Propagation of short gravity waves in shoaling water.

Refraction and diffraction.

I. Some approximation methods.

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Introduction.

In this report, a description is given of some approximation methods, to solve the problem of short wave propagation in shoaling water. The problem is stated as an elliptic initial value problem, which is in general not properly posed. After reduction of the wave equation to the Helmholtz equation, the transition to amplitude and phase is investigated, in particular at singular points. Perturbation theory is applied, by introducing expansions in powers of a small parameter.

Three approximation methods have been compared:
1) the phase-integral method, including the geometrical optics approximation as a special case,
2) Rytov's method,
3) a modification of Rytov's method.
Problem description.

The propagation of periodic, small amplitude, surface gravity waves in shoaling water is described by the two-dimensional reduced wave equation (ref. [2], [1]):

\[ \nabla \cdot (c g \nabla \phi) + c c k^2 \phi = 0 \]  

(1)

in which:

- \( \phi(x,y) \) = complex potential function,
- \( k \) = wavenumber, a function of the water-depth \( h \) through the dispersion relation
  \[ \omega^2 = gk \tanh(kh), \]
- \( g \) = acceleration of gravity,
- \( \omega \) = angular frequency,
- \( c \) = phase velocity,
- \( c_B \) = group velocity.

The area in which equation (1) should be solved is confined by (see fig. 1): a deep waterline (waterdepth \( > \frac{1}{2} \) half wavelength), a breakerline (defined by some breaker criterion), and two boundaries perpendicular to the deep waterline.
Reflection of the waves is neglected, restricting the bottom slope to small values in the whole area (say, less than 1 : 15). The problem is now stated as an initial value problem: on the deep waterline, the potential function represents an elementary progressive wave:

\[ \phi \approx \phi_0 = A_0 e^{ik(x \cos \alpha + y \sin \alpha)} \] (2)

in which:

- \( A_0 \) = constant amplitude,
- \( k_0 \) = wavenumber in deep water,
- \( \alpha \) = angle between wavefront and y-axis.

For \( x = 0 \) we have:

\[ \begin{aligned}
\phi &= \phi_0 \\
\frac{\partial \phi}{\partial x} &= \frac{\partial \phi_0}{\partial x}
\end{aligned} \] (3)

The linear elliptic Cauchy problem (1)-(3) is in general not properly posed (ref. [6]).

Two questions arise:

a) the solution should depend continuously on the data (requirement of stability),
b) the domain of influence of the data should be defined.

Continuous dependence on the data may be restored by restricting attention to those solutions satisfying a prescribed global bound (for further details, see ref. [8]).

Alternative methods are given in ref. [5].

**Approximate reduction of the reduced wave equation to the Helmholtz equation.**

In order to simplify equation (1) we introduce a scaling factor \( \sqrt{\frac{cc}{g}} \):

\[ \phi = \phi \sqrt{\frac{cc}{g}} \] (4)

Then (1) transforms into:

\[ \nabla^2 \phi + \left( k^2 - \frac{\nabla^2}{\sqrt{\frac{cc}{g}}} \right) \phi = 0 \] (5)
We compare the size of the term \( \frac{v^2/\sqrt{cc}}{\sqrt{cc}/g} \) with \( k^2 \) :

In Fig. 2 are shown \( c, \sqrt{cc}/g \) and \( \sqrt{cc}/g \) as a function of \( k_0 h \), with regard to the velocity \( c_0 = \frac{\omega}{k_0} \).

We have \( c_g \leq \sqrt{cc}/g \leq c \), in accordance with \( c_g \leq c \).

For \( k_0 h = 1 \), \( c_g \) has a maximum, and \( \sqrt{cc}/g \) equals its asymptotic value \( 1/\sqrt{2} \).

For \( k_0 h \approx 1.6 \), \( \sqrt{cc}/g \) has a maximum.

Using the identity:

\[
\frac{v^2/\sqrt{cc}}{\sqrt{cc}/g} = \frac{d}{dh} \sqrt{cc}/g \cdot v^2 h + \frac{d^2}{dh^2} \sqrt{cc}/g \cdot |\nu h|^2
\]

it follows, that this term can play a part with regard to \( k^2 \) only in shallow water. In that case one has \( k_0 h < 0.15 \), \( k^2 \approx \frac{k_0}{h} \),

\( c_g \approx c \approx \sqrt{2} \), and (6) is approximated by:

\[
\frac{v^2/\sqrt{cc}}{\sqrt{cc}/g} \approx \frac{v^2 h^2}{2h} - \frac{|\nu h|^2}{4h^2}
\]

This yields:

\[
k^2 - \frac{v^2/\sqrt{cc}}{\sqrt{cc}/g} \approx \frac{k_0}{h} \left( 1 - \frac{v^2 h}{2k_0} + \frac{|\nu h|^2}{4k_0 h} \right)
\]
The following assumptions are now made:

1) \( V^2 h << 2k_0 \), \hspace{1cm} (9a)
    implying short waves over a slowly varying bottom.

2) \( |\nabla h|^2 << 4k_0 h \), \hspace{1cm} (9b)
    implying a small bottomslope in case of shallow water.

The last two terms in (8) can then be neglected; and (5) reduces to the Helmholtz equation:

\[ \nabla^2 \phi + k^2 \phi = 0 \] \hspace{1cm} (10)

Defining

\[ n = \frac{k}{k_0} \] \hspace{1cm} (11)

(10) passes into:

\[ \nabla^2 \phi + k_0^2 n^2 \phi = 0 \] \hspace{1cm} (12)

In the theory of elasticity and electrodynamics for inhomogeneous media we meet exactly the same equation (12), valid for high frequencies, where \( n \) is the index of refraction of the medium (ref. [9]).

**Transition to amplitude and phase.**

**Singular points.**

For short waves, the potential function \( \phi \) is a rapidly oscillating function of \( x \) and \( y \), which is not easily to handle in a direct numerical approach (cf. ref. [2]).

Therefore, we turn to the amplitude \( A \) and the phase \( F \):

\[ \phi = Ae^{iF} = u + iv \] \hspace{1cm} (13)

Inserting equation (13) in equation (10) yields

\[ \frac{\nabla^2 A}{A} + k_0^2 - |\nabla F|^2 = 0 \] \hspace{1cm} (14a)

\[ \nabla \cdot (A^2 \nabla F) = 0 \] \hspace{1cm} (14b)
The Jacobi determinant of the transformation (13) amounts to:

\[
\frac{\partial (u,v)}{\partial (A,F)} = \begin{vmatrix} \frac{\partial u}{\partial A} & \frac{\partial u}{\partial F} \\ \frac{\partial v}{\partial A} & \frac{\partial v}{\partial F} \end{vmatrix} = A,
\]

(15)

so that (13) is not reversible in the neighbourhood of points where \(A = 0\). Let \(P\) be such a singular (isolated) point.

We introduce the energy streamfunction \(G\) and the energy vorticity \(w\):

\[
\begin{align*}
\frac{\partial G}{\partial y} &= A^2 \frac{\partial F}{\partial x} = u \frac{\partial v}{\partial x} - v \frac{\partial u}{\partial x} \\
\frac{\partial G}{\partial x} &= -A^2 \frac{\partial F}{\partial y} = -u \frac{\partial v}{\partial y} + v \frac{\partial u}{\partial y} \\
w &= \frac{1}{2} \frac{\partial^2 G}{\partial y^2} = \frac{\partial v}{\partial y} \frac{\partial u}{\partial y} - \frac{\partial u}{\partial x} \frac{\partial v}{\partial x}
\end{align*}
\]

(16a) (16b) (17)

Through (16), (14b) is satisfied automatically, while \(G\) must satisfy:

\[
\nabla \cdot \left( \frac{1}{A^2} \nabla G \right) = 0
\]

(18)

From (16) it follows further

\[
\nabla F \cdot \nabla G = 0
\]

(19)

so that lines of constant \(G\) (lines of average energy flow) are the orthogonals of lines of constant \(F\) (wavefronts). Cf. ref. [1].

The vorticity \(w\) is, in view of (17), a measure for the curvature of the surface \(G(x,y)\). Next, we investigate the behaviour of the functions \(G,F\) and \(w\) in the neighbourhood of the point \(P\).

At \(P\) we have \(A = 0\), so \(u = v = 0\), consequently

\[
\left. \frac{\partial G}{\partial x} \right|_P = \left. \frac{\partial G}{\partial y} \right|_P = 0
\]

(20)

Further,

\[
\left. \frac{\partial^2 G}{\partial x^2} \right|_P = \left. \frac{\partial^2 G}{\partial y^2} \right|_P = \left. \frac{\partial^2 G}{\partial x \partial y} \right|_P = 0,
\]

(21)

\[
\left. \frac{\partial^2 G}{\partial x^2} \right|_P = \left. \frac{\partial^2 G}{\partial y^2} \right|_P = \frac{\partial v}{\partial x} \frac{\partial u}{\partial y} - \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} = w
\]

(22)
so that:

\[
\begin{vmatrix}
\frac{\partial^2 G}{\partial x \partial y} & \frac{\partial^2 G}{\partial x^2} \\
\frac{\partial^2 G}{\partial y^2} & \frac{\partial^2 G}{\partial x \partial y}
\end{vmatrix}
= -w^2 \leq 0
\]  

(23)

From (20) and (23) we conclude:

At the point P, where \(A = 0\) and \(w \neq 0\), \(G\) has a (relative) extremum. For this situation the energy streamlines are shown in fig. 3 ("energy vortex").

In the nearest neighbourhood of \(P\) the streamlines are circles, because (22) is valid there. Within the loop all streamlines are closed, they don't contribute to the energy propagation of the waves. Along such a line, the phase \(F\) is a monotonous function, returning after one tour at the same point with a new value, differing from the old one by an amount of \(2\pi\). So, \(F\) is a multiple valued function and the point \(P\) is a branchpoint of a Riemann-surface of \(F\) with infinitely many sheets.

At the stagnation point \(Q\) equation (20) holds as well, but in general \(A \neq 0\), so that in view of (16):

\[
\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = 0 \quad Q
\]

(24)

This yields a homogeneous set of equations in \(u\) and \(v\), which has a non-trivial solution only if:

\[
\frac{\partial v}{\partial x} \frac{\partial u}{\partial y} - \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} = 0
\]

(25)
We conclude:

At the point Q, where G is stationary and \( \mathbf{A} \neq 0 \), F is stationary and \( w = 0 \).

It follows from the above, that on the phase contour QP the value of \( |VF| \) increases from zero (at Q) to infinite (at P), so somewhere on QP must hold (Cf. (14a)):

\[
|VF| = k
\]  

(26)

If we assume, that along the loop, on an average, (26) approximately holds, we can estimate the size of the loop:

\[
2\pi = \int \frac{2F}{ds} \, ds \approx k \int ds \geq k_0 \int ds = k_0 d,
\]

(27)

or, with \( k_0 = \frac{2\pi}{L_0} \) :

\[
d \leq \frac{L_0}{\pi}
\]

(28)

In which \( d \) = average diameter of the loop, \( L_0 \) = wavelength in deep water.

In physical optics, similar phenomena are found, for instance in the solution of Sommerfeld's diffraction problem (ref. [3]).

In our problem, the described phenomenon in general appears as a series of branchpoints, which in the limit of \( L_0 \rightarrow 0 \) passes into a caustic, a well known asymptotic phenomenon in geometrical optics.

Some perturbation methods.

In order to find an approximate solution of equation (12), we apply some methods of perturbation theory (ref. [10]). These consist in introducing an expansion in powers of a small parameter (e.g. \( 1/k_0 \)). The expansion does not in general converge, but it is an asymptotic expansion, which, if broken off after a finite number of terms, gives the solution to a good approximation if the parameter is sufficiently small: the terms first progressively decrease in magnitude, then reach a minimum and thereafter increase.

However, it frequently happens that the expansion is not valid in certain regions (transition regions or boundary layers), and special techniques must be applied to obtain uniform validity.
In ref. [7], a general survey of such asymptotic phenomena is given. In this report we investigate three methods:

1) The phase-integral (or: Liouville-Green, or: WKB) method, valid for large wavenumber \( k_0 \) (Geometrical optics can be considered as a special case).

2) Rytov's method, valid if the index of refraction \( n \) differs little from a constant.

3) A modification of Rytov's method.

**The phase-integral method.**

Let

\[ \phi = e^{k_0 \psi} \]

then \( \psi \) must satisfy:

\[ \nabla^2 \psi + k_0 \nabla \psi \cdot \nabla \psi + k_0 n^2 = 0 \]  

(30)

We expand \( \psi \) in powers of the parameter \( 1/k_0 \):

\[ \psi = \sum_{j=0}^{\infty} \frac{k_0^{-j}}{j!} \psi_j \]  

(31)

Inserting (31) in (30) and equating to zero coefficients of equal powers of \( k_0 \), yields the following series of equations in \( \psi_j \):

\[
\begin{align*}
\nabla \psi_0 \cdot \nabla \psi_0 + n^2 &= 0, \\
2 \nabla \psi_1 \cdot \nabla \psi_0 + \nabla^2 \psi_0 &= 0, \\
2 \nabla \psi_m \cdot \nabla \psi_0 + \nabla^2 \psi_{m-1} + \sum_{j=1}^{m-1} \nabla \psi_j \cdot \nabla \psi_{m-j} &= 0, \quad m > 1
\end{align*}
\]

(32)

Setting \( \psi_0 \) purely imaginary:

\[ \psi_0 = iS = i \bar{F}/k_0 \]  

(33)

one obtains from the first equation in (32) the eiconal equation of geometrical optics for the phase function \( S \):

\[ (\nabla S)^2 = n^2 \]  

(34)

(neglecting in equation (14a) the term \( \frac{V^2 A}{A} \), we get the same equation).
The remaining equations in (32) are equivalent to the transport equations of geometrical optics, from which the variation of wave amplitude can be evaluated (Cf. equ. (14b)).

The second order problem (30) is thus replaced by a series of first order problems (32). As a consequence, one initial condition will get lost: if we drop in (3) the condition \( \frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial x} \), we obtain the following initial values for the scheme (32):

\[
\psi_0 = i(x \cos \alpha + y \sin \alpha), \quad \{ \text{for } x = 0 \} \tag{35}
\]

\[\psi_m = 0, \quad m > 0\]

Let in general:

\[\psi_0 = w_0 + iv_0\]  \tag{36}

The equations of the scheme (32) can be integrated successively along the same characteristic curves (or wave rays):

\[\tau = \frac{\partial w}{\partial x} / \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} / \frac{\partial v}{\partial y}\]  \tag{37}

The process of losing an initial condition takes place through non-uniform convergence: it is well known that the described method fails at a caustic (envelope of wave rays), where adjacent rays intersect and the amplitude of the wave field becomes infinitely high. (Actually, wave rays represent an approximation of lines of average energy flow, which cannot intersect, as follows from the equation of energy conservation (14b)).

In order to construct a uniform asymptotic expansion, valid at a caustic, it is necessary to know the location and the nature of the caustic. The method consists then in introducing local boundary layer coordinates through a stretching transformation, and joining the boundary layer solution to the solution outside this region. A smooth caustic involves Airy functions, while at a cusped caustic Weber or parabolic cylinder functions are appropriate (for details, see ref. [10], [12]; ref. [4] deals with water waves).
The method of Rytov.

We introduce the non-dimensional variables

\[ \begin{align*}
    x' &= k_0 x, \\
y' &= k_0 y, \\
\n\end{align*} \tag{38} \]

Let \( \phi = e^\psi \) \tag{39} then \( \psi \) must satisfy:

\[ \nabla'^2 \psi + \nabla' \psi . \nabla' \psi + n^2 = 0 \] \tag{40}

By hypothesis, \( n^2 \) can be written as follows:

\[ n^2 = 1 + \varepsilon \chi(x,y) \tag{41} \]

with \( \varepsilon \ll 1 \), \( \chi(x,y) = O(1) \).

We expand \( \psi \) in powers of the parameter \( \varepsilon \):

\[ \psi = \sum_{j=0}^\infty \varepsilon^j \psi_j = \sum_{j=0}^\infty \psi_j \]

in which \( \psi_j = \varepsilon^j \psi_j \), \( j = 0, 1, \ldots \) \tag{42}

Inserting (42) in (40) and equating to zero coefficients of equal powers of \( \varepsilon \), yields the following series of equations in \( \psi_j \) (the symbol ' has been omitted):

\[ \begin{align*}
    \nabla'^2 \psi_0 + \nabla \psi_0 . \nabla \psi_0 + 1 &= 0, \\
    \nabla'^2 \psi_1 + 2 \nabla \psi_0 . \nabla \psi_1 + n^2 - 1 &= 0, \\
    \nabla'^2 \psi_m + 2 \nabla \psi_0 . \nabla \psi_m + \sum_{j=1}^{m-1} \nabla \psi_j . \nabla \psi_{m-j} &= 0, \quad m > 1 \tag{44} \]

From the initial condition (3), it follows that the exact solution of the first equation in (44) is given by:

\[ \psi_0 = i(x \cos \alpha + y \sin \alpha) \tag{45} \]
while for the remaining equations holds:

\[ \psi_m = \frac{\partial \psi_m}{\partial x} = 0, \quad m > 0, \quad \text{for} \quad x = 0 \]  \hspace{1cm} (46)

The non-linear problem (40) is thus replaced by a series of linear problems (44) of the same order. The validity of the method depends on the size of the perturbation term \( n^2 - 1 \), as well as on the size of the involved area.

The method has been applied to the problem of light propagation in a turbulent medium (for further references on this subject, see ref. [10]).

The modified method of Rytov.

If the perturbation term in (44) is not small, the first terms in the expansion (42) do not necessarily decrease in magnitude. Then it may happen that

\[ |\nabla \psi_1| \geq |\nabla \psi_0| \]  \hspace{1cm} (47)

We can modify the scheme (44) by including in each equation as many terms of higher order as possible, without introducing non-linear equations. The first two equations remain unchanged, the third one becomes:

\[ \nabla^2 \psi_2 + 2(\nabla \psi_0 + \nabla \psi_1) \cdot \nabla \psi_2 + \nabla \psi_1 \cdot \nabla \psi_1 = 0 \]  \hspace{1cm} (48)

This procedure leads to the scheme:

\[
\begin{align*}
\nabla^2 \psi_0 + \nabla \psi_0 \cdot \nabla \psi_0 + 1 &= 0, \\
\nabla^2 \psi_1 + 2 \nabla \psi_0 \cdot \nabla \psi_1 + n^2 - 1 &= 0, \\
\nabla^2 \psi_m + 2 \left( \sum_{j=0}^{m-1} \nabla \psi_j \right) \cdot \nabla \psi_m + \nabla \psi_{m-1} \cdot \nabla \psi_{m-1} &= 0, \quad m > 1
\end{align*}
\]  \hspace{1cm} (49)

Clearly, equations (45) and (46) remain valid. The scheme (49) does not, in general, define an asymptotic expansion, but it can be considered equally well as a method of approximation for solving equation (40).
Comparison of the described methods.

On account of the criterion of validity, the geometrical optics approximation appears appropriate in case of sufficiently short waves, propagating over a bottom with simple contours, so that the location and nature of caustics, if any, can easily be determined. The (modified) method of Rytov should be applied in case of irregular bottom contours (comparable with a turbulent medium). The modification of the method is necessary, if the index of refraction differs too much from a constant.

Taking into account the appearance of singular points in the transformations (29) and (39) (compare with (13)), we can distinguish the three following cases:

1) no branchpoints are present in the exact solution of the problem, and no crossing wave rays result from the geometrical optics approximation. Then, each of the methods can be applied, within their limits of validity.

2) no branchpoints are present, but crossing wave rays occur. The (modified) method of Rytov has in this case, within limits, a wider domain of validity, because it gives a better approximation to the lines of average energy flow.

3) branchpoints are present in the solution of the problem. The convergence of the (modified) method of Rytov should be investigated, together with a test on energy conservation.

As a preliminary conclusion, we may state that the (modified) method of Rytov has a wider field of application than the wave ray method, because diffraction effects are taken into account.
References.


