $F_3$ is the following. Breaker bars are created at the location where $F_2$ 
equals zero. This is the location where the offshore sediment transport due to the 
undertow equals the onshore transport due to the wave asymmetry. This location is the 
outer boundary of the analyzed profile, by V.d. Kerk referred to as the crest of the outer 
breaker bar. The increase in profile height at the upper boundary causes offshore 
sediment transport via $F_3$, the diffusion function. This sediment causes an offshore 
propagating wave, as can be seen in plot M3.

4. There is no persistent disturbed configuration of the equilibrium profile seawards of the 
outer breaker bar. All initial disturbances disappear. This means that the numerical results 
do not support the hypothesis that initial disturbances of the equilibrium profile can 
increase in time due to rapidly decreasing diffusion coefficients.
Chapter 9. Conclusions and recommendations

9.1. Introduction

In section 9.2, the main conclusions on this study are presented. It is remarked that the final section of each chapter includes separate conclusions on that chapter in particular. Furthermore, recommendations for a follow-up of this study are presented in section 9.3.

The main conclusions are better understood after reading section 1.3.

9.2. Main conclusions

General remarks

a. Investigated is the role of cross-shore diffusivity on the propagation and decay of breaker bars and the role of non-linear effects on the generation of those bars.

b. If cross-shore diffusivity would be uniform over the depth, a periodical boundary condition on the landward side of the profile leads to decaying bars, propagating in seaward direction, provided that the deepest part of the profile remains stationary. This results from well-known diffusivity theory.

c. In nature such a behaviour of breaker bars is found in the deeper part of the coastal profile. One of the subjects of the present study is to investigate, whether diffusivity on its own could be the explanation of this behaviour.

d. For this goal, first the diffusivity characteristics of the coastal profile are investigated, starting from the Bailard sediment transport theory, as elaborated by V.d.Kerk.

e. More sophisticated effects, which could lead as well to explanation of breaker bars, as effects of surf beat and edge waves, are explicitly left out of consideration.

Analytical

f. Diffusivity appears to be controlled by a proportionality constant between cross-shore transport and profile slope, which is called $F_v$. In the V.d.Kerk theory this constant depends only on deep water wave steepness $s_0$, on the dimensionless fall velocity $W_*=W\cdot T/H$ and on the dimensionless depth $k_0 h$.

Plot B5 in appendix B shows that $F_v$ is roughly inversely proportional to the seaward distance from a point, where seaward transport by undertow and landward transport caused by asymmetry of the waves, balance.

g. It is investigated, whether this fast decay of the diffusivity constant in seaward direction could lead to a kind of "shoaling" effect on the propagating bars, mentioned ad b.
This appears not to be the case. Instead of a negative-exponential decay of an harmonic wave the result is a wave according to an Airy-function. Two solutions are found, of the kind "Ai(u)" and "Bi(u)", see chapter 4.

The behaviour of the Ai-function is more or less similar to a sinewave. Whether the boundary conditions on the landward side can (or should) be such, that also the Bi-functions (increasing in seaward direction) can be important, remains a matter for further research.

h. The wave length of those Airy waves appears to be proportional to the water wave length, to which the coastal profile is exposed and to the third power root of the ratio $T_J/T$, where $T_J$ is the period of the harmonic motion and $T$ the period of the waves. Wave lengths of the order of 800 m are found for very coarse grains ($W^* = 1$). Effect of grain size and initial wave steepness has not yet been investigated.

i. Furthermore some de-stabilizing feature have been investigated:
- the rate of change of transport $F_2$, when the profile is horizontal, as function of water depth
- the rate of change of $F_3$ as function of water depth.

No definite conclusions can be drawn with respect to these aspects.

Numerical

a. The functions $F_2$ and $F_3$ form a stable equilibrium. There is no spontaneous increase in profile height at any location after a long term exposure to incident waves. This is proof of a correct implementation of the theory into the computer program as well.

b. A harmonic disturbance of the profile height at the upper boundary of the equilibrium profile as described by $F_2$ and $F_3$, where the profile slope becomes horizontal, induces an harmonic sediment transport through the upper boundary. The resulting disturbance of the equilibrium profile propagates in seawards direction which is in accordance with analytical results.

c. Every type of small initial harmonic disturbance of the equilibrium profile seawards of the outer breaker bar, which is the outer boundary of the equilibrium profile, disappears in time showing very small seawards convection. A harmonic disturbance, or another disturbance with which sediment is added to the equilibrium profile causes an offshore propagating wave. The propagation velocity decreases in offshore direction.

9.3. Recommendations

The present study can be seen as a setup for further study using the simplified sediment transport equation based on the functions $F_2$ and $F_3$. Two ways for further study have been proposed and investigated: analytical analysis and numerical analysis. The following recommendations try to favour both ways for further study.
The recommendations are:

1. Further analytical analysis on the stability of the sediment transport equation (4.12) in combination with the continuity equation. A mathematical technique was described in section 5.2.

2. Investigation of the sensitivity of the analytical and numerical results for other wave and sediment parameters $s_o$, $W$, $H$ and $T$.

3. Search for an indication of the values for $s_y$ using the numerical computer program based on the functions $F_2$ and $F_3$. A proposal for this is given in section 8.5.

4. Further improvement of the numerical program. It is recommended to implement the V.d. Kerk computer program in the present program to obtain more accurate results, and to improve the usability.

5. An extended analysis of the numerical results for all kinds of boundary conditions and initial conditions, for different wave and sediment parameters.
SYMBOLS

All symbols were defined at the moment of introduction as well. Local variables are not included in this list.

\[ A, B, C, D, E \] coefficients originating from fitting the functions \( F_2 \) and \( F_3 \) [-]
\[ c \] phase velocity of the waves [m/s]
\[ C_r \] shear coefficient [-]
\[ D \] diffusion coefficient [m²/s]
\[ D_{50} \] mean size of sediment particles [m]
\[ f_L \] function that defines the equilibrium profile [-]
\[ F_1 \] function describing the influence of wave asymmetry and undertow on sediment transport [m²/s]
\[ F_2, F_2(x) \] sediment transport by wave asymmetry and undertow for the equilibrium profile [m²/s]
\[ F_3, F_3(x) \] change in sediment transport by wave asymmetry and undertow in case of a disturbance of the equilibrium profile [m²/s]
\[ F_3 \] function describing the influence of the gravity force on sediment transport [m²/s]
\[ F_3, F_3(x) \] sediment transport by the gravity force for an equilibrium profile [m²/s]
\[ F_3, F_3(x) \] change in sediment transport by the gravity force in case of a disturbance of the equilibrium profile [m²/s]
\[ g \] gravitative acceleration [m/s²]
\[ h \] water depth [m]
\[ h_0 \] deep water depth [m]
\[ H_b \] wave height at bar crest [m]
\[ H_f \] fictive wave height [m]
\[ H_{rms} \] root-mean-square wave height [m]
\[ H_s \] significant wave height [m]
\[ H^* \] non-dimensional wave height [-]
\[ <i> \] time averaged total load sediment transport [m³/s]
\[ <i_b> \] time averaged bottom transport [m³/s]
\[ <i_s> \] time averaged sediment transport [m³/s]
\[ k_0 \] deep water wave number [1/m]
\[ L_0 \] deep water wave length [m]
\[ L_2 - L_1 \] distance between two contourlines in line modelling [m]
\[ N \] local volume concentration of solids [-]
\[ O_1 \] amount of supplied sand on the beach [m³]
\[ O_2 \] amount of supplied sand on the foreshore [m³]
\[ p \] pressure [N/m²]
\[ p_D \] dynamic pressure [N/m²]
\[ P(k,x) \] function describing a variable coefficient in the kinematic equation (chapter 5) [-]
\[ q \] sediment flux [m³/s]
\[ <q_{as}> \] time averaged sediment transport due to wave asymmetry [m³/s]
\[ <q_{un}> \] time averaged sediment transport due to the undertow [m³/s]
\[ <q_{sl}> \] time averaged sediment transport due to profile slope and gravity force [m³/s]
$$Q_b$$  fraction of breaking waves  
$$Q(k, \omega)$$  function describing a variable coefficient in the kinematic equation  
(chapter 5)  
$$R$$  difference between radiation stress and setup pressure gradient  
[N/m$^2$]  
$$s_0$$  deep water wave steepness  
[-]  
$$s_x, s_y$$  coastal constants resp. for the upper and the lower profile boundary  
[m/s]  
$$S_x$$  radiation stress  
[N/m$^2$]  
$$R(k, \omega)$$  function describing the amplitude of a disturbance (chapter 6)  
[-]  
$$s_e$$  coastal constant  
[m/s]  
$$S_x$$  cross-shore sediment transport  
[m$^2$/s]  
$$S_y$$  sediment transport for equilibrium profile (=0)  
[m$^2$/s]  
$$S_z$$  sediment transport due to a disturbance  
[m$^2$/s]  
$$S(k, \omega)$$  function describing the amplitude of a disturbance (chapter 6)  
[-]  
$$t$$  continues time  
[s]  
$$T$$  period of incoming short waves  
[s]  
$$T_b$$  period of a breaker bar  
[s]  
$$\tan \beta$$  slope of the profile  
[-]  
$$\tan \beta_e, \frac{\partial \bar{z}}{\partial y}$$  equilibrium slope of the profile  
[-]  
$$T_e, T_b$$  time constants resp. for the upper and the lower D-profile boundary  
[s]  
$$T_o$$  time constant for sediment motion in D-profile  
[s]  
$$\bar{u}$$  horizontal variable velocity  
[m/s]  
$$u_r$$  undertow  
[m/s]  
$$u_e, Re(u)$$  scale parameter of diffusion problem, chapter 4  
[-]  
$$u_n$$  undertow contribution for breaking waves  
[m/s]  
$$u_n$$  undertow contribution for non-breaking waves  
[m/s]  
$$u_m$$  amplitude of the orbital velocity near the bottom  
[m/s]  
$$w_s$$  sediment fall velocity in water  
[m/s]  
$$\bar{w}$$  vertical variable velocity  
[m/s]  
$$W$$  sediment fall velocity in water  
[m/s]  
$$W_e$$  equilibrium distance between two contourlines in line modelling  
[m]  
$$W^*$$  non-dimensional fall velocity of sediment in water  
[-]  
$$y$$  horizontal coordinate perpendicular to the coastline  
[m]  
$$y$$  amplitude of disturbances in horizontal direction  
[m]  
$$z$$  profile height  
[m]  
$$\bar{z}$$  equilibrium profile height  
[m]  
$$\bar{z}$$  amplitude of disturbances in vertical direction  
[m]  
$$\bar{z}_A, \bar{z}_B$$  solutions for amplitude of disturbance in vertical direction  
[m]  
$$\frac{\partial \bar{z}}{\partial y}$$  angle between the slope of the disturbance and the equilibrium slope  
[rad]  
$$Z$$  amplitude of a breaker bar  
[m]  
$$Z_A, Z_B$$  solutions for the amplitude of a breaker bar  
[m]  
$$Z_n$$  non-dimensional vertical coordinate in the bottom layer  
[m]  
$$Z_m$$  non-dimensional vertical coordinate in the middle layer (between bottom layer and wave trough level)  
[m]  
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<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Units</th>
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<tr>
<td>$\beta$</td>
<td>beach slope</td>
<td>[-]</td>
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<tr>
<td>$\phi$</td>
<td>internal shear angle of the sediment</td>
<td>[-]</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>breaking index</td>
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<td>$\lambda_n$</td>
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<tr>
<td>$\mu$</td>
<td>separation constant (chapter 7)</td>
<td>[-]</td>
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<tr>
<td>$\mu_n$</td>
<td>eigenvalues (chapter 7)</td>
<td>[-]</td>
</tr>
<tr>
<td>$v_i$</td>
<td>turbulent diffusivity coefficient</td>
<td>[m$^2$/s]</td>
</tr>
<tr>
<td>$\rho$</td>
<td>density of water</td>
<td>[kg/m$^3$]</td>
</tr>
<tr>
<td>$\tau_b$</td>
<td>mean bottom shear stress</td>
<td>[m$^2$/s]</td>
</tr>
<tr>
<td>$\tau(\mu_n)$</td>
<td>resulting shear stress at wave trough level</td>
<td>[N/m$^2$]</td>
</tr>
<tr>
<td>$\xi$</td>
<td>setup</td>
<td>[m]</td>
</tr>
<tr>
<td>$\omega_x, \omega$</td>
<td>angular velocity of a breaker bar</td>
<td>[rad/s]</td>
</tr>
<tr>
<td>$\Omega$</td>
<td>beach stage parameter</td>
<td>[-]</td>
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</tbody>
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REFERENCES


1. Design Report of the three Nourtec test sites.


ANNEX A

Derivation of expressions for $F_2$ and $F_3$
ANNEX A.

Derivations of expressions for $F_2$ and $F_3$

The starting point for the derivation of an expression for $F_2$ and $F_3$ in

$$S_y = F_2 - F_3 \left( \frac{\partial z}{\partial y} \right)$$  \hspace{1cm} (A.1)

is the Bailard formula (1981), reduced to the cross-shore section and treating the time averaged total load sediment transport (Stive 1986):

$$<i> = <i_b> + <i_v> =$$

$$= \frac{\rho C_f e_b}{\tan \phi} \left[ <|u|^2 u> - \left( \frac{\tan \beta}{\tan \phi} \right) <|u|^3> \right] +$$

$$+ \frac{\rho C_f e_s}{W} \left[ (<|u|^2 u> - \left( \frac{e_i}{W} \right) \tan \beta <|u|^3>) \right]$$  \hspace{1cm} (A.2)

where:

$u = \bar{u} + \bar{u}$ where $\bar{u}$ is the time averaged current velocity near the bottom and $\bar{u}$ is the variable component of the velocity,

$<i>$ = the total time-averaged "immersed weight" cross-shore transport,

$<i_b>$ = suspended transport,

$<i_v>$ = bottom transport,

$\rho$ = density of water,

$C_f$ = shear coefficient equal to 0.5 $f_w$, with $f_w$ defined by Swart (1974),

$\phi$ = internal shear angle of the sediment,

$\tan \beta$ = slope of the profile,

$W$ = fall velocity of the sediment in water.

$e_b$ and $e_s$ are efficiency factors, which indicate the amount of wave energy that is used for sediment transport.

Using

$$S_y = \frac{i}{(\rho_z - \rho)gN}, \quad \frac{\rho}{\rho_z - \rho} = \frac{1}{\Delta} \quad \text{and} \quad \tan \beta_{\text{Bailard}} = \tan \beta_{\text{Vd.Kerk}} = \frac{\partial z}{\partial y},$$

and writing the terms containing $\left( \frac{\partial z}{\partial y} \right)$ separately, the Bailard formula reads:

$$<S_y> = \frac{C_f}{N \Delta g} \left[ \left( \frac{e_b}{\tan \phi} \right) <|u|^2 u> + \left( \frac{e_i}{W} \right) <|u|^3> \right] -$$

$$- \left( \frac{\partial z}{\partial y} \right) \frac{C_f}{N \Delta g} \left[ \left( \frac{e_b}{\tan^2 \phi} \right) <|u|^3> + \left( \frac{e_i}{W} \right)^2 <|u|^5> \right].$$  \hspace{1cm} (A.3)
From this, the expressions for $F_2$ and $F_3$ yield:

$$F_2 = \frac{C_f}{N\Delta g} \left[ \left( \frac{\varepsilon_h}{\tan \phi} \right) < |u|^2 u > + \left( \frac{\varepsilon_s}{W} \right) < |u|^3 > \right] \quad (A.4)$$

$$F_3 = \frac{C_f}{N\Delta g} \left[ \left( \frac{\varepsilon_b}{\tan^2 \phi} \right) < |u|^3 > + \left( \frac{\varepsilon_m^2}{W} \right) < |u|^5 > \right] \quad (A.5)$$

In the following, these expressions will be rewritten according to the Stive (1986) notation.

First the total velocity $u$ is separated into a near bottom mean velocity and an oscillatory velocity:

$$u = \bar{u} + \tilde{u},$$

and furthermore it is assumed that

$$\tilde{u} = u_m \cos \omega t + u_{2m} \cos 2\omega t + ...,$$  \hspace{2cm} (A.6)

in which $u_m > u_{2m} > ...$.

Stive (1986) defines the following dimensionless parameters:

$$\delta_n = \bar{u}/u_m,$$ the relative current strength,  \hspace{2cm} (A.8)

$$\psi_1 = \frac{< \tilde{u}^3 >}{u_m^3} \quad \text{and} \quad \psi_2 = \frac{< |u|^3 \tilde{u} >}{u_m^4} ,$$  \hspace{2cm} (A.9) and (A.10)

representing non dimensional velocity moments, and

$$(u_3)^* = \frac{< |u|^3 >}{u_m^3} \quad \text{and} \quad (u_5)^* = \frac{< |u|^5 >}{u_m^5} \quad (A.11) \quad \text{and} \quad (A.12)$$

representing even velocity moments.

Using these notations, and neglecting higher order terms of $\tilde{u}$ respective of $\bar{u}$, Stive derives the following expressions:

$$< |u|^2 u > = u_m^3 (\psi_1 + \frac{3}{2} \delta_n),$$  \hspace{2cm} (A.13)

$$< |u|^3 u > = u_m^4 (\psi_2 + \delta_n (u_3)^*) ,$$  \hspace{2cm} (A.14)

$$< |u|^5 > = u_m^3 (u_3)^* \quad \text{and} \quad (A.15)$$
\[ <|u|^2> = u_m^5(u5)^* \] \hspace{1cm} (A.16)

Substituting this into (A.4) and (A.5) yields:

\[ F_2 = \frac{c_f}{N\Delta g} \left\{ \left( \frac{\epsilon_b}{\tan \phi} \right) (u_m^3(\psi_1 + \frac{3}{2} \delta_u)) + \left( \frac{\epsilon_l}{W} \right) (u_m^4(\psi_2 + \delta_u(u3)^*)) \right\} \] \hspace{1cm} (A.17)

\[ F_3 = \frac{c_f}{N\Delta g} \left\{ \left( \frac{\epsilon_b}{\tan^2 \phi} \right) (u_m^3(u3)^*) + \left( \frac{\epsilon_l}{W} \right)^2 (u_m^5(u5)^*) \right\} \] \hspace{1cm} (A.18)

In these equations, the following products can be simplified further:

\[ u_m^4(\psi_2) = <|\bar{u}|^3 \bar{u}> + 3 \bar{u} <|\bar{u}|^3>, \] \hspace{1cm} (A.19)

\[ u_m^3(u3)^* = <|\bar{u}|^3> + 3 \bar{u} <|\bar{u}| \bar{u}> \quad \text{and} \] \hspace{1cm} (A.20)

\[ u_m^5(u5)^* = <|\bar{u}|^5> + 5 \bar{u} <|\bar{u}|^3 \bar{u}> . \] \hspace{1cm} (A.21)

From this result it follows, that the important oscillatory velocity moments are:

the four even terms \( <\bar{u}^2>, <|\bar{u}|^3>, <\bar{u}^4>, <|\bar{u}|^5> \)

and two odd terms: \( <\bar{u}^2>, <|\bar{u}|^3 \bar{u}> \).

The odd terms are the most difficult to determine, as they are non-zero in case of asymmetric waves, like the waves in the coastal zone.

First the even terms will be treated, and second the odd terms.

**Even terms.**

Guza and Thornton (1985) give an expression based on a random, linear sea (Gaussian model). Their results indicate that the even moments do not critically depend on the asymmetry of the cross shore velocity. They write:

\[ |\bar{u}|^3 = 1.60 <\bar{u}^2>^{3/2} \] \hspace{1cm} (A.22a)

\[ |\bar{u}|^5 = 6.38 <\bar{u}^2>^{5/2} \] \hspace{1cm} (A.22b)

with \( <\bar{u}^2>^2 = s_u^2 \), where \( s_u^2 \) is the variance of the Gaussian distributions of \( \bar{u} \).

Stive (1986) gives for \( s_u \), applying the linear theory:

\[ s_u = u_{rms} = \frac{\pi H_{rms}}{T \sinh(kh)}, \] \hspace{1cm} (A.23)
which yields non-dimensional:
\[
\frac{u_{ms}}{gT} = \frac{1}{4\pi} \frac{H_{ms}}{h} \frac{k_0 h}{\sinh(kh)} , \quad \text{where} \quad k_0 = \frac{(2\pi)^2}{gT^2} .
\] (A.24)

Odd terms.

Stive (1986) assumes that
\[
\bar{u} = u_m \cos \omega t + u_{2m} \cos (2\omega t + \varphi_2)
\] (A.25)

where \( \varphi \neq 0 \) is the phase difference, and \( u_m > u_{2m} \).

After some algebra, Stive derives:
\[
\langle \bar{u}^3 \rangle = \frac{3}{4} u_m^2 u_{2m} \cos \varphi_2 , \quad \text{and}
\]
\[
\langle |\bar{u}|^3 \bar{u} \rangle = \frac{12}{5\pi} u_m^3 u_{2m} \cos \varphi_2 .
\] (A.26) (A.27)

Bowen (1981), however, states that
\[
\langle |\bar{u}|^3 \bar{u} \rangle = \frac{16}{5\pi} u_m^3 u_{2m} \cos \varphi_2 ,
\] (A.28)

which expression is also used by Stive in later publications. So the latter expression will be worked with.

Flick et al. (1981) point out that, from the breaker line to the coast, the relative phase of the second harmonic increases asymptotically towards the value \( \pi/2 \). So, in case of breaking waves, the odd terms vanish approaching the coast (see A.26) and (A.28).

This means that the odd moments become interesting for non-breaking waves, assuming that the addition of breaking waves can be neglected. In the following, the fraction of non-breaking waves is equal to \( (1 - Q_b) \), where \( Q_b \) is the fraction of breaking waves.

The conclusion is that the odd terms need only be calculated from the fraction of non-breaking waves. At this point, instead of (A.25), a non-linear approach is applied, using a second order Stokes estimation with \( \varphi_2 = 0 \):
\[
\bar{u} = u_m \cos(\omega t) + \frac{3}{4} u_m^2 \cos(2\omega t) \frac{\sinh(kh)}{\sinh^2(kh)} .
\] (A.29)

From this follows the relation between \( u_m \) and \( u_{2m} \):
\[
u_{2m} = \frac{3}{4} \frac{u_m^2}{C} \sinh^2(kh) .
\] (A.30)

Substitution of (A.30) into (A.26) and (A.28) yields:
\[
\langle \bar{u}^3 \rangle = \left( \frac{3}{4} \right)^3 \frac{u_m^4}{C} \frac{\cos \varphi_2}{\sinh^2(kh)}
\] (A.31)
\[ < |\tilde{u}^3|\tilde{u}| > = \left( \frac{12}{5\pi} \right) \frac{u_m^5 \cos \varphi_2}{C \sinh^2(kh)} \]  

(A.32)

In case of non-breaking waves, this becomes:

\[ < \tilde{u}^3 > = (1 - Q_b) \left( \frac{3}{4} \right) \frac{u_m^4 \cos \varphi_2}{C \sinh^2(kh)} \]  

(A.33)

\[ < |\tilde{u}^3|\tilde{u}| > = (1 - Q_b) \left( \frac{12}{5\pi} \right) \frac{u_m^5 \cos \varphi_2}{C \sinh^2(kh)} \]  

(A.34)

The results up to this point are summarised below:

* The expressions for \(F_2\) and \(F_3\):

\[
F_2 = \frac{c_f}{N\Delta g} \left\{ \left( \frac{\epsilon_b}{\tan \phi} \right) (u_m^3(\psi_1 + \frac{3}{2} \delta_2)) + \left( \frac{\epsilon_s}{W} \right) (u_m^4(\psi_2 + \delta_4(u3)^*)) \right\} \]  

(A.17 repeated)

\[
F_3 = \frac{c_f}{N\Delta g} \left\{ \left( \frac{\epsilon_b}{\tan \phi} \right) (u_m^5(u3)^*) + \left( \frac{\epsilon_s}{W} \right)^2 (u_m^5(u5)^*) \right\} . \]  

(A.18 repeated)

* The simplified products:

\[
u_m^4 \psi_2 = < |\tilde{u}|^3\tilde{u}| > + 3 \tilde{u} < |\tilde{u}|^3 > , \]  

(A.19 repeated)

\[
u_m^3(u3)^* = < |\tilde{u}|^3 > + 3 \tilde{u} < |\tilde{u}|\tilde{u}| > , \]  

(A.20 repeated)

\[
u_m^5(u5)^* = < |\tilde{u}|^5 > + 5 \tilde{u} < |\tilde{u}|^3\tilde{u}| > , \]  

(A.21 repeated)

\[
u_m^3 \psi_1 = < \tilde{u}^3 > \text{ (can not be sympified further).} \]  

(A.35)

An estimation on the term \(< |\tilde{u}|\tilde{u}| >\) in (A.20) is not presented in the literature. Because of this, (A.20) is rewritten according to Bailard (1981):

\[
u_m^3(u3)^* = < |\tilde{u}|^3 > \]  

(A.36)

* The even velocity moments:

\[ |\tilde{u}|^3 = 1.60 < \tilde{u}^2 >^{3/2} , \]  

(A.22* repeated)

\[ |\tilde{u}|^5 = 6.38 < \tilde{u}^2 >^{5/2} , \]  

(A.22b repeated)

in which \(< \tilde{u}^2 >\) is further specified assuming a linear Gaussian sea.

* The odd velocity moments:
\[
\langle \dot{u}^3 \rangle = (1 - Q_b) \left( \frac{3}{4} \right)^2 \frac{\mu_m^4 \cos \phi_2}{C \sinh^2(kh)} \]  
(A.33 repeated)

\[
\langle |\dot{u}|^3 \rangle = (1 - Q_b) \left( \frac{12}{5 \pi} \right) \frac{\mu_m^5 \cos \phi_2}{C \sinh^2(kh)} \]  
(A.34 repeated)

in which \( \mu_m = \mu_{rms} \) is estimated non-linear monochromatic.

The results summarised above can be made more transparent by using the notation of Stive.

Stive (1986) presents the following notations:

\[
F_B = \frac{C \mu_{rms}^3 \varepsilon_B}{\Delta B N \tan \phi} \quad \text{and} \quad (A.37)
\]

\[
F_s = \frac{C \mu_{rms}^4 \varepsilon_s}{\Delta B N W}, \quad (A.38)
\]

where \( \mu_{rms} = \mu_m \).

Applying this, (A.17) and (A.18) become:

\[
F_2 = F_B \left( \psi_1 + \frac{3}{2} \delta_u \right) + F_s \left( \psi_2 + \delta_u (u3)^* \right) \quad (A.39)
\]

\[
F_3 = F_B \left( \frac{(u3)^*}{\tan \phi} \right) + F_s \frac{\varepsilon_s (u5)^* u_m}{W} \quad (A.40)
\]

Using the following notations (Stive 1986):

\[
\langle q_{as} \rangle = F_B \psi_1 + F_s \psi_2 , \quad (A.41)
\]

\[
\langle q_{an} \rangle = F_B \frac{3}{2} \delta_u + F_s \delta_u (u3)^* \quad \text{and} \quad (A.42)
\]

\[
\langle q_{sl} \rangle = F_B \frac{\tan \beta}{\tan \phi} (u3)^* + F_s \frac{\varepsilon_s u_m}{W} \tan \beta (u5)^* \quad (A.43)
\]

\( F_2 \) and \( F_3 \) can be rewritten as follows:

\[
F_2 = \langle q_{as} \rangle + \langle q_{an} \rangle \quad , \quad (A.44)
\]

\[
F_3 = \langle q_{sl} \rangle / \frac{\partial z}{\partial y} \quad , \quad (A.45)
\]

where \( \tan \beta_{stive} = -\frac{\partial z}{\partial y} \).

In the following, similar to the theory of v.d. Kerk, the results are made non-dimensional.

The dimensions of the terms in equation:

\[
S_2 = F_2 - F_3 \left( \frac{\partial z}{\partial y} \right) \quad (A.1 \text{ repeated})
\]

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are $\frac{m^2}{s}$. Thus, in order to obtain a non-dimensional $F_2$ and $F_3$, the equation is written as follows:

$$\frac{S_y}{g^2T^3} = \frac{F_2}{g^2T^3} - \frac{F_3}{g^2T^3}\left(\frac{\partial z}{\partial y}\right)$$  \hspace{1cm} (A.46)

Consequently dividing the sequential parts that $F_2$ and $F_3$ consist of, yields the following:

$$(F_B)_{\text{dimless}} = \frac{C_f u_{\text{rms}}^2 e_B}{\Delta g N \tan \phi} \left(\frac{1}{g^2 T^3}\right) = \frac{C_f e_B u_{\text{rms}}^3}{\Delta N \tan \phi}$$

and

$$(F_S)_{\text{dimless}} = \frac{C_f u_{\text{rms}}^4 e_S}{g N W} \left(\frac{1}{g^2 T^3}\right) = \frac{C_f e_S u_{\text{rms}}^4}{\Delta N \left(\frac{g^3 T^3 W}{W}\right)}$$

$$= \frac{C_f e_S u_{\text{rms}}^4 g T}{\Delta N} \left(\frac{g T^2}{(2\pi)^2}\right) \frac{(2\pi)^2}{WT}$$

$$= \frac{C_f e_S u_{\text{rms}}^4}{\Delta N} \left(\frac{1}{k_0}\right) \frac{(2\pi)^2}{WT}$$

$$= \frac{C_f e_S u_{\text{rms}}^4}{\Delta N} \left(\frac{H_{rms}}{H_{rms}}\right) \frac{(2\pi)^2}{H_{rms}} \frac{h}{H_{rms}}$$

\hspace{1cm} (A.47)

\hspace{1cm} (A.48)

The last equation is now written as a function of $k_0 h$, which is the non-dimensional parameter used by v.d.Kerk in his computer program.

Furthermore, the equations (A.41), (A.42) and (A.43) become:

$$<q_{as\text{ dimless}} = (F_B)_{\text{dimless}} \psi_1 + (F_S)_{\text{dimless}} \psi_2 \hspace{1cm} \text{(A.49)}$$

$$<q_{un\text{ dimless}} = \frac{3}{2} \delta u + (F_S)_{\text{dimless}} \delta u' \hspace{1cm} \text{(A.50)}$$

$$<q_{st\text{ dimless}} = (F_B)_{\text{dimless}} \tan \beta \frac{u_3}{\tan \phi} + (F_S)_{\text{dimless}} \frac{u_m}{W} \epsilon_S \tan \beta \hspace{1cm} \text{(A.51)}$$

\hspace{1cm} (A.50)

\hspace{1cm} (A.51)
\[ (F_\phi)_{\text{dimless}} \frac{\tan\beta}{\tan\phi} (u_3)^* + (F_\psi)_{\text{dimless}} \frac{\varepsilon}{W} \varepsilon \tan\beta (u_5)^* \frac{u_{\text{rms}}}{g T} \frac{(2\pi)^2}{k_h} \frac{H_{\text{rms}}}{TW H_{\text{rms}}} \]

With this result, the equations (A.44) and (A.45) become:

\[(F_2)_{\text{dimless}} = <q_{as}>_{\text{dimless}} + <q_{un}>_{\text{dimless}}, \quad \text{(A.52)}\]

\[(F_3)_{\text{dimless}} = <q_{st}>_{\text{dimless}} + \frac{\partial z}{\partial y}, \quad \text{(A.53)}\]

and

\[(S_y)_{\text{dimless}} = <q_{as}>_{\text{dimless}} + <q_{un}>_{\text{dimless}} + <q_{st}>_{\text{dimless}}. \quad \text{(A.54)}\]

**Important remark:**

In the computer program of v.d. Kerk, the term \(- <q_{st}>_{\text{dimless}} / \tan\beta_{\text{Sive}}\) is called QSL. So, in the calculation of QSL, equation (A.51) was already divided by \(\tan\beta_{\text{Sive}}\), and was also multiplied with -1. Probably this was done to make sure that \(\tan\beta_{v.d.Kerk}\) has the right sign according to the definition of the slope.

Taking into account the definitions used in this report, the following yields for QSL:

\[QSL = - \frac{<q_{st}>_{\text{dimless}}}{\tan\beta_{\text{Sive}}}\]

\[= - \frac{<q_{st}>_{\text{dimless}} - \frac{\partial z}{\partial y}}{- \frac{\partial z}{\partial y}} = -(F_3)_{\text{dimless}}.\]

So \((F_3)_{\text{dimless}} = - QSL\).

Summarising the equations, it can be stated that:

\[S_y = f \left( \frac{\partial z}{\partial y}, k_h, s_q, \frac{H_{\text{rms}}}{WT} \right).\]
ANNEX B

Intermezzo's of chapter 4
ANNEX B. Intermezzo’s belonging to chapter 4.

Intermezzo 1:

Assume

\[ z = Ze^{i\omega t} \]

This can be rewritten to:

\[ z = Z\cos(\omega t) + iZ\sin(\omega t) \]

and \( \dot{z} = \text{Re}(Ze^{i\omega t}) = Z\cos(\omega t) \)

in which \( Z \) [m] is the amplitude of the periodic behaviour of \( z \), and is a function of \( y \). \( Z(y) \) is a real function. Computing the derivatives of \( z \) yields:

\[ \frac{\partial^2 z}{\partial y^2} = \frac{d^2Z}{dy^2} e^{i\omega t} \quad \text{and} \]

\[ \frac{\partial z}{\partial t} = Ze^{i\omega t}i\omega \]

After dividing these derivatives by \( e^{i\omega t} \), they result in:

\[ \frac{\partial^2 z}{\partial y^2} = \frac{d^2Z}{dy^2} \quad \text{and} \]

\[ \frac{\partial z}{\partial t} = Zi\omega \]

Substitution of this in equation (4.17) yields equation (4.21).

Intermezzo 2:

Equation (4.24) looks similar to the relation: \( Z'' - uZ = 0 \). Of this equation the solution is known.

Replacing " \( y \) " in (4.24) by " \( \alpha y_1 \) " reformulates (4.24) as:

\[ \frac{1}{\alpha^2} \frac{d^2Z}{dy_1^2} - \frac{i\omega\alpha}{A}k_0y_1Z = 0 \Leftrightarrow \frac{d^2Z}{dy_1^2} - \frac{i\omega\alpha^3}{A}k_0y_1Z = 0 \]

Thus it shows, that " \( y_1 \) " equals " \( u \)", provided that:

\[ \frac{i\omega\alpha^3}{A}k_0 = 1 \]

and thus, \( u = y/\alpha \) equals

\[ u = \left( \frac{i\omega k_0}{A} \right)^{1/3} y \]  \hspace{1cm} (4.25 repeated)

Intermezzo 3:
This result can easily be rewritten to equation (4.28).

**Intermezzo 4:**

\[
\xi = 2 e^{i \pi} \left( \frac{\omega k_0}{A} \right)^{1/2} y^{3/2}
\]

\[
= 2 \left( \frac{\omega k_0}{A} \right)^{1/2} y^{3/2} \left\{ \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right\}
\]

\[
= 2 \left( \frac{\omega k_0}{A} \right)^{1/2} y^{3/2} \left\{ \frac{1}{2} \sqrt{2} + i \cdot \frac{1}{2} \sqrt{2} \right\}
\]

\[
= \frac{\sqrt{2}}{3} \left( \frac{\omega k_0}{A} \right)^{1/2} y^{3/2} + i \cdot \frac{\sqrt{2}}{3} \left( \frac{\omega k_0}{A} \right)^{1/2} y^{3/2}
\]

**Intermezzo 5:**

\[
u = e^{i \frac{\pi}{6}} \left( \frac{\omega k_0}{A} \right)^{1/3} y \rightarrow \arg u = \frac{1}{6} \pi
\]

The condition is true if \( \left( \frac{\omega k_0}{A} \right)^{1/3} y \geq 0 \). As \( y \), \( k_0 \) and \( \omega \) are positive, \( A \) should be positive also. This condition is fulfilled, see (4.22). If \( A < 0 \) then \( \arg u = \frac{5}{6} \pi \).

**Intermezzo 6:**

\[
Z_A e^{i \omega t} = z_{A_r} e^{i \omega t} + i z_{A_i} e^{i \omega t}
\]

\[
= z_{A_r} (\cos(\omega t) + i \sin(\omega t)) + i z_{A_i} (\cos(\omega t) + i \sin(\omega t))
\]

\[
= z_{A_r} \cos(\omega t) + i z_{A_r} \sin(\omega t) + i z_{A_i} \cos(\omega t) - z_{A_i} \sin(\omega t)
\]

So, as \( \bar{z}_A = \text{Re}(Z_A e^{i \omega t}) \), the following results for \( \bar{z}_A \):

\[
\bar{z}_A = z_{A_r} \cos(\omega t) - z_{A_i} \sin(\omega t)
\]

(4.46 repeated)
ANNEX C

Derivation of the solutions $Z_A(u)$ and $Z_B(u)$
ANNEX C.

Derivation of the solutions $Ai(u)$ and $Bi(u)$

The following derivation of the solutions $Z_A(u)$ and $Z_B(u)$ is based on the presented solutions in the Handbook of Mathematical Functions [6]. In the Handbook, the solutions have a real variable. In this case, however, the variable has also an imaginary part. This difference leads to some changes in the plots of the solutions, compared with those presented in the Handbook. Especially the wave-length of the harmonical solutions is different.

First solution $Z_A(u)$ will be treated, afterwards $Z_B(u)$ is dealt with.

Solution $Z_A(u)$:

According to equation 10.4.2 [6], for $Z_A(u)$ it can be stated that:

$$Ai(u) = c_1 f(u) - c_2 g(u)$$

where:

$$f(u) = 1 + \frac{1}{3!} u^3 + \frac{1}{6!} u^6 + \frac{1}{9!} u^9 + \ldots$$

$$g(u) = u + \frac{2}{4!} u^4 + \frac{2.5}{7!} u^7 + \frac{2.5.8}{10!} u^{10} + \ldots$$

and, according to equation 10.4.4 and 10.4.5 [6],

$$c_1 = 0.35502 \ 80538 \ 87817 \ (C.4)$$
$$c_2 = 0.25881 \ 94037 \ 92807 \ (C.5)$$

The higher order terms are not included in this derivation, but the computer program calculating the solutions generates these terms automatically, as long as the accuracy is insufficient.

Substitution of (C.2) and (C.3) in (C.1) yields the following:

$$Ai(u) = c_1 + c_1 \frac{1}{3!} u^3 + c_1 \frac{1.4}{6!} u^6 + c_1 \frac{1.4.7}{9!} u^9$$
$$- c_2 u - c_2 \frac{2}{4!} u^4 - c_2 \frac{2.5}{7!} u^7 - c_2 \frac{2.5.8}{10!} u^{10}$$

The variable $u$ is an imaginary variable, according to:

$$u = e^{\frac{i}{6} \cdot u_r}$$

For the definition of $u$ and $u_r$ is referred to chapter 4.

Using (C.7), equation (C.6) changes to:

$$C1$$
After writing all terms in this expression separately, the terms containing "i" can be taken apart. This results in the following expression:

\[ Ai(u) = c_1 - c_1 \frac{14}{6!} u_r^6 - c_2 \frac{1}{2} \sqrt{3} u_r^7 + c_2 \frac{2}{4!} \left( \frac{1}{2} \sqrt{3} u_r^7 - c_2 2.5 \frac{1}{10!} \frac{1}{2} u_r^{10} \right) + i \left\{ c_1 \frac{14}{9!} u_r^9 - c_2 \frac{1}{2} \frac{4!}{2} \frac{1}{2} \sqrt{3} u_r^9 \right\} \]

From this it follows that:

\[ \text{Re}\{Ai(iu_r)\} = c_1 - c_1 \frac{14}{6!} u_r^6 - c_2 \frac{1}{2} \sqrt{3} u_r^7 + c_2 \frac{2}{4!} \frac{1}{2} u_r^{10} + c_2 \frac{2.5}{7!} \frac{1}{2} \sqrt{3} u_r^9 \]

\[ \text{Im}\{Ai(iu_r)\} = c_1 \frac{1}{3!} u_r^3 - c_1 \frac{14}{9!} u_r^9 - c_2 \frac{1}{2} \frac{4!}{2} \frac{1}{2} \sqrt{3} u_r^9 + c_2 \frac{2.5}{7!} \frac{1}{2} \sqrt{3} u_r^9 \]

Using these expressions, \( z_{A_r} \) and \( z_{A_j} \) are calculated.

Solution \( Z_6(u) \):

According to equation 10.4.3, for \( Z_6(u) \) it can be stated that:

\[ Bi(u) = \sqrt{3} \left( c_1 f(u) + c_2 g(u) \right) , \]

with \( f(u) \) and \( g(u) \) according to (C.2) and (C.3), and \( c_1 \) and \( c_2 \) according to (C.4) and (C.5).

Substitution of \( f(u) \) and \( g(u) \) into (C.13) yields the following:
\[ Bi(u) = \sqrt{3} \left\{ c_1 + c_1 \frac{1}{3!} u^3 + c_1 \frac{14}{6!} u^6 + c_1 \frac{14 \cdot 7}{9!} u^9 + 
+ c_2 u + c_2 \frac{2}{4!} u^4 + c_2 \frac{2 \cdot 5}{7!} u^7 + c_2 \frac{2 \cdot 5 \cdot 8}{10!} u^{10} \right\} \] (C.14)

Using equation (C.7) this becomes:

\[ Bi(u) = \sqrt{3} \left\{ c_1 + c_1 \frac{1}{3!} e^{i\frac{\pi}{2}} u^3 + c_1 \frac{14}{6!} e^{i\frac{3\pi}{2}} u^6 + c_1 \frac{14 \cdot 7}{9!} e^{i\frac{5\pi}{2}} u^9 + 
+ c_2 e^{i\frac{\pi}{6}} u_r + c_2 \frac{2}{4!} e^{i\frac{\pi}{2}} u_r^4 + c_2 \frac{2 \cdot 5}{7!} e^{i\frac{3\pi}{2}} u_r^7 + c_2 \frac{2 \cdot 5 \cdot 8}{10!} e^{i\frac{5\pi}{2}} u_r^{10} \right\} \] (C.15)

Rewriting the exponential terms yields:

\[ Bi(u) = \sqrt{3} c_1 + \sqrt{3} c_1 \frac{i}{3!} u_r^3 - \sqrt{3} c_1 \frac{1}{6!} u_r^6 - \sqrt{3} c_1 \frac{14 \cdot 7}{9!} i u_r^9 + \sqrt{3} c_2 u_r \left( \frac{1}{2} \sqrt{3} + i \frac{1}{2} \right) + 
+ \sqrt{3} c_2 \frac{2}{4!} u_r^4 \left( -\frac{1}{2} + i \frac{1}{2} \right) + \sqrt{3} c_2 \frac{2 \cdot 5}{7!} u_r^7 \left( -\frac{1}{2} \sqrt{3} - \frac{1}{2} \right) + \sqrt{3} c_2 \frac{2 \cdot 5 \cdot 8}{10!} u_r^{10} \left( \frac{1}{2} - i \frac{1}{2} \sqrt{3} \right) \] (C.16)

After writing all terms in this expression separately, the terms containing "i" can be taken apart. This results in the following expression:

\[ Bi(u) = \sqrt{3} c_1 - \sqrt{3} c_1 \frac{1}{3!} u_r^3 + \frac{3}{2} c_2 u_r - \frac{1}{2} \sqrt{3} c_2 \frac{2}{4!} u_r^4 - \frac{3}{2} c_2 \frac{2 \cdot 5}{7!} u_r^7 + \frac{1}{2} \sqrt{3} c_2 \frac{2 \cdot 5 \cdot 8}{10!} u_r^{10} + 
+ i \left\{ \sqrt{3} c_1 \frac{1}{3!} u_r^3 - \sqrt{3} c_1 \frac{14 \cdot 7}{9!} u_r^9 + \frac{1}{2} \sqrt{3} c_2 u_r + \frac{3}{2} c_2 \frac{2}{4!} u_r^4 - \frac{1}{2} \sqrt{3} c_2 \frac{2 \cdot 5}{7!} u_r^7 - \frac{3}{2} c_2 \frac{2 \cdot 5 \cdot 8}{10!} u_r^{10} \right\} \] (C.17)

From this it follows that:

\[ Re\{Bi(iu_r)\} = \sqrt{3} c_1 - \sqrt{3} c_1 \frac{1}{3!} u_r^3 + \frac{3}{2} c_2 u_r - \frac{1}{2} \sqrt{3} c_2 \frac{2}{4!} u_r^4 - 
- \frac{3}{2} c_2 \frac{2 \cdot 5}{7!} u_r^7 + \frac{1}{2} \sqrt{3} c_2 \frac{2 \cdot 5 \cdot 8}{10!} u_r^{10} \] (C.18)

\[ Im\{Bi(iu_r)\} = \sqrt{3} c_1 \frac{1}{3!} u_r^3 - \sqrt{3} c_1 \frac{14 \cdot 7}{9!} u_r^9 + \frac{1}{2} \sqrt{3} c_2 u_r + \frac{3}{2} c_2 \frac{2}{4!} u_r^4 - 
- \frac{1}{2} \sqrt{3} c_2 \frac{2 \cdot 5}{7!} u_r^7 - \frac{3}{2} c_2 \frac{2 \cdot 5 \cdot 8}{10!} u_r^{10} \] (C.19)

Using these expressions, \( z_{B_r} \) and \( z_{B_i} \) are calculated.
ANNEX D

The FORTRAN programs to compute $Z_A(u)$ and $Z_B(u)$
ANNEX D.

computation of Ai(u)

This program computes the values for
Re\{Ai(iur)\} and Im\{Ai(iur)\} for a given value
for c1 and c2 and an increasing value for ur,
and writes them to an ASCII file.

program rows

Declarations
parameter(c1 = 0.355028053887817)
parameter(c2 = 0.258819403792807)
integer urmax, clfac, c2fac,
c1exp, c2exp, c1sign,
c2sign, errcod
real Re_old, Re_new, Im_old, Im_new, Nur,
c1term, c2term, ur, urstep, ur_sum, Nmbrur
double precision cldown, c2down, clup, c2up
character*10 outfl
dimension Re_new(1000), Im_new(1000), urnmbr(1000)

Read the step with which ur increases and read the maximum ur.

write(6,3000)
3000 format(' + ','Type the step between ur(i) and ur(i+1):')
read(5,4000) urstep
4000 format(f5.2)
write(6,5000)
5000 format(' + ','Type the maximum value of ur')
read (5,6000) urmax
6000 format(i5)

Read name of output file
write(6,7000)
7000 format(' + ','type output filename')
read(5,8000) outfl
8000 format(a10)

Open output file, and connect file on unit 20
open(20, file = outfl, status = 'unknown',
iostat = errcod, err = 950)
rewind(20, iostat = errcod, err = 950)

Start loop for calculation Re and Im (loop is labelled 10).
For every ur the terms of Re and Im are determined
and summarised until the desired accuracy is reached.
The index for ur is i, and the index for the terms is j.
Nmbrur=ANINT(urmax/urstep)
write(6,8001) Nmbrur
8001 format(5x,'Number of ur-values for calculation = ',f5.2)
pause
urnmbr(1)=0
Re_new(1)=c1
Im_new(1)=0

do 10 i=2,Nmbrur
  ur_sum=(i-1)*urstep
  write(6,8002) i, ur_sum
8002 format(5x,'Start calculation of ur number ',i3,', ur = ',f5.2)
pause
ur=ur+urstep
urnmbr(i)=ur

C calculation Re(i)
  Re_new(i)=c1
  Re_old=999
  j=0
    c1exp=0
    c1fac=-2
    clup=1

C2exp=-2
C2up=-1
C2fac=-4

20 IF (ABS(Re_new(i) - Re_old) .GT. 1e-7) THEN
  Re_old=Re_new(i)
  j=j+1
  c1exp=c1exp+6
  c1fac=c1fac+3
  clup=clup*c1fac
  c1fac=c1fac+3
  clup=clup*c1fac
  cldown=1
  do 30 k=1,c1exp
    cldown=cldown*k
 30 continue
  clsign=(-1)**(j+2)
  clterm=clsign*c1*(clup/cldown)*(ur**c1exp)
  Re_new(i)=Re_new(i)+clterm
  c2exp=c2exp+3
c2fac = c2fac + 3

Do 40 k = 1, c2exp
   c2down = c2down * k
40 continue

c2sign = (-1) ** (j + 2)
c2term = 0.5 * SQRT(3) * c2sign * c2 * (c2up / c2down) * ur ** c2exp
Re_new(i) = Re_new(i) + c2term

Do 45 k = 1, c2exp
   c2down = c2down * k
45 continue

c2sign = c2sign * (-1)
c2term = 0.5 * c2sign * c2 * (c2up / c2down) * ur ** c2exp
Re_new(i) = Re_new(i) + c2term

Write(6, 8003) j, Re_new(i)
8003 format( ',', 'After three times ', i2, ', terms is Re(i) = ', e20.10)

GOTO 20
ENDIF

C calculation Im(i)

Im_new(i) = -c2 * 0.5 * ur
Im_old = 999
j = 0

c1exp = -3
c1fac = -5
c1up = -0.5

c2exp = 1
c2fac = -1
c2up = 1

50 IF (ABS(Im_new(i) - Im_old) .GT. 1e-7) THEN
   Im_old = Im_new(i)
   j = j + 1
   c1exp = c1exp + 6
clfac = clfac + 3
c1up = clup * clfac
c2fac = clfac + 3
c1up = clup * clfac
c1down = 1

do 60 k = 1, c1exp
    cldown = cldown * k
60 continue

clsign = (-1)**(j + 1)
c1term = clsign * c1 * (clup / cldown) * ur**c1exp
Im_new(i) = Im_new(i) + c1term


c2exp = c2exp + 3
c2fac = c2fac + 3
c2up = c2up * c2fac
c2down = 1

do 70 k = 1, c2exp
    c2down = c2down * k
70 continue

c2sign = (-1)**(j + 2)
c2term = 0.5 * SQRT(3) * c2sign * c2 * (c2up / c2down) * ur**c2exp
Im_new(i) = Im_new(i) + c2term

write(6, 8004) j, Im_new(i)
8004 format(’ ', 'After three times ', i2, ' terms is Im(i) = ', e20.10)

GOTO 50
ENDIF

pause

C Calculate with the next ur
10 continue

C Write a heading in the output file
write (20, 9000)
9000 format(’ ', 'number ur Re{Ai(iur)} Im{Ai(iur)}')
C Write the results in the output file
   do 80 n=1,Nmbrur
       write(20,10000) n, urnmbr(n), Re_new(n), Im_new(n)
10000 format(i6,4x,f20.10,4x,f20.10,4x,f20.10,4x)
   80 continue

C Place end file marker on outputfile
100 endfile(20,iostat=errcod,err=950)

C Error recovery
950 write(6,9002) errcod
   stop

C Formats errormessages
9002 format(/'/**** error #',i5,' on outputfile ***')

C End of program
   END
computation of Bi(u)

C This program computes the values for
C Re\{Bi(iur)\} and Im\{Bi(iur)\} for a given value
C for c1 and c2 and an increasing value for ur,
C and writes them to an ASCII file.

    program rows

C Declarations

    parameter(c1=0.355028053887817)
    parameter(c2=0.258819403792807)
    integer urmax, clfac, c2fac,
        $ c1exp, c2exp, c1sign,
        $ c2sign, errcod
    real Re_old, Re_new, Im_old, Im_new, Nur,
        $ clterm, c2term, ur, urstep, ur_sum, Nmbrur
    double precision c1down, c2down, clup, c2up
    character*10 outfl

    dimension Re_new(1000), Im_new(1000), urnumbr(1000)

C Read the step with which ur increases and read the maximum ur.

        write(6,3000)
    3000 format( '+' , 'Type the step between ur(i) and ur(i+1):')
        read(5,4000) urstep
    4000 format(f5.2)
        write(6,5000)
    5000 format( '+' , 'Type the maximum value of ur')
        read (5,6000) urmax
    6000 format(i5)

C Read name of output file

        write(6,7000)
    7000 format( '+' , 'type output filename')
        read(5,8000) outfl
    8000 format(A10)

C Open output file, and connect file on unit 20

        open(20, file=outfl, status='unknown',
            $ iostat=errcod, err=950)
        rewind(20, iostat=errcod, err=950)

C Start loop for calculation Re and Im (loop is labelled 10).
C For every ur the terms of Re and Im are determined
C and summarised until the desired accuracy is reached.
C The index for ur is i, and the index for the terms is j.

        Nmbrur=ANINT(urmax/urstep)
write(6,8001) Nmbrur

8001 format(’ ’,'Number of ur-values for calculation = ’,f5.2)
pause
urnmbr(1)=0
Re_new(1)=c1*sqrt(3)
Im_new(1)=0

do 10 i=2,Nmbrur
ur_sum=(i-1)*urstep
write(6,8002) i, ur_sum

8002 format(’ ’,'Start calculation of ur number ’,i3,’ ’, ur = ’,f5.2)
pause
ur=ur+urstep
urnmbr(i)=ur

C calculation Re(i)
Re_new(i)=SQRT(3)*c1
Re_old=999
j=0

c1exp=0
c1fac=-2
c1up=1

c2exp=-2
c2up=-1
c2fac=-4

20 IF (ABS(Re_new(i) - Re_old) .GT. 1e-7) THEN

Re_old=Re_new(i)

j=j+1
c1exp=c1exp+6
c1fac=c1fac+3
c1up=c1up*c1fac
c1fac=c1fac+3
c1up=c1up*c1fac
c1down=1

do 30 k=1,c1exp
   c1down=c1down*k
30 continue

c1sign=(-1)**(j+2)
c1term=SQRT(3)*c1sign*c1*(c1up/c1down)*(ur**c1exp)
Re_new(i)=Re_new(i)+c1term

C2 exp=c2exp+3
C2 fac=c2fac+3
c2up = c2up * c2fac

c2down = 1

do 40 k = 1, c2exp
   c2down = c2down * k
40 continue

   c2sign = (-1)**(j + 1)
   c2term = 1.5 * c2sign * c2*(c2up/c2down)**ur**c2exp
   Re_new(i) = Re_new(i) + c2term

c2exp = c2exp + 3

c2fac = c2fac + 3

c2up = c2up * c2fac

c2down = 1

do 45 k = 1, c2exp
   c2down = c2down * k
45 continue

   c2sign = c2sign * (-1)
   c2term = 0.5 * SQRT(3) * c2sign * c2*(c2up/c2down)**ur**c2exp
   Re_new(i) = Re_new(i) + c2term

write(6, 8003) j, Re_new(i)

8003 format(*After three times ',i2,' terms is Re(i) = ',e20.10)

GOTO 20
ENDIF

pause

C calculation Im(i)

Im_new(i) = c2 * 0.5 * SQRT(3) * ur

Im_old = 999

j = 0

c1exp = -3

c1fac = -5

c1up = -0.5

c2exp = 1

c2fac = -1

c2up = 1

50 IF (ABS(Im_new(i) - Im_old) .GT. 1e-7) THEN

   Im_old = Im_new(i)

   j = j + 1
   c1exp = c1exp + 6
   c1fac = c1fac + 3
clup=clup*clfac
clfac=clfac+3
clup=clup*clfac
cldown=1
do 60 k=1,c1exp
   cldown=cldown*k
60 continue
clsign=(-1)**(j+1)
c1term=SQRT(3)*c1sign*c1*(clup/cldown)*ur**c1exp
Im_new(i)=Im_new(i)+c1term

c2exp=c2exp+3
c2fac=c2fac+3
c2up=c2up*c2fac
c2down=1
do 70 k=1,c2exp
   c2down=c2down*k
70 continue
c2sign=(-1)**(j+1)
c2term=1.5*c2sign*c2*(c2up/c2down)*ur**c2exp
Im_new(i)=Im_new(i)+c2term

c2exp=c2exp+3
c2fac=c2fac+3
c2up=c2up*c2fac
c2down=1
do 75 k=1,c2exp
   c2down=c2down*k
75 continue

c2sign=c2sign*(-1)
c2term=0.5*SQRT(3)*c2sign*c2*(c2up/c2down)*ur**c2exp
Im_new(i)=Im_new(i)+c2term

write(6,8004) j,Im_new(i)
8004 format(’ ’,’After three times ’,i2,’ terms is Im(i) = ’,e20.10)
GOTO 50
ENDIF

pause

C Calculate with the next ur
10 continue

C Write a heading in the output file
write (20,9000)
9000 format(’ ’,’number ur Re{Bi(iur)} Im{Bi(iur)})")
C Write the results in the output file
   do 80 n=i,Nmbrur
       write(20,10000) n, urnmbr(n), Re_new(n), Im_new(n)
10000 format(i6,4x,f20.10,4x,f20.10,4x,f20.10,4x)
   80 continue

C Place end file marker on outputfile
100 endfile(20,iostat=errcod, err=950)

C Error recovery
   950 write(6,9002) errcod
      stop

C Formats error messages
   9002 format(//'**** error #',i5,' on outputfile ***')

C End of program
   END
ANNEX E

The PASCAL program calculating profile developments numerically
PROGRAM numerical_calculation_using_F2_and_F3 (input, output);

USES CRT, GRAPH;

TYPE
  matrix1 = array [1..500] of real;
  matrix2 = array [1..2, 1..500] of real;

VAR
  k0y, k0z, F2dl, F3dl, Sydl, Sydl_grad : matrix1;
  kOy, dtkdl, kOy_max, ampl, kOy_d, expon, kOy_start : real;
  imax : integer;
  n, nmax : longint;
  outkOy, outSydl, outgrad : text;
  again : char;

RUN
  A1 = 134;
  a2 = -1.9;
  pi4 = 97.409091;
  Bacc = 18.7;
  C = 5.5;
  Dacc = 2.65e-6;
  E = 10.72;
  pi = 3.1415927;

PROCEDURE initial_values;

VAR
  i, j : integer;

BEGIN
  clrscr;
  writeln ('Start program.'),
  writeln ('Initial values have been assigned to variables.'),
  writeln;
  for i:=1 to 500 do
  begin
    k0y[i]:=0; F2dl[i]:=0; F3dl[i]:=0; Sydl[i]:=0; Sydl_grad[i]:=0;
    for j:=1 to 2 do
      begin
        kOy[j,i]:=0;
      end;
  end;
  dkOy:= 3.0; {maximum number of nodes is 500}
  dtdl:= 250.0;
  kOy_max:= 146; {maximum value is 120}
  nmax:= 252288; {max. number of timesteps for quattro is 1000}
  kOy_d:= 15;
  ampl:= 0.3;
  expon:= -3.5e-2;
  writeln ('Press <enter> to proceed.'); readln;
END;

PROCEDURE input (var expon, kOy_d, ampl, dkOy, dtdl, kOy_max: real;
  var nmax: longint);

VAR
  a: char;
  b: real;
  c: longint;

BEGIN
  clrscr;
  a:= 'n';
  b:= 0;
  while a <> 'y' do
  begin
    writeln;
    writeln ('Type input values for the numerical calculation.');
    writeln ('Type 0 if default value is correct, otherwise type new value.'),
  writeln;
  writeln;
  writeln;
  writeln;
  a:= readln;
END;
writeln;
write ('Type the value for \( \Delta k_{0y} \): \( \Delta k_{0y}:4.2\)'); readln(b); if b>0 then \( \Delta k_{0y}:=b \);
write ('Type the value for \( \Delta t/\Lambda_{w} \): \( \Delta t/\Lambda_{w}:4.2\)'); readln(b); if b>0 then \( \Delta t/\Lambda_{w}:=b \);
write ('Type the value for \( k_{0y} \max \): \( k_{0y}\max :4.2\)'); readln(b); if (b>0) and (b<120) then \( k_{0y}\max :=b \);
write ('Type the number of timesteps: \( n_{\max }\):4\)'); readln(c); if c>0 then \( n_{\max }:=c \);
writeln;
write ('Type the initial harmonic distortion-amplitude: \( a_{m}:4.2\)'); readln(b); if b>0 then \( a_{m}:=b \);
write ('Type the initial harmonic wavelength \( \Lambda_{0Ld} \): \( \Lambda_{0Ld}:4.2\)'); readln(b); if b>0 then \( \Lambda_{0Ld}:=b \);
write ('Type the initial harmonic distortion-exponent: \( \alpha :4.2\)'); readln(b); if b>0 then \( \alpha :=b \);
writeln;
write ('Is the input correct? (y/n)'); readln(a);
clearscr;
writeln;
end;

procedure initial_conditions (\( a_{1}, a_{2}, k_{0y}\max , \Delta k_{0y} \): real;
var \( k_{0y} \\text{start} \): real;
var \( k_{0y} \) : matrix1;
var \( k_{0z} \) : matrix2;
var \( imax \) : integer);
{building up the \( k_{0y} \)-axis and calculating the initial equilibrium profile}

var \( start \): integer;
i : integer;

begin
write ('The \( k_{0y} \)-axis has been built-up, and '); writeln;
write ('the initial equilibrium profile has been calculated. '); writeln;
imax := round\( (k_{0y}\max /\Delta k_{0y}) \); {For \( k_{0y}=10 \) starts the equilibrium profile height}
start := round\( (10/\Delta k_{0y})+1 \); {at \( k_{0z}=1.647 \). Otherwise the fits of \( F_{2} \) and \( F_{3} \) are not valid.}
for i:=2 to imax do
begin
\( k_{0y}[i]:= k_{0y}[i-1] + \Delta k_{0y} \);
end;
for i:=start to imax do
begin
\( k_{0z}[1,i]:= (1/a_{2}) * \ln(k_{0y}[i]/a_{1}) \);
end;
\( k_{0z}\text{start}:= k_{0z}[1,start] \);
end;

procedure upwards_distortion (\( \Delta k_{0y} \): real; \( imax \): integer; \( k_{0y} \): matrix1;
var \( k_{0z} \): matrix2);

var \( start, i \): integer;
a : char;
const \( k_{0zup} = 0.1 \);

begin
a := 'n';
writeln;
write ('Should there be an equal upwards distortion (y/n)? '); readln(a);
if a='y' then
begin
\( start:= \text{round} \left( \left(10/\Delta k_{0y}\right)+1 \right) \);
for i:=\( \text{start}+1 \) to imax do
begin
\( k_{0z}[1,i]:= k_{0z}[1,i] + k_{0zup} \);
end;
write ('Press <enter> to proceed.'); readln;
end;

procedure equal_distortion (\( a_{1}, a_{2}, \Delta k_{0y} \): real; \( k_{0y} \): matrix1; \( imax \): integer;
var kOz: matrix2);

ir nri, start, i: integer;
  a: char;

listortion y tilde of 49.6 m for every location, kOy tilde =8.0

gin
nri:= round(8/dkOy);
start:= round((10/dkOy)+1);
a:= 'n';
writeln;
writeln ('Should there be an equal kOy tilde in every profile location (y/n)?'); readln(a);
if a='y' then
begin
  for i:= (start-t-nri) to imax do
  begin
    kOz [1,i] := (1/a2)*ln(kOy[i-nri]/al);
  end;
  for i:= start to (start+nri) do
  begin
    kOz [1,i] := kOz[1,start] ;
  end;
  writeln ('Press <enter> to proceed. ');
  readln;
end;

procedure harm_distortion (pi, expon, ampl, kOy, dkOy: real; imax: integer; kOy: matrix1;
  var kOz: matrix2);

ir kOztild; real;
i: integer;
a: char;
start: integer;
equil: integer; {denotes length upper part of the profile without distortion}

gin
equil:= 2;
a:= 'n';
writeln;
writeln ('Should there be a harmonic initial distortion (y/n)? ');
readln(a);
if a='y' then
begin
  start:= round((10/dk0y)+1+(equil/dkOy));
  for i:=start to imax do
  begin
    kOztild:=(ampl*sin(2*pi*(kOy[i+1-start]/kOy)+1.5*pi))*exp(expon*kOy[i+1-start]);
    kOz[1,i]:=kOz[1,i]+kOztild;
  end;
  writeln ('Press <enter> to start the calculation');
  readln;
clrscr;
end;

procedure open_files_with_headings;

ir i: integer;

gin
assign (outkOz, 'c:\stephan\prog\pascal\outkOz.prn');
reset (outkOz);
rewrite (outkOz);
writeln (outkOz, 'This file contains the bottom series for all timesteps. ');
writeln (outkOz);
assign (outSydl, 'c:\stephan\prog\pascal\outSydl.prn');
reset (outSydl);
rewrite (outSydl);
writeln (outSydl, 'This file contains the sediment transport series for all timesteps.' );
writeln (outSydl);

annex E3
assign (outgrad, 'c:\stephan\prog\pascal\outgrad.prn');
reset (outgrad);
rewrite (outgrad);
writeln (outgrad, 'This file contains sediment transport gradients for all timesteps.');
writeln (outgrad);
for i:= 1 to imax do
begin
  write (outkOz, i:3, ' ');
end;
i := 1;
while i<imax do
begin
  write (outSydl, i:3, ' ');
  write (outgrad, i:3, ' ');
i := i+3;
end;
writeln (outkOz);
writeln (outSydl);
writeln (outgrad);
for i:= 1 to imax do
begin
  write (outkOz, kOy[i]:3:0, ' ');
end;
i := 1;
while i<imax do
begin
  write (outSydl, kOy[i]:3:0, ' ');
  write (outgrad, kOy[i]:3:0, ' ');
i := i+3;
end;
writeln (outkOz); writeln (outkOz);
writeln (outSydl); writeln (outSydl);
writeln (outgrad); writeln (outgrad);
end;

procedure output_to_files (kOy, Sydl, Sydl_grad: matrixl; kOz: matrix2; imax: integer);
var i: integer;
begin
for i:=1 to imax do
begin
  write (outkOz, kOz[1,i]:3:2, ' ');
end;
i := 1;
while i<imax do
begin
  write (outSydl, Sydl[i]:3, ' ');
  write (outgrad, Sydl_grad[i]:3, ' ');
i := i+3;
end;
writeln (outkOz);
writeln (outSydl);
writeln (outgrad);
end;

procedure sediment_transport (Bacc, C, Dacc, E, dkOy : real;
imax : integer;
kOz : matrix2;
var F2dl, F3dl, Sydl, Sydl_grad : matrixl);
var i: integer;
start: integer;
Syplus, Symb: real;

{ calculate F2 and F3 for every kOz for this timestep }
begin
  writeln ('For every location in the profile F2 and F3 are calculated');
}
writeln ('Afterwards the local sediment transport is calculated.'); 
begin
    F2dl[i] := -Dace*(1.647-kOz[1,i])*exp(-E*(1.647-kOz[1,i]));
    F3dl[i] := Bacc*k0z[1,i]*k0z[1,i]*k0z[1,i]*k0z[1,i]*k0z[1,i]*sqrt(abs(kOz[1,i]));
end;

for i := start to imax do
    begin
        Syplus:= (1/2)*(F2dl[i]+F2dl[i+1]) - (1/2)*(F3dl[i]+F3dl[i+1])*((kOz[1,i+1]-kOz[1,i])/dkOy);
        Symin:= (1/2)*(F2dl[i-1]+F2dl[i]) - (1/2)*(F3dl[i-1]+F3dl[i])*((kOz[1,i]-kOz[1,i-1])/dkOy);
        if Syplus < 1.64 then
            begin
                k0z[2,i] := k0z[1,i] + 16*pi4*dtdl*(-Sydl_grad[i]);
            end;
        k0z[2,i] := k0z[1,i]; /* this is the upper boundary condition! */
        k0z[2,i] := k0z[1,i]; /* this is the lower boundary condition! */
end;

procedure shift_kOz_array (imax: integer; var kOz: matrix2);
begin
    for i := 1 to imax do
        begin
            if kOz[2,i] < 1.64 then
                begin
                    k0z[1,i] := k0z[2,i];
                    k0z[2,i] := 0;
                end;
            else
                begin
                    k0z[1,i] := 1.64;
                    k0z[2,i] := 0;
                end;
        end;
end;

{main program}
again := 'n';
while again <> 'y' do
begin

n:=1;
clearscr;
initial_values;
input (expon, kO1d, ampl, dkOy, dtdl, kOy, kOz, nmax);
initial_conditions (a1, a2, kOy, kOzstart, kOy, kOz, imax);
{equal_distortion (a1, a2, dkOy, kOy, imax, kOz);}
{upwards_distortion (dkOy, imax, kOy, kOz);}
harm_distortion (pi, expon, ampl, kO1d, dkOy, imax, kOy, kOz);
open_files_with_headings;
output_to_files (kOy, Sydl, Sydl_grad, kOz, imax); {prints first row kOz only}
for n:=1 to (nmax-1) do
begin
sediment_transport (Bacc,C,dacc,E,dkOy,imax,kOz,F2dl,F3dl, Sydl, Sydl_grad);
bottom_update (imax, dkOy, dtdl, pi, kOzstart, Sydl_grad, kOz, n);
shift_kOz_array (imax, kOz); {values second row kOz to first row}
if (n mod 1000) = 0 then
begin
output_to_files (kOy, Sydl, Sydl_grad, kOz, imax);
writeln ('Timestep: ',n:6,' of ',nmax:6);
end;
end;
end;
write ('Leave this program (y/n)? '); readln; {again);
end;
end.

annex E6
APPENDIX A

Plots of $<q_{st}>$, $<q_{st}>$ and $<q_{sn}>$
Components $P_2$ and $P_3$ (s0 = 0.02, W = 1.0)
Gas and Gun

Components $F_z$ $(s_0=0.02, \gamma=1.0$)
APPENDIX B

Plots of calculated values for $F_2$ and $F_3$ and profile characteristics
equilibrium profile
$s_0 = 0.02$ and $W^* = 1.0$
cross-shore distance L

non dimensional wave height H*

W = 0.02 and W* = 1.0

wave height

S0 = 0.02
F3 values
s0=0.02 and W*=1.0

cross-shore distance k0y

non dimensional F3

(Times 10E-6)
equilibrium slope
$s_0 = 0.02$ and $W^* = 1.0$
Sy = F2 - F3 * tan be; zero for equilibrium

(eq. 10E-14)
The function \( F_2(k_0z) \)
Sensitivity for sediment parameter

\[
\begin{align*}
(W*T)/H &= 1.0 \\
(W*T)/H &= 0.1 \\
(W*T)/H &= 0.01
\end{align*}
\]
The function $F_3(k_0y)$
sensitivity for sediment parameter

$\frac{(W*T)}{H} = 1.0$
$\frac{(W*T)}{H} = 0.1$
$\frac{(W*T)}{H} = 0.01$
APPENDIX C

Plots of fitting $F_3$ as a function of $y$
$A' = 2.45 \times 10^{-6}$

$S_0 = 0.02$ and $W^* = 1.0$

determination of $A'$
APPENDIX D

Plots illustrating the scale of $u_r$. 
ur versus y
Tw=10sec, Ts=10yr
ur versus y

$A' = 4.0 \times 10^{-6}, \quad T_w = 8 \text{ sec}$

$T_s = 8 \text{ yr}$

$T_s = 13 \text{ yr}$

cross-shore distance $y$


ur versus y

$A' = 4.0 \times 10^{-6}$, $T_s = 10 \text{ yr}$

Tw=2 sec

Tw=4 sec

Tw=6 sec

Tw=8 sec

Tw=10 sec

Tw=12 sec

ur

cross-shore distance $y$
APPENDIX E

Plots of the solutions for large $u_r$. 
impression of $\text{Im}(Ai(u))$
impression of $\text{Im}(\text{Bi}(u))$
impression of $\text{Re}(\text{Ai}(u))$
impression of $\text{Re}(\text{Bi}(u))$
APPENDIX F

Plots of $z_{A,r}$, $z_{A,i}$, $\tilde{z}_A$ and $z_{B,r}$, $z_{B,i}$ and $\tilde{z}_B$
Real part of solution $ZA$
$\text{urstep}=0.01$ and $\text{urmax}=3$
Imaginary part of solution $Z_A$

$\text{urstep}=0.01$ and $\text{urmax}=3$
$L_0 = 75, \ Tw = 10\ s, \ Ts = 10\ yr, \ A' = 3.5\ e^{-6}$
Real part of the solution ZB

with step=0.1 and urmax=1.0

PLOT F4
Imaginary part of solution ZB
urstep=0.1 and urmax=10
zb tilde

$\lambda_0 = 75$, $T_w = 10\, \text{s}$, $T_s = 10\, \text{yr}$, $\Lambda' = 3.5 \times 10^{-6}$
APPENDIX G

Plots of analytical and numerical results
numerical compared to analytical data
propagation of boundary condition

![Graph showing numerical and analytical data comparison for distortion vs. k0y. The graph illustrates different propagation times and shows a clear comparison between numerical and analytical solutions.]
numerical data compared to analytical

$Ts=10\text{yr},\ Tw=5\text{s},\ A'=2.49\times10^{-6}$
APPENDIX H

Plots of fitting $F_2$ and $F_3$ as a function of $k_{\sigma z}$, and their derivatives
Fit $F_2$ as a function of $k_0z$
(z is positive upwards)

\[ F_2/f_2^{-2}T^{-3} = -D' \times (1.647 - k_0z) \times \exp(-E \times (1.647 - k_0z)) \]

$D' = 2.65e-6$

$E = 10.72467$
$F_3 / g^{2} T^{3}$

(Times 10E-6)

$B' = 1E^{-7}$

$B' * kOz \sim C$

$C = 5.5$

$F_3$ as a function of $kOz$

(z is positive upwards)
plot of $dF^2/\text{dz}$

$T_w = 5$ sec (to calculate $k_0$)
plot of $\frac{dF_3}{dz}$

$Tw = 5$ sec (to calculate $k_0$)
APPENDIX I

Plots of the functions $P(k\xi)$ and $Q(k\xi)$
Plot of $P(k0z)$ using the fitted $F2$ & $F3$

$P(k0z) = (F2)^2 / F3$
Plot of $Q(k_{0z})$ using the fitted $F_2$ & $F_3$

$$Q(k_{0z}) = \frac{d}{dz}(F_2)^2 / F_3$$
APPENDIX J

Plot of fitting the equilibrium profile as a function of $k_f$. 
equilibrium profile
fit of profile shape as f(k0z)

$k_0y = a_1 \times \exp(a_2 \times k_0z)$

- $a_1 = 134$
- $a_2 = -1.9$
APPENDIX K

Plots of the functions $R(k_0 z)$ and $S(k_0 z)$, and the modulus
The functions $R(k0z)$ and $S(k0z)$

$Tw = 5\text{sec}$, $s0 = 0.02$, $a2 = -1.9$
Modulus of the right term

\[ \text{TW = 5sec, } a_1 = 1.9, \quad a_2 = 1.9, \quad a_3 = 0.25 \]
APPENDIX L

Plot of sediment transport gradients for an initial distortion
Initial sediment transport gradients

$T_w = 5 \text{s}, \ L_d = 1 \text{m}, \ a_1 = 134, \ a_2 = -1.9, \ a_3 = 0.25$
APPENDIX M

Output of numerical program
Profile development without distortion
equilibrium profile (timestep = 250)
Sediment transports without distortion
equilibrium profile (timestep = 250)
Sediment transport, harmonic distortion

$T_d = 10y$, $6 \times T_d$ plotted, $\delta t/T_w = 250$

Graph showing sediment transport with different $k_0y$ values:
- $k_0y = 9$
- $k_0y = 36$
- $k_0y = 63$
- $k_0y = 90$
- $k_0y = 117$
- $k_0y = 136$

$y / \Delta y$

nr. timesteps $n \times 1000$ ($\delta t/T_w = 250$)
Profile development after distortion
Td=10y, 2y plotted (delta t/Tw = 250)
Profile development, harmonic boundary
Td=10\(y\), 10\(y\) plotted (delta t/Tw=250)
Profile development, harmonic boundary

$T_d = 10^y$, $6 \times T_d$ plotted, $\Delta t/T_w = 250$

Graph showing the development of a profile with different values of $n$. The graph includes markers for $n = 0, 300,000, 600,000, 900,000, 1,200,000, 1,500,000$. The x-axis represents $k0y$ and the y-axis represents $k0z$. The graph demonstrates how the profile changes with different values of $n$. 
Profile shape after initial distortion

\[ \text{delta } t/T_w = 250, \ k_0 L_d = 15, \ 10y \text{ plotted} \]
Shape of distortion in time

delta t/Tw=250, k0Ld=15, 10y plotted

Plot M8
transport gradients, initial distortion

delta t/Tw=250, k0Ld=15, 10y plotted
Profile shape after initial distortion
timestep $t/T_w = 250$, wavelength $k_0 l_d = 20$
Profile shape after initial distortion
timestep $t/T_w=250$, wavelength $k_0 L_d = 5$

Instability occurs when $k_0 z > 1.647$

because in that case $F_2 > 0$
Profile shape after initial distortion
timestep $t/T_w=250$, wavelength $k_0 l_d = 5$

amplitude = 0.2 instead of 0.25
therefore $k_0 z < 1.647$ and $F_2 < 0$